Semi-Parametric Estimation and Simulation of Actively Managed Portfolios

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Abstract

In this paper we propose a copula-based technique to recover the distribution of actively managed funds. The copula is meant to represent the dependence structure between the market return (or in general the benchmark) and the investment strategy of the asset manager. The analysis is carried out in a rational investor economy with managed funds, such as that in Merton (1981) and Berk and Green (2004). The distribution of returns on any managed fund turns out to be represented by: i) a marginal distribution representing the asset management activity; ii) a copula function describing market timing activity.

Keywords: copula, inference for margins, market timing, stock-picking

1 Introduction

Consider a managed fund Z promising to yield an excess return with respect to a benchmark index X. Assume we know the dynamics and the distribution of the return of X. The distribution of the return on the managed fund Z will depend on the investment policy implemented by the fund manager that will add up to the return on the benchmark. Call Y this component of the return. In principle, Y contains all information needed to describe the management style of the fund. The mean of Y, for every level of the return X on the benchmark, is what is universally known as α of the fund, and it is a measure of the stock picking ability of the manager. The market timing activity instead is based on the

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change to the exposure of the benchmark, and then impacts on the dependence structure between the benchmark and the fund. The most straightforward way to see this is to think of active market timing as the introduction of a portfolio of options written on the benchmark, allowing to increase and reduce leverage in periods of higher or lower return on the benchmark. This option based approach was first proposed by Merton (1981) and Henriksson and Merton (1981).

A practical issue that is of utmost relevance is how the distribution of the investment strategy and its dependence with the benchmark impact on the distribution of the returns on the managed fund. In this paper we propose a semi-parametric approach based on copula functions to disentangle the effect of active asset management policies on the probability distribution of the returns. The methodology can be applied in two opposite instances: i) if you know the marginal distribution of the investment strategy and its dependence structure with the market (or the benchmark) you can recover both the marginal distribution of the return on the fund and its dependence on the market; ii) if you know the distribution of the return on the fund and its dependence with the market, you can extract the marginal distribution of the return from the investment strategy and its dependence with the market.

The analysis is developed within a model of a rational economy with managed funds, such as that assumed in the background of the Henriksson and Merton approach and that formalized in Berk and Green (2004). In this economy, excess returns of managed fund share the same statistical properties of publicly traded assets, that is they cannot be predicted using available information. In this framework of efficient market in semi-strong form we can exploit the statistical model proposed in Cherubini, Mulinacci and Romagnoli (2011).

The structure of the paper is as follows. Section 2 introduces the market model and the motivation of the paper. In Section 3 the copula concept is applied to our problem of disentangling the part of return due to market movements (passive return) from that due to the management strategy implemented by the asset manager. In Section 4 we recover the probability distribution of the return of a managed fund on a given investment horizon. Moreover the HM copula is obtained. In Section 5 we address the opposite problem of that of the Section 4: we assume we are given the time series of returns on a fund and that of the market and we back out the record of the asset manager that is running that fund. In Section 6 we show how to estimate the return of a managed fund. More precisely, we estimate via IFM technique the parameters of the copula and of the distribution of Y. In Section 7 we present an empirical application. Section 8 concludes and Section 9 contains the derivation of the HM copula.

2 Market Model and Motivation

Whenever we evaluate the performance of a managed fund we do so on the background of a model of the capital market. In this paper we use a standard model with rational investors and efficient markets. The rational market model with mutual funds that we have in mind was presented in Berk and Green

(2004). In that model, competitive equilibrium ensures that capital flows to efficient asset managers up to the point where expected risk adjusted excess returns equal zero across all possible investments, no matter whether passive and managed.

The starting point is a linear decomposition of the return Z of a managed portfolio in a passive component X and another one due to management decisions Y

$$Z = X + Y \tag{1}$$

The return component Y contains all information needed to characterize and measure the effects of the asset management strategy. In the standard literature on performance analysis, the asset management strategy is evaluated carrying out a regression. The most famous cases were proposed by Treynor and Matsuy (1966) and Henriksson and Merton (1981).

A problem with this kind of approach is that the regression representation is based on quite restrictive assumptions concerning both the kind of strategy followed by the manager and the institutional environment in which the asset manager operates, such as the kind of compensation scheme. So, for example, the Henriksson-Merton model above is based on the assumption that the asset manager switches the asset allocation between the risk-free investment and the risk-free rate or vice versa whenever she expects a rate of return on the asset higher than the risk-free rate (or vice versa). This is equivalent to a portfolio insurance strategy which uses put options. Of course, even though this representation is sufficient to induce non linearity in the relationship between the rate of return on the managed fund and the market, it remains too simple from the point of view of the management process, that can be path-dependent and may be inspired by different strategies in different periods. For these reasons, non-parametric tests (Jiang, 2003) and graphical (chartist) analysis (Leigh, Paz and Purvis, 2002) have been proposed to assess the timing activity. But beyond tests, one remains with the need to estimate and simulate the joint returns of a managed fund and its reference benchmark, and fully non parametric techniques are not very well suited to accomplish this.

A semi-parametric approach may strike a balance between the need to represent in a general way positive association between the returns on the market and the returns accrued to that by the management activity and a non-parametric representation of the marginal distribution of the managed fund. This semi-parametric approach (in the spirit of Chen and Fan, 2006) applies copula functions to model dependence among the variables and non parametric analysis to model their marginal distributions. As for copula functions we refer the interest reader to Nelsen (2006) for details and we only remind here that the copula technique allows to write every joint distribution as a function of marginal distributions. Then, we can represent the joint distribution of X and Y, say $\Pr(X \leq a, Y \leq b)$, with $a, b \in \Re$ as a function of $F_X(a) \equiv \Pr(X \leq a)$ and $F_Y(a) \equiv \Pr(Y \leq b)$. More formally, there exists a function $C_{X,Y}(u, v)$ such that

$$Pr(X \leq a, Y \leq b) = C_{X,Y}(F_X(a), F_Y(b)) \tag{2}$$

Conversely, given two distribution functions F_X and F_Y and a suitable bivariate function $C_{X,Y}$ we may build joint distribution for the returns. This one to one relationship between joint distributions and copula functions is known as Sklar theorem.

3 Convolution-Based Copulas and Performance Management

In this paper we present a new way of measuring the performance of an asset management strategy, with respect to a market index. The main idea is to use copulas to separate the specification of the marginal distribution of the asset management strategy, F_Y and its dependence structure with the market return, whose distribution is F_X. Dependence is represented by the copula function $C_{X,Y}$. The evaluation should result into a measure of the return on the asset management strategy and its co-movement with the market return. Differently from standard approaches, the result should be the entire probability distribution, rather than a description of it by its moments.

Let us notice the problem is more involved than a mere application of the copula function tool to returns X and Y. What makes the problem hard is that we are actually interested in specifying the bivariate dependence relationships of a set of three variables, of which one is defined as the sum of the other two. We are in fact investigating the relationship among: i) the return on the market; ii) the return on the investment strategy; iii) the return on the fund, which is defined as the sum of the two. Unfortunately, linearity is by no means a feature that simplifies the analysis, and actually is what makes the issue more involved. In fact, the distribution of the return on the managed fund is the convolution of the return on the market and the strategy, and the copula representing dependence between the market and the managed fund must take into account this feature. This specification problem problem was solved in Cherubini, Mulinacci and Romagnoli (2011a,b). We report here the proposition, referring to the paper for full proof.

Proposition 3.1. *Let X e Y be two real-valued random variables on the same probability space $(\Omega, \Im, \mathbb{P})$ with corresponding copula $C_{X,Y}$ and continuous marginals F_X and F_Y. With $D_1C_{X,Y}(u,v)$ we denote $\frac{\partial C_{X,Y}(u,v)}{\partial u}$. Then,*

$$F_{X+Y}(z) = \int_0^1 D_1C_{X,Y}\left(w, F_Y(z - F_X^{-1}(w))\right) dw \tag{3}$$

and

$$C_{X,X+Y}(u,v) = \int_0^u D_1C_{X,Y}\left(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))\right) dw. \tag{4}$$

Notice that equation (3) is a simple extension of the concept of convolution to the case in which the dependence between the variables is represented by a given copula function C. For this reason it is denoted *C-convolution*. As for equation (4), it is obtained by imposing the condition that the marginal distribution of the $X+Y$ variable be the convolution of X and Y: for this reason we could call this kind of copulas convolution-based.

