Convergence of Functionals of Sums of Fractionally Integrated Processes

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PRELIMINARY AND INCOMPLETE

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1 Introduction

Consider a sequence $\xi_j, -\infty < j < \infty$ of i.i.d. random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$. Define a stationary sequence X_k, $k \geq 1$, by $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, where $c_j, j \geq 0$, is a sequence of real numbers. Let

$$S_k = \sum_{j=1}^{k} X_j, \qquad k \geq 1.$$

Then under suitable conditions on the constants c_j it is known that for a suitable $0 < H < 1$ and for a suitable slowly varying function $u(n)$, the finite dimensional distributions of the partial sum process

$$\left(n^H u(n)\right)^{-1} S_{[nt]} \tag{1}$$

converge in distribution to those of what is called the Linear Fractional Stable Motion (LFSM) (indexed by α and H); it may however be noted that the convergence of in the Skorokhod space $D[0,1]$ do not hold in general. (What is known as a *fractional ARIMA process with possibly heavy tailed innovations* will be a special case of the process X_k.) When $\alpha = 2$, the LFSM reduces to the Fractional Brownian Motion (FBM), and when $H = 1/\alpha$ it is taken to be the α-stable Lévy motion. (Detailed definitions are given in Section 2 below.).

Now for simplicity let $\gamma_n = n^H u(n)$. In this paper, which is partly a review, we basically consider the convergence in distribution of two classes of nonlinear functionals of $\gamma_n^{-1} S_k$, as well as martingales whose quadratic variation processes are formed by these functionals. These are the functionals that has been used in Park and Phillips (1999, 2001) in the development of large sample theory for nonlinear cointegrated models. The first one is basically of the form

$$n^{-1} \sum_{k=1}^{n} f\left(\gamma_n^{-1} S_k\right). \tag{2}$$

If $\Lambda(t)$ is limit of the process $\left(n^H u(n)\right)^{-1} S_{[nt]}$, then it will be shown that the functional (2) converges in distribution to $\int_0^1 f(\Lambda(t))\, dt$, for a broad class of functions. A major point of departure here is that some of tools such as the weak convergence of $\left(n^H u(n)\right)^{-1} S_{[nt]}$ in the Skorokhod space $D[0,1]$ that are available in the usual situation are not available in general.

The second class consists of functionals (indexed by t and x) of the form

$$n^{-1} \beta_n \sum_{k=1}^{[nt]} f\left(\beta_n \left(\gamma_n^{-1} S_k + x\right)\right). \tag{3}$$

Here we show that the finite dimensional distributions converge in distribution to those of $\left(\int_{-\infty}^{\infty} f(y)\, dy\right) L(t, -x)$, where $L(t, x)$ is the local time of the LFSM, for a wide class of functions $f(y)$ that in particular includes the indicator functions of bounded intervals of the real line. Here $\beta_n \to \infty$ such that $n^{-1}\beta_n \to 0$. In particular β_n can be taken to be γ_n itself. These results are also extended to more general situations

$$n^{-1}\sum_{k=1}^{[nt]} f_n\left(\gamma_n^{-1}S_k + x\right)$$

obtained by replacing $\beta_n f(\beta_n y)$ in (2) by a general $f_n(y)$. In this case the limit will be of the form $\int_{-\infty}^{\infty} L(t, y-x)\, dF(y)$ for a suitable $F(y)$; when $f_n(y) = \beta_n f(\beta_n y)$, the $F(y)$ will be 'degenerate' at 0 with $F(0+) - F(0-) = \int_{-\infty}^{\infty} f(y)\, dy$.

The martingales we shall consider will be of the forms

$$\left.\begin{array}{l} \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} f\left(\gamma_n^{-1}S_{k-1}\right)\eta_k \\ \sqrt{n^{-1}\beta_n}\sum_{k=1}^{[nt]} f\left(\beta_n\gamma_n^{-1}S_{k-1}\right)\eta_k \end{array}\right\} \tag{4}$$

where $(\xi_k, \eta_k), k \geq 1$, are i.i.d. such that $((\xi_k, \eta_k), k \geq 1)$ will be independent of $\left(\xi_j, j \leq 0\right)$ and such that $E[\eta_1] = 0$ and $E[\eta_1^2] = 1$. The quadratic variation of the martingales in (4) will be respectively the functionals (2) and (3). It will be shown that the second martingale in (4) will converges in law to a mixture of normal distribution, jointly with (3).

For the particular situation where the limit of (1) is either a Brownian motion or a fractional Brownian motion, some partial results in some form are available in Akonom (1993), and in the works of Park and Phillips (1999, 2001) and Tyurin and Phillips (1999) where the motivation is an interesting development of a large sample theory in some important nonlinear econometric time series models that have functions of the form $f_n\left(\gamma_n^{-1}S_k\right)$ occurring as nonlinear regressions. The present paper has the same motivation.

In Section 2 the results on the convergence of finite dimensional distributions of the partial sum process (1) are recalled. Section 3 deals with the convergence of functionals (2), where we also review some known basic results in this direction. Section 4 deals with the functionals (3); the results in this section are reproductions of some of the results in Jeganathan (2002), which may be consulted for further details. Section 5 deals with the convergence of (4). In general the treatment of this paper parallels that of Park and Phillips (1999).

2 Weak convergence of partial sum process

Recall that $\xi_j, -\infty < j < \infty$, is a sequence of i.i.d. random variables belonging to the domain of attraction of a *strictly* stable law with index $0 < \alpha \leq 2$. This means the process defined by

$$Z_{n,\alpha}(t) = \begin{cases} b_n^{-1}\sum_{j=1}^{[nt]}\xi_j & \text{if } t > 0 \\ b_n^{-1}\sum_{j=-[nt]}^{1}\xi_j & \text{if } t < 0 \end{cases}$$

converges in law to the α-stable Levy motion $\{Z_\alpha(t), t \in R\}$, $0 < \alpha \leq 2$, for a normalizing constant b_n of the form $n^{1/\alpha}h(n)$ for some slowly varying function $h(n)$. (For the details of these and other related facts, see for instance Ibragimov and Linnik (1971, Chapter 2, Section 2) or Bingham et al (1987, page 344.) Explicitly the limit $Z_\alpha(t)$ is a process with stationary independent increments having a strictly α-stable distribution, that is, for each $s < t$, $s \in R$, $t \in R$, the increment $Z_\alpha(t) - Z_\alpha(s)$ has the characteristic function

$$E[e^{iu(Z_\alpha(t)-Z_\alpha(s))}] = \begin{cases} e^{-(t-s)|u|^\alpha\left(1+i\beta\,\mathrm{sign}(u)\,\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-(t-s)|u|} & \text{if } \alpha = 1. \end{cases}$$

with $|\beta| \leq 1$. (Note that this definition of strict α-stability for the case $\alpha = 1$ differs from the usual one in that we take the shift parameter to be 0.)

When $\alpha = 2$, $Z_2(t)$ becomes the Brownian Motion (BM) with variance 2.

Now let $c_j, j = 0, 1, \ldots,$ be a sequence of real numbers such that $c_0 = 1$. Let

$$X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}.$$

and define the partial sum process $S_{[nt]}$:

$$S_{[nt]} = \sum_{k=1}^{[nt]} X_k = \sum_{j=-\infty}^{0} \left(\sum_{i=1-j}^{[nt]-j} c_i \right) \xi_j + \sum_{j=1}^{[nt]} \left(\sum_{i=0}^{[nt]-j} c_i \right) \xi_j.$$

The following result is well known when the ξ_j has a finite second moment, and the general case is also known.

