A Nonlinear Regression Model with Integrated Time Series

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Preliminary version. This version does not include the conditional heteroscedasticity situation.

Introduction

In this paper we consider a model related in form to that treated in Park and Phillips (2001). To introduce the model, let $f(y,\theta)$ be a given function of y and θ (such that the conditions stated in Section 2 are satisfied; in particular it is assumed that $\int \left|\frac{\partial f(y,\theta)}{\partial\theta}\right|^2 dy < \infty$ for all θ).

Let (ε_j, η_j), $-\infty < j < \infty$, be iid such that $E[\varepsilon_1] = 0 = E[\eta_1]$, $0 < E[\varepsilon_1^2] < \infty$ and $0 < E[\eta_1^2] < \infty$.

Consider the model

$$X_t = f(Y_{t-1},\theta) + \zeta_t, \quad t = 1,...,n,$$

$$Y_t = Y_{t-1} + \xi_t, \quad t = 1,...,n,$$

for the observations (X_t, Y_t), $t = 1,...,n$, where Y_t, $t = 1,...,n$, form observable exogenous variables. The unknown parameter $\theta \in \Theta$, where Θ is a subset of the r-dimensional Euclidean space. Further, ξ_t is a linear process

$$\xi_t = \sum_{j=0}^{\infty} a_j \eta_{t-j}$$

with

$$\sum_{j=0}^{\infty} |a_j| < \infty, \qquad \sum_{j=0}^{\infty} a_j > 0, \tag{1}$$

and ζ_t is also a linear process

$$\zeta_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}$$

with

$$\sum_{j=0}^{\infty} |b_j| < \infty. \tag{2}$$

We shall suppose that ζ_t is invertible so that

$$\varepsilon_t = \sum_{j=0}^{\infty} c_j \zeta_{t-j}$$

with

$$\sum_{j=0}^{\infty} |c_j| < \infty. \tag{3}$$

It is convenient to mention at the outset itself that the coefficients c_j are assumed to satisfy the requirements that there are integers m_n such that

$$\frac{m_n}{\sqrt{n}} \to 0, \qquad n\left(\sum_{j=m_n}^{\infty} |c_j|\right)^2 \to 0, \qquad \text{and} \qquad \sum_{t=1}^{m_n}\left(\sum_{j=t}^{\infty}|c_j|\right)^2 = o\left(\sqrt{\frac{n}{m_n}}\right). \tag{4}$$

Two cases are particularly important. (a): In the case $|c_j|$ decays exponentially fast, one can choose m_n to be of the form $m_n = C\log n$. (b): In the case $|c_j| \sim Cj^{1+\lambda}$ with $\lambda > \frac{1}{2}$, one has $n\left(\sum_{j=m_n}^{\infty}|c_j|\right)^2 \sim Cnm_n^{-2\lambda}$. Hence in this case one can choose $m_n = n^{\frac{1}{2\lambda}+\epsilon}$ for some small ϵ with $0 < \epsilon < 1 - \frac{1}{2\lambda}$. Note that $\sum_{t=1}^{m_n}\left(\sum_{j=t}^{\infty}|c_j|\right)^2 \leq C$ because $\lambda > \frac{1}{2}$.

We shall consider estimation procedures based on the equation (with m_n as in (4))

$$\begin{aligned}
\varepsilon_t^* &= \sum_{j=0}^{(t-1)\wedge m_n} c_j \zeta_{t-j} = \sum_{j=0}^{(t-1)\wedge m_n} c_j \left(X_{t-j} - f\left(Y_{t-j-1},\theta\right)\right) \\
&= \sum_{j=0}^{(t-1)\wedge m_n} c_j X_{t-j} - \sum_{j=0}^{(t-1)\wedge m_n} c_j f\left(Y_{t-j-1},\theta\right) \\
&= X_t^* - f_t^*\left(\theta\right),
\end{aligned} \tag{5}$$

where we have let

$$X_t^* = \sum_{j=0}^{(t-1)\wedge m_n} c_j X_{t-j}, \qquad f_t^*\left(\theta\right) = \sum_{j=0}^{(t-1)\wedge m_n} c_j f\left(Y_{t-j-1},\theta\right).$$

More specifically, we shall consider $\widehat{\theta}_n$ defined by

$$\widehat{\theta}_n = \arg\sup_{\theta\in\Theta} \exp\left\{-\sum_{t=1}^{n}\rho\left(X_t^* - f_t^*\left(\theta\right)\right)\right\}$$

for a suitable function $\rho(x)$. The estimator $\widehat{\theta}_n$ corresponding to $\rho(x) = x^2$ is called the *Quasi MLE* or the nonlinear (generalized) least squares estimator. When $\rho(x) = |x|$ one obtains nonlinear least absolute deviation estimator.

More generally, the estimator $\widehat{\theta}_n$ may be called the nonlinear M-estimator, of which for example quantile regression estimators also become particular cases for suitable choices of $\rho(x)$.

The fact that $f(u,\theta)$ satisfies the conditions stated in Section 2, in particular the condition $\int \left|\frac{\partial f(y,\theta)}{\partial\theta}\right|^2 dy < \infty$ for all θ, makes the limit theory rather interesting and nonstandard, as will become clear below.

