A Copula-based Model of Speculative Price Dynamics in Discrete Time*

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Abstract

This paper suggests a new technique to construct first order Markov processes using products of copula functions, in the spirit of Darsow et al. (1992). The approach requires the definition of: i) a sequence of distribution functions of the increments of the process; ii) a sequence of copula functions representing dependence between each increment of the process and the corresponding level of the process before the increment. The paper shows how to use the approach to build several kinds of processes (stable, elliptical, Farlie-Gumbel-Morgernstern, Archimedean), martingale processes, and how to extend the analysis to the multivariate setting. The technique turns out to be well suited to provide a discrete time representation of the dynamics of innovations to financial prices under the restrictions imposed by the Efficient Market Hypothesis.

Keywords: Markov processes, Copula function, Efficient Market Hypothesis, Granger causality, H-condition.

1 Introduction

Finance and physics are the fields in which the theory of stochastic processes has experienced the largest development. In finance, the model of price dynamics first proposed by Bachelier (1900) has become well known as the *Efficient Market Hypothesis* (EMH). Simply, a market is called efficient if price changes are not predictable. In its modern version (see Fama (1970) and Samuelson (1963, 1973a, 1973b)), the model is applied to price logarithm instead of the prices themselves as in the original Bachelier work. In other words, in order to

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constrain a price S_i, (where i denotes time) to be non-negative, we model the process X_i, which is linked to the price by the relationship $S_i = e^{X_i}$. So, under the EMH, the innovation of the variable X_i cannot be predicted from current and past values of the variable itself: this is called *weak form efficiency*. If innovation cannot be predicted on the basis of other public information either, the market is said to exhibit *semi-strong efficiency*. If private information is also useless, the market is said to be *strongly efficient* (for a treatment of market efficiency from the point of view of financial econometrics see chapter 2 in Campbell et al.,1997). So, according to the EMH the price dynamics must be represented by the model $X_i = X_{i-1} + Y_i$, where Y_i represents the innovation, that is the increment of the log-price, at time i. *Weak form efficiency* results in two requirements: i) X is endowed with the first order Markov property, that is $\Pr(X_i \leq x \mid X_{i-1}, ..., X_0) = \Pr(X_i \leq x \mid X_{i-1})$ and ii) the conditional expectation of the increments is equal to zero, that is $E(Y_i \mid X_{i-1}) = 0, \forall i$, which is called the *martingale property*. In more general forms of efficiency the martingale condition is required to hold true with respect to larger filtrations: this is called *H-condition*.

In the literature, the EMH is enforced by assuming independent increments, namely Lévy and additive processes, which allow a synthetic representation in continuous time. In this paper we propose a general representation in discrete time that could exploit the flexibility of copulas. As for Markov property, a technique to represent first order Markov processes in terms of copulas was first proposed by Darsow et al. (1992). They showed that first order Markov processes are characterized by the following relation

$$C_{X_{j_1},X_{j_2},...,X_{j_n}} = C_{X_{j_1},X_{j_2}} \star C_{X_{j_2},X_{j_3}} \star ... \star C_{X_{j_{n-1}},X_{j_n}}$$

where $C_{X_{j_1},X_{j_2},...,X_{j_n}}$ is the copula associated to the vector $(X_{j_1}, X_{j_2}, \ldots, X_{j_n})$ and $C_{X_{j_k},X_{j_{k+1}}}$ stands for the copula function linking X_{j_k} and $X_{j_{k+1}}$ and the $\star$-product operator is defined as

$$A \star B(u, w, v) \equiv \int_0^w \frac{\partial A(u,t)}{\partial t} \frac{\partial B(t,v)}{\partial t} \, dt \tag{1}$$

for arbitrary bivariate copula functions A and B. This operator allows to express the Chapman-Kolmogorov equation in the language of copulas. Ibragimov (2005, 2009) extended the representation to the case of Markov processes of order k. The same results can obviously be applied to represent a Markov process in k dimensions, and it is in that sense that they will be used in this paper.

Building on the Darsow et al. (1992) idea much has already been done to explore the temporal dependence features of levels of Markov processes (see Joe (1997), Chen and Fan (2006), Abegaz and Naik-Nimbalkar (2008), Ibragimov and Lentzas (2008), Beare (2010) and Chen et al. (2010)). Already in Joe (1997) it was suggested that a mixture copula could provide a parametric family including “an iid sequence at one boundary and a perfectly dependent (or persistent) sequence at the other boundary”. In spite of this, we are not aware of any contribution addressing the problem of dependence of increments of processes, instead of levels. While on the boundaries of iid and persistent processes we may have a clear conjecture of what the answer would be, this is not so easy in general cases. This paper proposes a full-fledged method to discriminate Markov processes with independent and dependent increments. As the most simple example, take the copula extracted from Brownian motion, which is one of the cases provided in the original Darsow et al. (1992). This is a typical example of process with independent increments, even though levels have a specific kind of dependence, which can be represented by means of the Gaussian copula

$$C(u,v) = \int_0^u \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw$$

where $\Phi(x)$ denotes the standard normal distribution and ρ is a correlation parameter. Assuming that the variables in question are two observations of the level of the same process at different times t and s, with $s < t$ this copula naturally denotes a Gaussian dependence structure with $\rho = \sqrt{s/t}$. The question addressed in this paper is how this representation can be generalized to represent all Markov processes with independent increments and how the construction can be amended to extend the class to Markov processes with dependent increments. Apart from this theoretical innovation, we are then interested in investigating whether it is possible to design stochastic processes with dependent increments that abide by the requirements of the EMH, and how the construction can be extended to the multivariate case. From this point of view, this paper contributes to the econometric literature on the use of copula functions for the analysis of financial time series (see Chorós et al (2010) and Patton (2009) for a review). This new viewpoint also provides a different extension of financial application beyond that of time-changed Brownian copula, explored in Schmitz (2003) and Cherubini and Romagnoli (2010).
The outline of the paper is as follows. In section 2 we present the technique used to construct Markov processes with independent and dependent increments. In section 3 we show how to apply the technique to build several kinds of process. In section 4 we address the restriction that must be imposed to the dynamics of the process to make it a martingale. Section 5 provides the multivariate extension of the analysis. Section 6 concludes. Proofs that are too lengthy to be included in the text are found in the Appendix.

2 Copula-Based Markov Processes with (In)Dependent Increments

We assume a probability space $(\Omega, \Im, \mathbb{P})$ and a sequence of random variables $\{Y_n\}_{n\geq 1}$. We define a discrete time stochastic process $\{X_n\}_{n\geq 0}$ through $X_i = X_{i-1} + Y_i, i \geq 1$, assuming, for simplicity, $X_0 = 0$. Moreover, we endow the probability space with a filtration $\{\Im_n\}_{n\geq 0}$ (with $\Im_0$ trivial) to which $\{X_n\}_{n\geq 0}$ is adapted. We denote F_{Y_i} the cumulative distribution function of the increment Y_i and F_{X_i} the cumulative distribution function of X_i. Of course, we have $F_{Y_1} = F_{X_1}$. We also assume a set of copula functions C_{X_{i-1},Y_i} representing the dependence structure between the value of the process at the beginning of the period $[t_{i-1}, t_i]$ and its increment during that period. While a first version of the paper provided regularity conditions to ensure that the results would hold true in full generality, here for clarity, the paper will focus on absolute continuous copulas: this paper will derive the results in terms of copula density functions when it is simpler to do so. We also assume that all marginal cumulative distribution functions are strictly increasing. Given the copula function linking levels and increments, the task, then, is to determine the temporal dependence structure between X_{i-1} and X_i. The representation of bivariate distributions will allow the full exploitation of the flexibility granted by copula functions

$$\Pr(X_{i-1} \leq x, X_i \leq y) = C_{X_{i-1},X_i}(F_{X_{i-1}}(x), F_{X_i}(y)).$$

Finally, since the stochastic process $\{X_n\}_{n\geq 0}$ is assumed to be a first order Markov process, we may apply the result of Darsow et al. (1992) to give a complete copula based description of the law of the process. In order to apply this approach, we start by studying the problem of how to recover the distribution function of the sum of two dependent random variables and the copula function representing the dependence structure between this sum and one of the two variables.

2.1 Modelling the Dependence Structure of Increments

Let X and Y be two random variables with continuous c.d.f. F_X and F_Y, respectively, and let $C_{X,Y}(u, v)$ be the copula function that describes their mutual dependence.