4 The Probability Distribution of the Fund Returns

Using the convolution-based approach above, we address the following question. Assume a new fund management program is launched to the market. We would like to estimate the probability distribution of its return on a given investment horizon. Given the analysis above, in order to accomplish this we need two pieces of information:

1. **Asset Management Skill**. This is represented by the distribution of the return on the investment strategy.

2. **Market Timing Skill.** This is represented by a copula function linking the return from the investment strategy

Of course a problem is that if the product has to be launched, this information is not directly available. It is part of the fund analyst to make realistic hypothesis about this information or to use reasonable proxies for such information. Typically, an idea of the investment strategy can be extracted from

- Statements concerning the aims and the investment process issued by the corporate entity issuing the fund in official reports and in road shows to the investors.

- The record of the asset manager hired to run the fund in prior assignments. By record we mean actually the probability distribution of her investment strategy and its co-movement with the market.

- Automatic trading strategies. One could conceive very simplified strategies, such as the portfolio insurance strategy used in the Henriksson and Merton model, or more involved strategies based on simulation.

In the end, the analyst should come up with a probability distribution representing the investment strategy and a copula function representing market timing.

4.1 Passive Fund Management

If the fund management style is passive, the target is to replicate the market X as close as possible, and the asset management strategy should collapse to withe noise. So, passive strategies are consistent with the product copula $C_{X,Y}(u,v) = uv$ corresponding to independence, and we have

$$Pr(X \leq a, Y \leq b) = F_X(a)F_Y(b) \tag{5}$$

Using the convolution-based copula technique one could immediately provide the specific representation of the distribution of the return on the fund, and this will boil down to the standard convolution

$$F_{X+Y}(z) = \int_0^1 F_Y(z - F_X^{-1}(w))dw. \tag{6}$$

Furthermore, the dependence structure between the fund and the market will have the specific shape

$$C_{X,X+Y}(u,v) = \int_0^u F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))dw. \tag{7}$$

So, a passive management strategy could be subject to test by testing the shape of the copula function linking the market and the fund.

4.2 HM-copula

As a typical example of active fund management, consider the market timing strategy based on portfolio insurance as specified in Henriksson and Merton (1981). As it is well known, this model leads to a specification of the stratefy Y as

$$Y = \alpha + \gamma \max(-X, 0) + \epsilon$$

where ϵ is a zero mean disturbance, and the risk-free rate is assumed to be equal to zero for the sake of simplicity. As for the γ parameter, this is proportional to the concordance between the forecast of the manager and actual movements of the market.

Even though we extend a friendly advice to evaluate the distribution of the return on the fund by simulation, one could also derive the specific copula $C_{X,Y}$ representing the market timing activity in this model. We report here the copula, that we called *HM copula* for obvious reasons, referring the reader to the appendix for the details of the derivation

Now, the copula between X and Y is given by

$$C_{X,Y}(u,v) = G_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))\mathbf{1}_{\{u>F_X(0)\}} + H_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))\mathbf{1}_{\{u\leq F_X(0)\}}$$

where

$$G_{X,Y}(x,y) = F_X(x)F_\epsilon(y-\alpha) - \int_{-\infty}^{y-\alpha} F_X(\frac{e-y+\alpha}{\gamma})F_\epsilon(de)$$

and

$$H_{X,Y}(x,y) = F_X(x)F_\epsilon(\gamma x + y - \alpha) - \int_{-\infty}^{\gamma x + y - \alpha} F_X(\frac{e-y+\alpha}{\gamma})F_\epsilon(de).$$

The distribution of the fund is finally recovered as

$$F_Z(z) = \int_0^{F_X(0)} F_\epsilon(z + (\gamma - 1)F_X^{-1}(w) - \alpha)dw + \int_{F_X(0)}^1 F_\epsilon(z - F_X^{-1}(w) - \alpha)dw$$

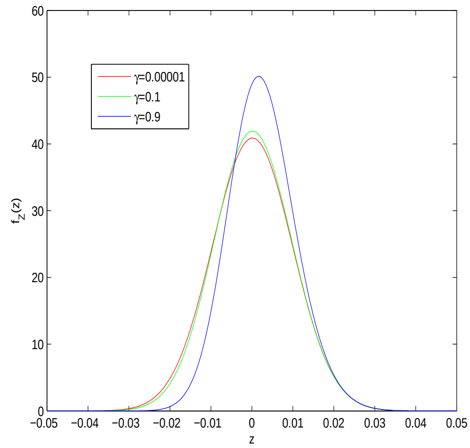

Figure 1: Probability density function of $Z = X + Y$ for different levels of the γ parameter.

However, the explicit derivation of the copula function enables to compute explicitly the shape of the density function and the dependence between the market and the investment strategy. In fig. 1 we report the shape of the probability density function for different levels of the γ parameter. As expected, the higher the parameter, the more effective the portfolio insurance strategy, and the lower the left tail of the distribution.

As for the dependence structure between investment strategy and the market, in fig. 2 we report the value of the corresponding Kendall τ statistic, as a function of different values of the γ parameter. Notice that the γ parameter in the Merton model is a measure of concordance between forecast of the asset manager and market movements. So, the figure shows how this concordance measure translates into a concordance measure (the Kendall τ) between the return on the investment strategy and the market. Since the market timing strategy implies a position in *protective* put options, it is not surprising that association is negative. The more effectively an asset manager is able to forecast future market movements the more effective the protective put position.

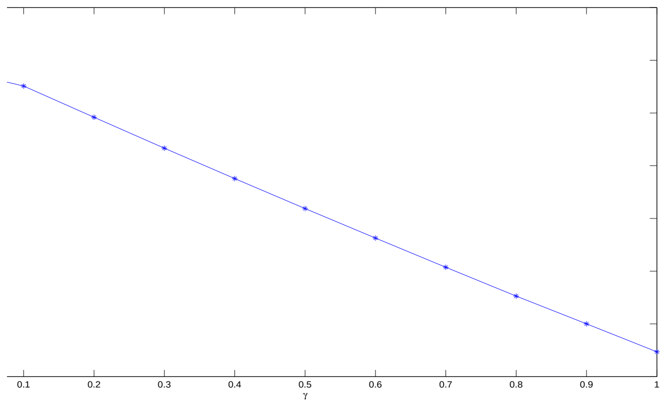

Figure 2: Kendall τ statistic as a function of different values of the γ parameter.

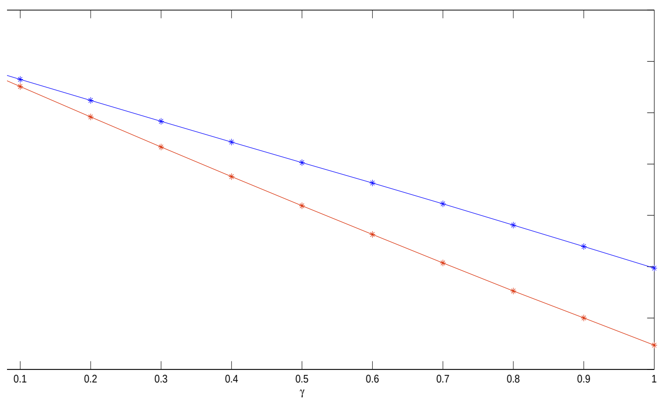

Figure 3: Kendall τ statistic as a function of different values of the γ parameter when σ_ϵ increases by 40 percent (blue line)

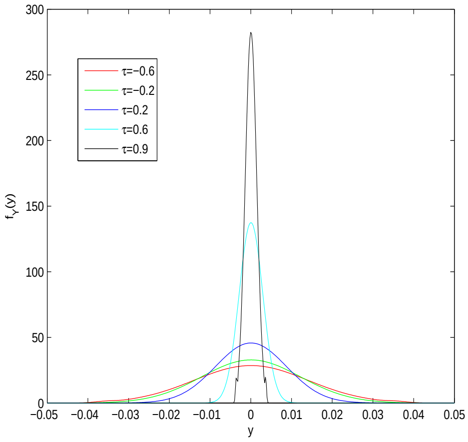

Figure 4: Probability density function of the management strategy Y consistent with selected dependence structures between Z and X: C_{ZX} is a Gaussian copula.