Note that in the equation (5), ε_t^* is in general not independent of $f_t^*(\theta) = \sum_{j=0}^{(t-1)\wedge m_n} c_j f(Y_{t-j-1},\theta)$, unfortunately, unless the sum $\varepsilon_t = \sum_{j=0}^{\infty} c_j\zeta_{t-j}$ is of finite order $\sum_{j=0}^{l} c_j\zeta_{t-j}$, for instance when ζ_t is an Autoregressive process, in which case $\varepsilon_t^* \equiv \varepsilon_t$ for all $(t-1)\wedge m_n \geq l$. But in many cases, for instance when ζ_t is a moving average process, the sum $\sum_{j=0}^{\infty} c_j\zeta_{t-j}$ will remain an infinite sum. Despite this we shall show that ε_t^* can effectively be treated independent of $\sum_{j=0}^{(t-1)\wedge m_n} c_j f(Y_{t-j-1},\theta)$ under the conditions in (4) on the coefficients c_j.

The model considered in Park and Phillips (2001) takes the form

$$X_t = f(Y_{t-1},\theta) + \varepsilon_t, \qquad Y_t = Y_{t-1} + \xi_t. \tag{6}$$

The estimator $\widehat{\theta}_n$ considered is then the ordinary LS estimator corresponding to $\rho(x) = x^2$. In this case (5) holds with $\varepsilon_t^* = \varepsilon_t$, $f_t^*(\theta) = f(Y_{t-1},\theta)$ and $X_t^* = X_t$. In addition ε_t is independent of regression function $f(Y_{t-1},\theta)$. (Also the conditions imposed in Park and Phillips (2001) are much stronger than the present ones.)

Now suppose that the regression function $f(Y_{t-1},\theta)$ in (6) is changed to $f(Y_t,\theta)$. Then ε_t is no longer independent of the regression function $f(Y_t,\theta)$ unless ε_j and η_j are independent. In addition, judging from the usual limit theory for linear time series models, one would expect that such a change would have no consequence, but rather interestingly it turns out that it indeed changes the forms of the limiting distributions of the estimators. The same problem arises when we consider the ordinary LS estimator of θ based on the equation of the form (6) but with ε_t replaced by the linear process ζ_t, even with the regression function $f(Y_{t-1},\theta)$. These problems will be treated separately in Jeganathan and Phillips (2007).

We let $\mathcal{F}_t$ = sigma-field of $(\varepsilon_s, -\infty < s \leq t,\ Y_0,\ Y_1, ..., Y_t)$. Note that ε_t is independent of $\mathcal{F}_{t-1}$.

The Main Results

Throughout below θ_0 stands for the true parameter, and all the probability statements are under θ_0.

Also, for convenience we restrict Θ to be a subset of the real line. With minor notational changes all the results will hold for the multidimensional parameter case.

Further we make the assumption, without further mentioning, that Θ is *a bounded subset.*

Regarding the function $f(y,\theta)$ involved in specifying the regression function

$f\left(Y_{t-1}, \theta\right)$, we suppose that for [Lebesgue] almost all y, the derivative

$$g(y, \theta)=\frac{\partial f(y, \theta)}{\partial \theta}$$

exists for all θ, satisfying the following assumptions:

$$\int\left(\sup_{\theta}|g(y, \theta)|\right)^{2} dy<\infty, \tag{7}$$

$$\inf_{\theta} \int|g(y, \theta)|^{2} dy>0, \tag{8}$$

$$\int\left(\sup_{\left|\theta-\theta^{\prime}\right| \leq \delta}\left|g(y, \theta)-g\left(y, \theta^{\prime}\right)\right|\right)^{2} dy \rightarrow 0 \quad \text { as } \delta \rightarrow 0, \tag{9}$$

In addition we shall assume that

$$\int|g(y, \theta)|^{4} dy<\infty \quad \text { for each } \theta \in \Theta, \tag{10}$$

$$\int\left|E\left[e^{i u \eta_{1}}\right]\right|^{2} du<\infty. \tag{11}$$

The condition (11) will be relaxed to the weaker (Cramér's) condition

$$\limsup_{|u| \rightarrow \infty}\left|E\left[e^{i u \eta_{1}}\right]\right|<1 \tag{12}$$

when the following additional conditions are satisfied:

$$\int\left|\sup_{|u-y| \leq \delta}|g(u, \theta)|\right|^{4} dy<\infty \quad \text { for some } \delta>0 \text { for each } \theta \in \Theta \tag{13}$$

and

$$\int\left|\sup_{|u-y| \leq \delta}|g(u, \theta)|-\inf_{|u-y| \leq \delta}|g(u, \theta)|\right| dy \rightarrow 0 \quad \text { as } \delta \rightarrow 0 \text { for each } \theta \in \Theta. \tag{14}$$

To state the main result recall that,

under the condition (1), the finite dimensional distributions of the process $s \longmapsto \frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \xi_{t}$ converge in distribution to those of the process $B(s)$, where $B(s)$ is a Brownian motion with variance $\left(\sum_{j=0}^{\infty} a_{j}\right)^{2}$.