We begin by reviewing a standard result of copula function literature, stating that the partial derivative of a copula function corresponds to conditional probability distribution. We will adopt the notation:

$$D_1C(u, v) = \frac{\partial C}{\partial u}(u, v) \quad D_2C(u, v) = \frac{\partial C}{\partial v}(u, v)$$

Formally, we have that, for every $x, y \in \mathbb{R}$,

$$D_1C_{X,Y}\left(F_X(x), F_Y(y)\right) = \mathbb{P}\left(Y \leq y | X = x\right).$$

Proposition 2.1. *Let X and Y be two real-valued random variables on the same probability space $(\Omega, \Im, \mathbb{P})$ with a dependence structure represented by the copula function $C_{X,Y}$ and continuous marginal distributions F_X and F_Y. Then,*

$$C_{X,X+Y}(u, v) = \int_0^u D_1C_{X,Y}\left(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))\right) dw \tag{2}$$

$$F_{X+Y}(t) = \int_0^1 D_1C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw. \tag{3}$$

Proof. Using the substitution $w = F_X(x) \in (0,1)$

$$\begin{aligned} F_{X,X+Y}(s,t) = \mathbb{P}(X \leq s, X+Y \leq t) &= \int_{-\infty}^{s} \mathbb{P}(X+Y \leq t | X = x)\, dF_X(x) = \\ &= \int_{-\infty}^{s} \mathbb{P}(Y \leq t - x | X = x)\, dF_X(x) = \\ &= \int_{-\infty}^{s} D_1 C_{X,Y}\left(F_X(x), F_Y(t-x)\right) dF_X(x) = \\ &= \int_{0}^{F_X(s)} D_1 C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw. \end{aligned}$$

Then, the copula function linking X and $X+Y$ is

$$C_{X,X+Y}(u,v) = \int_0^u D_1 C_{X,Y}\left(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))\right) dw.$$

Moreover

$$F_{X+Y}(t) = \lim_{s \to +\infty} F_{X,X+Y}(s,t) = \int_0^1 D_1 C_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) dw.$$

□

Remark 2.1. *It is straightforward to check that*

$$c_{X,X+Y}(u,v) = \frac{c_{X,Y}\left(u, F_Y\left(F_{X+Y}^{-1}(v) - F_X^{-1}(u)\right)\right) f_Y\left(F_{X+Y}^{-1}(v) - F_X^{-1}(u)\right)}{f_{X+Y}\left(F_{X+Y}^{-1}(v)\right)}$$

and

$$f_{X+Y}(t) = \int_0^1 c_{X,Y}\left(w, F_Y(t - F_X^{-1}(w))\right) f_Y(t - F_X^{-1}(w)) dw$$

where lower case letters denote the densities of copula functions and c.d.f's.

The copula function that is obtained from the proposition is explicitly constructed from the conditional distribution of the second term of the sum. We prove below that (2) is a copula only if the distribution functions that define it are linked through a relation of type (3). For this purpose, we formally provide an extended definition of the convolution operator.

Definition 2.1. *Let F, H be two continuous c.d.f's and C be a copula function. We define the* **C-convolution** *of H and F the c.d.f.*

$$H \overset{C}{*} F(t) = \int_0^1 D_1 C\left(w, F(t - H^{-1}(w))\right) dw$$

Alternatively, the **C-convolution** *can be expressed in terms of densities as*

$$h \overset{C}{*} f(t) = \int_0^1 c\left(w, F(t - H^{-1}(w))\right) f(t - H^{-1}(w)) dw$$

Proposition 2.2. *Let F, G, H be three continuous c.d.f's, $C(w,v)$ a copula function and*

$$\hat{C}(u,v) = \int_0^u D_1 C\left(w, F(G^{-1}(v) - H^{-1}(w))\right) dw.$$

$\hat{C}(u,v)$ is a copula function iff

$$G = H \overset{C}{*} F. \tag{4}$$

Proof. Let $\hat{C}$ be a copula function. Necessarily $\hat{C}(1,v) = v$ holds. But

$$\hat{C}(1,v) = \int_0^1 D_1 C\left(w, F\left(G^{-1}(v) - H^{-1}(w)\right)\right) dw = H \overset{C}{*} F\left(G^{-1}(v)\right) = v$$

for all $v \in (0,1)$ if and only if $G = H \overset{C}{*} F$. The converse is the content of Proposition 2.1. □

We may then formally define the class of copula functions that we are going to use to construct Markov processes as follows.

Definition 2.2. *Let F and H be two continuous c.d.f's and C a copula function. We define the copula function*

$$\hat{C}(u,v) = \int_0^u D_1 C\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw. \tag{5}$$

Remark 2.2. *The C-convolution operator is closed with respect to mixtures of copula functions. In fact, it is trivial to show that for all bivariate copula functions A and B, if $C(u,v) = \lambda A(u,v) + (1-\lambda)B(u,v)$ for $\lambda \in [0,1]$, then, for all c.d.f's H and F,*

$$H \overset{C}{*} F = H \overset{\lambda A + (1-\lambda)B}{*} F = \lambda H \overset{A}{*} F + (1-\lambda) H \overset{B}{*} F. \tag{6}$$

It is likewise trivial to observe that this is not true for the corresponding copula function $\hat{C}(u,v)$ defined through (5). However, we have

$$\begin{aligned}\hat{C}(u,v) = \lambda \int_0^u & D_1 A\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw + \\ &+ (1-\lambda) \int_0^u D_1 B\left(w, F((H \overset{C}{*} F)^{-1}(v) - H^{-1}(w))\right) dw\end{aligned}$$

with $H \overset{C}{*} F$ given by (6).

3 Building Markov processes by increments aggregation

The analysis in the previous sections allows us to characterize the law of a stochastic process by specifying the distribution of increments and the dependence structure between the process at any time i and its increment between time i and $i+1$. Here we also provide useful examples of such processes.

The stochastic process will be built using the results in Section 2. Namely, if H is the c.d.f. of X_{i-1}, F the c.d.f. of $Y_i = X_i - X_{i-1}$ and $C(u,v)$ the copula associated to the random vector (X_{i-1}, Y_i), then $H \overset{C}{*} F$ is the c.d.f. of X_i and $\hat{C}(u,v)$, given by (5), is the copula function associated to the random vector (X_{i-1}, X_i).

Following the notation introduced at the beginning of Section 2, as a consequence of the previous results, the temporal dependence structure between X_{i-1} and X_i is given by (see (2))

$$C_{X_{i-1},X_i}(u,v) = \int_0^u D_1 C_{X_{i-1},Y_i}\left(w, F_{Y_i}(F_{X_i}^{-1}(v) - F_{X_{i-1}}^{-1}(w))\right) dw,$$

where by (3)

$$F_{X_i}(x) = F_{X_{i-1}} \overset{C}{*} F_{Y_i}(x) = \int_0^1 D_1 C_{X_{i-1},Y_i}\left(w, F_{Y_i}(x - F_{X_{i-1}}^{-1}(w))\right) dw$$

Finally, if we assume that the process is first order Markov, its dynamics can then be completely described by the sequence of distributions F_{X_i} defined above and the sequence of copulas C_{X_{i-1},X_i}.

Before introducing the main examples, let us practice with the construction of these dynamics using the most standard examples of copula functions, namely that representing perfect dependence (the so called *upper Fréchet bound*) and that corresponding to independence (the so called *product copula*). The latter example is of utmost importance, since it describes the class of processes with independent increments. Furthermore, these examples show the main contribution of our approach. We will see that perfectly dependent increments bring about persistent processes (in the sense used in Joe (1997), section 8.1.,page 245), but independent increments do not lead to independent processes.

Example 3.1. The co-monotonic case *In the case* $C(w,v) = w \wedge v = \min(w,v)$*, it is easy to verify*

$$F_{X_{i-1}} \overset{C}{*} F_{Y_i}(t) = \sup\left\{w \in (0,1) : F_{Y_i}^{-1}(w) + F_{X_{i-1}}^{-1}(w) < t\right\}$$

that implies the well known result (see Prop. 6.15 in McNeil et al. (2005))

$$F_{Y_i}^{-1}(F_{X_{i-1}} \overset{C}{*} F_{Y_i}(t)) + F_{X_{i-1}}^{-1}(F_{X_{i-1}} \overset{C}{*} F_{Y_i}(t)) = t.$$

Moreover in this case the time series is deterministic (Chen and Fan (2006))

$$C_{X_{i-1},X_i}(u,v) = u \wedge \sup\left\{w \in (0,1) : F_{Y_i}^{-1}(w) + F_{X_{i-1}}^{-1}(w) < (F_{X_{i-1}} \overset{C}{*} F_{Y_i})^{-1}(v)\right\} =$$
$$= u \wedge v = min(u,v).$$

Example 3.2. The independence case. *If C is the product copula, the C-convolution of $F_{X_{i-1}}$ and F_{Y_i} coincides with the convolution $F_{X_{i-1}} * F_{Y_i}$ of $F_{X_{i-1}}$ and F_{Y_i}, while the copula C_{X_{i-1},X_i} defined through (5) takes the form*

$$C_{X_{i-1},X_i}(u,v) = \int_0^u F_{Y_i}((F_{X_{i-1}} * F_{Y_i})^{-1}(v) - F_{X_{i-1}}^{-1}(w))dw.$$

In this case, through our construction, we recover the law of all random walks.

