Second Order Limits of Functionals of Sums of Linear Processes that Converge to Fractional Stable Motions

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Abstract. Consider a sequence $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, where c_j, $j \geq 0$, is a sequence of constants and ξ_j, $-\infty < j < \infty$, is a sequence of i.i.d. random variables belonging to the domain of normal attraction of a strictly stable law with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^{k} X_j$. Under certain conditions on the constants c_j, it is known that for a suitable slowly varying function $u(n)$ and for a suitable constant $0 < H < 1$, the partial sum process $\left(n^H u(n)\right)^{-1} S_{[nt]}$ converges in distribution to a Linear Fractional Stable Motion (indexed by α and H). In addition, it has been established elsewhere that if $f(y)$ is Lebesgue integrable then under certain further conditions on the distribution of ξ_1, the sequence $n^{-(1-H)} u(n) \sum_{k=1}^{n} f(S_k)$ converges in distribution to $\left(\int_{-\infty}^{\infty} f(y)\, dy\right) L(1,0)$, where $L(t,x)$ is the local time of the linear fractional stable motion at x upto time t. In this paper it is shown that if in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then under the further restriction $H > 1/3$ (which probably cannot be relaxed), $\sqrt{n^{-(1-H)} u(n)} \sum_{k=1}^{n} f(S_k)$ converges in distribution to $W \sqrt{bL(1,0)}$, where W is standard normal, independent of $L(1,0)$, and b is a positive constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$. The result has motivation in large sample theory for certain nonlinear time series models.

1 Introduction and motivation

Consider a sequence $\xi_j, -\infty < j < \infty$, of i.i.d. random variables belonging to the domain of *normal* attraction of a strictly stable law with index $0 < \alpha \leq 2$. This means the process $n^{-1/\alpha} \sum_{j=1}^{[nt]} \xi_j \Longrightarrow Z_\alpha(t), t > 0$, where $\{Z_\alpha(t), t \in R\}$ has stationary independent increments such that, for each $-\infty < s < t < \infty$, $E[e^{iu(Z_\alpha(t)-Z_\alpha(s))}] = e^{-(t-s)|u|^\alpha \left(1+i\beta\, \mathrm{sign}(u)\, \tan\left(\frac{\pi\alpha}{2}\right)\right)}$ with $\beta = 0$ when $\alpha = 1$ and $|\beta| \leq 1$ otherwise. $\{Z_\alpha(t), t \in R\}$ is called an *α-stable Levy motion*. When $\alpha = 2$, $Z_2(t)$ becomes the Brownian Motion with variance 2. (Above and in the rest of the paper, the **notation** $\Longrightarrow$ signifies the convergence in distribution, in the sense of convergence in distribution of all *finite dimensional distributions*.)

Now suppose that $1 < \alpha \leq 2$, so that the *local time* $L(t, x)$ of $Z_\alpha(t)$ at x upto the time t exists in a suitable sense. (When $0 < \alpha \leq 1$, the local time of $Z_\alpha(t)$ does not exist.) Define $S_k = \sum_{j=1}^k \xi_j$. Suppose that $\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty$ for some $p > 0$. Then, for a function $f(y)$ such that both $f(y)$ and $f^2(y)$ are Lebesgue integrable, the following two results are particular cases of the results obtained in Borodin and Ibragimov (1995, Section 3 of Chapter III and Theorem 3.3 of Chapter IV). (a): The sequence $n^{-1+\frac{1}{\alpha}} \sum_{k=1}^n f(S_k) \Longrightarrow \left(\int f(y)dy\right) L(1,0)$. (b): On the other hand, if in addition $\int f(y)dy = 0$ and $\int |yf(y)|\, dy < \infty$, then $\sqrt{n^{-1+\frac{1}{\alpha}}} \sum_{k=1}^n f(S_k) \Longrightarrow W\sqrt{bL(1,0)}$ where W is standard normal independent of $L(1,0)$ and the constant $b = \frac{1}{2\pi} \int \left|\widehat{f}(\lambda)\right|^2 \frac{1+E\left[e^{i\lambda\xi_1}\right]}{1-E\left[e^{i\lambda\xi_1}\right]} d\lambda$ with $\widehat{f}(\lambda) = \int e^{i\lambda y} f(y)dy$. According to Borodin and Ibragimov (1995), the statement (b) was discovered by Dobrushin (1955) in the case $\xi_1 = \pm 1$ with probability 1/2, and was extended to the case $E\left[|\xi_1|^5\right] < \infty$ by Skorokhod and Slobodenyuk (1970, Chapter VI).

Now consider the linear process $X_k = \sum_{j=0}^\infty c_j \xi_{k-j}$, $k \geq 1$, where c_j, $j \geq 0$, are constants and ξ_j, $-\infty < j < \infty$, is as earlier with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^k X_j$. Under suitable conditions on the constants c_j it is known that for a suitable H, $0 < H < 1$, and for a slowly varying $u(n)$ the process $\left(n^H u(n)\right)^{-1} S_{[nt]}$ converges in distribution to a *Linear Fractional Stable Motion* (LFSM) $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ defined by

$$\Lambda_{\alpha,H}(t) = c\int_{-\infty}^0 \left\{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\right\} Z_\alpha(du) + c\int_0^t (t-u)^{H-1/\alpha} Z_\alpha(du)$$

where $Z_\alpha(t)$ is an α-stable Levy motion as before and c is a non-zero constant. When $\alpha = 2$, the LFSM reduces to the *Fractional Brownian Motion*. See Samorodnitsky and Taqqu (1994) and Maejima (1989) for the details of LFSM. *We make the convention that in the case $H = 1/\alpha$, in which case the restriction $0 < H < 1$ reduces to $1 < \alpha \leq 2$, the LFSM $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is taken to be $\{Z_\alpha(t), t \geq 0\}$.*

It follows from Jeganathan (2003, Statement (ii) of Theorem 3) that the earlier statement (a) generalizes to the case $S_k = \sum_{j=1}^{k} X_j$ (when both $f(y)$ and $f^2(y)$ are Lebesgue integrable and $\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty$ for some $p > 0$). Specifically, $n^{-(1-H)}u(n)\sum_{k=1}^{n} f(S_k) \Longrightarrow \left(\int_{-\infty}^{\infty} f(y)\,dy\right) L(1,0)$, where $L(1,0)$ is now the local time of the LFSM $\Lambda_{\alpha,H}(t)$ at 0 upto the time $t = 1$. (It may be noted that the earlier statements (a) and (b), as well as the results in Jeganathan (2003) assumed only that ξ_j belong to the domain of attraction, instead of the present restrictive domain of normal attraction, see the Remark in Section 4 below in this connection.) In this paper we obtain a similar generalization of the earlier statement (b) (Theorem 1 below), under the restriction $H > 1/3$ (which probably cannot be relaxed because it cannot be relaxed in the continuous time situation, see below). It may be noted that no particular cases of such a generalization seem to be available in the literature, even under the restricted situation in which ξ_j are i.i.d. Gaussian and c_j are such that (the Gaussian process) $\frac{1}{\sqrt{n}}S_{[nt]}$ converges in distribution to a Brownian motion.

We now indicate the motivation of this paper. To illustrate the phenomenon, it is enough to consider the specific case $S_k = \sum_{j=1}^{k} \xi_j$ with ξ_j as before. (Full treatment of the general situation of the present paper will be available in Jeganathan and Phillips (2003), from which the present illustration is borrowed.) Consider the observations (Y_k, S_k) modeled by $Y_k = \theta g(S_k) + \eta_k$ where (η_k, ξ_k) are i.i.d. with η_1 and ξ_1 assumed for simplicity to have 0 mean values and finite variances. Then the least squares estimator $\widehat{\theta}_n$ of θ based on (Y_k, S_k), $k = 1, ..., n$, has the representation $\widehat{\theta}_n - \theta = \left(\sum g^2(S_k)\right)^{-1} \sum g(S_k)\eta_k$. According to the statement (a) stated earlier, the denominator $\frac{1}{\sqrt{n}}\sum g^2(S_k) \Longrightarrow \left(\int g^2(y)dy\right) L(1,0)$. To deal with the numerator $\sum g(S_k)\eta_k = \sum E_{k-1}[g(S_k)\eta_k] + \sum\left(g(S_k)\eta_k - E_{k-1}[g(S_k)\eta_k]\right)$, where $E_{k-1}[g(S_k)\eta_k]$ is the conditional expectation of $g(S_k)\eta_k$ given $(\eta_j, \xi_j, j = 1, ..., k-1)$, note that $g(S_k)\eta_k - E_{k-1}[g(S_k)\eta_k]$ form martingale differences and the sum of their conditional variances is given by $\sum g_1(S_{k-1})$ where $g_1(y) = E\left[g^2(y+\xi_1)\eta_1^2\right] - E^2\left[g(y+\xi_1)\eta_1\right]$. According to the earlier statement (a), $\frac{1}{\sqrt{n}}\sum g_1(S_{k-1}) \Longrightarrow \left(\int g_1(y)dy\right) L(1,0)$. It can be shown further that $\frac{1}{n^{1/4}}\sum\left(g(S_k)\eta_k - E_{k-1}[g(S_k)\eta_k]\right) \Longrightarrow W_1\sqrt{\left(\int g_1 dy\right) L(1,0)}$ where W_1 is standard normal independent of $L(1,0)$ (see Proposition 2 below).

On the other hand, the earlier statement (b) will be required in order to obtain the asymptotic behavior of $\frac{1}{n^{1/4}}\sum E_{k-1}[g(S_k)\eta_k]$. To see this note that $E_{k-1}[g(S_k)\eta_k] = g_2(S_{k-1})$ with $g_2(y) = E[g(y+\xi_1)\eta_1]$, where $\int g_2(y)\,dy = E\left[\eta_1 \int g(y+\xi_1)\,dy\right] = E[\eta_1]\int g(y)\,dy = 0$. From the above facts, it will follow that $\frac{1}{n^{1/4}}\sum g(S_k)\eta_k \Longrightarrow W_1\sqrt{\left(\int g_1(y)dy\right) L(1,0)} + W_2\sqrt{bL(1,0)}$ where the constant b is as defined earlier in the statement (b) with $f(y) = g_2(y)$, and the vector (W_1, W_2) is standard bivariate normal independent of $L(1,0)$.

The plan of the paper is as follows. The required assumptions as well as the statement of the main result will be stated in Section 2, where it is also noted that the main result is a consequence of a martingale CLT, the proof of which is also presented in Section 2. The proof of the main result will then consists of the verification of the conditions of this martingale CLT, which verification will be done in Sections 3 - 5.

Throughout the paper the notation C stands for a generic constant that may take different values at different places of even the same proof or the same expression.

We hope to present elsewhere the continuous time analogues of the present main result, in the forms of generalizations of the appropriate results in for instance Yor (1983). In this situation, it may be noted that the restriction $1/3 < H < 1$ cannot be relaxed, as can be seen from the known regularity properties of $L(1, x)$ with respect to x when $L(1, x)$ is the local time of the fractional Brownian motion (see Geman and Horowitz (1980, Table 2)).

2 The main result and its reduction to a martingale CLT

One of the following mutually exclusive conditions will be imposed on the coefficients c_j of the process $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, where $c_0 = 1$.

(C.1): $c_j = j^{H-1-1/\alpha} u(j)$, with $H \neq 1/\alpha$, $0 < H < 1$, where $u(j)$ is slowly varying at infinity, satisfying $\sum_{j=0}^{\infty} c_j = 0$ when $H - 1/\alpha < 0$.

(C.2): $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} c_j \neq 0$.

Let

$$\gamma_n = \begin{cases} n^H u(n) & \text{if (C.1) is satisfied} \\ (\sum_{j=0}^{\infty} c_j) n^{1/\alpha} & \text{if (C.2) is satisfied.} \end{cases}$$

Then it is known (see Kasahara and Maejima (1988, Theorems 5.1, 5.2 and 5.3)) that when (C.1) is satisfied, the process $\gamma_n^{-1} S_{[nt]} \Longrightarrow \Lambda_{\alpha,H}(t)$, $H \neq 1/\alpha$, and similarly when $1 < \alpha \leq 2$ and (C.2) is satisfied, $\gamma_n^{-1} S_{[nt]} \Longrightarrow Z_\alpha(t)$. *In view of our convention that $Z_\alpha(t) = \Lambda_{\alpha,1/\alpha}(t)$ when $1 < \alpha \leq 2$, the preceding statements will be combined in the form $\gamma_n^{-1} S_{[nt]} \Longrightarrow \Lambda_{\alpha,H}(t)$, with the understanding that when (C.2) is satisfied the limit is $Z_\alpha(t)$ with $1 < \alpha \leq 2$.*

We shall impose the following additional **assumptions,** where

$$\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right] \text{ and } \widehat{f}(\lambda) = \int e^{i\lambda y} f(y) dy.$$

(A1): The function $f(y)$ is such that both $f(y)$ and $f^2(y)$ are Lebesgue integrable, $\int f(y)\, dy = 0$ and $\int |y f(y)|\, dy < \infty$.

(A2): $\int |\psi(\lambda)|^2 d\lambda < \infty$. Further, $|\psi(\lambda)| \leq C |\lambda|^{-\beta}$ for some $\beta > 0$ as $\lambda \to \infty$. Moreover, as $\lambda \to \infty$, $\left|\widehat{f}(\lambda)\psi(\lambda)\right| \leq C |\lambda|^{-(2+\omega)}$ for some $0 < \omega < 1$.

(The tail restrictions in (A2) will be invoked only in obtaining the appropriate bound for (16) of Section 5 below). Note that (A1) entails that $\left|\widehat{f}(\lambda_1) - \widehat{f}(\lambda_2)\right| \leq C|\lambda_1 - \lambda_2|$ because $\int |yf(y)|\,dy < \infty$, and hence $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$ because $\widehat{f}(0) = \int f(y)\,dy = 0$. In addition, $\sup_\lambda \left|\widehat{f}(\lambda)\right| \leq \int |f(y)|\,dy < \infty$ and $\int \left|\widehat{f}(\lambda)\right|^2 d\lambda = \int |f(y)|^2\,dy < \infty$.