Proposition 1 *: Assume that the constants $c_j, j = 0, 1, \ldots,$ satisfy the conditions*

$$\sum_{j=0}^{\infty} |c_j|^a < \infty \quad \text{for some a such that } \; 0 < a < \alpha, \quad c \leq 1,$$

and

$$\sum_{j=0}^{\infty} c_j \neq 0.$$

Then the finite dimensional distributions of the partial sum process

$$S_n(t) = b_n^{-1} \left(\sum_{j=0}^{\infty} c_j \right)^{-1} \sum_{k=1}^{[nt]} X_k$$

converge in distribution to those of the α-stable Levy motion $Z_\alpha(t)$.

Note that when $1 < \alpha \leq 2$, the required condition takes form

$$\sum_{j=0}^{\infty} |c_j| < \infty.$$

To see the plausibility of this result note that if we let

$$g(j) = \sum_{k=0}^{j} c_k,$$

then

$$\begin{aligned} \frac{1}{b_n}\sum_{k=1}^{[nt]} X_k &= \sum_{j=-\infty}^{0}(g([nt]-j)-g(1-j))\frac{\xi_j}{b_n}+\sum_{j=1}^{[nt]} g([nt]-j)\frac{\xi_j}{b_n} \\ &= \int_{-\infty}^{0}((g([nt]-[nu])-g(1-[nu]))\,dZ_{n,\alpha}(u) \\ &\quad +\int_0^t g([nt]-[nu])dZ_{n,\alpha}(u) \end{aligned} \tag{5}$$

Now note that when $0<u<t<1$, $g([nt]-[nu]) \to \sum_{k=0}^{\infty} c_k$ as $n\to\infty$ and when $0<t<1$ and $u<0$, $g([nt]-[nu])-g(1-[nu]) \to 0$. Thus one would expect that $\frac{1}{b_n}\sum_{k=1}^{[nt]} X_k$ converges in distribution to $\left(\sum_{k=0}^{\infty} c_k\right)\int_0^t dZ_\alpha(u) = \left(\sum_{k=0}^{\infty} c_k\right) Z_\alpha(t)$. A justification of this step can be found for instance in Astrauskas (1983) and Kasahara and Maejima (1988, Theorem 5.3).

Note that Proposition 1 asserts only the convergence of finite dimensional distributions. When the limit $Z_\alpha(t)$ is Gaussian (the case $\alpha=2$), convergence in the Skorokhod space is known under stronger conditions on the coefficients c_j, see Phillips and Solo (1992) and Wang et al (2002). We are not aware of the corresponding result when the limit is α-stable Levy motion with $0<\alpha<2$.

We next consider the case in which the conditions of Proposition 1 will be violated. Specifically assume that

$$c_j = j^{H-1-1/\alpha}R(j) \tag{6}$$

where $R(j)$ is slowly varying at infinity and $0<H<1$ with $H\neq 1/\alpha$. The familiar case in which this situation arises is when X_k is a fractional ARIMA $(0,d,0)$ process

$$(1-B)^d X_k = \xi_k$$

where B is back-shift operator. Here $d = H-\frac{1}{\alpha}$ so that $-\frac{1}{\alpha}<d<1-\frac{1}{\alpha}$. In this case

$$c_j = (d)(1+d)\cdots(j-1+d)/j! \sim \frac{1}{(d-1)!\,j^{1-H+\frac{1}{\alpha}}}.$$

(Here $(\alpha-1)!$ stands for the Gamma function $\Gamma(\alpha)$.) More generally one may take X_k to be a fractional ARIMA (p,d,q) process, see Hosking (1981).

One can check that (see Bingham et al (1987, Chapter 1))

$$\frac{g([nt])}{n^{H-1/\alpha}R(n)} = \frac{\sum_{k=0}^{[nt]} c_k}{n^{H-1/\alpha}R(n)} \to a t^{H-1/\alpha}$$

where the constant a is obtained by taking $t=1$. Hence as in the previous situation (see (5)) it becomes plausible that

$$\frac{1}{n^{H-1/\alpha}R(n)b_n}\sum_{k=1}^{[nt]} X_k$$

will converge in distribution to

$$\Lambda_{\alpha,H}(t) = a\int_{-\infty}^{0}\left\{(t-u)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z_\alpha(du) + a\int_0^t (t-u)^{H-1/\alpha}Z_\alpha(du)$$

where a is a non-zero constant. The process $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is called a Linear Fractional Stable Motion (LFSM) with Hurst parameter H, $0 < H < 1$. See Samorodnitsky and Taqqu (1994) for a detailed treatment of LFSM. As noted earlier when $\alpha = 2$, the LFSM reduces to the Fractional Brownian Motion (FBM).

We make the convention that in the case $H = 1/\alpha$ the LFSM $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is taken to be $\{Z_\alpha(t), t \geq 0\}$. It is important to note however that in this case the restriction $0 < H < 1$ is equivalent to that of $1 < \alpha \leq 2$.

We thus have the following result, a rigorous justification of which can be found in Astrauskas (1983) and Kasahara and Maejima (1988, Theorems 5.1 and 5.2.).

Proposition 2 *: Assume that the constants c_j satisfy (6) with $H \neq 1/\alpha$. Further assume that whenever $H - 1/\alpha < 0$, the sum $\sum_{j=0}^{\infty} c_j = 0$. (The case $\sum_{j=0}^{\infty} c_j \neq 0$ is covered by the preceding Proposition 1.) Then the finite dimensional distributions of the partial sum process*

$$S_n(t) = \frac{1}{n^{H-1/\alpha} R(n) b_n} \sum_{k=1}^{[nt]} X_k, \qquad t > 0$$

converge in distribution to those of LFSM process $\Lambda_{\alpha,H}(t)$. When $H - 1/\alpha > 0$ almost all trajectories of $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ belong to $C[0,1]$, and $S_n(t)$ converges in distribution to $\Lambda_{\alpha,H}(t)$ in the Skorokhod space $D[0,1]$

The last statement on the convergence in $D[0,1]$ is due to Astrauskas (1983). When the limit is a FBM ($\alpha = 2$) and when $H - 1/2 < 0$, convergence in $D[0,1]$ holds when $E\left[|\xi_1|^q\right]$ for $q > 1/H$. When $0 < \alpha < 2$, the trajectories of $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ do not belong $D[0,1]$ with probability one; in fact they have discontinuities of the second kind, see Astrauskas (1983).

3 Convergence of functionals of integral form I

In this section we consider the convergence of the functional (2). When the convergence in Skorokhod space $D[0,1]$ is available, a basic result is the following.

Theorem 3 *: Let $\zeta_n(t)$ and $\zeta(t)$ denote (measurable) processes, belonging to $D[0,1]$ almost surely. Assume that $\zeta_n(t)$ converges in distribution to $\zeta(t)$ in $D[0,1]$. If $f(x)$ is a continuous function, then $\int_0^1 f(\zeta_n(t))\,dt$ converges in distribution to $\int_0^1 f(\zeta(t))\,dt$.*

When $f(x)$ has a compact support, and hence is uniformly continuous, it is easy to see that the functional $\int_0^1 f(z(t))\,dt$, $z(.) \in D[0,1]$, is continuous with respect to the Skorokhod metric, and hence the result follows; there is no loss of generality in assuming that $f(x)$ has a compact support because convergence of $\zeta_n(t)$ to $\zeta(t)$ in $D[0,1]$ entails that $\lim_{T\to\infty} \operatorname{limsup}_{n\to\infty} P\left[\sup_{t,n} |\zeta_n(t)| > T\right] = 0$ and $\lim_{T\to\infty} P\left[\sup_t |\zeta(t)| > T\right] = 0$.

