

# Stochastic Volatility, Event Risk and Transaction Costs

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Even though the effects of transaction costs are well documented, the majority of the literature on optimal trading policies and portfolio allocation fails to take these effects into account. Probably the main reason for this is the increased complexity of the Hamilton-Jacobi-Bellmann equation when transaction costs are included. In this paper we show that it is possible to circumvent this problem. We find the optimal trading strategies for a model where the risky asset is allowed to have stochastic volatility and jumps in both prices and volatility and taking transaction costs into account. We find that the liquidity premium is up to ten times higher than results in standard literature and highly depending on the size of the transaction costs. The presence of transaction costs can partly explain the equity premium puzzle.

Transaction costs have substantial effects on the optimal trading and portfolio allocation of an investor, since its presence changes the trading strategy from continuous to more infrequent trading. This is shown by several authors, including Constantinides (1986), Akian et al. (1996), Framstad et al. (2001) and Liu and Loewenstein (2002). The reason for this is that contin-

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uous trading would imply an instant bankruptcy. This implies that if the investor faces transaction costs, he will let the stock-to-wealth ratio fluctuate around its optimum before a trade is initiated. This fluctuation can be illustrated by an interval where the investor does not trade as long as the ratio of the risky asset to wealth is within this interval. If the stock-to-wealth ratio reaches or jumps beyond the boundaries, then the investor trades the minimal amount in order to keep the portfolio value inside this interval. The size of this interval is not dependent on the level of wealth. In the following we will require the portfolio to be self-financing and its value is not allowed to be negative. In other words, it is not possible to add or withdraw funds from the portfolio so any increase or decrease of the wealth comes entirely from the trading strategies and thus from how the portfolio is reallocated. Classical literature that accounts for transaction costs, e.g. Constantinides (1986), shows that the trading strategies tend to become more buy-and-hold, in line with our intuition of investor behavior. Davis and Norman (1990) also include transaction costs in their portfolio optimization problem, and show that the non-intervention region is a wedge-shaped region which starts in the origin. The implication of this is that the optimal portfolio allocation is no longer a scalar, but rather an interval. We will denote this interval as the above mentioned non-intervention region. The width of this interval depends on the size of the transaction costs, so when the costs converge to zero, the non-intervention interval converges to the *Merton proportion*. The *Merton proportion* is the optimal fraction of stock-to-wealth in the no-cost case, and this proportion is usually inside the no-transaction region. As the name of this interval suggests, the investor does not reallocate the portfolio if the fraction of stock-to-wealth is inside this interval. The Merton proportion will hereafter be denoted as  $\pi^*$  and the boundaries of the non-intervention region will be  $\underline{\pi}$  and  $\bar{\pi}$  for the lower and upper boundary, respectively. We will allow the stock price to have path-wise discontinuities, which will be

denoted as jumps or *events*. As is shown in Jang et al. (2007), even small transaction costs can have strong effects on the optimal strategy of trading. This holds true also for the liquidity premium. Mehra and Prescott (1985) failed to explain the excess returns on equity over virtually risk-free bonds and they concluded that the most likely model to explain this well known equity premium puzzle, is one that accounts for some kind of market friction, for example transaction costs. A paper by Kocherlakota (1996) also concludes that a model which is to resolve the equity premium puzzle, must abandon some of the usual assumptions of the market, such as completeness and costless trading. Further, Balduzzi and Lynch (1999) show that the utility loss of ignoring transaction costs is significant. Together with evidence from empirical work, see for example Eleswarapu (1997), this strongly supports the inclusion of transaction costs in portfolio optimization problems. A drawback in the majority of studies that incorporate transaction costs, is that it is typically assumed that the investment opportunity set is constant. This is inconsistent with the heteroskedastic features of the volatility of returns found in almost all asset markets. An important exception from the constant opportunity set framework, is the above mentioned paper by Jang et al. (2007). Here the authors solve a regime switching model, which implicitly makes the opportunity set stochastic, since the intensity of going from one regime to the other is given by a stochastic process. This model does not, however, take into account the correlation between the stochastic processes driving the prices and the volatility. The optimal allocation will therefore not include an intertemporal hedging component, but as they show, the importance of stochastic volatility is crucial when finding the liquidity premium. In fact, the authors prove that the liquidity premium, as defined in Constantinides (1986), is much higher than previous thought. Here we include instantaneous stochastic volatility when transaction costs are present and we show that this has several major implications for how we analyze investor

behavior. To our knowledge this has previously not been addressed for cases where market frictions such as transaction costs are included. Jumps, on the other hand, have been studied by several authors in a portfolio optimization setting. Framstad et al. (2001) found that, in a constant opportunity set framework, the no-transaction region will tilt towards the risk-less asset in presence of jumps. Liu and Loewenstein (2007) come to similar conclusions. We will use a model where the volatility is allowed to change both continuously and discontinuously in time. Since both the volatility and the asset prices are affected by jumps, we will use the double jump framework introduced by Duffie et al. (2000). Our first contribution is that we find the optimal trading strategy when the risky asset has instantaneous stochastic volatility and co-jumps in the prices and the volatility. This means that we give explicit expressions for the boundaries of the no-transaction region. In order to do this, we have to rewrite the Hamilton-Jacobi-Bellmann equation, and for this we will use some of the conditions of optimality for the investment problem. These conditions are given in a verification theorem and must hold in order to conclude that the proposed solution is in fact the optimal one. A thorough treatment of this can be found in Øksendal and Sulem (2005), and then especially Chapter 5 on singular control theory. We show that the method we propose significantly simplifies the calculations needed to find the optimal trading strategy. We give two examples; First we solve the standard problem when the risky asset follows a geometric Brownian motion. We verify that the solution we find equals the solution of Jang et al. (2007). We claim that our method can simplify the calculations slightly and also that we are able to find solutions to problems with a more general framework. The second example illustrates this and we apply a model with stochastic volatility and jumps. A similar model without transaction costs is described in Liu et al. (2003). Using this model we find solutions for the optimal trading strategy when transaction costs are included. In addition

we find upper and lower bounds for the value function. These bounds reflect that the experienced value of the wealth for the investor is different depending on the amount of wealth in the risky asset. Specifically, we find that, when time to terminal time is 0, then the lower and upper boundaries both decrease with 23% when a jump of -20% is taken into account. If time to terminal time is 1, then the decrease of the boundaries will be approximately 25%.