When using our approach based on increment dependence to construct Markov processes one is quite naturally led to address two questions that are extremely useful in building parsimonious copula-based (or semi-parametric) representations of stochastic processes. A reference to Brownian motion may serve to introduce the issue. This process features three peculiar properties:

- the dependence structure between levels and increments is represented by the same kind of copula, namely the product copula
- the copula linking levels and increments, that is the product copula, is of the same family as the copula linking levels, which is Gaussian
- the copula function linking levels is closed under the product operator, namely $C_{X_i,X_{i+j}}$ is a copula of the same kind as $C_{X_i,X_{i+1}}$, that is the Gaussian copula, with a suitable parameter ρ^j.

Actually, the example could be extended to the general case of Gaussian Markov processes. Also in this case, the dependence structure between the increments and the levels will be the same, that is Gaussian, as that representing the dependence structure among levels. A natural question then emerges: whether this approach can allow the building of processes that preserve some of these properties. Some useful examples of this are provided here.

3.1 Stable Processes

Since the Gaussian process is a special instance in the class of stable processes, this is the first extension that comes to mind. If one considers a process with independent increments, such as those reported in Example 3.2, and specifies the distribution F as a stable distribution, the copula linking levels preserves a specific shape that generalizes the Gaussian copula.

More formally, let Z be a random variable. This variable will be endowed with an α-stable distribution with parameters $\alpha, \mu, \beta, \gamma$ (with $0 < \alpha \leq 2, \mu \in \mathbb{R}, \beta \in [-1,1], \gamma \geq 0$) if its characteristic function is of the following type

$$\phi_Z(\lambda) = e^{i\mu\lambda - \gamma^\alpha|\lambda|^\alpha(1-i\beta sign(\lambda)W(\alpha,\lambda))}$$

with

$$W(\alpha,\lambda) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log|\lambda|, & \text{if } \alpha = 1 \end{cases}$$

The case $\alpha = 2$ corresponds to the normal distribution and the α-stable distribution is symmetric around the origin if $\mu = 0, \beta = 0$.

Let X be an α-stable symmetrically distributed random variable, then one among the several possible expressions of its characteristic function is

$$\phi_X(\lambda) = e^{-\gamma^\alpha|\lambda|^\alpha} = e^{-(\gamma|\lambda|)^\alpha} = \phi_{\gamma Z}(\lambda)$$

where Z is α-stable distributed with $\mu = 0, \beta = 0, \gamma = 1$. This way $X =^d \gamma Z$ and $F_X(x) = \Phi_Z\left(\frac{x}{\gamma}\right)$, where Φ_Z is the c.d.f of Z. If (X_{i-1}, Y_i) is an α-stable vector with independent components, then by simply applying the standard convolution formula it is easy to find that, if $\rho = \frac{\gamma_{X_{i-1}}}{\gamma_{X_i}}$

$$C_{X_{i-1},X_i}(u,v) = \int_0^u \Phi_Z\left(\frac{\Phi_Z^{-1}(v) - \rho\Phi_Z^{-1}(w)}{(1-\rho^\alpha)^{\frac{1}{\alpha}}}\right) dw. \tag{7}$$

Remark 3.1. *For $\alpha = 2$ we get the Gaussian copula generating all Gaussian processes. To get (7) we assumed that X_{i-1} and Y_i were independent. As noticed above, in the case $\alpha = 2$, we get a copula of type (7) even if we do not assume independence. Unfortunately in the case $\alpha < 2$, this fact does not hold true anymore and examples of α-stable dependent vectors (X_{i-1}, Y_i) can be constructed so that the copula associated to (X_{i-1}, X_i) is no longer of the same type as (7).*

3.2 Elliptical Processes

The Gaussian distribution is not only a specific instance of stable distributions: it is also the main reference case of elliptical distributions (see Fang et al., 2002, 2005). We show below that in the case of elliptical distribution of increments and elliptical dependence structure between levels and increments, the dependence structure between levels turns out to be elliptical as well.
Formally, let $F(x,y)$ be a bivariate elliptical distribution. Without any loss of generality we can assume that both marginal distributions have zero mean. It is well known that the associated density is of the type

$$\sqrt{ac-b^2}\,g(ax^2 + 2bxy + cy^2)$$

where $g(\cdot)$ is a positive function of a scalar variable such that $\int_0^{+\infty} g(y)dy < +\infty$ (see (2.19) in Fang et al. (1990)) and the parameters a, b and c are such that the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite.

We recall standard choices of the function $g(.)$. It is trivial to check that the Gaussian and Student-t with m degrees of freedom distributions can be recovered by considering respectively

$$g(z) = \frac{1}{2\pi} e^{-\frac{z^2}{2}} \quad \text{and} \quad g(z) = \frac{\Gamma(\frac{m+2}{2})}{\pi m \Gamma(\frac{m}{2})} \left(1 + \frac{z^2}{m}\right)^{-\frac{m+2}{2}}.$$

Let $F_{X_{i-1}}$ and F_{Y_i} be the corresponding marginal cumulative distribution functions. By Sklar's Theorem we can define the copula function

$$C_{X_{i-1},Y_i}(u,v) = \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{Y_i}^{-1}(v)} \sqrt{ac-b^2}\, g(as^2 + 2bst + ct^2)\, ds\, dt. \tag{8}$$

The conditional probability is

$$D_1 C_{X_{i-1},Y_i}(u,v) = \sqrt{ac-b^2} \frac{1}{f_{X_{i-1}}(F_{X_{i-1}}^{-1}(u))} \int_{-\infty}^{F_{Y_i}^{-1}(v)} g(aF_{X_{i-1}}^{-1}(u)^2 + 2bF_{X_{i-1}}^{-1}(u)t + ct^2)\, dt,$$

from which we obtain the convolution

$$\begin{aligned} & F_{X_{i-1}} \overset{C}{*} F_{Y_i}(z) = \\ = \quad & \sqrt{ac-b^2} \int_0^1 \frac{1}{f_{X_{i-1}}(F_{X_{i-1}}^{-1}(w))} \int_{-\infty}^{F_{Y_i}^{-1}(F_{Y_i}(z-F_{X_{i-1}}^{-1}(w)))} g(aF_{X_{i-1}}^{-1}(w)^2 + 2bF_{X_{i-1}}^{-1}(w)t + ct^2)\, dt dw \end{aligned}$$

Since $F_{X_{i-1}} \overset{C}{*} F_{Y_i} = F_{X_i}$, we have

$$\begin{aligned} & C_{X_{i-1},X_i}(u,v) = \sqrt{ac-b^2} \times \\ & \int_0^u \frac{1}{f_{X_{i-1}}(F_{X_{i-1}}^{-1}(w))} \int_{-\infty}^{F_{X_i}^{-1}(v) - F_{X_{i-1}}^{-1}(w)} g(aF_{X_{i-1}}^{-1}(w)^2 + 2bF_{X_{i-1}}^{-1}(w)t + ct^2)\, dt dw = \\ & = \sqrt{ac-b^2} \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{X_i}^{-1}(v)-s} g(as^2 + 2bst + ct^2)\, dt dw = \\ & = \sqrt{ac-b^2} \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{X_i}^{-1}(v)} g((a+c-2b)s^2 + 2\hat{t}s(b-c) + c\hat{t}^2)\, d\hat{t} dw \end{aligned}$$

and this is of the same type as (8) with associated matrix $\begin{pmatrix} a+c-2b & b-c \\ b-c & c \end{pmatrix}$.

3.3 Farlie-Gumbel-Morgenstern Processes

An important question is how to extract the dependence structure between level and increments given an assigned dependence structure between levels. The question is particularly useful for copula functions that are known to be closed under the product operator, for example for Markov processes for which the dependence structure of any couple of observations is described by copulas of the same family. Besides the Gaussian copula, another case is the Farlie-Gumbel-Morgenstern copula, defined as

$$C(u,v) = uv + \theta uv(1-u)(1-v), \quad \theta \in [0,1]$$

The following proposition shows how to choose the dependence structure between levels and increments to generate a Farlie-Gumbel-Morgenstern stochastic process.