As noted earlier, in the context of Section 2, convergence in $D[0,1]$, which the preceding result assumes, is not available in general. Also, the continuity of $f(x)$ itself may not be available in applications. Thus we next search for results that do not require these two restrictions. A particularly simple result, due to Gikhmann and Skorokhod (1969, Theorem 1, page 485), is the next result. Though as it stands it has limited applications, we shall see that it will form a basis to obtain more satisfactory results.

Theorem 4 *: Let $\zeta_n(t)$ and $\zeta(t)$ denote processes defined on $[0,1]$. Assume that the finite dimensional distributions of $\zeta_n(t)$ converge in distribution to those of $\zeta(t)$. If*

$$sup_{t,n} E\left[|\zeta_n(t)|\right] < \infty$$

and if

$$lim_{h\to 0} limsup_{n\to\infty} sup_{|t-s|\le h} E\left[|\zeta_n(t) - \zeta_n(s)|\right] = 0,$$

then $\int_0^1 \zeta_n(t)\,dt$ converges in distribution to $\int_0^1 \zeta(t)\,dt$.

The idea of this result is very simple and is worth recalling. First note that $\int_0^1 \zeta_n(t)\,dt$ can be approximated by the Riemann sum $\frac{1}{m}\sum_{l=1}^{m}\zeta_n\left(\frac{l}{m}\right)$ in the sense that

$$E\left[\left|\int_0^1 \zeta_n(t)\,dt - \frac{1}{m}\sum_{l=1}^{m}\zeta_n\left(\frac{l}{m}\right)\right|\right] \le \sup_{|t-s|\le 1/m} E\left[|\zeta_n(t) - \zeta_n(s)|\right].$$

Now for each fixed m, $\frac{1}{m}\sum_{l=1}^{m}\zeta_n\left(\frac{l}{m}\right)$ converges in distribution to $\frac{1}{m}\sum_{l=1}^{m}\zeta\left(\frac{l}{m}\right)$, which limit as in the previous step can be approximated by $\int_0^1 \zeta(t)\,dt$ because by Fatou's lemma

$$\sup_{|t-s|\le 1/m} E\left[|\zeta(t) - \zeta(s)|\right] \le \text{limsup}_{n\to\infty}\sup_{|t-s|\le 1/m} E\left[|\zeta_n(t) - \zeta_n(s)|\right].$$

We now give a simple application of this result to a situation of Section2 in order to indicate its limitations. We shall see later (see Proposition 7 below) that a much more general result becomes available by a direct method.

Corollary 5 *: Let $S_n(t)$ be as defined in Section 2. Assume that ξ_1 has a finite second moment (which will entail that the limit $\Lambda_{\alpha,H}(t)$ of $S_n(t)$ is either a BM or FBM ($\alpha = 2$)). Then $\int_0^1 S_n^2(t)\,dt$ converges in distribution to $\int_0^1 \Lambda_{\alpha,H}^2(t)dt$.*

To obtain this result from Theorem 4, take $\zeta_n(t) = S_n^2(t)$ and first note that it can be verified easily by direct computations that

$$\lim_{h\to 0}\text{limsup}_{n\to\infty}\sup_{|t-s|\le h} E\left[|S_n(t) - S_n(s)|^2\right] = 0 \text{ and } \sup_{t,n} E\left[S_n^2(t)\right] < \infty.$$

Next, using the inequality $|a^2 - b^2| \leq (L+1)(a-b)^2 + L^{-1}b^2$ for every $L > 1$, one has

$$\sup_{|t-s|\leq h} E\left[\left|S_n^2(t) - S_n^2(s)\right|\right] \leq (L+1)\sup_{|t-s|\leq h} E\left[|S_n(t) - S_n(s)|^2\right] + L^{-1}\sup_{t,n} E\left[S_n^2(t)\right]$$

which verifies the requirement of Theorem 4.

The next result is related to a result in Borodin and Ibragimov (1995, Theorem 1.1, chapter 1) and in fact is implicit in the arguments of their proof, though it may also be considered implicit in some form in Gikhmann and Skorokhod (1969, see for example the proof of Theorem 3, page 456).

Theorem 6 *: Let $\zeta_n(t)$ and $\zeta(t)$ denote processes defined on $[0,1]$. Assume that the finite dimensional distributions of $\zeta_n(t)$ converge in distribution to those of $\zeta(t)$ such that $\zeta(t)$ has a continuous distribution. Further assume that for every $\delta > 0$*

$$lim_{h\to 0} limsup_{n\to\infty} sup_{|t-s|\leq h} P\left[|\zeta_n(t) - \zeta_n(s)| > \delta\right] = 0. \tag{7}$$

If $f(x)$ is a locally Riemann integrable function such that for every $\delta > 0$

$$\begin{aligned} &lim_{T\to\infty} limsup_{n\to\infty} P\left[\int_0^1 |f(\zeta_n(t))|\, \mathbb{I}(|\zeta_n(t)| > T)\, dt > \delta\right] = 0 \\ &lim_{T\to\infty} P\left[\int_0^1 |f(\zeta(t))|\, \mathbb{I}(|\zeta(t)| > T)\, dt > \delta\right] = 0 \end{aligned} \tag{8}$$

(where $\mathrm{I}(.)$ stands for the indicator function), then $\int_0^1 f(\zeta_n(t))\, dt$ converges in distribution to $\int_0^1 f(\zeta(t))\, dt$.

The intent of the requirement (8) is just to reduce to the situation of $f(x)$ with a compact support. This requirement is clearly satisfied when

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left[\sup_{t,n} |\zeta_n(t)| > T\right] = 0 \text{ and } \lim_{T\to\infty} P\left[\sup_t |\zeta(t)| > T\right] = 0$$

but in general will be unavailable in situations where the convergence in $D[0,1]$ is unavailable. In the context of Section 2, we shall verify the requirement (8) directly, see Proposition 7 below.

To indicate the reasoning behind this result, first note that as indicated above one can restrict to an $f(x)$ with a compact support. Assume in addition for the moment that $f(x)$ is continuous, which will entail that $f(x)$ is in addition uniformly continuous together with $\sup_x |f(x)| < \infty$. These together with the assumption (7) will then entail that the requirements of Theorem 4 are satisfied for the process $f(\zeta_n(t))$, establishing the result (when $f(x)$ is continuous).