Let $\mathcal{L}$ be the *local time* of $B(s)$ at 0 up to the time $s=1$. It is known that

$$\mathcal{L}>0 \text { almost surely. } \tag{15}$$

Further, define the constant

$$Q(\theta) = \frac{1}{2\pi} \int |\widehat{g}(\mu,\theta)|^2 \, E\left[\left|\sum_{j=0}^{\infty} c_j e^{i\mu Y_j}\right|^2\right] d\mu, \tag{16}$$

where $\widehat{g}(\mu,\theta)$ is the Fourier transform of $g(y,\theta)$. Here we note that

$$\begin{aligned} E\left[\left|\sum_{j=0}^{\infty} c_j e^{i\mu Y_j}\right|^2\right] &= \sum_{j=0}^{\infty}\sum_{r=0}^{\infty} c_j c_r E\left[e^{-i\mu(Y_r - Y_j)}\right] \\ &= \sum_{j=0}^{\infty} c_j^2 + 2\sum_{j=0}^{\infty}\sum_{r>j}^{\infty} E\left[e^{-i\mu Y_{r-j}}\right], \end{aligned}$$

where we have used the fact that $E\left[e^{-i\mu(Y_r-Y_j)}\right] = E\left[e^{-i\mu Y_{r-j}}\right]$, which holds in view of the stationarity of the increments of the process Y_r, $r \geq 0$.

Note that, in view of the equality $\int |\widehat{g}(\mu,\theta)|^2 d\mu = \int |g(y,\theta)|^2 dy$, the condition (8) is the same as the requirement

$$\inf_{\theta} Q(\theta) > 0. \tag{17}$$

We shall let

$$S_{nt}(\theta) = \sum_{j=0}^{(t-1)\wedge m_n} c_j g(Y_{t-j-1},\theta)$$

and

$$\delta_n = n^{-\frac{1}{4}}.$$

Proposition 1. *Let $Q(\theta)$ and the local time L be as before. Assume that θ is such that either $\int |g(y,\theta)|^2 dy < \infty$ together with (10) and (11) hold or $\int \left|\sup_{|u-y|\leq\delta} |g(u,\theta)|\right|^2 dy < \infty$ for some $\delta > 0$ together with (12) - (14) hold. Then*

$$\left(\delta_n^2 \sum_{t=1}^{n} S_{nt}^2(\theta),\ \delta_n \sum_{t=1}^{n} S_{nt}(\theta)\varepsilon_t\right) \Longrightarrow \left(\mathcal{L}Q(\theta), Z\sqrt{\mathcal{L}Q(\theta)}\right),$$

where Z is normal with mean 0 and variance $E\left[\varepsilon_1^2\right]$ and is independent of $\mathcal{L}$.

To state the next main result, let

$$U_{n,\theta_0} = \delta_n^{-1}(\Theta - \theta_0),$$

and assume that

$$U_{n,\theta_0} \uparrow U. \tag{18}$$

Note that in the case θ_0 is in the interior of Θ, one has $U = \mathbb{R}^r$.

Theorem 2. *Suppose that the estimator $\widehat{\theta}_n$ is a Quasi MLE, that is, corresponds to $\rho(x) = x^2$. Assume that the conditions (7) - (9) hold. In addition assume that the conditions of Proposition 1 hold.*

Then if θ_0 is an interior point of Θ, the difference

$$\delta_n^{-1}\left(\widehat{\theta}_n - \theta_0\right) - \left(\delta_n^2 \sum_{t=1}^n S_{nt}^2(\theta_0)\right)^{-1} \delta_n \sum_{t=1}^n S_{nt}(\theta_0)\,\varepsilon_t \xrightarrow{p} 0.$$

In general when (18) holds, one has

$$\delta_n^{-1}\left(\widehat{\theta}_n - \theta_0\right) \Longrightarrow \arg\sup_{u \in U}\left\{uZ\sqrt{\mathcal{L}Q(\theta_0)} - \frac{u^2}{2}\mathcal{L}Q(\theta_0)\right\}$$

where the random quantities involved on the right hand side are as in Proposition 1.

A main step in establishing Theorem 2 consists of establishing the fact that

$$\delta_n^{-1}\left(\widehat{\theta}_n - \theta_0\right) = O_p(1). \tag{19}$$

According to the Manuscript (2007, Theorems 1 and 2, and Remark 3), we shall need to verify the following conditions (A1) - (A3). Recall that $U_{n,\theta_0} = \delta_n^{-1}(\Theta - \theta_0)$.

