Bootstrapping realized multivariate volatility measures*

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Abstract

We propose bootstrap methods for statistics that are a function of multivariate high frequency returns such as realized regression coefficients and realized covariances and correlations. For these measures of covariation, the Monte Carlo simulation results of Barndorff-Nielsen and Shephard (2004) show that finite sample distortions may arise if the sampling is not too frequent. Therefore, our methods are an alternative tool of inference to the asymptotic theory of Barndorff-Nielsen and Shephard (2004).

We consider an i.i.d. bootstrap applied to the vector of returns. We show that the finite sample performance of the bootstrap is superior to the first-order asymptotic theory of Barndorff-Nielsen and Shephard (2004). However, and contrary to the existing results in the bootstrap literature for regression models subject to heteroskedasticity in the error term, we show that the i.i.d. bootstrap is not second order accurate. We provide an explanation for this difference.

Keywords: Realized regression, realized beta, realized correlation, bootstrap, Edgeworth expansions.

1 Introduction

Realized statistics based on high frequency returns have become very popular in financial economics. Realized volatility is perhaps the most well known example, providing a consistent estimator of the integrated volatility under certain conditions (including the absence of microstructure noise). Its multivariate analogue is the realized covariance matrix, defined as the sum of the outer product of the vector of high frequency returns. Two economically interesting functions of the realized covariance matrix are the realized correlation and the realized regression coefficients. In particular, realized regression coefficients are obtained by regressing high frequency returns for one asset on high frequency returns for another asset. When one of the assets is the market portfolio, the result is a realized beta coefficient. A beta coefficient measures the asset's systematic risk as assessed by its correlation with

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the market portfolio. Recent examples of papers that have obtained empirical estimates of realized betas include Andersen, Bollerslev, Diebold and Wu (2005a, 2005b) and Viceira (2007).

Recently, Barndorff-Nielsen and Shephard (2004) (henceforth BN-S(2004)) (see also Jacod (1994) and Jacod and Protter (1998)) have proposed an asymptotic distribution theory for realized covariation measures based on multivariate high frequency returns. Their simulation results show that asymptotic theory-based confidence intervals for regression and correlation coefficients between two assets returns can be severely distorted if the sampling horizon is not small enough. To improve the finite sample performance of their feasible asymptotic theory approach, BN-S (2004) propose the Fisher-z transformation for realized correlation. This analytical transformation does not apply to realized regression coefficients, which in particular can be negative and larger than one in absolute value.

In this paper we propose bootstrap methods for statistics based on multivariate high frequency returns, including the realized covariance, the realized regression and the realized correlation coefficients. Our aim is to improve upon the first order asymptotic theory of BN-S (2004). We consider an i.i.d. bootstrap applied to the vector of realized returns. Gonçalves and Meddahi (2008) have recently applied this method to realized volatility in the univariate context. They also proposed a wild bootstrap for realized volatility with the motivation that intraday returns are (conditionally on the volatility path) independent but heteroskedastic when log prices are driven by a stochastic volatility model. In this paper we focus only on the i.i.d. bootstrap for three reasons. First, the results in Gonçalves and Meddahi (2008) show that the i.i.d. bootstrap dominates the wild bootstrap in Monte Carlo simulations even when volatility is time varying. Second, the i.i.d. bootstrap is easier to apply than the wild bootstrap: the wild bootstrap requires choosing an external random variable used to construct the bootstrap data whereas the i.i.d. bootstrap does not involve the choice of any tuning parameter. Third, the i.i.d. bootstrap is a natural candidate in the context of realized regressions driven by heteroskedastic errors. Indeed, the i.i.d. bootstrap applied to the vector of returns corresponds to a pairs bootstrap, as proposed by Freedman (1981). His results show that the pairs bootstrap is robust to heteroskedasticity in the error term of cross section regression models. Mammen (1993) shows that the pairs bootstrap is not only first order asymptotically valid under heteroskedasticity in the error term, but it is also second-order correct (i.e. the error incurred by the bootstrap approximation converges more rapidly to zero than the error incurred by the standard normal approximation).

We can summarize our main contributions as follows. We show the first order asymptotic validity of the i.i.d. bootstrap for estimating the distribution function of the realized covariance matrix and smooth functions of it such as the realized covariance, the realized regression and the realized correlation coefficients. We assess the finite sample performance of bootstrap confidence intervals for these three covariation measures by simulation. Our simulation results show that the bootstrap outperforms the feasible first order asymptotic theory of BN-S (2004).

The ability of the bootstrap to provide higher order asymptotic refinements over the standard normal approximation is usually established via Edgeworth expansions. In a related paper (Dovonon, Gonçalves and Meddahi (2007)), we develop the Edgeworth expansions of the distribution of the t-statistics associated with the three covariation measures studied here. These expansions are then used to construct analytical transformations of the raw statistics with improved finite sample properties (in particular, we propose transformations aimed at eliminating the bias or the skewness of the transformed statistics). By developing similar expansions for the bootstrap statistics, we can compare the accuracy of the bootstrap approximation with that of the normal approximation.

In this paper, we develop the Edgeworth expansion for the i.i.d. (or pairs) bootstrap distribution of the realized regression estimator. The existing results in the statistics literature (see Mammen (1993)) suggest that the pairs bootstrap can be second order correct in the realized regression context analyzed here even under stochastic volatility. This is not the case for the two other statistics (covariance and correlation coefficients), where the i.i.d. bootstrap cannot be expected to provide second order refinements due to the fact that it does not replicate the conditional heteroskedasticity in the data. For this reason, we do not analyze the higher order properties of the i.i.d. bootstrap for the covariance and the correlation coefficients and focus only on the regression estimator.

Contrary to our expectations based on the existing theory for the pairs bootstrap in the statistics literature, we show that the pairs bootstrap does not provide an asymptotic refinement over the standard first order asymptotic theory in the context of realized regressions. We contrast our application of the pairs bootstrap to realized regressions with the application of the pairs bootstrap in standard cross section regressions. We show that there is a main difference between these two applications, namely the fact that the score of the underlying realized regression model is heterogeneous and does not have mean zero (although the mean of the sum of the scores is zero). This heterogeneity implies that the standard Eicker-White heteroskedasticity robust variance estimator is not consistent in the realized regression context, which justifies the need for the more involved variance estimator proposed by BN-S (2004). The pairs bootstrap variance coincides with the Eicker-White robust variance estimator and therefore it does not provide a consistent estimator of the variance of the scaled average of the scores. This is in contrast with the results of Freedman (1981) and Mammen (1993), where the score has mean zero by assumption. Nevertheless, the pairs bootstrap is first order asymptotically valid when applied to a bootstrap t-statistic which is studentized with a variance estimator that is consistent for the population bootstrap variance of the scaled average of the scores. Because the bootstrap scores have mean zero, the Eicker-White robust variance estimator can be used for this effect. This implies that the bootstrap statistic is not of the same form as the statistic based on the original data, which explains why we do not get second order refinements for the pairs bootstrap in our context.

The remainder of this paper is organized as follows. In Section 2, we introduce the setup, review the existing first order asymptotic theory and state regularity conditions. We also present some Monte Carlo simulation results that illustrate the finite sample performance of the existing theory. In Section 3, we introduce the bootstrap methods and establish their first-order asymptotic validity for the three statistics of interest in this paper under the regularity conditions stated in Section 2. We also compare the finite sample performance of the bootstrap method with the existing first order asymptotic theory. Section 4 provides a detailed study of the pairs bootstrap for realized regressions. We first revisit the first order asymptotic theory of the realized regression estimator, comparing the standard Eicker-White robust variance estimator with the more involved estimator of the variance proposed by BN-S (2004). We then contrast the theoretical properties of the pairs bootstrap, in particular its asymptotic variance, with the properties of the pairs bootstrap in a standard cross section regression. We also discuss the second order accuracy of this bootstrap method based on the Edgeworth expansion that we develop here. Section 5 contains two empirical applications and Section 6 concludes. Appendix A contains the tables and figures. Appendix B contains the proofs of results appearing in Section 3 whereas the proofs of results in Section 4 are collected in Appendix C.

2 Setup and first-order asymptotic theory

2.1 Setup

Let $p(t)$, for $t \geq 0$, denote the log-price of a bivariate vector of assets[1]. We assume that $dp(t) = \Theta(t)dW(t)$, where $p(0) = 0$ and where $\Sigma(t) = \Theta(t)\Theta(t)'$ denotes the spot covariance matrix. Here, W denotes a bivariate vector standard Brownian motion and Θ is the spot covolatility process. As in Gonçalves and Meddahi (2008), we suppose the absence of drift. Following BN-S (2004), we make the following additional assumptions.

Assumption 1 Θ has elements that are all pathwise càdlàg, the instantaneous covariance Σ is independent of W and, for all $t < \infty$, $\int_0^t \Sigma_{kk}(u)du < \infty$, $k = 1, 2$, where $\Sigma_{kl}(t)$ denotes the (k, l)th element of the $\Sigma(t)$ process.

Assumption 2 For $k = 1, 2$, and $i = 1, \ldots, 1/h$, $h^{-1} \int_{(i-1)h}^{ih} \Sigma_{kk}(u)du$ are bounded away from 0 and infinity, uniformly in i and h.

The results in this paper are derived regarding the paths of Σ as fixed. Assumption 1 rules out the presence of leverage effects. Let $y_i' = \begin{pmatrix} y_{1i} & y_{2i} \end{pmatrix}$, $i = 1, \ldots, 1/h$, be the h-horizon intraday returns on a given day on the two assets. We can write $y_i = \int_{(i-1)h}^{ih} \Theta(u)dW(u)$. The integrated covariance matrix of the daily return y is given by $\Gamma \equiv \int_0^1 \Sigma(u)du = \int_0^1 \Theta(u)\Theta'(u)du$, with typical element (k, l) given by $\Gamma_{kl} \equiv \int_0^1 \Sigma_{kl}(u)du$. For $i = 1, \ldots, 1/h$, let $\Gamma_i \equiv \int_{(i-1)h}^{ih} \Sigma(u)du$ and note that $\Gamma = \sum_{i=1}^{1/h} \Gamma_i$. Note that conditionally on the volatility path, $y_i \sim N(0, \Gamma_i)$ independently across i. Thus the data are (conditionally on Σ) heterogeneous, but independent.

The parameters of interest in this paper are elements of Γ and smooth functions of these.

[1] For notational simplicity, we focus on the bivariate case, but the results could be extended to the general case in a straightforward manner.

2.2 The realized covariance matrix

The realized covariance matrix is defined $\hat{\Gamma} = \sum_{i=1}^{1/h} y_i y_i'$. It contains realized volatilities for each asset on its main diagonal and realized covolatilities between the two assets outside the main diagonal. Conditionally on Σ, the theory of quadratic variation implies that $\hat{\Gamma} = \sum_{i=1}^{1/h} y_i y_i' \xrightarrow{P} \Gamma$.

Let $vech\left(\hat{\Gamma}\right)$ denote the vector that stacks the lower triangular elements of the columns of the matrix $\hat{\Gamma}$ into a vector. BN-S (2004) (see also Jacod (1994) and Jacod and Protter (1998)) show that under Assumptions 1 and 2, conditionally on Σ,

$$\sqrt{h^{-1}}\left(vech\left(\hat{\Gamma}\right) - vech\left(\Gamma\right)\right) \equiv \sqrt{h^{-1}} \begin{pmatrix} \sum_{i=1}^{1/h} y_{1i}^2 - \int_0^1 \Sigma_{11}(u)\,du \\ \sum_{i=1}^{1/h} y_{1i}y_{2i} - \int_0^1 \Sigma_{12}(u)\,du \\ \sum_{i=1}^{1/h} y_{2i}^2 - \int_0^1 \Sigma_{22}(u)\,du \end{pmatrix} \to^d N\left(0, V\right), \tag{1}$$

where

$$V = \int_0^1 \left\{ \begin{array}{ccc} 2\Sigma_{11}^2(u) & 2\Sigma_{11}(u)\Sigma_{12}(u) & 2\Sigma_{12}^2(u) \\ 2\Sigma_{11}(u)\Sigma_{12}(u) & \Sigma_{11}(u)\Sigma_{22}(u) + \Sigma_{12}^2(u) & 2\Sigma_{22}(u)\Sigma_{12}(u) \\ 2\Sigma_{12}^2(u) & 2\Sigma_{22}(u)\Sigma_{12}(u) & 2\Sigma_{22}^2(u) \end{array} \right\} du.$$

Let $x_i = vech\left(y_i y_i'\right)$. Then, Corollary 2 of BN-S (2004) shows that

$$\hat{V} = h^{-1}\sum_{i=1}^{1/h} x_i x_i' - \frac{1}{2}h^{-1}\sum_{i=1}^{1/h-1}\left(x_i x_{i+1}' + x_{i+1}x_i'\right) \xrightarrow{P} V.$$

As BN-S (2004) remark, $\hat{V}$ is a substantially different estimator than that used by Barndorff-Nielsen and Shephard (2002) in the univariate context, in which case letting $x_i = y_{1i}^2$, it corresponds to

$$\hat{V} = h^{-1}\sum_{i=1}^{1/h} y_{1i}^4 - h^{-1}\sum_{i=1}^{1/h-1} y_{1i}^2 y_{1,i+1}^2,$$

as opposed to $\frac{2}{3}\sum_{i=1}^{1/h} y_{1i}^4$, the estimator proposed by BN-S (2002). The main feature of notice is the presence of lags of returns in the second piece. One of our contributions is to provide a new interpretation for this estimator in the context of the realized regression estimator (see Section 4.1).

2.3 The realized covariance

Let $\hat{\Gamma}_{12} = \sum_{i=1}^{1/h} y_{1i}y_{2i}$ be the realized covariance between assets 1 and 2, and let $\Gamma_{12} = \int_0^1 \Sigma_{12}(u)\,du$ be the corresponding integrated covariance. From (1), it follows that as $h \to 0$,

$$S_{\Gamma,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\Gamma}_{12} - \Gamma_{12}\right)}{\sqrt{V_\Gamma}} \to^d N\left(0,1\right),$$

where $V_\Gamma = \int_0^1 \left\{\Sigma_{11}(u)\Sigma_{22}(u) + \Sigma_{12}^2(u)\right\} du$ is the asymptotic variance of $\hat{\Gamma}_{12}$.

The corresponding feasible limit theory is

$$T_{\Gamma,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\Gamma}_{12} - \Gamma_{12}\right)}{\sqrt{\hat{V}_\Gamma}} \to^d N\left(0,1\right),$$

where

$$\hat{V}_\Gamma = h^{-1} \sum_{i=1}^{1/h} y_{1i}^2 y_{2i}^2 - h^{-1} \sum_{i=1}^{1/h-1} y_{1i} y_{2i} y_{1,i+1} y_{2,i+1}, \tag{2}$$

is a consistent estimator of V_Γ.

2.4 The realized correlation

The realized correlation between assets 1 and 2 is given by

$$\hat{\rho} = \frac{\sum_{i=1}^{1/h} y_{1i} y_{2i}}{\sqrt{\sum_{i=1}^{1/h} y_{1i}^2} \sqrt{\sum_{i=1}^{1/h} y_{2i}^2}} \xrightarrow{P} \rho \equiv \frac{\int_0^1 \Sigma_{12}(u) du}{\sqrt{\int_0^1 \Sigma_{11}(u) du} \sqrt{\int_0^1 \Sigma_{22}(u) du}}.$$

Its asymptotic variance is $V_\rho = \left(\int_0^1 \Sigma_{11}(u) du \int_0^1 \Sigma_{22}(u) du \right)^{-1} g_\rho$, where $g_\rho = d_{12}' V d_{12}$, and $d_{12} = \left(-\frac{\beta_{12}}{2}, 1, -\frac{\beta_{21}}{2} \right)'$, with V defined as above and where β_{kl} denotes the population regression coefficient of regressing asset k on asset l. The feasible statistic is then

$$T_{\rho,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\rho} - \rho)}{\sqrt{\hat{V}_\rho}},$$

where $\hat{V}_\rho = \left(\sum_{i=1}^{1/h} y_{1i}^2 \sum_{i=1}^{1/h} y_{2i}^2 \right)^{-1} h^{-1} \hat{g}_\rho$, with $\hat{g}_\rho = \sum_{i=1}^{1/h} x_{\rho i}^2 - \sum_{i=1}^{1/h-1} x_{\rho i} x_{\rho,i+1}$, $x_{\rho i} = y_{2i}(y_{1i} - \hat{\beta}_{12} y_{2i})/2 + y_{1i}(y_{2i} - \hat{\beta}_{21} y_{1i})/2$, and $\hat{\beta}_{kl} = \sum_{i=1}^{1/h} y_{ki} y_{li} / \sum_{i=1}^{1/h} y_{li}^2$, for $k, l = 1, 2$.