Our second contribution is to show that the liquidity premium, as defined in Constantinides (1986), is larger than what is found in standard literature. We find that, in our model, the liquidity premium is strongly varying with the transaction costs. For a transaction cost of 1% of the size of the trade, the liquidity premium can be as much as ten times larger than what is found in Jang et al. (2007) and similar to this paper we conclude that the transaction costs have *first-order* effect on the liquidity premium. The bounds on the value function implies that the liquidity premium also will have a lower and an upper bound. The liquidity premium will be different for the two boundaries, which means that we find an upper and lower bound for the liquidity premium. Further, the liquidity premium decrease in the allocation to the risky asset and therefore the liquidity premium is highest for the lower bound on the value function. Surprisingly, we find that the size of the liquidity premium is almost constant in the risk aversion when we account for the change in optimal bounds for the non-intervention region. We say almost, since the upper bound is increasing slightly in risk aversion, from 7.2% to 7.3% for a risk aversion of 2 and 10 respectively. The lower bound is more rigid against changes in risk aversion, and the change is only 0.03 percentage points. However, for a given allocation of stock-to-wealth, the liquidity premium is decreasing in risk aversion. Specifically, these bounds are found to be (3.5%, 3.6%) for a transaction cost of 1% and (6.9%, 7.2%) when the transaction costs are 2%. The size of this liquidity premium implies that

transaction costs, together with an incomplete market, might be a possible solution of the equity premium puzzle, as described and analyzed in Mehra and Prescott (1985). The model used in this paper implies that there will be a equity premium as long as markets are incomplete and trading incur costs.

The paper is organized as follows. In section 1 we introduce the asset model. This is similar to what Liu et al. (2003) used in their paper with the additional market frictions of transaction costs. We highlight the approach we intend to use for the optimization procedure. In section 2 we study the problem of portfolio optimization with costs. In a first example we show that the approach proposed in section 1 replicate results in standard literature. We will then state results for our setting with stochastic volatility and jumps and then highlight some important differences that arise when jumps are included in a transaction cost model. The liquidity premium is calculated and a short discussion of the size and implications for the equity premium puzzle is given. Section 3 concludes.

## 1 The Basic Model

### 1.1 The Asset Market

Throughout this paper we are assuming a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)$ . Uncertainty in the model comes from the standard Brownian motion  $W$  and a Poisson lump process  $N$ . The investor can invest in two assets, the risk-less asset gives an annual interest of  $r$ . The risky asset follows the double jump process introduced in Duffie et al. (2000) and used in Liu et al. (2003). The dynamics of the assets are given by

$$(1.1) \quad dP_t = rP_t dt$$

$$(1.2) \quad dS_t = (r + (\eta - \mu\xi)\nu_t)S_t dt + S_t \sqrt{\nu_t} dW_1 + Y_1(t)S_{t-} dN_t$$

$$(1.3) \quad d\nu_t = (\alpha - (\beta + \kappa\xi)\nu_t)dt + \sigma \sqrt{\nu_t} dW_2 + Y_2(t)dN_t$$

where the price of the risk-less and the risky assets are  $P$  and  $S$ , respectively. The correlation between the Brownian motions  $W_1$  and  $W_2$  is  $\rho$ ,  $\nu$  is the instantaneous variance of diffusive returns and  $N$  is a Poisson process with stochastic intensity  $\xi\nu$ . The jump sizes,  $Y_1$  and  $Y_2$ , are two independent Poisson processes where the mean price jump size is  $\mu$  and the mean volatility jump size is  $\kappa$ . As we know that the volatility is nonnegative,  $\kappa$  is positive and has support on  $(0, \infty)$  and likewise since the price of a stock cannot decrease more than 100%, the jump size has support on  $(-1, \infty)$ . The jump sizes are assumed to be independent of jump times and also of the Brownian motions. All parameters in the volatility dynamics are positive. The risky asset is therefore subject to three sources of uncertainty: 1) The price shocks, 2) the shocks in volatility and 3) the jumps in prices and volatility. The last uncertainty factor is due to the fact that a jump is reflected in both the price and in the volatility. The sizes of the jumps however, are stochastic and independent of  $W_1$ ,  $W_2$  and  $N$ . Since the volatility is affected by a jump in the prices, a realization of  $N$  is echoed in both prices and volatility. The transaction costs are subtracted from the safe bank account when a trade is done. If the investor sells the risky asset, the excess money after costs goes into the bank account. If the investor buys the risky asset, the purchase price and all expenses are withdrawn from the bank account. Without loss of generality, we let the costs for buying and selling be equal. We can then illustrate the amount invested in the risk-less asset and the amount invested in the risky asset by the following equations

$$(1.4) \quad dx_t = rx_t dt - (1 + \lambda)d\mathcal{I}_t + (1 - \lambda)d\mathcal{D}_t$$

$$(1.5) \quad dy_t = (r + (\eta - \mu\xi)\nu_t)y_t dt + y_t\sqrt{\nu_t}dW_1 + Y_1(t)y_t - dN_t + d\mathcal{I}_t - d\mathcal{D}_t$$

where the processes  $\mathcal{I}$  and  $\mathcal{D}$  represents the cumulative purchase and sale of the risky asset. In order to solve this problem, we use the singular control theory and a treatment of this theory is given in Øksendal and Sulem (2005), Chapter 5. The investor should also be able to reallocate or liquidate the portfolio without its value becoming negative. This can be illustrated by a set,  $\mathcal{SR}$ , in which the

portfolio is said to be solvent. Let the solvency region be denoted as

$$\mathcal{SR} = \{(x, y) : y \geq \frac{-x}{1 + \lambda}, \quad y \geq \frac{-x}{1 - \lambda}\}$$

As we see from this, we do not impose any short selling constraints on the risky asset. The only constraint is that the value of the portfolio must at any time be positive, even after a possible liquidation. Liu and Loewenstein (2002) use a slightly different approach, by solving the problem using the *bond-to-stock* ratio. They are able to find the value function explicitly over the solvency region they find, however, a drawback of this procedure, is that it excludes any short selling of the risky asset. This is because whenever the amount in the risky asset is close to zero, the bond-to-stock ratio explodes and eventually have a discontinuity in zero. The solvency region is divided into three regions, the *Buy Region*,  $\mathcal{B}$ , the *Non-Intervention Region*,  $\mathcal{N}$  and the *Sell Region*,  $\mathcal{S}$ . As the names suggest, the investor buys the risky asset if the value of the portfolio is in  $\mathcal{B}$ , sells it if the value is in  $\mathcal{S}$  and does not trade if the value is in  $\mathcal{N}$ . The non-intervention interval will be denoted as  $[\underline{\pi}, \bar{\pi}]$ , and as can be seen, this interval is equal to the boundaries of  $\mathcal{N}$ .