Theorem 1. *Assume that $1/3 < H < 1$. Further assume that $f(y)$ and $\psi(\lambda)$ satisfy (A1) and (A2). Let $h(y)$ be such that $h(y)$ and $h^2(y)$ are Lebesgue integrable. Then the process $\left(\gamma_n^{-1} S_{[nt]}, n^{-1}\gamma_n \sum_{k=1}^n h(S_k), \sqrt{n^{-1}\gamma_n}\sum_{k=1}^n f(S_k)\right)$ converges in distribution to $\left(\Lambda_{\alpha,H}(t), \left(\int h(y)\,dy\right) L(1,0), W\sqrt{bL(1,0)}\right)$, where $L(1,0)$ is the local time of $\Lambda_{\alpha,H}(t)$ as before, W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ and*

$$0 < b = \frac{1}{2\pi}\int \left|\widehat{f}(\lambda)\right|^2 \left(1 + 2\sum_{r=1}^{\infty} E\left[e^{i\lambda S_r}\right]\right) d\lambda < \infty.$$

We now reduce Theorem 1 to that of a martingale CLT. For this purpose define, for a sequence of positive integers $m_n \uparrow \infty$ as $n \to \infty$,

$$\zeta_{m_n k} = \sqrt{n^{-1}\gamma_n} \sum_{l=\left[n\frac{k-1}{m_n}\right]+1}^{\left[n\frac{k}{m_n}\right]} f(S_l), \quad k = 1, 2, ...$$

Note that $\sqrt{n^{-1}\gamma_n}\sum_{l=1}^n f(S_l) = \sum_{k=1}^{m_n} \zeta_{m_n k}$. Using *the* **notation** $E_{m_n,l}$ *for the conditional expectation given the* σ-field $\sigma\left(\xi_j;\ j \leq \left[n\frac{l}{m_n}\right]\right)$, in Sections 3 - 5 below we shall establish that there is a sequence $m_n \uparrow \infty$, $\frac{m_n}{n} \to 0$, as $n \to \infty$ such that the following facts hold for each fixed integer $s > 0$ (with the limits taken as $n \to \infty$):

(R1): There is a nonrandom $\Delta(m_n, s)$ such that $\sup_{1\leq k\leq sm_n} \sum_{k=1}^q \left|E_{m_n,k-1}\left[\zeta_{m_n k}\right]\right| \leq \Delta(m_n, s) \to 0$.

(R2): $\sum_{k=1}^{sm_n} E_{m_n,k-1}\left[\zeta_{m_n k}^2\right] \Longrightarrow sbL(1,0)$, where b and $L(1,0)$ are as in Theorem 1.

(R3): $\sum_{k=1}^{sm_n} E\left[\zeta_{m_n k}^4\right] \to 0$.

(R4): When $\alpha = 2$, $\sup_{1\leq q\leq sm_n} \left|\sum_{k=1}^q E_{m_n,k-1}\left[\zeta_{m_n k}\chi_{m_n k}\right]\right| \xrightarrow{p} 0$, $\chi_{m_n k} = \frac{1}{\sqrt{n}}\sum_{l=\left[n\frac{k-1}{m_n}\right]+1}^{\left[n\frac{k}{m_n}\right]} \xi_l$.

Note that (R4) pertains only to the case $\alpha = 2$. (R3) entails the conditional Lindeberg condition $\sum_{k=1}^{sm_n} E_{m_n,k-1}\left[\zeta_{m_n k}^2 \mathbb{I}\left(|\zeta_{m_n k}| > \varepsilon\right)\right] \xrightarrow{p} 0$ for all $\varepsilon > 0$, which can be seen to be sufficient below. ($\mathbb{I}(A)$ is the indicator function of the event A.) The later condition entails

$$(\text{R5}): \sup_{1\leq k\leq sm_n} E_{m_n,k-1}\left[\zeta_{m_n k}^2\right] \xrightarrow{p} 0.$$

Proposition 2. *Assume that the requirements (R1)-(R4) are satisfied. Then the convergence statement of Theorem 1 holds.*

Proof. Define the martingale differences $\zeta^*_{m_nk} = \zeta_{m_nk} - E_{m_n,k-1}\left[\zeta_{m_nk}\right], \quad k = 1, 2, \ldots$. Note that, in view of (R1),

$$\text{(R2) - (R5) hold with } \zeta_{m_nk} \text{ replacedby } \zeta^*_{m_nk}. \tag{1}$$

Now for each fixed $t > 0$ define

$$\tau_{m_n}(t,\delta) = \inf\left\{q : \sum_{k=1}^{q}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right) \geq t\right\}, \quad \delta > 0.$$

Then $\{\tau_{m_n}(t,\delta) \leq l\} = \left\{\sum_{k=1}^{l}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right) \geq t\right\} \in \sigma\left(\xi_j;\ j \leq \left[n\frac{l-1}{m_n}\right]\right)$, $l = 1, 2, \ldots$, so that with respect to these σ-fields $\tau_{m_n}(t,\delta)$ is a stopping time for each $t > 0$. Note that $P\left[\frac{\tau_{m_n}(t,\delta)}{m_n} > J\right] \leq P\left[\sum_{k=1}^{m_nJ}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right) \leq t\right]$. Hence because of (R2) and (1), $\lim_{J\to\infty}\limsup_{n\to\infty} P\left[\frac{\tau_{m_n}(t,\delta)}{m_n} > J\right] = 0$ for each $\delta > 0$. (We do not know if $P(L(1,0) > 0) = 1$ is true; if this is true then we can take $\delta = 0$.)

Thus $\sum_{k=1}^{\tau_{m_n}(t,\delta)} E_{m_n,k-1}\left[\zeta_{m_nk}\right] \overset{p}{\to} 0$ in view of (R1), and $\sum_{k=1}^{\tau_{m_n}(t,\delta)} E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^4\right] \overset{p}{\to} 0$ in view of (R3) and (1). Similarly, because of (R4) and (1), $\sum_{k=1}^{\tau_{m_n}(t,\delta)} E_{m_n,k-1}\left[\zeta^*_{m_nk}\chi_{m_nk}\right] \to_p 0$. Further, because of (R5) and (1), $E_{m_n,\tau_{m_n}(t,\delta)-1}\left[\left|\zeta^*_{m_n,\tau_{m_n}(t,\delta)}\right|^2\right] \overset{p}{\to} 0$. Hence $\sum_{k=1}^{\tau_{m_n}(t,\delta)}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right) \overset{p}{\to} t$ in view of $\sum_{k=1}^{\tau_{m_n}(t)}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right) \geq t \geq \sum_{k=1}^{\tau_{m_n}(t)-1}\left(E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] + \frac{\delta}{m_n}\right)$. Now by a standard diagonal argument one can choose a sequence $\delta_n \downarrow 0$ as $n \to \infty$ such that these convergencies will remain true also when δ is replaced by δ_n, in addition to $\frac{\delta_n\tau_{m_n}(t,\delta_n)}{m_n} \overset{p}{\to} 0$ because $\lim_{\delta\to 0}\limsup_{n\to\infty} P\left[\frac{\delta\tau_{m_n}(t,\delta)}{m_n} > \varepsilon\right] = 0$ for all $\varepsilon > 0$. Letting $\tau_{m_n}(t) = \tau_{m_n}(t,\delta_n)$ we thus have established

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^2\right] \overset{p}{\to} t \ \text{ and } \ \sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta^*_{m_nk}\right|^4\right] \overset{p}{\to} 0 \tag{2}$$

together with $\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\zeta_{m_nk}\right] \overset{p}{\to} 0$ and, if $\alpha = 2$, $\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\zeta^*_{m_nk}\chi_{m_nk}\right] \to_p 0$.

We now show that for $0 < \alpha \leq 2$ and for every integer $l > 0$,

(*): The process $\left(\sum_{k=1}^{\tau_{m_n}(t)}\zeta^*_{m_nk},\ n^{-1/\alpha}\sum_{j=-nl}^{[nu]}\xi_j\right) \Longrightarrow \left(B(t), Z_\alpha(u) - Z_\alpha(-l)\right)$, $0 \leq t < \infty, -l \leq u < \infty$, where $B(t)$ is a Brownian motion independent of $Z_\alpha(u)$.

For this purpose define $\chi_{m_nk} = n^{-1/\alpha}\sum_{l=\left[n\frac{k-1}{m_n}\right]+1}^{\left[n\frac{k}{m_n}\right]}\xi_l, \quad k = \ldots -2, -1, 0, 1, 2\ldots$. First suppose that $0 < \alpha < 2$. Note that $\sum_{k=-m_n(l+1)}^{[m_nu]}\chi_{m_nk} = n^{-1/\alpha}\sum_{j=-nl}^{\left[n\frac{[m_nu]}{m_n}\right]}\xi_j$, the difference between which and $n^{-1/\alpha}\sum_{j=-nl}^{[nu]}\xi_j$ converges to 0 in probability. (l is an integer). Thus the process $\sum_{k=-m_n(l+1)}^{[m_nu]}\chi_{m_nk} \Longrightarrow Z_\alpha(u) - Z_\alpha(-l)$ which is an infinitely divisible process whose Lévy canonical measure will have no mass at the origin, whereas, because of (2), the process $\sum_{k=1}^{\tau_{m_n}(t)} \zeta^*_{m_n k}$ converges in distribution to the Brownian motion which is also an infinitely divisible process but whose Lévy canonical measure is degenerate at the origin. Further, because $\frac{m_n}{n} \to 0$, $\sup_{-m_n J \le k \le m_n J} P(|\chi_{m_n k}| > \varepsilon) \to 0$ for every $\varepsilon > 0$ and for every integer $J > 0$, and the same is of course true for $\zeta^*_{m_n k}$'s. Hence the claim (*) follows. (This is known when $\zeta^*_{m_n k}, k = 1, 2 ...$ are i.i.d. and $\tau_{m_n}(t)$ is nonrandom (Resnick and Greenwood (1979)), and it can be easily seen that it remains true for the present case also.)

Similarly (*) holds when $\alpha = 2$ in view of (R4) and (2), by invoking the CLT.

Because (*) is true for every $l > 0$, it entails (see Kasahara and Maejima (1988)) $\left(\sum_{k=1}^{\tau_{m_n}(t)} \zeta^*_{m_n k}, \gamma_n^{-1} S_{[nu]}\right) \Longrightarrow (B(t), \Lambda_{\alpha,H}(u))$ where the processes $B(t)$ and $\Lambda_{\alpha,H}(u)$ are independent. Further, in Section 4 below $\sum_{k=1}^{m_n} E_{m_n,k-1}\left[\left|\zeta^*_{m_n k}\right|^2\right]$ is approximated by a functional of the process $\gamma_n^{-1} S_{[nt]}$ that converges in distribution if the finite dimensional distributions of $\gamma_n^{-1} S_{[nt]}$ converge in distribution. Letting $T_n = \sum_{k=1}^{m_n} E_{m_n,k-1}\left[\left|\zeta^*_{m_n k}\right|^2\right] + \frac{\delta_n \tau_{m_n}(t,\delta_n)}{m_n}$ (recall $\frac{\delta_n \tau_{m_n}(t,\delta_n)}{m_n} \to_p 0$), we then have

(**) The process $\left(\sum_{k=1}^{\tau_{m_n}(t)} \zeta^*_{m_n k}, \gamma_n^{-1} S_{[nu]}, T_n\right) \Longrightarrow (B(t), \Lambda_{\alpha,H}(u), bL(1,0))$, $0 \le t < \infty, 0 \le u < \infty$, where $B(t)$ and $\Lambda_{\alpha,H}(u)$ are independent.

Now with q a positive integer and $J > 0$, let $0 = \tau_{q0} < \tau_{q1} < ...\tau_{q,q-1} < \tau_{qq} = J$ be such that $\tau_{q1} \downarrow 0$ and $\sup_{1 \le i \le q} |\tau_{qi} - \tau_{q,i-1}| \to 0$ as $q \to \infty$. Define

$$T_{n,q,J} = \begin{cases} \tau_{qi} & \text{if } \tau_{qi} \le T_n < \tau_{q,i+1},\ i = 0, 1, ...q-1, \\ J & \text{if } T_n \ge J. \end{cases}$$

Letting $T = bL(1,0)$, define $T_{q,J}$ analogously. Now the event $\left\{\sum_{k=1}^{\tau_{m_n}\left(T_{n,q,J}\right)} \zeta^*_{m_n k} \le c\right\}$ is the union $\cup_{i=0}^{q} \left\{\sum_{k=1}^{\tau_{m_n}(\tau_{qi})} \zeta^*_{m_n k} \le c, \tau_{qi} \le T_n < \tau_{q,i+1}\right\}$ of disjoint events, where we take $\tau_{q,q+1} = \infty$. One can assume without loss of generality that $\tau_{q1}, ...\tau_{qq}$ are continuity points of T. Then (**) entails that $\left(\sum_{k=1}^{\tau_{m_n}\left(T_{n,q,J}\right)} \zeta^*_{m_n k}, \gamma_n^{-1} S_{[nu]}\right) \Longrightarrow (B(T_{q,J}),\ \Lambda_{\alpha,H}(u))$. (Note that $T_{q,J}$ is a function of $L(1,0)$, which, being a functional of $\Lambda_{\alpha,H}(t)$, is independent of $B(t)$ by (**).) In addition, for each $J > 0$, (2) entails that

$$\lim_{h\to 0} \limsup_{n\to\infty} P\left[\sup_{|t-s|\le h, t,s\in[0,J]} \left|\sum_{k=1}^{\tau_{m_n}(t)} \zeta^*_{m_n k} - \sum_{k=1}^{\tau_{m_n}(s)} \zeta^*_{m_n k}\right| > \varepsilon\right] = 0 \quad \text{for all } \varepsilon > 0.$$

Hence $\lim_{J\to\infty} \lim_{q\to\infty} \limsup_{n\to\infty} P\left[\left|\sum_{k=1}^{\tau_{m_n}\left(T_{n,q,J}\right)} \zeta^*_{m_n k} - \sum_{k=1}^{\tau_{m_n}(T_n)} \zeta^*_{m_n k}\right| > \varepsilon\right] = 0$. Similarly $\lim_{J\to\infty} \lim_{q\to\infty} P[|B(T_{q,J}) - B(T)| > \varepsilon] = 0$. It follows that $\left(\sum_{k=1}^{\tau_{m_n}(T_n)} \zeta^*_{m_n k}, \gamma_n^{-1} S_{[nu]}\right) \Longrightarrow (B(T),\ \Lambda_{\alpha,H}(u))$. Noting that $T = bL(1,0)$ and $\tau_{m_n}(T_n) = m_n$, and in view of the independence of the processes $B(t)$ and $(\Lambda_{\alpha,H}(u), L(1,0))$, we get $\left(\sum_{k=1}^{m_n} \zeta_{m_n k}, \gamma_n^{-1} S_{[nu]}\right) \Longrightarrow$

$\left(W\sqrt{bL(1,0)}, \Lambda_{\alpha,H}(u)\right)$ where W is standard normal independent of the process $\Lambda_{\alpha,H}(u)$. (Recall that $\sum_{k=1}^{m_n}\zeta_{m_nk} = \sqrt{n^{-1}\gamma_n}\sum_{k=1}^{n} f(S_k)$.) Now note that in Jeganathan (2003, Section 4) the quantity $n^{-1}\gamma_n\sum_{k=1}^{n} h(S_k)$ occurring the statement of Theorem 1 is approximated by a functional of the process $\gamma_n^{-1}S_{[nt]}$ that converges in distribution if the finite dimensional distributions of $\gamma_n^{-1}S_{[nt]}$ converge. Hence the proof. □

3 Some preliminaries and the verification of (R1)

Recall that ξ_1 is in the domain of *normal* attraction of a strictly stable law as specified earlier with $0 < \alpha \le 2$ entails that there are constants $\eta > 0$ and $d > 0$ such that

$$|\psi(u)| \le e^{-d|u|^\alpha} \quad \text{for all} \quad |u| \le \eta, \quad \psi(u) = E\left[e^{iu\xi_1}\right]. \tag{3}$$

Note that when $\int |\psi(u)|^2 du < \infty$ the *Cramér's condition* $\limsup_{|u|\to\infty}|\psi(u)| < 1$ involved in the next result holds. Also recall that $\prod_{k=i}^{j}\psi\left(\gamma_j^{-1}g(k)u\right)$ is the characteristic function of $\gamma_j^{-1}\sum_{k=i}^{j} g(k)\xi_k$, where

$$g(k) = \sum_{s=0}^{k} c_s, \text{ with } c_s \text{ as in (C1) or C2)},$$

and $\gamma_n = n^H u(n)$ is as defined earlier. (Recall $g(k) \sim Ck^{H-1/\alpha}u(k)$ under (C1).)

