Efficient Estimation of Nonparametric Regression with Autocorrelated Errors

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Introduction

- We discuss the efficient estimation of the model
$$B(L)Y_t = A(L)m(X_t) + \varepsilon_t,$$
where
  - ε_t is an i.i.d. sequence with finite variance σ^2_ε and independent of the regressors X_t
  - the regressors X_t are assumed to follow some stationary process
  - $A(L) = \sum_{j=0}^{\infty} a_j L^j$ and $B(L) = \sum_{j=0}^{\infty} b_j L^j$
  - The function $m(.)$ is assumed to be unknown but smooth.
  - It is of interest to estimate m, and $A(L), B(L)$.

- Leading case of interest is the nonparametric regression model
$$Y_t = m(X_t) + u_t,$$
$t = 1, \ldots, T$, where the residual process u_t satisfies
$$A(L)u_t = \varepsilon_t = \sum_{j=0}^{\infty} a_j u_{t-j}.$$

- In this case
$$A(L)Y_t = A(L)m(X_t) + \varepsilon_t,$$
which is in the above form with $A(L) = B(L)$.

- This is the model considered by Xiao, Linton, Carroll, and Mammen (JASA, 2003) XLCM.

Parametric Case

- Suppose that
$$m(x) = \beta^\top x$$
for some unknown parameters. This classic model has been extensively treated in the econometrics literature.
  - The variance of OLS is proportional to the long run variance of the process $\{X_t u_t\}$ and least squares standard errors that ignore this fact are inconsistent and need to be modified in a non-trivial way.
  - Also, one can generally improve efficiency of least squares estimators by using a GLS weighting scheme that reflects the error autocorrelation function.
  - Cochrane-Orcutt, Hildreth-Lu, Prais-Winsten, and Durbin-Watson et als.
  - Some cases where OLS=GLS. Specifically, when the regressors are polynomials in time, see for example Andersen

(1971, p581).

- Compare this with the case where $m(.)$ is nonparametric, which has been analyzed in Robinson (1983), Masry (1996ab) for example.
  - In this case, standard kernel regression smoothers do not take account of the correlation structure in X_t or u_t and estimate the regression function in the same way as if these processes were independent.
  - Furthermore, the variance of such estimators is proportional to the short run variance of u_t, $\sigma_u^2 = \mathrm{var}(u_t)$ and does not depend on the regressor or error covariance functions $\gamma_X(j) = \mathrm{cov}(X_t, X_{t-j})$, $\gamma_u(j) = \mathrm{cov}(u_t, u_{t-j})$, $j \neq 0$.
- Practitioners accustomed to correcting standard errors for dependence believe that the standard errors in nonparametric regression are therefore suspect.

- XLCM introduced an alternative non-parametric estimator of m that was more efficient than the usual estimators and took account of the autocorrelation structure.

- The estimation method of XLCM is unattractive because it requires several degrees of smoothing.

- We propose an alternative estimator of the function m based on solving a type 2 linear integral equation.

- We show that it has attractive theoretical and finite sample properties.
  - In particular, it has smaller asymptotic variance than the main method of XLCM
  - It also works in the case where u_t is a unit root process, whereas standard kernel regression and XLCM procedures do not.

Estimation Method

- First suppose that $A(L), B(L)$ are known.
- Letting $Z_t = B(L)Y_t$ we have
$$Z_t = \sum_{j=1}^{\infty} a_j m(X_{t-j}) + \varepsilon_t,$$
which is an additive autoregression with i.i.d. errors and with a restriction on the additive components.
- Define m as the minimizer of the criterion
$$E\left[\left\{Z_t - \sum_{j=1}^{\infty} a_j m(X_{t-j})\right\}^2\right].$$
- It follows that m satisfies the first order condition
$$E\left[\left\{Z_t - \sum_{j=1}^{\infty} a_j m(X_{t-j})\right\}\sum_{k=1}^{\infty} a_k h(X_{t-k})\right] = 0$$
for any measurable function h.
- This implies that, taking $h(.)$ to be the