2.5 The realized regression

Suppose we regress asset 1 on asset 2 to obtain the realized regression estimator $\hat{\beta}_{12} = \frac{\sum_{i=1}^{1/h} y_{1i} y_{2i}}{\sum_{i=1}^{1/h} y_{2i}^2}$. BN-S (Proposition 1, 2004) show that as $h \to 0$,

$$S_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12} - \beta_{12})}{\sqrt{V_\beta}} \to^d N(0,1),$$

where $V_\beta = (\Gamma_{22})^{-2} g_{12}$, $g_{12} = d_{12}' \Psi_{12} d_{12}$, $d_{12} = (1, -\beta_{12})'$, and

$$\Psi_{12} = \int_0^1 \left\{ \begin{array}{cc} \Sigma_{11}(u) \Sigma_{22}(u) + \Sigma_{12}^2(u) & 2\Sigma_{22}(u) \Sigma_{12}(u) \\ 2\Sigma_{22}(u) \Sigma_{12}(u) & 2\Sigma_{22}^2(u) \end{array} \right\} du.$$

BN-S (2004) propose

$$\hat{V}_\beta = \left(\sum_{i=1}^{1/h} y_{2i}^2 \right)^{-2} h^{-1} \hat{g}_\beta, \tag{3}$$

where $\hat{g}_\beta = \sum_{i=1}^{1/h} x_{\beta i}^2 - \sum_{i=1}^{1/h-1} x_{\beta i} x_{\beta,i+1}$, and $x_{\beta i} = y_{1i} y_{2i} - \hat{\beta}_{12} y_{2i}^2$, and show that $\hat{V}_\beta \xrightarrow{P} V_\beta$, and therefore it follows that

$$T_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12} - \beta_{12})}{\sqrt{\hat{V}_\beta}} \to^d N(0,1).$$

2.6 Monte Carlo results for the first-order asymptotic theory

In this section we assess the finite sample performance of confidence intervals for the three covariation measures (covariance, regression and correlation) based on the existing first order asymptotic theory. We present results for two data generating processes. The first model (henceforth Design 1) is the same as that used by BN-S (2004). In particular, we let $dp(t) = \Theta(t)dW(t)$, $\Sigma(t) = \Theta(t)\Theta(t)'$, where

$$\Sigma(t) = \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{12}(t) & \Sigma_{22}(t) \end{pmatrix} = \begin{pmatrix} \sigma_1^2(t) & \sigma_{12}(t) \\ \sigma_{12}(t) & \sigma_2^2(t) \end{pmatrix},$$

and $\sigma_{12}(t) = \sigma_1(t)\sigma_2(t)\rho(t)$.

Following BN-S (2004), we let $\sigma_1^2(t)$ be the sum of two uncorrelated CIR processes: $\sigma_1^2(t) = \sigma_1^{2(1)}(t) + \sigma_1^{2(2)}(t)$. For $s = 1, 2$, $d\sigma_1^{2(s)}(t) = -\lambda_s(\sigma_1^{2(s)}(t) - \xi_s)dt + \omega_s\sigma_1^{(s)}(t)\sqrt{\lambda_s}db_s(t)$, where b_i is the i-th component of a vector of standard Brownian motions, independent from W. We let $\lambda_1 = 0.0429$, $\xi_1 = 0.110$, $\omega_1 = 1.346$, $\lambda_2 = 3.74$, $\xi_2 = 0.398$, and $\omega_2 = 1.346$.

Similarly to BN-S (2004), our model for $\sigma_2^2(t)$ is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998): $d\sigma_2^2(t) = -0.035(\sigma_2^2(t) - 0.636)dt + 0.236\sigma_2^2(t)db_3(t)$. The model we specify for $\rho(t)$ is the same as the one proposed by BN-S(2004): $\rho(t) = (e^{2x(t)} - 1)/(e^{2x(t)} + 1)$, where x follows the GARCH diffusion: $dx(t) = -0.03(x(t) - 0.64)dt + 0.118x(t)db_4(t)$. Our second model (Design 2) specifies $\sigma_2^2(t)$ and $\rho(t)$ exactly as Design 1, with the only difference being in the model used to generate $\sigma_1^2(t)$. In particular, for $\sigma_1^2(t)$ we consider the two-factor diffusion model studied by Chernov et al. (2003) (see also Huang and Tauchen (2005)): $\sigma_1(t) = \text{s-exp}(-1.2 + 0.04v_1(t) + 1.5v_2(t))$, $dv_1(t) = -0.00137v_1(t)dt + db_1(t)$, and $dv_2(t) = -1.386v_2(t)dt + (1 + 0.25v_2(t))db_2(t)$ This diffusion model has continuous sample paths but can imply sample paths for the price process that look like jumps[2]. Although our theory does not allow for a non zero correlation between the price process and the volatility, in our simulations, we allow for these leverage effects. In particular, we let $Corr(dW_1, db_1) = -0.3$ and $Corr(dW_1, db_2) = -0.3$.

We study the finite sample performance of (lower) one-sided and two-sided (symmetric) confidence intervals for each of the three measures of covariation: Γ_{12}, β_{12}, and ρ. We present results for three nominal levels: 95% (i.e. $\alpha = 0.05$), 90% ($\alpha = 0.10$) and 99% ($\alpha = 0.01$).

We compute the actual coverage probabilities of these confidence intervals for each of the stochastic volatility models described above. We report results across 10,000 replications for five different sample sizes: $1/h = 1152, 288, 48, 24$ and 12, corresponding to "1.25-minute", "5-minute", "15-minute", "half-hour", "1-hour", and "2-hour" returns. Table 1 contains results for $\alpha = 0.05$, for each of the two designs, for both one-sided and two-sided symmetric intervals. Table 2 contains results for $\alpha = 0.10$ whereas Table 3 refers to $\alpha = 0.01$. (These tables also include results for the bootstrap method but those results will be discussed later.)

We start with Table 1. For the two DGP's, both one-sided and two-sided intervals tend to un-

[2] The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: $\text{s-exp}(x) = \exp(x)$ if $x \leq x_0$ and $\text{s-exp}(x) = \frac{\exp(x_0)}{\sqrt{x_0}}\sqrt{x_0 - x_0^2 + x^2}$ if $x > x_0$, with $x_0 = \log(1.5)$.

dercover. The degree of undercoverage is especially large for larger values of h, when sampling is not too frequent. For the covariance measure and the regression coefficient, one-sided intervals tend to perform worse than two-sided intervals. The opposite is true for the correlation coefficient, which is surprising when analyzed from the viewpoint of the theory of Edgeworth expansions (the analysis based on Edgeworth expansions suggests that the error of one-sided intervals is of the order $O\left(\sqrt{h}\right)$ whereas the error of symmetric two-sided intervals is usually of the order $O(h)$). For one-sided intervals, the covariance measure is associated with the largest distortions, followed by the regression coefficient, which in turn is worse than the correlation coefficient. For two-sided intervals, this ranking is changed, with the correlation coefficient performing worst, followed by the covariance and by the regression coefficient. The degree of undercoverage can be quite substantial at the smallest sample sizes. For instance, a lower 95% nominal level for the covariance measure between the two assets for Design 1 is equal to 80.74% when we sample every two hours ($h = 1/12$). For the regression coefficient, it is equal to 86.04% and for the correlation coefficient is is equal to 91.43%. The corresponding coverage rates for two-sided intervals based on the BN-S asymptotics are 83.90%, 85.27% and 81.04% for the covariance, the regression and the correlation coefficients, respectively. For this last measure of dependence, we also report the coverage rates of confidence intervals based on the Fisher-z transform, as proposed by BN-S (2004). For one-sided intervals, the 95% interval based on the Fisher transform covers the correlation coefficient 90.28% percent of the time whereas for two-sided intervals, the actual coverage rate is equal to 85.44%. Compared to the intervals based on the raw statistic, the Fisher-z transform outperforms the raw statistic only for the two-sided intervals and not for the one-sided interval. In both cases, however, it is clear that finite sample distortions remain for the Fisher-z transform, thus motivating the use of the bootstrap and/or of alternative analytical corrections.

The results for Design 2 are qualitatively similar to those discussed for Design 1. Quantitatively, the degree of undercoverage is smaller for Design 2, which suggests that contrary to the univariate case (see Gonçalves and Meddahi (2008)) the asymptotic theory handles well the presence of the two-factor diffusion model of Chernov et al. (2003) in one of the volatility processes. The results for Design 2 also suggest that the theory of BN-S (2004) is robust to the introduction of leverage effects.

Tables 2 and 3 show that the performance of the asymptotic theory of BN-S (2004) for the 90% and 99% confidence intervals is qualitatively similar to the performance of the 95% level intervals.

3 The bootstrap

In this section we propose bootstrap methods for smooth functions of the realized covariance matrix. The bootstrap method we consider is the i.i.d. bootstrap applied to the vector of returns.

3.1 The bootstrap realized covariance matrix

We first state the first order asymptotic validity of the bootstrap for the realized covariance matrix and smooth functions of its elements. We then specialize our results to the three statistics of interest: realized covariance, realized correlation and realized regression.

Let $x_i = vech\left(y_i y_i'\right) = \begin{pmatrix} y_{1i}^2 & y_{1i}y_{2i} & y_{2i}^2 \end{pmatrix}'$, and recall that

$$T_h \equiv \hat{V}^{-1/2}\sqrt{h^{-1}}\sum_{i=1}^{1/h}\left(x_i - E\left(x_i\right)\right) \xrightarrow{d} N\left(0, I_3\right),$$

where $\sum_{i=1}^{1/h} x_i = vech\left(\hat{\Gamma}\right)$ denotes the vectorized realized covariance matrix $\hat{\Gamma} = \sum_{i=1}^{1/h} y_i y_i'$, and $\hat{V}$ is a consistent estimator of $V = \lim_{h\to 0} Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} x_i\right)$.

We apply the i.i.d. bootstrap to x_i. In particular, let $x_i^* = x_{I_i} = \begin{pmatrix} y_{1I_i}^2 & y_{1I_i}y_{2I_i} & y_{2I_i}^2 \end{pmatrix}'$, where I_i is i.i.d. on $\{1, \ldots, 1/h\}$. Notice that this is equivalent to bootstrapping the bivariate vector of assets returns $y_i = \left(y_{1i}, y_{2i}\right)'$. Define the (scaled) vectorized bootstrap realized covariance matrix as $\sqrt{h^{-1}}\sum_{i=1}^{1/h} x_i^* = \sqrt{h^{-1}}\sum_{i=1}^{1/h} vech\left(y_i^* y_i^{*\prime}\right) \equiv \sqrt{h^{-1}} vech\left(\hat{\Gamma}^*\right)$. As usual in the bootstrap literature, we let E^* (and Var^*) denote the expectation (and the variance) with respect to bootstrap data, conditional on the original data. It is easy to show that $E^*\left(\sqrt{h^{-1}} vech\left(\hat{\Gamma}^*\right)\right) = \sqrt{h^{-1}} vech\left(\hat{\Gamma}\right)$, and

$$V^* \equiv Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} x_i^*\right) = h^{-1}\sum_{i=1}^{1/h} x_i x_i' - \left(\sum_{i=1}^{1/h} x_i\right)\left(\sum_{i=1}^{1/h} x_i\right)'.$$

We can show that

$$V^* \to^P V + \int_0^1 vech\left(\Sigma\left(u\right)\right) vech\left(\Sigma\left(u\right)\right)' du - \left(\int_0^1 vech\left(\Sigma\left(u\right)\right) du\right)\left(\int_0^1 vech\left(\Sigma\left(u\right)\right) du\right)',$$

which is not equal to V (one exception is when $\Sigma\left(u\right) = \Sigma$ for all u). Although V^* does not consistently estimate V, the i.i.d. bootstrap is still asymptotically valid when applied to the following studentized statistic

$$T_h^* \equiv \hat{V}^{*-1/2}\sqrt{h^{-1}}\left(vech\left(\hat{\Gamma}^*\right) - vech\left(\hat{\Gamma}\right)\right),$$

where

$$\hat{V}^* = h^{-1}\sum_{i=1}^{1/h} x_i^* x_i^{*\prime} - \left(\sum_{i=1}^{1/h} x_i^*\right)\left(\sum_{i=1}^{1/h} x_i^*\right)'$$

is a consistent estimator of V^*. The following theorem states formally these results.

Theorem 3.1 *Let Assumptions 1 and 2 hold. Then, as* $h \to 0$, *(a)* $\hat{V}^* - V^* \xrightarrow{P^*} 0$, *in probability, and (b)* $sup_{x\in\mathbb{R}}\left|P^*\left(T_h^* \leq x\right) - P\left(T_h \leq x\right)\right| \to 0$ *in probability.*

The proofs of all the results in this section appear in Appendix B.

Several statistics of interest can be written as smooth functions of the realized covariance matrix. Examples include the realized covariance measure between two assets, the realized regression coefficient, and the realized correlation coefficient. The following theorem proves that the i.i.d. bootstrap is first order asymptotically valid when applied to smooth functions of the (appropriately centered and studentized verison of) the vectorized realized covariance matrix.

Let $f(\theta) : \mathbb{R}^3 \to \mathbb{R}$ denote a real valued function with continuous derivatives, and let $\nabla f(\theta) = \begin{pmatrix} \partial f/\partial\theta_1 & \partial f/\partial\theta_2 & \partial f/\partial\theta_3 \end{pmatrix}'$ denote its gradient. We suppose that $\nabla f(\theta)$ is nonzero at θ_0, the true value of θ. The statistic of interest is defined as

$$T_{f,h} = \frac{\sqrt{h^{-1}}\left(f\left(vech\left(\hat{\Gamma}\right)\right) - f\left(vech\left(\Gamma\right)\right)\right)}{\sqrt{\hat{V}_f}},$$

where $\hat{V}_{f,h} = \left(\nabla' f\left(vech\left(\hat{\Gamma}\right)\right)\hat{V}\ \nabla f\left(vech\left(\hat{\Gamma}\right)\right)\right)$. The i.i.d. bootstrap version of $T_{f,h}$ is $T^*_{f,h}$, which replaces $\hat{\Gamma}$ with $\hat{\Gamma}^*$, Γ with $\hat{\Gamma}$, and $\hat{V}_f$ with $\hat{V}^*_f = \left(\nabla' f\left(vech\left(\hat{\Gamma}^*\right)\right)\hat{V}^*\nabla f\left(vech\left(\hat{\Gamma}^*\right)\right)\right)$, which is a consistent estimator of the bootstrap asymptotic variance $V^*_f \equiv \left(\nabla' f\left(vech\left(\hat{\Gamma}\right)\right)V^*\nabla f\left(vech\left(\hat{\Gamma}\right)\right)\right)$.

Theorem 3.2 *Under the same conditions of Theorem 3.1, as* $h \to 0$,$\sup_{x\in\mathbb{R}}\left|P^*\left(T^*_{f,h} \le x\right) - P\left(T_{f,h} \le x\right)\right| \to 0$, *in probability.*

The next sections give explicitly the bootstrap statistics for the three cases of interest, namely the covariance measure Γ_{12}, the correlation coefficient ρ and the regression coefficient β.

3.2 The bootstrap realized covariance

The bootstrap realized covariance measure is defined as $\hat{\Gamma}^*_{12} = \sum_{i=1}^{1/h} y^*_{1i}y^*_{2i}$, which corresponds to taking $f\left(vech\left(\hat{\Gamma}\right)\right)$ with $f(\theta) = \theta_2$, with $\theta = (\theta_1, \theta_2, \theta_3)$. Thus, the bootstrap statistic is defined as

$$T^*_{\Gamma,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\Gamma}^*_{12} - \hat{\Gamma}_{12}\right)}{\sqrt{\hat{V}^*_{\Gamma}}},$$

where $\hat{V}^*_{\Gamma} = h^{-1}\sum_{i=1}^{1/h} y^{*2}_{1i}y^{*2}_{2i} - \left(\sum_{i=1}^{1/h} y^*_{1i}y^*_{2i}\right)^2$. Theorem 3.2 above proves the first order asymptotic validity of the bootstrap when applied to $T^*_{\Gamma,h}$.

3.3 The bootstrap realized correlation

The bootstrap realized correlation coefficient $\hat{\rho}^*$ is defined in the same fashion as $\hat{\rho}$ but with the bootstrap data replacing the original data, i.e. $\hat{\rho}^* = \frac{\sum y^*_{1i}y^*_{2i}}{\sqrt{\sum y^{*2}_{1i}}\sqrt{\sum y^{*2}_{2i}}}$. The corresponding t-statistic is given by

$$T^*_{\rho,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\rho}^* - \hat{\rho}\right)}{\sqrt{\hat{V}^*_{\rho}}},$$

where $\hat{V}^*_{\rho} = \left(\hat{\Gamma}^*_{11}\hat{\Gamma}^*_{22}\right)^{-1}\hat{B}^*_{\rho}$, $\hat{B}^*_{\rho} = h^{-1}\sum x^{*2}_{\rho i}$, and $x^*_{\rho i} = y^*_{2i}\left(y^*_{1i} - \hat{\beta}^*_{12}y^*_{2i}\right)/2 + y^*_{1i}\left(y^*_{2i} - \hat{\beta}^*_{21}y^*_{1i}\right)/2$. Here $\hat{\beta}^*_{kl}$ denotes the bootstrap OLS regression estimator of the realized regression of y^*_k on y^*_l, for

$k, l = 1, 2$. We note that $\rho = f\left(vech\left(\Gamma\right)\right)$, with $f\left(\theta\right) = \frac{\theta_2}{\sqrt{\theta_1\theta_3}}$. Thus, the first order asymptotic validity of the i.i.d. bootstrap for the correlation coefficient follows from Theorem 3.2.

