ON ROBUST MEASUREMENT OF RISK, INEQUALITY AND CONCENTRATION

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PRELIMINARY AND INCOMPLETE

ABSTRACT

Many risk, inequality, poverty and concentration measures are extremely sensitive to outliers, dependence, heterogeneity and heavy tails. In this paper we focus on robust measurement of risk, inequality, poverty and concentration under heterogeneity, dependence and heavy-tailedness of largely unknown form using the recent results on $t-$statistic based heterogeneity and correlation robust inference in Ibragimov & Müller (2007). The robust large sample inference on risk, inequality, poverty and concentration measures is conducted as follows: partition the observations into $q \geq 2$ groups, calculate the empirical measures for each group and conduct a standard $t-$test with the resulting q estimators of the population measures. Numerical results confirm the appealing properties of $t-$statistic based robust inference method in this context, and its applicability to many widely used risk, inequality, poverty and concentration measures, including Sharpe ratio; value at risk and expected shortfall; Gini coefficient; Theil index, mean logarithmic deviation and generalized entropy measures; Atkinson measures; coefficient of variation and Herfindahl-Hirschman index; head count, poverty gap and squared poverty gap indices and other Foster-Greer-Thorbecke measures of poverty, among others. The results discussed in the paper further indicate a strong link between the $t-$statistic based robust inference methods and stochastic analogues of the majorization conditions that are usually imposed on risk, inequality, poverty and concentration measures related to self-normalized sums or their transforms, as in the case of Sharpe ratio, coefficient of variation and Herfindahl-Hirschman index.

JEL Classification: C1; C12; C13; D31; D63; L11

KEYWORDS: Risk; Inequality; Poverty; Concentration; Measures; Robustness; Income inequality; Wealth inequality; Firm size; Distribution; Dependence; Correlation; Heterogeneity; Heavy tails; Extremes; Outliers; Heterogeneity and correlation robust inference; $t-$statistic based inference; Majorization; Dalton-Pigou condition; Financial crisis

1 Introduction

1.1 Heavy tails, heterogeneity and dependence

Empirical analyses on risk, inequality, poverty and concentration measurement often face the difficulty that the data is correlated, heterogeneous or heavy-tailed in some unknown fashion.

For instance, as has been documented in numerous studies, observations on many variables of interest in economics and finance, including financial returns, insurance risks, income and wealth, and firm sizes typically exhibit dependence and heavy tails as in the case of commonly observed Pareto or power laws (see, among others, the discussion and reviews in Embrechts, Klüppelberg & Mikosch, 1997, Gabaix, 2009, Ibragimov, 2009, and references therein).

Heavy-tailed random variables (r.v.'s) X with distribution that has power tails satisfy

$$P(|X| > x) \asymp \frac{C}{x^{\zeta}}, \quad \zeta > 0,\ C > 0, \quad \text{as} \quad x \to +\infty \tag{1}$$

(here and throughout the article, $f(x) \asymp g(x)$ as $x \to +\infty$ means that $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$). The parameter ζ in (1) is referred to as the tail index, or the tail exponent, of the distribution of X. An important property of r.v.'s X satisfying (1) is that the absolute moments of X are finite if and only if their order is less than the tail index ζ : $E|X|^p < \infty$ if $p < \zeta$ and $E|X|^p = \infty$ if $p \geq \zeta$.

Empirical results on Pareto laws of income and wealth indicate that distributions of these variables typically satisfy (1) with the tail index ζ that varies between 1.5 and 3 for income and is rather stable, perhaps around 1.5, for wealth (see Persky, 1992, Atkinson & Piketty, 2007, Gabaix, 2009, Kirman, 2008, and references therein). The following is a sample of estimates of the tail index ζ in distributions satisfying (1) for returns on various stocks and stock indices: $3 < \zeta < 5$ (Jansen & de Vries, 1991), $2 < \zeta < 4$ (Loretan & Phillips, 1994), $1.5 < \zeta < 2$ (McCulloch, 1997), $0.9 < \zeta < 2$ (Rachev & Mittnik, 2000), $\zeta \approx 3$ (Gabaix et al., 2003). Power laws (1) with $\zeta \approx 1$ (Zipf laws) have been found to hold for firm sizes and city sizes (see Gabaix 1999; Axtell, 2001). The empirical results in Ibragimov, Ibragimov & Kattuman (2009) indicate that exchange rates in emerging economies are typically much more heavy-tailed than those in developed markets. In particular, the estimates in Ibragimov, Ibragimov & Kattuman (2009) imply that the tail indices for exchange rates in many emerging economies may be less than two thus implying infinite variances. As discussed by Nešlehova, Embrechts & Chavez-Demoulin (2006), tail indices less than one are observed for empirical loss distributions of a number of operational risks. Silverberg & Verspagen (2007) report the tail indices ζ to be significantly less than one for financial returns from technological innovations. The analysis in Ibragimov, Jaffee & Walden (2009) indicates that the tail indices may be considerably less than one for economic losses from earthquakes and other natural disasters. Anderson (2006) discusses the heavy-tailedness paradigm in many modern economic and financial markets transformed by the Internet and the development of technology. Taleb (2007) proposes the "Black Swan" concept to theorize the phenomena of high-impact rare events in the context of heavy-tailed distributions generating extreme outliers.

1.2 Implications for risk, inequality and concentration measures

Applicability of many measures of risk, inequality, poverty and concentration becomes problematic under heterogeneity and correlation in the data generating process. Several recent works in the literature, for instance, have emphasized robustness as an important aspect in the choice of measures used in assessing economic inequality and poverty and estimation and inference methods for them (see, among others, Cowell & Flachaire, 2007, Davidson & Flachaire, 2007, Zandvakili, 2008 and references therein). The interest in robust inequality and poverty assessment is motivated, in part, by sensitivity of many commonly used income inequality and poverty measures to changes in different parts of the underlying income distribution, including sensitivity to extremes and outliers generated by heavy-tailedness.

In addition, some studies in the literature (e.g., Mandelbrot, 1997, Ch. E7) have criticized the use of one of the commonly used measures of concentration, the Herfindahl-Hirschman index (HHI), by arguing that its distributional limits can be random. This is the case whenever the firm sizes have distributions that satisfy the empirically documented power Zipf laws (1) with $\zeta \approx 1$. One can show that similar lack of consistency and non-Gaussian asymptotics under infinite variances also hold for a number of risk and inequality measures such as coefficient of variation and Sharpe ratio, that are related, similar to HHI, to self-normalized sums or their transforms (see the discussion in Section 2). This, in turn, naturally leads to poor small sample performance of estimators of these measures in the case when the risks, returns or income distributions in consideration have finite variances but infinite higher (e.g., fourth) moments, as typically the case in economic, financial and insurance markets, according to the discussion in Section 1.1.