Lemma 3. *(i): There are constants $B > 0, \eta > 0$ and $d > 0$ such that* $\prod_{k=i}^{j}\left|\psi\left(\gamma_j^{-1}g(k)u\right)\right| \le B\exp\left(-\left(\frac{j-i+1}{j+1}\right)d|u|^\alpha\right)$ *for all $|u| \le \eta j^{1/\alpha}$ and $j \ge i \ge 0$.*

(ii): Assume that $\psi(u)$ satisfies the Cramér's condition. Then for any $\eta > 0$ there is a $B > 0$ and a $0 < \rho < 1$ such that $\sup_{|u|\ge\eta j^{1/\alpha}}\left|\prod_{k=i}^{j}\psi\left(\gamma_j^{-1}g(k)u\right)\right| \le B\rho^{j-i+1}$ *for all $j \ge i \ge 0$.*

Lemma 4. *Let q_n and r_n be positive integers such that $q_n < r_n$ and $q_n \uparrow \infty$. Further, let $a_k, k \ge 0$, be arbitrary constants such that $\sup_{q_n\le k\le r_n}|a_k| \to 0$ as $n \to \infty$. Then*

(i): There are constants $B > 0, \eta > 0$ and $d > 0$ and an integer $n_0 > 0$ such that $\prod_{k=i}^{j}\left|\psi\left(\gamma_j^{-1}g(k)u - a_k\right)\right| \le B\exp\left\{-\left(\frac{j-i+1}{j+1}\right)d|u|^\alpha + d\sum_{k=i}^{j}|a_k|^\alpha\right\}$ *for all $|u| \le \eta j^{1/\alpha}$ and $r_n \ge j \ge i \ge q_n$, $n \ge n_0$.*

(ii): Assume that ξ_1 satisfies the Cramér's condition. Then for any $\eta > 0$ there is a $B > 0$ and a $0 < \rho < 1$ and an integer $n_0 > 0$ such that $\sup_{|u|\ge\eta j^{1/\alpha}}\left|\prod_{k=i}^{j}\psi\left(\gamma_j^{-1}g(k)u - a_k\right)\right| \le B\rho^{j-i+1}$ *for all $r_n \ge j \ge i \ge q_n$, $n \ge n_0$.*

Remark. The proofs of the preceding Lemmas 3 and 4 are essentially contained in Jeganathan (1983, Lemmas 12 and 13), from which it follows that Lemma 3 as well as the Statement (ii) of Lemma 4 hold also for the case of ξ_j belonging to the domain of attraction (instead of the present domain of restrictive normal attraction), with α in the exponent of the bounds in Statement (i) of Lemmas 3 and 4 replaced by some c, $0 < c < \alpha$. Unfortunately, we are unable to establish the same for the Statement (i) of Lemma 4. The reason is that in the general case one has, instead of (3), $|\psi(u)| \le e^{-d|u|^{\alpha}G(u)}$ for all $|u| \le \eta$ where $G(u)$ is slowly varying at 0. This will lead to an inequality of the form $\prod \left|\psi\left(\gamma_j^{-1}g(k)u - a_k\right)\right| \le \exp\left(-d\sum|\gamma_j^{-1}g(k)u - a_k|^{\alpha}G\left(\gamma_j^{-1}g(k)u - a_k\right)\right)$ when $\left(j^{1/\alpha}h(j)\right)^{-1}u$ is in a suitable neighborhood of 0 for a suitable slowly varying $h(j)$, see the proof of Lemma 12 of Jeganathan (2003). In the present situation $G(u) \equiv 1$ and $h(j) \equiv 1$ so that the Statement (i) of Lemma 4 will follow in view of the inequality $\sum|\gamma_j^{-1}g(k)u - a_k|^{\alpha} \ge C\sum|\gamma_j^{-1}g(k)u|^{\alpha} - C\sum|a_k|^{\alpha}$. In the general case the occurrence of $G\left(\gamma_j^{-1}g(k)u - a_k\right)$ appears to pose some additional problem which we are unable to resolve. □

The next result will be invoked repeatedly.

Lemma 5. *Let $d_j, j \ge 0$, be arbitrary constants. Then there is an integer $l_0 \ge 0$ such that $\int_{-\infty}^{\infty}\prod_{j=\left[\frac{l}{2}\right]}^{l}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right| d\lambda \le C$ if $l \ge l_0$. Further, for any integer $l \ge 0$ and constant d, $\frac{1}{\gamma_l}\int_{-\infty}^{\infty}\prod_{j=0}^{l}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l} - d\right)\right| d\lambda \le C$.*

Proof. $\int_{-\infty}^{\infty}\prod_{j=q}^{l}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right| d\lambda \le \prod_{j=q}^{l}\left(\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right|^{l-q+1} d\lambda\right)^{\frac{1}{l-q+1}}$ (by Hölder's inequality) which becomes $\left(\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^{l-q+1} d\lambda\right)\prod_{j=q}^{l}\left(\frac{l^{H-1/\alpha}u(l)}{g(j)}\right)^{\frac{1}{l-q+1}}$ because $\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right|^{l-q+1} d\lambda = \frac{l^{H-1/\alpha}u(l)}{g(j)}\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^{l-q+1} d\lambda$ if $g(j) \ne 0$ ($\gamma_l = l^H u(l)$). By (3), $\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^{l-q+1} \le Ce^{-d\frac{l-q+1}{l}|\lambda|^{\alpha}}$ for all $|\lambda| \le \eta l^{1/\alpha}$. Further $\sup_{|\lambda| \ge \eta l^{1/\alpha}}\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right| \le \rho < 1$ by the Cramér's condition. Therefore $\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^{l-q+1} d\lambda$ is bounded by $C\int e^{-d\frac{l-q+1}{l}|\lambda|^{\alpha}} d\lambda +$ $C\rho^{l-q-3}\int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^2 d\lambda$ which is bounded when $q = \left[\frac{l}{2}\right]$. Now $\prod_{j=q}^{[l]}\left(\frac{l^{H-1/\alpha}u(l)}{g(j)}\right)^{\frac{1}{l-q+1}} \le C$ when $q = \left[\frac{l}{2}\right]$ using $g(j) \sim j^{H-1/\alpha}u(j)$. This proves the first part.

Because $\int_{-\infty}^{\infty}\prod_{j=0}^{l}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - d_j\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l} - d\right)\right| d\lambda \le \int_{-\infty}^{\infty}\left|\psi\left(\frac{\lambda}{\gamma_l} - d_0\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l} - d\right)\right| d\lambda$ $\le \gamma_l\sqrt{\int_{-\infty}^{\infty}|\psi(\lambda)|^2 d\lambda\int_{-\infty}^{\infty}\left|\widehat{f}(\lambda)\right|^2 d\lambda}$, the second part also follows. □

It is convenient to restate the conditions (R1) - (R4) in a different but equivalent form. For this purpose define, for a fixed positive integer m,

$$\zeta_{nmk} = \sqrt{n^{-1}\gamma_n}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f(S_l), \quad k = 1, 2, \ldots.$$

Further, henceforth E_l *stands for the conditional expectation given* $\sigma(\xi_j;\ j \le l)$. Then, for every integer $s > 0$, (R1): There is a nonrandom $\Delta(n,m,s)$ such that $\sup_{1\le q\le sm}\sum_{k=1}^{q}\left|E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}]\right|\le\Delta(n,m,s)\to 0$ as $n\to\infty$ for each fixed m and s.

(R2): $\lim_{m\to\infty}\limsup_{n\to\infty}P\left[\sum_{k=1}^{sm}E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}^{2}]\le a\right]=P[sbL(1,0)\le a]$ for all continuity points of the distribution of $sbL(1,0)$, where b and $L(1,0)$ are as in Theorem 1.

(R3): $\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=1}^{sm}E[\zeta_{nmk}^{4}]=0$.

(R4): When $\alpha=2$, $\lim_{m\to\infty}\limsup_{n\to\infty}P\left[\sup_{1\le q\le sm}\left|\sum_{k=1}^{q}E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}\chi_{nmk}]\right|>\varepsilon\right]=0$ for all $\varepsilon>0$, where $\chi_{nmk}=n^{-1/2}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\xi_l$.

It is clear that these conditions are equivalent to the ones stated earlier because by a standard diagonal argument one can choose $m_n\uparrow\infty$ as $n\to\infty$ such that the preceding convergencies hold with m replaced by m_n, in which case they become identical to those stated earlier.

(***) *For notational simplification of the computations, we shall henceforth restrict to the situation where* $g(j)=\sum_{s=0}^{j}c_s\sim Cj^{H-1/\alpha}$ (and hence $\gamma_n=n^H$).

We now verify (R1). Note that $S_k=\sum_{l=-\infty}^{0}(g(k-l)-g(1-l))\xi_l+\sum_{l=1}^{k}g(k-l)\xi_l$, with $g(j)=\sum_{s=0}^{j}c_s$. We write

$$S_k=S_{k,j}+\gamma_{k-j}S_{k-j}^{*}\tag{4}$$

where $S_{k,j}=\sum_{l=-\infty}^{0}(g(k-l)-g(1-l))\xi_l+\sum_{l=1}^{j}g(k-l)\xi_l$ and $S_{k-j}^{*}=\gamma_{k-j}^{-1}\sum_{l=j+1}^{k}g(k-l)\xi_l=\gamma_{k-j}^{-1}\sum_{q=0}^{k-j-1}g(q)\xi_{k-q}$. Here it is important to note that S_{k-j}^{*} and $S_{k,j}$ are independent. In addition recall that under (A1), $\left|\widehat{f}(\lambda)\right|\le C|\lambda|$, $\sup_\lambda\left|\widehat{f}(\lambda)\right|\le C$ and $\int\left|\widehat{f}(\lambda)\right|^2d\lambda\le C$.

Lemma 6. *There is a* $0<\rho<1$ *such that* $|E_j[f(S_k)]|\le C\left(\gamma_{k-j}^{-2}+\rho^{k-j}\right)$ *for all* $k>j$, *where* E_j *stands for the conditional expectation given* $\{\xi_k,k\le j\}$.

Proof. $f(y)=\frac{1}{2\pi}\int e^{-i\lambda y}\widehat{f}(\lambda)d\lambda$, so that $f(S_k)=\frac{1}{2\pi}\int e^{-i\lambda\left(S_{kj}+\gamma_{k-j}S_{k-j}^{*}\right)}\widehat{f}(\lambda)d\lambda$ using (4). Because S_{kj} and S_{k-j}^{*} are independent, $|E_j[f(S_k)]|\le\frac{C}{\gamma_{k-j}}\int\left|E\left[e^{-i\lambda S_{k-j}^{*}}\right]\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_{k-j}}\right)\right|d\lambda$. Lemma 3 gives $\int\left|E\left[e^{-i\lambda S_l^{*}}\right]\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|d\lambda\le\frac{C}{\gamma_l}\int|\lambda|e^{-d|\lambda|^\alpha}d\lambda+C\rho^l\int\left|\psi\left(\frac{\lambda}{\gamma_l}\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|d\lambda$, where over $\left\{|\lambda|\le\eta l^{1/\alpha}\right\}$ we have used $\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|\le C\left|\frac{\lambda}{\gamma_l}\right|$ and Statement (i) of Lemma 3, and over $\left\{|\lambda|>\eta l^{1/\alpha}\right\}$ we have used Statement (ii) of Lemma 3. Now $\int\left|\psi\left(\frac{\lambda}{\gamma_l}\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|d\lambda\le\gamma_l\sqrt{\int|\psi(\lambda)|^2d\lambda\int\left|\widehat{f}(\lambda)\right|^2d\lambda}$. Hence the proof. □

Now $E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}]=n^{-(1-H)/2}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}E_{\left[n\frac{k-1}{m}\right]}[f(S_l)]$ and hence by the preceding Lemma 6 (and taking into account the simplification (***) above),

$$\left|E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}]\right|\le\frac{C}{n^{(1-H)/2}}\sum_{l=1}^{n}\left(\frac{1}{l^{2H}}+\rho^l\right)\le\begin{cases}\frac{C\log n}{n^{(1-H)/2}} & \text{if } H\ge 1/2\\ \frac{C}{n^{(3H-1)/2}} & \text{if } H<1/2.\end{cases}$$

Noting that $1/3 < H < 1$, this verifies the Condition (R1).