(A1). There are nonnegative random variables B_n such that

$$\sum_{t=1}^n \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0)\right)^2 \geq B_n |u|^2$$

for all u in U_{n,θ_0}, with

$$B_n^{-1} = O_p(1)$$

(A2). There are nonnegative $\mathcal{F}_{t-1}$ measurable random variables Δ_{nt} such that

$$|f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')|^2 \leq (|u| + |u'|)\min\left(1, |u - u'|^2\right)\Delta_{nt}$$

for all u and u' in U_{n,θ_0}, with

$$\sum_{t=1}^n \Delta_{nt} = O_p(1) \qquad \text{and} \qquad \max_{1 \leq t \leq n} \Delta_{nt} = o_p(1).$$

(A3). There are random variables C_n such that

$$\left|\sum_{t=1}^{n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)\left(\varepsilon_t-\varepsilon_t^*\right)\right| \leq \left|u-u'\right| C_n \qquad (20)$$

for all u and u' in U_{n,θ_0}, with

$$C_n = O_p\left(1\right).$$

Further, once the Proposition 1 is obtained, it and (19) together with the verification of the following requirement (A4) will give the limiting statements in Theorem 2. Note that in view of (18), every u in U will be in U_{n,θ_0} for all sufficiently large n.

(A4). (20) holds but with $C_n = o_p\left(1\right)$. In addition, for all u in U,

$$\sum_{t=1}^{n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)-u\delta_n S_{nt}\left(\theta_0\right)\right)^2 \xrightarrow{p} 0. \qquad (21)$$

Proofs

We start with the following result, which is essentially Theorem 3 in Jeganathan (2006).

Proposition 3. *Assume that $\theta_1,...,\theta_s$ are such that the assumptions of Proposition 1 hold for each of $\theta_1,...,\theta_s$. Let*

$$\ell_k(y) = g\left(y,\theta_k\right), \quad k=1,...,p.$$

Then for any constants $\alpha_1,...,\alpha_r$, $\beta_1,...,\beta_\nu$.

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left\{\sum_{p=1}^{\nu}\alpha_p\left(\sum_{k=1}^{s}\ell_k^2(Y_{j-p})\right)+\sum_{p=1}^{\nu}\sum_{q=1}^{p-1}\beta_p\beta_r\left(\sum_{k=1}^{s}\ell_k(Y_{j-p})\ell_k(Y_{j-q})\right)\right\}$$
$$\Longrightarrow \quad \mathcal{L}\frac{1}{2\pi}\sum_{k=1}^{s}\int\left|\widehat{\ell_k}\left(\mu\right)\right|^2\left(\sum_{p=1}^{\nu}\alpha_p+2\sum_{p=1}^{\nu}\sum_{q=1}^{p-1}\beta_p\beta_q E\left[e^{-i\mu Y_{p-q}}\right]\right)d\mu,$$

where $\widehat{\ell_k}\left(\mu\right)$ is the Fourier transform of $\ell_k\left(y\right)$. (Recall that $Y_j=\sum_{t=1}^{j}\xi_t+Y_0$)

Proof. First note that for any $1<p\leq\nu$ and $1<k\leq s$, $\sum_{j=1}^{n}\ell_k^2(Y_{j-p}) = \sum_{j=2-p}^{n-p+1}\ell_k^2(Y_{j-1})$, and hence the difference $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\ell_k(Y_{j-p})-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\ell_k(Y_{j-1}) \xrightarrow{p} 0$. Similarly, the difference $\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\ell_k(Y_{j-p})\ell_k(Y_{j-q})-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\ell_k(Y_{j-1})\ell_k(Y_{j-q+p-1}) \xrightarrow{p} 0$. Therefore, it is enough to consider the asymptotic distribution of

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left\{\sum_{p=1}^{\nu}\alpha_p\left(\sum_{k=1}^{s}\ell_k^2(Y_{j-1})\right)+\sum_{p=1}^{\nu}\sum_{q=1}^{p-1}\beta_p\beta_q\left(\sum_{k=1}^{s}\ell_k(Y_{j-1})\ell_k(Y_{j-(p-q)+1})\right)\right\}.$$

We have

$$\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\begin{pmatrix}\ell_k^2(Y_{j-1})\\ \ell_k(Y_{j-2})\ell_k(Y_{j-1})\\ \vdots\\ \ell_k(Y_{j-r})\ell_k(Y_{j-1})\end{pmatrix},,\quad k=1,...,s\right)$$

$$\Longrightarrow\quad \mathcal{L}\frac{1}{2\pi}\left(\begin{pmatrix}\int\left|\widehat{\ell_k}\left(\mu\right)\right|^2 d\mu\\ \int\left|\widehat{\ell_k}\left(\mu\right)\right|^2 E\left[e^{-i\mu Y_1}\right]d\mu\\ \vdots\\ \int\left|\widehat{\ell_k}\left(\mu\right)\right|^2 E\left[e^{-i\mu Y_{r-1}}\right]d\mu\end{pmatrix},\quad k=1,...,p\right).$$

The marginal convergence in this statement follows directly from Jeganathan (2006, Theorem 3) and the joint convergence follows in exactly the same way. (a remark to this effect to be added in the revision (for ET) of this paper)). This completes the proof. ■