3.4 The bootstrap realized regression

Let $\{y_i^* = (y_{1i}^*, y_{2i}^*) : i = 1, \ldots, 1/h\}$ be an i.i.d. bootstrap sample from $\{y_i\}$. The boostrap OLS estimator that we obtain by regressing y_{1i}^* on y_{2i}^* is given by $\hat{\beta}_{12}^* = \frac{\sum_{i=1}^{1/h} y_{1i}^* y_{2i}^*}{\sum_{i=1}^{1/h} y_{2i}^{*2}}$. The corresponding t-statistic is

$$T_{\beta,h}^* \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12}^* - \hat{\beta}_{12})}{\sqrt{\hat{V}_\beta^*}}, \tag{4}$$

where

$$\hat{V}_\beta^* = \left(\sum_{i=1}^{1/h} y_{2i}^{*2}\right)^{-2} h^{-1} \sum_{i=1}^{1/h} y_{2i}^{*2} \hat{\varepsilon}_i^{*2} \equiv \left(\hat{\Gamma}_{22}^*\right)^{-2} \hat{B}_{1h}^*. \tag{5}$$

Theorem 3.2 covers the case of realized regression when $f\left(\theta\right) = \frac{\theta_2}{\theta_3}$, with $\theta = (\theta_1, \theta_2, \theta_3)'$ by noting that $\hat{\beta}_{12} = f\left(vech\left(\hat{\Gamma}\right)\right)$.

3.5 Monte Carlo results for the bootstrap

Our theoretical results suggest the first order asymptotic validity of the i.i.d. bootstrap. Thus we can build bootstrap confidence intervals for each measure of covariation. We concentrate our discussion on Table 1, which contains results for 95% level confidence intervals. Tables 2 and 3 contain the corresponding results for 90% and 99% level intervals, respectively. Since the results are qualitatively similar, we do not discuss these in detail here. Our results suggest that the i.i.d. bootstrap intervals outperform the asymptotic theory based intervals for the two DGP's and for both one-sided and two-sided intervals, for all three measures of dependence. Symmetric intervals are generally better than equal-tailed intervals (this is consistent with the theory based on Edgeworth expansions) and both improve upon the first order asymptotic theory based intervals. The gains associated with the i.i.d. bootstrap can be quite substantial, especially for the smaller sample sizes, when distortions of the BN-S intervals are larger. For instance, for the regression coefficient, the coverage rate for a symmetric bootstrap interval is equal to 93.48% when $1/h = 12$, whereas it is equal to 85.27% for the feasible asymptotic theory of BN-S (2004) (the corresponding equal-tailed interval yields a coverage rate of 90.42%, better than BN-S (2004) but worse than the symmetric bootstrap interval). The gains are especially important for the two-sided intervals for the correlation coefficient, when the asymptotic theory of BN-S (2004) does worst. For $1/h = 12$, the bootstrap symmetric interval has a rate of 93.69% (the equal tailed interval is in this case even better behaved, with a rate equal to 94.60) whereas the BN-S interval based on the raw statistic has a rate of 81.04% and the interval based on the Fisher-z transform has a rate of 85.44%. For the correlation coefficient, the bootstrap essentially removes all finite sample bias associated with the first order asymptotic theory of BN-S (2004).

4 A detailed study of realized regressions

The realized regression estimator is one of the most popular measures of covariation between two assets. In this section we study in more detail the application of the i.i.d. bootstrap to realized regression. We first provide a new interpretation for the feasible approach of BN-S (2004). In particular, we establish a link between the standard Eicker-White heteroskedasticity robust variance estimator and the variance estimator proposed by BN-S (2004). We then exploit the special structure of the regression model to obtain the asymptotic distribution of the bootstrap realized regression estimator. We relate the bootstrap variance with the Eicker-White robust variance estimator. We end this section with a discussion of the second order accuracy of the i.i.d. bootstrap in this context.

4.1 The first order asymptotic theory revisited

Given Assumptions 1 and 2, and conditionally on Σ, we can write

$$y_{1i} = \beta_{12,i} y_{2i} + u_i, \tag{6}$$

where independently across $i = 1, \ldots, h^{-1}$, $u_i | y_{2i} \sim N(0, V_i)$, with $V_i \equiv \Gamma_{11,i} - \frac{\Gamma_{12,i}^2}{\Gamma_{22,i}}$, and $\beta_{12,i} \equiv \frac{\Gamma_{12,i}}{\Gamma_{22,i}}$. Here $\Gamma_{kl,i} = \int_{(i-1)h}^{ih} \Sigma_{kl}(u)\, du$. Thus, the regression coefficient in the true DGP describing the relationship between y_{1i} and y_{2i} is heterogeneous (it depends on i) and the true error term in this model is heteroskedastic.

When we regress y_{1i} on y_{2i} to obtain $\hat{\beta}_{12}$, we get that $\hat{\beta}_{12} \stackrel{P}{\to} \beta_{12} \equiv \frac{\Gamma_{12}}{\Gamma_{22}}$. Thus, $\hat{\beta}_{12}$ does not estimate $\beta_{12,i}$ but instead β_{12}, which can be thought of as a weighted average of $\beta_{12,i}$. We can write the underlying regression model as follows:

$$y_{1i} = \beta_{12} y_{2i} + \varepsilon_i, \tag{7}$$

where $\varepsilon_i = \left(\beta_{12,i} - \beta_{12}\right) y_{2i} + u_i$. It follows that $\varepsilon_i | y_{2i} \sim N\left(\left(\beta_{12,i} - \beta_{12}\right) y_{2i}, V_i\right)$, independently across i. Moreover, noting that $E(y_{2i}) = 0$,

$$Cov(y_{2i}, \varepsilon_i) = E(y_{2i}\varepsilon_i) = \left(\beta_{12,i} - \beta_{12}\right)\Gamma_{22,i} = \Gamma_{12,i} - \beta_{12}\Gamma_{22,i},$$

which in general is not equal to zero (unless volatility is constant). However, $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) = 0$, and therefore $\hat{\beta}_{12}$ converges in probability to β_{12}. The fact that $E(y_{2i}\varepsilon_i) \neq 0$ is crucial to understand several properties of $\hat{\beta}_{12}$ (and of its bootstrap analogue).

To find the asymptotic distribution of $\hat{\beta}_{12}$, we can write

$$\sqrt{h^{-1}}\left(\hat{\beta}_{12} - \beta_{12}\right) = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\sum_{i=1}^{1/h} y_{2i}^2} = (\Gamma_{22})^{-1}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i + o_P(1).$$

The asymptotic variance of $\sqrt{h^{-1}}\hat{\beta}_{12}$ is thus of the usual sandwich form $V_\beta \equiv Var\left(\sqrt{h^{-1}}\hat{\beta}_{12}\right) = (\Gamma_{22})^{-1} B (\Gamma_{22})^{-1}$, where $B = \lim_{h\to 0} B_h$, and $B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)$. Because $E(y_{2i}\varepsilon_i) \neq 0$, we have that

$$B_h = h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}^2\varepsilon_i^2\right) - h^{-1}\sum_{i=1}^{1/h}\left(E\left(y_{2i}\varepsilon_i\right)\right)^2 \equiv B_{1h} - B_{2h}.$$

We can easily show that

$$B = \lim_{h\to 0} B_h = \int_0^1 \left(\Sigma_{12}^2(u) + \Sigma_{11}(u)\Sigma_{22}(u) - 4\beta_{12}\Sigma_{12}(u)\Sigma_{22}(u) + 2\beta_{12}^2\Sigma_{22}^2(u)\right) du.$$

It follows that

$$S_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12} - \beta_{12})}{\sqrt{V_\beta}} \to^d N(0,1),$$

where $V_\beta = (\Gamma_{22})^{-2} B$. We can contrast this result with Proposition 1 of BN-S (2004). It is easy to check that $g_{12} = B$.

It is helpful to contrast the BN-S (2004) variance estimator of V_β with the Eicker-White heteroskedasticity-robust variance estimator that one would typically use in a cross section regression context. Let $\hat{\varepsilon}_i$ denote the OLS residual underlying the regression model (7). Then, the Eicker-White robust variance estimator of B is given by $\hat{B}_{1h} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$. In contrast, noting that $x_{\beta i} = y_{2i}\hat{\varepsilon}_i$, BN-S (2004)'s estimator of B corresponds to

$$h^{-1}\hat{g}_\beta = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2 - h^{-1}\sum_{i=1}^{1/h-1} y_{2i}\hat{\varepsilon}_i y_{2,i+1}\hat{\varepsilon}_{i+1} \equiv \hat{B}_{1h} - \hat{B}_{2h}. \quad (8)$$

We can see that $h^{-1}\hat{g}_\beta = \hat{B}_{1h} - \hat{B}_{2h}$, where $\hat{B}_{1h}$ is the usual Eicker-White robust variance estimator, and $\hat{B}_{2h} = h^{-1}\sum_{i=1}^{1/h-1} y_{2i}\hat{\varepsilon}_i y_{2,i+1}\hat{\varepsilon}_{i+1}$. This extra term is needed to correct for the fact that $E(y_{2i}\varepsilon_i) \neq 0$, as we noted above. In particular, $\hat{B}_{1h} \to B_{1h}$ and $\hat{B}_{2h} \to B_{2h}$ in probability.

4.2 First order asymptotic properties of the pairs bootstrap

The i.i.d. bootstrap applied to the vector of returns $y_i = (y_{1i}, y_{2i})'$ is equivalent to the so-called pairs bootstrap, a popular bootstrap method in the context of cross section regression models. Freedman (1981) proves the consistency of the pairs bootstrap for possibly heteroskedastic regression models when the dimension p of the regressor vector is fixed. Mammen (1993) treats the case where $p \to \infty$ as the sample size grows to infinity. Mammen (1993) also discusses the second order accuracy of the pairs bootstrap in this context. His results specialized to the case where p is fixed show that the pairs bootstrap is not only first order asymptotically valid under heteroskedasticity in the error term, but it is also second-order correct.

It is easy to check that $\hat{\beta}_{12}^*$ converges in probability (under the bootstrap probability measure P^*) to $\hat{\beta}_{12} = \frac{\sum_{i=1}^{1/h} E^*\left(y_{1i}^* y_{2i}^*\right)}{\sum_{i=1}^{1/h} E^*\left(y_{2i}^{*2}\right)}$. The bootstrap analogue of the regression error ε_i in model (7) is thus $\varepsilon_i^* = y_{1i}^* - \hat{\beta}_{12} y_{2i}^*$, whereas the bootstrap OLS residuals are defined as $\hat{\varepsilon}_i^* = y_{1i}^* - \hat{\beta}_{12}^* y_{2i}^*$.

Our next Theorem provides the first order asymptotic properties of $\hat{\beta}_{12}^*$.

Theorem 4.1 *Under the conditions of Theorem 3.1, as $h \to 0$,*

a) $\sqrt{h^{-1}}\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) \to^{d^*} N\left(0, V_\beta^*\right)$, *in probability, where* $V_\beta^* = \left(\hat{\Gamma}_{22}\right)^{-2} B_h^*$.

b) $B_h^* = Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right) = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2 \equiv \hat{B}_{1h}$.

c) $V_\beta^* \to^P (\Gamma_{22})^{-2} B^* \neq V_\beta$, *where* $B^* = B + \int_0^1 (\Sigma_{12}(u) - \beta_{12}\Sigma_{22}(u))^2\, du$.

Part (a) of Theorem 4.1 states that the bootstrap OLS estimator has a first order asymptotic normal distribution with mean zero and covariance matrix V_β^*. Its proof follows from Theorem 3.2. Parts (b) and (c) show that the pairs bootstrap variance estimator is not consistent for V_β in the general context of stochastic volatility. One exception is when volatility is constant, in which case $B^* = B$ and $V_\beta^* \to^P V_\beta$.

To understand the form of V_β^*, note that we can write

$$\sqrt{h^{-1}}\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) = \left(\sum_{i=1}^{1/h} y_{2i}^{*2}\right)^{-1}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*.$$

Since $\sum_{i=1}^{1/h} y_{2i}^{*2} \to^{P^*} \sum_{i=1}^{1/h} y_{2i}^2 = \hat{\Gamma}_{22}$, in probability, it follows that

$$\sqrt{h^{-1}}\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) = \left(\hat{\Gamma}_{22}\right)^{-1}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^* + o_{P^*}(1),$$

in probability. We can now apply a central limit theorem to $\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*$ to obtain the limiting normal distribution for $\sqrt{h^{-1}}\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right)$. It follows that

$$\sqrt{h^{-1}}\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) \to^{d^*} N\left(0, V_\beta^*\right),$$

in probability, where $V_\beta^* = \left(\hat{\Gamma}_{22}\right)^{-2} B_h^*$, with $B_h^* = Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)$. Part (b) of Theorem 4.1 follows easily from the properties of the i.i.d. bootstrap. In particular, we can show that $B_h^* = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$, since $\sum_{i=1}^{1/h} y_{2i}\hat{\varepsilon}_{2i} = 0$ by construction of $\hat{\beta}_{12}$. Thus, the i.i.d. bootstrap variance of the scaled average of the bootstrap scores $y_{2i}^*\varepsilon_i^*$ is equal to $\hat{B}_{1h}$, the Eicker-White heteroskedasticity robust variance estimator of the scaled average of the scores $y_{2i}\varepsilon_i$.

Theorem 4.1 (part c) shows that the pairs bootstrap does not in general consistently estimate the asymptotic variance of $\hat{\beta}_{12}$. An exception is when volatility is constant. This is in contrast with the existing results in the cross section regression context, where the pairs bootstrap variance estimator of the least squares estimator is robust to heteroskedasticity in the error term. This failure of the pairs bootstrap to provide a consistent estimator of the variance of $\hat{\beta}_{12}$ is related to the fact that, as we explained in in the previous section, we cannot in general assume that $E(y_{2i}\varepsilon_i) = 0$, unless for instance when volatility is constant. When the the scores have mean zero, i.e. $E(y_{2i}\varepsilon_i) = 0$, the Eicker-White robust variance estimator, and therefore the i.i.d. bootstrap variance estimator, are consistent estimators of the asymptotic variance of the scaled average of the scores. Both Freedman (1981) and Mammen (1993) make this assumption. The fact that $E(y_{2i}\varepsilon_i) \neq 0$ creates a bias term in $\hat{B}_{1h}$, which is eliminated with the variance estimator proposed by BN-S (2004). Because $B_h^* = \hat{B}_{1h}$, the i.i.d. bootstrap variance estimator is not a consistent estimator of $B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)$. The non zero mean property of the scores in our context is crucial in understanding the differences between the realized regression and the usual cross section regression.

The i.i.d. bootstrap is nevertheless first order asymptotically valid when applied to the t-statistic T_h, as our Theorem 3.2 proves. This first order asymptotic validity occurs despite the fact that V_β^* does not consistently estimate V_β. The key aspect is that we studentize the bootstrap OLS estimator with $\hat{V}_\beta^*$ (defined in (5)), a consistent estimator of V_β^*, implying that the asymptotic variance of the bootstrap t-statistic is one.

4.3 Second order asymptotic properties of the pairs bootstrap

In this section, we study the second order accuracy of the pairs bootstrap for realized regressions. In particular, we compare the rates of convergence of the error of the bootstrap and the normal approximation when estimating the distribution function of $T_{\beta,h}$. This is accomplished via a comparison of the Edgeworth expansion of the distribution of $T_{\beta,h}$ derived by Dovonon, Gonçalves and Meddahi (2007) with the bootstrap Edgeworth expansion of $T_{\beta,h}^*$, which we derive here. See Gonçalves and Meddahi (2006) and Zhang et al. (2005b) for two recent papers that have used Edgeworth expansions for realized volatility as a means to improve upon the first order asymptotic theory.

For $i = 1, 3$, we denote by $\kappa_i\left(T_{\beta,h}\right)$ the first and third order cumulant of $T_{\beta,h}$, respectively. The second order Edgeworth expansion of the distribution of $T_{\beta,h}$ is given by (see e.g. Hall, 1992, p. 47)

$$P\left(T_{\beta,h} \leq x\right) = \Phi\left(x\right) + \sqrt{h} q\left(x\right)\phi\left(x\right) + o\left(\sqrt{h}\right),$$

where for any $x \in \mathbb{R}$, $\Phi\left(x\right)$ and $\phi\left(x\right)$ denote the cumulative distribution function and the density function of a standard normal random variable. The correction term $q\left(x\right)$ is defined as

$$q\left(x\right) = -\left(\kappa_1 + \frac{1}{6}\kappa_3\left(x^2 - 1\right)\right),$$

where κ_1 and κ_3 are the coefficients of the leading terms of $\kappa_1\left(T_{\beta,h}\right)$ and $\kappa_3\left(T_{\beta,h}\right)$, respectively. In particular, up to order $O\left(\sqrt{h}\right)$, as $h \to 0$, $\kappa_1\left(T_{\beta,h}\right) = \sqrt{h}\kappa_1$ and $\kappa_3\left(T_{\beta,h}\right) = \sqrt{h}\kappa_3$.