More generally, pronounced heavy-tailedness naturally presents a challenge for applications of standard statistical and econometric methods. In particular, as pointed out by Granger & Orr (1972) and in a number of more recent studies (see, among others, Ch. 7 in Embrechts et al., 1997, and references therein) many classical approaches to inference based on variances and (auto)correlations such as regression and spectral analysis, least squares methods and autoregressive models may not apply directly for heavy-tailed observations with infinite second or higher moments as in the case of many key economic and financial variables discussed in Section 1.1.

Recent results in the literature further indicate that, besides the properties of many empirical risk, inequality, poverty and concentration indices, heavy-tailedness, extremes and outliers may have dramatic effects on their population analogues, as in the case of the value at risk (VaR) and other risk measures and the properties of a number of economic models (see Ibragimov, 2009, and references therein). In particular, as discussed in Ibragimov (2009), for extremely heavy-tailed risks that have power tails (1) with $\zeta < 1$ and infinite first moments, diversification may increase value at risk. Furthermore, it is difficult to construct an appropriate risk measure for such distributions since, for instance, such coherent alternatives to the VaR as the expected shortfall become infinite for extremely heavy-tailed distributions (1) with $\zeta < 1$. Interestingly, the (non-)diversification results in heavy-tailed value at risk models continue to hold for bounded risks (Ibragimov, Jaffee & Walden 2009, Ibragimov 2009,and references therein).

1.3 Robust $t-$statistic based inference on risk, inequality, poverty and concentration measures

In this paper we focus on robust measurement of risk, inequality and concentration under heterogeneity, dependence and heavy-tailedness of largely unknown form using the recent results on $t-$statistic based heterogeneity and correlation robust inference in Ibragimov & Müller (2007). The robust large sample inference on risk, inequality, poverty and concentration measures is conducted as follows: partition the observations into $q \geq 2$ groups, calculate the empirical measures for each group and conduct a standard $t-$test with the resulting q estimators of the population measures. As follows from the general results in Ibragimov & Müller (2007), this provides asymptotically valid inference when the empirical measures in the q groups are asymptotically independent, unbiased and Gaussian of possibly different variances, as is typically the case in applications of many indices of risk, inequality, poverty and concentration. In addition, the $t-$statistic based approach to robust inference is valid in the case when the group estimators of the population measures of risk, inequality, poverty or concentration are asymptotically independent and converge (at an arbitrary rate) to scale mixtures of normals such as stable distributions that arise as distributional limits in heavy-tailed models with infinite variances.

Numerical results confirm the appealing properties of $t-$statistic based robust inference method in this context, and its applicability to many widely used risk, inequality, poverty and concentration measures, including Sharpe ratio; value at risk and expected shortfall; Gini coefficient; Theil index, mean logarithmic deviation and generalized entropy measures; Atkinson measures; coefficient of variation and Herfindahl-Hirschman index; head count, poverty gap and squared poverty gap indices and other Foster-Greer-Thorbecke measures of poverty, among others. The results discussed in the paper further indicate a strong link between the $t-$statistic based robust inference methods and stochastic analogues of the majorization conditions that are usually imposed on risk, inequality, poverty and concentration measures related to self-normalized sums or their transforms, as in the case of Sharpe ratio, coefficient of variation and Herfindahl-Hirschman index.

1.4 Organization of the paper

The paper is organized as follows. Section 2 discusses implications of conservativeness results for $t-$statistics and self-normalized sums for small sample properties of a number of risk, inequality and concentration measures under heterogeneity and heavy tailedness in observations. Section 3.1 describes asymptotic robust inference for risk, inequality, poverty and concentration measures using the $t-$statistic based approach to robust statistical analysis under heterogeneity and dependence. Section 3.2 provides numerical results on the performance of the $t-$statistic based approach to robust inference for risk, inequality, poverty and concentration measures and its comparisons with the alternative asymptotic procedures. Section 4 makes concluding remarks. Appendix A reviews the commonly used measures of risk, inequality, poverty and concentration and discusses their properties. Appendix B discusses the general results on the $t-$statistic based approach to robust inference in Ibragimov & Müller (2007) that are needed for applications considered in this paper.

2 Small sample properties of risk, inequality and concentration indices based on self-normalized sums

Throughout the paper, given a r.v. Z, we denote by $\mu_Z = EZ$ and $\sigma_Z^2 = E(Z-\mu_Z)^2$ its mean and variance, respectively. In addition, as usual, for a sample $Z_1, ..., Z_q$, $\overline{Z} = q^{-1}\sum_{j=1}^q X_j$ will denote the sample mean of the observations Z_j's and $s_Y^2 = (q-1)^{-1}\sum_{j=1}^q (Z_j - \overline{Z})^2$ will denote their sample variance.

Consider a sample of observations $X_1, ..., X_q$, $q \geq 2$. Let $cv_q(\alpha)$ denote the $(1-\alpha/2)-$quantile of Student-t distribution with $(q-1)$ degrees of freedom: $P(|T_{q-1}| > cv_q(\alpha)) = \alpha$.

Representations similar to $t-$statistic $t = \sqrt{q}\overline{X}/s_X$, and self-normalized sums $S_q = \frac{\sum_{j=1}^q X_j}{\sqrt{\sum_{j=1}^q X_j^2}}$ in (14) and (16) hold for a wide range of variables of interest in economics and finance. These variables include, for instance, the empirical coefficient of variation $\hat{CV} = s_X/\overline{X} = \sqrt{q}/t$ and the estimators of Sharpe ratio $\hat{SR}$ for excess returns X_j, $j = 1, ..., q$. In addition, this is the case for the Herfindahl-Hirschman Index of market concentration that has the form $HHI = \sum_{j=1}^q X_j^2/\big(\sum_{j=1}^q X_j\big)^2$ and is, thus, the inverse of the square of the self-normalized ratio in (16) for firm sizes X_j, $j = 1, ..., q$.[1] These representations, together with the conservativeness results for $t-$statistics and self-normalized sums given by (15) and (17) imply similar results for the tail probabilities of the empirical Sharpe ratio $\hat{SR}$, the empirical coefficient of variation $\hat{CV}$, HHI, and a number of other variables in economics and finance. These comparisons for the empirical risk, inequality and concentration measures such as $\hat{SR}$, $\hat{CV}$, HHI and their analogues for transformations (such as logarithms) of the observations X_j provide comparisons between the tail probabilities and the cdf's of these measures under heterogeneity and heavy-tailedness and those in the standard homogeneous Gaussian case.