Dirac delta function,

$$\sum_{j=1}^{\infty} a_j E[Z_t|X_{t-j} = x] = \sum_{j=1}^{\infty} a_j^2 m(x)$$
$$+\sum\sum_{j\neq k} a_j a_k E[m(X_{t-j})|X_{t-k} = x].$$

- This can be re-expressed as

$$m(x) = m^*(x) + \int \mathcal{H}(x,y)m(y)f_0(y)dy$$
$$m = m^* + \mathcal{H}m,$$

which is a linear type 2 integral equation in $L_2(f_0)$ with

$$m^*(x) = \sum_{j=1}^{\infty} a_j^* E[Z_t|X_{t-j} = x]$$

$$\mathcal{H}(x,y) = -\sum_{j=\pm 1}^{\pm\infty} a_j^+ \frac{f_{0,j}(y,x)}{f_0(y)f_0(x)},$$

where $a_j^* = a_j / \sum_{j=1}^{\infty} a_j^2$ and $a_j^+ = \sum_{k\neq 0} a_{j+k}a_j / \sum_{l=1}^{\infty} a_l^2$, while f_0 is the density of X_t and $f_{0,j}$ is the joint density of (X_t, X_{t-j}).

Assumption A1. *The operator* $\mathcal{H}(x,y)$ *is Hilbert-Schmidt i.e.,*

$$\int\int \mathcal{H}(x,y)^2 f_0(x) f_0(y) dx dy < \infty.$$

- Under assumption A1, $\mathcal{H}$ is a self-adjoint bounded compact linear operator on the Hilbert space of functions $L_2(p_0)$, and therefore has a countable number of eigenvalues:
$$\infty > |\lambda_1| \geq |\lambda_2| \geq ...,$$
with $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$.

Assumption A2. *There exist no* $m \in \mathcal{M}$ *with* $\|m\|_2 = 1$ *such that* $\sum_{j=1}^{\infty} a_j m(y_{t-j}) = 0$ *with probability one.*

- This condition rules out a certain 'concurvity' in the stochastic process. That is, the data cannot be functionally related in this particular way.
- Under A1-A2 there exists a unique solution to the integral equation that satisfies
$$m = (I - \mathcal{H})^{-1} m^*.$$

Further Details for a special case

- Suppose that X_t is a scalar absolutely continuous random variable and that
$$A(L) = B(L) = 1 - \rho_0 L,$$

- i.e., this is nonparametric regression with AR(1) disturbances.
$$Y_t = m(X_t) + u_t,$$
$$u_t = \rho_0 u_{t-1} + \varepsilon_t$$

- Then we write
$$Z_t(\rho) = Y_t - \rho Y_{t-1}$$
for any ρ.

The estimation method is

1. For each ρ compute estimators of
$$\widehat{m}^*_\rho, \widehat{\mathcal{H}}_\rho$$
of $m^*_\rho, \mathcal{H}_\rho$

2. Solve an empirical version of the integral equation to obtain an estimator
$$\widehat{m}_\rho$$
of m_ρ

3. Choose $\widehat{\rho}$ to minimize the profiled negative log likelihood or least squares criterion with respect to ρ.

4. Let
$$\widehat{m}(x) = \widehat{m}_{\widehat{\rho}}(x).$$

- For any $\{Z_t(\rho)\}$ and lag j define $\widehat{g}_j(x;\rho) = \widehat{a}_0$, where $(\widehat{a}_0, \widehat{a}_1)$ are the minimizers of

$$\sum_{t=j+1}^{T} \{Z_t(\rho) - a_0 - a_1(X_{t-j} - x)\}^2 K_h(X_{t-j} - x)$$

  with respect to (a_0, a_1), where K is a symmetric probability density function, h is a positive bandwidth, and $K_h(.) = K(./h)/h$.