To remove the restriction that $f(x)$ is continuous, note that when $f(x)$ is Riemann integrable with a compact support, one can find for every $\varepsilon > 0$, compactly supported continuous functions $f_\varepsilon^+(x)$ and $f_\varepsilon^-(x)$ such that

$$f_\varepsilon^+(x) \geq f(x) \geq f_\varepsilon^-(x), \quad \int \left[f_\varepsilon^+(x) - f_\varepsilon^-(x)\right] dx < \varepsilon, \quad sup_{x,\varepsilon} \left|f_\varepsilon^+(x)\right| < \infty.$$

Because the convergence holds for $f_\varepsilon^+(x)$ and $f_\varepsilon^-(x)$, passing to the limit it thus remains only to show that, with $p_t(x)$ denoting the Lebesgue density of $\zeta(t)$,

$$E\left[\int_0^1 \left[f_\varepsilon^+(\zeta(t)) - f_\varepsilon^-(\zeta(t))\right] dt\right] \le \int_0^1 dt \int \left[f_\varepsilon^+(x) - f_\varepsilon^-(x)\right] p_t(x)\, dx \to 0$$

as $\varepsilon \to 0$.This is true because, with $M = 2\sup_{x,\varepsilon}|f_\varepsilon^+(x)|$, for every $L > 0$

$$\int \left[f_\varepsilon^+(x) - f_\varepsilon^-(x)\right] p_t(x)\, dx \le L \int \left[f_\varepsilon^+(x) - f_\varepsilon^-(x)\right] dx + M \int_{\{p_t(x)>L\}} p_t(x)\, dx.$$

We shall now consider the situation of Section 2 and apply the preceding result to obtain a satisfactory result. For this purpose we need the following direct result, which will in particular show that the requirement (8) is satisfied. For the case $H - 1/\alpha > 0$ the result is a direct corollary to the preceding result because convergence in $D[0,1]$ is available. We have not yet obtained a complete proof of this result for the case $H - 1/\alpha < 0$ but we shall indicate plausible arguments below, valid at least when the limit is a FBM.

Proposition 7 *: Let $S_n(t)$ and $\Lambda_{\alpha,H}(t)$ be as defined in Section 2, where recall that LFSM $\Lambda_{\alpha,H}(t)$ is the limit of $S_n(t)$. In the case $H - 1/\alpha < 0$ assume that ξ_1 is in the domain of attraction of a normal law ($\alpha = 2$). Then $\int_0^1 S_n^r(t)\, dt$ converges in distribution to $\int_0^1 \Lambda_{\alpha,H}^r(t)dt$, for every integer $r \ge 1$, where the limit $\int_0^1 \Lambda_{\alpha,H}^r(t)dt$ is a random variable.*

The next result, again dealing with the situation of Section 2, is a consequence of Theorem 6 and Proposition 7.

Theorem 8 *: Let $S_n(t)$ and $\Lambda_{\alpha,H}(t)$ be as defined in Section 2. . Let $f(x)$ be locally Riemann integrable. In the case $H - 1/\alpha < 0$ assume that ξ_1 is in the domain of attraction of a normal law ($\alpha = 2$) and that limsup$_{|x|\to\infty} \frac{|f(x)|}{|x|^r} < \infty$ for some $r > 0$. Then $\int_0^1 f(S_n(t))\, dt$ converges in distribution to $\int_0^1 f(\Lambda_{\alpha,H}(t))\, dt$.*

The requirement (7) of Theorem 6 for $\zeta_n(t) = S_n(t)$ has been verified (for general $0 < \alpha < 2$) in Jeganathan (2002, see the proof of Lemma 12). Thus it only remains to verify the requirement (8) of Theorem 6 when ξ_1 is in the domain of attraction of a normal law. Let $q \ge 1$ be an integer such that $2q \ge r$. Then the condition $\text{limsup}_{|x|\to\infty} \frac{|f(x)|}{|x|^r} < \infty$ entails that for all sufficiently large T

$$\begin{aligned}\int_0^1 |f(\zeta_n(t))|\, \mathbb{I}(|\zeta_n(t)| > T)\, dt &\le C \int_0^1 |\zeta_n(t)|^r\, \mathbb{I}(|\zeta_n(t)| > T)\, dt \\ &\le CT^{r-2q} \int_0^1 |\zeta_n(t)|^{2q}\, dt\end{aligned}$$

from which the (8) follows in view of Proposition 7.

We now indicate the plausibility of Proposition 7; we consider in detail $\int_0^1 S_n^2(t)\,dt$, the case $r=2$, though the general integer $r \geq 1$ will be more complex. First note that

$$\int_0^1 S_n^2(t)\,dt = \frac{1}{n}\sum_{k=1}^{n} {S_n}^2\left(\frac{k}{n}\right)$$

where recall that

$$S_n\left(\frac{k}{n}\right) = \left(n^{H-1/\alpha}R(n)b_n\right)^{-1}\sum_{j=1}^{k} X_j.$$

Then, letting $g_n(j) = \left(n^{H-1/\alpha}R(n)\right)^{-1} g(j)$ (where recall that $g(j)=\sum_{k=0}^{j} c_k$), note that in view of (5),

$$\begin{aligned} S_n\left(\frac{k}{n}\right) &= \sum_{j=-\infty}^{0}\left(g_n(k-j)-g_n(1-j)\right)\frac{\xi_j}{b_n} + \sum_{j=1}^{k} g_n(k-j)\frac{\xi_j}{b_n} \\ &= S_{n1}\left(\frac{k}{n}\right) + S_{n2}\left(\frac{k}{n}\right), \text{ say,} \end{aligned}$$

where we let

$$S_{n1}\left(\frac{k}{n}\right) = \sum_{j=1}^{k} g_n(k-j)\frac{\xi_j}{b_n}.$$

Then

$$\begin{aligned} S_{n1}^2\left(\frac{k}{n}\right) &= 2\sum_{j=1}^{k} g_n(k-j)\left(\sum_{l=1}^{j-1} g_n(k-l)\frac{\xi_l}{b_n}\right)\frac{\xi_j}{b_n} + \sum_{j=1}^{k} g_n^2(k-j)\frac{\xi_j^2}{b_n^2} \\ &= 2I_{nk}^{(1)} + I_{nk}^{(2)}, \text{ say.} \end{aligned}$$

Now note that

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} I_{nk}^{(1)} &= \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} g_n(k-j)\left(\sum_{l=1}^{j-1} g_n(k-l)\frac{\xi_l}{b_n}\right)\frac{\xi_j}{b_n} \\ &= \sum_{j=1}^{n}\left(\sum_{l=1}^{j-1}\left(\frac{1}{n}\sum_{k=j}^{n} g_n(k-j)g_n(k-l)\right)\frac{\xi_l}{b_n}\right)\frac{\xi_j}{b_n}. \end{aligned}$$

Here $\sum_{l=1}^{j-1}\left(\sum_{k=j}^{n} g_n(k-j)g_n(k-l)\right)\xi_l$ depends only on $\left(\xi_1,...,\xi_{j-1}\right)$ and hence is independent of ξ_j. Further recall that $g_n([nt]) \sim t^{H-1/\alpha}$. Then we believe that the preceding quantity can be shown (for all $0 < \alpha \le 2$) to converge in law to

$$\int_0^1 \int_0^u \int_u^1 (t-u)^{H-1/\alpha} (t-v)^{H-1/\alpha} \, dt Z_\alpha(dv) Z_\alpha(du)$$
$$= \int_0^1 \int_0^t \int_0^u (t-u)^{H-1/\alpha} (t-v)^{H-1/\alpha} \, Z_\alpha(dv) Z_\alpha(du) dt.$$

Now consider

$$\frac{1}{n} \sum_{k=1}^n I_{nk}^{(2)} = \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k g_n^2(k-j) \frac{\xi_j^2}{b_n^2}.$$

When ξ_j are in the domain of attraction to a normal law ($\alpha = 2$), this will converge in law to $\int_0^1 \int_0^t (t-u)^{2(H-1/2)} \, dudt$ (which is well defined). Thus $\frac{1}{n} \sum_{k=1}^n S_{n1}^2\left(\frac{k}{n}\right)$ will converge in law to

$$\int_0^1 \left(\int_0^t (t-u)^{H-1/\alpha} \, Z_\alpha(du) \right)^2 dt$$

(when $\alpha = 2$). In the same way the remaining quantities $\frac{1}{n} \sum_{k=1}^n S_{n2}^2\left(\frac{k}{n}\right)$ and $\frac{1}{n} \sum_{k=1}^n S_{n2}\left(\frac{k}{n}\right) S_{n1}\left(\frac{k}{n}\right)$ of $\frac{1}{n} \sum_{k=1}^n S_n^2\left(\frac{k}{n}\right)$ can be dealt with. (What about the case $0 < \alpha < 2$?).