## 1.2 The Investment Problem

Let the investor start with positive wealth,  $w_0$ . We can assume without loss of generality that the investor starts with wealth allocated with respect to  $\pi^*$ . Given this opportunity to invest in the risky and the risk-less asset, the investor verifies at each time  $t$  whether the fraction in the risky asset is inside  $(\underline{\pi}, \bar{\pi})$ , that is whether  $\pi_t \in \mathcal{N}$ . This is done in order to maximize the expected utility of the terminal wealth  $w_T$ ,

$$(1.6) \quad \max_{\pi_t, 0 \leq t \leq T} E_0[U(w_T)]$$



where we use the power utility function

$$(1.7) \quad U(u) = \frac{u^{1-\gamma}}{1-\gamma}.$$

Here  $\gamma > 0$  is the risk aversion. The problem of the investor with constant relative risk aversion preferences, is to choose the trading policy in order to maximize  $E[U(w_T)]$  subject to the equations (1.1), (1.2) and (1.3). To solve the problem, we define the value function at time  $t$  as

$$(1.8) \quad V(t, w, \nu) = \max_{w \in \mathcal{W}} E_t[U(w_{T \wedge \tau_{SR}})].$$

Here  $T \wedge \tau_{SR}$  is whatever comes first of terminal time  $T$  or the first time the value of the portfolio is outside the solvency region, the time of bankruptcy.  $\mathcal{W}$  is the set of all possible levels of wealth. As usual, the expectation conditional upon information at time  $t$  is denoted  $E_t[\cdot]$  and we will optimize the terminal wealth.

From the above dynamics of the risk-less asset and the risky asset, the investor should solve the corresponding Hamilton-Jacobi-Bellmann equation in order to find the trading strategies. In our case it is given by

$$(1.9) \quad 0 = \sup_{\mathcal{I}, \mathcal{D}} \{V_t + rxV_x + (r + (\eta - \mu\xi)\nu)yV_y + (\alpha - (\beta + \kappa\xi)\nu)V_\nu + \frac{1}{2}y^2\nu V_{yy} + \frac{\sigma^2}{2}\nu V_{\nu\nu} + \rho\sigma\nu V_{y\nu} + \xi\nu[E[V(t, y(1 + Y_1), \nu + Y_2)] - V]\}.$$

Here  $\mathcal{I}$  and  $\mathcal{D}$  are the Markov controls needed to optimize this equation. This equation is found by using a differential operator,  $\mathcal{A}$ , on the state equations and it is defined in Øksendal and Sulem (2005). Further, as we can see from the equation (1.9), it includes four variables,  $t, x, y$  and  $\nu$ , with corresponding partial derivatives of order one or two. A solution to this PDE is restricted to cases where the risky asset follows a geometric Brownian motion and this is the main reason for why there are relatively few papers that are taking transaction costs into account. If we now take a closer look at what happens when the risky asset is traded, we see that each time the investor trade, one of the processes  $\mathcal{I}$  or  $\mathcal{D}$  will denote the size

of the trade. However, since the costs are accounted for in the risk-less asset, the amount of change in the risky asset,  $dy$ , will be equal to either  $d\mathcal{I}$  or  $d\mathcal{D}$ . Because of this, we argue that, despite the transaction costs, it is possible to denote the wealth as the sum of the risk-less and the risky asset, and in turn optimize over the fraction in the risky asset,  $\pi$ . This fraction however, will have the costs of trading incorporated, and we will therefore need a notation for the wealth with costs included. To make sure that the wealth always stays positive and within the solvency region, Akian et al. (1996) proposed the following notation for the net wealth

$$(1.10) \quad w = x + \min\{(1 - \lambda)y, (1 + \lambda)y\}$$

where  $1 - \lambda$  is the sell cost, and  $1 + \lambda$  is the buy cost. This can easily be generalized to have different costs for selling and buying the risky asset, the quantitative solutions to the problem would not change. Without loss of generality and simplification of the reading experience, we will use  $\theta$  as the proportional cost constant, both for buying or selling the risky asset. Note that if there are no costs involved, i.e.  $\lambda = 0$ , then  $\theta = 1$ . We will use the notation of  $\theta$  when a result holds true for both selling and buying the risky asset. The notation of net wealth is equal to the approach in Akian et al. (1996), Framstad et al. (2001) and Jang et al. (2007). The ratio of the risky asset to wealth is  $\pi = \frac{y}{w}$  and the ratio risk-less asset to wealth will be  $1 - \pi = \frac{w-y}{w} = \frac{x+y(\theta-1)}{w}$ . Using this notation, we can rewrite the Hamilton-Jacobi-Bellmann equation which is highlighted in the following theorem

**Theorem 1.1** *If the value function is  $C^2(\mathcal{S})$  and (1) and (3) or (2) and (3) in Theorem (1.2) hold with equalities, then equation (1.9) can be written as*

$$(1.11) \quad \begin{aligned} 0 = & \sup_{\pi} \{V_t + (r + \pi\theta(\eta - \mu\xi)\nu)wV_w + (\alpha - (\beta + \kappa\xi)\nu)V_{\nu} \\ & + \frac{\pi^2}{2}\theta^2w^2\nu V_{ww} + \frac{\sigma^2}{2}\nu V_{\nu\nu} + \theta\pi\rho\sigma\nu wV_{w\nu} \\ & + \xi\nu(E[V(t, w(1 + \pi\theta Y_1), \nu + Y_2)] - V)\}. \end{aligned}$$

*Proof is found in the appendix.*

Here  $V_i$  is the derivative of  $V$  with respect to the variable  $i \in \{t, w, \nu\}$  and the value function in this case will be denoted as  $V(t, w, \nu)$ . We see that this Bellmann equation is the same as if we used the differential operator  $\mathcal{A}$  on the wealth dynamics denoted by

$$(1.12) \quad dw = (r + \pi\theta(\eta - \mu\xi)\nu)w dt + \pi\theta\sqrt{\nu}w dZ_1 + \pi\theta Y_1 w dN.$$

This equation again, is the same as writing

$$(1.13) \quad dw = dx + \theta dy.$$

where  $\theta$  is as above,  $1 \pm \lambda$ . In deriving Theorem 1.11, we have used some properties that are not yet defined. These properties, or conditions, are necessary in optimization problems in order to conclude that the function found is in fact the optimal one and is usually collected in a verification theorem. In our procedure to find equation (1.11), we make use of several conditions in the verification theorem for singular control theory. As a matter of fact, the Hamilton-Jacobi-Bellmann equation (1.9) is only one of several such conditions that must be fulfilled in order to ensure that the optimal solution of the investment problem is found. The value function and the optimal trading strategy of the optimization problem is found by verifying the conditions in the following verification theorem<sup>a</sup>

**Theorem 1.2** *Suppose the value function is  $C^2$  on  $\mathcal{SR}$ . If*

1.  $(1 - \lambda)V_x - V_y \leq 0$
2.  $-(1 + \lambda)V_x + V_y \leq 0$
3.  $\mathcal{A}V(\mathbf{q}) \leq 0$ ,

*then  $V$  is the value function, and the optimal policy is to transact the minimum amount in order to keep the value of the portfolio inside  $\mathcal{N}$ .*

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<sup>a</sup>We only state the most prominent conditions. A complete verification theorem is found in Øksendal and Sulem (2005) Chapter 5, Theorem 5.2.