Proposition 4. *Assume that the conditions of Proposition 1 hold. Then for each fixed positive integer* ν,

$$\left(n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(\sum_{j=0}^{\nu}c_j g\left(Y_{t-j-1},\theta\right)\right)^2,\ n^{-\frac{1}{4}}\sum_{t=1}^{n}\left(\sum_{j=0}^{\nu}c_j g\left(Y_{t-j-1},\theta\right)\right)\varepsilon_t\right)$$
$$\Longrightarrow\quad\left(\mathcal{L}Q_\nu\left(\theta\right),Z\sqrt{\mathcal{L}Q_\nu\left(\theta\right)}\right),$$

where Z *is normal with mean* 0 *and variance* $E\left[\varepsilon_1^2\right]$ *) and is independent of* $\mathcal{L}$, *and*

$$Q_\nu\left(\theta\right)=\frac{1}{2\pi}\int\left|\widehat{g}\left(\mu,\theta\right)\right|^2 E\left[\left|\sum_{j=0}^{\nu}c_j e^{i\mu S_j}\right|^2\right]d\mu.$$

Proof. In view of the stationarity of the process Y_j, the distribution of $Y_r - Y_j$ is the same as that of Y_{r-j} for $r > j$. Hence

$$E\left[\left|\sum_{j=0}^{q}c_j e^{i\mu Y_j}\right|^2\right]=\sum_{j=0}^{q}c_j^2+2\sum_{j=0}^{q}\sum_{r>j}^{q}E\left[e^{-i\mu Y_{r-j}}\right],$$

so that the marginal convergence of the first component $n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(\sum_{j=0}^{\nu}c_j g\left(Y_{t-j-1},\theta\right)\right)^2$ is a consequence of Proposition 3. The joint convergence is contained in Jeganathan (2006). ■

To proceed further we shall also need the following

Lemma 5. *One has*

$$E\left[|g(Y_j,\theta)|^2\right] \le \frac{C}{\sqrt{j}}, \qquad E\left[|g(Y_j,\theta)|\,|g(Y_r,\theta)|\right] \le \frac{C}{\sqrt{j}\sqrt{r-j}} \text{ for } r > j.$$

Proof. These bounds are contained in Jeganathan (2006, see (??) and (??)).
■

Lemma 6. *For each* θ*,*

$$\lim_{\nu\to\infty} \limsup_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g(Y_{t-j-1},\theta) - \sum_{j=0}^{\nu} c_j g(Y_{t-j-1},\theta)\right|^2 > \delta\right] = 0$$

for all $\delta > 0$*.*

Proof. Note that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{t=1}^{2m_n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g(Y_{t-j-1},\theta)\right|^2 &= \frac{1}{\sqrt{n}}\sum_{t=1}^{m_n}\left|\sum_{j=0}^{t-1} c_j g(Y_{t-j-1},\theta)\right|^2 \\ &\quad + \frac{1}{\sqrt{n}}\sum_{t=m_n+1}^{2m_n}\left|\sum_{j=0}^{m_n} c_j g(Y_{t-j-1},\theta)\right|^2\end{aligned}$$

We have, using Lemma 5,

$$E\left[\left|\sum_{j=0}^{t} c_j g(Y_{t-j},\theta)\right|^2\right] \le \left|\sum_{j=0}^{t}|c_j|\right|^2 \le C,$$

and hence

$$E\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{m_n}\left|\sum_{j=0}^{t-1} c_j g(Y_{t-j-1},\theta)\right|^2\right] \le \frac{m_n}{\sqrt{n}} \to 0.$$

In the same way

$$E\left[\frac{1}{\sqrt{n}}\sum_{t=m_n+1}^{2m_n}\left|\sum_{j=0}^{m_n} c_j g(Y_{t-j-1},\theta)\right|^2\right] \le \frac{m_n}{\sqrt{n}} \to 0,$$

and $E\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{2m_n}\left|\sum_{j=0}^{\nu} c_j g(Y_{t-j-1},\theta)\right|^2\right] \to 0$ for each ν.

Thus, noting that

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{t=2m_n+1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g(Y_{t-j-1},\theta) - \sum_{j=0}^{\nu} c_j g(Y_{t-j-1},\theta)\right|^2 \\ =\ &\frac{1}{\sqrt{n}}\sum_{t=2m_n+1}^{n}\left|\sum_{j=\nu+1}^{m_n} c_j g(Y_{t-j-1},\theta)\right|^2,\end{aligned}$$

it is enough to show that

$$\lim_{\nu\to\infty}\limsup_{n\to\infty} E\left[\frac{1}{\sqrt{n}}\sum_{t=2m_n+1}^{n}\left|\sum_{j=\nu}^{m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2\right] = 0. \tag{22}$$

For this purpose, we have, using Lemma 5,

$$\begin{aligned}
& E\left[\left|\sum_{j=\nu}^{m_n} c_j g\left(Y_{t-j},\theta\right)\right|^2\right] \\
\leq\ & \sum_{j=\nu}^{m_n}\left|c_j\right| E\left[\left|g\left(Y_{t-j},\theta\right)\right|^2\right] + \sum_{j=\nu}^{m_n}\sum_{r=\nu}^{j-1}\left|c_j\right|\left|c_r\right| E\left[\left|g\left(Y_{t-j},\theta\right)\right|\left|g\left(Y_{t-r},\theta\right)\right|\right] \\
\leq\ & C\sum_{j=\nu}^{m_n}\left|c_j\right|\frac{1}{\sqrt{t-j}} + \sum_{j=\nu}^{m_n}\sum_{r=\nu}^{j-1}\left|c_j\right|\left|c_r\right|\frac{1}{\sqrt{t-j}}\frac{1}{\sqrt{j-r}}.
\end{aligned}$$