[1]Similar representations also hold, for instance, for commonly used sample split prediction test statistics employed in testing for time series stationarity (see Loretan & Phillips 1994, and references therein).

Below, $Y_j = \log X_j$ denote the logarithms of the observations, provided that $X_j > 0$. In addition, $\tilde{X}_1, ..., \tilde{X}_q$ denote the i.i.d. standard normal r.v.'s: $\tilde{X}_j \sim \mathcal{N}(0, 1)$.

Proposition 1 *If $X_1, ..., X_q$ are independent heterogenous normal r.v.'s $X_j \sim \mathcal{N}(0, \sigma_j^2)$ or are scale mixtures of normals (for instance, independent not necessarily identically distributed stable r.v.'s), then*

$$P(SR_X > y) \leq P(SR_{\tilde{X}} > y) \tag{2}$$

$$P(|SR_X| > y) \leq P(|SR_{\tilde{X}}| > y) \tag{3}$$

for all $y > cv_{q-1}(0.05)\sqrt{q}$,

$$P(0 < CV_X < y) \leq P(0 < CV_{\tilde{X}} < y) \tag{4}$$

$$P(|CV_X| < y) \leq P(|CV_{\tilde{X}}| < y) \tag{5}$$

for all $y < 1/(cv_{q-1}(0.05)\sqrt{q})$. If $Y_j = \log X_j$, $j = 1, ..., q$ are independent heterogenous normal r.v.'s $X_j \sim \mathcal{N}(0, \sigma_j^2)$ or are scale mixtures of normals (for instance, independent not necessarily identically distributed stable r.v.'s), then

$$P(HHI_Y \leq y) \leq P(HHI_{\tilde{X}} \leq y) \tag{6}$$

$$P(|HHI_Y| \leq y) \leq P(|HHI_{\tilde{X}}| \leq y) \tag{7}$$

for all $y > \frac{qcv_{q-1}^2(0.05)}{cv_{q-1}^2(0.05)+q-1}$. In general, inequalities (2), (3) do not hold for $y > cv_{q-1}(0.1)\sqrt{q}$; inequalities (4), (5) do not hold for $y < 1/(cv_{q-1}(0.1)\sqrt{q})$; and inequalities (6), (7) do not hold for $y > \sqrt{\frac{qcv_{q-1}^2(0.1)}{cv_{q-1}^2(0.1)+q-1}}$.

Inequalities (2), (3) imply that homogeneity and thin-tailed risks (such as the i.i.d. standard normal ones) are more likely to produce high values of the Sharpe ratio far out in tails, than the heterogeneous or heavy-tailed ones (say, heterogeneous normal risks or heavy-tailed stable risks with infinite variances). This, however, is not true in general. Similarly, inequalities (4), (5) imply that homogeneity and thin-tailedness (such as normality) reduces the inequality and disparity, as measured by the coefficient of variation, in the region of their small values. However, in general, this does not hold, that may be viewed as an indicator that the coefficient of variation is a poor measure of inequality for some parts of the income or wealth distribution, including the middle and high income and wealth ranges. Inequalities (6), (7) imply that homogeneity and thin-tailedness in the firm size distribution are likely to reduce concentration, as measured by HHI, in the region of its small values. However, this does not hold in general, and homogeneity and thin-tailedness may produce higher values of HHI than heterogeneous and heavy-tailed observations in some parts of the firm size distribution, including the middle range.

3 Robust inference on risk, inequality, poverty and concentration measures using the $t-$statistic based approach

3.1 Description of the approach

For an illustration of the $t-$statistic based approach to robust risk, inequality, poverty and concentration measurement, consider the problem of statistical inference on a risk measure R (for instance, Sharpe ratio for log returns, the value at risk or the expected shortfall). As is common in many contexts where risk, inequality and concentration are important, let the data generating processes exhibit heavy tails, heterogeneity or dependence. The $t-$statistic based robust test of level $\alpha \leq 5\%$ of the hypothesis $H_0 : R = R_0$ against the alternative $H_a : R \neq R_0$ is performed as follows: partition the observations on risks or returns into $q \geq 2$ groups, estimate the risk measure R for each group thus obtaining the empirical risk measures $\hat{R}_j$, $j = 1, ..., q$, and reject H_0 in favor of H_a when t_R exceeds the $(1-\alpha/2)-$percentile of the Student$-t$ distribution with $q-1$ degrees of freedom, where t_R is the usual $t-$statistic $t_R = \sqrt{q}\frac{\overline{\hat{R}}-R_0}{s_{\hat{R}}}$ with $\overline{\hat{R}} = q^{-1}\sum_{j=1}^{q}\hat{R}_j$ and $s_{\hat{R}}^2 = (q-1)^{-1}\sum_{j=1}^{q}(\hat{R}_j - \overline{\hat{R}})^2$. As follows from the general results in Ibragimov and Müller (2007), the above procedure provides asymptotically valid and in some sense efficient inference when the groups are chosen in a way that ensures the empirical risk measures $\hat{R}_j$, $j = 1, ..., q$, to be asymptotically independent, unbiased and Gaussian of possibly different variances. Furthermore, the asymptotic validity of the $t-$statistic based inference approach continues to hold even when the group estimators $\hat{R}_j$ of R converge (at an arbitrary rate) to independent but potentially heterogeneous mixed normal distributions, such as the family of stable symmetric distributions, or to conditionally normal variates which are unconditionally dependent through their second moments. In particular, the $t-$statistic based robust inference on R can thus be applied under heavy tails, extremes and outliers in observations and, among others, dependence structures that include models with multiplicative common shocks and their convolutions (see Ibragimov 2007).

3.2 Small sample properties and comparisons with alternative inference procedures

This section provides the numerical results on small sample performance of the $t-$statistic based approach to robust inference on commonly used inequality measures in comparison with the alternative procedures. In particular, we focus on the comparison of the error in rejection probabilities (ERP) of the $t-$statistic based tests on inequality indices with those of the standard asymptotic and bootstrap tests for inequality and non-standard bootstrap inference procedures, including the m out of n bootstrap (also known as the *moon* bootstrap) and a semiparametric bootstrap (see Cowell & Flachaire, 2007 and Davidson & Flachaire, 2007). We also provide analogous comparisons of the small sample properties of methods discussed in this paper with those of asymptotic tests based on semiparametric estimation of the income distribution (semiparametric inequality measures) discussed in Cowell & Flachaire (2007). As indicated in Cowell & Flachaire (2007), outliers and heavy-tailedness in income distribution have dramatic effects on performance of empirical inequality measures, even when the standard bootstrap procedures are employed. According to the results presented in Cowell & Flachaire (2007), semiparametric inference approaches, such as asymptotic tests based on semiparametric inequality measures and semiparametric bootstrap, can greatly improve the performance of many commonly used empirical inequality indices.