Note that the boldface  $\mathbf{q}$  is the vector  $\mathbf{q} = (t, x, y, \nu)$ . Further, inside  $\mathcal{N}$ , condition (3) holds with equality. On  $\mathcal{B}$ , condition (2) holds with equality and on  $\mathcal{S}$ , (1) holds with equality. Note that the upper boundary of  $\mathcal{N}$  is the boundary that divides the non-intervention region and the sell region. Similarly, the lower boundary of  $\mathcal{N}$  is the boundary that divides the non-intervention region and the buy region. This is important and very useful in the outline of our approach to find a solution to the portfolio optimization problem, since we can exploit the fact that the derivative of the value function with respect to  $y$  is nothing else than the derivative of the value function with respect to  $x$  times the costs of trading. The following Corollary provides boundaries for the value function and the optimal trading strategies. These boundaries are general, and facilitate the possibility to numerically find the value function approximately for any model applied.

**Corollary 1.1** *If we let  $\theta_1 = 1 - \lambda$  and  $\theta_2 = 1 + \lambda$ , then the bounds for the value function are*

$$(1.14) \quad V(t, x, (1 - \lambda)y, \nu) \leq V(t, x, y, \nu) \leq V(t, x, (1 + \lambda)y, \nu)$$

*The optimal upper and lower boundaries for  $\mathcal{N}$  are*

$$(1.15) \quad \bar{\pi} = \frac{-((\eta - \mu\xi)V_w + \rho\sigma V_{w\nu} + \xi E[Y_1 \frac{\partial V(t, w(1 + \bar{\pi}(1 - \lambda)Y_1), \nu + Y_2)}{\partial \pi}])}{w(1 - \lambda)V_{ww}}$$

$$(1.16) \quad \underline{\pi} = \frac{-((\eta - \mu\xi)V_w + \rho\sigma V_{w\nu} + \xi E[Y_1 \frac{\partial V(t, w(1 + \underline{\pi}(1 + \lambda)Y_1), \nu + Y_2)}{\partial \pi}])}{w(1 - \lambda)V_{ww}}$$

Equations (1.15) and (1.16) show the optimal portfolio bounds on the risky asset. The bounds are divided in three terms, the usual mean-variance term, the intertemporal hedging component and the term stemming from the jumps. If we let  $\lambda = 0$ , we get the Merton proportion, and in this case it equals the fraction found in Liu et al. (2003). However, when  $\lambda \neq 0$  we see that  $\underline{\pi}$  and  $\bar{\pi}$  is decreasing and increasing, respectively, with respect to  $\lambda$ . This means that the non-intervention region,  $\mathcal{N}$ , is becoming wider in the transaction costs. Moreover, we can see from these expressions that the cost is included linearly in the first two terms and non-

linearly in the last, the term stemming from the jump. This nonlinear function of  $\lambda$  will affect the relative length from the Merton proportion to the upper or lower boundaries. It is easy to see this if we first assume that there are no jumps. Then the last term in (1.15) and (1.16) vanishes, and we are thus able to factor out the cost constant  $(1 \pm \lambda)$ . But then  $\bar{\pi}$  and  $\underline{\pi}$  are nothing else than the Merton proportion,  $\pi^*$  multiplied by a function of  $\lambda$ . Thus for some  $f$ ,  $\bar{\pi} = f(\lambda)\pi^*$ , and hence the ratio  $\frac{\bar{\pi}}{\pi^*} = f(\lambda)$ , which is constant. This is not possible if we include jumps, since we are not able to factor out the cost function. To see this more explicitly, we need to obtain the value function  $V$ .

## 2 Examples and numerical solutions

In this section we present the solution to our problem. We start by giving a simple example where the risky asset follows a geometric Brownian motion. This is studied by numerous authors and explicit solutions for a special case are given in Jang et al. (2007). Estimation of similar models for the risky asset is given thorough treatment in Andersen et al. (2003) and Eraker et al. (2003). The former applies an EMM method to estimate a model with jumps in the price returns and the latter applies the Markov Chain Monte Carlo method to a model with jumps in both returns and volatility. We will borrow the parameter estimates from these papers to illustrate our findings.

### 2.1 Constant volatility and no jumps

We will use theorem (1.1) to derive the optimal trading strategies for the investor when the opportunity set is constant. This will be done by letting the risky asset be modeled as the geometric Brownian motion  $dy = \mu y dt + \nu_0 y dW$ . This implies that the corresponding Hamilton-Jacobi-Bellmann equation is

$$(2.1) \quad 0 = V_t + (r + \theta\pi(\mu - r))wV_w + \frac{\nu_0^2}{2}\theta^2\pi^2w^2V_{ww}$$

In this standard case, several authors have proven that the value function is  $V(w) = \exp(-\delta t) \frac{w^{1-\gamma}}{1-\gamma}$ , for some discount factor  $\delta$ . First order conditions of the equation (2.1) will therefore result in the following boundaries for  $\mathcal{N}$

**Corollary 2.1** *The no-transaction region boundaries are*

$$\begin{aligned}\underline{\pi} &= \frac{\mu - r}{(1 + \lambda)\nu_0^2\gamma} \\ \bar{\pi} &= \frac{\mu - r}{(1 - \lambda)\nu_0^2\gamma}\end{aligned}$$

which implies the bond-stock ratios

$$(2.2) \quad \underline{z} = (1 - \lambda)\left(\frac{\nu_0^2\gamma}{\mu - r} - 1\right)$$

$$(2.3) \quad \bar{z} = (1 + \lambda)\left(\frac{\nu_0^2\gamma}{\mu - r} - 1\right)$$

*Proof is found in the appendix.*

The boundaries (2.2) and (2.3) is equivalent to the boundaries found in the paper by Jang et al. (2007). We conclude that in this case our method significantly simplifies the calculations. Note that  $z$  denotes the *bond-to-stock* ratio, and therefore the lowest ratio,  $\underline{z}$ , is achieved for the highest fraction in the risky asset,  $\bar{\pi}$ .

## 2.2 Stochastic volatility and jumps

In this section we will derive the details of the optimal portfolio when the risky asset includes stochastic volatility and jumps. The problem without transaction costs has been analyzed by Liu et al. (2003), and we conjecture for this reason that the value function will be similar to what these authors found. However, if the no cost policy is sought for, we can just let the proportional cost constant  $\lambda$  equal zero, and the policies from Liu et al. (2003) are obtained. The value function will be

$$(2.4) \quad V(\tau, w, \nu) = \frac{w^{1-\gamma}}{1-\gamma} \exp(A_\tau + B_\tau \nu).$$

Here  $\tau = T - t$ , which is the time left to terminal time.