5 The distribution of the return on the investment strategy

In this section we address the opposite problem to that of the previous section. We assume we are given the time series of returns on a fund and that of the market, and we want to back out the record of the asset manager that is running that fund. As in the section before, the record is made up by two components: i) a probability distribution representing the active management skill; ii) a copula function representing market timing skills.

The solution to this problem can be again found in the convolution-based copula concept reported in section 2. Trivially, the problem is to find the probability distribution of $Y = Z - X$ given the dependence structure between Z and X. We then have:

$$F_Y(t) = \int_0^1 D_1 C_{Z,-X}\left(w, F_{-X}(t - F_Z^{-1}(w))\right) dw. \tag{8}$$

It is well known that $C_{Z,-X}$ can be recovered directly from $C_{Z,X}$ using the invariance relationship $C_{Z,-X}(u, v) = u - C_{Z,X}(u, 1 - v)$. Therefore, the return on the asset management strategy can be recovered as

$$F_Y(t) = 1 - \int_0^1 D_1 C_{Z,X}\left(w, F_X(F_Z^{-1}(w) - t)\right) dw.$$

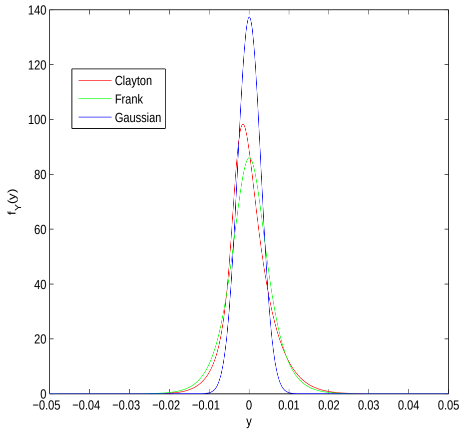

Figure 5: Probability density function of Y for the Gaussian copula, the Clayton copula and the Frank copula, with parameters consistent with the Kendall τ equal to 0.6.

Furthermore, the market timing activity of the asset manager will be described by the copula function $C_{X,Y}$, that is

$$C_{X,Y}(u,v) = u - \int_0^u D_1 C_{Z,X}\left(w, F_X(F_Z^{-1}(w) - F_Y^{-1}(v))\right) dw \tag{9}$$

In figure 4 we report the density of the management strategy Y consistent with selected dependence structures between Z and X: we use Gaussian copulas with increasing dependence (Kendall $\tau = -0.6, ..., 0.9$). In figure 5 we compare the density of Y for the Gaussian copula, the Clayton copula and the Frank copula, with parameters consistent with the same figure of Kendall τ equal to 0.6.

5.1 Market neutral investment (hedge funds?)

A very interesting specific case of the analysis above applies to so called market neutral funds. It is well known that hedge funds should by definition be endowed with this feature. Market neutrality has been recently questioned on empirical grounds, and for this reason we use a question mark in the title. Anyway, if we were facing a genuine market neutral fund management policy, we could use the technique above to measure the skill of the market neutral asset manager. As a matter of fact, one would simply have to set $C_{Z,X} = uv$ and we would simply obtain

$$F_Y(t) = 1 - \int_0^1 F_X(F_Z^{-1}(w) - t)dw.$$

for the hedge fund strategy and

$$C_{X,Y}(u,v) = u - \int_0^u F_X(F_Z^{-1}(w) - t)dw.$$

for the market timing activity.
It is also quite straightforward to describe the distribution of the so called *core-satellite* strategies, in which a small investment in market neutral funds is added to a position in mutual funds to add some α while decreasing correlation in the portfolio. Namely, if a hedge fund H is added to the fund Z, the distribution of the return would be given by

$$F_{\overline{Z}}(t) = \int_0^1 F_H(t - F_Z^{-1}(w)))dw. \tag{10}$$

where $\overline{Z}$ denotes the *core-satellite* portfolio. The gain in diversification can be gauged measuring.

$$C_{Z,H}(u,v) = \int_0^u F_H(F_{\overline{Z}}^{-1}(v) - F_Z^{-1}(w)))dw. \tag{11}$$

6 Estimation

6.1 Maximum likelihood estimation

In the time series framework we need to extend the C-convolution operator to the case of the conditional copulas (Patton, 2006). Let the conditional distribution $(X_t, Y_t)|\mathcal{F}_{t-1}$ be parametrized by

$$H_t(x_t, y_t; \varphi) = C_t(F_{X_t|\mathcal{F}_{t-1}}(x_t; \psi_x), F_{Y_t|\mathcal{F}_{t-1}}(y_t; \psi_y); \theta),$$

where C_t is the conditional copula, $F_{X_t|\mathcal{F}_{t-1}}(\cdot; \psi_x)$ and $F_{Y_t|\mathcal{F}_{t-1}}(\cdot; \psi_y)$ are the conditional marginal distributions of X_t and Y_t, θ, ψ_x and ψ_y are vectors of parameters and finally $\varphi = (\psi_x, \psi_y, \theta)$. Let Ψ_x, Ψ_y and Θ be the parameter spaces. The log-likelihood function may be obtained from the joint density of $(X_t, Y_t)|\mathcal{F}_{t-1}$ which is

$$h_t(x_t, y_t; \varphi) = c_t((F_{X_t|\mathcal{F}_{t-1}}(x_t; \psi_x), F_{Y_t|\mathcal{F}_{t-1}}(y_t; \psi_y); \theta))f_{X_t|\mathcal{F}_{t-1}}(x_t; \psi_x)f_{Y_t|\mathcal{F}_{t-1}}(y_t; \psi_y),$$

where we denote by c_t the copula density and by $f_{X_t|\mathcal{F}_{t-1}}$ and $f_{Y_t|\mathcal{F}_{t-1}}$ the densities of marginal distributions. Then

$$\ell(\varphi) = \sum_{t=1}^{T} \log h_t(x_t, y_t; \varphi) =$$

$$= \sum_{t=1}^{T} \log c_t((F_{X_t|\mathcal{F}_{t-1}}(x_t;\psi_x), F_{Y_t|\mathcal{F}_{t-1}}(y_t;\psi_y);\theta)) +$$

$$+ \sum_{t=1}^{T} f_{X_t|\mathcal{F}_{t-1}}(x_t;\psi_x) + \sum_{t=1}^{T} f_{Y_t|\mathcal{F}_{t-1}}(y_t;\psi_y).$$

We notice that the log-likelihood may be decomposed into the sum of three terms: $\sum_{t=1}^{T} f_{X_t|\mathcal{F}_{t-1}}(x_t;\psi_x)$ which contains the parameter associated with the first margin only, $\sum_{t=1}^{T} f_{Y_t|\mathcal{F}_{t-1}}(y_t;\psi_y)$ which contains the parameter associated with the second margin only and $\sum_{t=1}^{T} \log c_t((F_{X_t|\mathcal{F}_{t-1}}(x_t;\psi_x), F_{Y_t|\mathcal{F}_{t-1}}(y_t;\psi_y);\theta))$ which also contains the parameter associated with the copula function. Then

$$\ell(\varphi) = \ell_x(\psi_x) + \ell_y(\psi_y) + \ell_c(\psi_x, \psi_y, \theta).$$

Therefore, if we denote by $\hat{\varphi}$ the (two-stage) maximum likelihood estimator its components are given by

$$\hat{\psi}_x = \arg\max_{\Psi_x} \ell_x(\psi_x),$$
$$\hat{\psi}_y = \arg\max_{\Psi_y} \ell_y(\psi_y),$$
$$\hat{\theta} = \arg\max_{\Theta} \ell_c(\hat{\psi}_x, \hat{\psi}_y, \theta).$$

7 Empirical application

In this section we present an empirical application of our model based on data on Italian mutual funds provided by Prometeia s.p.a, an Italian consulting company. We use three different bivariate time series representing the main categories of mutual funds recognized under the Italian regulation. The first time series is the benchmark X and the second one is the excess return Y due to the ability of the manager. They are in turn:

- "Bilanciato" (BIL), referring to investment equally split between equity and bonds;
- "Azionario Multidivisa" (AzMul), representing investment in multi-currency equity;
- "Corporate IG" (CIG), representing investment in corporate bonds.