As in the case of non-parametric and semiparametric asymptotic and bootstrap procedures considered in Cowell & Flachaire (2007), the data used in the analysis of the ERP reported in this section are simulated using the Singh-Maddala, Pareto and log-normal cdf's that are widely used in modeling observed income distributions (see the discussion and references in Cowell & Flachaire, 2007, Davidson & Flachaire, 2007 and Section 1.1 in this paper).

R.v.'s X with Singh-Maddala distribution satisfy $P(X > x) = \frac{1}{(1+ax^b)^c}$, $x > 0$, with $a, b, c > 0$ and thus follow power law (1) with the tail index $\zeta = bc$. The true values of the GE measures for Singh-Maddala income distributions follow from definition (8) and the following formulas for moments of r.v.'s X with such distributions (see Section 2.1 in Cowell & Flachaire 2007): $E(X^\alpha) = a^{-\alpha/b}\frac{\Gamma(1+\alpha b^{-1})\Gamma(c-\alpha b^{-1})}{\Gamma(c)}$, where $\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)dt$, $x \in \mathbf{R}$, is the Gamma function (in particular, $E[X] = a^{-1/b}\frac{\Gamma(1+b^{-1})\Gamma(c-b^{-1})}{\Gamma(c)}$,. The true values of the mean logarithmic deviation and the Theil measure for Singh-Maddala distribution can be found from (9) and (10), together with the following formulas: $E[X\log(X)] = E[X]b^{-1}(\gamma(b^{-1}+1) - \gamma(c-b^{-1}) - \log a]$, $E[\log(X)] = b^{-1}(\gamma(1) - \gamma(c) - \log a]$, where $\gamma(x) = \frac{d\log(\Gamma(x))}{dx}$ is the digamma function.

R.v.'s X with Pareto distributions satisfy (1) with the exact equality for $x \geq x_0$, where $x_0 = C^{1/\zeta}$: $P(X > x) = \frac{C}{x^\zeta}$, $x \geq x_0$. The Theil index and for Pareto income distribution with the tail index $\zeta > 1$ are given by (see Section 4 in Cowell & Flachaire 2007) $\mathcal{I}^1 = \frac{1}{\alpha-1} + \log\frac{\alpha-1}{\alpha}$; and the mean logarithmic deviation for Pareto income distribution with $\alpha > 0$ is given by $\mathcal{I}^0 = -\frac{1}{\alpha} - \log\frac{\alpha-1}{\alpha}$.

The density of the log-normal distribution is given by $\frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{1}{2\sigma^2}(\log x - \mu)^2\right)$. The tails of log-normal distributions are thinner than those of power laws (1): in particular, all power moments of r.v.'s X with log-normal distributions are finite: $E|X|^p < \infty$ for all $p > 0$. However, similar to power laws, the moment generating function of X is infinite in any neighborhood of zero: $E\exp(cX) = \infty$ for all $c > 0$. In part because of this reason, log-concave distributions are difficult to distinguish from power laws in empirical applications (see the discussion in Perline 2005). The Theil index and the mean logarithmic deviation for log-normal income distribution are both equal to $\mathcal{I}^1 = \mathcal{I}^0 = \sigma^2/2$.

In the numerical results discussed in this section, we use the same parameters for Singh-Maddala, Pareto and log-normal distributions as in Section 4 of Cowell & Flachaire (2007). The parameters for the Singh-Maddala distributions are $a = 100, b = 2.8$, and $c = 0.7, 1.2, 1.7$.[2] The corresponding tail indices $\zeta = bc$ in asymptotic relation (1) for these distributions equal to, respectively, $\zeta = 1.96$ (implying infinite first moments and finite variances), $\zeta = 3.36$ (finite variances and infinite fourth moments) and $\zeta = 4.76$ (finite fourth moments but infinite moments of order greater than ζ). For

[2]As indicated in Cowell & Flachaire (2007), the choice of the parameter values $a = 100$, $b = 2.8$ and $c = 1.7$ is motivated by the fact that the Singh-Maddala distribution with these parameters closely approximates the net income distribution of German households, up to a scale factor.

the choice of the parameters $a = 100, b = 2.8$ and $c = 1.7$ as in Section 3 in Cowell & Flachaire (2007) and Table 1 and Figures 1 and 2 in this section of the paper, the true values of the inequality measures are given by (see Cowell & Flachaire 2007) $\mathcal{I}_E^2 = 0.1620$, $\mathcal{I}_E^1 = 0.1401$, $\mathcal{I}_E^{0.5} = 0.1397$, $\mathcal{I}_E^0 = 0.1460$, $\mathcal{I}_E^{-1} = 0.1898$, $\mathcal{I}_E^{-2} = 0.3866$, $\mathcal{I}_{LV} = 0.3321$ and $\mathcal{I}_{Gini} = 0.2887$. Table 1 presents the true values of these measures for other choices of the parameters in Singh-Maddala distributions considered.

The simulations for Pareto distributions use the threshold value $x_0 = 0.1$ and the tail index parameters ζ equal to $\zeta = 1.5, 2$ (finite means and infinite variances) and $\zeta = 2.5$ (finite variances and infinite fourth moments).[3] The true values of the Theil index and the mean logarithmic deviation for the above values of the tail index ζ in Pareto distributions are given in Table 2.

The simulations for log-concave distributions use $\mu = -2$ and $\sigma = 1, 0.7, 0.5$. Table 3 contains the true values of the Theil index and the mean logarithmic deviation for these choices of the parameter σ.

As in Cowell & Flachaire (2007), we first focus on the analysis of performance of the $t-$statistic based tests on inequality measures under Singh-Maddala income distributions with parameters $a = 100$, $b = 2.8$ and $c = 1.7$. Table 1 presents the ERP of the $t-$statistic based tests with $q = 2, 4$ at nominal level 0.05, that is, the difference between the actual and nominal probabilities of rejection, for different GE measures, the Gini index and the logarithmic variance. The numerical results in the table are also presented in Figures 1 and 2 for the ease of comparisons with the ERP of the alternative asymptotic and bootstrap procedures reported in Cowell & Flachaire (2007).