### 2.2.1 Optimal policies

With the value function above, equation (1.11) and the corresponding optimal boundaries for  $\mathcal{N}$  are summarized in the following proposition

**Proposition 2.1** *If the value function is of the form (2.4), then the HJB equation becomes*

$$\begin{aligned}
 0 &= -(A'_\tau + B'_\tau \nu) + (r + \pi\theta(\eta - \mu\xi)\nu)(1 - \gamma) + (\alpha - (\beta + \kappa\xi)\nu)B_\tau \\
 &\quad - \frac{\gamma}{2}(1 - \gamma)\pi^2\theta^2\nu + \frac{\sigma^2}{2}\nu B_\tau^2 + \rho\pi\theta\sigma\nu(1 - \gamma)B_\tau \\
 (2.5) \quad &+ \xi\nu(E[(1 + \pi\theta Y_1)^{1-\gamma} \exp(Y_2 B_\tau)] - 1)
 \end{aligned}$$

and the first order conditions with respect to  $\pi$ , evaluated at  $\bar{\pi}$  and  $\underline{\pi}$ , are

$$(2.6) \quad \bar{\pi} = \frac{(\eta - \mu\xi)}{\gamma(1 - \lambda)} + \frac{\sigma\rho}{\gamma(1 - \lambda)}B_\tau + \frac{\xi E[Y_1(1 + (1 - \lambda)\bar{\pi}Y_1)^{-\gamma} e^{B_\tau Y_2}]}{\gamma(1 - \lambda)}$$

$$(2.7) \quad \underline{\pi} = \frac{(\eta - \mu\xi)}{\gamma(1 + \lambda)} + \frac{\sigma\rho}{\gamma(1 + \lambda)}B_\tau + \frac{\xi E[Y_1(1 + (1 + \lambda)\underline{\pi}Y_1)^{-\gamma} e^{B_\tau Y_2}]}{\gamma(1 + \lambda)}.$$

where  $A_\tau$  and  $B_\tau$  are functions of time but not of the state variables and solve the equations:

$$(2.8) \quad \underline{A}'_\tau = r(1 - \gamma) + \alpha \underline{B}_\tau$$

$$(2.9) \quad \bar{A}'_\tau = r(1 - \gamma) + \alpha \bar{B}_\tau$$

$$\begin{aligned}
 (2.10) \quad \underline{B}'_\tau &= \frac{\sigma^2}{2}\underline{B}_\tau^2 + (\pi\theta\sigma\rho(1 - \gamma) - (\beta + \kappa\xi))\underline{B}_\tau + \pi\theta(\eta - \mu\xi)(1 - \gamma) \\
 &\quad - \frac{\gamma}{2}(1 - \gamma)\underline{\pi}^2\theta^2 + \xi[E[(1 + \underline{\pi}\theta Y_1)^{1-\gamma} e^{\underline{B}_\tau Y_2}] - 1]
 \end{aligned}$$

$$\begin{aligned}
 (2.11) \quad \bar{B}'_\tau &= \frac{\sigma^2}{2}\bar{B}_\tau^2 + (\bar{\pi}\theta\sigma\rho(1 - \gamma) - (\beta + \kappa\xi))\bar{B}_\tau + \bar{\pi}\theta(\eta - \mu\xi)(1 - \gamma) \\
 &\quad - \frac{\gamma}{2}(1 - \gamma)\bar{\pi}^2\theta^2 + \xi[E[(1 + \bar{\pi}\theta Y_1)^{1-\gamma} e^{\bar{B}_\tau Y_2}] - 1].
 \end{aligned}$$

The equations (2.8), (2.9), (2.11) and (2.12) ensure that the value function will have an upper and lower boundary and is therefore different from the value function in the case without transaction costs. The optimal buy and sell boundaries have three terms, all of which contain the transaction cost constant. An implication of this is that the optimal trading strategy is no longer myopic. The equations for  $\underline{\pi}$  and  $\bar{\pi}$  are for some cases solvable together with the corresponding equations for  $A$  and  $B$ . However, the solutions might require numerical procedures for the most general cases. For example, if the risk aversion is higher than 3, the functions of  $\pi$  will include an exponent of the power higher than four. Elementary algebra now states that there is in general no explicit solution for such equations and so we will not be able to find a solution analytically. Further, as the equations for  $\underline{\pi}$  and  $\bar{\pi}$  are given above, the expectation in the last term makes it fairly difficult to find explicit solutions for the optimal allocation to the risky asset no matter which size the risk aversion is. If we want closed form expressions, we need to rely on some simplifications, e.g. that the jump sizes are constant and that the risk aversion is an integer less than or equal to 3. The non-intervention region,  $\mathcal{N}$ , is given in the two figures below. The first illustrates the change in the boundaries with respect to transaction costs.

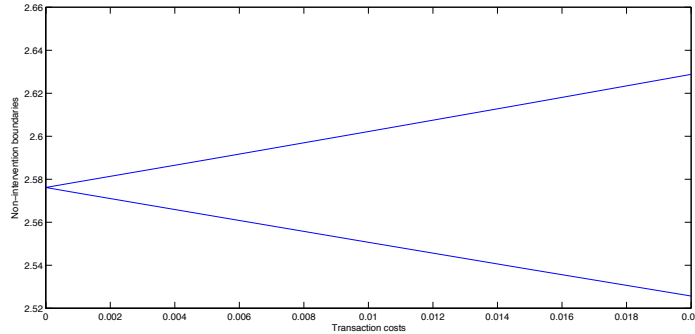


Figure 2.1: The non-intervention region,  $\mathcal{N}$ , is widening as a function of the transaction costs. We have used a risk aversion of  $\gamma = 2$ , a mean jump size in the price returns of  $\mu = -0.2$  and mean jump size of the volatility of  $\kappa = 0.226$ .



As we see from Figure 2.1, the region  $\mathcal{N}$  is widening in the transaction costs, in line with standard literature. The next figure illustrates how the region  $\mathcal{N}$  changes with the risk aversion,  $\gamma$ . First of all, we see that the boundaries of  $\mathcal{N}$  are decreasing, which confirms the fact that as risk aversion increases, the investor keeps less in risky assets.

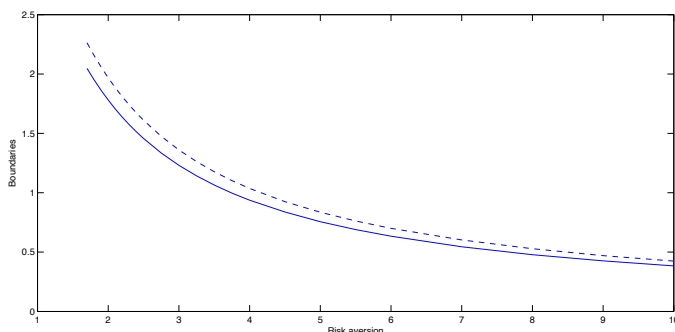


Figure 2.2: The optimal allocation to the risky asset is decreasing in the risk aversion, and the non-intervention region will therefore have decreasing boundaries. The length between the upper and lower boundaries also decreases.