4 Verification of (R2) and (R4)

$E_{\left[n\frac{k-1}{m}\right]}[\zeta_{nmk}^2]$ is given by (letting $n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right]$)

$$\frac{1}{n^{(1-H)}}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)\right]+\frac{2}{n^{(1-H)}}\sum_{l=1}^{n_{mk}}\sum_{r=1}^{n_{mk}-l} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)f\left(S_{l+r+\left[n\frac{k-1}{m}\right]}\right)\right].$$

We shall verify (R2) with $s = 1$. We first obtain

Proposition 7. *For $r \geq 0$, $n^{-(1-H)}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)f\left(S_{l+r+\left[n\frac{k-1}{m}\right]}\right)\right]$ converges in distribution to $\left(\frac{1}{2\pi}\int \psi_{S_r}(\mu)\left|\widehat{f}(\mu)\right|^2 d\mu\right)L(1,0)$ (where $S_0 = 0$) as $n \to \infty$ first and then $m \to \infty$.*

By (4), $S_i = S_{i,\left[n\frac{k-1}{m}\right]} + \sum_{q=0}^{i-\left[n\frac{k-1}{m}\right]-1} g(q)\,\xi_{i-q}$ when $\left[n\frac{k-1}{m}\right] < i \leq \left[n\frac{k}{m}\right]$, where $\left\{S_{i,\left[n\frac{k-1}{m}\right]};\left[n\frac{k-1}{m}\right] < i \leq \left[n\frac{k}{m}\right]\right\}$ and $\left\{\sum_{q=0}^{i-\left[n\frac{k-1}{m}\right]-1} g(q)\,\xi_{i-q};\left[n\frac{k-1}{m}\right] < i \leq \left[n\frac{k}{m}\right]\right\}$ are independent. (Recall $S_{i,\left[n\frac{k-1}{m}\right]} = \sum_{l=-\infty}^{0}(g(i-l)-g(1-l))\xi_l + \sum_{l=1}^{\left[n\frac{k-1}{m}\right]} g(i-l)\xi_l$.) Now $E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)f\left(S_{l+r+\left[n\frac{k-1}{m}\right]}\right)\right] = \varphi\left(S_{l+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}, S_{l+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}\right)$ where

$$\varphi(y_1,y_2) = E\left[f(y_1+T_l)\,f(y_2+T_{l+r})\right], \qquad T_l = \sum_{j=1}^{l} g(l-j)\,\xi_j. \tag{5}$$

Letting, for any $0 \leq \nu_n < l$,

$$T_{nl}^* = \sum_{j=1}^{l-\nu_n} g(l-j)\xi_j \quad\text{and}\quad T_{nl,r}^* = \sum_{j=1}^{l-\nu_n} g(l+r-j)\xi_j,$$

we have

$$T_l = T_{nl}^* + \sum_{j=l-\nu_n+1}^{l} g(l-j)\xi_j \text{ and } T_{l+r} = T_{nl,r}^* + \sum_{j=l-\nu_n+1}^{l+r} g(l+r-j)\xi_j.$$

Hence,

$$\begin{aligned}
& (2\pi)^2 E\left[f(y_1+T_l)\,f(y_2+T_{l+r})\right] \\
= & \int e^{-i\lambda y_1 - i\mu y_2} E\left[e^{-i\lambda T_l - i\mu T_{l+r}}\right]\widehat{f}(\lambda)\,\widehat{f}(\mu)\,d\lambda d\mu \\
= & \int e^{-i\lambda y_1 - i\mu y_2} E\left[e^{-i(\lambda+\mu)T_{nl}^* - i\mu\left(T_{nl,r}^* - T_{nl}^*\right)}\right] E\left[e^{-i\lambda\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^*\right)}\right]\widehat{f}(\lambda)\,\widehat{f}(\mu)\,d\lambda d\mu \\
= & \frac{1}{n^H}\int e^{-i\frac{\lambda}{n^H}y_1 - i\mu(y_2-y_1)} E\left[e^{-i\lambda\frac{T_{nl}^*}{n^H} - i\mu\left(T_{nl}^* - T_{nl,r}^*\right)}\right] E\left[e^{-i\frac{\lambda}{n^H}\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^* - T_l + T_{nl}^*\right)}\right] \\
& \times\widehat{f}\left(\frac{\lambda}{n^H} - \mu\right)\widehat{f}(\mu)\,d\lambda d\mu
\end{aligned} \tag{6}$$

for any $0 \leq \nu_n < l$. Now, $E\left[e^{-i\frac{\lambda}{n^H}\left(T_l - T^*_{nl}\right) - i\mu\left(T_{l+r} - T^*_{nl,r} - T_l + T^*_{nl}\right)}\right]$ is given by

$$\begin{aligned} & E\left[e^{-i\frac{\lambda}{n^H}\left(T_l - T^*_{nl}\right) - i\mu\sum_{j=l-\nu_n+1}^{l+r}(g(l+r-j)-g(l-j))\xi_j}\right] \\ = \; & E\left[e^{-i\frac{\lambda}{n^H}\left(T_l - T^*_{nl}\right) - i\mu\sum_{j=l-\nu_n+1}^{l}\left(c_{l+1-j}+..+c_{l+r-j}\right)\xi_j}\right]\prod_{j=0}^{r-1}\psi\left(-g\left(j\right)\mu\right). \end{aligned}$$

where we let $c_j = 0$ for $j < 0$. Similarly

$$E\left[e^{-i\lambda\frac{T^*_{nl}}{n^H} - i\mu\left(T^*_{nl,r} - T^*_{nl}\right)}\right] = \prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{n^H} - \mu\left(c_{j+1} + .. + c_{j+r}\right)\right). \quad (7)$$

Hence

$$\begin{aligned} & \left|E\left[e^{-i\lambda\frac{T^*_{nl}}{n^H} - i\mu\left(T^*_{nl} - T^*_{nl,r}\right)}\right] E\left[e^{-i\frac{\lambda}{n^H}\left(T_l - T^*_{nl}\right) - i\mu\left(T_{l+r} - T^*_{nl,r} - T_l + T^*_{nl}\right)}\right]\right| \\ \leq \; & \prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{n^H} - \mu\left(c_{j+1} + .. + c_{j+r}\right)\right)\prod_{j_1=0}^{r-1}\left|\psi\left(-g\left(j_1\right)\mu\right)\right|\right| \quad (8) \end{aligned}$$

To proceed further we obtain

Lemma 8. *(i): For all $a \geq 0$ and $T^*_{nl,r}$ and T^*_{nl} corresponding to $\nu_n \leq \left[\frac{l-1}{2}\right]$, $\nu_n \uparrow \infty$,*

$$\int_{R\times\{|\mu|\geq a\}}\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{n^H} - i\mu\left(T^*_{nl,r} - T^*_{nl}\right)}\right]\psi\left(-\mu\right)\widehat{f}\left(\mu\right)\right| d\lambda d\mu \leq C\left(\frac{n}{l}\right)^H \int_{\{|\mu|>a\}}\left|\psi\left(\mu\right)\widehat{f}\left(\mu\right)\right| d\mu.$$

(ii): For every $a \geq 0$ and $\nu_n < [n\delta]$, $\nu_n \uparrow \infty$, with $\delta > 0$,

$$n^{-1}\sum_{l=[n\delta]}^{n}\int_{\{|\lambda|>b,|\mu|\leq a\}}\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{n^H} - i\mu\left(T^*_{nl,r} - T^*_{nl}\right)}\right]\psi\left(-\mu\right)\right| d\lambda d\mu \leq R\left(a,b\right)$$

where $R(a,b) \to 0$ as $b \to \infty$ for every fixed $a > 0$.

Proof. We use (7). The first part is an immediate consequence of Lemma 5.

Take $r = 1$ in the second part. Now $\int_{\{|\lambda|>b,|\mu|\leq a\}}\left|\psi\left(-\mu\right)\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{n^H} - \mu c_{j+1}\right)\right| d\lambda d\mu$ is the sum of the integrals of $\left(\frac{n}{l}\right)^H\left|\psi\left(-\mu\right)\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{l^H} - \mu c_{j+1}\right)\right|$ over $\left\{|\lambda| > \eta l^{1/\alpha}, |\mu| \leq a\right\}$ and over $\left\{\eta l^{1/\alpha} > |\lambda| > b\left(\frac{l}{n}\right)^H, |\mu| \leq a\right\}$. Further $\sup_{\nu_n\leq j\leq n}|\mu c_j| \to 0$ when $|\mu| \leq a$. Hence by the Statement (ii) of Lemma 4, there is a $\eta > 0$ such that the integral of $\left|\psi\left(-\mu\right)\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{l^H} - \mu c_{j+1}\right)\right|$ over $\left\{\eta l^{1/\alpha} > |\lambda| > b\left(\frac{l}{n}\right)^H, |\mu| \leq a\right\}$ is bounded by $\left(\int_{\{\infty>|\lambda|>b\kappa_n\}} e^{-d|\lambda|^\alpha}\right)\int_{\{|\mu|\leq a\}} e^{d|\mu|^\alpha\sum_{j=\nu_n}^{l-1}|c_{j+1}|^\alpha}\left|\psi\left(-\mu\right)\right| d\mu$, where $\kappa_n = \left(\frac{[n\delta]}{n}\right)^H$. In the same way, by the Statement (ii) of Lemma 4, the integral of $\left|\psi\left(-\mu\right)\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{l^H} - \mu c_{j+1}\right)\right|$ over $\left\{|\lambda| > \eta l^{1/\alpha}, |\mu| \leq a\right\}$ is bounded by $C\rho^{l-\nu_n-2}\left(\int_{\{|\mu|\leq a\}}\left|\psi\left(-\mu\right)\right| d\mu\right)\int\left|\psi\left(\frac{\lambda}{l^{1/\alpha}}\right)\right|^2 d\lambda$ for some $0 < \rho < 1$. Hence the proof follows. □

Lemma 9. *The difference between* $n_{mk}^{-(1-H)}\sum_{l=1}^{n_{mk}} E\left[f\left(y_1+T_l\right)f\left(y_2+T_{l+r}\right)\right]$ *and*

$$\frac{1}{n_{mk}\left(2\pi\right)^2}\sum_{l=1}^{n_{mk}}\int_{\{|(\lambda,\mu)|\le a\}} h\left(y_1,y_2\right)E\left[e^{-i\lambda n_{mk}^{-H}T_l}\right]\psi_{S_r}\left(-\mu\right)\left|\widehat{f}\left(\mu\right)\right|^2 d\lambda d\mu \tag{9}$$

converges to 0*, uniformly in* y_1 *and* y_2*, as* $n\to\infty$ *first and then* $a\to\infty$*, where* $h\left(y_1,y_2\right)=e^{-i\lambda n_{mk}^{-H}y_1-i\mu(y_2-y_1)}$ *and* $\psi_{S_r}\left(\mu\right)$ *is the characteristic function of* S_r*.*

Proof. Take $n_{mk}=n$ for simplicity. Applying Cauchy-Schwarz inequality twice $\sup_{y_1,y_2}\left|n^{-(1-H)}\sum_{l=1}^{n^\theta}E\left[f\left(y_1+T_l\right)f\left(y_2+T_{l+r}\right)\right]\right| \le \sup_y\left|n^{-(1-H)}\sum_{l=1}^{n^\theta+r}E\left[f^2\left(y_1+T_l\right)\right]\right|$ which converges to 0 if $0<\theta<1$ because $\sup_y n^{-\theta(1-H)}\sum_{l=1}^{n^\theta+r}E\left[f^2\left(y_1+T_l\right)\right]\le C$ by Jeganathan (1983). Similarly, using Lemma 3, (9) with its sum restricted to $\left\{1\le l\le n^\theta\right\}$ converges to 0. Now, by (6) and (8), it follows easily from Lemma 8, with $\nu_n=n^\theta-2$, that the difference between $n^{-(1-H)}\left(2\pi\right)^2\sum_{l=n^\theta}^{n}E\left[f\left(y_1+T_l\right)f\left(y_2+T_{l+r}\right)\right]$ and

$$\frac{1}{n}\sum_{l=n^\theta}^{n}\int_{\{|(\lambda,\mu)|\le a\}} h\left(y_1,y_2\right)E\left[e^{-i\lambda\frac{T_{nl}^*}{n^H}-i\mu\left(T_{nl,r}^*-T_{nl}^*\right)}\right]E\left[e^{-i\frac{\lambda}{n^H}\left(T_l-T_{nl}^*\right)-i\mu\left(T_{l+1}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]$$
$$\times\widehat{f}\left(\frac{\lambda}{n^H}-\mu\right)\widehat{f}\left(\mu\right)d\lambda d\mu$$

is bounded, uniformly in y_1 and y_2, in absolute value by a quantity that converges to 0 as $n\to\infty$ first and then $a\to\infty$. (Statement (i) of Lemma 8 holds for $a=0$ also.) Because $T_l-T_{nl}^*$ and $\sum_{s=0}^{\nu_n-1}g\left(s\right)\xi_s$ have the same distribution, $E\left[e^{-i\frac{\lambda}{n^H}\left(T_l-T_{nl}^*\right)-i\mu\left(T_{l+1}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]$ can be approximated by $E\left[e^{-i\mu\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]$ as $n\to\infty$ uniformly over $\left\{|(\lambda,\mu)|\le a\right\}$ and $\left\{n^\theta\le l\le n\right\}$, which reduces to $\prod_{j=0}^{\nu_n+r-1}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)$ having the approximation $\prod_{j=0}^{\infty}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)=\psi_{S_r}\left(-\mu\right)$ uniformly over $\left\{|(\lambda,\mu)|\le a\right\}$. Now according to Kasahara and Maejima (1988, Theorems 2.2 and 4.1), for $0<\tau<\alpha$

$$\begin{aligned}\sup_{n^\theta<l<\infty}P\left(\left|T_{nl,r}^*-T_{nl}^*\right|>\varepsilon\right) &= \sup_{n^\theta<l<\infty}P\left(\left|\sum_{j=\nu_n}^{l-1}(c_{j+1}+..+c_{j+r})\xi_j\right|>\varepsilon\right)\\ &\le Cr\left(1+\varepsilon^{-2}\right)\sum_{j=\nu_n}^{\infty}\left|c_j\right|^\tau\to 0\end{aligned} \tag{10}$$

when τ is suitably close to α. Hence $\left|E\left[e^{-i\lambda\frac{T_{nl}^*}{n^H}-i\mu\left(T_{nl,r}^*-T_{nl}^*\right)}\right]-E\left[e^{-i\lambda\frac{T_l}{n^H}}\right]\right|\to 0$ as $n\to\infty$ uniformly over $\left\{|(\lambda,\mu)|\le a\right\}$ and $\left\{n^\theta\le l\le n\right\}$. Hence the proof. □

It follows from Lemma 9 that $\left(2\pi\right)^2 n_{mk}^{-(1-H)}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)f\left(S_{l+r+\left[n\frac{k-1}{m}\right]}\right)\right]$ is approximated by (9) with $\left(y_1,y_2\right)=\left(S_{l+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]},S_{l+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}\right)$, see (4) and (5).