- Then define

$$\widehat{m}^*_\rho(x) = \frac{1}{1+\rho^2}\{\widehat{g}_0(x;\rho) - \rho\widehat{g}_1(x;\rho)\}$$

$$\widehat{\mathcal{H}}_\rho(x,y) = \frac{-\rho}{1+\rho^2}\frac{\widehat{f}_{0,1}(x,y) + \widehat{f}_{0,1}(y,x)}{\widehat{f}_0(x)\widehat{f}_0(y)},$$

  where

$$\widehat{f}_{0,1}(y,x) = \frac{1}{T-1}\sum_{t=2}^{T} K_h(y - X_t)K_h(x - X_{t-1}),$$

$$\widehat{f}_0(x) = \frac{1}{T}\sum_{t=1}^{T} K_h(x - X_t).$$

- Define $\widehat{m}_\rho$ as any solution to the equation
$$m = \widehat{m}^*_\rho + \widehat{\mathcal{H}}_\rho m,$$
in $L_2(\widehat{f}_0)$

- Let
$$\widehat{\rho} = \arg \min_{\rho \in [-1,1]} \widehat{S}_T(\rho),$$
where
$$\widehat{S}_T(\rho) = \sum_{t=2}^{T} \left\{ Y_t - \rho Y_{t-1} - \widehat{m}_\rho(X_t) + \rho \widehat{m}_\rho(X_{t-1}) \right\}^2 .$$

- Finally, let
$$\widehat{m}(x) = \widehat{m}_{\widehat{\rho}}(x).$$

- How we solve the integral equation in practice. Note that one can rewrite as an integral equation on $[0,1]^2$ as
$$m_\rho^\dagger(s) = m_\rho^{*\dagger}(s) + \int_0^1 \mathcal{H}_\rho^\dagger(s,t) m_\rho(t) dt,$$
where $\mathcal{H}_\rho^\dagger(s,t) = \mathcal{H}_\rho^\dagger(F_0^{-1}(s), F_0^{-1}(t))$ with $y = F_0^{-1}(s), x = F_0^{-1}(t)$ and $m_\rho^\dagger(t) = m_\rho(F_0^{-1}(t))$ and $m_\rho^{*\dagger}(t) = m_\rho^*(F_0^{-1}(t))$ and F_0 is the c.d.f. of X_t. For simplicity we drop the superfluous $\dagger$ superscript in the sequel.
- Let $\{t_{j,n}, j = 1, \ldots, n\}$ be some equally spaced grid of points in $[0,1]$, and let $q_{j,n} = \widehat{F}_0^{-1}(t_{j,n})$ be the empirical $t_{j,n}$ quantile of X_t. Now approximate the origirnal equation by
$$\widehat{m}_\rho(q_{i,n}) = \widehat{m}_\rho^*(q_{i,n}) + \sum_{j=1}^n \widehat{\mathcal{H}}_\rho(q_{i,n}, q_{j,n}) \widehat{m}_\rho(q_{j,n}),$$
$$i = 1, \ldots, n.$$

- The linear system can be written in matrix notation
$$(I_n - \widehat{\mathbf{H}}_\rho)\widehat{\mathbf{m}}_\rho = \widehat{\mathbf{m}}^*_\rho,$$
where I_n is the $n \times n$ identity,
$$\widehat{\mathbf{m}}_\rho = (\widehat{m}_\rho(q_{1,n}), \ldots, \widehat{m}_\rho(q_{n,n}))^\top$$
$$\widehat{\mathbf{m}}^*_\rho = (\widehat{m}^*_\rho(q_{1,n}), \ldots, \widehat{m}^*_\rho(q_{n,n}))^\top$$
$$\widehat{\mathbf{H}}_\rho = -\left[\frac{\rho}{1+\rho^2}\frac{\widehat{f}_{0,1}(q_{i,n}, q_{j,n}) + \widehat{f}_{0,1}(q_{j,n}, q_{i,n})}{\widehat{f}_0(q_{i,n})\widehat{f}_0(q_{j,n})}\right]_{i,j=1}^{n}$$
is an $n \times n$ matrix.
- We then find the solution values
$$\widehat{\mathbf{m}}_\rho = (\widehat{m}_\rho(q_{1,n}), \ldots, \widehat{m}_\rho(q_{n,n}))^\top$$
to this system by direct inversion when n is less than say 2000.