In Figure 2.2 the cost of a transaction is 1% of the size of the transaction. Further, as Figure 2.2 illustrates, the optimal allocation to the risky asset is converging to zero as the risk aversion increases. The distance between the upper and lower boundaries ranges from 0.22 to 0.04 for risk aversion of 2 and 10 respectively. As we will see in the next section, this is because the more risk avert the investor is, the more sensitive he is to deviations from the optimal portfolio and will therefore trade frequently in order to prevent potential losses.

### 2.2.2 Investor Sensitivity

A model with constant opportunity set is incapable of including two measures of sensitivities of the investor with respect to the optimal trading strategy. First, the ratios  $\frac{\pi}{\pi^*}$  and  $\frac{\bar{\pi}}{\pi^*}$  will be constants, e.g.  $\frac{1}{1 \pm \lambda}$  for a model using a geometric

Brownian motion for the risky asset. In case the opportunity set is stochastic, and especially if one includes jumps, this ratio will change according to whether the Merton proportion is high or low. To illustrate how this affects the optimal trading strategies, assume for simplicity that the two first terms in (2.7) are zero and that  $Y_1 = \mu$  and  $Y_2 = \kappa$ . Consider the ratio  $g(\pi^*) = \frac{\pi}{\pi^*}$ , which will become

$$(2.12) \quad g(\pi^*) = \frac{(1 + \mu\pi^*)^\gamma \exp(\underline{B}_\tau \kappa)}{(1 - \lambda)(1 + (1 - \lambda)\mu\underline{\pi})^\gamma \exp(B_\tau^* \kappa)}.$$

If we now find the first order condition with respect to  $\pi^*$ , we get

$$\frac{\partial g}{\partial \pi^*} = \frac{\mu\gamma(1 + \mu\pi^*)^{\gamma-1} \exp(\underline{B}_\tau \kappa)}{(1 - \lambda)(1 + (1 - \lambda)\mu\underline{\pi})^\gamma \exp(B_\tau^* \kappa)}.$$

For reasonable parameter choices, the lower boundary  $\underline{\pi}$  will be too small for the term  $1 + \mu\underline{\pi}$  to become negative. This implies that the sign of the derivative follows the sign of  $\mu$ , such that if  $\mu < 0$ , then the relative length from the Merton proportion to the lower boundary is decreasing for an increasing proportion in the risky asset. In other words, when the optimal amount in the risky asset is relatively high, then  $g(\pi^*)$  is low. This again implies that a relatively small decrease in the stock price is seen as a buying opportunity. On the other hand, if the optimal allocation to the risky asset is fairly low, then  $g(\pi^*)$  is high, and the investor would have to see a greater decline in the stock price before it would be considered a buying opportunity. Second, the sensitivity towards deviations from the optimal portfolio can be measured by the second derivative the Hamilton-Jacobi-Bellmann equation (2.5). If there are no jumps, the second derivative with respect to  $\pi$  will be a constant, and the investor is..... equally sensitive to whether his portfolio is far away from the optimal no matter if his risk aversion is high or low. A property from (2.5) is that, for a general  $\gamma$ , we can find the second order derivative, and analyze how it changes with the different parameters. Let for the moment equation (2.5) by denoted as  $f(\pi)$ , then the second order condition is

$$(2.13) \quad f_{\pi\pi} = -\gamma(1 - \gamma)\theta^2\nu - \xi\nu\gamma(1 - \gamma)\theta^2 E[Y_1^2 \exp(B_\tau Y_2)(1 + \pi\theta Y_1)^{-\gamma-1}]$$

As we can see from this equation, the behavior of the investor with respect to risk aversion, depend on the stochastic jump size of the price.  $f_{\pi\pi}$  will in fact always be positive as long as  $(1 + \pi\theta Y_1)$  is greater than zero and  $\gamma > 1$ . This means that the more risk averse the investor is, the more sensitive he is to deviations from the optimal allocation. The investor will therefore refrain from investing too much in the risky asset if there is a probability of negative events.

### 2.2.3 The Liquidity Premium

The liquidity premium, LP hereafter, is defined as the maximum expected rate of return an investor is willing to exchange a costly asset against a cost-less asset. It is a measure of the effect transaction costs have on expected returns. To quantify this we follow the procedure in Constantinides (1986) and compare the value functions of two assets, one asset that is free to trade and one asset that the investor has to pay to trade. To find the LP with this procedure, we subtract a  $\delta$  from the expected returns of the cost-less asset, and find how big this  $\delta$  must be in order for the size of the value function of the cost-less asset to equal the size of the value function of the costly asset. Further, transaction costs have first order effect on the LP if the derived utility is sensitive to deviations from the optimal allocation. As we showed in the last section, the sensitivity to deviations from the optimal portfolio increases when transaction costs are included in a model which allows for jumps. The size of the ratio liquidity premium to transaction costs is decreasing in risk aversion, but is always, in our model, above one. These features imply that transaction costs have first-order effect on the liquidity premium. The LP will decrease with risk aversion if we do not account for the fact that investments in the risky asset decrease in risk aversion. In Figure 2.3 we illustrate how the LP varies with RA when we have used that the optimal allocation to wealth is between 0.756 and 0.836: Interestingly, we find that if we take into account the fact that investments in the risky assets decline with risk aversion, then the liquidity premium will be virtually constant for all levels of risk aversion. This means that for an investor with risk aversion 2 and another with risk aversion 10, with upper and lower boundaries of the non intervention region equal to (1.968,1.781)

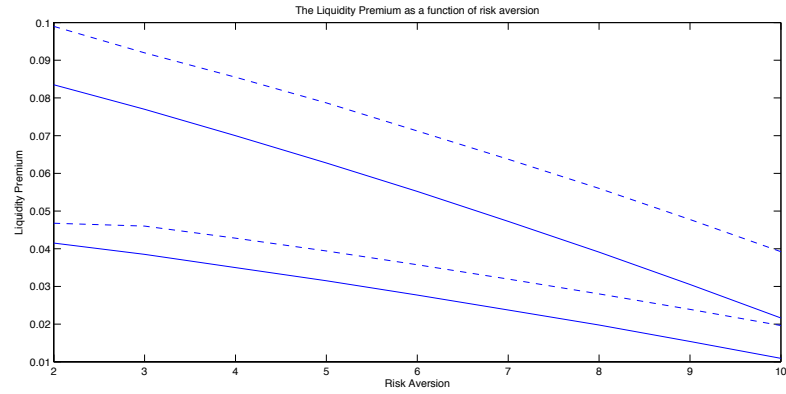


Figure 2.3: The liquidity premium as a function of risk aversion. The upper solid line is the liquidity premium when transaction costs are 2% but no jumps and the upper dashed line is when jumps are included. The low solid and low dashed line are with and without jumps, respectively, band with a transaction cost of 1%.

and  $(0.3839, 0.4243)$  respectively, the liquidity premium will be 6.7% if transaction costs are 2%. The size of the liquidity premium will be highly dependent on the size of the transaction costs, as Figure 2.4 shows. The dashed line in Figure 2.4 is for the upper boundary of  $\mathcal{N}$ .