4 Convergence of functionals of integral form II

In this section we deal with the convergence of the functionals of the form (3) of Section 1. Here we essentially reproduce the statements of the results and some of the discussions in Jeganathan (2002), and therefore it may be consulted for further details and the proofs.

As noted in Section 1, the possible limit will involve what is call the local time of the LFSM $\Lambda_{\alpha,H}(t)$. To understand the nature of this limit note that (by the so called 'invariance principle') the possible limit will coincide with that of the particular case in which $S_k = \Lambda_{\alpha,H}(k)$ (that is $X_k = \Lambda_{\alpha,H}(k) - \Lambda_{\alpha,H}(k-1)$). Note that the distribution of $\Lambda_{\alpha,H}(k)$ is the same as that of $n^H \Lambda_{\alpha,H}(k/n)$. Thus, with $f(y) = \mathbb{I}_{[0,1)}(y)$ and $\gamma_n = n^H$, the functional (3) of Section 1 is given by

$$n^{-1} \beta_n \sum_{k=1}^{[nt]} f\left(\beta_n \left(\gamma_n^{-1} S_k - x\right)\right) =^{\mathcal{D}} n^{-1} \beta_n \sum_{k=1}^{[nt]} \mathbb{I}_{[0,1)}\left(\beta_n \left(\Lambda_{\alpha,H}(k/n) - x\right)\right)$$
$$= n^{-1} \beta_n \sum_{k=1}^{[nt]} \mathbb{I}_{[x, x+\beta_n^{-1})}\left(\Lambda_{\alpha,H}(k/n)\right).$$

where recall that $\beta_n \to \infty$ such that $n^{-1}\beta_n \to 0$. This may be viewed (see Theorem 12 below) as an approximate Riemann sum of

$$\beta_n \int_0^t \mathbb{I}_{[x, x+\beta_n^{-1})}\left(\Lambda_{\alpha,H}(s)\right) ds.$$

That the limit, called the local time of $\Lambda_{\alpha,H}(t)$, of this is well behaved is given by the following result.

Theorem 9 *: For a LFSM with $0 < \alpha \leq 2$ and $0 < H < 1$, there is a local time $L(t,x)$ such that for each t and x,*

$$\lim_{\eta \downarrow 0} \frac{1}{\eta} \int_0^t \mathbb{I}_{[x,x+\eta)}(\Lambda_{\alpha,H}(s))ds = L(t,x) \qquad in \;\; \mathbf{L}^2. \tag{9}$$

In addition $L(t,x)$ has the representation

$$L(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \int_0^t e^{iu\Lambda_{\alpha,H}(s)} dsdu \quad a.s. \tag{10}$$

(Note that $H = 1/\alpha$ in the case of α-stable Levy motion $Z_\alpha(t)$, and hence the restriction $0 < H < 1$ entails the restriction $1 < \alpha \leq 2$; in fact it known that the local time of α-stable Levy motion does not exist when $0 < \alpha \leq 1$.)

The preceding result will actually be a particular case of a result to be stated later, see the Remark 3 below. It may be noted that one can choose $L(t,x)$ such that almost all trajectories $(t,x) \longmapsto L(t,x)$ will be continuous.

The definition of the local time in the preceding result is perhaps a natural one in the context of LFSM. However it has also the interpretation of 'occupation density' of a suitable 'occupation measure'. To see this, note that another way of writing (10) is

$$\int_{-\infty}^{\infty} e^{iux} L(t,x)dx = \int_0^t e^{iu\Lambda_{\alpha,H}(s)} ds \quad \text{a.s.}$$

This will mean

$$\int_{-\infty}^{\infty} \mathbb{I}_A(x) L(t,x)dx = \int_0^t \mathbb{I}_A(\zeta(s))ds \quad \text{a.s.}$$

whenever $A = (a,b]$, and hence for all Borel subset A. If we denote the r.h.s by $\mu_t(A)$, that is

$$\mu_t(A) = \int_0^t \mathbb{I}_A(\zeta(s))ds,$$

clearly, for each $t > 0$ and for each trajectory of $\{\Lambda_{\alpha,H}(s), s \geq 0\}$, μ_t is a measure on R, and the identity

$$\mu_t(A) = \int_{-\infty}^{\infty} \mathbb{I}_A(x) L(t,x)dx \quad \text{a.s.}$$

means, for each $t > 0$, the measure μ_t is absolutely continuous with respect to the Lebesgue measure with Lebesgue density $L(t,x)$.

Note that the preceding identities hold only almost surely and the exceptional event may depend on t, but this can be remedied when $L(t,x)$ is chosen to be a jointly continuous version. In such a case (9) will hold, with probability one (in addition to the $\mathbf{L}^2$ sense), for all t and x.

The measure $\mu_t(A)$ is called the *occupation measure or occupation time* of $\Lambda_{\alpha,H}(t)$ in a Borel set A of R up to the time $t \geq 0$. Then $L(t,x)$ 'characterizes' the amount of time $\Lambda_{\alpha,H}(t)$ spends at x up to time t, and for this reason $L(t,x)$ is called the *local time or occupation density* of $\Lambda_{\alpha,H}(t)$.

Next we state and discuss the results on the convergence of the functionals of the form (3) of Section 1. $L(t,x)$ will stand for the local time of LFSM, with the understanding that when $H = 1/\alpha$ it stands for the local time of the α-stable Levy motion $Z_\alpha(t)$ *in which case (that is, in the case of Proposition 1), the restriction* $1 < \alpha \leq 2$ *is always imposed with out any further mentioning* (because the local time of α-stable Levy motion does not exist when $0 < \alpha \leq 1$) . Also recall that γ_n stands for the normalizing constant involved in Propositions 1 or 2, that is,

$$\gamma_n = \begin{cases} n^{H-1/\alpha} R(n) b_n & \text{if the conditions of Proposition 1 are satisfied} \\ (\sum_{j=0}^{\infty} c_j) b_n & \text{if the conditions of Proposition 2 are satisfied.} \end{cases}$$

To state the next result, for any function $h\,(y)$ we define

$$M_{h,\eta}(y) = \sup\{h(u) : |u-y| \leq \eta\} \quad \text{and} \quad m_{h,\eta}(y) = \inf\{h(u) : |u-y| \leq \eta\}.$$

Theorem 10 *: Assume that the conditions of either one of the Propositions 1 or 2 are satisfied. In addition assume that the distribution of* ξ_1 *satisfies the Cramér's condition* $\limsup_{|u|\to\infty} \left|E\left[e^{iu\xi_1}\right]\right| < 1$. *Let* $f(y)$ *be such that* $M_{|f|,\eta}(y)$ *and* $M_{f^2,\eta}(y)$ *are Lebesgue integrable for some* $\eta > 0$ *and*

$$\int_{-\infty}^{\infty} \left(M_{|f|,\delta}(y) - m_{|f|,\delta}(y)\right) dy \to 0 \quad \text{as} \quad \delta \to 0. \tag{11}$$

Then the finite dimensional distributions of the process (indexed by (t,x)*)*

$$n^{-1}\beta_n \sum_{k=1}^{[nt]} f\left(\beta_n\left(\gamma_n^{-1} S_k + x\right)\right)$$

converge in distribution to those of

$$\left(\int_{-\infty}^{\infty} f(y)dy\right) L(t,-x).$$

We remark that there are other alternative requirements on the function $f(y)$ that will imply those stated in the preceding statement. For example one possibility is to assume that the set of discontinuity points of $f\,(y)$ is of Lebesgue measure zero together with the integrability of $M_{|f|,\eta}(y)$ and $M_{f^2,\eta}(y)$. (It is clear that the condition (11) is then implied by the Lebesgue dominated convergence theorem).