Main Result

- We shall assume that the error process $\{u_t\}$ is independent of the process $\{X_t\}$ and we assume that $\{X_t\}$ is a α-mixing process.
- Suppose that Assumptions 1 to 7 hold. Then,

$$\sqrt{Th}\left(\widehat{m}(x) - m(x) - h^2 b(x)\right)$$
$$\Longrightarrow N\left(0, \frac{\sigma_\varepsilon^2}{f_0(x)(1+\rho_0^2)}||K||_2^2\right).$$

- Compare this with
  - the usual kernel estimator, which has variance
  $$||K||_2^2 \frac{\sigma_u^2}{f_0(x)} = ||K||_2^2 \frac{\sigma_\varepsilon^2}{f_0(x)(1-\rho_0^2)},$$
  - the estimator of XLCM, which has variance
  $$||K||_2^2 \frac{\sigma_\varepsilon^2}{f_0(x)}.$$

- Note that the standard nonparametric regression estimator is not consistent when $\rho_0 = 1$, whereas our procedure is.

- The XLCM procedure also does not work in the unit root case because it relies on the initial standard nonparametric regression estimator.

The parametric part

- When $|\rho_0| < 1$
$$\sqrt{T}(\widehat{\rho} - \rho_0) \Longrightarrow N(0, V),$$
- Consistency is shown by
$$\sup_{|\rho - \rho_0| \leq \epsilon} \left| T^{-1} \widehat{S}_T(\rho) - S(\rho) \right| = o_p(1)$$
for some $\epsilon > 0$; and unique minimization of limit $S(\rho)$.
- Distribution theory follows from Taylor expansion.
- When $\rho_0 = 1$ we conjecture that
$$T(\widehat{\rho} - \rho_0) \Longrightarrow D$$
for some unit root distribution.
- Consistency is shown by
$$T^{-1} \widehat{S}_T(1) \xrightarrow{P} \sigma_\varepsilon^2$$
and for all $\epsilon > 0$
$$\liminf_{T \to \infty} \inf_{1 - \rho \geq \epsilon} \left| T^{-1} \left(\widehat{S}_T(\rho) - \widehat{S}_T(1) \right) \right| > 0.$$

Heuristic Argument

- For each ρ

$$\begin{aligned}
\widehat{m}_\rho - m_\rho &= (I - \widehat{\mathcal{H}}_\rho)^{-1}\widehat{m}^*_\rho - (I - \mathcal{H}_\rho)^{-1}m^*_\rho \\
&= (I - \widehat{\mathcal{H}}_\rho)^{-1}(\widehat{m}^*_\rho - m^*_\rho) \\
&\quad + \left[(I - \widehat{\mathcal{H}}_\rho)^{-1} - (I - \mathcal{H}_\rho)^{-1}\right] m^*_\rho \\
&= (I - \widehat{\mathcal{H}}_\rho)^{-1} \\
&\quad \times \left[(\widehat{m}^*_\rho - m^*_\rho) + (\widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho)(I - \mathcal{H}_\rho)^{-1}m^*_\rho\right] \\
&= (I - \widehat{\mathcal{H}}_\rho)^{-1}\left[(\widehat{m}^*_\rho - m^*_\rho) + (\widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho)m^*_\rho\right].
\end{aligned}$$

- Therefore, provided

$$\widehat{m}^*_\rho - m^*_\rho \text{ and } \widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho$$

are small in an appropriate sense

$$\widehat{m}_\rho - m_\rho \simeq (I - \mathcal{H}_\rho)^{-1}\left[(\widehat{m}^*_\rho - m^*_\rho) + (\widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho)m^*_\rho\right].$$

- It can be further shown that
$\widehat{m}_\rho - m_\rho \simeq (\widehat{m}^*_\rho - m^*_\rho) + (\widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho)m^*_\rho$.

- When $\|\mathcal{H}_\rho\| < 1$ this can be seen as follows since
$$(I - \mathcal{H}_\rho)^{-1} = I + \mathcal{H}_\rho + \mathcal{H}^2_\rho + \ldots$$
and
$$\mathcal{H}_\rho(\widehat{m}^*_\rho - m^*_\rho) = o_p(\widehat{m}^*_\rho - m^*_\rho)$$
because applying the operator $\mathcal{H}_\rho$ involves averaging.