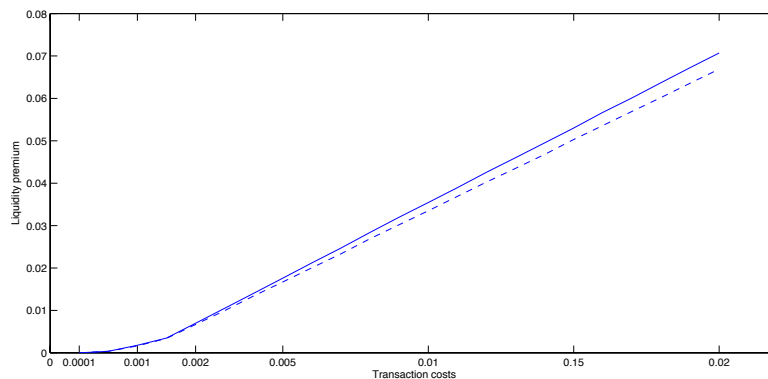


Figure 2.4: The liquidity premium as a function of the transaction costs. This result holds for all levels of risk aversion, when the decline in risky investments is accounted for. The solid line is for the lower boundary of the non-intervention region and the dashed line for the upper boundary.

### 2.2.4 The Equity Premium Puzzle

A vast number of papers have analyzed why the average stock returns are so much higher than bond returns in US data. This is denoted as the equity premium puzzle and was proposed by Mehra and Prescott (1985). In this paper we will not discuss whether the puzzle is real or not, but a thorough treatment is found in Mehra and Prescott (2003). Kocherlakota (1996) concludes that in order to explain the puzzle, one needs to relax at least one of three assumptions often made in the economic models. We relax two of the standard assumptions; we have a model with an incomplete market and we have frictions in the form of transaction costs. And as the figures and numbers in the above sections illustrate, the size of the liquidity premium indicates that incompleteness and transaction costs can in fact explain large parts of the equity premium puzzle. This is contrary to standard literature which concludes that transaction costs are not enough to explain the puzzle. In Figure 2.5 we analyze how the liquidity premium changes with respect to the jump intensity and the standard deviation of the volatility. Note that in order to be able

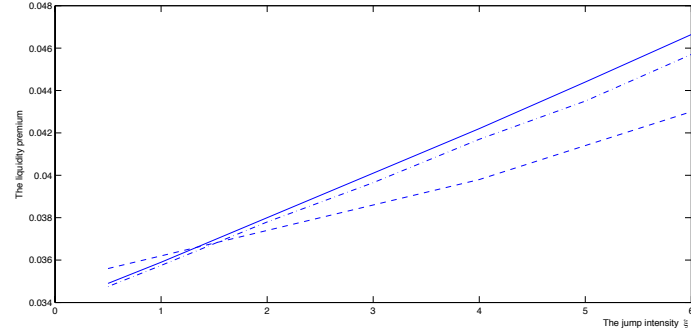


Figure 2.5: The liquidity premium as a function of the  $\xi$ ,  $\mu$  and  $\sigma$ . The solid line is the liquidity premium with respect to the jump intensity  $\xi$ , the dash-dotted line is with respect to the mean jump size of returns and the dashed line is the liquidity premium with respect to  $\sigma$ .

to plot the changes in the liquidity premium with respect to these three parameters, we have chosen to use  $\mu = -\frac{\xi}{10}$  and  $\sigma = \frac{\xi}{10}$ . The liquidity premium is increasing approximately with two percentage points in each case, which means that the size of these parameters is important in order to explain the equity premium puzzle. Now, if we consider a market of high volatility with many large jumps, then the liquidity premium can be as high as 9%. This reflects that in order to find an equity premium, the estimation of parameters is crucial.

### 2.2.5 The mean Price Jump size

As is highlighted in several other papers, e.g. Liu et al. (2003) and Øksendal and Sulem (2005), the optimal amount of wealth in the risky asset is sensitive to jumps, and then especially negative jumps in the returns of the stock price. Figure 2.6 shows how the optimal allocation in the risky asset behaves when the mean jump sizes change. The figure plots both the case for when there is one jump every year, and when the jumps are more infrequent. The investor keeps a smaller fraction of his wealth when the jumps are more frequent. If the jumps are less frequent, the prospect of small jumps is not changing his portfolio much.

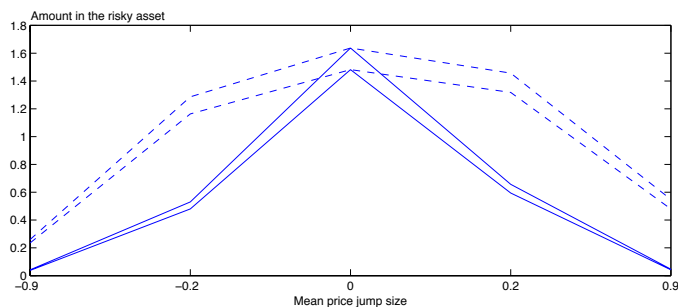


Figure 2.6: The upper and lower boundaries of the no-transaction region is concave in the mean jump size. The solid lines are for the case when there is one jump each year. The dashed lines when there is a jump every ten years. The proportional cost are 5%.

As we can see from Figure 2.6, the investor quickly changes the amount of money in the risky asset when there is a probability for jumps. Moreover, while this holds for both negative and positive jumps, the reaction to negative jumps is more severe. Also, when the jumps are relatively infrequent, small jumps do not have great effects on the allocation to the risky asset.

### 3 Conclusion

In this paper we solve an optimal portfolio problem where the market is incomplete and contains restrictions in the form of transaction costs. The risk-less asset is a bank account and the risky asset is modeled with stochastic volatility and jumps which implies that our opportunity set is stochastic. Despite the transaction costs we find, in addition to boundaries for the value function, explicit trading strategies for the investor. We show that the transaction costs are included in a non-linear fashion in the boundaries of the non-intervention region. The implication of this is that the transaction costs have much higher impact on the trading strategies than previously thought. Specifically we show that the non-intervention region widens in the transaction costs and gets tilted towards the risk-less asset if the

risk aversion increases.

We analyze the liquidity premium and find it to be a factor of ten times higher than standard literature on the area. The size of the liquidity premium is highly dependent on the size of the transaction costs. Other factors that contribute much to the size of the liquidity premium are the jump intensity, the mean size of jumps in the returns and the standard deviation of the volatility. In some cases the size of the liquidity premium is enough to explain the equity premium puzzle.

## A Appendix

we will here prove some of the statements made throughout this paper.