Proposition 3.1. *Let F_{Y_i}, $F_{X_{i-1}}$ and F_{X_i} three cumulative distribution functions on $\mathbb{R}$, with corresponding densities f_{Y_i}, $f_{X_{i-1}}$ and f_{X_i}. Let us consider a copula C_{X_{i-1},Y_i} whose density is*

$$c_{X_{i-1},Y_i}(u,u') = \left[1+\theta(1-2u)(1-2F_{X_i}(F_{Y_i}^{-1}(u')+F_{X_{i-1}}^{-1}(u)))\right]\frac{f_{X_i}(F_{Y_i}^{-1}(u')+F_{X_{i-1}}^{-1}(u))}{f_{Y_i}(F_{Y_i}^{-1}(u'))}.$$

Then

$$C_{X_{i-1},X_i}(u,v) \quad = \quad uv(1+\theta(1-u)(1-v)), \quad F_{X_i} = F_{X_{i-1}} \overset{C}{*} F_{Y_i}.$$

Proof. By Remark 2.1 and the assumption

$$\begin{aligned}
c_{X_{i-1},X_i}(u,v) &= c_{X_{i-1},Y_i}(u, F_{Y_i}(F_{X_i}^{-1}(v) - F_{X_{i-1}}^{-1}(u)))\frac{f_{Y_i}(F_{X_i}^{-1}(v) - F_{X_{i-1}}^{-1}(u))}{f_{X_i}(F_{X_i}^{-1}(v))} \\
&= 1+\theta(1-2u)(1-2v)
\end{aligned}$$

which is the density of the Farlie-Gumbel-Morgenstern copula. Moreover,

$$\begin{aligned}
(F_{X_{i-1}} \overset{C}{*} F_{Y_i})'(t) &= \int_0^1 c_{X_{i-1},Y_i}(w, F_{Y_i}(t-F_{X_{i-1}}^{-1}(w)))f_{Y_i}(t-F_{X_{i-1}}^{-1}(w))dw = \\
&= \int_0^1 \left[1+\theta(1-2w)(1-2F_{X_i}(t))\right] f_{X_i}(t)dw = \\
&= f_{X_i}(t)\left[1+\theta(1-2F_{X_i}(t))\int_0^1 (1-2w)dw\right] = f_{X_i}(t).
\end{aligned}$$

□

3.4 Archimedean Processes

Unfortunately, in many cases the copula function is not closed either under the convolution or the product operator. An example is given by Archimedean copulas, which are largely used in the representation of financial price dynamics, with special mention of the Clayton copula (see in particular Ibragimov and Lentzas (2008)). This means that if the Clayton copula is used to represent the dependence structure between the level of a stochastic process and its increment, the dependence structure between the levels of the process before and after the increment will not be represented by a Clayton copula. By the same token, in a Markov process built using the Clayton copula linking levels at adjacent dates, the dependence structure between the levels of the process in more distant dates will not be described by a Clayton copula. In these cases, though, the only solution is to simulate the Markov process. Luckily, in our approach the use of conditional sampling makes it easy to generate these trajectories, as the algorithm below illustrates.

Remark 3.2. *Simulation of a process with dependent increments. We provide here a pseudo-algorithm of the simulation of the dynamics of a process with dependent increments. The input is given by a sequence of distributions of increments that for the sake of simplicity we assume stationary $F_{Y_i} = F_Y$ and a temporal dependence structure that we consider stationary as well, $C_{X_i,Y_{i+1}}(u,v) = C(u,v)$. We also assume $X_0 = 0$. We describe a procedure to generate an iteration of a n-step trajectory.*

1. *For $i = 1$ to n*
2. *Generate u from the uniform distribution*
3. *Compute $X_i = F_Y^{-1}(u)$*
4. *Use conditional sampling to generate v from $D_1C(u,v)$*
5. *Compute $Y_{i+1} = F_Y^{-1}(v)$*
6. *$X_{i+1} = X_i + Y_{i+1}$*
7. *Compute the distribution $F_{X_{i+1}}(t)$ by C-convolution*
8. *Compute $u = F_{X_{i+1}}(X_{i+1})$*

Figure 1 reports examples of trajectories generated assuming Clayton dependence between levels and increments and Gaussian distributions for the increments.

3.5 Symmetric Processes

A natural question is how to build processes endowed with particular features, such as symmetry. The issue of building symmetric processes in the Darsow et al. (1992) framework was addressed in Cherubini and Romagnoli (2010). Here we provide the corresponding characterization for our construction based on increments. The result is quite straightforward, but in order to introduce it we must first briefly digress on the concepts of symmetry. As it is well known, in the univariate setting the symmetry with respect to the origin of a distribution is represented by the condition

$$F(k) = \overline{F}(-k) \equiv 1 - F(-k) \tag{9}$$

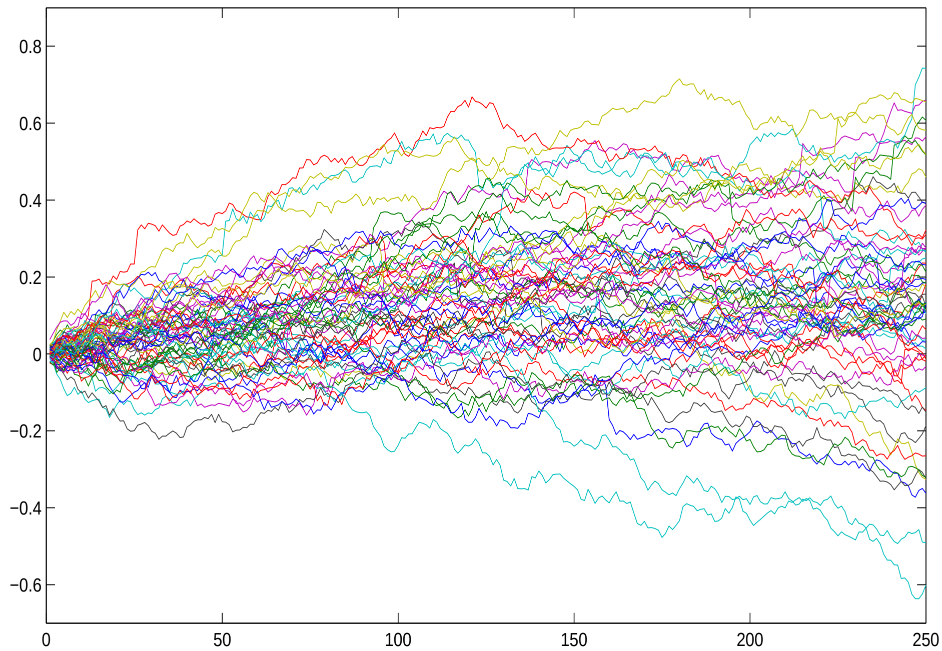

Figure 1: Simulation of trajectories with Clayton temporal dependence ($\tau = 0.1$) and Normal distribution of increments with volatility equal to 0.2

In a multivariate setting, several definitions of symmetry are available, and in this paper we will use two of them. One of them is that of *radial symmetry*. To introduce it, we need the concept of *survival copula*, that is the copula associated with the complements of the events. In a bivariate setting, given a copula $C(u,v)$ we define

$$\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

the corresponding survival copula (see Nelsen (2006), p.32). A copula function is said to be *radially symmetric*, if $\overline{C}(u,v) = C(u,v)$. Standard examples are the Gaussian and Frank copulas. It is now straightforward to prove that symmetry of the distribution of increments and radial symmetry of the dependence structure between levels and increments imply symmetry of the stochastic process.

Proposition 3.2. *Let $\overline{C}$ be the survival copula that is $\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ and $\overline{F}(t) = 1 - F(t)$. If $\overline{C}_{X_{i-1},Y_i}(u,v) = C_{X_{i-1},Y_i}(u,v)$ and $\overline{F}_{X_{i-1}}(t) = F_{X_{i-1}}(-t), \overline{F}_{Y_i}(t) = F_{Y_i}(-t)$, then $\overline{F}_{X_i}(t) = F_{X_i}(-t)$.*

Proof. Since $\overline{F}^{-1}_{X_{i-1}}(w) = F^{-1}_{X_{i-1}}(1-w) = -F^{-1}_{X_{i-1}}(w)$,

$$\begin{aligned}
\overline{F}_{X_i}(t) &= \int_0^1 D_1\overline{C}_{X_{i-1},Y_i}\left(w, \overline{F}_{Y_i}(t - \overline{F}^{-1}_{X_{i-1}}(w))\right) dw \\
&= \int_0^1 D_1 C_{X_{i-1},Y_i}\left(w, F_{Y_i}(-t + \overline{F}^{-1}_{X_{i-1}}(w))\right) dw \\
&= \int_0^1 D_1 C_{X_{i-1},Y_i}\left(w, F_{Y_i}(-t - F^{-1}_{X_{i-1}}(w))\right) dw = F_{X_i}(-t).
\end{aligned}$$