Given this Edgeworth expansion, the error (conditional on Σ) incurred by the normal approximation in estimating the distribution of $T_{\beta,h}$ is given by

$$\sup_{x\in\mathbb{R}}\left|P\left(T_{\beta,h} \leq x\right) - \Phi\left(x\right)\right| = \sqrt{h}\sup_{x\in\mathbb{R}}\left|q\left(x\right)\phi\left(x\right)\right| + O\left(h\right).$$

Thus, $\sup_{x\in\mathbb{R}}\left|q\left(x\right)\phi\left(x\right)\right|$ is the contribution of order $O\left(\sqrt{h}\right)$ to the normal error.

Similarly, we can write a one-term Edgeworth expansion for the conditional distribution of $T_{\beta,h}^*$ as follows

$$P^*(T_{\beta,h}^* \leq x) = \Phi(x) + \sqrt{h}q_h^*(x)\phi(x) + O_P(h),$$

where q_h^* is defined as

$$q_h^*(x) = -(\kappa_{1,h}^* + \kappa_{3,h}^*(x^2 - 1)/6),$$

and where $\kappa_{1,h}^*$ and $\kappa_{3,h}^*$ are the leading terms of the first and the third order cumulants of $T_{\beta,h}^*$. In particular, $\kappa_1^*\left(T_{\beta,h}^*\right) = \sqrt{h}\kappa_{1,h}^*$ and $\kappa_3^*\left(T_{\beta,h}^*\right) = \sqrt{h}\kappa_{3,h}^*$, up to order $O\left(\sqrt{h}\right)$.

The bootstrap error implicit in the bootstrap approximation of $P\left(T_{\beta,h}\leq x\right)$ (conditional on Σ) is given by

$$\begin{aligned}
P^{*}\left(T_{\beta,h}^{*}\leq x\right)-P\left(T_{\beta,h}\leq x\right) &= \sqrt{h}\ \left(q_{h}^{*}\left(x\right)-q\left(x\right)\right)\phi\left(x\right)+O_{P}\left(h\right)\\
&= \sqrt{h}\left(\underset{h\to0}{\text{plim}}\,q_{h}^{*}\left(x\right)-q\left(x\right)\right)\phi\left(x\right)+o_{P}\left(\sqrt{h}\right)\\
&= -\sqrt{h}\left[\left(\kappa_{1}^{*}-\kappa_{1}\right)+\frac{1}{6}\left(\kappa_{3}^{*}-\kappa_{3}\right)\left(x^{2}-1\right)\right]+o_{P}\left(\sqrt{h}\right),
\end{aligned}$$

where $\kappa_1^*\equiv\text{plim}_{h\to0}\,\kappa_{1,h}^*$ and $\kappa_3^*\equiv\text{plim}_{h\to0}\,\kappa_{3,h}^*$. If $\kappa_1^*=\kappa_1$ and $\kappa_3^*=\kappa_3$, $P^*\left(T_{\beta,h}^*\leq x\right)-P\left(T_{\beta,h}\leq x\right)=o_P\left(\sqrt{h}\right)$, and the bootstrap error is of a smaller order of magnitude than the normal error which is equal to $O\left(\sqrt{h}\right)$. If this is the case, the bootstrap is said to be second-order correct and to provide an asymptotic refinement over the standard normal approximation.

The following result gives the expressions of the leading terms of the first and third order cumulants for the original statistic and for its bootstrap analogue. We need to introduce some notation. For simplicity, we will henceforth write Σ instead of $\Sigma\left(u\right).$

Let

$$\begin{aligned}
A_{0} &= \int_{0}^{1}\left(\Sigma_{22}\Sigma_{12}-\beta_{12}\Sigma_{22}^{2}\right)du,\\
A_{1} &= \int_{0}^{1}\left(2\Sigma_{12}^{3}+6\Sigma_{11}\Sigma_{12}\Sigma_{22}-18\beta_{12}\Sigma_{12}^{2}\Sigma_{22}-6\beta_{12}\Sigma_{22}^{2}\Sigma_{11}+24\beta_{12}^{2}\Sigma_{12}\Sigma_{22}^{2}-8\beta_{12}^{3}\Sigma_{22}^{3}\right)du,\\
B &= \int_{0}^{1}\left(\Sigma_{12}^{2}+\Sigma_{11}\Sigma_{22}-4\beta_{12}\Sigma_{12}\Sigma_{22}+2\beta_{12}^{2}\Sigma_{22}^{2}\right)du,\\
H_{1} &= \frac{4A_{0}}{\Gamma_{22}\sqrt{B}},\quad\text{and}\quad H_{2}=\frac{A_{1}}{B^{3/2}}.
\end{aligned}$$

Similarly, let

$$\begin{aligned}
B^{*} &= B+\int_{0}^{1}\left(\Sigma_{12}-\beta_{12}\Sigma_{22}\right)^{2}du,\\
A_{1}^{*} &= A_{1}+2\int_{0}^{1}\left(\Sigma_{12}-\beta_{12}\Sigma_{22}\right)^{3}du,\\
H_{1}^{*} &= \frac{4A_{0}}{\Gamma_{22}\sqrt{B^{*}}},\ \text{and}\ H_{2}^{*}=\frac{A_{1}^{*}}{B^{*3/2}}.
\end{aligned}$$

In order to obtain the higher order results in this section, we add the following additional assumption. A more primitive assumption such as a multivariate analogue of Assumption V in Gonçalves and Meddahi (2008) may be sufficient to ensure Assumption 3, but we have not yet confirmed this.

Assumption 3 For $k,l,k',l'=1,2$, $h^{-1}\sum_{i=1}^{1/h}\Gamma_{kl,i}\Gamma_{k'l',i}-\int_0^1\Sigma_{kl}(u)\Sigma_{k'l'}(u)du=o_P(\sqrt{h})$, and $h^{-1}\sum_{i=1}^{1/h-1}\Gamma_{kl,i}\Gamma_{k'l',i+1}-\int_0^1\Sigma_{kl}(u)\Sigma_{k'l'}(u)du=o_P(\sqrt{h})$.

Proposition 4.1 *Under Assumptions 1, 2 and 3, (a)* $\kappa_1=\frac{1}{2}\left(H_1-H_2\right)$ *and* $\kappa_3=3H_1-2H_2$; *and* $\kappa_1^*=\frac{3}{4}\left(H_1^*-H_2^*\right)$ *and* $\kappa_3^*=3\left(\frac{3}{2}H_1^*-H_2^*\right).$

Part (a) of Proposition 4.1 is derived in Dovonon, Gonçalves and Meddahi (2007). (We reproduce the proof in Appendix C for completeness.) The proof of part (b) is in Appendix C. A comparison of the two parts reveals a disagreement between the two sets of cumulants. Notice in particular that $B \neq B^*$ contributes to this discrepancy. B here denotes the limiting variance of the scaled average of the scores whereas B^* denotes its bootstrap analogue. As we noted before, under general stochastic volatility, the pairs bootstrap does not consistently estimate B and the bias term is exactly equal to the difference between B and B^*, i.e. $B^* - B = \int_0^1 (\Sigma_{12} - \beta_{12}\Sigma_{22})^2 \, du = \text{plim}_{h\to 0} B_{2h}$, where $B_{2h} = h^{-1} \sum_{i=1}^{1/h} (E(y_{2i}\varepsilon_i))^2$. An exception is when volatility is constant, where $B_{2h} = 0$ and therefore $B^* = B$. In this case, we also have that $A_1^* = A_1 = A_0 = 0$, implying that both the bootstrap and the normal approximations have an error of the order $O(h)$. We need a higher order expansion to be able to discriminate the two approximations. In the general stochastic volatility case, the pairs bootstrap error is of order $O\left(\sqrt{h}\right)$, similar to the error incurred by the normal approximation.

The lack of second order refinements of the pairs bootstrap in the context of realized regressions is in contrast with the results available in the bootstrap literature for standard regression models (see Mammen 1993). One explanation for this difference lies in the fact that $E(y_{2i}\varepsilon_i) \neq 0$, as we noted above. This implies that $T_{\beta,h}$ must rely on a variance estimator that contains a bias correction term, as proposed by BN-S (2004). Instead, in the bootstrap regression, $E^*(y_{2i}^*\varepsilon_i^*) = h \sum_{i=1}^{1/h} y_{2i}\hat{\varepsilon}_i = 0$, and therefore there is no need for the bias correction proposed by BN-S (2004). This implies that the bootstrap t-statistic $T_{\beta,h}^*$ is not of the same form as $T_{\beta,h}$, relying on a bootstrap variance estimator $\hat{V}_\beta^*$ that depends on an Eicker-White type variance estimator $\hat{B}_{1h}^*$.

5 Empirical application

A well documented empirical fact in finance is the time variability of bonds risk, as recently documented by Viceira (2007) for the US market. As suggested by the CAPM, the bond risk is often measured by its beta over the return on the market portfolio. With a positive beta, bonds are considered as risky as the market while a bond with a negative beta could be used to hedge the market risk.

Following Merton (1980) and French, Schwert and Stambaugh (1987), Viceira (2007) studies the bond risk for the US market by considering the 3-month (monthly) rolling realized beta as measured by the ratio of the realized covariance of daily log-returns on bonds and stocks and the realized volatility of the daily log-return on stocks over the same period. Following the standard practice, the number of days in a month is normalized to 22 such that the 3-month realized beta is computed considering sub-samples of 66 days. From July 1962 through December 2003, Viceira (2007) reports a strong variability of US bond CAPM betas, which may switch sign even though the average over the full sample is positive. Nevertheless, in his analysis Viceira (2007) does not discuss the precision of the realized betas as a measure of the actual covariation between bonds and stock returns.

The aim of this section is to illustrate the usefulness of our approach as a method of inference for realized covariation measures in the context of measuring the time variation of bonds risk. We consider both the US bonds market, as in Viceira (2007), and the UK bonds market.

Our data set includes the daily 7-to-10-year maturity government bond index for the US and the UK markets as released by JP Morgan from January 2, 1986 through August 24, 2007. As a proxy for the US and the UK market portfolio returns, we consider the log-return on the S&P500 and the FTSE 100 indices, respectively. The S&P500 index is designed to measure performance of the broad domestic economy through changes in the aggregate market value of 500 stocks representing all major industries. The FTSE 100 index is a capitalization-weighted index of the 100 most highly capitalized companies traded on the London Stock Exchange. Both indices are commonly used in scientific researches as well as in the finance industry as a proxy for the market portfolio. The first two series have a shorter history and therefore constrained the sample we consider in this study.

From the estimates presented in Table 4 (Appendix A), the full-sample beta for bonds in the US is about 0.024, slightly smaller than the UK bond beta, which is about 0.030. Both the bootstrap and the asymptotic theory based confidence intervals display support that the true values of the betas in both countries are positive.

A closer analysis of Figures 1 and 2 shows that the average positivity of the betas hides considerable time variation in both countries, a fact already documented by Viceira (2007) for the US market. Furthermore, the betas for these two countries follow similar dynamics. We can distinguish two patterns for the 3-month betas. For the period before April 1997, the betas are mostly significantly positive or, in few cases, non-significantly different from 0. This period is also characterized by betas of larger magnitude, with a maximum value of 0.500 at the end of July 1994 for the US and 0.438 in August 1994 for the UK. The period after April 1997 is characterized by a drop of the magnitude of the bonds betas in both countries. They are often not significantly different from 0. For this whole sub-period, the betas for the US and UK bonds are significantly negative only between June 2002 and July 2003, but in these cases their magnitude is small. We conclude that bonds are riskier in the period before April 1997, while in the recent periods they appear to be non risky or at most a hedging instrument against shocks on market portfolio returns.

A comparison of the bootstrap intervals with the intervals based on the asymptotic theory of BN-S (2004) suggests that they two types of intervals tend to be similar, but there are instances where the bootstrap intervals are wider than the asymptotic theory-based intervals (see Tables 5 and 6 for a detailed comparison of the two types of intervals for a selected set of dates). This is specially true for the first part of the sample for the UK bond market, where the width of the bootstrap intervals can be much larger than the width of the BN-S (2004) intervals. In this empirical application, the gain in accuracy of the bootstrap intervals in terms of coverage probability appears to be associated with a deterioration of length of the bootstrap intervals.

6 Conclusion

This paper proposes bootstrap methods for inference on measures of multivariate volatility such as integrated covariance, integrated correlation and integrated regression coefficients. We show the first order asymptotic validity of a particular bootstrap scheme, the i.i.d. bootstrap applied to the vector of returns, for the three statistics of interest. Our simulation results show that the bootstrap outperforms the feasible first order asymptotic approach of BN-S(2004).

For the special case of the realized regression estimator, our i.i.d. bootstrap corresponds to a pairs bootstrap as proposed by Freedman (1981) and further studied by Mammen (1993). We analyze the second order accuracy of this bootstrap method and conclude that it is not second order accurate. This contrasts with the existing literature on the pairs bootstrap for cross section models, which shows that this method is not only robust to heteroskedasticity in the error term but it is also second order accurate. We provide a detailed analysis of the pairs bootstrap in the context of realized regressions which allows us to highlight some key differences with respect to the usual application of the pairs bootstrap in standard cross section regression models. These differences explain why the pairs bootstrap does not provide second order refinements in this context.

An important characteristic of high frequency financial data that our theory ignores is the presence of microstructure effects: the prices are observed with contamination errors called noise due to the presence of bid-ask bounds, rounding errors, etc, and prices are asynchronous, i.e., the prices of two assets are often not observed at the same time. The first problem is well addressed by the literature in the univariate context, in particular, Zhang, Mykland, and Ait-Sahalia (2005a), Zhang (2006), and Barndorff-Nielsen, Hansen, Lunde and Shephard (2007) provide consistent estimators of the integrated volatility. Likewise, Hayashi and Yoshida (2005) provide a consistent estimator of the covariation of two assets when they are asynchronous, but their analysis rules out the presence of noise. Little is known when the two effects are present; see however the analysis in Zhang (2006), Griffin and Oomen (2006) and Voev and Lunde (2007). Another feature that our theory ignores is the possible presence of jumps and co-jumps. This is a difficult problem that the literature has only started recently to address (see Jacod and Todorov (2007) and Bollerslev and Todorov (2007)). The extension of our bootstrap theory to these important problems is left for future research.

Appendix A

Table 1. Coverage rates of nominal 95% intervals for covariation measures

	Covariance					Regression					Correlation						
	One-sided		Two-sided			One-sided		Two-sided			One-sided			Two-sided			
$1/h$	BN-S	Boot	BN-S	Boot		BN-S	Boot	BN-S	Boot		BN-S	Fisher	Boot	BN-S	Fisher	Boot	
				Sym	Eq-T.				Sym	Eq-T						Sym	Eq-T
								Design 1									
12	80.76	87.30	83.98	90.58	89.22	86.01	90.05	85.20	93.51	90.72	92.02	90.47	94.91	81.47	85.90	93.82	94.57
24	84.74	89.55	87.59	91.37	90.65	89.34	91.90	89.57	93.84	91.91	93.55	92.15	94.51	86.90	89.15	93.59	93.96
48	88.09	92.28	90.39	93.01	92.36	91.08	93.05	91.37	94.05	93.04	94.86	93.75	95.15	90.24	91.62	93.97	94.43
288	92.22	94.61	93.89	94.76	94.44	93.57	94.80	94.00	94.73	94.52	95.42	94.75	94.94	93.87	94.12	94.71	94.69
1152	93.69	94.98	94.62	94.90	94.59	94.30	95.07	94.57	94.70	94.68	95.27	94.92	95.03	94.63	94.74	94.90	94.90
								Design 2									
12	81.27	87.89	84.15	90.95	90.11	86.88	90.73	85.80	93.78	91.31	92.60	90.96	95.37	82.39	86.70	93.93	95.28
24	85.38	90.42	88.09	91.73	91.39	89.73	91.93	89.93	93.98	92.30	94.34	92.83	95.09	87.56	89.86	93.90	94.58
48	88.49	92.22	91.05	93.17	92.92	91.28	92.99	91.77	94.37	93.03	95.40	94.17	95.33	90.98	92.30	94.42	95.00
288	92.64	94.69	94.15	94.95	94.46	93.61	94.53	94.29	94.90	94.41	95.57	94.95	95.11	94.20	94.43	94.95	94.90
1152	93.78	94.99	94.94	95.15	94.91	94.14	94.75	94.44	94.56	94.54	95.30	94.87	94.89	94.87	94.83	95.07	94.94

Note: 10,000 replications, with 999 bootstrap replications each.