Here

$$\sum_{j=\nu}^{m_n}\left|c_j\right|\frac{1}{\sqrt{t-j}} \leq \sum_{j=\nu}^{m_n}\left|c_j\right|\frac{1}{\sqrt{t-m_n}} \leq C\frac{1}{\sqrt{t}}\sum_{j=\nu}^{m_n}\left|c_j\right| \qquad \text{because } m_n \leq \frac{t}{2},$$

and, using this inequality and the fact $\sum_{r=\nu}^{j-1}\left|c_r\right|\frac{1}{\sqrt{j-r}} \leq \sum_{r=\nu}^{j-1}\left|c_r\right| \leq C$,

$$\begin{aligned}
\sum_{j=\nu}^{m_n}\sum_{r=\nu}^{j-1}\left|c_j\right|\left|c_r\right|\frac{1}{\sqrt{t-j}}\frac{1}{\sqrt{j-r}} &\leq \sum_{j=\nu}^{m_n}\left|c_j\right|\frac{1}{\sqrt{t-j}}\sum_{r=\nu}^{j-1}\left|c_r\right|\frac{1}{\sqrt{j-r}} \\
&\leq C\sum_{j=\nu}^{m_n}\left|c_j\right|\frac{1}{\sqrt{t-j}} \leq C\frac{1}{\sqrt{t}}\sum_{j=\nu}^{m_n}\left|c_j\right| \qquad \text{because } m_n \leq \frac{t}{2}.
\end{aligned}$$

Thus

$$E\left[\frac{1}{\sqrt{n}}\sum_{t=2m_n+1}^{n}\left|\sum_{j=\nu}^{m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2\right] \leq C\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{t}}\right)\sum_{j=\nu}^{\infty}\left|c_j\right|.$$

Because $\sum_{j=q}^{\infty}\left|c_j\right| \to 0$ as $q \to \infty$ by (3), and $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{t}} \leq C$, this proves (22), completing the proof of the Lemma. ■

For the proof of the next result, we note that for any real a and b, and $\kappa > 0$,

$$\begin{aligned}
2\left|a^2 - b^2\right| &= 2\left|a-b\right|\left|a+b\right| \leq \kappa\left|a-b\right|^2 + \frac{1}{\kappa}\left|a+b\right|^2 \\
&\leq \left(\kappa + \frac{2}{\kappa}\right)\left|a-b\right|^2 + \frac{8}{\kappa}b^2.
\end{aligned} \tag{23}$$

Lemma 7. *For each θ,*

$$\lim_{\nu\to\infty}\limsup_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2-\left|\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right|^2\right|>\varepsilon\right]=0.$$

Proof. Taking $a=\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)$ and $b=\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)$ in (23), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2-\left|\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right|^2\right| \\
\leq\ & \frac{1}{2}\left(\kappa+1\right)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)-\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right|^2 \\
&+\frac{4}{\kappa}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right|^2 \\
=\ & C_n\left(\kappa,\nu\right)+D_n\left(\kappa,\nu\right), \text{ say..}
\end{aligned}$$

According to Proposition 4, $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right|^2$ converges in distribution to $\mathcal{L}Q_\nu\left(\theta\right)$ as $n\to\infty$, where $Q_\nu\left(\theta\right)\to Q\left(\theta\right)$ as $\nu\to\infty$. Hence

$$\lim_{\kappa\to\infty}\lim_{\nu\to\infty}\limsup_{n\to\infty} P\left[D_n\left(\kappa,\nu\right)>\delta\right]=0 \quad \text{for each } \delta>0.$$

According to Lemma 6, $\lim\limits_{\nu\to\infty}\limsup\limits_{n\to\infty} P\left[C_n\left(\kappa,\nu\right)>\delta\right]=0$ for each $\kappa>0$ and $\delta>0$. It then follows that

$$\lim_{\kappa\to\infty}\lim_{\nu\to\infty}\limsup_{n\to\infty} P\left[C_n\left(\kappa,\nu\right)+D_n\left(\kappa,\nu\right)>\delta\right]=0 \quad \text{for each } \delta>0.$$

Because the left side does not depend on κ, the proof of the lemma is complete. ■

Proposition 8. *Let $Q\left(\theta\right)$ be as in (16) and $\mathcal{L}$ be the local time as before. Then for each finite θ_1,...,θ_p,*

$$\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta_k\right)\right|^2; k=1,...,p\right)\Longrightarrow\left(\mathcal{L}Q\left(\theta_k\right); k=1,...,p\right).$$

Proof. The result follows in view of Lemma 7, Proposition 4 and the fact that

$$E\left[\left|\sum_{j=0}^{\nu} c_j e^{i\mu Y_j}\right|^2\right]\to E\left[\left|\sum_{j=0}^{\infty} c_j e^{i\mu Y_j}\right|^2\right] \qquad \text{as } \nu\to\infty.$$

■

We can now present the proof of Proposition 1

Proof of Proposition 1. Because $\left(\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right) - \sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right)\varepsilon_t$ are martingale differences, Lemma 7 entails that

$$\lim_{\nu\to\infty}\ \limsup_{n\to\infty} P\left[\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right) - \sum_{j=0}^{\nu} c_j g\left(Y_{t-j-1},\theta\right)\right)\varepsilon_t\right| > \delta\right] = 0.$$

This together with Lemma 8 gives the result in view of Proposition 4, in the same manner as in the proof of Proposition 8. ■

Next, in order to verify the conditions (A1) - (A4), we need the following result.