- The full argument allowing also for the case that $\|\mathcal{H}_\rho\| \geq 1$ is given in Linton and Mammen (2003, Proposition 1).

- We proceed in the case that $\rho = \rho_0$. We have

$$\widehat{m}^*_{\rho_0}(x) - m^*_{\rho_0}(x) = \widehat{m}^{*,A}_{\rho_0}(x) + \widehat{m}^{*,B}_{\rho_0}(x) + R_T(x),$$

where

$$\sup_x |R_T(x)| = o_p(T^{-2/5}),$$

while

$$\widehat{m}^{*,A}_{\rho_0}(x) = \frac{1}{1+\rho_0^2} \frac{1}{Thf_0(x)} \times$$

$$\sum_{t=1}^{T} K\left(\frac{x - X_t}{h}\right) \{Z_t(\rho_0) - E[Z_t(\rho_0)|X_t]\}$$

$$-\rho_0 \sum_{t=2}^{T} K\left(\frac{x - X_{t-1}}{h}\right) \{Z_t(\rho_0) - E[Z_t(\rho_0)|X_{t-1}]\},$$

where

$$E[Z_t(\rho_0)|X_t] = m(X_t) - \rho_0 E[m(X_{t-1})|X_t]$$
$$E[Z_t(\rho_0)|X_{t-1}] = E[m(X_t)|X_{t-1}] - \rho_0 m(X_{t-1}).$$

- Likewise

$$(\widehat{\mathcal{H}}_\rho - \mathcal{H}_\rho) m^*_\rho(x) = \widehat{m}^{*,C}_{\rho_0}(x) + \widehat{m}^{*,D}_{\rho_0}(x) + R_{HT}(x),$$

where

$$\sup_x |R_{HT}(x)| = o_p(T^{-2/5}).$$

- It follows that

$$\widehat{m}^{*,A}_{\rho_0}(x) + \widehat{m}^{*,C}_{\rho_0}(x) \simeq \frac{1}{1+\rho_0^2} \frac{1}{Th f_0(x)} \times$$

$$\left[\sum_{t=1}^{T} K\left(\frac{x - X_t}{h}\right) \varepsilon_t - \rho_0 \sum_{t=2}^{T} K\left(\frac{x - X_{t-1}}{h}\right) \varepsilon_t \right].$$

- This term is asymptotically normal with zero mean and variance given in the theorem.

Numerical Results

- We suppose that
$$m(x) = \beta_0 x^2/2,$$
where
$$X_t \sim N(0,1) \text{ and } \varepsilon_t \sim N(0,\sigma).$$
- We examine the cases $T \in \{800, 400, 200\}$ and $\rho_0 \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$, and use $ns = 1000$ replications.
- We compute our estimator $\widehat{m}$ using 200 grid points and assuming in the first instance that ρ_0 is known.
- We also compute the standard local linear estimator $\widetilde{m}$, in both cases the Gaussian kernel was used.

- We chose bandwidth to be optimal according to (asymptotic) weighted mean squared error

$$P_\infty(\widehat{m}) = \operatorname*{plim}_{T\to\infty} T^{4/5} \int_{-c}^{c} [\widehat{m}(x) - m(x)]^2 f_0(x) dx,$$