The comparison of the results on the of ERP of the $t-$statistic based tests on inequality measures reported in Table 2 and Figures 1 and 2 with those in Figure 7 in Cowell & Flachaire (2007) indicates that the size properties of the $t-$statistic based robust tests with $q = 2$ and $q = 4$ in small size are uniformly better than those of the asymptotic tests. In addition, the small sample size properties of the $t-$statistic based tests on inequality measures, especially that with $q = 2$, are at least comparable to and in many cases dominate the size properties of the computationally expensive alternatives, including the standard and non-standard bootstrap methods, as well as those of the asymptotic tests based on semiparametric inequality measures and semiparametric boostrap (see

[3] The corresponding values of the constant $C = x_0^\zeta$ in (1) equal to, respectively, $C = 0.0316, 0.01$ and $C = 0.0032$.

Figures 8-11 in Cowell & Flachaire 2007 and the discussion in Section 3 therein).

Similar to the analysis of the alternative inference procedures in Cowell & Flachaire (2007), Tables 2 and 3 provide the results on performance of the $t-$statistic based robust tests on inequality for different parameters in the Singh-Maddala distributions for incomes as well as Pareto and log-normal distributions. As in Cowell & Flachaire (2007), the results are provided for the ERP of the $t-$statistic based tests on the Theil and mean logarithmic deviation (MLD) measures (that is, the generalized entropy measures with $\alpha = 1$ and $\alpha = 0$, respectively).

Comparison of the ERP of the $t-$statistic based tests on the Theil measure and the mean logarithmic deviation in Tables 2 and 3 with the corresponding results in Tables 5 and 6 in Cowell & Flachaire (2007) for the alternative procedures leads to the following conclusions. In essentially all choices of the sample sizes and the parameter values for the distributions considered, the small sample properties of the $t-$statistic based tests with $q = 2$ and $q = 4$ on the Theil index and the mean logarithmic deviation are much better than those of the alternative procedures, including the asymptotic inference methods (where the better small sample performance of the $t-$statistic based robust tests is especially pronounced), standard, *moon* and semiparametric bootstrap tests as well as the asymptotic tests with semiparametric inequality measures. In addition, according to the results in Tables 5 and 6, the choice of the smaller number of blocks $q = 2$ is to be preferred, in terms of the small sample size performance of the $t-$statistic based tests, to $q = 4$. According to the unreported simulation results, the choice of $q = 2$ leads to better performance of the $t-$statistic based tests comparing to the number of blocks greater than 4 in the samples considered.

Table 4 provides the results on the effective rejection probabilities for the $t-$test applied to the Sharpe ratio under heavy-tailedness. The simulations are based on Student-t distributions with the number of degrees of freedom indicated in the table. As is seen from the results in the table, the size control of the $t-$test is remarkable, even in the case of pronounced heavy-tailedness.

4 Concluding remarks

Many risk, inequality, poverty and concentration measures are extremely sensitive to outliers, dependence, heterogeneity and heavy tails typically observed in empirical data. This paper focuses on robust measurement of risk, inequality, poverty and concentration under heterogeneity, dependence and heavy-tailedness of largely unknown form using the recent results on $t-$statistic based heterogeneity and correlation robust inference in Ibragimov & Müller (2007). Numerical results confirm the appealing properties of $t-$statistic based robust inference approach, and its applicability to many widely used risk, inequality, poverty and concentration measures, including Sharpe ratio; value at risk and expected shortfall; Gini coefficient; Theil index, mean logarithmic deviation and generalized entropy measures; Atkinson measures; coefficient of variation and Herfindahl-Hirschman index; head count, poverty gap and squared poverty gap indices and other Foster-Greer-Thorbecke measures of poverty, among others. The results discussed in the paper further indicate a strong link between the $t-$statistic based robust inference methods and tail probability comparisons for risk, inequality, poverty and concentration measures related to self-normalized sums or their transforms, as in the case of Sharpe ratio, coefficient of variation and Herfindahl-Hirschman index.

A Risk, inequality, poverty and concentration measures and their properties

In this appendix, we review the definitions of the risk, inequality, poverty and concentration measures considered in the paper. The detailed discussions of the properties of these and other measures are available, for instance, in Section 13.F in Marshall & Olkin, 1979, Cowell & Flachaire, 2007, Davidson & Flachaire, 2007, and references therein.

Variance The variance σ_X^2 and the sample variance s_X^2 are standard examples of the population and empirical measures of dispersion.

Coefficient of variation The coefficient of variation is the normalized sample standard deviation defined by $CV_X = \sigma_X/\mu_X$.

Variance and coefficient of variation of logarithms and related measures The variance and the coefficient of variation of logarithms have the form $\sigma_Y^2 = E(Y - EY)^2$, $CV_Y = \sigma_Y/\mu_Y$, where $Y = \log X$. The commonly used estimators of these measures are provided by the empirical sample variance s_Y^2 and the empirical coefficient of variation of logarithms given by

$CV_Y = s_Y/\overline{Y}_q.$

Generalized entropy (GE) measures, $\alpha \neq 0, 1$

$$\mathcal{I}_E^{\alpha} = \frac{1}{\alpha(\alpha-1)}\Big(\frac{EX^{\alpha}}{\mu_X} - 1\Big). \tag{8}$$

Mean logarithmic deviation (MLD) is a limiting case of GE measures as $\alpha \to 0$:

$$MLD = \mathcal{I}_E^0 = \log(\mu_X) - \mu_Y. \tag{9}$$

Theil measure is a limiting case of GE indices as $\alpha \to 1$:

$$\mathcal{I}_E^1 = \frac{EX\log(X)}{\mu_X} - \log(\mu_X). \tag{10}$$

Gini coefficient The Gini coefficient has the form $Gini_X = E|X - X'|$, where X' is an independent copy of X. The empirical Gini coefficient is calculated using the formulas

$$\hat{Gini}_X = \frac{1}{2q^2\overline{X}}\sum_{i=1}^{q}\sum_{j=1}^{q}|X_i - X_j| = 1 - \frac{1}{q^2\overline{X}}\sum_{i=1}^{q}\sum_{j=1}^{q}\min(X_i, X_j) = 1 + \frac{1}{q} - \frac{2}{q^2\overline{X}}\sum_{i=1}^{q} iX_{[i]},$$

where $X_{[1]} \geq ... \geq X_{[q]}$ are the components of $X = (X_1, ..., X_q)$ arranged in non-increasing order.