Lemma 9. *For each* $\epsilon > 0$,

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty} P\left[\begin{array}{c}\sup_{|\theta-\theta'|\le\delta}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2\right.\\ \left.-\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta'\right)\right|^2\right| > \epsilon\end{array}\right] = 0.$$

Proof. First consider

$$\begin{aligned}&\sup_{|\theta-\theta'|\le\delta}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j\left(g\left(Y_{t-j-1},\theta\right) - g\left(Y_{t-j-1},\theta'\right)\right)\right|^2\\ \le\ &\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}\left|c_j\right| g_\delta^*\left(Y_{t-j-1}\right)\right|^2,\end{aligned}$$

where we let

$$g_\delta^*\left(y\right) = \sup_{|\theta-\theta'|\le\delta}\left|g\left(y,\theta\right) - g\left(y,\theta'\right)\right|.$$

For each $\delta > 0$, we have in the same way as in Proposition 8

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}\left|c_j\right| g_\delta^*\left(Y_{t-j-1}\right)\right|^2 \Longrightarrow_{\mathcal{L}} \frac{1}{2\pi}\int\left|\widehat{g_\delta^*}\left(\mu\right)\right|^2 E\left[\left|\sum_{j=0}^{\infty}\left|c_j\right| e^{i\mu Y_j}\right|^2\right] d\mu. \tag{24}$$

Here

$$\begin{aligned}\frac{1}{2\pi}\int\left|\widehat{g_\delta^*}\left(\mu\right)\right|^2 E\left[\left|\sum_{j=0}^{\infty}\left|c_j\right| e^{i\mu Y_j}\right|^2\right] d\mu\ &\le\ \frac{C}{2\pi}\int\left|\widehat{g_\delta^*}\left(\mu\right)\right|^2 d\mu\\ &=\ C\int\left|g_\delta^*\left(y\right)\right|^2 dy \to 0\ \text{ as } \delta\to 0,\quad \text{by (9)}.\end{aligned}$$

Thus, for each $\epsilon > 0$,

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}|c_j|\, g_\delta^*\left(Y_{t-j-1}\right)\right|^2 > \epsilon\right] = 0. \tag{25}$$

Then taking $a = \sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)$ and $b = \sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta'\right)$ in the inequality (23), we see that

$$\begin{aligned}
&\sup_{|\theta-\theta'|\le\delta}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2 - \left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta'\right)\right|^2\right| \\
\le\ & \frac{1}{2}\left(\kappa+1\right)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}|c_j|\, g_\delta^*\left(Y_{t-j-1}\right)\right|^2 \\
& + \frac{4}{\kappa}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}|c_j|\sup_\theta\left|g\left(Y_{t-j-1},\theta\right)\right|\right|^2 \\
=\ & C_n\left(\delta,\kappa\right) + D_n\left(\kappa\right),\ \text{say.}
\end{aligned}$$

Now similar to (24), and in view of (7),

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n}|c_j|\sup_\theta\left|g\left(Y_{t-j-1},\theta\right)\right|\right|^2 \text{ converges in distribution,} \tag{26}$$

and hence $\lim_{\kappa\to\infty}\ \limsup_{n\to\infty} P\left[D_n\left(\kappa\right) > \epsilon\right] = 0$. In view of (25), $\lim_{\delta\to 0}\ \limsup_{n\to\infty} P\left[C_n\left(\delta,\kappa\right) > \epsilon\right] = 0$ for each $\kappa > 0$. It then follows that

$$\lim_{\kappa\to\infty}\ \lim_{\delta\to 0}\ \limsup_{n\to\infty} P\left[C_n\left(\delta,\kappa\right) + D_n\left(\kappa\right) > \epsilon\right] = 0.$$

This completes the proof of the lemma. ■

We can now verify the requirements (A1) and (A2). Note that as a consequence of Proposition 8 and Lemma 9 (recall that the parameter space Θ is assumed to be a bounded set), taking into account (15) and (17),

$$\inf_\theta \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta\right)\right|^2 \Longrightarrow \mathcal{L}\inf_\theta Q\left(\theta\right) > 0 \text{ almost surely.}$$

This verifies (A1) with $\delta_n = n^{-\frac{1}{4}}$, in view of

$$\left|f_t^*\left(\theta\right) - f_t^*\left(\theta'\right)\right| = \left|\theta-\theta'\right|\left|\sum_{j=0}^{(t-1)\wedge m_n} c_j g\left(Y_{t-j-1},\theta^*\right)\right| \tag{27}$$

for a θ^* lying in between θ and θ'.