- This gives

$$h_{opt} = c_K c_M T^{-1/5},$$

  where

$$c_K = (2c||K||_2^2/\mu_2^2(K))^{1/5}$$

  is to do with the kernel and

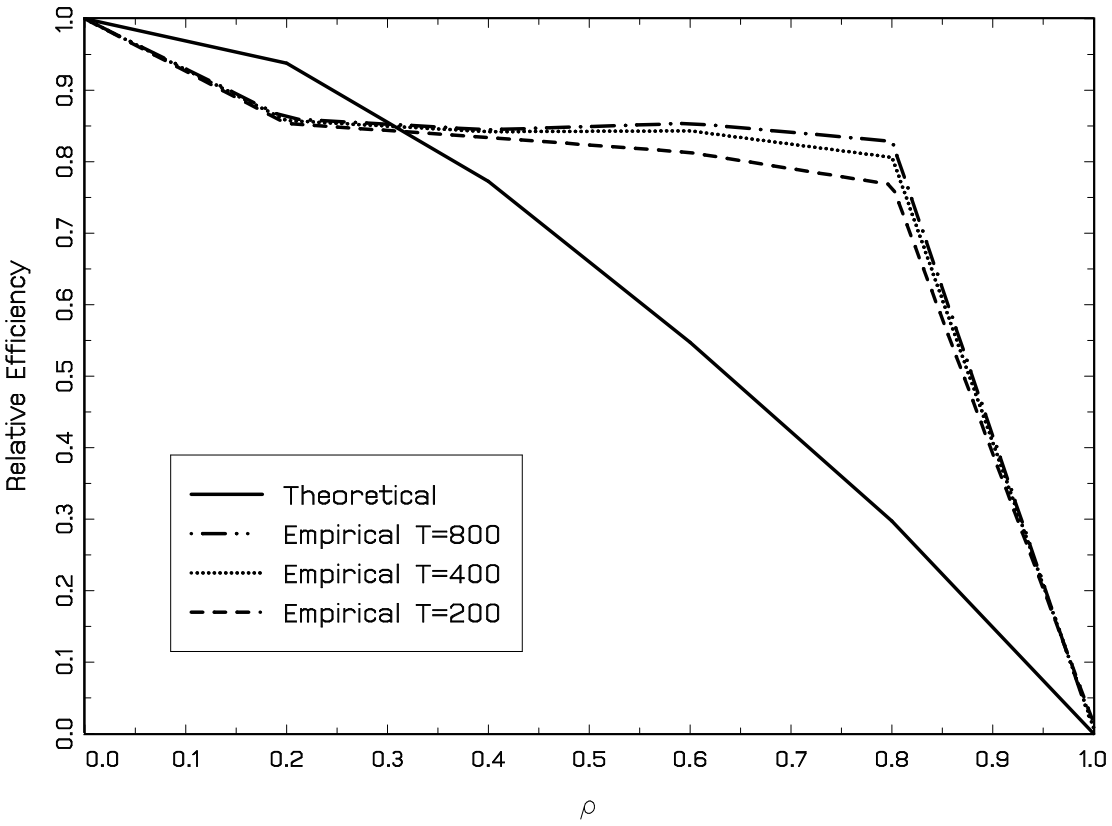
$$c_M = (\sigma_\varepsilon^2/(1+\rho_0^2)\beta_0^2(F_0(c) - F_0(-c)))^{1/5},$$

  where $F_0(x)$ is the c.d.f. of the covariate, is to do with the model.