□

3.6 Strictly Stationary Processes

From (2) and (3) it is trivial to recover the copula between levels and increments from that between levels. In fact, one gets

$$C_{X_{i-1},Y_i}(u,v) = \int_0^u D_1 C_{X_{i-1},X_i}\left(w, F_{X_i}(F^{-1}_{Y_i}(v) + F^{-1}_{X_{i-1}}(w))\right) dw$$

where $F_{X_{i-1}}$, F_{X_i} and F_{Y_i} must satisfy

$$F_{Y_i}(t) = \int_0^1 D_1 C_{X_{i-1},X_i}\left(w, F_{X_i}(t + F^{-1}_{X_{i-1}}(w))\right) dw.$$

A strictly stationary Markov process is characterized by assuming $C_{X_{i-1},X_i} \equiv C, \forall i$ and $F_{X_i} \equiv F, \forall i$. This kind of process can be recovered in our framework, for any given C and F, by setting

$$F_{Y_i}(t) = \int_0^1 D_1 C\left(w, F(t + F^{-1}(w))\right) dw \equiv G(t)$$

and

$$C_{X_{i-1},Y_i}(u,v) = \int_0^u D_1 C\left(w, F(G^{-1}(t) + F^{-1}(w))\right) dw \equiv A(u,v).$$

Notice that both the distribution of the increments and the copula between the level and the increments are stationary.

4 The Martingale Condition

We can now come to impose the necessary restriction to make our approach applicable to financial time series that are consistent with the Efficient Market

Hypothesis. This requires that price movements are unpredictable. While in most of the literature this has been accomplished by assuming independent increments (in the famous random walk hypothesis), here we show that this restriction is not necessary, and that we may build a model of an efficient market with dependent increments. The technical issue is then to impose the martingale restriction to Markov processes. To the best of our knowledge, this topic was first introduced in the Darsow et al. (1992) framework by Ibragimov (2005). It is straightforward to see that the representation of the process in terms of increments makes it easier to address the problem. Formally, we want to choose the stochastic process for $\{X_i\}_{i\geq 0}$ such that, for $i \geq 1$ and all Borel measurable functions f

$$E\left(f(X_{i-1})(X_i - X_{i-1})\right) = 0. \tag{10}$$

We need to work out the restrictions that must be imposed on copula based representations of Markov processes in order to ensure that the condition in (10) holds. Actually, our strategy to model increments makes the analysis tractable for some class of processes. It is definitely immediate for processes with independent increments (see Example 3.2), in which case the restriction to be imposed follows directly from the result desired.

Proposition 4.1. *Any process whose increments $Y_i \equiv X_i - X_{i-1}$, are independent of X_{i-1} ($C_{X_{i-1},Y_i}(u,v) \equiv uv$) and whose distributions F_{Y_i} have zero mean is a martingale.*

Furthermore, our choice to model the dependence structure between increments and levels provides a straightforward extension to the more general case, in which the independence assumption is dropped. Actually, our entire strategy for the construction of Markov processes is built upon the idea of modelling

$$\mathbb{P}\left(X_i - X_{i-1} \leq x | X_{i-1}\right). \tag{11}$$

It is for this reason that it suffices to concentrate on the copula function $C_{X_{i-1},Y_i}(u,v)$ and its density $c_{X_{i-1},Y_i}(u,v)$.

Theorem 4.1. *Let $X = \{X_i\}_{i\geq 0}$ be a Markov process and set $Y_i = X_i - X_{i-1}$. X is a martingale if and only if:*

1. *F_{Y_i} has finite mean for every i;*
2. *for $i \geq 1$, $\int_0^1 F_{Y_i}^{-1}(v) c_{X_{i-1},Y_i}(u,v) dv = 0, \ \ \forall u \in [0,1]$ a.e..*

Proof. X is a Markov process and it is a martingale if and only if $\mathbb{E}\left[X_i - X_{i-1} | X_{i-1}\right] = 0$ for every $i \geq 1$. But

$$\begin{aligned}\mathbb{E}\left[X_i - X_{i-1}|X_{i-1}\right] &= \int_{-\infty}^{+\infty} z c_{X_{i-1},Y_i}(F_{X_{i-1}}(X_{i-1}), F_{Y_i}(z)) f_{Y_i}(z) dz = \\ &= \int_0^1 F_{Y_i}^{-1}(v) c_{X_{i-1},Y_i}\left(F_{X_{i-1}}(X_{i-1}), v\right) dv.\end{aligned}$$

Hence X is a martingale if and only if $\int_0^1 F_{Y_i}^{-1}(v) c_{X_{i-1},Y_i}\left(F_{X_{i-1}}(X_{i-1}), v\right) dv = 0$. □

Remark 4.1. *An analogous result is given in Ibragimov (2009) for the martingale difference, that, in that paper, is assumed to be a Markov process. Unfortunately, even if the martingale difference is a Markov process, in general the induced martingale may not be a Markov process anymore. The above Theorem is motivated by the fact that we are interested in constructing a process X that is both a Markov process and a martingale.*

4.1 Martingale with Symmetric Increments

The above theorem provides the set of necessary and sufficient requirements that have to be imposed on the Markov process to make it a martingale. An interesting question is whether this definition accommodates other classes of processes beyond the independent increment class. In order to construct other cases we first define a class of copula functions, according to a concept of symmetry different from that applied above.

Definition 4.2. *A copula function $C(u,v)$ is said to be "symmetric around the first coordinate", if*

$$\widetilde{C}(u,v) \equiv u - C(u,1-v) = C(u,v).$$

so that the pairs (U,V) and $(U,1-V)$ have the same joint distribution C.

This concept of symmetry, coupled with symmetry of the distribution of increments, enables us to define an interesting class of martingale processes.

Proposition 4.2. *The martingale condition stated in Theorem 4.1 within the class of Markov processes, is satisfied for every symmetric distribution of increments F_{Y_i} if and only if the copula between the increments and the levels is symmetric (around the first coordinate).*

Remark 4.2. *An analogous result for the Martingale difference is obtained in Theorem 4 in Ibragimov (2009).*

A question remains as to how large is the class of copulas which is encompassed by Proposition 4.2. Actually this class may be quite large, since, as we prove below, a copula with the required symmetry feature can be built starting from any arbitrary copula. The same result is found in an even more general setting in Klement et al. (2002), who show that this technique can be further extended to all concepts of symmetry (or invariance, how they are called in that paper), including radial symmetry.

Proposition 4.3. *Take any bivariate copula $A(u,v)$ and its symmetric part $\widetilde{A}(u,v) \equiv u - A(u,1-v)$. Define: $C(u,v) \equiv 0.5A(u,v) + 0.5\widetilde{A}(u,v)$. Then, $C(u,v)$ is a copula and it is symmetric in the sense that $C(u,v) = \widetilde{C}(u,v)$.*

Proof. First, notice that it is easy to show that $\widetilde{A}(u,v)$ is a copula (see Nelsen (2006)). Second, $C(u,v)$ is a copula because it is a mixture of copulas. It may be in fact immediately verified that $C(0,v) = C(u,0) = 0$, $C(1,v) = v$,

$C(u,1)=u$. It is 2-increasing because it is the sum of two 2-increasing elements. Having proved that it is a copula, the symmetry property of $C(u,v)$ can be easily checked

$$\begin{aligned}\widetilde{C}(u,v) &= u - C(u,1-v) = u - (0.5A(u,1-v) + 0.5u - 0.5A(u,v)) \\ &= 0.5A(u,v) + 0.5u - 0.5A(u,1-v) = C(u,v)\end{aligned}$$

□

Proposition 4.3 states that all symmetric copulas (in our sense) can be obtained in this way. For every choice of the class of symmetric distributions of increments we can then choose a symmetric copula function $C(u,v)$ corresponding to an arbitrary copula function $A(u,v)$. Furthermore, all the copulas endowed with this symmetry property can be represented by this procedure. As for the simulation of martingale processes, it is easy to see that a minor modification of the procedure presented in Remark 3.2 would do the job.