The data runs from December 2000 to September 2009 and we will analyse the log-returns. Table 1 contains the summary statistics.

7.1 The models for the marginal distributions

As for the marginal distributions we need to take into account that almost all economic time series present some temporal dependence. So, we cannot model them by an invariant marginal distribution, i.e., $F_{X_t|\mathcal{F}_{t-1}} = F_X$ and $F_{Y_t|\mathcal{F}_{t-1}} = F_Y$ for all t. Our model for marginal distributions is, in fact, the following t-Garch(1,1) model (see Bollerslev (1987) and Patton (2005))

$$\begin{aligned} X_t &= \alpha_x + \epsilon_t \\ \sigma^2_{x,t} &= \gamma_{x,0} + \gamma_{x,1} X^2_{t-1} + \gamma_{x,2}\sigma^2_{x,t-1} \\ h_t \epsilon_t|\mathcal{F}_{t-1} &\overset{iid}{\sim} t_{(\nu_x)}, \end{aligned}$$

where $h_t = \frac{1}{\sigma_{x,t}}\sqrt{\frac{\nu_x}{\nu_x - 2}}$

$$\begin{aligned} Y_t &= \alpha_y + \eta_t \\ \sigma^2_{y,t} &= \gamma_{y,0} + \gamma_{y,1} Y^2_{t-1} + \gamma_{y,2}\sigma^2_{y,t-1} \\ k_t \eta_t|\mathcal{F}_{t-1} &\overset{iid}{\sim} t_{(\nu_y)}, \end{aligned}$$

where $k_t = \frac{1}{\sigma_{y,t}}\sqrt{\frac{\nu_y}{\nu_y - 2}}$. Here we have $\psi_x = (\alpha_x, \gamma_{x,0}, \gamma_{x,1}, \gamma_{x,2}, \nu_x)$ and $\psi_y = (\alpha_y, \gamma_{y,0}, \gamma_{y,1}, \gamma_{y,2}, \nu_y)$. In this model the conditional means α_x and α_y are constants while the conditional variances $\sigma^2_{x,t}$ and $\sigma^2_{y,t}$ are given by a Garch(1,1) model (Engle (1982) and Bollerslev (1986)). The parameters ν_x and ν_y provide the degrees of freedom of the conditional Student's t innovations. For the purpose of comparison we have also estimated two Normal Garch(1,1) for each data set. Tables 2, 3 and 4 provide the results of the evaluation of the goodness of fit of the models of the marginal distributions. If the estimated marginal distribution is adequate the transformed r.vs. $U_t = F_{X_t|\mathcal{F}t-1}(X_t; \hat{\psi}_x)$ and $V_t = F_{Y_t|\mathcal{F}t-1}(Y_t; \hat{\psi}_y)$, for $t = 1, .., T$, would be i.i.d $U(0, 1)$. We test such an hypothesis in two stages as in Diebold et al. (1998) and Patton (2006): firstly testing the i.i.d. hypothesis via LM test by regressing $(U_t - \bar{U}_T)^d$ and $(V_t - \bar{V}_T)^d$ on 20 lags of both variables for $d = 1, 2, 3, 4$ and secondly testing the $U(0, 1)$ hypothesis via the classical Kolmogorov-Smirnov test. The results show that for each data set the benchmark and the excess return cannot be modelled by an invariant marginal distribution: both the Gaussian and the Student's t distribution are rejected. Our t-Grach(1,1) model is the best choice for the marginal behavior for both the time series in each data set. The p-value of the KS test is very high. The parameter estimates and relative standard errors for the marginal distribution models are reported in table 5. We notice that all the conditional variance parameters are significant both for the benchmark and for the excess return. The conditional mean of the benchmark is significantly different from zero in each data set whereas the conditional mean of the excess return (stock picking ability) is never significant. Figures 6, 7 and 8 show the log-returns, the squared log-returns and the estimated conditional volatility for each data set.

7.2 Estimation of the conditional copula parameter

Since models for the marginal distributions of the benchmark and of the excess return seems to be adequate, we proceed to the modelling the dependence structure by using a number of copula families. In this paper we assume that the conditional copula C_t is invariant, that is $C_t = C$ for all t. For the purpose of comparison we will estimate the following copulas: Clayton (only for the data set BIL), Frank, Gaussian and Student's t. Table 6 reports the estimated parameters with the relative standard errors. We notice that only the parameter of the Clayton copula is significantly different from zero in the case of the data set BIL whereas in the other two cases all the parameters are significant. We also notice that only the corporate mutual fund CIG is essentially gaussian, since the degrees of freedom of the Student's t copula are quite high. The investment in multi-currency equity (AzMul) shows tail dependence (due to low degrees of freedom of the Student's t copula). As for the sign of the dependence, only in the case of mutual funds equally invested in both equity and bonds we find a positive, albeit weak, dependence. In the other cases, we find evidence of negative dependence, consistent with some portfolio insurance behavior like in Henriksson and Merton (1981).

To test the goodness of fit we use a procedure due to Diebold et al. (1999) and Patton (2006), similar to those already used for the marginal distributions. We construct the following two time series $(U_t, D_1C(U_t, V_t))$ and $(U_t, D_1C(U_t, V_t))$ for $t = 1, ..., T$. If the copula C is well specified both the time series are i.i.d. $U(0, 1)$. Tables 7, 8 and 9 contain the results relative to the LM test of independence and to the KS test. The Clayton copula in the case of mutual funds equally invested in both equity and bonds (BIL) is adequate. In the case of investment in multi-currency equity (AzMul) the Student's t copula seems to be preferable to the other even if the Frank copula and the Gaussian copula also pass the test. The most problematic situation we find in the case of corporate mutual funds (CIG) where none of the selected copulas pass the test. Further investigation would probably be necessary.

7.3 Estimation of the conditional C-convolution

We are now able to get the conditional C-convolution which gives us the conditional distribution of the managed fund return $Z_t|\mathcal{F}_{t-1} = X_t|\mathcal{F}_{t-1} + Y_t|\mathcal{F}_{t-1}$. Firstly, we derive the marginal distributions of $X_t|\mathcal{F}_{t-1}$ and $Y_t|\mathcal{F}_{t-1}$. from our t-Garch(1,1) models it is not difficult to show that

$$F_{X_t|\mathcal{F}_{t-1}}(x_t) = t_{(\nu_x)}\left((x_t - \alpha_x)h_t\right),$$
$$F_{Y_t|\mathcal{F}_{t-1}}(y_t) = t_{(\nu_y)}\left((y_t - \alpha_y)k_t\right).$$

Now

$$F_{Z_t|\mathcal{F}_{t-1}} = F_{X_t|\mathcal{F}_{t-1}} \overset{C}{*} F_{Y_t|\mathcal{F}_{t-1}} =$$
$$= \int_0^1 D_1C\left(w, F_{Y_t|\mathcal{F}_{t-1}}\left(z_t - F^{-1}_{X_t|\mathcal{F}_{t-1}}(w)\right)\right) dw.$$