Regarding the verification of (A2), one can take

$$\Delta_{nt} = \frac{1}{\sqrt{n}} \left| \sum_{j=0}^{(t-1)\wedge m_n} |c_j| \sup_{\theta} |g(Y_{t-j-1}, \theta)| \right|^2 \tag{28}$$

and then invoke (26).

The next result verifies (A3).

Lemma 10. *The requirement (A3) holds with* $C_n = o_p(1)$.

Proof. First note that

$$\begin{aligned} & \left| \sum_{t=1}^{n} \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')\right) \left(\sum_{j=(t-1)\wedge m_n+1}^{\infty} c_j \zeta_{t-j} \right) \right|^2 \\ \leq & \left| \sum_{t=1}^{m_n} \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')\right) \left(\sum_{j=t}^{\infty} c_j \zeta_{t-j} \right) \right|^2 \\ & \left| \sum_{t=m_n+1}^{n} \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')\right) \left(\sum_{j=m_n+1}^{\infty} c_j \zeta_{t-j} \right) \right|^2 . \end{aligned}$$

We have, using Cauchy - Schwarz inequality,

$$\begin{aligned} & \left| \sum_{t=m_n+1}^{n} \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')\right) \left(\sum_{j=m_n+1}^{\infty} c_j \zeta_{t-j} \right) \right|^2 \\ \leq & \left(\sum_{t=m_n+1}^{n} \left(f_t^*(\theta_0 + \delta_n u) - f_t^*(\theta_0 + \delta_n u')\right)^2 \right) \left(\sum_{t=m_n+1}^{n} \left(\sum_{j=m_n+1}^{\infty} c_j \zeta_{t-j} \right)^2 \right). \end{aligned}$$

Now recall that for any integer $m \geq 0$,

$$E\left[\left(\sum_{j=m}^{\infty} c_j \zeta_{t-j} \right)^2 \right] = C \int_{-\pi}^{\pi} \left(\sum_{j=m}^{\infty} c_j e^{ij\phi} \right)^2 |h(\phi)|^2 \, d\phi,$$

where $h(\phi)$ is the spectral density function of the process ζ_t, that is, $h(\phi) = \sum_{j=0}^{\infty} b_j e^{ij\phi}$. In view of (2), we have $|h(\phi)| \leq C$. Thus

$$E\left[\left(\sum_{j=m}^{\infty} c_j \zeta_{t-j} \right)^2 \right] \leq C \left(\sum_{j=m}^{\infty} |c_j| \right)^2 . \tag{29}$$

Then, in view of second condition in (4),

$$E\left[\sum_{t=m_n+1}^{n}\left(\sum_{j=m_n+1}^{\infty}c_j\zeta_{t-j}\right)^2\right] = \sum_{t=m_n+1}^{n}E\left[\left(\sum_{j=m_n+1}^{\infty}c_j\zeta_{t-j}\right)^2\right]$$

$$\leq nE\left[\left(\sum_{j=m_n+1}^{\infty}c_j\zeta_{1-j}\right)^2\right] = o\left(1\right).$$

Further, with Δ_{nt} as defined in (28),

$$\sum_{t=1}^{n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)^2 \leq \left|u-u'\right|^2\sum_{t=1}^{n}\Delta_{nt} = \left|u-u'\right|^2 O_p\left(1\right).$$

Thus

$$\left|\sum_{t=m_n+1}^{n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)\left(\sum_{j=m_n+1}^{\infty}c_j\zeta_{t-j}\right)\right|^2 \leq \left|u-u'\right|^2 o_p\left(1\right). \tag{30}$$

In the same way

$$\left|\sum_{t=1}^{m_n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)\left(\sum_{j=t}^{\infty}c_j\zeta_{t-j}\right)\right|^2$$

$$\leq \left(\sum_{t=1}^{m_n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)^2\right)\left(\sum_{t=1}^{m_n}\left(\sum_{j=t}^{\infty}c_j\zeta_{t-j}\right)^2\right).$$

Here

$$\sum_{t=1}^{m_n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)^2 \leq \left|u-u'\right|^2\sum_{t=1}^{m_n}\Delta_{nt} = \left|u-u'\right|^2 O_p\left(\sqrt{\frac{m_n}{n}}\right),$$

using the fact $\sum_{t=1}^{m_n}\Delta_{nt} = O_p\left(\sqrt{\frac{m_n}{n}}\right)$. Further, according to (29) and the last condition in (4)

$$\sum_{t=1}^{m_n}\left(\sum_{j=t}^{\infty}c_j\zeta_{t-j}\right)^2 = o_p\left(\sqrt{\frac{n}{m_n}}\right).$$

Therefore it follows that

$$\left|\sum_{t=1}^{m_n}\left(f_t^*\left(\theta_0+\delta_n u\right)-f_t^*\left(\theta_0+\delta_n u'\right)\right)\left(\sum_{j=t}^{\infty}c_j\zeta_{t-j}\right)\right|^2 \leq \left|u-u'\right|^2 o_p\left(1\right),$$

This together with (30) verifies (A3). ∎

It remains to verify (A4). The fact that (20) holds with $C_n = o_p(1)$ is already verified in Lemma 10. The verification of (21) is an easy consequence of (27) and Lemma 9.

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