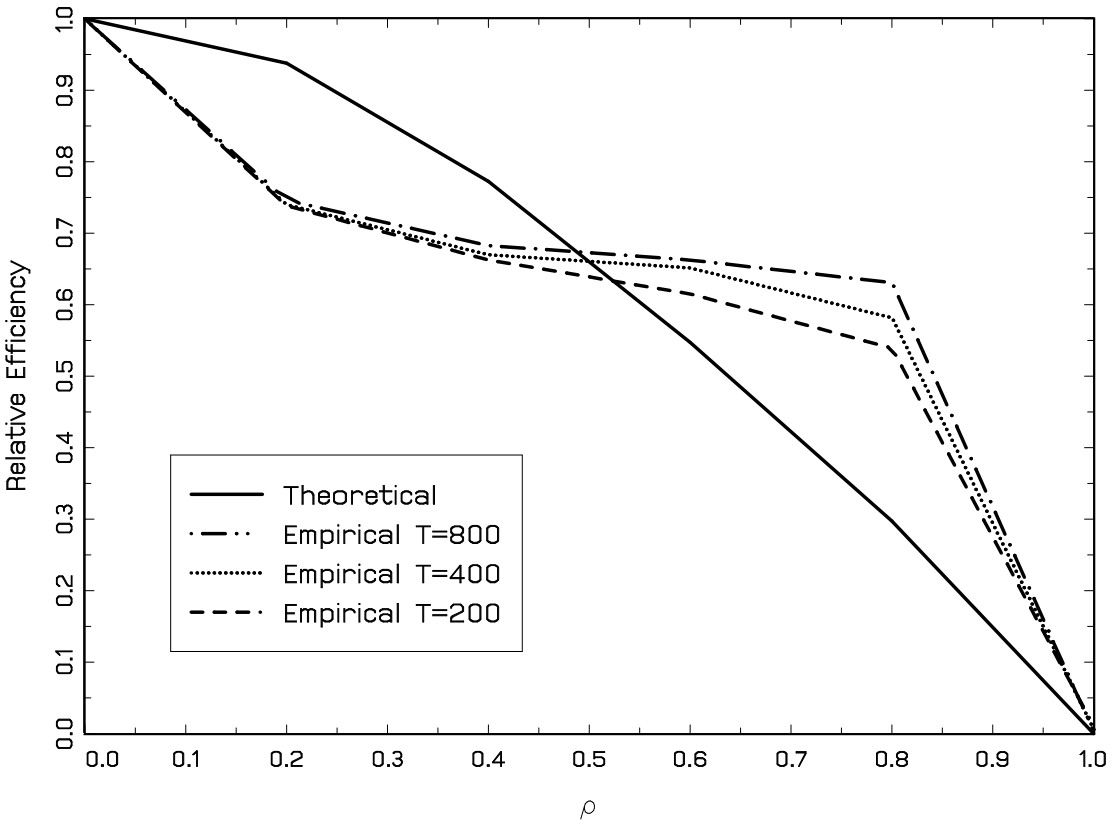
- We have taken $c = 2$, which corresponds to an interval containing almost 95% of the covariate distribution.

- For the standard local linear estimator the optimal bandwidth is
$$c_K c_M^* T^{-1/5}$$
with
$c_M^* = (\sigma_\varepsilon^2/(1-\rho_0^2)\beta_0^2(F_0(c)-F_0(-c))^{1/5}$
provided $\rho_0 \neq 1$ (when $\rho_0 = 1$ we set ρ_0 in the formula arbitrarily to 0.9).