Since $F^{-1}_{X_t|\mathcal{F}_{t-1}}(w) = h_t^{-1} t^{-1}_{(\nu_x)}(w) + \alpha_x$ we get

$$F_{Z_t|\mathcal{F}_{t-1}}(z_t) = \\ = \int_0^1 D_1 C\left(w, t_{(\nu_y)}\left(\left(z_t - \alpha_x - \alpha_y - h_t^{-1} t^{-1}_{(\nu_x)}(w)\right) k_t\right)\right) dw.$$

The estimated conditional distribution of Z_t is

$$\hat{F}_{Z_t|\mathcal{F}_{t-1}}(z_t; \hat{\varphi}) = \\ = \int_0^1 D_1 C\left(w, t_{(\hat{\nu}_y)}\left(\left(z_t - \hat{\alpha}_x - \hat{\alpha}_y - \hat{h}_t^{-1} t^{-1}_{(\hat{\nu}_x)}(w)\right) \hat{k}_t\right); \hat{\theta}\right) dw.$$

An interesting question is to compare the distribution of $Z_t|\mathcal{F}_{t-1}$ when the conditional variance of $X_t|\mathcal{F}_{t-1}$ and $Y_t|\mathcal{F}_{t-1}$ is very different. Figures **??**, **??** and **??** display, for each data set, the estimated distributions $\hat{F}_{Z_t|\mathcal{F}_{t-1}}(\cdot; \hat{\varphi})$ when two two opposite situations happens: the market volatility, measured by $\hat{h}_t$, is maximum or minimum. For the purpose of comparison we also report the invariant Gaussian distribution.

Another interesting problem is to study the estimated conditional VaR over time and its dependence on the conditional volatility. We can write

$$VaR_{Y_t|\mathcal{F}_{t-1}}(p) = \hat{k}_t^{-1} t^{-1}_{(\hat{\nu}_y)}(p) + \hat{\alpha}_y, \quad t = 1, ..., T,$$

while the conditional VaR of Z_t may be obtained numerically

$$VaR_{Z_t|\mathcal{F}_{t-1}}(p) = \hat{F}^{-1}_{Z_t|\mathcal{F}_{t-1}}(p; \hat{\varphi}), \quad t = 1, ..., T.$$

Figures 9, 10 and 11 report the conditional VaR of Y_t and of Z_t over time for each data set. We notice that the proximity between $VaR_{Y_t|\mathcal{F}_{t-1}}$ and $VaR_{Z_t|\mathcal{F}_{t-1}}$ in the case of corporate mutual fund. Finally, tables 10, 11 and 12 compare the quantiles for different percentage levels in the same two scenarios.

8 Conclusion

In this paper we propose a flexible model to specify and estimate the contribution of an investment strategy to the return of a managed fund. More specifically, the value added by the asset manager skill is represented by a probability distribution, while her market timing ability is portrayed by the dependence structure between this distribution and the market, represented by a copula function. We show how to estimate: i) the probability distribution of a managed fund, given the asset management ability (a marginal distribution) and the market timing ability (a copula function) and ii) how to extract the asset management ability given the returns on the managed fund and those on the market.

For the sake of illustration we provide an application of the method to four

cases of Italian mutual funds. We use marginal Student t and non-central t distributions to represent asset management strategy returns from stock picking and elliptical copulas to represent market timing. In all cases we find evidence of market timing activity. While in one case we find that the joint distribution is gaussian, in the other cases we find evidence of tail dependence of the market timing activity. In some cases, this turns out in higher leptokurtosis of the managed fund returns with respect to returns on the strategy. In one other case, we find that the leptokurtosis of the managed fund is actually decreased: interestingly, this result corresponds to a case in which there is strong evidence of negative dependence between the market and the strategy corresponding to a portfolio insurance activity of the kind portrayed in the classical Henriksson and Merton (1981) framework.
Future research will include extension of the model to cases in which liquidity issues make more involved the dynamic structure of dependence between managed returns and the market and applications to several classes of investment, included multi-factor models.

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9 Appendix: Derivation of the HM-copula

In this appendix we derive the copula that links X and Y in the Henriksson-Merton model presented in Section 4. We study the case where $\gamma > 0$. Suppose that marginal distributions F_X and F_ϵ are assigned. To construct this copula function we need the joint distribution between X and Y, say $F_{X,Y}$. We have

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x, \alpha + \gamma \max\{K, -X\} + \epsilon \leq y) =$$

$$\int_{-\infty}^{+\infty} \mathbb{P}(X \leq x, \alpha + \gamma \max\{K, -X\} + e \leq y | \epsilon = e) F_\epsilon(de) =$$

$$\int_{-\infty}^{+\infty} \mathbb{P}(X \leq x, \max\{K, -X\} \leq \frac{y - e - \alpha}{\gamma}) F_\epsilon(de) =$$

$$\int_{-\infty}^{+\infty} \Big[\mathbb{P}(X \le x, X \ge \frac{e-y+\alpha}{\gamma}, X < -K) + \mathbb{P}(X \le x, K \ge \frac{y-e-\alpha}{\gamma}, X \ge -K)\Big] F_\epsilon(de).$$

We have to distinguish two cases: $x > -K$ and $x \le -K$. In the first case

$$\mathbb{P}(X \le x, X \ge \frac{e-y+\alpha}{\gamma}, X < -K) + \mathbb{P}(X \le x, K \ge \frac{y-e-\alpha}{\gamma}, X \ge -K) =$$

$$\mathbb{P}(X \le -K, X \ge \frac{e-y+\alpha}{\gamma}) + \mathbb{P}(-K \le X \le x, e \ge y - \alpha - \gamma K),$$

therefore, if we indicate by $G_{X,Y}$ the joint distribution in this case

$$G_{X,Y}(x,y) =$$

$$\int_{-\infty}^{y-\alpha-\gamma K} \Big[\mathbb{P}(X \le -K, X \ge \frac{e-y+\alpha}{\gamma}) + \mathbb{P}(-K \le X \le x, e \ge y-\alpha-\gamma K)\Big] F_\epsilon(de) =$$

$$F_X(x) F_\epsilon(y - \alpha - \gamma K) - \int_{-\infty}^{y-\alpha-\gamma K} F_X(\frac{e-y+\alpha}{\gamma}) F_\epsilon(de).$$

in the case where $x \le -K$ we have

$$\mathbb{P}(X \le x, X \ge \frac{e-y+\alpha}{\gamma}, X < -K) + \mathbb{P}(X \le x, K \ge \frac{y-e-\alpha}{\gamma}, X \ge -K) =$$

$$\mathbb{P}(X \le x, X \ge \frac{e-y+\alpha}{\gamma}),$$

then, the only relevant case is when $\frac{e-y+\alpha}{\gamma} \le x$ and therefore if we indicate by $H_{X,Y}$ the joint distribution in this case

$$H_{X,Y}(x,y) = \int_{-\infty}^{\gamma x + y - \alpha} \mathbb{P}(X \le x, X \ge \frac{e-y+\alpha}{\gamma}) F_\epsilon(de) =$$

$$F_X(x) F_\epsilon(\gamma x + y - \alpha) - \int_{-\infty}^{\gamma x + y - \alpha} F_X(\frac{e-y+\alpha}{\gamma}) F_\epsilon(de).$$

Briefly, we can conclude that

$$F_{X,Y}(x,y) = G_{X,Y}(x,y) \mathbf{1}_{\{x > -K\}} + H_{X,Y}(x,y) \mathbf{1}_{\{x \le -K\}}$$