Remark 4.3. *Simulation of a martingale process with dependent increments. We provide here a pseudo-algorithm of the simulation of the dynamics of a process with dependent increments with symmetric marginal distributions and temporal dependence. The input is given by a sequence of symmetric distributions of increments that for the sake of simplicity we assume stationary $F_{Y_i} = F_Y$ and a mixture copula representing temporal dependence that we consider stationary as well: $C_{X_i,Y_{i+1}}(u,v) = 0.5C(u,v) + 0.5(u - C(u,1-v))$. We also assume $X_0 = 0$. We describe a procedure to generate an iteration of a n-step trajectory.*

1. *For $i = 1$ to n*
2. *Generate u from the uniform distribution*
3. *Compute $X_i = F_Y^{-1}(u)$*
4. *Generate ξ from the uniform distribution*
5. *Use conditional sampling to generate v from $D_1C(u,v)$*
6. *Compute $Y_{i+1} = F_Y^{-1}(v)$*
7. *If $\xi \leq 0.5$, $Y_{i+1} = -Y_{i+1}$*
8. *$X_{i+1} = X_i + Y_{i+1}$*
9. *Compute the distribution $F_{X_{i+1}}(t)$ by C-convolution*
10. *Compute $u = F_{X_{i+1}}(X_{i+1})$*

In Figure 2 we report examples of trajectories of a martingale process generated assuming Clayton dependence between levels and increments and Gaussian distributions for the increments.

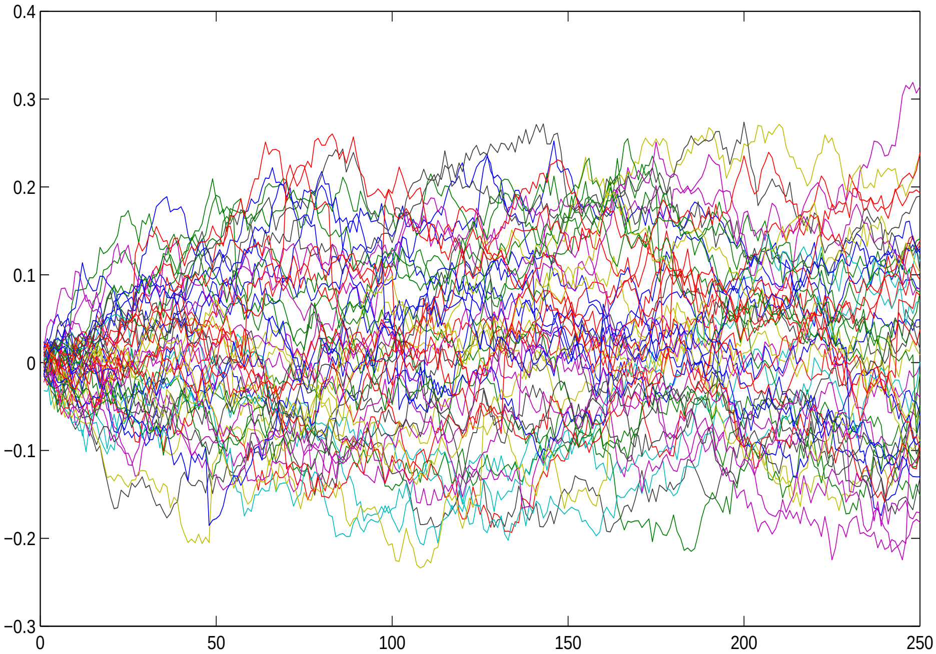

Figure 2: Simulation of martingale trajectories with Clayton temporal dependence ($\tau = 0.1$) and Normal distribution of increments with volatility equal to 0.2

5 Copula characterization of bivariate Markov processes

Multivariate Markov processes have already been studied in Remillard et al. (2010) and Yi and Liao (2010). In this paper we are interested in extending the above analysis to the multivariate setting. We first provide the multivariate extension of the copula approach to Markov processes following Ibragimov (2009).

Let $m, n \geq 2$ and A and B be, respectively, m- and n- dimensional copulas.

Set

$$A_{1,\dots,m|m-1,m}(u_1,\dots,u_{m-2},\xi,\eta) = \frac{\frac{\partial^2 A(u_1,\dots,u_{m-2},\xi,\eta)}{\partial\xi\partial\eta}}{\frac{\partial^2 A(1,\dots,1,\xi,\eta)}{\partial\xi\partial\eta}}, \quad \text{and}$$

$$B_{1,\dots,n|1,2}(\xi,\eta,u_3,\dots,u_n) = \frac{\frac{\partial^2 B(\xi,\eta,u_3,\dots,u_n)}{\partial\xi\partial\eta}}{\frac{\partial^2 B(\xi,\eta,1\dots,1)}{\partial\xi\partial\eta}}.$$

If $A(1,\dots,1,\xi,\eta) = B(\xi,\eta,1,\dots,1) = C(\xi,\eta)$, where C is a bivariate copula, we can define the $\star^2$-product of the copulas A and B as the copula $D = A\star^2 B : [0,1]^{m+n-2} \to [0,1]$ given by

$$D(u_1,\dots,u_{m+n-2}) = \int_0^{u_{m-1}}\int_0^{u_m} A_{1,\dots,m|m-1,m}(u_1,\dots,u_{m-2},\xi,\eta)\cdot \\ \cdot B_{1,\dots,n|1,2}(\xi,\eta,u_3,\dots,u_n)dC(\xi,\eta).$$

The $\star^2$ operator is a particular case of the $\star^k$ operator in Ibragimov (2009).

Recall that, if $(Y_1,\dots,Y_n)$ is a random vector with associated copula function $C(u_1,\dots,u_n)$ and margins F_i for $i=1,\dots,n$,

$$\mathbb{P}\left(Y_1 \le y_1,\dots,Y_{n-2}\le y_{n-2}|Y_{n-1}=x, Y_n=y\right) = \frac{\frac{\partial^2 C(F_1(y_1),\dots,F_{n-2}(y_{n-2}),F_{n-1}(x),F_n(y))}{\partial u_{n-1}\partial u_n}}{\frac{\partial^2 C(1,1,\dots,1,F_{n-1}(x),F_n(y))}{\partial u_{n-1}\partial u_n}}$$

that is

$$\mathbb{P}\left(Y_1 \le y_1,\dots,Y_{n-2}\le y_{n-2}|Y_{n-1}=x, Y_n=y\right) = C_{1,\dots,n|n-1,n}(F_1(y_1),\dots,F_{n-1}(x),F_n(y))$$

and, similarly,

$$\mathbb{P}\left(Y_3 \le y_3,\dots,Y_n\le y_n|Y_1=x, Y_2=y\right) = C_{1,\dots,n|1,2}(F_1(x),F_2(y),\dots,F_{n-1}(y_{n-1}),F_n(y_n)).$$

Let $(X,Z) = \{(X_i,Z_i)\}_{i\ge 0}$ be an $\mathbb{R}^2$-valued stochastic process defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $(\mathcal{F}_i^{X,Z})_{i\ge 0}$ be its natural filtration.

By definition (X,Z) is a Markov process if, for all $(x_{n+1},z_{n+1})\in\mathbb{R}^2$

$$\mathbb{P}\left(X_{n+1}\le x_{n+1}, Z_{n+1}\le z_{n+1}|X_n,Z_n,X_{n-1},Z_{n-1},\dots,X_1,Z_1\right) = \\ = \mathbb{P}\left(X_{n+1}\le x_{n+1}, Z_{n+1}\le z_{n+1}|X_n,Z_n\right). \tag{12}$$

Let $C_{j_1,j_2,\dots,j_n}(u_1,v_1,\dots,u_n,v_n)$ denote the $2n$-dimensional copulas corresponding to the joint distribution of the random vector $(X_1,Z_1,X_2,Z_2,\dots,X_n,Z_n)$. We set

$$C_{j_1,\dots,j_n|j_n}(u_1,v_1,\dots,\xi,\eta) = \frac{\frac{\partial^2 C(u_1,v_1,\dots,u_{n-1},v_{n-1},\xi,\eta)}{\partial\xi\partial\eta}}{\frac{\partial^2 C(1,\dots,1,\xi,\eta)}{\partial\xi\partial\eta}}, \quad \text{and}$$

$$C_{j_1,\dots,j_n|j_1}(\xi,\eta,\dots,u_n,v_n) = \frac{\frac{\partial^2 C(\xi,\eta,\dots,u_n,v_n)}{\partial\xi\partial\eta}}{\frac{\partial^2 C(\xi,\eta,1,\dots,1)}{\partial\xi\partial\eta}}.$$

Theorem 5.1. *An $\mathbb{R}^2$-valued stochastic process (X, Z) is a Markov process if and only if for all $n \geq 2$ and all $j_1 < \ldots < j_i < \ldots < j_n$*

$$C_{j_1,j_2,\ldots,j_n} = C_{j_1,j_2} \star^2 C_{j_2,j_3} \star^2 \cdots \star^2 C_{j_{n-1},j_n}. \tag{13}$$

5.1 The Martingale Condition

We provide here the multivariate extension of the martingale condition.
By definition, (X, Z) is a martingale with respect to $\mathcal{F}^{X,Z}$ iff

$$\mathbb{E}\left[X_{i+1} - X_i | X_i, Z_i\right] = 0 \quad \text{and} \quad \mathbb{E}\left[Z_{i+1} - Z_i | X_i, Z_i\right] = 0, \qquad \forall i \geq 0.$$

Let $\Delta X_i = X_{i+1} - X_i$, $\Delta Z_i = Z_{i+1} - Z_i$ and $A_{i,i+1}(u, v, w, \lambda)$ be the copula function of the random vector $(X_i, Z_i, \Delta X_i, \Delta Z_i)$ with $F_{X_i}, F_{Z_i}, F_{\Delta X_i}, F_{\Delta Z_i}$ the corresponding marginal c.d.f.'s. We set $a_{i,i+1}(u, v, w, 1)$ the density of the copula $A_{i,i+1}(u, v, w, 1)$ and $a_{i,i+1}(u, v, 1, w)$ the density of the copula $A_{i,i+1}(u, v, 1, w)$.