B $t-$statistic based correlation and heterogeneity robust inference

Suppose we want to do inference on a scalar parameter β of an econometric model in a large data set with n observations. For a wide range of models and estimators $\hat{\beta}$, it is known that $\sqrt{n}(\hat{\beta} - \beta) \Rightarrow \mathcal{N}(0, \sigma^2)$ as $n \to \infty$, where "$\Rightarrow$" denotes convergence in distribution. Suppose further that the observations exhibit correlations of largely unknown form. If such correlations are pervasive and pronounced enough, then it will be very challenging to consistently estimate σ^2, and inference procedures for β that ignore the sampling variability of a candidate consistent estimator $\hat{\sigma}^2$ will have poor small sample properties.

Ibragimov & Müller (2007) propose the following general approach to robust inference about the parameter β under heterogeneity and correlation of a largely unknown form. consider a partition the original data set into $q \geq 2$ groups, with n_j observations in group j, and $\sum_{j=1}^{q} n_j = n$. Denote by $\hat{\beta}_j$ the estimator of β using observations in group j only. Suppose the groups are chosen such that $\sqrt{n}(\hat{\beta}_j - \beta) \Rightarrow \mathcal{N}(0, \sigma_j^2)$ for all j, and, crucially, such that $\sqrt{n}(\hat{\beta}_j - \beta)$ and $\sqrt{n}(\hat{\beta}_i - \beta)$ are asymptotically independent for $i \neq j$—this amounts to the convergence in distribution

$$\sqrt{n}(\hat{\beta}_1 - \beta, \cdots, \hat{\beta}_q - \beta)' \Rightarrow \mathcal{N}(0, \text{diag}(\sigma_1^2, \cdots, \sigma_q^2)), \quad \max_{1 \leq j \leq q} \sigma_j^2 > 0 \tag{11}$$

and $\{\sigma_j^2\}_{j=1}^{q}$ are, of course, unknown. The asymptotic Gaussianity of $\sqrt{n}(\hat{\beta}_j - \beta)$, $j = 1, \cdots, q$, typically follows from the same reasoning as the asymptotic Gaussianity of the full sample estimator $\hat{\beta}$. The argument for an asymptotic independence of $\hat{\beta}_j$ and $\hat{\beta}_i$ for $i \neq j$, on the other hand, depends on the choice of groups and the details of the application (see Section 4 in Ibragimov & Müller 2007 for the discussion of such arguments for several common econometric models, including time series, panel, clustered and spatially correlated settings, and Section 3.1 in this paper for the case of inference on risk, inequality, poverty and concentration measures).

As discussed in Ibragimov & Müller (2007), one can perform an asymptotically valid test of level α, $\alpha \leq 0.05$ of $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$ by rejecting H_0 when $|t_\beta|$ exceeds the $(1 - \alpha/2)$ percentile of the Student$-t$ distribution with $q - 1$ degrees of freedom, where t_β is the usual $t-$statistic

$$t_\beta = \sqrt{q} \frac{\overline{\hat{\beta}} - \beta_0}{s_{\hat{\beta}}} \tag{12}$$

with $\overline{\hat{\beta}} = q^{-1} \sum_{j=1}^{q} \hat{\beta}_j$, the sample mean of the group estimators $\hat{\beta}_j$, $j = 1, ..., q$, and $s_{\hat{\beta}}^2 = (q - 1)^{-1} \sum_{j=1}^{q} (\hat{\beta}_j - \overline{\hat{\beta}})^2$, the sample variance of $\hat{\beta}_j$, $j = 1, ..., q$.

In other words, the usual $t-$tests can be used in the presence of asymptotic heteroskedasticity in group estimators as long the level of the tests is not greater than the typically used 5% threshold. As discussed in Ibragimov & Müller (2007), the $t-$statistic approach provides a number of important advantages over the existing methods. In particular, it can be employed when data are potentially heterogeneous and correlated in a largely unknown way. In addition, the approach is simple to implement and does not need new tables of critical values. The assumptions of asymptotic normality for group estimators in the approach are explicit and easy to interpret, in contrast to conditions that imply validity of alternative procedures. Furthermore, as shown in Ibragimov & Müller (2007), the $t-$statistic based approach to robust inference efficiently exploits the information contained in these regularity assumptions, both in the small sample settings (uniformly most powerful scale invariant test against a benchmark alternative with equal variances) and also in the asymptotic frameworks. It is important to emphasize that the asymptotic efficiency results for $t-$statistic based robust inference further imply that it is not possible to use data dependent methods to determine the optimal number of groups q to be used in the approach when the only assumption imposed on the data generating process is that of asymptotic normality for the group estimators $\hat{\beta}_j$. The numerical results presented in Ibragimov & Müller (2007) demonstrate that, for many dependence and heterogeneity settings considered in the literature and typically encountered in practice for time series, panel, clustered and spatially correlated data, the choice $q = 8$ or $q = 16$ leads to robust tests with attractive finite sample performance.

One should also note that the $t-$statistic approach described provides a formal justification for the widespread Fama-MacBeth method for inference in panel regressions with heteroskedasticity (see Fama & MacBeth 1973). In the approach, one estimates the regression separately for each year, and then tests hypotheses about the coefficient of interest using the $t-$statistic of the resulting yearly coefficient estimates. The Fama-MacBeth approach is a special case of the $t-$statistic based approach to inference, with observations of the same year collected in a group.

In addition, the same approach remains valid under deviations from normality as in the case of heavy-tailed models, as long as the estimators $\hat{\beta}_j$, $j = 1, ..., q$, are asymptotically independent and converge (at an arbitrary rate) to scale mixtures of normals. Namely, the approach is asymptotically valid if

$$\{m_n(\hat{\beta}_j - \beta)\}_{j=1}^q \Rightarrow \{Z_j V_j\}_{j=1}^q \tag{13}$$

for some real sequence m_n, where $Z_j \sim i.i.d.\ \mathcal{N}(0, 1)$, the random vector $\{V_j\}_{j=1}^q$ is independent of the vector $\{Z_j\}_{j=1}^q$ and $\max_j |V_j| > 0$ almost surely. The class of limiting scale mixtures of normals in (13) is a rather large class of distributions: it includes, for instance, the Student$-t$ distribution with arbitrary degrees of freedom (including the Cauchy distribution), the double exponential distribution, the logistic distribution and all symmetric stable distributions that typically arise as distributional limits of estimators in econometric models under heavy-tailedness with infinite variances.