Now, the copula between X and Y is given by

$$C_{X,Y}(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) =$$

$$G_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \mathbf{1}_{\{u > F_X(-K)\}} + H_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \mathbf{1}_{\{u \le F_X(-K)\}},$$

or more explicitly

$$C_{X,Y}(u,v) =$$

$$\Big[u F_\epsilon(F_Y^{-1}(v) - \alpha - \gamma K) - \int_{-\infty}^{F_Y^{-1}(v) - \alpha - \gamma K} F_X(\frac{e - F_Y^{-1}(v) + \alpha}{\gamma}) F_\epsilon(de)\Big] \mathbf{1}_{\{u > F_X(-K)\}} +$$

$$+\Big[uF_\epsilon(\gamma F_X^{-1}(u)+F_Y^{-1}(v)-\alpha)-\int_{-\infty}^{\gamma F_X^{-1}(u)+F_Y^{-1}(v)-\alpha} F_X(\frac{e-F_Y^{-1}(v)+\alpha}{\gamma})F_\epsilon(de)\Big]\mathbf{1}_{\{u\leq F_X(-K)\}}.$$

We call this copula function the Henriksson-Merton copula. It remains to determine the marginal distribution F_Y. We have

$$F_Y(y)=\lim_{x\to+\infty}F_{X,Y}(x,y)=\lim_{x\to+\infty}G_{X,Y}(x,y)=$$

$$F_\epsilon(y-\alpha-\gamma K)-\int_{-\infty}^{y-\alpha-\gamma K}F_X(\frac{e-y+\alpha}{\gamma})F_\epsilon(de).$$

Moreover, we determine its density function by derivation

$$f_Y(y)=f_\epsilon(y-\alpha-\gamma K)(1-F_X(-K))+\int_{-\infty}^{y-\alpha-\gamma K}f_X(\frac{e-y+\alpha}{\gamma})F_\epsilon(de).$$

Finally, we can find the distribution of $Z=\beta X+Y$ by using a generalization of (3). Since we assume that $\beta>0$

$$F_Z(z)=\int_0^1 D_1C_{X,Y}\left(w,F_Y(z-\beta F_X^{-1}(w)\right),$$

where

$$D_1C_{X,Y}(u,v)=$$

$$F_\epsilon(F_Y^{-1}(v)-\alpha-\gamma K)\mathbf{1}_{\{u>F_X(-K)\}}+F_\epsilon(\gamma F_X^{-1}(u)+F_Y^{-1}(v)-\alpha)\mathbf{1}_{\{u\leq F_X(-K)\}}.$$

So, more explicitly

$$F_Z(z)=$$

$$\int_0^{F_X(-K)}F_\epsilon(z+(\gamma-\beta)F_X^{-1}(w)-\alpha)dw+\int_{F_X(-K)}^1 F_\epsilon(z-\beta F_X^{-1}(w)-\alpha-\gamma K)dw.$$

As a consequence we can compute the density of Z, which is

$$f_Z(z)=$$

$$\int_0^{F_X(-K)}f_\epsilon(z+(\gamma-\beta)F_X^{-1}(w)-\alpha)dw+\int_{F_X(-K)}^1 f_\epsilon(z-\beta F_X^{-1}(w)-\alpha-\gamma K)dw$$

where f_X and f_ϵ are the probability density functions of X and ϵ respectively.

Tables

	BIL	
	Benchmark	Excess return
Mean	0.0001	0.0001
Standard deviation	0.0069	0.0037
Skewness	-0.2234	0.2368
Kurtosis	10.4972	14.2359
	AzMul	
	Benchmark	Excess return
Mean	0.0003	-0.0001
Standard deviation	0.0146	0.0090
Skewness	-0.1717	0.0845
Kurtosis	9.8433	9.7496
	CIG	
	Benchmark	Excess return
Mean	0.0000	0.0001
Standard deviation	0.0017	0.0044
Skewness	-0.1910	-0.7115
Kurtosis	4.1144	9.7726

Table 1: Summary Statistics.

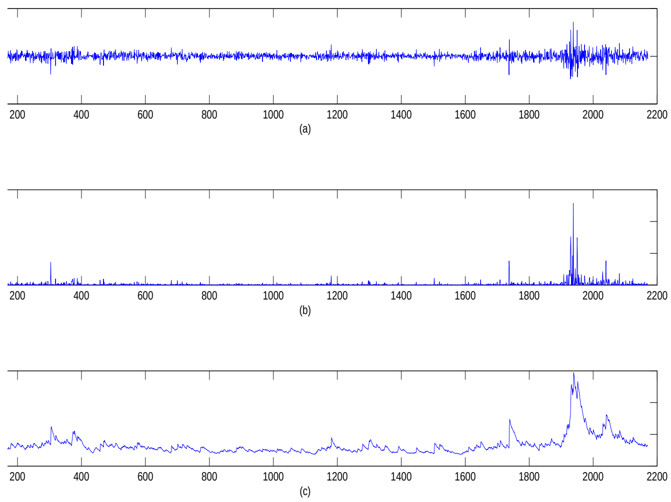

Figure 6: Data source: BIL. (a) Daily excess returns over the sample period; (b) Squared returns; (c) Estimated volatility implied by the $\mathcal{N}$-Garch(1,1).

Marginal distribution	Normal	
	Benchmark	Excess return
First moment	60.0947(0.0214)	79.8325(0.0000)
Second moment	399.9922(0.0000)	287.3777(0.0000)
Third moment	63.4950(0.0104)	131.1184(0.0000)
Fourth moment	501.0621(0.0000)	354.1134(0.0000)
K-S test	0.0686(0.0000)	0.0703(0.0000)
Marginal distribution	Student's t	
	Benchmark	Excess return
First moment	60.7844(0.0186)	80.7309(0.0000)
Second moment	417.2121(0.0000)	297.5998(0.0000)
Third moment	68.2470(0.0035)	139.2980(0.0000)
Fourth moment	530.9333(0.0000)	374.7113(0.0000)
K-S test	0.0873(0.0000)	0.0768(0.0000)
Marginal distribution	$\mathcal{N}$-Garch(1,1)	
	Benchmark	Excess return
First moment	58.7920(0.0279)	68.1747(0.0036)
Second moment	18.9275(0.9981)	39.0614(0.5124)
Third moment	42.9847(0.3446)	82.1434(0.0000)
Fourth moment	15.9999(0.9997)	39.8940(0.4750)
K-S test	0.0239(0.1634)	0.0310(0.0301)
Marginal distribution	t-Garch(1,1)	
	Benchmark	Excess return
First moment	58.9578(0.0270)	65.3524(0.0069)
Second moment	18.8370(0.9982)	37.6655(0.5758)
Third moment	43.9974(0.3061)	83.0822(0.0000)
Fourth moment	16.1038(0.9997)	40.1104(0.4654)
K-S test	0.0159(0.6380)	0.0183(0.4588)

Table 2: Data source: BIL. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the marginal distributions.

Marginal distribution	Normal	
	Benchmark	Excess return
First moment	47.0754(0.2055)	48.2051(0.1751)
Second moment	328.1762(0.0000)	241.9759(0.0000)
Third moment	42.2624(0.3735)	80.2028(0.0000)
Fourth moment	382.5929(0.0000)	251.5926(0.0000)
K-S test	0.0764(0.0000)	0.0657(0.0000)
Marginal distribution	Student's t	
	Benchmark	Excess return
First moment	47.1190(0.2042)	48.8889(0.1582)
Second moment	340.6647(0.0000)	250.1982(0.0000)
Third moment	41.6056(0.4007)	85.0243(0.0000)
Fourth moment	407.0406(0.0000)	246.4893(0.0000)
K-S test	0.0825(0.0000)	0.0733(0.0000)
Marginal distribution	$\mathcal{N}$-Garch(1,1)	
	Benchmark	Excess return
First moment	46.0848(0.2351)	31.6064(0.8257)
Second moment	35.3200(0.6807)	30.5034(0.8607)
Third moment	39.0087(0.5148)	36.8707(0.6119)
Fourth moment	26.4191(0.9514)	29.6055(0.8859)
K-S test	0.0389(0.0020)	0.0408(0.0000)
Marginal distribution	t-Garch(1,1)	
	Benchmark	Excess return
First moment	45.3749(0.2579)	31.2383(0.8379)
Second moment	33.8351(0.7430)	31.9793(0.8130)
Third moment	40.1695(0.4627)	38.7953(0.5244)
Fourth moment	26.6849(0.9463)	29.5848(0.8865)
K-S test	0.0196(0.3429)	0.0157(0.6222)

Table 3: Data source: AzMul. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the marginal distributions.