Theorem 5.2. *The Markov process (X, Z) is a martingale with respect to the filtration $\mathcal{F}^{X,Z}$ iff:*

1. *$F_{\Delta X_i}$ and $F_{\Delta Z_i}$ have finite mean for every i;*
2. *for every i,*

$$\begin{aligned} &\int_0^1 F^{-1}_{\Delta X_i}(w) a_{i,i+1}(u, v, w, 1) dw = 0 \quad \textit{and} \\ &\int_0^1 F^{-1}_{\Delta Z_i}(w) a_{i,i+1}(u, v, 1, w) dw = 0, \qquad \forall u, v \in [0, 1]. \end{aligned} \tag{14}$$

5.2 No-Granger causality

We now show that once the martingale condition has been proved for each process, the multivariate extension can be recovered by simply applying a concept that is standard in econometrics and is known as the Granger causality.

Definition 5.3. *Let $X = \{X_i\}_{i\geq 0}$ and $Z = \{Z_i\}_{i\geq 0}$ be two stochastic processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}^X_i$ and $\mathcal{F}^Z_i$ be their respective natural filtrations, while $\mathcal{F}^{X,Z}_i$ denotes the natural filtration of the bivariate process $(X, Z) = \{(X_i, Z_i)\}_{i\geq 0}$.*

We say that Z_i doesn't cause X_i with respect to $\mathcal{F}^{X,Z}_i$ if, for any i and x

$$\mathbb{P}[X_{i+1} \leq x | \mathcal{F}^{X,Z}_i] = \mathbb{P}[X_{i+1} \leq x | \mathcal{F}^X_i].$$

Remark 5.1. *Definition 5.3 is the usual concept of no-causality in the sense of Granger and for this reason we call it "no-Granger causality".*

Let us now restrict the analysis to the class of Markov processes. Remember that a process, Markov with respect to a given filtration, is not in general Markov with respect to a larger filtration. We show that this is in fact guaranteed by no-Granger causality.

Theorem 5.4. *The following are equivalent:*

1. *Z does not Granger cause X for every i;*
2. *if X is an $\mathcal{F}^X$-Markov process, then it is an $\mathcal{F}^{X,Z}$-Markov process, as well.*

Proof. 1. $\Rightarrow$ 2.:1. implies $\mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^{X,Z}] = \mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^X]$, for every $x \in \mathbb{R}$. By hypothesis $\mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x|X_i]$ and the thesis follows. The other implication is trivial. □

We saw that no-Granger causality and Markov property of each process with respect to its natural filtration together imply the Markov structure of the system as a whole. The converse does not hold true as the following Remark shows.

Remark 5.2. *Let (X, Z) be a Markov process with respect to its natural filtration so that*

$$\mathbb{P}\left(X_{i+1} \leq x|\mathcal{F}_i^{X,Z}\right) = \mathbb{P}\left(X_{i+1} \leq x, |X_i, Z_i\right) \tag{15}$$

If Z does not Granger cause X for every i and $x \in \mathbb{R}$,

$$\mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^{X,Z}] = \mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x|X_i, X_{i-1}, X_{i-2}, \dots, X_0]. \tag{16}$$

(15) and (16) do not imply that X is a Markov process with respect to its natural filtration. Take $X_{i+1} = Z_i + X_i$ and $Z_{i+1} = X_i$ as a counterexample: (X, Z) is a Markov process, X satisfies (15) and (16), but

$$\mathbb{P}[X_{i+1} \leq x|\mathcal{F}_i^X] = \mathbb{P}[X_{i+1} \leq x|X_i, X_{i-1}, X_{i-2}, \dots, X_0] \neq \mathbb{P}[X_{i+1} \leq x|X_i].$$

In order to guarantee that, given a multivariate Markov process, each of its components be a Markov process with respect to its own natural filtration as well, it is necessary to introduce an adequate restriction to the law of the processes involved.

Proposition 5.1. *Let (X, Z) be a bivariate Markov process and assume that Z does not Granger cause X for every i.*

If X is an $\mathcal{F}^X$-Markov process, then

$$\mathbb{P}\left(X_{i+1} \leq x|\, X_i, Z_i\right) = \mathbb{P}\left(X_{i+1} \leq x|\, X_i\right).$$

Proof. By hypothesis

$$\mathbb{P}\left(X_{i+1} \leq x|\, \mathcal{F}_i^{X,Z}\right) = \mathbb{P}\left(X_{i+1} \leq x|\, \mathcal{F}_i^X\right),$$

$$\mathbb{P}\left(X_{i+1} \leq x \middle| \mathcal{F}_i^{X,Z}\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i, Z_i\right)$$

and

$$\mathbb{P}\left(X_{i+1} \leq x \middle| \mathcal{F}_i^{X}\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right).$$

The statement follows trivially. □

Proposition 5.2. *Let (X, Z) be a bivariate Markov process. If*

$$\mathbb{P}\left(X_{i+1} \leq x \middle| X_i, Z_i\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right)$$

then X is an $\mathcal{F}^X$-Markov process.

Proof. By hypothesis

$$\mathbb{P}\left(X_{i+1} \leq x \middle| \mathcal{F}_i^{X,Z}\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i, Z_i\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right).$$

But

$$\begin{aligned}\mathbb{P}\left(X_{i+1} \leq x \middle| \mathcal{F}_i^{X}\right) &= \mathbb{E}\left[\mathbb{P}\left(X_{i+1} \leq x \middle| \mathcal{F}_i^{X,Z}\right) \middle| \mathcal{F}_i^X\right] = \\ &= \mathbb{E}\left[\mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right) \middle| \mathcal{F}_i^X\right] = \\ &= \mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right).\end{aligned}$$

□

Propositions 5.1 and 5.2 are equivalent to Lemma 3.5 in (Florens et al. (1996)).

Theorem 5.5. *Let (X, Z) be a bivariate Markov process and $C_{i,i+1}(u_1, v_1, u_2, v_2) = C_{X_i,Z_i,X_{i+1},Z_{i+1}}(u_1, v_1, u_2, v_2)$.*

$$\mathbb{P}\left(X_{i+1} \leq x \middle| X_i, Z_i\right) = \mathbb{P}\left(X_{i+1} \leq x \middle| X_i\right)$$

iff

$$C_{i,i+1}(u_1, v_1, u_2, 1) = C_{Z_i,X_i} \star C_{X_i,X_{i+1}}(v_1, u_1, u_2).$$

Similarly we obtain that X does not Granger cause Z iff

$$C_{i,i+1}(u_1, v_1, 1, v_2) = C_{X_i,Z_i} \star C_{Z_i,Z_{i+1}}(u_1, v_1, u_2).$$

The importance of the concept of no-causality relies on the fact that it permits ensuring the stability of the martingale property with respect to enlarged filtrations (see Florens and Fougère (1993)). In fact, it is a known result that if X is both an $\mathcal{F}^X$ and an $\mathcal{F}^{X,Y}$-Markov process and it is furthermore an $\mathcal{F}^X$-martingale, it turns out to be an $\mathcal{F}^{X,Y}$-martingale as well (see Brémaud and Yor (1978)). This, together with Theorem 5.4, implies that the no-Granger causality induces that the martingale property extends from the natural filtration of each process to the filtration generated by the whole multivariate Markov process.

6 Conclusions

In this paper we have addressed the problem of constructing Markov processes for speculative prices, in the spirit of the copula-based representation that was first introduced by Darsow et al. (1992). The approach requires defining:

- a sequence of distribution functions of the increments of the process;
- a sequence of copula functions representing dependence between each increment of the process and the corresponding level of the process before the increment.

We show that this construction is very well suited to impose restrictions that are consistent with the speculative price dynamics expected under the Efficient Market Hypothesis. Namely, we specify conditions under which innovations of log prices are unpredictable. More precisely, we single out two classes of Markov processes that satisfy this martingale condition:

- processes with independent increments with zero mean distributions;
- processes with symmetric increments linked to the initial levels by a symmetric copula.