Table 2. Coverage rates of nominal 90% intervals for covariation measures

	Covariance					Regression					Correlation						
	One-sided		Two-sided			One-sided		Two-sided			One-sided			Two-sided			
$1/h$	BN-S	Boot	BN-S	Boot		BN-S	Boot	BN-S	Boot		BN-S	Fisher	Boot	BN-S	Fisher	Boot	
				Sym	Eq-T				Sym	Eq-T						Sym	Eq-T
							Design 1										
12	75.15	82.41	79.16	86.14	84.64	80.55	85.27	78.65	88.97	85.51	87.42	85.61	90.12	75.53	79.04	88.54	89.58
24	79.40	85.08	82.78	87.06	85.63	83.95	87.33	83.97	89.47	87.17	89.08	87.41	89.78	81.01	83.28	88.15	88.75
48	82.27	87.20	85.51	88.46	87.43	85.85	88.44	85.77	89.07	87.32	90.23	88.71	90.15	84.72	86.10	89.20	89.36
288	86.97	89.60	88.83	89.95	89.23	88.53	90.03	88.91	89.77	89.35	90.21	89.47	89.64	89.00	89.20	89.91	89.75
1152	88.72	90.06	89.19	89.59	89.37	89.71	90.44	89.65	89.98	89.85	90.36	90.06	89.87	89.65	89.74	90.01	89.98
							Design 2										
12	76.02	82.92	79.64	86.13	85.18	81.30	85.61	79.44	88.93	86.12	88.13	86.43	90.56	76.40	80.02	88.63	90.34
24	80.00	85.63	83.17	87.28	86.13	84.36	87.19	84.08	89.40	87.23	89.95	88.24	90.63	81.74	84.03	88.46	89.54
48	83.00	87.57	86.17	88.76	87.42	85.60	88.06	86.14	89.30	87.77	90.63	89.24	90.18	85.62	86.61	89.51	89.85
288	87.64	89.86	89.28	90.11	89.59	89.01	90.06	89.09	90.05	89.35	90.98	90.42	90.27	89.35	89.58	90.20	90.20
1152	88.66	89.86	89.80	90.03	89.95	89.23	89.96	89.69	89.80	89.67	90.64	90.27	90.16	89.80	89.80	89.96	89.96

Note: 10,000 replications, with 999 bootstrap replications each.

Table 3. Coverage rates of nominal 99% intervals for covariation measures

	Covariance					Regression					Correlation						
	One-sided		Two-sided			One-sided		Two-sided			One-sided			Two-sided			
$1/h$	BN-S	Boot	BN-S	Boot		BN-S	Boot	BN-S	Boot		BN-S	Fisher	Boot	BN-S	Fisher	Boot	
				Sym	Eq-T				Sym	Eq-T						Sym	Eq-T
	Design 1																
12	88.24	92.75	89.99	95.58	94.12	92.82	95.62	92.42	97.93	95.99	96.49	95.74	98.62	88.94	93.46	98.73	98.65
24	91.49	94.84	93.07	96.18	95.69	94.95	96.17	95.41	97.75	96.35	97.69	96.85	98.55	93.50	95.56	98.41	98.44
48	94.35	96.89	95.54	97.33	97.10	96.58	97.60	96.97	98.39	97.75	98.52	97.85	98.81	96.09	97.25	98.66	98.64
288	97.37	98.69	98.25	98.79	98.70	98.22	98.89	98.64	99.08	98.94	99.05	98.85	99.04	98.30	98.53	98.75	98.83
1152	98.28	98.79	98.66	98.85	98.88	98.66	98.98	98.76	98.97	98.90	99.04	98.91	99.01	98.91	98.97	99.02	98.99
	Design 2																
12	88.76	93.34	90.36	95.68	94.57	93.66	96.46	92.98	97.90	96.51	96.97	96.11	98.91	89.84	94.00	98.95	98.95
24	92.22	95.66	93.72	96.49	96.35	95.43	96.79	95.59	97.91	96.97	98.11	97.28	98.87	94.03	96.05	98.62	98.81
48	94.44	97.25	95.93	97.65	97.55	96.77	97.69	97.31	98.51	97.98	98.84	98.31	99.10	96.45	97.71	98.77	98.85
288	97.58	98.60	98.29	98.71	98.63	98.18	98.65	98.41	98.69	98.47	99.10	98.76	98.92	98.30	98.56	98.72	98.80
1152	98.17	98.86	98.71	98.93	98.94	98.59	98.86	98.82	98.90	98.84	99.17	99.03	99.08	98.81	98.83	98.97	98.91

Note: 10,000 replications, with 999 bootstrap replications each.

Figure 1: Symmetric bootstrap and BN-S (2004) asymptotic theory based 95% two-sided confidence intervals for the CAPM 3-month (monthly) rolling realized beta of US bond. April 1986 through July 2007.

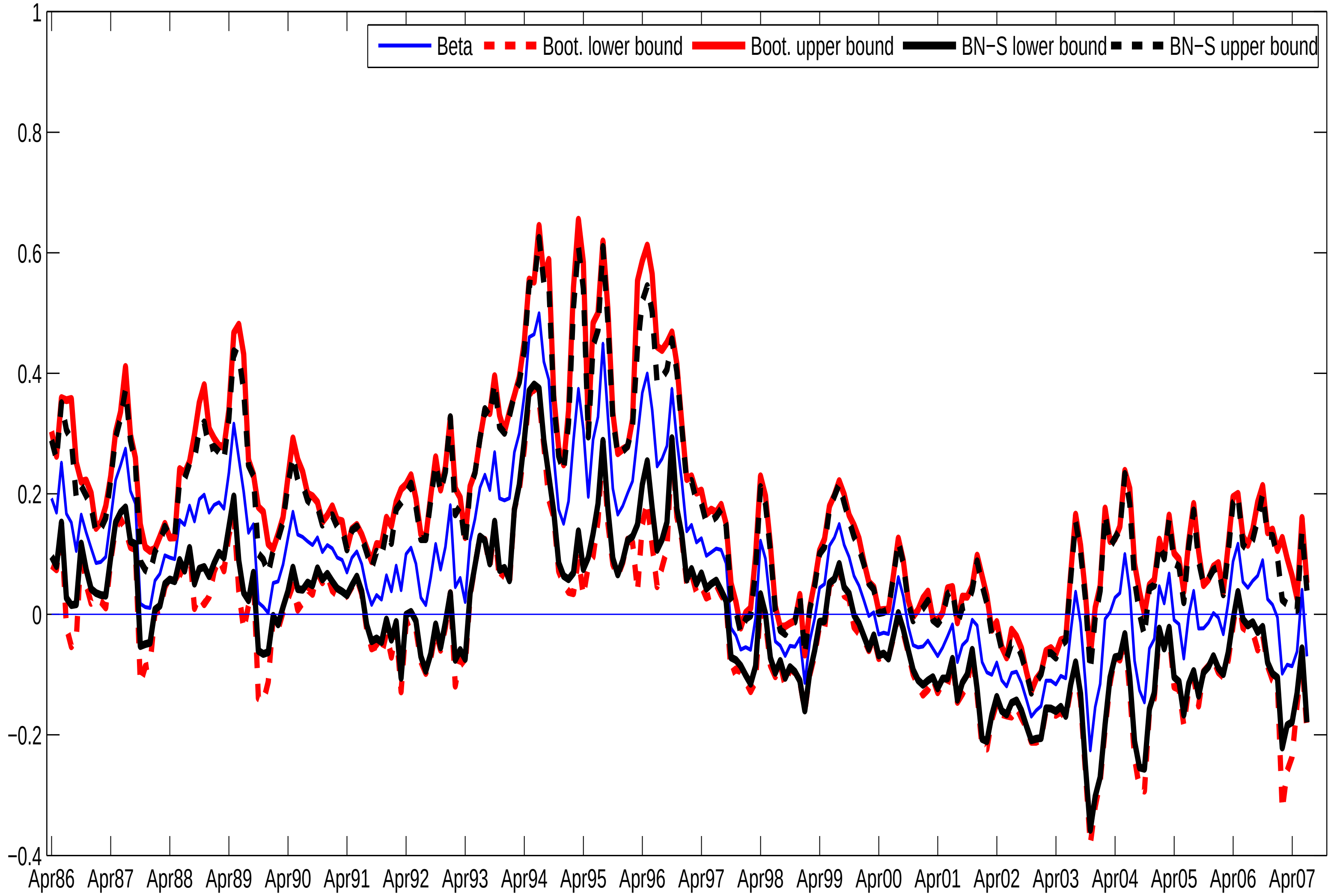

Figure 2: Symmetric bootstrap and BN-S (2004) asymptotic theory based 95% two-sided confidence intervals for the CAPM 3-month (monthly) rolling realized beta of UK bond. April 1986 through July 2007.

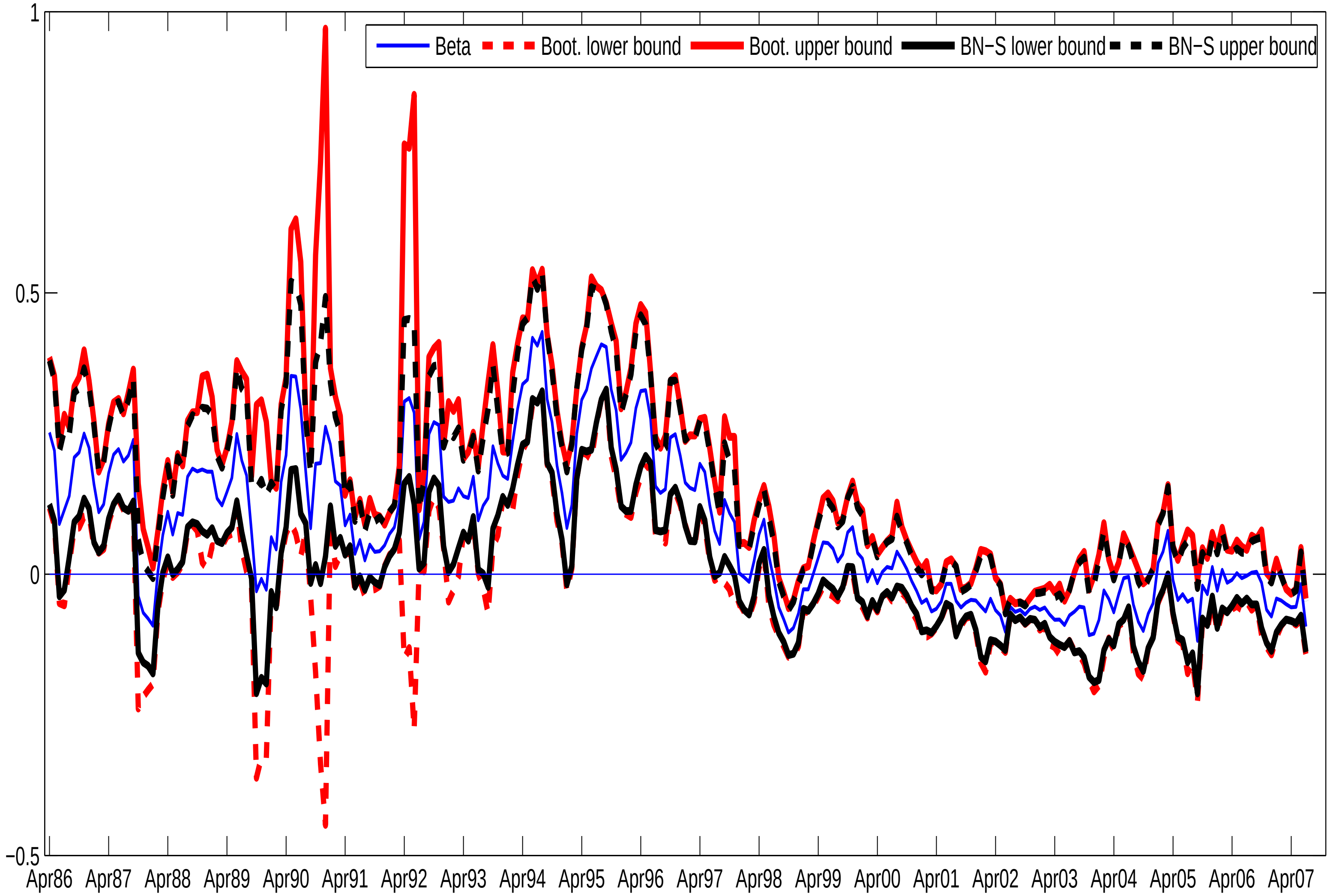

Table 4. Full-sample estimates of bonds betas for the US and the UK
from January 2, 1986 through August 24, 2007

	Beta	BN-S 95% 2-sided CI	Boot. symm. 95% CI
US			
	0.024	$[0.010, 0.038]$	$[0.009, 0.038]$
UK			
	0.030	$[0.016, 0.045]$	$[0.015, 0.046]$

Table 5. Divergence between BN-S and bootstrap confidence intervals for the US

Date	Beta	BN-S	Bootstrap
31-Jul-86	0.167	$[0.027, 0.306]$	$[-0.022, 0.355]$
29-Aug-86	0.152	$[0.015, 0.289]$	$[-0.053, 0.357]$
30-Sep-86	0.106	$[0.017, 0.194]$	$[-0.041, 0.252]$
31-Jul-89	0.204	$[0.036, 0.371]$	$[-0.025, 0.432]$
29-May-92	0.111	$[0.004, 0.217]$	$[-0.010, 0.231]$
29-May-98	0.093	$[0.001, 0.184]$	$[-0.012, 0.197]$
31-Aug-00	0.062	$[0.002, 0.121]$	$[-0.002, 0.126]$
30-Jan-98	-0.054	$[-0.101, -0.008]$	$[-0.111, 0.003]$
27-Feb-98	-0.059	$[-0.115, -0.002]$	$[-0.128, 0.010]$
29-Dec-00	-0.055	$[-0.109, -0.000]$	$[-0.117, 0.008]$
31-May-01	-0.055	$[-0.107, -0.004]$	$[-0.113, 0.003]$
31-Dec-03	-0.154	$[-0.302, -0.005]$	$[-0.319, 0.011]$
29-Oct-04	-0.146	$[-0.256, -0.036]$	$[-0.293, 0.001]$

Table 6. Divergence between BN-S and bootstrap confidence intervals for the UK

Date	Beta	BN-S	Bootstrap
31-Mar-88	0.070	$[0.003, 0.137]$	$[-0.010, 0.150]$
31-Oct-90	0.197	$[0.016, 0.377]$	$[-0.175, 0.568]$
31-Dec-90	0.262	$[0.031, 0.493]$	$[-0.446, 0.970]$
30-Apr-92	0.307	$[0.162, 0.452]$	$[-0.151, 0.764]$
29-May-92	0.314	$[0.173, 0.454]$	$[-0.131, 0.758]$
30-Jun-92	0.288	$[0.125, 0.450]$	$[-0.277, 0.852]$
29-Jan-93	0.129	$[0.003, 0.254]$	$[-0.049, 0.306]$
26-Feb-93	0.131	$[0.018, 0.243]$	$[-0.029, 0.290]$
31-Mar-93	0.153	$[0.046, 0.259]$	$[-0.004, 0.309]$
31-Aug-93	0.122	$[0.001, 0.242]$	$[-0.025, 0.268]$
29-Aug-97	0.054	$[0.002, 0.105]$	$[-0.003, 0.111]$
30-Sep-97	0.132	$[0.031, 0.233]$	$[-0.015, 0.279]$
31-Oct-97	0.109	$[0.015, 0.202]$	$[-0.027, 0.244]$
29-Jan-88	-0.092	$[-0.177, -0.007]$	$[-0.195, 0.012]$
31-Jan-01	-0.052	$[-0.102, -0.001]$	$[-0.111, 0.008]$
30-Sep-04	-0.085	$[-0.156, -0.014]$	$[-0.177, 0.008]$
30-Nov-06	-0.064	$[-0.122, -0.005]$	$[-0.129, 0.002]$

Appendix B

This Appendix contains the proofs of the results in Section 3. We first present two auxiliary lemmas.

Lemma B.1 *Let y_{ji} denote the jth component of y_i. Under Assumptions 1 and 2, for any $q_1, q_2 \geq 0$ such that $q_1 + q_2 > 0$, $h^{1-(q_1+q_2)/2} \sum_{i=1}^{1/h} |y_{1i}|^{q_1} |y_{2i}|^{q_2} = O_P(1)$.*

Proof of Lemma B.1. $\sum_{i=1}^{1/h} |y_{1i}|^{q_1} |y_{2i}|^{q_2} \leq \left(\sum_{i=1}^{1/h} |y_{1i}|^{2q_1}\right)^{1/2} \left(\sum_{i=1}^{1/h} |y_{2i}|^{2q_2}\right)^{1/2}$ by the Cauchy-Schwarz inequality. From Theorem 1 of BN-S (2004), $\sum_{i=1}^{1/h} |y_{1i}|^{2q_1} = O_P(h^{-1+q_1})$ and $\sum_{i=1}^{1/h} |y_{2i}|^{2q_2} = O_P(h^{-1+q_2})$, which proves the result.