Marginal distribution	Normal	
	Benchmark	Excess return
First moment	53.1894(0.0792)	54.7697(0.0684)
Second moment	123.4249(0.0000)	100.8551(0.0000)
Third moment	47.9170(0.1824)	48.6469(0.1640)
Fourth moment	138.9579(0.0000)	97.0204(0.0000)
K-S test	0.0389(0.0020)	0.0514(0.0000)
Marginal distribution	Student's t	
	Benchmark	Excess return
First moment	53.3240(0.0773)	74.3387(0.0000)
Second moment	124.6638(0.0000)	101.3888(0.0000)
Third moment	48.3706(0.1708)	47.2432(0.2007)
Fourth moment	140.0741(0.0000)	97.5445(0.0000)
K-S test	0.0429(0.0000)	0.0558(0.0000)
Marginal distribution	$\mathcal{N}$-Garch(1,1)	
	Benchmark	Excess return
First moment	54.6943(0.0607)	72.6717(0.0012)
Second moment	29.6102(0.8858)	58.8454(0.0276)
Third moment	43.2926(0.3326)	43.4041(0.3282)
Fourth moment	26.8180(0.9451)	50.5352(0.1228)
K-S test	0.0268(0.0745)	0.0478(0.0000)
Marginal distribution	t-Garch(1,1)	
	Benchmark	Excess return
First moment	54.4280(0.0637)	73.2722(0.0010)
Second moment	29.3825(0.8917)	45.8545(0.2423)
Third moment	44.8708(0.2750)	52.2816(0.0924)
Fourth moment	27.2058(0.9386)	39.9062(0.4744)
K-S test	0.0227(0.1874)	0.0173(0.4949)

Table 4: Data source: CIG. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the marginal distributions.

	BIL	AzMul	CIG
Benchmark	$\hat{c}_x$=3.6825×10^{-4*}	$\hat{c}_x$=9.5191×10^{-4*}	$\hat{c}_x$=3.1610×10^{-5}
	$\hat{\gamma}_{x,0}$=4.0506×10^{-7*}	$\hat{\gamma}_{x,0}$=1.8391×10^{-6*}	$\hat{\gamma}_{x,0}$=2.000×10^{-7*}
	$\hat{\gamma}_{x,1}$=0.0782*	$\hat{\gamma}_{x,1}$=0.0746*	$\hat{\gamma}_{x,1}$=0.0686*
	$\hat{\gamma}_{x,2}$=0.9111*	$\hat{\gamma}_{x,2}$=0.9172*	$\hat{\gamma}_{x,2}$=0.8651*
	$\hat{\nu}_x$=18.3767*	$\hat{\nu}_x$=7.2720*	$\hat{\nu}_x$=9.4144*
Excess return	$\hat{c}_y$=9.3841×10^{-4}	$\hat{c}_y$=-1.8976×10^{-4}	$\hat{c}_y$=1.4571×10^{-4}
	$\hat{\gamma}_{y,0}$=2.000×10^{-7*}	$\hat{\gamma}_{y,0}$=1.4560×10^{-6*}	$\hat{\gamma}_{y,0}$=2.000×10^{-7}
	$\hat{\gamma}_{y,1}$=0.0701*	$\hat{\gamma}_{y,1}$=0.0696*	$\hat{\gamma}_{y,1}$=0.0210*
	$\hat{\gamma}_{y,2}$= 0.9116*	$\hat{\gamma}_{y,2}$=0.9109*	$\hat{\gamma}_{y,2}$= 0.9680*
	$\hat{\nu}_y$=7.8250*	$\hat{\nu}_y$=6.0917*	$\hat{\nu}_y$=5.1826*

Table 5: Estimated marginal distribution parameters in the case where X and Y are t-Garch(1,1). The asterisk denotes the parameters which are significantly different from zero at the 5% level.

	BIL	
Clayton	0.1207*(0.0565)	
Frank	0.3859(0.3600)	
Gaussian	0.0785(0.0590)	
Student's t	(0.0686, 7.6123*)(0.0604, 2.5326)	
	AzMul	CIG
Frank	-0.9222*(0.4158)	-1.9946*(0.3995)
Gaussian	-0.1543*(0.0616)	-0.3014*(0.0591)
Student's t	(-0.1522*, 5.4658*)(0.0640, 0.7340)	(-0.3072*, 24.1033)(0.0212, n.a.)

Table 6: Estimated copula parameter and relative standard error when X and Y are t-Garch(1,1). The asterisk denotes the parameters which are significantly different from zero at the 5% level.

Copula	Clayton	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	47.0897(0.2050)	70.1116(0.0023)
Second moment	37.2146(0.5963)	53.3021(0.0776)
Third moment	55.0284(0.0572)	77.3443(0.0000)
Fourth moment	43.5031(0.3946)	55.1092(0.0563)
K-S test	0.0144(0.3260)	0.0137(0.3857)

Table 7: Data source: BIL. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the estimated copulas.

Copula	Frank	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	28.0557(0.9224)	21.5643(0.9924)
Second moment	72.3212(0.0013)	40.8644(0.4323)
Third moment	31.8695(0.8168)	23.9682(0.9790)
Fourth moment	81.3893(0.0000)	53.7005(0.0724)
K-S test	0.0143(0.3044)	0.0137(0.3540)
Copula	Gaussian	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	26.2197(0.9543)	21.8308(0.9914)
Second moment	69.2833(0.0028)	38.3887(0.5429)
Third moment	36.9036(0.6104)	21.1831(0.9937)
Fourth moment	78.5768(0.0000)	48.4505(0.1688)
K-S test	0.0140(0.3305)	0.0142(0.3170)
Copula	Student's t	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	27.5758(0.9318)	21.5613(0.9924)
Second moment	30.4010(0.8637)	23.7147(0.9809)
Third moment	36.7477(0.6175)	21.1047(0.9939)
Fourth moment	30.9754(0.8463)	29.2007(0.8963)
K-S test	0.0150(0.2580)	0.0134(0.3852)

Table 8: Data source: AzMul. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the estimated copulas.

Copula	Frank	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	223.1007(0.0000)	155.7995(0.0000)
Second moment	82.1191(0.0000)	110.0141(0.0000)
Third moment	225.1985(0.0000)	168.5809(0.0000)
Fourth moment	106.5103(0.0000)	98.4169(0.0000)
K-S test	0.0266(0.0031)	0.0105(0.6896)
Copula	Gaussian	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	225.4825(0.0000)	139.8936(0.0000)
Second moment	86.4071(0.0000)	105.9532(0.0000)
Third moment	226.2261(0.0000)	179.2675(0.0000)
Fourth moment	110.2071(0.0000)	95.1074(0.0000)
K-S test	0.0242(0.0096)	0.0114(0.5968)
Copula	Student's t	
	$(U_t, D_1C(U_t, V_t))$	$(V_t, D_2C(U_t, V_t))$
First moment	225.7516(0.0000)	140.6675(0.0000)
Second moment	86.2091(0.0000)	92.2386(0.0000)
Third moment	226.6428(0.0000)	176.1901(0.0000)
Fourth moment	111.2439(0.0000)	80.2042(0.0000)
K-S test	0.0255(0.0053)	0.0111(0.6217)

Table 9: Data source: CIG. LM test (p-value) of temporal independence and Kolmogorov-Smirnov test (p-value) of the estimated copulas.

	Max market volatility		Min market volatility	
%	Excess return	Managed fund	Excess return	Managed fund
1	-2.46	-6.75	-0.54	-0.92
5	-1.57	-4.53	-0.35	-0.60
10	-1.18	-3.44	-0.26	-0.44
50	0.0094	0.0821	0.0094	0.0514
90	1.19	3.50	0.28	0.53
95	1.59	4.52	0.36	0.67
99	2.48	6.57	0.56	0.95

Table 10: Quantiles of the estimated distributions of $Y_t|\mathcal{F}_{t-1}$ and $Z_t|\mathcal{F}_{t-1}$ (percentage values). Data source: BIL.