Also, as will be indicated later, it is possible to relax the integrability of $M_{f^2,\eta}(y)$ to that of local integrability. Thus the second alternative that will also imply the stated requirement in Theorem 10 is to assume the local Riemann integrability of $f(y)$ together with the integrability of $M_{|f|,\eta}(y)$.

We further note that $L(t,-x)$ will have the same distribution as that of $L(t,x)$ only when the process $Z_\alpha(t)$ involved in the definition of LFSM is symmetric around zero (which is always true in the case of FBM, the case $\alpha = 2$.).

Theorem 10 in particular holds for the important situation in which $f(y) = I_{(c,d)}(y)$. (Here the limit will remain the same even if the open interval (c,d) is replaced by the closed interval $[c,d]$ or by a semi open interval.)

For the earlier results in the direction of the results of this section, see Akonom (1993) and Park and Phillips (1999,2001).

Now coming back to Theorem 10 above, according to an example in Borodin and Ibragimov (1995, Chapter IV, page 143), the requirement (11) on $f(y)$ stated in Theorem 10 cannot be avoided entirely. The next result relaxes that requirement but assumes conditions stronger than the Cramér's condition.

Theorem 11 *: Assume that the conditions of either one of the Propositions 1 or 2 are satisfied.*

(i): Suppose that for some integer n_0 the n_0-fold convolution of the distribution of ξ_1 has a non-zero absolutely continuous (with respect to Lebesgue measure on R) component. Let $f(y)$ be Lebesgue integrable such that $\sup_{y\in R}|f(y)| < \infty$. Then the conclusion of Theorem 10 holds.

(ii): Suppose that the distribution of ξ_1 is such that $\int \left|E\left[e^{iu\xi_1}\right]\right|^{n_0} du < \infty$ for some integer $n_0 > 0$ and $n^{-1}\beta_n \sum_{k=1}^{n_0-1} f(\beta_n(\gamma_n^{-1}S_k + x))$ converges in probability to 0. Assume further that both $f(y)$ and $f^2(y)$ are Lebesgue integrable. Then the conclusion of Theorem 10 holds.

Note that the requirements on $f(y)$ in the first statement is stronger than those in the second statement; consider for instance the example $f(y) \sim |y|^\tau$ as $|y| \to 0$ with $0 > \tau > -1/2$. Also, as in Theorem 10, the integrability of $f^2(y)$ in the second statement can be relaxed to that of local integrability.

Regarding the requirement on $n^{-1}\beta_n \sum_{k=1}^{n_0-1} f(\beta_n(\gamma_n^{-1}S_k + x))$, first note that it is redundant when $n_0 = 1$. In the case $x = 0$ and $\beta_n = \gamma_n$, an important case in applications, the requirement is satisfied because the quantity reduces to $n^{-1}\beta_n \sum_{k=1}^{n_0-1} f(S_k)$ which clearly converges in probability to 0 in view of $n^{-1}\beta_n \to 0$.

Suppose that $x \neq 0$. Then the additional condition $\lim_{a\to\infty} \sup_{|y|\geq a} |f(y)| < \infty$ is sufficient because with probability tending to one $\beta_n(\gamma_n^{-1}S_k + x)$ will be supported in a neighborhood of $\pm\infty$. The same is the case when $x = 0$ and $\beta_n\gamma_n^{-1} \to \infty$.

In the remaining case $x = 0$ and $\beta_n\gamma_n^{-1} \to 0$, the condition $\lim_{\eta\to 0} \sup_{|y|\leq\eta} |f(y)| < \infty$ will be sufficient but is too strong in some important special cases. For instance suppose that $f(y) \sim |y|^\tau$ as $|y| \to 0$ with $\tau > -1/2$. Then this condition is not satisfied when $0 > \tau > -1/2$, but the requirement in question itself is satisfied because with probability tending to one $n^{-1}\beta_n |f(\beta_n\gamma_n^{-1}S_k)| \leq Cn^{-1}\beta_n |\beta_n\gamma_n^{-1}|^\tau \to 0$.

We obtain the next result as a by-product to the statement (ii) of Theorem 11 (whose requirements are satisfied with $n_0 = 1$). The result may be viewed as a discrete approximation to the local time of the LFSM. Note that the approximation involved is in $\mathbf{L}^2$, in contrast to the distributional approximation of the preceding two results. Note also that in this case $\gamma_n = n^H$.

Theorem 12 : *Suppose that $f(y)$ and $f^2(y)$ are Lebesgue integrable. Then for the LFSM process $\Lambda_{\alpha,H}(t)$, one has*

$$\frac{\beta_n}{n}\sum_{k=1}^{[nt]} f\left(\beta_n\left(\Lambda_{\alpha,H}\left(k/n\right)-x\right)\right) \rightarrow \left(\int_{-\infty}^{\infty} f(y)dy\right) L(t,x) \quad \textit{in} \quad \mathbf{L}^2.$$

The next result is a continuous analogue of the preceding result.

Theorem 13 : *Suppose that $f(y)$ and $f^2(y)$ are Lebesgue integrable. Then for the LFSM process $\Lambda_{\alpha,H}(t)$, one has, for each t and x,*

$$\frac{\beta_\kappa}{\kappa}\int_0^{\kappa t} f\left(\beta_\kappa\left(\Lambda_{\alpha,H}(s/\kappa)-x\right)\right) ds \rightarrow \left(\int_{-\infty}^{\infty} f(y)dy\right) L(t,x) \quad \textit{in} \quad \mathbf{L}^2$$

as $\kappa \rightarrow \infty$, where $\beta_\kappa \rightarrow \infty$ such that $\frac{\beta_\kappa}{\kappa} \rightarrow 0$ as $\kappa \rightarrow \infty$.

As noted in connection with the statement of Theorem 10, the Lebesgue integrability of $f^2(y)$ in Theorems 12 and 13 can be relaxed to that of local integrability.

Remark 1. Note that, because the distribution of $n^H \Lambda_{\alpha,H}(k/n)$ is the same as that of $\Lambda_{\alpha,H}(k)$, it follows from theorem 11 that $\frac{1}{n^{1-H}}\sum_{k=1}^{[nt]} f\left(\Lambda_{\alpha,H}\left(k\right)-xn^H\right)$ converges in distribution to $\left(\int_{-\infty}^{\infty} f(y)dy\right) L(t,x)$ as $n \rightarrow \infty$. Similarly Theorem 13 entails that $\frac{1}{\kappa^{1-H}}\int_0^{\kappa t} f\left(\Lambda_{\alpha,H}(s)-x\kappa^H\right) ds$ converges in distribution to $\left(\int_{-\infty}^{\infty} f(y)dy\right) L(t,x)$ as $\kappa \rightarrow \infty$.