Note that $h\left(S_{[n_{mk}t]+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}, S_{[n_{mk}t]+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}\right)$ involves $n_{mk}^{-H}S_{[n_{mk}t]+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}$ and $S_{[n_{mk}t]+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]} - S_{[n_{mk}t]+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}$. Let

$$\begin{aligned} S_{mk}\left(\frac{t}{m}\right) &= c\int_{-\infty}^{0}\left\{\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z_\alpha(du) \\ &\quad +c\int_{0}^{\frac{k-1}{m}}\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}Z_\alpha(du) \end{aligned} \tag{11}$$

and

$$T(t)=\int_0^t (t-u)^{H-1/\alpha}Z_\alpha(du).$$

Then $\left(n_{mk}^{-H}S_{[n_{mk}t]+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}, n_{mk}^{-H}T_{[n_{mk}t]}\right) \Longrightarrow \left(m^H S_{mk}\left(\frac{t}{m}\right), T(t)\right)$. Further note that $S_{l+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]} - S_{l+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]} = \sum_{j=-\infty}^{\left[n\frac{k-1}{m}\right]}(c_{l+\left[n\frac{k-1}{m}\right]+1-j}+..+c_{l+\left[n\frac{k-1}{m}\right]+r-j})\xi_j$, and hence, similar to (10), $\sup_{\tau_n<l<\infty}P\left(\left|S_{l+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]} - S_{l+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}\right|>\varepsilon\right)\to 0$ for any $\tau_n\uparrow\infty$. It then follows in the same way as in Jeganathan (2003, Lemma 8) that (9) with $(y_1,y_2)=\left(S_{l+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}, S_{l+r+\left[n\frac{k-1}{m}\right],\left[n\frac{k-1}{m}\right]}\right)$ converges in distribution to

$$\frac{1}{(2\pi)^2}\int_{\{|(\lambda,\mu)|\le a\}}\left\{\int_0^1 e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dt\right\}\psi_{S_r}(-\mu)\left|\widehat{f}(\mu)\right|^2 d\lambda d\mu$$

which limit converges in probability, as $a\to\infty$, to

$$\left(\frac{1}{2\pi}\int\psi_{S_r}(\mu)\left|\widehat{f}(\mu)\right|^2d\mu\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda.$$

Thus we have shown that $n^{-(1-H)}\sum_{k=1}^m\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{l+\left[n\frac{k-1}{m}\right]}\right)f\left(S_{l+r+\left[n\frac{k-1}{m}\right]}\right)\right]$ converges in distribution to

$$\left(\frac{1}{2\pi}\int\psi_{S_r}(\mu)\left|\widehat{f}(\mu)\right|^2d\mu\right)\frac{1}{m^{1-H}}\sum_{k=1}^m\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda.$$

To complete the proof of Proposition 7, it thus remains to obtain

Lemma 10. $\frac{1}{m^{1-H}}\sum_{k=1}^m\int_0^1\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt \Longrightarrow L(1,0).$

Proof. The quantity in the lemma takes the form $\int_0^1\frac{1}{m^{1-H}}\sum_{k=1}^m h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)dt$, where $h_t(y)\ge 0$ is the density function of $T(t)$, i.e., $h_t(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda y}\widehat{h}_t(\lambda)d\lambda$ where $\widehat{h}_t(\lambda)=E\left[e^{-i\lambda T(t)}\right]$. Now $\int_0^\delta\frac{1}{m^{1-H}}\sum_{k=2}^m E\left[h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)\right]dt$ is, by the preceding arguments, bounded by $\int_0^\delta\left[\frac{1}{m}\sum_{k=2}^m\int\left|E\left[e^{i\lambda S_{mk}\left(\frac{t}{m}\right)}\right]\right|\left|\widehat{h}_t\left(\frac{\lambda}{m^H}\right)\right|d\lambda\right]dt$. In view of (11), $S_{mk}\left(\frac{t}{m}\right)$ is α-stable with scale parameter σ_{tmk} bounded below by $C\left|\frac{t}{m}+\frac{k-1}{m}\right|^H$ (see Samorodnitsky and Taqqu (1994, page 345)). Hence $\frac{1}{m}\sum_{k=2}^m\int\left|E\left[e^{i\lambda S_{mk}\left(\frac{t}{m}\right)}\right]\right|\left|\widehat{h}_t\left(\frac{\lambda}{m^H}\right)\right|d\lambda$ is bounded by $\frac{1}{m}\sum_{k=2}^{m}\frac{1}{\sigma_{mk}}\int e^{-|\lambda|^{\alpha}}\widehat{h}_t\left(\frac{\lambda}{\sigma_{mk}m^H}\right)d\lambda \leq \frac{C}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^H\int e^{-|\lambda|^{\alpha}}d\lambda \leq C$. Thus $\int_0^{\delta}\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)dt \xrightarrow{p} 0$ as $m\to\infty$ first and then $\delta\to 0$.

Next consider $\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)dt$. Note that for each fixed t, $S_{mk}\left(\frac{t}{m}\right)$ has the same structure as that of $\Lambda_{\alpha,H}\left(\frac{k}{m}\right)$, $0\leq k\leq m$, and hence Jeganathan (2003, Proposition 6) contains the fact that the difference between $\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)$ and $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H\left(S_{mk}\left(\frac{t}{m}\right)+\varepsilon z\right)\right)e^{-z^2/2}dz$ converges to 0 in mean-square, as $m\to\infty$ first and then $\varepsilon\to 0$. In addition it is easy to see this convergence is uniform over $\delta\leq t\leq 1$. Now, because $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H(y+\varepsilon z)\right)e^{-z^2/2}dz$ is sufficiently smooth in y, it can be seen that $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H\left(S_{mk}\left(\frac{t}{m}\right)+\varepsilon z\right)\right)e^{-z^2/2}dz$ can be approximated, for each $\varepsilon>0$, by $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H\left(S_{mk}(0)+\varepsilon z\right)\right)e^{-z^2/2}dz$ uniformly over $\delta\leq t\leq 1$, which in turn is approximated by $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}h_t\left(-m^H S_{mk}(0)\right)$ as before as $m\to\infty$ first and then $\varepsilon\to 0$. Noting that $S_{mk}(0)=\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)$, we thus have approximated $\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S_{mk}\left(\frac{t}{m}\right)\right)dt$ by $\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)dt$, which in turn is approximated as before by $\int_0^1\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)dt$ as $m\to\infty$ first and then $\delta\to 0$. Now, noting that $\int_0^1 h_t(y)\,dt = E[L(1,y)] = g(y)$, say, $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\int_0^1 h_t\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)=\frac{1}{m^{1-H}}\sum_{k=1}^{m}g\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)\Longrightarrow\left(\int g(y)\,dy\right)L(1,0)$ $=L(1,0)$ (Jeganathan (2003, Theorem 4)). Hence the proof. □

Lemma 11. $\sum_{r=1}^{\infty}\int\prod_{j=1}^{r}|\psi(g(j)\mu)|\left|\widehat{f}(\mu)\right|^2 d\mu<\infty$. *In particular the quantity b defined in Theorem 1 is finite.*

Proof. Note that $|\psi_{S_r}(\mu)|\leq\prod_{j=0}^{r-1}|\psi(g(j)\mu)|$. Further $\int\prod_{j=0}^{r}|\psi(g(j)\mu)|\left|\widehat{f}(\mu)\right|^2 d\mu$ $=\frac{1}{r^H}\int\prod_{j=0}^{r}\left|\psi\left(g(j)\frac{\mu}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|^2 d\mu$. By $\left|\widehat{f}(\lambda)\right|\leq C|\lambda|$ and by the first part of Lemma 3, $\frac{1}{r^H}\int_{\{|\mu|\leq\eta r^{1/\alpha}\}}\prod_{j=0}^{r}\left|\psi\left(g(j)\frac{\mu}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|^2 d\mu\leq\frac{C}{r^{3H}}\int_{\{|\mu|\leq\eta r^{1/\alpha}\}}|\mu|^2 e^{-\sigma|\mu|^{\alpha}}d\mu\leq\frac{C}{r^{3H}}$.

Now $\frac{1}{r^H}\int_{\{|\mu|\geq\eta r^{1/\alpha}\}}\prod_{j=0}^{r}\left|\psi\left(g(j)\frac{\mu}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|d\mu\leq\frac{C\rho^r}{r^H}\int\left|\psi\left(\frac{\mu}{r^H}\right)\widehat{f}\left(\frac{\mu}{r^H}\right)\right|d\mu\leq\frac{C\rho^r}{r^H}$ by the second part of Lemma 3. Because $0<\rho<1$ and $3H>1$, the proof follows. □

By Proposition 7 and Lemma 11, the next result will complete the verification (R2).

Lemma 12. $\frac{1}{n^{(1-H)}}\sum_{l=1}^{n_{mk}}\sum_{r=q}^{n_{mk}}\left|E_{[n\frac{k-1}{m}]}\left[f\left(S_{l+[n\frac{k-1}{m}]}\right)f\left(S_{l+r+[n\frac{k-1}{m}]}\right)\right]\right|$ *is bounded by a nonrandom quantity that converges to 0 as $n\to\infty$ first and then $q\to\infty$, for each $1\leq k\leq m$.*

Note that $\frac{1}{n^{(1-H)}}\left|E_{[n\frac{k-1}{m}]}\left[f\left(S_{l+[n\frac{k-1}{m}]}\right)f\left(S_{l+r+[n\frac{k-1}{m}]}\right)\right]\right|$ is bounded by the integral of

$$\frac{1}{n^{1-H}l^H r^H}\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{l^H}-\frac{\mu}{r^H}(c_{j+1}+..+c_{j+r})\right)\right|\prod_{j_1=0}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu}{r^H}\right)\right|$$
$$\times\left|\widehat{f}\left(\frac{\lambda}{l^H}-\frac{\mu}{r^H}\right)\widehat{f}\left(\frac{\mu}{r^H}\right)\right|,\qquad 0\leq\nu_n\leq l,\tag{12}$$

by (6) and (8). Hence we need to show that the sum over $\{1 \le l \le n_{mk}, q \le r \le n_{mk}\}$ of the of the integral of (12) converges to 0 as $n \to \infty$ first and then $q \to \infty$. The next three Lemmas 13 - 15 are intended to show that the integral of (12) can be restricted to $\left\{|\lambda| \le \eta l^{2\vartheta}, |\mu| \le \eta r^{\vartheta}\right\}$ for a suitable $\vartheta > 0$ and $\eta > 0$.

Lemma 13. *Let $0 < \tau < 1$, $0 < \vartheta < 1/\alpha$ and $0 < \delta < \min\left\{\frac{\left(\frac{1}{\alpha}-\vartheta\right)}{2(1-H)+1/\alpha}, \frac{\vartheta}{2(1-H)+\vartheta}\right\}$. Then the sum over $\left\{[n^{\tau}] \le l \le \left[n^{\tau(1+\delta)}\right], q \le r \le \left[n^{\tau(1+\delta)}\right]\right\}$ of the integral of (12) over $\left(\left\{|\lambda| \le \eta l^{2\vartheta}, |\mu| \le \eta r^{\vartheta}\right\}\right)^{c}$ (the complement of $\left\{|\lambda| \le \eta l^{2\vartheta}, |\mu| \le \eta r^{\vartheta}\right\}$) is bounded by $Ce^{-dn^{c}} + Ce^{-dq^{c}}\frac{1}{n}\sum_{l=1}^{n_{mk}}\left(\frac{n}{l}\right)^{H}$ for suitable constants $\eta > 0$, $c > 0$ and $d > 0$.*

Proof. The constant $0 \le \eta \le 1$ below is as in Lemma 4. Note that in (12), $\left|\widehat{f}\left(\frac{\lambda}{l^H} - \frac{\mu}{r^H}\right)\widehat{f}\left(\frac{\mu}{r^H}\right)\right| \le C$.

Step I. In view of Lemma 5, the integral of (12) over $\left\{|\lambda| < \infty, |\mu| > \eta r^{\vartheta}\right\}$ is bounded by $\frac{C}{n^{1-H}l^{H}r^{H}}\int_{\left\{|\mu|>\eta r^{\vartheta}\right\}}\left|\prod_{j_1=0}^{r-1}\psi\left(-g\left(j_1\right)\frac{\mu}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|d\mu$. According to the second part of the proof of Lemma 11, $\frac{1}{r^H}\int_{\left\{|\mu|>\eta r^{1/\alpha}\right\}}\left|\prod_{j_1=0}^{r-1}\psi\left(-g\left(j_1\right)\frac{\mu}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|d\mu \le C\rho^{r}$, $0 < \rho < 1$. Now $\frac{1}{r^H}\int_{\left\{\eta r^{\vartheta} \le |\nu| \le \eta r^{1/\alpha}\right\}}\prod_{j=0}^{r}\left|\psi\left(g\left(j\right)\frac{\nu}{r^H}\right)\right|d\nu \le \frac{C}{r^H}\int_{\left\{|\nu| \ge \eta r^{\vartheta}\right\}}e^{-\sigma|\nu|^{\alpha}}d\nu \le \frac{C}{r^H}e^{-\frac{\sigma\eta^{\alpha}r^{\vartheta\alpha}}{2}}$ where we use $\int_{\{|\nu|\ge\tau\}}e^{-\sigma|\nu|^{\alpha}}d\nu \le Ce^{-\frac{\sigma\tau^{\alpha}}{2}}$. Thus we get the bound $\frac{C}{n^{1-H}l^{H}}e^{-dr^{c}}$ for suitable $c > 0$ and $d > 0$, the sum of which over the required range of (l, r) is bounded by $Ce^{-dq^{c}}\frac{1}{n}\sum_{l=1}^{\left[n^{\gamma(1+\delta)}\right]}\left(\frac{n}{l}\right)^{H}$.

Step II. Consider the integral of (12) over $\left\{|\lambda| > \eta l^{1/\alpha}, |\mu| \le \eta r^{\vartheta}\right\}$. Taking $\nu_n = \left[n^{\tau(1-\delta)}\right]$, $\left|\frac{\mu}{r^H}\left(c_{j+1} + .. + c_{j+r}\right)\right| \le \left|\frac{C\mu}{r^H}\right|\left|(j+1)^{H-1-1/\alpha} + ... + (j+r)^{H-1-1/\alpha}\right| \le Cr^{1-H+\vartheta}n^{\tau(1-\delta)(H-1-1/\alpha)}$ when $j \ge \nu_n = \left[n^{\tau(1-\delta)}\right]$ and $|\mu| \le \eta r^{\vartheta}$, so that, when $0 < \delta < \frac{\left(\frac{1}{\alpha}-\vartheta\right)}{2(1-H)+1/\alpha}$,

$$\sup\left\{\left|\frac{\mu}{r^H}\left(c_{j+1} + .. + c_{j+r}\right)\right| : \ |\mu| \le \eta r^{\vartheta},\ r \le \left[n^{\tau(1+\delta)}\right],\ \ j \ge \nu_n\right\} \to 0. \qquad (13)$$

Hence, by Statement (ii) of Lemma 4, $\int_{\left\{|\lambda|>\eta l^{1/\alpha}\right\}}\left|\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{l^H} - \frac{\mu}{r^H}\left(c_{j+1} + .. + c_{j+r}\right)\right)\right|d\lambda \le C\rho^{l-\nu_n}$ when $|\mu| \le \eta r^{\vartheta}$, $r \le \left[n^{\tau(1+\delta)}\right]$. Further $\int_{\left\{|\mu|\le\eta r^{\vartheta}\right\}}\prod_{j_1=0}^{r-1}\left|\psi\left(\frac{\mu g(j_1)}{r^H}\right)\right|\left|\widehat{f}\left(\frac{\mu}{r^H}\right)\right|d\mu \le C$ by Lemma 3 as before. Hence the integral of (12) over $\left\{|\lambda| > \eta l^{1/\alpha}, |\mu| \le \eta r^{\vartheta}\right\}$ is bounded by $\frac{C\rho^{l-\nu_n}}{n^{1-H}l^{H}r^{H}}$, the sum of which over $\{[n^{\tau}] \le l \le n_{mk}, 1 \le r \le n_{mk}\}$ is bounded by $Ce^{-dn^{\delta\tau}}$ for some $d > 0$, because $\nu_n = \left[n^{\tau(1-\delta)}\right]$.