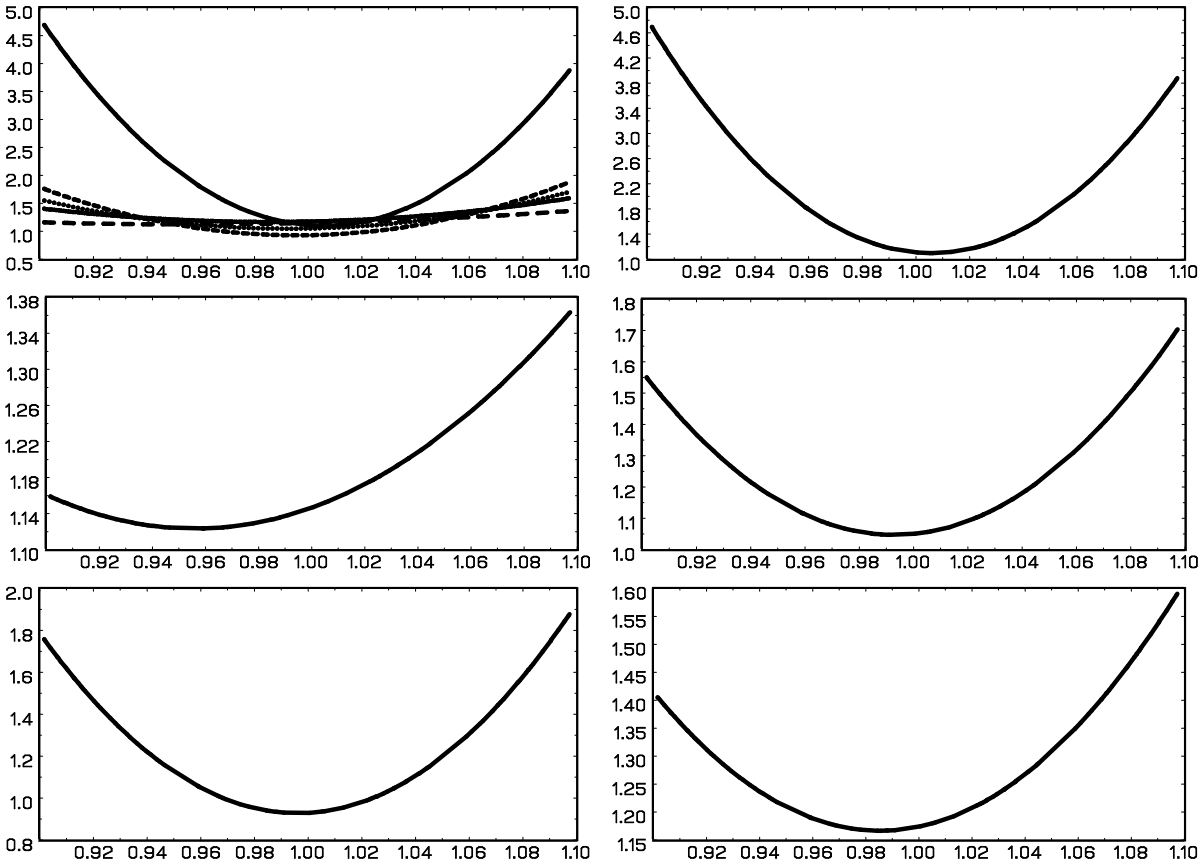
- In Figure 1 below we report the relative value of the performance measure
$$P_T(\widehat{m}) = E\int_{-c}^{c}\left[\widehat{m}(x)-m(x)\right]^2 f_0(x)dx$$
to $P_T(\widetilde{m})$, where E is computed by the average over Monte Carlo simulations. Both estimators use their optimal bandwidths, and consequently their theoretical relative efficiency is
$$\left(\frac{1-\rho_0^2}{1+\rho_0^2}\right)^{4/5}.$$



Shows the empirical performance ratio $P_T(\widehat{m})/P_T(\widetilde{m})$ for different sample sizes along with the asymptotic value $P_\infty(\widehat{m})/P_\infty(\widetilde{m})$ predicted from the asymptotic theory. X_t iid $N(0,1)$.



Shows the empirical performance ratio $P_T(\widehat{m})/P_T(\widetilde{m})$ for different sample sizes along with the asymptotic value $P_\infty(\widehat{m})/P_\infty(\widetilde{m})$ predicted from the asymptotic theory. X_t iid $N(0,1)$.
$X_t = 0.95X_{t-1} + u_t$ with $X_t \sim N(0,1)$.



Shows Average Sum of Square Residuals against parameter value in the unit root case from five simulations

Concluding Remarks

- The more general model with $A(L) \neq B(L)$ can be treated similarly.
- Estimation works fine when A, B are low order. Otherwise need to parameterize in some way, so that
$$A(L) = \sum_{j=0}^{\infty} a_j(\theta) L^j$$
and $\theta \in \mathbb{R}^p$
- Can allow for heteroskedasticity with not much change in methods and results.
- For example suppose
$$Y_t = m(X_t) + \varepsilon_t \sigma_t$$
with σ_t^2 some volatility process where ε_t is independent of X_t. We could have
$$\sigma_t^2 = f(X_t, X_{t-1}, \ldots, Y_{t-1}, \ldots).$$
- Suppose that σ_t^2 is known. Then find m to minimize
$$E\left[\left(\frac{Y_t - m(X_t)}{\sigma_t}\right)^2 + \ln \sigma_t^2\right].$$

The solution is

$$m(x) = \frac{E\left[\frac{Y_t}{\sigma_t^2}|X_t\right]}{E\left[\frac{1}{\sigma_t^2}|X_t\right]}.$$

In practice σ_t^2 has to be estimated from residuals.

- If for some smooth unknown function f

$$\sigma_t^2 = f(X_t)$$

then there is no gain, but otherwise there is.