The robust approach to asymptotic inference proposed in Ibragimov & Müller (2007) relies on the following powerful result on small sample properties of the $t-$statistic in heteroskedastic normal observations due to Bakirov & Székely (2005) (see also the independent proof of the result in Ibragimov & Müller 2007).

Let $X_j, j = 1, \cdots, q$, with $q \geq 2$, be independent Gaussian random variables with common mean $E[X_j] = \mu$ and variances $V[X_j] = \sigma_j^2$. Consider the usual $t-$statistic for the hypothesis test $H_0 : \mu = 0$ against the alternative $H_a : \mu \neq 0 :$

$$t = \sqrt{q}\frac{\bar{X}}{s_X}. \tag{14}$$

If the variances σ_j^2 are the same: $\sigma_j^2 = \sigma^2$ for all j, by definition, the critical value cv of $|t|$ is given by the appropriate percentile of the distribution of a Student$-t$ distributed random variable T_{q-1} with $q-1$ degrees of freedom.

The case of equal variances is extremal for the $t-$statistic in (1) in the following sense (see Bakirov & Székely 2005 and Theorem 1 in Ibragimov & Müller 2007). Let $cv_q(\alpha)$ be the critical value of the usual two-sided $t-$test of H_0 against H_a of level $\alpha \leq 0.05 : P(|T_{q-1}| > cv_q(\alpha)) = \alpha$. Then then for all $q \geq 2$,

$$\sup_{\{\sigma_1^2, \cdots, \sigma_q^2\}} P(|t| > cv_q(\alpha)|H_0) = P(|T_{q-1}| > cv_q(\alpha)) = \alpha. \tag{15}$$

The conservativeness result in (15) does not hold for 10% level with $\alpha = 0.1$.

The conservativeness properties of $t-$statistic given by (15) imply analogous results for the tail probabilities self-normalized sums

$$S_q = \sum_{j=1}^{q} X_j / \big(\sum_{j=1}^{q} X_j^2\big)^{1/2} \tag{16}$$

and their squares using the equality

$$P(|t| > y) = P(S_q^2 > \frac{qy^2}{y^2 + q - 1}) \tag{17}$$

for all $y > 0$.

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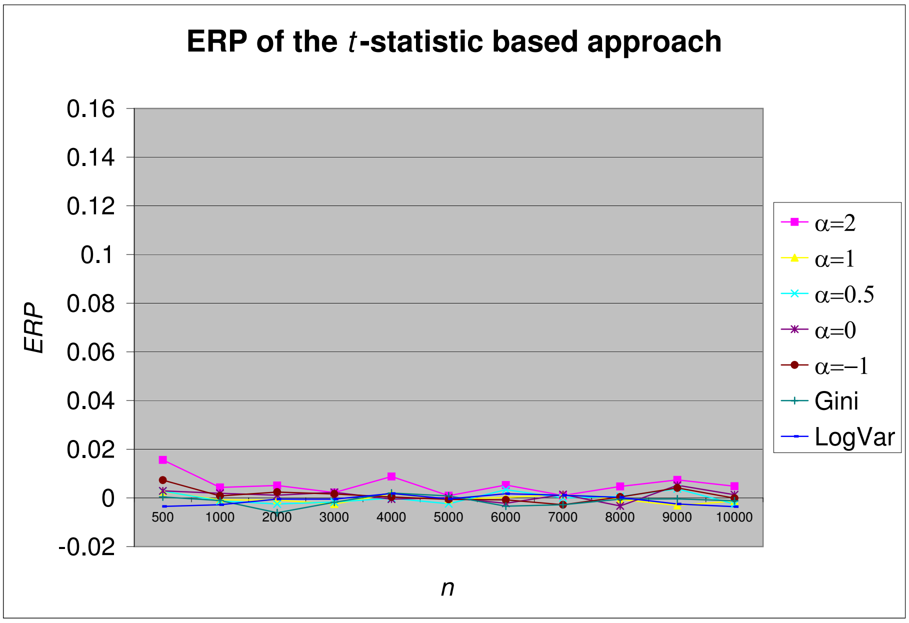