	Max market volatility		Min market volatility	
%	Excess return	Managed fund	Excess return	Managed fund
1	-5.18	-14.38	-1.95	-2.26
5	-3.22	-9.09	-1.22	-1.38
10	-2.39	-6.77	-0.09	-1.00
50	-0.019	0.078	-0.019	0.078
90	2.35	6.94	0.87	1.16
95	3.18	9.28	1.18	1.54
99	5.14	14.71	1.91	2.45

Table 11: Quantiles of the estimated distributions of $Y_t|\mathcal{F}_{t-1}$ and $Z_t|\mathcal{F}_{t-1}$ (percentage values). Data source: AzMul.

	Max market volatility		Min market volatility	
%	Excess return	Managed fund	Excess return	Managed fund
1	-1.44	-1.38	-1.00	-0.94
5	-0.87	-0.87	-0.59	-0.56
10	-0.63	-0.65	-0.43	-0.41
50	0.014	0.017	0.015	0.017
90	0.66	0.69	0.46	0.45
95	0.89	0.91	0.62	0.60
99	1.47	1.44	1.02	0.98

Table 12: Quantiles of the estimated distributions of $Y_t|\mathcal{F}_{t-1}$ and $Z_t|\mathcal{F}_{t-1}$ (percentage values). Data source: CIG.

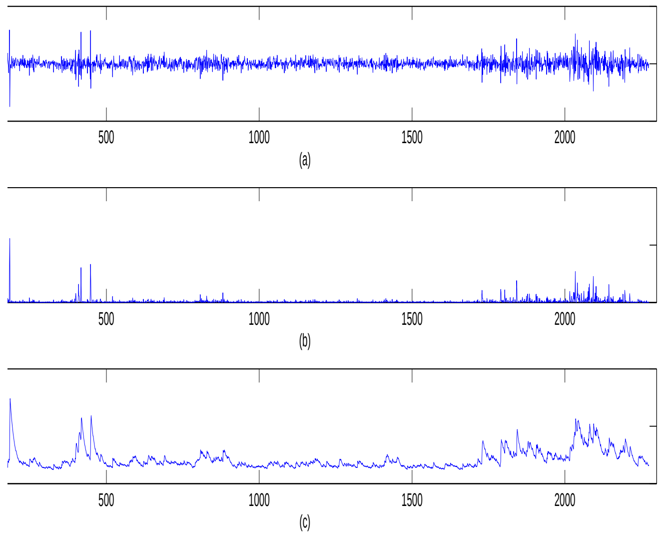

Figure 7: Data source: AzMul. (a) Daily excess returns over the sample period; (b) Squared returns; (c) Estimated volatility implied by the $\mathcal{N}$-Garch(1,1).

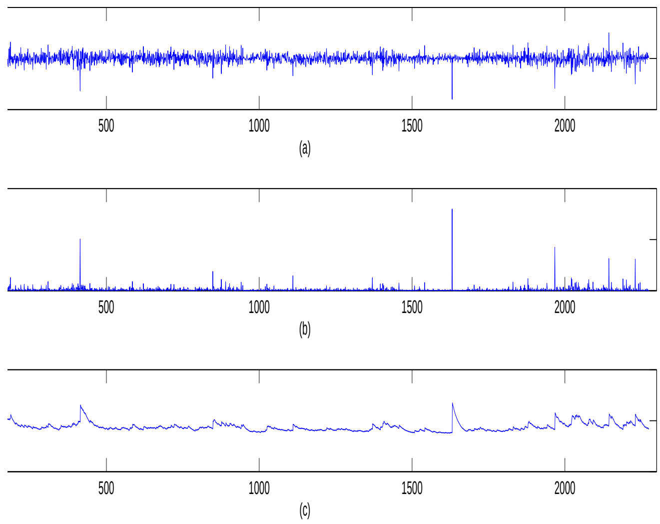

Figure 8: Data source: CIG. (a) Daily excess returns over the sample period; (b) Squared returns; (c) Estimated volatility implied by the $\mathcal{N}$-Garch(1,1).

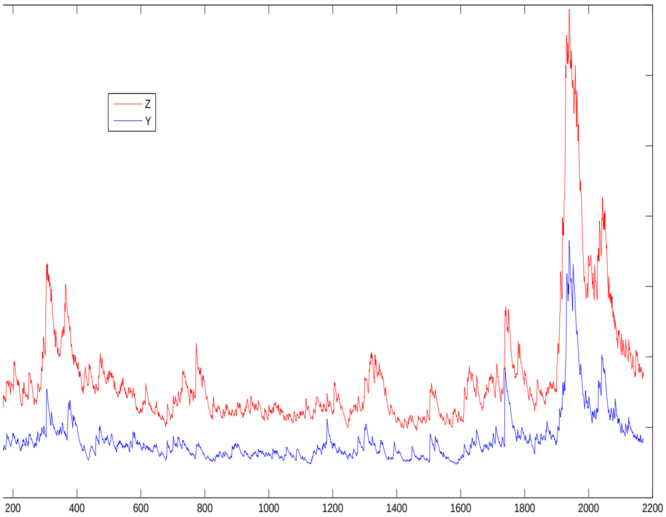

Figure 9: Data source: BIL. Comparison between the estimated conditional VaR of the excess return and the managed fund.

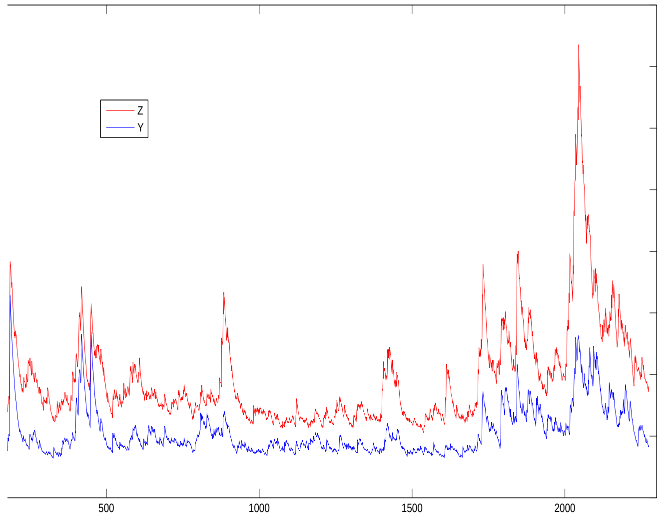

Figure 10: Data source: AzMul. Comparison between the estimated conditional VaR of the excess return and the managed fund.

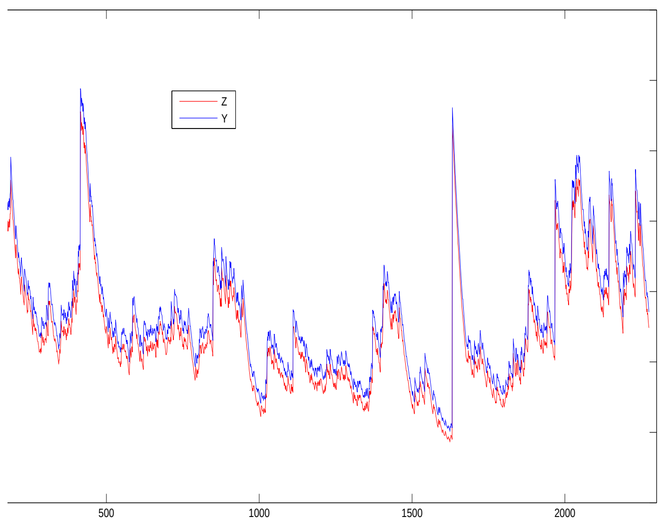

Figure 11: Data source: CIG. Comparison between the estimated conditional VaR of the excess return and the managed fund.