Remark 2. Also note that if the existence and the continuity of $x \mapsto L(t,x)$ is known, then the statement of Theorem 13 *but with convergence in $\mathbf{L}^2$ replaced by convergence almost surely* can be shown to hold directly, using the fact

$$\begin{aligned}\frac{\beta_\kappa}{\kappa}\int_0^{\kappa t} f\left(\beta_\kappa\left(\Lambda_{\alpha,H}(s/\kappa)-x\right)\right) ds &= \beta_\kappa\int_0^{t} f\left(\beta_\kappa\left(\Lambda_{\alpha,H}(s)-x\right)\right) ds \\ &= \beta_\kappa\int_{-\infty}^{\infty} f\left(\beta_\kappa\left(u-x\right)\right) L(t,u)du \\ &= \int_{-\infty}^{\infty} f(y)L(t,y\beta_\kappa^{-1}+x)dy.\end{aligned}$$

Remark 3. We remark that Theorem 13 contains Theorem 9 (stated at the beginning of this section). To see this, taking $\beta_\kappa = \kappa^H$ in the preceding remark,

$$\frac{1}{\kappa^{1-H}} \int_0^{\kappa t} f\left(\kappa^H \left(\Lambda_{\alpha,H}(s/\kappa) - x\right)\right) ds = \kappa^H \int_0^t f\left(\kappa^H \left(\Lambda_{\alpha,H}(s) - x\right)\right) ds.$$

Then when $f(y) = \mathbb{I}_{[0,1)}(y)$ and $\kappa^H = 1/\eta$ right hand side here reduces to

$$\frac{1}{\eta} \int_0^t \mathbb{I}_{[x,x+\eta)}\left(\Lambda_{\alpha,H}(s)\right) ds$$

which is exactly the one occurring in the statement of Theorem 9.

So far our results are for fixed $f(y)$. Under appropriate conditions it is possible to replace it by varying $f_n(y)$. To motivate the conditions, consider the specific $f_n(y)$ defined by $f_n(y) = \beta_n f(\beta_n y)$ based on a fixed $f(y)$. Then note that

$$F_n(y) = \int_{-\infty}^{y} f_n(u)\, du = \int_{-\infty}^{\beta_n y} f(u)\, du \to \begin{cases} \int_{-\infty}^{\infty} f(u)\, du & \text{if} \quad y > 0 \\ 0 & \text{if} \quad y < 0. \end{cases}$$

In the general case we define

$$F_n(y) = \begin{cases} \int_0^y f_n(u)\, du & \text{if} \quad y \geq 0 \\ -\int_y^0 f_n(u)\, du & \text{if} \quad y < 0. \end{cases}$$

and impose the following Condition (*).

Condition (*). The function $f_n(y)$ is said to satisfy the condition (*) if the following requirements are satisfied.

1. There is an $F(y)$ such that $F_n(y) \to F(y)$ at all continuity points of $F(y)$,
2. $\lim_{\kappa\to\infty} \sup_n \int_{\{|y|\geq\kappa\}} |f_n(y)|\, dy = 0$,
3. $\sup_n \int |f_n(y)|\, dy < \infty$, and
4. $\sup_n \frac{1}{n} \int_{-\infty}^{\infty} |f_n^2(y)|\, dy \to 0$.

Some of the requirements of this condition, especially 2, are restrictive in applications. We shall discuss it after the statement of the next result.

Theorem 14 *: (i): Assume that the distribution of* ξ_1 *satisfies the assumptions of the statement (i) of Theorem 11. Assume further that* $f_n(y)$ *satisfies the Condition (*) for a suitable limit* $F(y)$*, together with the requirement that* $\sup_{\{n,y\}} \sigma_n^{-1} |f_n(y)| < \infty$ *for some* $\sigma_n \to \infty$ *with* $\frac{\sigma_n}{n} \to 0$.

Then the finite dimensional distributions of

$$n^{-1} \sum_{k=1}^{[nt]} f_n(\gamma_n^{-1} S_k + x)$$

converge in distribution to those of

$$\int_{-\infty}^{\infty} L(t, y-x)\, dF(y).$$

(ii): Assume that the distribution of ξ_1 *satisfies the assumptions of the statement (ii) of Theorem 11 for some integer* $n_0 \geq 1$*. Assume further that* $f_n(y)$ *satisfies the Condition (*) for a suitable* $F(y)$*, together with the requirement that* $n^{-1}\sum_{k=1}^{n_0-1} f_n(\gamma_n^{-1}S_k + x)$ *converges in probability to* 0*. Then the conclusion of the preceding statement holds.*

Regarding the Condition (*), while its requirements are quite reasonable in many applications, it is also easy to find interesting situations where requirement 2 is not satisfied. It simply amounts to the statement that the functions $f_n(y)$ can be assumed to have a common compact support (in which case the requirements 3 and 4 become natural). A familiar way this is achieved is by imposing the requirement that

$$\lim_{\kappa\to\infty}\text{limsup}_n P\left(\sup_{1\leq k\leq n}\left|\gamma_n^{-1}S_k\right| \geq \kappa\right) = 0.$$

This is available whenever the convergence in $D[0,1]$ is available, but not in general in the present context unfortunately as discussed in connection with Theorem 6 of Section 3. However, as noted there, the same purpose is achieved if the condition $\text{limsup}_{|x|\to\infty}\sup_n \frac{|f_n(x)|}{|x|^r} < \infty$ for some $r > 0$ holds, at least in the FBM situation..

A generalization for Theorem 11 similar to Theorem 14 can be formulated for varying $f_n(y)$, see Jeganathan (2002, Theorem 8).

5 Convergence of martingales (4)

In this section we consider the convergence of martingales defined in (4) of Section 1. Recall that $(\xi_k, \eta_k), k \geq 1$, are i.i.d. such that $((\xi_k, \eta_k), k \geq 1)$ is independent of $\left(\xi_j, j \leq 0\right)$ and such that $E[\eta_1] = 0$ and $E[\eta_1^2] = 1$. Then note that the process $\left(\gamma_n^{-1}S_{[nt]}, n^{-1/2}\sum_{k=1}^{[nt]}\eta_k\right)$ will converge in law to the process $(\Lambda_{\alpha,H}(t), B(t))$ where $B(t)$ is a Brownian motion. In the case $0 < \alpha < 2$, the process $\Lambda_{\alpha,H}(t)$ will be independent of the process $B(t)$; in the case $\alpha = 2$, they will be jointly Gaussian. The next result deals with the martingale where the functional (2) arises as the quadratic variation process.

Theorem 15 *Let* $S_n(t)$ *and* $\Lambda_{\alpha,H}(t)$ *be as defined in Section 2. Assume that* $f(x)$ *is locally Riemann integrable. In the case* $H - 1/\alpha < 0$*, assume further that* $\alpha = 2$*, that is,* ξ_1 *is in the domain of attraction of a normal law and that* $\text{limsup}_{|x|\to\infty}\frac{f^2(x)}{|x|^r} < \infty$ *for some* $r > 0$*. Then*

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} f\left(\gamma_n^{-1}S_{k-1}\right)\eta_k,\ n^{-1}\sum_{k=1}^{n} f^2\left(\gamma_n^{-1}S_{k-1}\right)\right)$$

converges in law to

$$\left(\int_0^1 f\left(\Lambda_{\alpha,H}(t)\right) dB(t), \int_0^1 f^2\left(\Lambda_{\alpha,H}(t)\right) dt\right).$$

In the case $0 < \alpha < 2$*, the distribution of this limit will be the same as that of*

$$\left(Z\sqrt{\int_0^1 f^2\left(\Lambda_{\alpha,H}(t)\right) dt}, \int_0^1 f^2\left(\Lambda_{\alpha,H}(t)\right) dt\right)$$

where Z *is standard normal independent of the process* $\Lambda_{\alpha,H}(t)$*.*

To see that this result is true, as in Theorem 8, one can restrict to f that is compactly supported and continuous. Then the result can be obtained using the arguments of Theorem 4, as was done in the proof of Theorem 8.