Figure 1. ERP of the $t-$statistic based test with $q = 2$

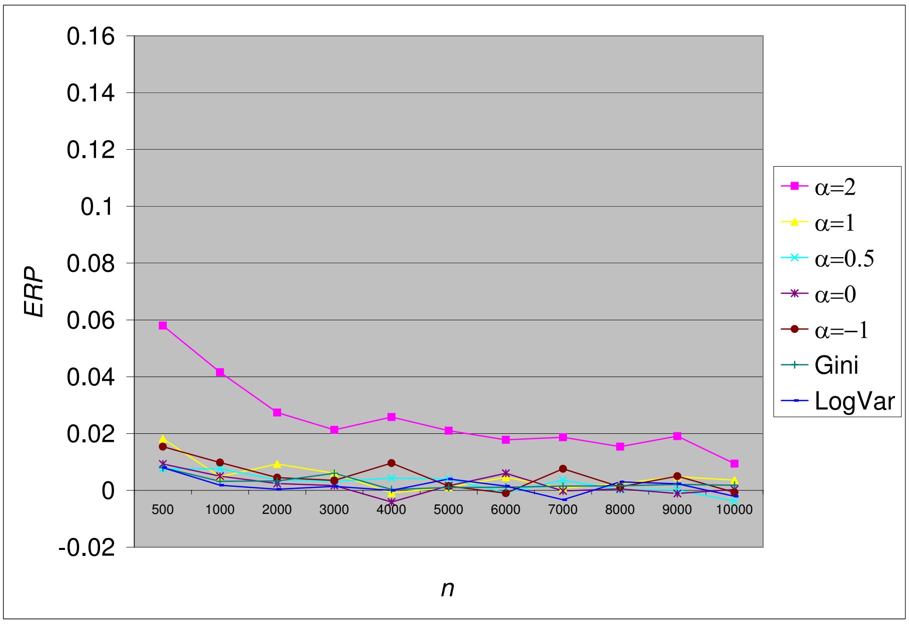

Figure 2. ERP of the $t-$statistic based tests with $q = 4$

Table 1: ERP of the $t-$statistic based tests on inequality measures

N	$\alpha=-1$	$\alpha=0$	$\alpha=0.5$	$\alpha=1$	$\alpha=2$	Gini	LogVar
				$q=2$			
500	0.0073	0.0029	0.0028	0.0026	0.0156	0.0004	−0.0035
1000	0.0008	0.0019	−0.0012	−0.0008	0.0043	−0.0011	−0.0028
2000	0.0024	0.0011	−0.0025	−0.0009	0.0051	−0.0061	−0.0005
3000	0.0016	0.0023	−0.0017	−0.0025	0.0022	−0.0018	−0.0005
4000	0.0004	−0.0005	−0.0001	0.0017	0.0088	0.0020	0.0018
5000	−0.0006	0.0001	−0.0024	−0.0006	0.0009	0.0008	−0.0004
6000	−0.0008	−0.0022	0.0034	0.0002	0.0053	−0.0034	0.0017
7000	−0.0028	0.0014	0.000	0.0014	0.0010	−0.0028	0.0011
8000	0.0004	−0.0033	0.0006	−0.0006	0.0047	−0.0005	0.0002
9000	0.0041	0.0052	0.0037	−0.0031	0.0074	−0.0004	−0.0025
10000	−0.0001	0.0013	−0.0025	−0.0013	0.0048	−0.0012	−0.0036
500	0.0154	0.0093	0.0076	0.0182	0.0580	0.0080	0.0080
1000	0.0098	0.0050	0.0076	0.0048	0.0416	0.0032	0.0018
2000	0.0045	0.0024	0.0037	0.0093	0.0274	0.0033	0.0004
3000	0.0035	0.0017	0.0032	0.0060	0.0213	0.0060	0.0014
4000	0.0096	−0.0041	0.0043	−0.0009	0.0258	0.0003	0.0000
5000	0.0015	0.0016	0.0040	0.0011	0.0210	0.0010	0.0040
6000	−0.0010	0.0060	−0.0008	0.0046	0.0178	0.0011	0.0015
7000	0.0076	−0.0002	0.0036	0.0008	0.0187	0.0015	−0.0033
8000	0.0012	0.0005	0.0002	0.0024	0.0154	0.0016	0.0030
9000	0.0050	−0.0011	0.0001	0.0048	0.0191	0.0021	0.0023
10000	−0.0007	0.0003	-0.0038	0.0036	0.0094	0.0018	−0.0021

Table 2: ERP of the $t-$statistic based test on the MLD measure ($\alpha = 0$)

	Singh-Maddala			Pareto			Lognormal		
N	$c = 0.7$	$c = 1.2$	$c = 1.7$	$\zeta = 1.5$	$\zeta = 2$	$\zeta = 2.5$	$\sigma = 1$	$\sigma = 0.7$	$\sigma = 0.5$
					$q = 2$				
500	0.0144	0.0018	0.0025	0.0416	0.0145	0.0124	0.0037	0.0027	0.0040
1000	0.0141	0.0041	0.0031	0.0387	0.0099	0.0091	0.0004	−0.0045	0.0044
2000	0.0114	0.0023	0.0003	0.0283	0.0080	0.0067	−0.0004	−0.0006	0.0026
3000	0.0066	0.0004	0.0002	0.0293	0.0075	0.0011	−0.0022	0.0031	0.0014
4000	0.0081	0.0004	0.0022	0.0310	0.0055	−0.0018	−0.0007	−0.0027	−0.0034
5000	0.0012	−0.0004	−0.0006	0.0266	0.0035	0.0035	−0.0004	0.0016	−0.0006
					$q = 4$				
500	0.0684	0.0125	0.0131	0.1754	0.0835	0.0516	0.0114	0.0084	0.0084
1000	0.0519	0.0091	0.0053	0.1553	0.0637	0.0301	0.0050	0.0015	0.0058
2000	0.0369	0.0066	0.0018	0.1282	0.0448	0.0232	0.0037	0.0037	0.0004
3000	0.0344	0.0030	−0.0004	0.1201	0.0387	0.0181	0.0014	0.0017	−0.0019
4000	0.0312	0.0030	0.0029	0.1093	0.0361	0.0177	0.0045	0.0010	−0.0008
5000	0.0316	0.0026	−0.0007	0.1115	0.0321	0.0182	0.0012	−0.0008	−0.0017

Table 3: ERP of the $t-$statistic based test on the Theil measure ($\alpha = 1$)

	Singh-Maddala			Pareto			Lognormal		
N	$c = 0.7$	$c = 1.2$	$c = 1.7$	$\zeta = 1.5$	$\zeta = 2$	$\zeta = 2.5$	$\sigma = 1$	$\sigma = 0.7$	$\sigma = 0.5$
					$q = 2$				
500	0.0502	0.0138	0.0037	0.1211	0.0475	0.0221	0.0033	0.0033	0.0020
1000	0.0387	0.0069	−0.0021	0.1033	0.0362	0.0191	0.0018	0.0055	0.0013
2000	0.0301	0.0053	0.0016	0.0960	0.0298	0.0114	0.0038	0.0010	0.0009
3000	0.0322	0.0060	0.0000	0.0858	0.0293	0.0136	0.0008	0.0022	0.0000
4000	0.0264	0.0041	0.0006	0.0898	0.0231	0.0110	0.0028	−0.0020	0.0001
5000	0.0244	0.0040	−0.0026	0.0827	0.0183	0.0065	0.0016	0.0018	−0.0010
					$q = 4$				
500	0.2103	0.0480	0.0186	0.4415	0.1948	0.1160	0.0428	0.0178	0.0119
1000	0.1647	0.0348	0.0138	0.3841	0.1605	0.0945	0.0227	0.0111	0.0043
2000	0.1437	0.0228	0.0030	0.3532	0.1368	0.0669	0.0153	0.0055	−0.0004
3000	0.1288	0.0187	0.0052	0.3325	0.1260	0.0601	0.0044	0.0016	0.0001
4000	0.1201	0.0168	0.0067	0.3149	0.1186	0.0555	0.0077	0.0051	−0.0040
5000	0.1080	0.0127	0.0013	0.3187	0.1055	0.0414	0.0038	0.0035	−0.0003

Table 4: ERP of the $t-$statistic based test on the Sharpe ratio

DF	$\zeta = 2$		$\zeta = 3$		$\zeta = 4$		$\zeta = 5$	
N	500	1000	500	1000	500	1000	500	1000
				Size				
$t-$statistic $q = 2$	5.14	5.28	5.38	4.88	5.05	5.45	5.17	5.31
$t-$statistic $q = 4$	5.73	5.76	4.85	5.01	4.82	5.39	4.9	4.92
$t-$statistic $q = 8$	5.04	5.23	5.09	5.23	4.82	5.15	5.03	5.12
$t-$statistic $q = 16$	5.29	5.33	5.03	4.74	5.07	4.93	5.01	4.86