Notice that the latter class actually provides an extension of the standard independent increment class used in most of the literature. We find that the extension of the martingale restriction to a multivariate setting involves a concept which is very well known in econometrics and is called *Granger causality*. We show how to express this concept in our copula based framework.
We also provide algorithms for the generation of paths of Markov processes with dependent increments, both with and without the martingale restriction. An evaluation and optimization of these procedures is the main topic that we leave for future research.

7 Appendix

Proof of Proposition 4.2. For simplicity we set $C_{X_{i-1},Y_i} = C$, $c_{X_{i-1},Y_i} = c$ and $F_i = F$. Being F a symmetric distribution,

$$\begin{aligned}
\int_0^1 F^{-1}(v)c(u,v)dv &= \int_0^{\frac{1}{2}} F^{-1}(v)c(u,v)dv + \int_{\frac{1}{2}}^1 F^{-1}(v)c(u,v)dv = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)c(u,v)dv - \int_{\frac{1}{2}}^0 F^{-1}(1-\rho)c(u,1-\rho)d\rho = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)c(u,v)dv - \int_0^{\frac{1}{2}} F^{-1}(\rho)c(u,1-\rho)d\rho = \\
&= \int_0^{\frac{1}{2}} F^{-1}(v)[c(u,v) - c(u,1-v)]dv = 0, \quad \forall u \in (0,1).
\end{aligned}$$

Last condition is satisfied for every symmetric distribution F iff (notice that, in last integral, $F^{-1}(v) < 0$ in some not empty interval)

$$c(u,v) - c(u,1-v) = 0 \quad \forall u, v \in (0,1).$$

It may be easily verified that this condition is satisfied if and only if

$$C(u,v) + C(u,1-v) = u$$

which is the symmetry condition assumed for the copula. □

Proof of Theorem 5.1. Similarly as done in the proof of Theorem 1 in Ibragimov (2009), it is easily obtainable that property (12) holds if and only if

$$\begin{aligned}
&\mathbb{P}\left(X_i \leq x_i, Z_i \leq y_i,\, i = 1, \ldots, n+1 \mid X_n, Z_n\right) = \\
&= \mathbb{P}\left(X_i \leq x_i, Z_i \leq y_, i = 1, \ldots, n-1 \mid X_n, Z_n\right) \mathbb{P}\left(X_{n+1} \leq x_{n+1}, Z_{n+1} \leq y_{n+1} \mid X_n, Z_n\right)
\end{aligned} \tag{17}$$

that is

$$\begin{aligned}
&\mathbb{P}\left(X_i \leq x_i, Z_i \leq y_i,\, i = 1, \ldots, n+1 \mid X_n, Z_n\right) = \\
&= C_{1,\ldots,n|t_n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), F_{X_n}(X_n), F_{Z_n}(Z_n)\right) \cdot \\
&\cdot C_{n,n+1|n}\left(F_{X_n}(X_n), F_{Z_n}(Z_n), F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right).
\end{aligned} \tag{18}$$

Integrating (18) over $X_n^{-1}\left((-\infty, x_n)\right) \times Z_n^{-1}\left((-\infty, y_n)\right)$, we get

$$\begin{aligned}
&C_{1,\ldots,n,n+1}(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})) = \\
&= \int_{-\infty}^{x_n}\int_{-\infty}^{y_n} C_{1,\ldots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), F_{X_n}(x), F_{Z_n}(y)\right) \cdot \\
&\cdot C_{n,n+1|n}\left(F_{X_n}(x), F_{Z_n}(y), F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dF_{(X_n,Z_n)}(x,y) = \\
&= \int_{-\infty}^{F_{X_n}(x_n)}\int_{-\infty}^{F_{Z_n}(y_n)} C_{1,\ldots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), \xi, \eta\right) \cdot \\
&\cdot C_{n,n+1|n}\left(\xi, \eta, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dC_n(\xi,\eta) = \\
&= C_{1,\ldots,n} \star^2 C_{n,n+1}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right).
\end{aligned}$$

By induction, we obtain (13).

Conversely, suppose that (13) holds. We have

$$\begin{aligned}
&\mathbb{P}\left(X_i \leq x_i, Z_i \leq y_i, i = 1, \ldots n+1\right) = \\
&= C_{1,\ldots n,n+1}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) = \\
&= \int_{-\infty}^{F_{X_n}(x_n)}\int_{-\infty}^{F_{Z_n}(y_n)} C_{1,\ldots,n|n}\left(F_{X_1}(x_1), F_{Z_1}(y_1), \ldots, F_{X_{n-1}}(x_{n-1}), F_{Z_{n-1}}(y_{n-1}), \xi, \eta\right) \cdot \\
&\cdot C_{n,n+1|n}\left(\xi, \eta, F_{X_{n+1}}(x_{n+1}), F_{Z_{n+1}}(y_{n+1})\right) dC_n(\xi,\eta) = \\
&= \mathbb{E}\left[\mathbb{P}\left(X_1 \leq x_1, Z_1 \leq y_1, \ldots, X_{n-1} \leq x_{n-1}, Z_{n-1} \leq y_{n-1} \mid X_n, Z_n\right) \cdot\right. \\
&\left.\cdot\, \mathbb{P}\left(X_{n+1} \leq x_{n+1}, Z_{n+1} \leq y_{n+1} \mid X_n, Z_n\right) \mathbb{I}_{\{X_n \leq x_n, Z_n \leq y_n\}}\right]
\end{aligned}$$

from which (17) follows. $\square$

Proof of Theorem 5.2. Since

$$\mathbb{P}\left(\Delta X_i \leq z | X_i, Z_i\right) = \frac{\frac{\partial^2}{\partial u \partial v} A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1)}{\frac{\partial^2}{\partial u \partial v} A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)},$$

$$\mathbb{E}\left[X_{i+1} - X_i | X_i, Z_i\right] = \int_{-\infty}^{+\infty} z \frac{a_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1) F'_{\Delta X_i}(z)}{\frac{\partial^2}{\partial u \partial v} A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)} dz =$$
$$= \frac{1}{\frac{\partial^2}{\partial u \partial v} A_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1)} \int_{-\infty}^{+\infty} z a_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1) F'_{\Delta X_i}(z) dz = 0$$

if and only if

$$\int_{-\infty}^{+\infty} z a_{i,i+1}(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{\Delta X_i}(z), 1) F'_{\Delta X_i}(z) dz = 0. \tag{19}$$

Now, setting $F_{X_i}(X_i) = u$, $F_{Z_i}(Z_i) = v$ and $F_{\Delta X_i}(z) = w$, (19) is equivalent to

$$\int_0^1 F^{-1}_{\Delta X_t}(w) a_{i,i+1}(u, v, w, 1) dw = 0, \quad \forall u, v \in [0,1].$$

(14) is obtainable similarly. $\square$

Proof of Theorem 5.5. Since

$$\mathbb{P}\left(X_{i+1} \leq x | X_i, Z_i\right) = \frac{\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(F_{X_i}(X_i), F_{Z_i}(Z_i), F_{X_{i+1}}(x), 1\right)}{\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(F_{X_i}(X_i), F_{Z_i}(Z_i), 1, 1\right)}$$

and

$$\mathbb{P}\left(X_{i+1} \leq x | X_i\right) = \frac{\partial}{\partial u_1} C_{i,i+1}\left(F_{X_i}(X_i), 1, F_{X_{i+1}}(x), 1\right),$$

the no-Granger causality holds iff

$$\frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(u_1, v_1, u_2, 1\right) = \frac{\partial^2}{\partial u_1 \partial v_1} C_{i,i+1}\left(u_1, v_1, 1, 1\right) \frac{\partial}{\partial u_1} C_{i,i+1}\left(u_1, 1, u_2, 1\right).$$

Integrating we obtain

$$\begin{aligned} C_{i,i+1}(u_1, v_1, u_2, 1) &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{i,i+1}(u', v_1, 1, 1) \frac{\partial}{\partial u'} C_{i,i+1}(u', 1, u_2, 1) du' = \\ &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{X_i, Z_i}(u', v_1) \frac{\partial}{\partial u'} C_{X_i, X_{i+1}}(u', u_2) du' = \\ &= \int_0^{u_1} \frac{\partial}{\partial u'} C_{Z_i, X_i}(v_1, u') \frac{\partial}{\partial u'} C_{X_i, X_{i+1}}(u', u_2) du' = \\ &= C_{Z_i, X_i} \star C_{X_i, X_{i+1}}(v_1, u_1, u_2). \end{aligned}$$

$\square$

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