Lemma B.2 *Let $\{y_i^* : i = 1, \ldots, 1/h\}$ denote an i.i.d. bootstrap sample of intraday returns $\{y_i : i = 1, \ldots, 1/h\}$ and assume that Assumptions 1 and 2 hold. Then, for k, l, k', $l' = 1, 2$, with probability approaching one, (i) $\sum_{i=1}^{1/h} y_{ki}^* y_{li}^* \xrightarrow{P^*} \sum_{i=1}^{1/h} y_{ki} y_{li}$, and (ii) $h^{-1} \sum_{i=1}^{1/h} y_{ki}^* y_{li}^* y_{k'i}^* y_{l'i}^* \xrightarrow{P^*} h^{-1} \sum_{i=1}^{1/h} y_{ki} y_{li} y_{k'i} y_{l'i}$.*

Proof of Lemma B.2. We show that the results hold in quadratic mean with respect to the bootstrap measure, with probability approaching one. This ensures that the bootstrap convergence also holds in probability. For (i), we have $E^* \left(\sum_{i=1}^{1/h} y_{ki}^* y_{li}^*\right) = h^{-1} E^* \left(y_{k1}^* y_{l1}^*\right) = h^{-1} h \sum_{i=1}^{1/h} y_{ki} y_{li} = \sum_{i=1}^{1/h} y_{ki} y_{li}$. Similarly,

$$Var^* \left(\sum_{i=1}^{1/h} y_{ki}^* y_{li}^*\right) = h^{-1} Var^* \left(y_{k1}^* y_{l1}^*\right) = h^{-1} \left(E^* (y_{k1}^* y_{l1}^*)^2 - (E^* y_{k1}^* y_{l1}^*)^2\right)$$

$$= h^{-1} \left(h \sum_{i=1}^{1/h} (y_{ki} y_{li})^2 - \left(h \sum_{i=1}^{1/h} y_{ki} y_{li}\right)^2\right) = \sum_{i=1}^{1/h} (y_{ki} y_{li})^2 - h \left(\sum_{i=1}^{1/h} y_{ki} y_{li}\right)^2 = o_P(1),$$

given that Lemma B.1 implies that $\sum_{i=1}^{1/h} (y_{ki} y_{li})^2 = O_P(h) = o_P(1)$ and $\sum_{i=1}^{1/h} y_{ki} y_{li} = O_P(1)$. This proves the result. The proof of (ii) follows similarly and therefore we omit the details.

Proof of Theorem 3.1. The proof of (a) follows from Lemma B.2 by noting that the elements of $x_i^* x_i^{*\prime}$ are of all of the form $y_{ki}^* y_{li}^* y_{k'i}^* y_{l'i}^*$, for $k, l, k', l' = 1, 2$.

To prove (b), we first show that both $\hat{V}^*$ and V^* are non singular in large samples with probability approaching one, as the sample size grows. The probability limit of V^* follows from Theorem 4 of BN-S (2004) and is equal to

$$\begin{pmatrix} 3\int_0^1 \Sigma_{11}^2 du - \Gamma_{11}^2 & 3\int_0^1 \Sigma_{11}\Sigma_{12} du - \Gamma_{11}\Gamma_{12} & \int_0^1 (\Sigma_{11}\Sigma_{22} + 2\Sigma_{12}^2) du - \Gamma_{11}\Gamma_{22} \\ & \int_0^1 (\Sigma_{11}\Sigma_{22} + 2\Sigma_{12}^2) du - \Gamma_{12}^2 & 3\int_0^1 \Sigma_{12}\Sigma_{22} du - \Gamma_{12}\Gamma_{22} \\ & & 3\int_0^1 \Sigma_{22}^2 du - \Gamma_{22}^2 \end{pmatrix},$$

which can be written as $V + V_1$ where

$$V_1 = \begin{pmatrix} \int_0^1 \Sigma_{11}^2 du - \Gamma_{11}^2 & \int_0^1 \Sigma_{11}\Sigma_{12} du - \Gamma_{11}\Gamma_{12} & \int_0^1 \Sigma_{11}\Sigma_{22} du - \Gamma_{11}\Gamma_{22} \\ & \int_0^1 \Sigma_{11}\Sigma_{22} du - \Gamma_{12}^2 & \int_0^1 \Sigma_{12}\Sigma_{22} du - \Gamma_{12}\Gamma_{22} \\ & & \int_0^1 \Sigma_{22}^2 du - \Gamma_{22}^2 \end{pmatrix}.$$

V is the asymptotic variance of $\sqrt{h^{-1}} \sum_{i=1}^{1/h} x_i$ and it is pathwise symmetric positive definite by assumption. We show that V_1 is positive semidefinite, which guarantees the positive definiteness of $V + V_1$. For any $\lambda \in \mathbb{R}^3$, by straightforward calculation,

$$\lambda' V_1 \lambda = \int_0^1 (\lambda_1 \Sigma_{11}(u) + \lambda_2 \Sigma_{12}(u) + \lambda_3 \Sigma_{22}(u))^2 du - (\lambda_1 \Gamma_{11} + \lambda_2 \Gamma_{12} + \lambda_3 \Gamma_{22})^2 \geq 0$$

by the Jensen inequality. Thus, V_1 is positive-semidefinite and therefore V^* is positive definite.

Now, let $S_h^* = V^{*-1/2}\sqrt{h^{-1}}(\sum_{i=1}^{1/h} x_i^* - \sum_{i=1}^{1/h} x_i)$. Clearly, $T_h^* = \hat{V}^{*^{-1/2}} V^{*1/2} S_h^*$. As we just showed, $\hat{V}^{*-1}V^* \xrightarrow{P^*} I_3$, in probability. Thus, the proof of (b) follows from showing that for any $\lambda \in \mathbb{R}^3$ such that $\lambda'\lambda = 1$, $\sup_{x\in\mathbb{R}} |P^*(\sum_{i=1}^{1/h} \tilde{x}_i^* \leq x) - \Phi(x)| \xrightarrow{P} 0$, where $\tilde{x}_i^* = \left(\lambda' V^* \lambda\right)^{-1/2} \sqrt{h^{-1}}\lambda'(x_i^* - E^*(x_i^*))$, and where $\Phi(x)$ is the standard Gaussian cumulative distribution function. Clearly, $E^*\left(\sum_{i=1}^{1/h} \tilde{x}_i^*\right) = 0$ and $Var^*\left(\sum_{i=1}^{1/h} \tilde{x}_i^*\right) = 1$. Thus, by Katz's (1963) Berry-Essen Bound, for some small $\epsilon > 0$ and some constant $K > 0$, $\sup_{x\in\mathbb{R}} \left|P^*\left(\sum_{i=1}^{1/h} \tilde{x}_i^* \leq x\right) - \Phi(x)\right| \leq K \sum_{i=1}^{1/h} E^*|\tilde{x}_i^*|^{2+\epsilon}$. Next, we show that $\sum_{i=1}^{1/h} E^*|\tilde{x}_i^*|^{2+\epsilon} = o_p(1)$. We have

$$\begin{aligned}
\sum_{i=1}^{1/h} E^*|\tilde{x}_i^*|^{2+\epsilon} &= h^{-1}E^*|\tilde{x}_1^*|^{2+\epsilon} = h^{-1}E^*\left|\left(\lambda' V^* \lambda\right)^{-1/2} h^{-1/2}\lambda'(x_1^* - E^*\left(x_1^*\right))\right|^{2+\epsilon} \\
&= h^{-1}h^{-(2+\epsilon)/2}|\lambda' V^* \lambda|^{-(2+\epsilon)/2} E^*|\lambda'\left(x_1^* - E^*\left(x_1^*\right)\right)|^{2+\epsilon} \\
&\leq 2^{2+\epsilon}h^{-(2+\epsilon/2)}|\lambda' V^* \lambda|^{-(1+\epsilon/2)} E^*|\lambda' x_1^*|^{2+\epsilon} \leq 2^{2+\epsilon}h^{-(2+\epsilon/2)}|\lambda' V^* \lambda|^{-(1+\epsilon/2)} E^*|x_1^*|^{2+\epsilon} \\
&= 2^{2+\epsilon}h^{-1-\epsilon/2}|\lambda' V^* \lambda|^{-(1+\epsilon/2)} \sum_{i=1}^{1/h} |x_i|^{2+\epsilon},
\end{aligned}$$

where the first inequality follows from the C_r and the Jensen inequalities and the second inequality follows from the Cauchy-Schwarz inequality and the fact that $\lambda'\lambda = 1$. We let $|z| = (z'z)^{1/2}$ for any vector z. It follows that $\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} = \sum_{i=1}^{1/h} |x_i|^{2(1+\epsilon/2)} \leq \sum_{i=1}^{1/h} \left(y_{1i}^2 + y_{2i}^2\right)^{2(1+\epsilon/2)}$, since $|x_i|^2 = \left(y_{1i}^4 + y_{1i}^2 y_{2i}^2 + y_{2i}^4\right)^2 \leq y_{1i}^4 + 2y_{1i}^2 y_{2i}^2 + y_{2i}^4 = \left(y_{1i}^2 + y_{2i}^2\right)^2$. Thus, $\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} \leq \sum_{i=1}^{1/h} (y_{1i}^2 + y_{2i}^2)^{2+\epsilon}$. By the Minkowski's inequality,

$$\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} \leq \left\{\left(\sum_{i=1}^{1/h} |y_{1i}|^{4+2\epsilon}\right)^{1/(2+\epsilon)} + \left(\sum_{i=1}^{1/h} |y_{2i}|^{4+2\epsilon}\right)^{1/(2+\epsilon)}\right\}^{2+\epsilon}.$$

By Lemma B.1, $\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} = O_P(h^{1+\epsilon})$. Therefore, $\sum_{i=1}^{1/h} E^*|\tilde{x}_i^*|^{2+\epsilon} = O_P(h^{\epsilon/2}) = o_P(1)$.

Proof of Theorem 3.2. Since $T_h \xrightarrow{d} N(0, I_3)$, by the standard delta method, $T_{f,h} \xrightarrow{d} N(0,1)$. Similarly, by a mean value expansion, and conditionally on the original sample,

$$\sqrt{h^{-1}}\left(f(vech(\hat{\Gamma}^*)) - f(vech(\hat{\Gamma}))\right) = \sqrt{h^{-1}}\nabla' f\left(vech(\hat{\Gamma})\right)\left(vech(\hat{\Gamma}^*) - vech(\hat{\Gamma})\right) + o_{P^*}(1),$$

since $\hat{\Gamma}^* \to^{P^*} \hat{\Gamma}$ in probability. Let

$$S_{f,h}^* \equiv \frac{\sqrt{h^{-1}}\left(f(vech(\hat{\Gamma}^*)) - f(vech(\hat{\Gamma}))\right)}{\sqrt{V_f^*}},$$

with $V_f^* \equiv \nabla' f(vech(\hat{\Gamma}))V^* \nabla f(vech(\hat{\Gamma}))$. It follows that $S_{f,h}^* \to^{d^*} N\left(0,1\right)$ in probability, given Theorem 3.1 (b). Next note that $T_{f,h}^* = \sqrt{\frac{V_f^*}{\hat{V}_f^*}} S_{f,h}^*$, where $\hat{V}_f^* \to^{P^*} V_f^*$. The result follows from Polya's theorem (e.g. Serfling, 1980) given that the normal distribution is continuous.

Proof of Theorem 4.1. Part (a) follows from Theorem 3.2 with $f(\theta) = \theta_2/\theta_3$. To derive V_β^*, let $\hat{\theta}_1 = \sum_{i=1}^{1/h} y_{1i}^2$, $\hat{\theta}_2 = \sum_{i=1}^{1/h} y_{1i}y_{2i}$, $\hat{\theta}_3 = \sum_{i=1}^{1/h} y_{2i}^2$. Clearly, $\hat{\beta}_{12} = f(\hat{\theta})$ and $\nabla f(\hat{\theta}) = \begin{pmatrix} 0 & \frac{1}{\hat{\theta}_3} & -\frac{\hat{\theta}_2}{\hat{\theta}_3^2} \end{pmatrix}'$. Then V_β^* is given by $V_\beta^* = \nabla' f(\hat{\theta}))V^* \nabla f(\hat{\theta})$, with $V^* = h^{-1}\sum_{i=1}^{1/h} x_i x_i' - \left(\sum_{i=1}^{1/h} x_i\right)\left(\sum_{i=1}^{1/h} x_i\right)'$.

Straightforward calculations show that
$\nabla' f(\hat{\theta}))\left(h^{-1}\sum_{i=1}^{1/h} x_i x_i'\right)\nabla f(\hat{\theta}) = \left(\hat{\Gamma}_{22}\right)^{-2}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$ whereas $\nabla' f(\hat{\theta}))\left[\left(\sum_{i=1}^{1/h} x_i\right)\left(\sum_{i=1}^{1/h} x_i\right)'\right]\nabla f(\hat{\theta}) =$ 0. Thus $V_\beta^* = \left(\hat{\Gamma}_{22}\right)^{-2}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$. Part (b) is proven in the text. Part (c) follows from Theorem 4 of BN-S (2004) and the fact that $\hat{\beta}_{12} \to^P \beta_{12}$.

Appendix C

In this Appendix we prove the results appearing in Section 4. Appendix C.1 contains the proof of the asymptotic expansions of the cumulants of $T_{\beta,h}$ appearing in Proposition 4.1.(a). A number of auxiliary lemmas are also presented and proved. Appendix C.2 contains the proof of the asymptotic expansion of the bootstrap cumulants of of $T_{\beta,h}^*$ appearing in Proposition 4.1.(b) as well as some useful lemmas.

Note that the statistic of interest can be written as follows

$$T_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12} - \beta_{12})}{\sqrt{\left(\sum_{i=1}^{1/h} y_{2i}^2\right)^{-2} h^{-1}\hat{g}_\beta}} = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\sqrt{h^{-1}\hat{g}_\beta}} = S_h\left(\frac{h^{-1}\hat{g}_\beta}{B_h}\right)^{-1/2},$$

where $\hat{g}_\beta$ and B_h are defined in the text, and $S_h = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\sqrt{B_h}}$.

Throughout this Appendix, we use the convention that $z_{1+1/h} = 0$ for any random variable z.

C.1 Asymptotic expansions of the cumulants of $T_{\beta,h}$

In this subsection, we first provide a set of lemmas that are useful to deriving the asymptotic expansions of the cumulants of $T_{\beta,h}$ through order $O\left(\sqrt{h}\right)$. Next, we prove these lemmas and at the end we prove Proposition 4.1 a). We introduce the following notations.

$$\begin{aligned}
u_i &= h^{-1}\left(y_{2i}^2\varepsilon_i^2 - E\left(y_{2i}^2\varepsilon_i^2\right)\right),\\
u_{i,i+1} &= h^{-1}\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1} - E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)\right),\\
A_{1h}^1 &= h^{-2}\sum_{i=1}^{1/h}\left(2\Gamma_{12,i}^3 - 18\beta_{12}\Gamma_{22,i}\Gamma_{12,i}^2 + 24\beta_{12}^2\Gamma_{22,i}^2\Gamma_{12,i} + 6\Gamma_{11,i}\Gamma_{22,i}\Gamma_{12,i} - 8\beta_{12}^3\Gamma_{22,i}^3 - 6\beta_{12}\Gamma_{11,i}\Gamma_{22,i}^2\right)\\
A_{1h}^2 &= h^{-2}\sum_{i=1}^{1/h}(-12\Gamma_{22,i}^3\beta_{12}^3 + 2\Gamma_{22,i}\Gamma_{22,i+1}^2\beta_{12}^3 + 2\Gamma_{22,i}^2\Gamma_{22,i+1}\beta_{12}^3 + 36\Gamma_{12,i}\Gamma_{22,i}^2\beta_{12}^2\\
&\quad -2\Gamma_{12,i+1}\Gamma_{22,i}^2\beta_{12}^2 - 2\Gamma_{12,i}\Gamma_{22,i+1}^2\beta_{12}^2 - 4\Gamma_{12,i}\Gamma_{22,i}\Gamma_{22,i+1}\beta_{12}^2 - 4\Gamma_{12,i+1}\Gamma_{22,i}\Gamma_{22,i+1}\beta_{12}^2\\
&\quad -8\Gamma_{11,i}\Gamma_{22,i}^2\beta_{12} - 28\Gamma_{12,i}^2\Gamma_{22,i}\beta_{12} + \Gamma_{12,i+1}^2\Gamma_{22,i}\beta_{12} + 4\Gamma_{12,i}\Gamma_{12,i+1}\Gamma_{22,i}\beta_{12} + \Gamma_{12,i}^2\Gamma_{22,i+1}\beta_{12}\\
&\quad +4\Gamma_{12,i}\Gamma_{12,i+1}\Gamma_{22,i+1}\beta_{12} + \Gamma_{11,i}\Gamma_{22,i}\Gamma_{22,i+1}\beta_{12} + \Gamma_{11,i+1}\Gamma_{22,i}\Gamma_{22,i+1}\beta_{12} + 4\Gamma_{12,i}^3 - \Gamma_{12,i}\Gamma_{12,i+1}^2\\
&\quad -\Gamma_{12,i}^2\Gamma_{12,i+1} + 8\Gamma_{11,i}\Gamma_{12,i}\Gamma_{22,i} - \Gamma_{11,i}\Gamma_{12,i+1}\Gamma_{22,i} - \Gamma_{11,i+1}\Gamma_{12,i}\Gamma_{22,i+1}).
\end{aligned}$$

Similarly, let

$$\begin{aligned}
A_{0h}^1 &= h^{-1}\sum_{i=1}^{1/h} E(y_{2i}^3\varepsilon_i), \quad A_{0h}^2 = h^{-1}\sum_{i=1}^{1/h} E(y_{2i}^2 y_{2,i+1}\varepsilon_{i+1}), \quad A_{0h}^3 = h^{-1}\sum_{i=1}^{1/h} E(y_{2,i+1}^2 y_{2i}\varepsilon_i),\\
A_{0h} &= \frac{1}{4}(2A_{0h}^1 - A_{0h}^2 - A_{0h}^3), \quad \text{and recall that } B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right).
\end{aligned}$$

Lemma C.3 *Let $k,l,k',l',k'',l'',m,n,m',n',m'',n''=1,2$ and let n_1, n_2, n_3, n_4, n_5 and n_6, be any non negative integers. Under Assumptions 1 and 2, and conditionally on the volatility path Σ,*

$$h^{1-(n_1+n_2+n_3+n_4+n_5+n_6)}\sum_{i=1}^{1/h}\Gamma_{kl,i}^{n_1}\Gamma_{k'l',i}^{n_2}\Gamma_{k'l',i}^{n_3}\Gamma_{mn,i+1}^{n_4}\Gamma_{m'n',i+1}^{n_5}\Gamma_{m''n'',i+1}^{n_6}$$
$$\to\int_0^1\Sigma_{kl}^{n_1}(u)\Sigma_{k'l'}^{n_2}(u)\Sigma_{k'l'}^{n_3}(u)\Sigma_{mn}^{n_4}(u)\Sigma_{m'n'}^{n_5}(u)\Sigma_{m''n''}^{n_6}(u)du,$$

as $h\to 0$.