Step III. For the case $\left\{\eta l^{1/\alpha} \ge |\lambda| \ge \eta l^{2\vartheta}, |\mu| \le \eta r^{\vartheta}\right\}$, note that because of (13) and by the Statement (i) of Lemma 4, $\int_{\left\{\eta l^{1/\alpha}>|\lambda|>c\right\}}\left|\prod_{j=\nu_n}^{l-1}\psi\left(\frac{\lambda g(j)}{l^H} - \frac{\mu}{r^H}\left(c_{j+1} + .. + c_{j+r}\right)\right)\right|d\lambda$ is bounded, for any $c \le \eta l^{1/\alpha}$, by

$$\left(\int_{\left\{\eta l^{1/\alpha}>|\lambda|>c\right\}}\exp\left(-\left(\frac{l-\nu_n}{l}\right)d\left|\lambda\right|^{\alpha}\right)d\lambda\right)\exp\left(d\left|\frac{\mu}{r^H}\right|^{\alpha}\sum_{j=\nu_n}^{l-1}\left|c_{j+1} + .. + c_{j+r}\right|^{\alpha}\right)$$

Noting that $\frac{\nu_n}{l} \leq Cn^{-\tau\delta}$ when $l \geq [n^\tau]$, the integral of (12) over $\left\{\eta l^{1/\alpha} \geq |\lambda| \geq \eta l^{2\vartheta}, |\mu| \leq \eta r^\vartheta\right\}$ is thus bounded by $\frac{1}{n^{1-H}l^H r^H}\int_{\{|\lambda|\geq\eta l^{2\vartheta}\}} e^{-d|\lambda|^\alpha}d\lambda$ times

$$\begin{aligned}
&\int_{\{|\mu|\leq\eta r^\vartheta\}} \exp\left(d\left|\frac{\mu}{r^H}\right|^\alpha \sum_{j=\nu_n}^{l-1}|c_{j+1}+..+c_{j+r}|^\alpha\right)\prod_{j=0}^{r}\left|\psi\left(g(j)\frac{\mu}{r^H}\right)\right|d\mu \\
\leq\; &\int_{\{|\mu|\leq\eta r^\vartheta\}} \exp\left(C|\mu|^\alpha r^{\alpha(1-H)}n^{\tau(1-\delta)\alpha(H-1)}\right)\exp\left(-c|\mu|^\alpha\right)d\mu \\
\leq\; &\int_{\{|\mu|\leq\eta r^\vartheta\}} \exp\left(C|\mu|^\alpha n^{2\tau\delta\alpha(1-H)}\right)\exp\left(-c|\mu|^\alpha\right)d\mu \leq C\exp\left(n^{2\tau\delta\alpha(1-H)+\tau\alpha\vartheta(1+\delta)}\right)
\end{aligned}$$

where we have used $\sum_{j=\nu_n}^{l-1}|c_{j+1}+..+c_{j+r}|^\alpha \leq Cr^\alpha\sum_{j=\nu_n}^{l-1}(j+1)^{\alpha H-\alpha-1} \leq Cr^\alpha n^{\tau(1-\delta)\alpha(H-1)}$ with $r \leq \left[n^{\tau(1+\delta)}\right]$. Now $\frac{1}{l^H}\int_{\{|\lambda|>\eta l^{2\vartheta}\}} e^{-d|\lambda|^\alpha}d\lambda \leq Ce^{-cl^{2\vartheta\alpha}}$ for some $c>0$, the sum of which over $\left\{[n^\tau] \leq l \leq \left[n^{\tau(1+\delta)}\right]\right\}$ is bounded by $C\exp\left(-cn^{\tau2\vartheta\alpha}\right)$. Hence if $2\delta(1-H)+\vartheta(1+\delta) < 2\vartheta$, i.e., if $\delta < \frac{\vartheta}{2(1-H)+\vartheta}$, one has $\exp\left(-cn^{\tau2\vartheta\alpha}\right)\exp\left(n^{2\tau\delta\alpha(1-H)+\tau\alpha\vartheta(1+\delta)}\right) \leq \exp\left(-Cn^{\tau2\vartheta\alpha}\right)$. Hence the proof. □

Lemma 14. *Let $0<\tau<\tau'<1$ and $0<\epsilon<\tau$. Then ($n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right]$ as before) the sum over $\left\{[n^\epsilon] \leq l \leq [n^\tau], \left[n^{\tau'}\right] \leq r \leq n_{mk}\right\}$ of the integral of (12) over $\left(\left\{|\lambda| \leq \eta l^{2\vartheta}, |\mu| \leq \eta r^\vartheta\right\}\right)^c$ is bounded by $Ce^{-dn^c}\left(1+\frac{1}{n}\sum_{l=1}^{n_{mk}}\left(\frac{n}{l}\right)^H\right)$ for suitable $\eta>0$, $c>0$, $d>0$, when $0<\vartheta<\min\left\{H,\frac{\tau}{\alpha}\right\}$.*

Proof. Step I of Lemma 13 applies here, obtaining the bound $Ce^{-dn^{c\tau}}\frac{1}{n}\sum_{l=1}^{[n^\gamma]}\left(\frac{n}{l}\right)^H$ (by taking $q=[n^\tau]$). For Step II, we now verify, in analogy with (13) (for any ν_n),

$$\sup\left\{\left|\frac{\mu}{r^H}(c_{j+1}+..+c_{j+r})\right| : \;|\mu| \leq \eta r^\vartheta,\;\left[n^{\tau'}\right] \leq r \leq n,\; j \geq \nu_n = 1\right\} \to 0.$$

If $H-1/\alpha>0$ and $j \leq r$, then $|c_{j+1}+..+c_{j+r}| \leq \sum_{i=1}^{2r}|c_i| \leq Cr^{H-1/\alpha}$, so that because $r \geq \left[n^{\tau'}\right]$, $\left|\frac{\mu}{r^H}(c_{j+1}+..+c_{j+r})\right| \leq Cn^{-\frac{\tau}{\alpha}+\vartheta} \to 0$. When $H-1/\alpha<0$, $|c_{j+1}+..+c_{j+r}| \leq C$, and hence $\left|\frac{\mu}{r^H}(c_{j+1}+..+c_{j+r})\right| \leq \frac{C}{r^{H-\vartheta}} \leq Cn^{-\tau'(H-\vartheta)} \to 0$ because $r \geq \left[n^{\tau'}\right]$. The same holds when $H=1/\alpha$. Hence, in analogy with Step II of Lemma 13, the required bound, which is the sum of $\frac{C\rho^{l-\nu_n}}{n^{1-H}l^H r^H}$ over $\left\{[n^\epsilon] \leq l \leq [n^\tau], \left[n^{\tau'}\right] \leq r \leq n_{mk}\right\}$, can be taken to be Ce^{-dn^ϵ} for some $d>0$.

Next, noting that $j<l<r$,

$$\sum_{j=1}^{l}|c_{j+1}+..+c_{j+r}|^\alpha \leq \begin{cases} C\sum_{j=1}^{l}r^{\alpha H-1} \leq Clr^{\alpha H-1} & \text{if } H-1/\alpha \geq 0 \\ C\sum_{j=1}^{l}j^{\alpha H-1} \leq Cl^{\alpha H} & \text{if } H-1/\alpha < 0. \end{cases}$$

Hence $\left|\frac{\mu}{r^H}\right|^\alpha\sum_{j=1}^{l-1}|c_{j+1}+..+c_{j+r}|^\alpha \leq C|\mu|^\alpha n^{-(\tau'-\tau)\min\{1,\alpha H\}}$ because $r \geq \left[n^{\tau'}\right]$ and $l \leq [n^\tau]$. Hence, in analogy with Step III of Lemma 13, the required bound, which is now the sum of $\frac{1}{n^{1-H}r^H}C\exp\left(-cl^{2\vartheta\alpha}\right)$ over the required region of (l,r), is given by $C\exp\left(-cn^{2\epsilon\vartheta\alpha}\right)\frac{1}{n}\sum_{r=1}^{n_{mk}}\left(\frac{n}{r}\right)^H$. Hence the proof. □

Lemma 15. *Let $0<\epsilon<1/2$. Then the sum over $\{(l,r):1\le l\le[n^\epsilon],1\le r\le[n^\epsilon]\}$ of the integral of (12) with $\nu_n=1$, is bounded by $\frac{C}{n^{(1-2\epsilon)(1-H)}}$.*

Proof. In view of Lemma 5, the required sum is bounded by $\frac{1}{n^{1-H}}\sum_{r=1}^{[n^\varepsilon]}\frac{1}{l^H}\sum_{r=1}^{[n^\varepsilon]}\frac{1}{r^H}$ $\le C\frac{n^{2\varepsilon(1-H)}}{n^{1-H}}$. Hence the proof. □

It follows from Lemmas 13 - 15 that the sum over $\{1\le l\le n_{mk},q\le r\le n_{mk}\}$ of the integral of (12) over $\left(\{|\lambda|\le\eta l^{2\vartheta},|\mu|\le\eta r^\vartheta\}\right)^c$ is bounded by $\frac{C}{n^\delta}+C\left(e^{-dq^c}+\frac{1}{n^\delta}\right)\frac{1}{n}\sum_{l=1}^{n_{mk}}\left(\frac{n}{l}\right)^H$ for suitable $\vartheta>0$, $c>0$, $d>0$ and $\delta>0$, which converges to 0 by letting $n\to\infty$ first and then $q\to\infty$. Now by applying the first part of Lemma 5 and using $\left|\widehat{f}\left(\frac{\lambda}{l^H}-\frac{\mu}{r^H}\right)\widehat{f}\left(\frac{\mu}{r^H}\right)\right|\le$ $\left(\frac{1}{l^{H-2\vartheta}}+\frac{1}{r^{H-\vartheta}}\right)\frac{C}{r^{H-\vartheta}}$ over $\left\{|\lambda|\le\eta l^{2\vartheta},|\mu|\le\eta r^\vartheta\right\}$, the sum over $\{J\le l\le n_{mk},q\le r\le n_{mk}\}$, for a suitable J, of the integral of (12) over $\left\{|\lambda|\le\eta l^{2\vartheta},|\mu|\le\eta r^\vartheta\right\}$ is bounded by $\sum_{r=1}^{n_{mk}}\sum_{r=q}^{n_{mk}}\frac{C}{n^{1-H}l^Hr^H}\left(\frac{1}{l^{H-2\vartheta}}+\frac{1}{r^{H-\vartheta}}\right)\frac{1}{r^{H-\vartheta}}$, which converges to 0 by letting $n\to\infty$ first and then $q\to\infty$, if ϑ further satisfies $H-2\vartheta>0$ and $3H-2\vartheta>1$ (which is possible because $H>0$ and $3H>1$). Now, according to the second part of Lemma 5, the sum over $\{1\le l\le J,q\le r\le n_{mk}\}$ has the bound $C\sum_{r=1}^{n_{mk}}\frac{C}{n^{1-H}r^H}\frac{1}{r^{H-\vartheta}}$ which also converges to 0 as above. Hence the proof of Lemma 12 follows. □

It remains to verify (R4). First write $n^{\frac{1}{2}+\frac{1-H}{2}}\zeta_{nmk}\chi_{nmk}=I_{1,nmk}+I_{2,nmk}+I_{3,nmk}$ where $I_{1,nmk}=\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=l+1}^{\left[n\frac{k}{m}\right]}f(S_l)\xi_r$, $I_{2,nmk}=\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=l+1}^{\left[n\frac{k}{m}\right]}\xi_lf(S_r)$ and $I_{3,nmk}=$ $\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}f(S_l)\xi_l$, Clearly $E_{\left[n\frac{k-1}{m}\right]}[I_{1,nmk}]=0$. Now $E_{\left[n\frac{k-1}{m}\right]}[f(S_l)\xi_l]=E_{\left[n\frac{k-1}{m}\right]}[f_1(S_l^*)]$ where $S_l^*=S_l-\xi_l$ and $f_1(y)=E[\xi_1f(y+\xi_1)]$. Note that $\int f_1(y)\,dy=0$ and similarly other restrictions in (A1) stated for $f(y)$ are satisfied for $f_1(y)$. Then the earlier arguments of this section contains the fact $E_{\left[n\frac{k-1}{m}\right]}\left[\left(n^{-\frac{1-H}{2}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}f_1(S_l^*)\right)^2\right]\le C$. (Actually the arguments contains this fact when $f_1(S_l)$ is involved in place of $f_1(S_l^*)$ but the same apply for $f_1(S_l^*)$ also.) Hence $\left|E_{\left[n\frac{k-1}{m}\right]}\left[n^{-\frac{1}{2}-\frac{1-H}{2}}I_{3,nmk}\right]\right|\le Cn^{-\frac{1}{2}}$. To deal with $I_{2,nmk}$ we have $f(S_r)=\frac{1}{2\pi}\int e^{-i\lambda S_r}\widehat{f}(\lambda)\,d\lambda$ with $S_r=S_{r,\left[n\frac{k-1}{m}\right]}+\sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1}g(q)\xi_{r-q}$ with $S_{r,\left[n\frac{k-1}{m}\right]}$ independent of $\sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1}g(q)\xi_{r-q}$, see (4). Hence $\left|E_{\left[n\frac{k-1}{m}\right]}[\xi_lf(S_r)]\right|$ is bounded by $\frac{1}{2\pi}\int\left|E\left[\xi_le^{-i\lambda\sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1}g(q)\xi_{r-q}}\right]\right|\left|\widehat{f}(\lambda)\right|d\lambda$, so that $E_{\left[n\frac{k-1}{m}\right]}[I_{2,nmk}]$, which takes the form $\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]}f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]$, is bounded in absolute value by

$$\frac{1}{2\pi}\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\gamma_r^{-1}\int\left|E\left[\xi_1e^{-i\lambda\gamma_r^{-1}g(r-l)\xi_1}\right]\right|\prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\lambda\gamma_r^{-1}g(q)\right)\right|\left|\widehat{f}\left(\lambda\gamma_r^{-1}\right)\right|d\lambda.$$

Now $\left|E\left[\xi_1 e^{-i\lambda\gamma_r^{-1}g(r-l)\xi_1}\right]\right| \leq C|\lambda|\left|\gamma_r^{-1}g(r-l)\right|$ because $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$. (Note that (R4) pertains only to the case $\alpha=2$.) Further $\left|\widehat{f}\left(\lambda\gamma_r^{-1}\right)\right| \leq C|\lambda|\left|\gamma_r^{-1}\right|$. Thus by an application of Lemma 3 as before, $\left|E_{\left[n\frac{k-1}{m}\right]}\left[I_{2,nmk}\right]\right|$ is bounded by $C\sum_{r=1}^{n_{mk}}\sum_{l=1}^{r-1}\gamma_r^{-3}\left|g(r-l)\right|$. We have $\sum_{l=1}^{r-1}|g(r-l)| \sim Cr^{H+1-1/2}$ because $g(s)\sim Cs^{H-1/2}$. Hence, noting that $\gamma_r=r^H$, $\left|E_{\left[n\frac{k-1}{m}\right]}\left[I_{2,nmk}\right]\right| \leq Cn^{-2H+\frac{3}{2}}$. Therefore $\left|E_{\left[n\frac{k-1}{m}\right]}\left[n^{-\frac{1}{2}-\frac{1-H}{2}}I_{2,nmk}\right]\right| \leq Cn^{-\frac{3H-1}{2}}$. Because $3H-1>0$, this completes the verification of (R4). □

5 Verification of (R3)

We now show that $E\left[\zeta_{nmk}^4\right]$ is bounded by

$$\frac{C}{n^\delta}+C\left(\frac{1}{n}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\left(\frac{n}{l}\right)^H\right)\left(\frac{1}{n^\delta}+\frac{1}{n}\sum_{j=1}^{n_{mk}}\left(\frac{n}{j}\right)^H\right), \text{ for some } \delta>0, \tag{14}$$

which will verify (R3). (Recall $n_{mk}=\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]$). We shall show in detail that

$$\frac{1}{n^{2(1-H)}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\left|E\left[f(S_l)f(S_{l+r})f^2(S_{l+r+q})\right]\right| \tag{15}$$

and

$$\frac{1}{n^{2(1-H)}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\sum_{s=1}^{n_{mk}}\left|E\left[f(S_l)f(S_{l+r})f(S_{l+r+q})f(S_{l+r+q+s})\right]\right| \tag{16}$$

is bounded by (14). The same can be similarly shown to be true for the remaining analogues in the expansion of $E\left[\zeta_{nmk}^4\right]=E\left[\left(\frac{1}{n^{(1-H)/2}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}f(S_l)\right)^4\right]$. We shall use ideas similar to those of Lemmas 13 - 15 but the tail restrictions involved in the assumption (A2) will be crucially employed in dealing with (16).