*Proof of Theorem (1.1).*

$V(t, w, \nu)$  is the solution of the HJB equation

(A.1)

$$0 = V_t + [(r + \pi\theta(\mu - r))w]V_w + (\alpha - (\beta + \kappa\xi)\nu)V_\nu + \frac{\pi^2}{2}\theta^2 w^2 \nu V_{ww} + \frac{\sigma^2}{2}\nu V_{\nu\nu} + \rho\sigma w\nu\theta V_{w\nu}$$

*From the conditions of existence and uniqueness of the value function in the singular control theory, we have*

$$(A.2) \quad V_y = \theta V_x$$

$$(A.3) \quad V(\cdot) \in C^2(\mathcal{S})$$

*From these conditions it follows that*

$$(A.4) \quad \begin{aligned} V_{yy} &= \frac{\partial}{\partial y} \frac{\partial}{\partial y} V(x, y, \nu) \\ &= \frac{\partial}{\partial y} \theta \frac{\partial}{\partial x} V(x, y, \nu) \\ &= \theta^2 V_{xx} \end{aligned}$$

$$(A.5) \quad \begin{aligned} V_{y\nu} &= \frac{\partial}{\partial \nu} \frac{\partial}{\partial y} V(x, y, \nu) \\ &= \theta V_{x\nu} \end{aligned}$$



We can therefore write the HJB of (1.1), (1.2) and (1.3) as follows

$$\begin{aligned}
 0 &= V_t + rxV_x + \mu yV_y + \kappa(\alpha - \nu)V_\nu + \frac{y^2\nu}{2}V_{yy} + \frac{\sigma^2\nu}{2}V_{\nu\nu} + \rho\sigma\nu yV_{y\nu} \\
 &= V_t + rxV_x + \mu y\theta V_x + \kappa(\alpha - \nu)V_\nu + \frac{y^2\nu}{2}\theta^2 V_{xx} + \frac{\sigma^2\nu}{2}V_{\nu\nu} + \rho\sigma\nu\theta yV_{x\nu} \\
 &= V_t + \left(r\frac{x}{w} + \theta\mu\frac{y}{w}\right)wV_x + \kappa(\alpha - \nu)V_\nu + \frac{\nu}{2}\frac{y^2}{w^2}w^2\theta^2 V_{xx} + \frac{\sigma^2\nu}{2}V_{\nu\nu} + \rho\sigma\nu\theta\frac{y}{w}wV_{x\nu} \\
 &= V_t + (r + \theta\pi(\mu - r))wV_x + \kappa(\alpha - \nu)V_\nu + \frac{\nu}{2}\pi^2 w^2\theta^2 V_{xx} + \frac{\sigma^2\nu}{2}V_{\nu\nu} + \rho\sigma\nu\theta\frac{y}{w}wV_{x\nu}.
 \end{aligned}$$

When  $x + \theta y$  is substituted for  $w$  in a homogeneous function we have that  $V_x = V_w$  and (1.11) follows  $\square$

*Proof of Corollary 1.1*

By taking the first order conditions of equation (1.11) the boundaries follows directly.  $\square$

Proof of Corollary (2.1):

Remember that  $dw = (r + \theta\pi(\mu - r))w dt + \pi\theta w\sigma dW$ .

From 2.1 we get that

$$\pi^* = -\frac{(\mu - r)J_w}{\theta\sigma^2 w J_{ww}}.$$

Thus, for the value function,  $J(w) = \frac{w^{1-\gamma}}{1-\gamma}$ ,

$$\begin{aligned}
 \pi^* &= \frac{\mu - r}{\theta\sigma^2} \\
 1 - \pi &= \frac{\theta\sigma^2\gamma - (\mu - r)}{\theta\sigma^2\gamma} \\
 \Rightarrow \frac{1 - \pi}{\pi} &= \frac{\theta\sigma^2\gamma - (\mu - r)}{\mu - r} = \frac{\theta\sigma^2\gamma}{\mu - r} - 1.
 \end{aligned}$$

It now follows

$$\begin{aligned}
 z &= \frac{x}{y} = \frac{\frac{x}{w}}{\frac{y}{w}} = \frac{\frac{x+y(\theta-1)}{w}}{\frac{y}{w}} - \frac{y(\theta-1)}{w} = \frac{1 - \pi}{\pi} - (\theta - 1) \\
 &= \theta\left(\frac{\sigma^2\gamma}{\mu - r} - 1\right)
 \end{aligned}$$

which concludes the proof  $\square$

We can with this proof see that the approach we propose gives that same results as Jang et al. (2007). in addition to this, our method also holds true for more general dynamics for the risky asset, as the next proof shows.

*Proof of proposition 2.1*

If  $V(w, \nu) = \frac{w^{1-\gamma}}{1-\gamma} \exp(A_\tau + B_\tau \nu)$ ,  $\bar{\pi}$  is given as in (2.6) and the HJB equation (1.11), then we get the following

$$\begin{aligned}
V_t &= -(A'_\tau + B'_\tau \nu)V \\
V_w &= \frac{1-\gamma}{w}V \\
V_{ww} &= -\gamma \frac{1-\gamma}{w^2}V \\
V_\nu &= B_\tau V \\
V_{\nu\nu} &= B_\tau^2 V \\
V_{w\nu} &= B_\tau \frac{1-\gamma}{w}V \\
V(t, w(1 + \pi\theta Y_1), \nu + Y_2) &= (1 + \pi\theta Y_1)^{1-\gamma} \exp(B_\tau Y_2)V
\end{aligned}$$

Which, when substituted into equation (1.11), gives us

$$\begin{aligned}
0 &= -(A'_\tau + B'_\tau \nu)V + (r + \pi\theta(\eta - \mu\xi)\nu)w \frac{1-\gamma}{w}V + (\alpha - (\beta + \kappa\xi)\nu)B_t V - \frac{\pi^2}{2}\theta_1^2 \nu w^2 \gamma \frac{1-\gamma}{w^2}V \\
&+ \frac{\sigma^2}{2}\nu B_t^2 V + \rho\sigma\nu\theta_1\pi w B_t \frac{1-\gamma}{w}V + \xi\nu[E[(1 + \pi\theta Y_1)^{1-\gamma} \exp(B_\tau Y_2)V] - V] \\
&= (A'_t + B'_t \nu) + (r + \pi\theta(\eta - \mu\xi)\nu)(1 - \gamma) + (\alpha - (\beta + \kappa\xi)\nu)B_t - \frac{\pi^2}{2}\theta_1^2 \nu \gamma(1 - \gamma) \\
&+ \frac{\sigma^2}{2}\nu B_t^2 + \rho\sigma\nu\theta_1\pi B_t(1 - \gamma) + \xi\nu[E[(1 + \pi\theta Y_1)^{1-\gamma} \exp(B_\tau Y_2)] - 1].
\end{aligned}$$

Let us now separate these expressions into two brackets, one for the terms including  $\nu$  and one for the terms which does not include  $\nu$ . From elementary algebra, we now know that both brackets must equal zero, since the HJB equation must be zero for all  $t$ . This gives us the equations for  $A_\tau$  and  $B_\tau$  as in (2.8) and (2.11) such that the conditions for optimality hold.  $\square$

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