Lemma C.4 *Under Assumptions 1 and 2, and conditionally on Σ, as $h\to 0$, $A_{1h}^j\to A_1$, for $j=1,2$; $B_h=h^{-1}\sum_{i=1}^{1/h}\left(\Gamma_{12,i}-4\beta_{12}\Gamma_{22,i}\Gamma_{12,i}+2\beta_{12}^2\Gamma_{22,i}^2+\Gamma_{11,i}\Gamma_{22,i}\right)\to B$; $A_{0h}^1=3h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12,i}\Gamma_{22,i}-\beta_{12}\Gamma_{22,i}^2)\to 3A_0$; $A_{0h}^2=h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12,i+1}\Gamma_{22,i}-\beta_{12}\Gamma_{22,i}\Gamma_{22,i+1})\to A_0$; $A_{0h}^3=h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12,i}\Gamma_{22,i+1}-\beta_{12}\Gamma_{22,i}\Gamma_{22,i+1})\to A_0$.*

Lemma C.5 *Under Assumptions 1 and 2, and conditionally on Σ, $E\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)=0$; $E\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)^2=hB_h$; $E\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)^3=h^2A_{1h}^1$; $E\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)^4=3h^2B_h^2+O(h)$; $E\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=hA_{1h}^2$; $E\left(\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)^2\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=O\left(h^2\right)$; $E\left(\left(\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)^3\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=3h^2B_hA_{1h}^2+O(h^3)$.*

Lemma C.6 *Under Assumptions 1 and 2, and conditionally on Σ, $E\left(S_h\right)=0$, $E\left(S_h^2\right)=1$, $E\left(S_h^3\right)=\sqrt{h}\frac{A_{1h}^1}{B_h^{3/2}}$, $E\left(S_h^4\right)=3+O(h)$, $E\left(S_h\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=\frac{A_{1h}^2}{\sqrt{B_h}}$, $E\left(S_h^2\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=O\left(\sqrt{h}\right)$, ans $E\left(S_h^3\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i-u_{i,i+1})\right)=3\frac{A_{1h}^2}{\sqrt{B_h}}+O(h)$.*

Lemma C.7 *Under Assumptions 1, 2 and 3, and conditionally on Σ,*

$$h^{-1}\hat{g}_\beta=B_h\left(1+\frac{1}{B_h}\sum_{i=1}^{1/h}(u_i-u_{i,i+1})-\frac{4A_{0h}}{B_h\Gamma_{22}}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)+o_P(\sqrt{h}).$$

Proof of Lemma C.3. This result follows from the boundedness of $\Sigma_{kk}(u)$ and the Reimann integrability of $\Sigma_{kl}^n(u)$ for any $k,l=1,2$ and for any non negative integer n_i.

Proof of Lemma C.4. The convergence results follow from Lemma C.3. To derive the expressions of the moments, we use the fact that under our assumptions $y_1,\ldots,y_{1/h}$ are pairs independent and $y_i\sim N(0,\Gamma_i)$ with $\Gamma_i=\int_{(i-1)h}^{ih}\Sigma(u)du$. Let C_i be the Cholesky decomposition of Γ_i. Note that $y_i\overset{d}{=}C_iu_i$: $u_i\sim iidN(0,I_2)$ where I_2 is the 2×2-identity matrix and '$\overset{d}{=}$' expresses equivalence in distribution. Let $\Gamma_{kl,i}$ and $C_{kl,i}$ be the (k,l)-th element of Γ_i and C_i, respectively. We have that

$$C_i=\begin{pmatrix}\sqrt{\Gamma_{11,i}} & 0\\ \frac{\Gamma_{12,i}}{\sqrt{\Gamma_{11,i}}} & \sqrt{\Gamma_{22,i}-\frac{\Gamma_{12,i}^2}{\Gamma_{11,i}}}\end{pmatrix},$$

and $y_{1i}\overset{d}{=}C_{11,i}u_{1i}$ and $y_{2i}\overset{d}{=}C_{21,i}u_{1i}+C_{22,i}u_{2i}$. For the second result, let $z_i=y_{2i}\varepsilon_i-E(y_{2i}\varepsilon_i)$ and note that by definition, the $z_i's$ are independent with $Ez_i=0$. It follows that

$$B_h=Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i\right)=h^{-1}E\left(\sum_{i=1}^{1/h}(y_{2i}\varepsilon_i-E(y_{2i}\varepsilon_i))\right)^2=h^{-1}\sum_{i=1}^{1/h}E\left(z_i^2\right).$$

Now, $E\left(z_i^2\right) = E\left(y_{2i}^2\varepsilon_i^2\right) - \left(E\left(y_{2i}\varepsilon_i\right)\right)^2$. Since $\varepsilon_i = y_{1i} - \beta_{12}y_{2i}$, we get that

$$\begin{aligned} E\left(y_{2i}\varepsilon_i\right) &= E\left(y_{1i}y_{2i}\right) - \beta_{12}E\left(y_{2i}^2\right) = \Gamma_{12,i} - \beta_{12}\Gamma_{22,i}, \\ E\left(y_{2i}^2\varepsilon_i^2\right) &= E\left(y_{2i}^2\left(y_{1i} - \beta_{12}y_{2i}\right)^2\right) = E\left(y_{2i}^2y_{1i}^2\right) - 2\beta_{12}E\left(y_{1i}y_{2i}^3\right) + \beta_{12}^2E\left(y_{2i}^4\right). \end{aligned}$$

We now use the Cholesky decomposition to get that

$$\begin{aligned} E\left(y_{2i}^2y_{1i}^2\right) &= E\left(\left(C_{11,i}u_{1i}\right)^2\left(C_{21,i}u_{1i} + C_{22,i}u_{2i}\right)^2\right) = E\left(C_{11,i}^2u_{1i}^2\right)\left(C_{21,i}^2u_{1i}^2 + 2C_{21,i}C_{22,i}u_{1i}u_{2i} + C_{22,i}^2u_{2i}^2\right) \\ &= 3C_{11,i}^2C_{21,i}^2 + C_{11,i}^2C_{22,i}^2 = 2\Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i}; \\ E\left(y_{1i}y_{2i}^3\right) &= E\left(\left(C_{11,i}u_{1i}\right)\left(C_{21,i}u_{1i} + C_{22,i}u_{2i}\right)^3\right) = 3C_{11,i}C_{21,i}^3 + 3C_{11,i}C_{21,i}C_{22,i}^2 = 3\Gamma_{12,i}\Gamma_{22,i};\text{ and} \\ E\left(y_{2i}^4\right) &= E\left(\left(C_{21,i}u_{1i} + C_{22,i}u_{2i}\right)^4\right) = 3C_{21,i}^4 + 6C_{21,i}^2C_{22,i}^2 + 3C_{22,i}^4 = 3\Gamma_{22,i}, \text{ implying that} \\ E\left(y_{2i}^2\varepsilon_i^2\right) &= 2\Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i} - 6\beta_{12}\Gamma_{12,i}\Gamma_{22,i} + 3\beta_{12}^2\Gamma_{22,i}^2. \end{aligned}$$

Thus, $E\left(y_{2i}^2\varepsilon_i^2\right) = 2\Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i} - 6\beta_{12}\Gamma_{12,i}\Gamma_{22,i} + 3\beta_{12}^2\Gamma_{22,i}^2$, and

$$\begin{aligned} E\left(z_i^2\right) &= 2\Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i} - 6\beta_{12}\Gamma_{12,i}\Gamma_{22,i} + 3\beta_{12}^2\Gamma_{22,i}^2 - \left(\Gamma_{12,i} - \beta_{12}\Gamma_{22,i}\right)^2 \\ &= \Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i} - 4\beta_{12}\Gamma_{12,i}\Gamma_{22,i} + 2\beta_{12}^2\Gamma_{22,i}^2, \end{aligned}$$

which implies $B_h = h^{-1}\sum_{i=1}^{1/h}\left(\Gamma_{12,i}^2 + \Gamma_{11,i}\Gamma_{22,i} - 4\beta_{12}\Gamma_{12,i}\Gamma_{22,i} + 2\beta_{12}^2\Gamma_{22,i}^2\right)$, proving the second result. The proofs of the remaining results is similar and we omit the details.

Proof of Lemma C.5. The first result follows by definition of β_{12} whereas the second result follows by the definition of B_h. For the remaining results, write $z_i = y_{2i}\varepsilon_i - E(y_{2i}\varepsilon_i)$ and note that by definition, the $z_i's$ are independent with $Ez_i = 0$. Note also that $\sum_{i=1}^{1/h} z_i = \sum_{i=1}^{1/h} y_{2i}\varepsilon_i$ since by $\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i\right) = 0$. For the third result, note that

$$E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^3 = E\left(\sum_{i=1}^{1/h} z_i\right)^3 = \sum_{i,j,k=1}^{1/h} E\left(z_iz_jz_k\right) = \sum_{i=1}^{1/h} E\left(z_i^3\right).$$

We now compute $E\left(z_i^3\right)$ using the Cholesky decomposition as in the proof of Lemma C.4 to show that $\sum_{i=1}^{1/h} E\left(z_i^3\right) = h^2A_{1h}^1$, with A_{1h}^1 as defined above. For the fourth result, note that $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^4 = \sum_{i=1}^{1/h} E\left(z_i^4\right) + 3\sum_{i\neq j} E\left(z_i^2\right)E\left(z_j^2\right) = 3\left(\sum_{i=1}^{1/h} E\left(z_i^2\right)\right)^2 + O(h^3)$ and use the definition of B_h to prove the result. For the fifth result, note that

$$E\left(\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = \sum_{i=1}^{1/h} E(z_iu_i) - \sum_{i=1}^{1/h} E(z_iu_{i,i+1}) - \sum_{i=1}^{1/h} E(z_{i+1}u_{i,i+1}).$$

Useing the definitions of u_i and $u_{i,i+1}$, the result follows from simple but tedious algebra using the Cholesky decomposition. The remaining results follow similarly and therefore we omit the details.

Proof of Lemma C.6. The proof follows straightforwardly by using Lemma C.5.

Proof of Lemma C.7. Using the definition of $\hat{g}_\beta$ in the text, we can write

$$\begin{aligned} h^{-1}\hat{g}_\beta &= h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}^2\varepsilon_i^2 + (\hat{\beta}_{12} - \beta_{12})^2y_{2i}^4 - 2(\hat{\beta}_{12} - \beta_{12})y_{2i}^3\varepsilon_i\right) \\ &\quad -h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}y_{2,i+1}\varepsilon_i\varepsilon_{i+1} + (\hat{\beta}_{12} - \beta_{12})^2y_{2i}^2y_{2,i+1}^2 - (\hat{\beta}_{12} - \beta_{12})(y_{2i}^2y_{2,i+1}\varepsilon_{i+1} + y_{2,i+1}^2y_{2i}\varepsilon_i)\right). \end{aligned}$$

Adding and subtracting appropriately, it follows that

$$\begin{aligned}
h^{-1}\hat{g}_\beta &= h^{-1}\sum_{i=1}^{1/h} E(y_{2i}\varepsilon_i)^2 - h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) + \left(h^{-1}\sum_{i=1}^{1/h}((y_{2i}\varepsilon_i)^2 - E(y_{2i}\varepsilon_i)^2)\right) \\
&\quad - \left(h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1} - E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)\right)\right) - (\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h} E(2y_{2i}^3\varepsilon_i) \\
&\quad +(\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h} E(y_{2i}^2 y_{2,i+1}\varepsilon_{i+1}) + (\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h} E(y_{2,i+1}^2 y_{2i}\varepsilon_i) + O_P(h), \\
&= B_h + h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) + \sum_{i=1}^{1/h}\left(u_i - u_{i,i+1}\right) - (\hat{\beta}_{12}-\beta_{12})2A_{0h}^1 \\
&\quad +(\hat{\beta}_{12}-\beta_{12})A_{0h}^2 + (\hat{\beta}_{12}-\beta_{12})A_{0h}^3 + O_P\left(h\right) \\
&= B_h + \sum_{i=1}^{1/h}\left(u_i - u_{i,i+1}\right) - \frac{2A_{0h}^1}{\Gamma_{22}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i + \frac{A_{0h}^2}{\Gamma_{22}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i + \frac{A_{0h}^3}{\Gamma_{22}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i + o_P\left(\sqrt{h}\right),
\end{aligned}$$

where the remainder term is of order $o_P(\sqrt{h})$ given that $\hat{\beta}_{12}-\beta_{12} = O_P(\sqrt{h})$, $h^{-1}\sum_{i=1}^{1/h} y_{2i}^2 y_{2,i+1}^2 = O_P(1)$, and given that $h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}^3\varepsilon_i - E\left(y_{2i}^3\varepsilon_i\right)\right) = O_P(\sqrt{h})$ and $h^{-1}\sum_{i=1}^{1/h}\left(y_{2,i+1}^2 y_{2i}\varepsilon_i - E\left(y_{2,i+1}^2 y_{2i}\varepsilon_i\right)\right) = O_P(\sqrt{h})$ by a verifying a CLT condition. Note also that the last equality uses the fact that $\hat{\beta}_{12}-\beta_{12} = \frac{\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\Gamma_{22}} + O_P\left(h\right)$. By Lemma C.1, $h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2$ and $h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)$ have the same probability limit and by Assumption 3,
$h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - p\lim h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 = o_P\left(\sqrt{h}\right)$ and
$h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) - p\lim h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) = o_p\left(\sqrt{h}\right)$.
Therefore, $h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - h^{-1}\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) = o_P\left(\sqrt{h}\right)$.