We first deal with (15). As in Section 4, define $T_l=\sum_{j=1}^{l}g(l-j)\xi_j$, $T_{nl}^*=\sum_{j=1}^{l-\nu_n}g(l-j)\xi_j$, and $T_{nl,r}^*=\sum_{j=1}^{l-\nu_n}g(l+r-j)\xi_j$. Let $R_l=\sum_{j=-\infty}^{0}(g(l-j)-g(1-j))\xi_j$ so that $S_l=R_l+T_l$. Note that R_l and T_l are independent. We have, similar to (6),

$$\begin{aligned}
&(2\pi)^3E\left[f(S_l)f(S_{l+r})f^2(S_{l+r+q})\right]\\
=&\int E\left[e^{-i\lambda_1R_l-\lambda_2R_{l+r}-\lambda_3R_{l+r+q}}\right]E\left[e^{-i(\lambda_1+\lambda_2+\lambda_3)T_{nl}^*-i(\lambda_2+\lambda_3)\left(T_{nl,r}^*-T_{nl}^*\right)-i\lambda_3\left(T_{nl,r+q}^*-T_{nl,r}^*\right)}\right]\\
&\times E\left[e^{-i\lambda_1\left(T_l-T_{nl}^*\right)-i\lambda_2\left(T_{l+r}-T_{nl,r}^*\right)-i\lambda_3\left(T_{l+r+q}-T_{nl,r+q}^*\right)}\right]\widehat{f}(\lambda_1)\widehat{f}(\lambda_2)\widehat{f}^2(\lambda_3)\,d\lambda_1d\lambda_2d\lambda_3.
\end{aligned}$$

Hence, as in (8), $\frac{1}{n^{2(1-H)}}(2\pi)^3\left|E\left[f(S_l)f(S_{l+r})f^2(S_{l+r+q})\right]\right|$ is bounded by the integral of

$$\frac{1}{n^{2(1-H)}l^Hr^Hq^H}\prod_{j_1=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda_1 g(j_1)}{l^H}+\frac{\lambda_2 g(j_1,r)}{r^H}+\frac{\lambda_3 g(j_1+r,q)}{q^H}\right)\right|$$
$$\times\left|\prod_{j_2=0}^{r-1}\psi\left(\frac{\lambda_2 g(j_2)}{r^H}+\frac{\lambda_3 g(j_2,q)}{q^H}\right)\prod_{j_3=0}^{q-1}\psi\left(\frac{\lambda_3 g(j_3)}{q^H}\right)\widehat{f}\left(\frac{\lambda_1}{l^H}-\frac{\lambda_2}{r^H}\right)\widehat{f}\left(\frac{\lambda_2}{r^H}-\frac{\lambda_3}{q^H}\right)\right| \quad (17)$$

for any $0\le\nu_n\le\left[n\frac{k-1}{m}\right]$, where we let $g(j,r)=g(j+r)-g(j)$. The integral over λ_1 of

$$\left|\widehat{f}\left(\frac{\lambda_1}{l^H}-\frac{\lambda_2}{r^H}\right)\right|\prod_{j=0}^{l-1}\left|\psi\left(\frac{\lambda_1 g(j)}{l^H}+\frac{\lambda_2 g(j,r)}{r^H}+\frac{\lambda_3 g(j+r,q)}{q^H}\right)\right| \quad (18)$$

is, by Lemma 5, bounded by C. Further the sum over $\{1\le r,q\le n_{mk}\}$ of the integral of $\frac{1}{n^{1-H}r^Hq^H}\prod_{j_1=0}^{r-1}\left|\psi\left(\frac{\lambda_2 g(j_1)}{r^H}+\frac{\lambda_3 g(j_1,q)}{q^H}\right)\right|\prod_{j_2=0}^{q-1}\left|\psi\left(\frac{\lambda_3 g(j_2)}{q^H}\right)\right|$ over $\left(\left\{|\lambda_2|\le\eta r^{2\vartheta},|\lambda_3|\le\eta q^{\vartheta}\right\}\right)^c$ is bounded by $C\frac{1}{n^\delta}+\frac{C}{n}\sum_{l=1}^{n_{mk}}\left(\frac{n}{l}\right)^H$, $\delta>0$, by Lemmas 13 - 15. Thus sum over

$$\left\{\left[n\frac{k-1}{m}\right]<l\le\left[n\frac{k}{m}\right],1\le r\le n_{mk},1\le q\le n_{mk}\right\} \quad (19)$$

of the integral of (17) over $R\times\left(\left\{|\lambda_2|\le\eta r^{2\vartheta},|\lambda_3|\le\eta q^{\vartheta}\right\}\right)^c$ is bounded by (14), for a suitable $\vartheta>0$.

Now, following the proofs of Lemmas 13 - 15, we show that the sum over (19) of the integral of (17) over $\left\{|\lambda_1|>\eta l^{3\vartheta},|\lambda_2|\le\eta r^{2\vartheta},|\lambda_3|\le\eta q^{\vartheta}\right\}$ is bounded by (14). First consider $k\ge 2$, in which case we can choose $\nu_n=\left[m^{-1}n^{1-\delta}\right]$. Noting that $r\le n_{mk}$ and $|\lambda_3|\le\eta q^{\vartheta}\le\eta n_{mk}^{\vartheta}$, $\left|\frac{\lambda_3}{q^H}g(j+r,q)\right|=\left|\frac{\lambda_3}{q^H}\left(g(j+r+q)-g(j+r)\right)\right|\le C|\lambda_3|q^{1-H}(j+r)^{H-1-1/\alpha}$ $\le Cn^{1-H+\vartheta}n^{(1-\delta)(H-1-1/\alpha)}\le Cn^{-\upsilon}$, $\upsilon>0$, when $j\ge\nu_n$ for suitable $\delta>0$ and $\vartheta>0$. Similarly $\left|\frac{\lambda_2}{r^H}g(j,r)\right|\le Cn^{-\upsilon}$. Hence, in analogy with Step II of Lemma 13, the integral of (18) over $\left\{|\lambda_1|>\eta l^{1/\alpha}\right\}$ is bounded by $C\rho^{l-\left[m^{-1}n^{1-\delta}\right]}$, $0<\rho<1$. Hence, noting that $\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\rho^{l-\left[m^{-1}n^{1-\delta}\right]}\le C\rho^{\left[n\frac{k-1}{m}\right]-\left[m^{-1}n^{1-\delta}\right]}$ with $k\ge 2$, the sum over (19) of the integral of (17) over $\left\{|\lambda_1|>\eta l^{1/\alpha},|\lambda_2|\le\eta r^{2\vartheta},|\lambda_3|\le\eta q^{\vartheta}\right\}$ is bounded by (14).

Noting that $\sum_{j=\nu_n}^{l-1}|g(j+r,q)|^\alpha\le Cq^\alpha\nu_n^{\alpha(H-1)}\le Cq^\alpha n^{\alpha(1-\delta)(H-1)}\le Cn^{\alpha\delta(H-1)}$ and $\sum_{j=\nu_n}^{l-1}|g(j,r)|^\alpha\le Cn^{\alpha\delta(H-1)}$, the integral of (18) over $\left\{\eta l^{1/\alpha}>|\lambda_1|>\eta l^{3\vartheta}\right\}$ is, by Lemma 4, bounded by $\left(\int_{\left\{\eta l^{1/\alpha}>|\lambda_1|>\eta l^{3\vartheta}\right\}}e^{-d|\lambda_1|^\alpha}d\lambda_1\right)\exp\left(n^{\alpha\delta(H-1)}\left(C_1|\lambda_2|^\alpha+C_2|\lambda_3|^\alpha\right)\right)$. Therefore the integral of (17) over $\left\{\eta l^{1/\alpha}>|\lambda_1|>l^{3\vartheta},|\lambda_2|\le r^{2\vartheta},|\lambda_3|\le q^{\vartheta}\right\}$ is, by an application of Lemma 5, bounded by

$$\frac{C}{n^{2(1-H)}l^Hr^Hq^H}\left(\int_{\left\{\eta l^{1/\alpha}>|\lambda_1|>\eta l^{3\vartheta}\right\}}e^{-d|\lambda_1|^\alpha}d\lambda_1\right)\int_{\left\{|\lambda|\le\eta l^{2\vartheta}\right\}}e^{Cn^{\alpha\delta(H-1)}|\lambda|^\alpha}d\lambda$$
$$\le\frac{C}{n^{2(1-H)}l^Hr^Hq^H}\exp\left(-C_1l^{3\alpha\vartheta}+C_2n^{\alpha\delta(H-1)}l^{2\alpha\vartheta}\right)$$

Thus, noting that $l \geq [m^{-1}n]$, for a suitable $\delta > 0$ the sum over (19) of the integral of (17) over $\left\{\eta l^{1/\alpha} > |\lambda_1| > \eta l^{3\vartheta}, |\lambda_2| \leq \eta r^{2\vartheta}, |\lambda_3| \leq \eta q^{\vartheta}\right\}$ is bounded by Ce^{-dn^c} for suitable $c > 0$ and $d > 0$.

In the case $k = 1$, it is easy to see that Steps II and III of Lemma 13 extend to the present situation, i.e., the sum over $\left\{[n^{\tau}] \leq l \leq \left[n^{\tau(1+\delta)}\right], 1 \leq r \leq \left[n^{\tau(1+\delta)}\right], 1 \leq q \leq \left[n^{\tau(1+\delta)}\right]\right\}$ of the integral of (17) over $\left\{|\lambda_1| > \eta l^{3\vartheta}, |\lambda_2| \leq \eta r^{2\vartheta}, |\lambda_3| \leq \eta q^{\vartheta}\right\}$ is bounded by Ce^{-dn^c} for suitable constants $\vartheta > 0$, $c > 0$ and $d > 0$. Similar remark applies to Lemmas 14 and 15. Combining the ideas of Lemmas 13 and 14, the same remark holds, for instance to the sum over $\left\{[n^{\tau}] \leq l \leq \left[n^{\tau(1+\delta)}\right], \left[n^{\tau'(1+\delta)}\right] \leq r \leq n_{mk}, 1 \leq q \leq \left[n^{\tau(1+\delta)}\right]\right\}$, $0 < \tau < \tau' < 1$.

From the above steps we see that the required sum of the integral of (17) over the complement $\left(\left\{|\lambda_1| \leq \eta l^{3\vartheta}, |\lambda_2| \leq \eta r^{2\vartheta}, |\lambda_3| \leq \eta q^{\vartheta}\right\}\right)^c$ is bounded by (14) for all $1 \leq k \leq m$. Now sum, over (19) with the restriction $l, r, q \geq J$ for a suitable integer J, of the integral of (17) over $\left\{|\lambda_1| \leq \eta l^{3\vartheta}, |\lambda_2| \leq \eta r^{2\vartheta}, |\lambda_3| \leq \eta q^{\vartheta}\right\}$ is bounded by

$$\frac{C}{n^{2(1-H)}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \frac{1}{l^H r^H q^H} \left(\frac{1}{l^{H-3\vartheta}} + \frac{1}{r^{H-2\vartheta}}\right) \left(\frac{1}{r^{H-2\vartheta}} + \frac{1}{q^{H-\vartheta}}\right), \tag{20}$$

by the repeated use of the first part of Lemma 5 after replacing $\left|\widehat{f}\left(\frac{\lambda_1}{l^H} - \frac{\lambda_2}{r^H}\right) \widehat{f}\left(\frac{\lambda_2}{r^H} - \frac{\lambda_3}{q^H}\right)\right|$ in (17) by its bound $\left(\frac{1}{l^{H-3\vartheta}} + \frac{1}{r^{H-2\vartheta}}\right)\left(\frac{1}{r^{H-2\vartheta}} + \frac{1}{q^{H-\vartheta}}\right)$ over $\left\{|\lambda_1| \leq \eta l^{3\vartheta}, |\lambda_2| \leq \eta r^{2\vartheta}, |\lambda_3| \leq \eta q^{\vartheta}\right\}$. Now (20) is easily seen to be bounded by (14) when $H - 3\vartheta > 0$ and $3H - 3\vartheta > 1$. Similarly the sum over (19) under for instance $q \leq J$ and $l, r \geq J$, we get, using in addition the second part of Lemma 5, the bound $\frac{C}{n^{2(1-H)}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \frac{1}{l^H r^H} \left(\frac{1}{l^{H-3\vartheta}} + \frac{1}{r^{H-2\vartheta}}\right)$ which is again seen to be bounded by (14) for a suitable $\vartheta > 0$. Thus (15) is bounded by (14).

We next consider (16). Using the same arguments used in obtaining the bound (17), $\frac{1}{n^{2(1-H)}} (2\pi)^4 \left|E\left[f(S_l) f(S_{l+r}) f(S_{l+r+q}) f(S_{l+r+q+s})\right]\right|$ is, for any $0 \leq \nu_n < l$, bounded by the integral of

$$\begin{aligned} &\frac{1}{n^{2(1-H)} l^H r^H q^H s^H} \prod_{j_1=\nu_n}^{l-1} \left|\psi\left(\frac{\lambda_1 g(j_1)}{l^H} + \frac{\lambda_2 g(j_1, r)}{r^H} + \frac{\lambda_3 g(j_1 + r, q)}{q^H} + \frac{\lambda_4 g(j_1 + r + q, s)}{s^H}\right)\right| \\ &\times \prod_{j_2=0}^{r-1} \left|\psi\left(\frac{\lambda_2 g(j_2)}{r^H} + \frac{\lambda_3 g(j_2, q)}{q^H} + \frac{\lambda_4 g(j_2 + q, s)}{s^H}\right)\right| \prod_{j_3=0}^{q-1} \left|\psi\left(\frac{\lambda_3 g(j_3)}{q^H} + \frac{\lambda_4 g(j_3, s)}{s^H}\right)\right| \\ &\times \prod_{j_4=0}^{s-1} \left|\psi\left(\frac{\lambda_4 g(j_4)}{s^H}\right)\right| \left|\widehat{f}\left(\frac{\lambda_1}{l^H} - \frac{\lambda_2}{r^H}\right) \widehat{f}\left(\frac{\lambda_2}{r^H} - \frac{\lambda_3}{q^H}\right) \widehat{f}\left(\frac{\lambda_3}{q^H} - \frac{\lambda_4}{s^H}\right) \widehat{f}\left(\frac{\lambda_4}{s^H}\right)\right| \end{aligned} \tag{21}$$

where recall $g(j, r) = g(j + r) - g(j)$. Fix k and m, $1 \leq k \leq m$, and let, *with* ω *as in (A2)*,

$$A_n = \left\{i : \left[n_{mk}^{\theta}\right] \leq i \leq n_{mk}\right\} \text{ and } B_n = \left\{i : 1 \leq i \leq \left[n_{mk}^{\theta}\right]\right\}, \; 1/(1+\omega) < \theta < 1.$$

We shall first deal with the situation $2 \leq k \leq m$, in which case we consider four cases.