The next result deals with the martingale where the functional (3) now arises as the quadratic variation process. Here recall the restriction that in the situation of Proposition 1, α is restricted to $1 < \alpha \leq 2$ in view of the fact that when $0 < \alpha \leq 1$ the local time of the α-stable motion $Z_\alpha(t)$ does not exist.

Theorem 16 *Under suitable conditions obtainable from either Theorem 10 or Theorem 11,*

$$\left(\sqrt{n^{-1}\beta_n}\sum_{k=1}^{n} f\left(\beta_n\gamma_n^{-1}S_{k-1}\right)\eta_k,\ n^{-1}\beta_n\sum_{k=1}^{n} f^2\left(\beta_n\gamma_n^{-1}S_{k-1}\right)\right)$$

converges in law to $\left(Z\sqrt{L(1,0)}, L(1,0)\right)$ *where* Z *is standard normal independent of the process* $\Lambda_{\alpha,H}(t)$*. (Recall that* $L(t,x)$ *is the local time, as defined earlier, of the process* $\Lambda_{\alpha,H}(t)$*.)*

The idea of this result consists of following steps:

I. For each $t > 0$, define

$$\tau_n(t) = \inf\left\{m : n^{-1}\beta_n\sum_{k=1}^{m} f^2\left(\beta_n\gamma_n^{-1}S_{k-1}\right) \geq t\right\}.$$

Then observe that $\tau_n(t)$ is a stopping time with respect to the σ-fields $\mathcal{G}_j = \sigma(S_1, ...S_{j-1})$, that is, for each n and t, the event $\{\tau_n(t) \leq j\} \in \mathcal{G}_j$ for every nonnegative integer j. This will in particular imply that for each fixed t,

$$\sum_{k=1}^{m} f\left(\beta_n\gamma_n^{-1}S_{k-1}\right)\eta_k,\ m = 1, ..., \tau_n(t)$$

is a martingale, and for each n,

$$\sum_{k=1}^{\tau_n(t)} f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) \eta_k$$

is a martingale with respect to the index t.

In addition $\tau_n(t)$ will have the properties that for each $t > 0$, $n^{-1}\tau_n(t)$ is bounded in probability and

$$n^{-1}\beta_n \sum_{k=1}^{\tau_n(t)} f^2\left(\beta_n \gamma_n^{-1} S_{k-1}\right) \to_P \ t.$$

Actually, because the left hand side here increases in t for each fixed n, one has, for every $M > 0$,

$$\sup_{0\le t\le M} \left| n^{-1}\beta_n \sum_{k=1}^{\tau_n(t)} f^2\left(\beta_n \gamma_n^{-1} S_{k-1}\right) - t \right| \to_P \ 0.$$

II. Consider the process

$$\left(\sqrt{n^{-1}\beta_n} \sum_{k=1}^{\tau_n(t)} f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) \eta_k, \ \gamma_n^{-1} S_{[ns]}, \ n^{-1}\beta_n \sum_{k=1}^{n} f^2\left(\beta_n \gamma_n^{-1} S_{k-1}\right) \right).$$

The finite dimensional distribution of this process will converge to that of $(B(t), \Lambda_{\alpha,H}(s), L(1,0))$, where $B(t)$ will be standard Brownian motion, *independent* of $\Lambda_{\alpha,H}(s)$ (and hence independent of $L(1,0)$ also). As noted earlier, the claimed independence is a standard result when $0 < \alpha < 2$. In the case $\alpha = 2$, it is a consequence of the fact that

$$\begin{aligned}
& \operatorname{Cov}\left(\sqrt{n^{-1}\beta_n} \sum_{k=1}^{\tau_n(t)} f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) \eta_k, \ \gamma_n^{-1} S_{[ns]} \right) \\
= & \sqrt{n^{-1}\beta_n} \sum_{k=1}^{\min(\tau_n(t),[ns])} f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) b_n^{-1} g_n([ns]-k) \\
= & b_n^{-1}\sqrt{n\beta_n^{-1}} \left(\frac{1}{n} \sum_{k=1}^{\min(\tau_n(t),[ns])} \beta_n f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) b_n^{-1} g_n([ns]-k) \right) \\
\to & 0.
\end{aligned}$$

(where recall that $g_n(j) = \left(n^{H-1/\alpha} R(n)\right)^{-1} g(j)$ with $g(j) = \sum_{k=0}^{j} c_k$). This is true because it can be shown that

$$\left(\frac{1}{n} \sum_{k=1}^{\min(\tau_n(t),[ns])} \beta_n f\left(\beta_n \gamma_n^{-1} S_{k-1}\right) b_n^{-1} g_n([ns]-k) \right) = O_p(1)$$

and because $b_n^{-1}\sqrt{n\beta_n^{-1}} \to 0$.

III. Let

$$T_n = n^{-1}\beta_n \sum_{k=1}^{n} f^2\left(\gamma_n^{-1} S_{k-1}\right).$$

With m a positive integer and $L > 0$, let $0 = \tau_{m0} < \tau_{m1} < ...\tau_{m,m-1} < \tau_{mm} = L$ be such that $\sup_i |\tau_{mi} - \tau_{m,i-1}| \to 0$. Define

$$T_{n,m,L} = \begin{cases} \tau_{mi} & \text{if } \tau_{mi} \leq T_n < \tau_{m,i+1},\ i = 0, 1, ...m \\ L & \text{if } T_n \geq L. \end{cases}$$

Letting $T = L(1,0)$, define $T_{m,L}$ analogously. One can assume with out loss of generality that $\tau_{m0}, \tau_{m1}, ...\tau_{mm}$ are continuity points of T. Then step II entails that

$$\left(\sqrt{n^{-1}\beta_n} \sum_{k=1}^{\tau_n\left(T_{n,m,L}\right)} f\left(\gamma_n^{-1} S_{k-1}\right)\eta_k,\ T_n\right)$$

converges in distribution to $(B(T_{m,L}),\ L(1,0))$. (Note that $T_{m,L}$ is a function of $L(1,0)$, and $L(1,0)$ is independent of $B(t)$.)

IV. It can be shown that $\sqrt{n^{-1}\beta_n}\sum_{k=1}^{\tau_n(t)} f\left(\gamma_n^{-1} S_{k-1}\right)\eta_k$ is tight in $D[0,1]$, (with its limit $B(t)$ belonging to $C[0,1]$ almost surely). From this it follows easily, $T_{n,m,L}$ and $T_{n,m,L}$ being respective approximations, in the above sense, to T_n and $T = L(1,0)$ respectively, that

$$\left(\sqrt{n^{-1}\beta_n} \sum_{k=1}^{\tau_n(T_n)} f\left(\gamma_n^{-1} S_{k-1}\right)\eta_k,\ T_n\right)$$

converges in distribution to $(B(L(1,0)),\ L(1,0))$. Noting that $\tau_n(T_n) = n$, and in view of the independence of the process $B(t)$ and $L(1,0)$, the result follows.

6 References

1. Hosking, J.R.M. (1981): fractional differencing. *Biometrica* **68**, 165-176.

2. Jeganathan P. (2002): Convergence of functionals of weighted sums of independent r.v.s to local times of fractional Brownian and stable motions. Preprint.

3. Wang et al (2002). *Econometric Theory.*

4. For the remaining references, see the item 2 above.