Proof of Proposition 4.1 (a). Given Lemma C.7, we can write

$$T_{\beta,h} = S_h\left(1 + \frac{1}{B_h}\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) - \frac{4A_{0h}}{B_h\Gamma_{22}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i + o_P(\sqrt{h})\right)^{-1/2}.$$

The first and third cumulants of $T_{\beta,h}$ are given by (see e.g., Hall, 1992, p. 42) $\kappa_1(T_{\beta,h}) = E(T_{\beta,h})$ and

$$\kappa_3(T_{\beta,h}) = E(T_{\beta,h}^3) - 3E(T_{\beta,h}^2)E(T_{\beta,h}) + 2[E(T_{\beta,h})]^3.$$

Our goal is to identify the terms of order up to $O(\sqrt{h})$ of the asymptotic expansions of these two cumulants. We will first provide asymptotic expansions through order $O(\sqrt{h})$ for the first three moments of $T_{\beta,h}$. Note that for a given fixed value of k, a first-order Taylor expansion of $f(x) = (1+x)^{-k/2}$ around 0 yields $f(x) = 1 - \frac{k}{2}x + O(x^2)$. Provided that $\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) = O_P(\sqrt{h})$, we have for any fixed integer k,

$$T_{\beta,h}^k = S_h^k\left(1 - \sqrt{h}\frac{k}{2}\frac{\sqrt{h^{-1}}}{B_h}\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) + \sqrt{h}k\frac{2A_{0h}}{B_h\Gamma_{22}}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) + o(\sqrt{h}) = \tilde{T}_h^k + o(\sqrt{h}).$$

For $k = 1, 2, 3$, the moments of $\tilde{T}_h^k$ are given by

$$
\begin{aligned}
E(\tilde{T}_h) &= E\left(S_h\right) - \sqrt{h}\frac{1}{2}\frac{1}{B_h}E\left(S_h\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{2A_{0h}}{\sqrt{B_h}\Gamma_{22}}E(S_h^2),\\
E(\tilde{T}_h^2) &= E\left(S_h^2\right) - \sqrt{h}\frac{1}{B_h}E\left(S_h^2\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{4A_{0h}}{\sqrt{B_h}\Gamma_{22}}E(S_h^3),\\
E(\tilde{T}_h^3) &= E(S_h^3) - \frac{3}{2}\sqrt{h}\frac{1}{B_h}E\left(S_h^3\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{6A_{0h}}{\sqrt{B_h}\Gamma_{22}}E(S_h^4).
\end{aligned}
$$

Given Lemma C.6, $E(\tilde{T}_h) = -\sqrt{h}\frac{1}{2B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \sqrt{h}\frac{2A_{0h}}{\sqrt{B_h}\Gamma_{22}}$; $E(\tilde{T}_h^2) = 1 + O(h)$; and $E(\tilde{T}_h^3) = \sqrt{h}\frac{A_{1h}^1}{B_h^{3/2}} - \frac{3}{2B_h}\sqrt{h}\,3\times\frac{A_{1h}^2}{\sqrt{B_h}} + \sqrt{h}\frac{18A_{0h}}{\sqrt{B_h}\Gamma_{22}} + O(h)$. Thus $\kappa_1(T_{\beta,h}) = \sqrt{h}\left(-\frac{1}{2B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \frac{2A_{0h}}{\sqrt{B_h}\Gamma_{22}}\right) + o(\sqrt{h}) \equiv \sqrt{h}\kappa_{1,h} + o\left(\sqrt{h}\right)$ and $\kappa_3\left(T_{\beta,h}\right) = \sqrt{h}\left(\frac{A_{1h}^1}{B_h^{3/2}} - \frac{3}{B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \frac{12A_{0h}}{\sqrt{B_h}\Gamma_{22}}\right) + o(\sqrt{h}) \equiv \sqrt{h}\kappa_{3,h} + o\left(\sqrt{h}\right)$. By Lemma C.4, we can now show that $\lim_{h\to 0}\kappa_{1,h} = -\frac{1}{2}\frac{A_1}{B^{3/2}} + \frac{1}{2}\frac{4A_0}{\sqrt{B}\Gamma_{22}} \equiv \frac{1}{2}\left(H_1 - H_2\right)$ and $\lim_{h\to 0}\kappa_{3,h} = -2\frac{A_1}{B^{3/2}} + 3\frac{4A_0}{\sqrt{B}\Gamma_{22}} \equiv 3H_1 - 2H_2$, where A_0, A_1, B, H_1 and H_2 are as defined in the text.

C.2 Asymptotic expansions of the bootstrap cumulants of $T_{\beta,h}^*$

In this section we provide the asymptotic expansions through $O_P\left(\sqrt{h}\right)$ of the first and third cumulants of the bootstrap statistic $T_{\beta,h}^*$. Let $\varepsilon_i^* = y_{1i}^* - \hat{\beta}_{12}y_{2i}^* = \hat{\varepsilon}_{I_i}$, with I_i a uniform draw from $\{1,\ldots,n\}$, and let $\hat{\varepsilon}_i^* = y_{1i}^* - \hat{\beta}_{12}^* y_{2i}^*$ be the bootstrap OLS residual. Note that

$$
T_{\beta,h}^* \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12}^* - \hat{\beta}_{12})}{\sqrt{\left(\sum_{i=1}^{1/h} y_{2i}^{*2}\right)^{-2}\hat{B}_{1h}^*}} = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\sqrt{\hat{B}_{1h}^*}} = S_h^*\left(\frac{\hat{B}_{1h}^*}{\hat{B}_{1h}}\right)^{-1/2},
$$

where $S_h^* = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\sqrt{\hat{B}_{1h}}}$, where $\hat{B}_{1h} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$, $\hat{B}_{1h}^* = h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*2}\hat{\varepsilon}_i^{*2}$, $\hat{\varepsilon}_i^* = y_{1i}^* - \hat{\beta}_{12}^* y_{2i}^*$. Let $\hat{A}_{0h} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^3\hat{\varepsilon}_i$, $\hat{A}_{1h} = h^{-2}\sum_{i=1}^{1/h}(y_{2i}\hat{\varepsilon}_i)^3$, and $\tilde{B}_{1h}^* = h^{-1}\sum_{i=1}^{1/h}(y_{2i}^*\varepsilon_i^*)^2$.

Proposition C.1 *Under Assumptions 1 and 2 and conditionally on Σ, as $h \to 0$, $\kappa_1^*(T_{\beta,h}^*) = \sqrt{h}\left(-\frac{\hat{A}_{1h}}{2\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}\right) \equiv \sqrt{h}\kappa_{1,h}^*$, and $\kappa_3^*(T_{\beta,h}^*) = \sqrt{h}\left(-\frac{2\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{6\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}\right) + O_P(h) \equiv \sqrt{h}\kappa_{3,h}^* + O_P(h)$.*

Proposition C.1 is used to prove Proposition 4.1 (b). The proofs of these two propositions are given after the following set of auxiliary lemmas, whose proofs follow the proofs of the propositions.

Lemma C.8 *Under Assumptions 1 and 2, and conditionally on Σ, $E^*\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right) = 0$; $E^*\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)^2 = h\hat{B}_{1h}$; $E^*\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)^3 = h^2\hat{A}_{1h}$; $E^*\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)^4 = 3h^2\left(\hat{B}_{1h}\right)^2 + O_P(h^3)$; $E^*\left(\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) = h\hat{A}_{1h}$; $E^*\left(\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)^2\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) = O_P\left(h^2\right)$; $E^*\left(\left(\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right)^3\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) = 3h^2\hat{B}_{1h}\hat{A}_{1h} + O_P\left(h^3\right)$.*

Lemma C.9 *Under Assumptions 1 and 2, and conditionally on* Σ, $E^*\left(S_h^*\right) = 0$; $E^*\left(S_h^{*2}\right) = 1$; $E^*\left(S_h^{*3}\right) = \sqrt{h}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}}$; $E^*\left(S_h^{*4}\right) = 3+O_P(h)$; $E^*\left(S_h^*\sqrt{h^{-1}}(\tilde{B}_{1h}^* - \hat{B}_{1h})\right) = \frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}}$; $E^*\left(S_h^{*2}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) = O_P\left(\sqrt{h}\right)$; *and* $E^*\left(S_h^{*3}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) = 3\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}} + O_P(h)$.

Lemma C.10 *Under Assumptions 1 and 2, and conditionally on* Σ,

$$\hat{B}_{1h}^* = \hat{B}_{1h}\left(1 + \frac{\tilde{B}_{1h}^* - \hat{B}_{1h}}{\hat{B}_{1h}} - \frac{2\hat{A}_{0h}}{\hat{B}_{1h}\hat{\Gamma}_{22}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right) + O_{P^*}(h),$$

in probability.

Proof of Proposition C.1. By Lemma C.10,

$$T_{\beta,h}^* = S_{\beta,h}^*\left(1 + \frac{\tilde{B}_{1h}^* - \hat{B}_{1h}}{\hat{B}_{1h}} - \frac{2\hat{A}_{0h}}{\hat{B}_{1h}\hat{\Gamma}_{22}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^* + O_{P^*}(h)\right)^{-1/2}.$$

Following the proof of Proposition 4.1.(a), for any fixed integer k, we have that

$$T_{\beta,h}^{*^k} = S_h^{*^k}\left(1 - \sqrt{h}\frac{k}{2}\frac{\sqrt{h^{-1}}}{\hat{B}_{1h}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right) + \sqrt{h}k\frac{\hat{A}_{0h}}{\hat{B}_{1h}\hat{\Gamma}_{22}}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*\right) + O_P(h) \equiv \tilde{T}_{\beta,h}^{*^k} + O_P(h).$$

For $k = 1, 2, 3$, the moments of $\tilde{T}_h^{*k}$ are given by

$$\begin{aligned}
E^*(\tilde{T}_{\beta,h}^*) &= 0 - \sqrt{h}\frac{1}{2}\frac{1}{\hat{B}_{1h}}E^*\left(S_h^*\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) + \sqrt{h}\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}E^*(S_h^{*^2}),\\
E^*(\tilde{T}_{\beta,h}^{*^2}) &= 1 - \sqrt{h}\frac{1}{\hat{B}_{1h}}E^*\left(S_h^{*^2}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) + \sqrt{h}\frac{2\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}E^*(S_h^{*^3}),\\
E^*(\tilde{T}_{\beta,h}^{*^3}) &= E(S_h^{*^3}) - \sqrt{h}\frac{3}{2}\frac{1}{\hat{B}_{1h}}E^*\left(S_h^{*^3}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)\right) + \sqrt{h}\frac{3\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}E^*(S_h^{*^4}).
\end{aligned}$$

Lemma C.9 implies that

$$\begin{aligned}
E^*(\tilde{T}_{\beta,h}^*) &= -\sqrt{h}\frac{1}{2}\frac{1}{\hat{B}_{1h}}\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}} + \sqrt{h}\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}} = \sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}\right)\\
E^*\left(\tilde{T}_{\beta,h}^*\right) &= 1 + O_P\left(h\right),\\
E^*(\tilde{T}_{\beta,h}^{*^3}) &= \sqrt{h}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} - \sqrt{h}\frac{9}{2}\frac{1}{\hat{B}_{1h}}\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}} + \sqrt{h}\frac{9\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}} = \sqrt{h}\left(-\frac{7}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 9\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}\right).
\end{aligned}$$

Thus $\kappa_1^*(T_{\beta,h}^*) = E^*(\tilde{T}_{\beta,h}^*) = \sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_{22}}\right) \equiv \sqrt{h}\kappa_{1,h}^*$, and

$$\begin{aligned}
\kappa_3^*(T^*_{\beta,h}) &= E^*(\tilde{T}^{*^3}_{\beta,h}) - 3E^*(\tilde{T}^{*^2}_{\beta,h})E^*(\tilde{T}^*_{\beta,h}) + 2[E^*(\tilde{T}^*_{\beta,h})]^3 \\
&= \sqrt{h}\left(-\frac{7}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 9\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{22}}}\right) - 3\sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{22}}}\right) + O_P(h) \\
&= \sqrt{h}\left(-2\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 6\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{22}}}\right) + O_P(h).
\end{aligned}$$

Proof of Proposition 4.1 b). By Theorem 4 by BN-S (2004), and because $\hat{\beta}_{12} \xrightarrow{P} \beta_{12}$, we have that $\hat{B}_{1h} \xrightarrow{P} B^*$, and $\hat{A}_{0h} \xrightarrow{P} 3\int_0^1 \left(\Sigma_{12}(u)\Sigma_{12}(u) - \beta_{12}\Sigma_{22}^2(u)\right) du = 3A_0$. Similarly, we can show that

$$\hat{A}_{1h} = h^{-2}\sum_{i=1}^{1/h}(\varepsilon_i y_{2i})^3 + o_P(1) = h^{-2}\sum_{i=1}^{1/h} E\left((\varepsilon_i y_{2i})^3\right) + R_h + o_P(1),$$

where $R_h = h^{-2}\sum_{i=1}^{1/h}(\varepsilon_i y_{2i})^3 - E\left((\varepsilon_i y_{2i})^3\right)$. $E(R_h) = 0$ and by straightforward calculations, $Var\left(h^{-2}\sum_{i=1}^{1/h}(\varepsilon_i y_{2i})^3\right) = O(h) = o(1)$, which implies that $R_h = o_P(1)$. By tedious but simple algebra we can verify that

$$h^{-2}\sum_{i=1}^{1/h} E\left((\varepsilon_i y_{2i})^3\right) = h^{-2}\sum_{i=1}^{1/h}\left(\begin{array}{c} 6\Gamma_{12,i}^3 + 9\Gamma_{11,i}\Gamma_{12,i}\Gamma_{22,i} - 36\beta_{12}\Gamma_{12,i}^2\Gamma_{22,i} \\ -9\beta_{12}\Gamma_{11,i}\Gamma_{22,i}^2 + 45\beta_{12}^2\Gamma_{12,i}\Gamma_{22,i}^2 - 15\beta_{12}^3\Gamma_{22,i}^3 \end{array}\right).$$

By Lemma C.3, this last expression converges to

$$\int_0^1 (6\Sigma_{12}^3 + 9\Sigma_{11}\Sigma_{12}\Sigma_{22} - 36\beta_{12}\Sigma_{12}^2\Sigma_{22} - 9\beta_{12}\Sigma_{11}\Sigma_{22}^2 + 45\beta_{12}^2\Sigma_{12}\Sigma_{22}^2 - 15\beta_{12}^3\Sigma_{22}^3)du = \frac{3}{2}A_1^*,$$

proving that $\hat{A}_{1h} \to^P \frac{3}{2}A_1^*$. Thus, using Proposition C.1, we get that

$$p\lim \kappa_{1,h}^* = p\lim\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{22}}}\right) = -\frac{1}{2}\frac{\frac{3}{2}A_1^*}{B^{*3/2}} + \frac{3A_0}{\sqrt{B^*}\Gamma_{22}} = \frac{3}{4}\left(\frac{4A_0}{\sqrt{B^*}\Gamma_{22}} - \frac{A_1^*}{B^{*3/2}}\right) \equiv \frac{3}{4}(H_1^* - H_2^*).$$

Similarly,

$$p\lim \kappa_{3,h}^* = \left(-2\frac{\frac{3}{2}A_1^*}{B^{*3/2}} + 6\frac{3A_0}{\sqrt{B^*}\Gamma_{22}}\right) = \left(\frac{3*3}{2}H_1^* - 3H_2^*\right) = 3\left(\frac{3}{2}H_1^* - H_2^*\right).$$

Proof of Lemma C.8. The first result follows by noting that $E^*\left(\sum_{i=1}^{1/h}\varepsilon_i^* y_{2i}^*\right) = h^{-1}h\sum_{i=1}^{1/h}\hat{\varepsilon}_i y_{2i} = 0$ by the first order OLS equations. Note in particular that $E^*(\varepsilon_i^* y_{2i}^*) = 0$. The remaining results follow by using the independence between $y_{2i}^*\varepsilon_i^*$ and $y_{2j}^*\varepsilon_j^*$ for $i \neq j$.

Proof of Lemma C.9. The proof follows easily from Lemma C.8.

Proof of Lemma C.10. Since $\hat{\beta}_{12}^* - \hat{\beta}_{12} = O_{P^*}\left(\sqrt{h}\right)$, in probability, it follows that

$$\begin{aligned}
\hat{B}_{1h}^* &= h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}\hat{\varepsilon}_i^{*^2} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}(y_{1i}^* - \hat{\beta}_{12}^* y_{2i}^*)^2 \\
&= h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}(y_{1i}^* - \hat{\beta}_{12}y_{2i}^*)^2 - 2(\hat{\beta}_{12}^* - \hat{\beta}_{12})h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^3}(y_{1i}^* - \hat{\beta}_{12}y_{2i}^*) + O_{P^*}(h) \\
&= h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}(y_{1i}^* - \hat{\beta}_{12}y_{2i}^*)^2 - 2\frac{\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\hat{\Gamma}_{22}^*}h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^3}(y_{1i}^* - \hat{\beta}_{12}y_{2i}^*) + O_{P^*}(h).
\end{aligned}$$

By a CLT for i.i.d random variables, we can prove that

$$\begin{aligned} h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*2}(y_{1i}^{*}-\hat{\beta}_{12}y_{2i}^{*})^2 - h^{-1}\sum_{i=1}^{1/h} y_{2i}^{2}(y_{1i}-\hat{\beta}_{12}y_{2i})^2 &= O_{P^*}(\sqrt{h}), \\ h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*3}(y_{1i}^{*}-\hat{\beta}_{12}y_{2i}^{*}) - h^{-1}\sum_{i=1}^{1/h} y_{2i}^{3}(y_{1i}-\hat{\beta}_{12}y_{2i}) &= O_{P^*}(\sqrt{h}), \quad \text{and} \\ \hat{\Gamma}_{22}^{*} - \hat{\Gamma}_{22} &= O_{P^*}(\sqrt{h}), \end{aligned}$$

in probability. Adding and subtracting appropriately gives the result.

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