Case I: Here we deal with the sum over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, q, s \in A_n\right\}$ of the integral of (21). Because $r, q, s \geq n^{\theta}_{mk}$, it follows from the earlier arguments that the integral can be restricted to $\left\{|\lambda_1| \leq \eta l^{4\vartheta}, |\lambda_2| \leq \eta r^{3\vartheta}, |\lambda_3| \leq \eta q^{2\vartheta}, |\lambda_4| \leq \eta s^{\vartheta}\right\}$ for a suitable $\vartheta > 0$. The integral of (21) over this restriction is, similar to (20), bounded by

$$\frac{C}{n^{2(1-H)} l^H r^H q^H s^H}\left(\frac{1}{l^{H-4\vartheta}} + \frac{1}{r^{H-3\vartheta}}\right)\left(\frac{1}{r^{H-3\vartheta}} + \frac{1}{q^{H-2\vartheta}}\right)\left(\frac{1}{q^{H-2\vartheta}} + \frac{1}{s^{H-\vartheta}}\right)\frac{1}{s^{H-\vartheta}} \tag{22}$$

by the repeated use of the first part of Lemma 5 (note $r, q, s \geq n^{\theta}_{mk}$). The sum of (22), similar to (20), is seen to be bounded by (14) for a suitable $\vartheta > 0$.

Case II: We consider the sum over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, q \in A_n, s \in B_n\right\}$ of the integral of (21). *With ω as in (A2),* define θ^* such that

$$(1-H)/(1+\omega) < \theta^* < \theta(1-H).$$

When the integral is restricted to $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| < \infty, |\lambda_4| \geq n^{\theta^*}\right\}$, the sum is, by Lemma 5, bounded by the sum over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, q \in A_n\right\}$) of

$$\frac{1}{n^{2(1-H)}}\frac{1}{l^H r^H q^H}\sum_{s=1}^{\left[n^{\theta}_{mk}\right]}\frac{1}{s^H}\int_{\left\{|\lambda| \geq n^{\theta^*}\right\}}\prod_{j=0}^{s-1}\left|\psi\left(\frac{\lambda g(j)}{s^H}\right)\right|\left|\widehat{f}\left(\frac{\lambda}{s^H}\right)\right| d\lambda. \tag{23}$$

For $s > s_1$, the integral in the preceding (23) is bounded by $\int_{\left\{|\lambda| \geq n^{\theta^*}\right\}}\prod_{j=s_1}^{s-1}\left|\psi\left(\frac{\lambda}{s^H} g(j)\right)\right| d\lambda$ $\leq \prod_{j=s_1}^{s-1}\left(\left|\frac{s^H}{g(j)}\right|\int_{\left\{\left|\lambda\frac{s^H}{g(j)}\right| \geq n^{\theta^*}\right\}}|\psi(\lambda)|^{s-s_1} d\lambda\right)^{\frac{1}{s-s_1}}$ (by Hölder's), which itself is bounded by $\prod_{j=s_1}^{s-1}\left(\left|\frac{s^H}{g(j)}\right|^3 n^{-2\theta^*}\int|\lambda|^2|\psi(\lambda)|^{s-s_1} d\lambda\right)^{\frac{1}{s-s_1}} = D(s, s_1)\, s^{-3/\alpha} n^{-2\theta^*}\int|\lambda|^2|\psi(\lambda)|^{s-s_1} d\lambda$ if $g(j) \neq 0$ for all $j \geq s_1$, where $D(s, s_1) = \prod_{j=s_1}^{s-1}\left(\left|\frac{s^{H-1/\alpha}}{g(j)}\right|^3\right)^{\frac{1}{s-s_1}}$. Because $g(j) \sim Cj^{H-1/\alpha}$ there is an s_0 such that if $s - \left[\frac{s}{2}\right] \geq s_0$ and $s_1 = \left[\frac{s}{2}\right]$, then $D(s, s_1) \leq C$. Thus $\sum_{s=s_0}^{\left[n^{\theta}_{mk}\right]}\frac{1}{s^H}\int_{\left\{|\lambda| \geq n^{\theta^*}\right\}}\prod_{j=0}^{s-1}\left|\psi\left(\frac{\lambda}{s^H} g(j)\right)\right| d\lambda \leq C\sum_{s=s_0}^{\left[n^{\theta}_{mk}\right]} s^{-3/\alpha} n^{-2\theta^*}\int|\lambda|^2|\psi(\lambda)|^{s_0} d\lambda \leq Cn^{-(1-H)}$ because $2\theta^* \geq \theta^*(1+\omega) \geq 1 - H$ and $\int|\lambda|^2|\psi(\lambda)|^{s_0} d\lambda < \infty$, by (A2), for a suitable s_0. Now $\frac{1}{s^H}\int_{\left\{|\lambda| \geq n^{\theta^*}\right\}}\prod_{j=0}^{s-1}\left|\psi\left(\frac{\lambda}{s^H} g(j)\right)\right|\left|\widehat{f}\left(\frac{\lambda}{s^H}\right)\right| d\lambda \leq \int_{\left\{s_0|\lambda| \geq n^{\theta^*}\right\}}|\psi(\lambda)|\left|\widehat{f}(\lambda)\right| d\lambda$ $\leq Cn^{-\theta^*(1+\omega)} \leq Cn^{-(1-H)}$, by (A2), for $1 \leq s \leq s_0$. Thus combining the preceding two steps the sum of (23) over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, q \in A_n\right\}$ is bounded by (14).

Hence the integral can be restricted to $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| < \infty, |\lambda_4| \leq n^{\theta^*}\right\}$. Further, according to Case I, the conclusion of which is true for any $0 < \theta < 1$, the sum can be restricted to $\Pi_n = \left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, q \in A_n, 1 \leq s \leq \left[n_{mk}^{\epsilon}\right]\right\}$ with $0 < \epsilon < \theta - \frac{\theta^*}{1-H}$. Because $s \leq \left[n_{mk}^{\epsilon}\right]$ and $|\lambda_4| \leq n^{\theta^*}$, when $j_3 \geq \left[3^{-1}n_{mk}^{\theta}\right]$ we have $\left|g(j_3, s)\frac{\lambda_4}{s^H}\right| \leq C|\lambda_4| s^{1-H} n^{\theta(H-1-1/\alpha)} \leq Cn^{\theta^*} n^{\epsilon(1-H)} n^{\theta(H-1-1/\alpha)} = Cn^{-1/\alpha-\upsilon}$, where $\upsilon = \theta(1-H) - \theta^* - \epsilon(1-H) > 0$. (Recall $g(j_3, s) = c_{j_3+1} + .. + c_{j_3+s}$). Similarly $\sum_{j_3=\left[3^{-1}n_{mk}^{\theta}\right]}^{q-1}\left|g(j_3, s)\frac{\lambda_4}{s^H}\right|^{\alpha} \leq C|\lambda_4|^{\alpha} s^{\alpha(1-H)} \sum_{j_3=\left[3^{-1}n_{mk}^{\theta}\right]}^{l-1} (j_3+1)^{\alpha H-\alpha-1} \leq Cn^{\alpha\theta^*} n^{\alpha\epsilon(1-H)} n^{-\theta\alpha(1-H)} = Cn^{-\alpha\upsilon}$, $\upsilon > 0$. Then using the Steps II and III of Lemma 13 (note $q \geq \left[n_{mk}^{\theta}\right]$ and $\left(q - \left[3^{-1}n_{mk}^{\theta}\right]\right)/q \geq 2/3$), when the integral is restricted to $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| \geq \eta q^{2\vartheta}, |\lambda_4| \leq n^{\theta^*}\right\}$, the sum over $(l, r, q, s) \in \Pi_n$ is bounded by Ce^{-dn^c} for some $c > 0$ and $d > 0$. Hence the integral can be restricted to $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| \leq \eta q^{2\vartheta}, |\lambda_4| \leq n^{\theta^*}\right\}$. As before the integral can be further restricted to $\left\{|\lambda_1| \leq \eta l^{4\vartheta}, |\lambda_2| \leq \eta r^{3\vartheta}, |\lambda_3| \leq \eta q^{2\vartheta}, |\lambda_4| \leq n^{\theta^*}\right\}$ by repeating the preceding procedure using the ideas of Lemmas 13 and 14. Now, using Lemma 5, the integral of (21) over $\left\{|\lambda_1| \leq \eta l^{4\vartheta}, |\lambda_2| \leq \eta r^{3\vartheta}, |\lambda_3| \leq \eta q^{2\vartheta}, \eta s^{1/\alpha} \leq |\lambda_4| \leq n^{\theta^*}\right\}$ is bounded by the integral over $\left\{\eta s^{1/\alpha} \leq |\lambda_4| \leq n^{\theta^*}\right\}$ of

$$\frac{C}{n^{2(1-H)}l^H r^H q^H s^H}\left(\frac{1}{l^{H-4\vartheta}} + \frac{1}{r^{H-3\vartheta}}\right)\left(\frac{1}{r^{H-3\vartheta}} + \frac{1}{q^{H-2\vartheta}}\right)\prod_{j_4=0}^{s-1}\left|\psi\left(\frac{\lambda_4 g(j_4)}{s^H}\right)\right|\left|\widehat{f}\left(\frac{\lambda_4}{s^H}\right)\right|,$$

which integral is, by Lemma 3, bounded by $\frac{C}{n^{2(1-H)}l^H r^H q^H s^H}\left(\frac{1}{l^{H-4\vartheta}} + \frac{1}{r^{H-3\vartheta}}\right)\left(\frac{1}{r^{H-3\vartheta}} + \frac{1}{q^{H-2\vartheta}}\right)\rho^s$, the sum of which is bounded by (14) (see (20)) for a suitable $\vartheta > 0$. Hence the integral of (21) can be restricted to $\left\{|\lambda_1| \leq \eta l^{4\vartheta}, |\lambda_2| \leq \eta r^{3\vartheta}, |\lambda_3| \leq \eta q^{2\vartheta}, |\lambda_4| \leq \eta s^{1/\alpha}\right\}$, in which case we get the bound (22). This completes the Case II.

Case III: The sum over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r, s \in A_n, q \in B_n\right\}$ of the integral of (21). First, the integral of (21) can be restricted to $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| < \infty, |\lambda_4| \leq \eta s^{\vartheta}\right\}$ as before because $s \geq n_{mk}^{\theta}$. In addition, as in the Case II, the range of q can be further restricted to $1 \leq q \leq [n^{\epsilon}]$ with $0 < \epsilon < \theta - \frac{\theta^*}{1-H}$. Then the rest of the arguments are similar to those of Case II, except that, instead of the integrals of $\prod_{j_4=0}^{s-1}\left|\psi\left(\frac{\lambda_4 g(j_4)}{s^H}\right)\right|\left|\widehat{f}\left(\frac{\lambda_4}{s^H}\right)\right|$ of the Case II, the integrals of $\prod_{j_3=0}^{q-1}\left|\psi\left(\frac{\lambda_3 g(j_3)}{q^H} + \frac{\lambda_4 g(j_3,s)}{s^H}\right)\right|\left|\widehat{f}\left(\frac{\lambda_3}{q^H} - \frac{\lambda_4}{s^H}\right)\right|$ over appropriate ranges of λ_3 will now be involved. However, because $j_3 < q \leq [n^{\epsilon}]$, $\left[n_{mk}^{\theta}\right] \leq s$ and $|\lambda_4| \leq \eta s^{\vartheta} \leq \eta n^{\vartheta}$, it follows from Steps II and III of the proof of Lemma 14 that, for a suitable $\vartheta > 0$, $\left|g(j_3, s)\frac{\lambda_4}{s^H}\right| \leq Cn^{-\upsilon}$ and $\sum_{j_3=0}^{q-1}\left|g(j_3, s)\frac{\lambda_4}{s^H}\right| \leq Cn^{-\upsilon}, \upsilon > 0$, and hence the arguments for both cases will be essentially the same. For instance, the required sum of the integral of (21) over $\left\{|\lambda_1| < \infty, |\lambda_2| < \infty, |\lambda_3| \geq n^{\theta^*}, |\lambda_4| \leq \eta s^{\vartheta}\right\}$ is, by Lemma 5, bounded by the sum of the integral of $\frac{1}{n^{2(1-H)}l^H r^H q^H s^H}\left|\prod_{j_3=0}^{q-1}\psi\left(\frac{\lambda_3 g(j_3)}{q^H} + \frac{\lambda_4 g(j_3,s)}{s^H}\right)\widehat{f}\left(\frac{\lambda_3}{q^H} - \frac{\lambda_4}{s^H}\right)\prod_{j_4=0}^{s-1}\psi\left(\frac{\lambda_4 g(j_4)}{s^H}\right)\right|$ over $\left\{|\lambda_3| \geq n^{\theta^*}, |\lambda_4| \leq \eta s^{\vartheta}\right\}$, which sum is, by essentially the same arguments used in (23) , bounded by the sum of $\frac{Cn^{-(1-H)}}{n^{2(1-H)}l^H r^H}\sum_{s=\left[n_{mk}^{\theta}\right]}^{n_{mk}}\frac{1}{s^H}\int_{\left\{|\lambda_4| \leq \eta s^{\vartheta}\right\}}\prod_{j_4=0}^{s-1}\left|\psi\left(\frac{\lambda_4 g(j_4)}{s^H}\right)\right| d\lambda_4$ over

$\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], r \in A_n\right\}$. Using Lemma 3, the last sum is bounded by (14).

The *CaseIV* where the sum is over $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right], q, s \in A_n, r \in B_n\right\}$ is almost the same as the previous Case III. This completes the verification when $2 \leq k \leq m$.

For the case $k = 1$, all the preceding cases go through if the range of l is $\left\{\left[n_{m1}^{\theta}\right] \leq l \leq [n_{m1}]\right\}$ instead of the earlier $\left\{\left[n\frac{k-1}{m}\right] < l \leq \left[n\frac{k}{m}\right]\right\}$. (Note that now $A_n = \left\{\left[n_{m1}^{\theta}\right] \leq i \leq n_{m1}\right\}$ and $B_n = \left\{1 \leq i \leq \left[n_{m1}^{\theta}\right]\right\}$). The remaining case $\left\{1 \leq l \leq \left[n_{m1}^{\theta}\right], r, q, s \in A_n\right\}$ is essentially the same as for instance the earlier Case III. This completes the verification of (R3).

6 References

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