Rate-optimal Tests for Jumps in Diffusion Processes

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Abstract

Suppose one has given a sample of high-frequency intra-day discrete observations of a continuous-time random process (e.g. stock market data) and wants to test for the presence of jumps. We show that the power of any test of this hypothesis depends on the frequency of observation. In particular, we show that if the process is observed at intervals of length $1/n$ and the instantaneous volatility of the process is given by σ_t, at best one can detect jumps of height no smaller than $\sigma_t \sqrt{2\log(n)/n}$. We construct a test which achieves this rate in the case for diffusion-type processes.

Keywords : High Frequency Data, Jump, Likelihood Test.

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1 Introduction

Continuous diffusion models have provided a simple, flexible, and powerful tool to analyze economic and financial data since the time before high frequency data become available. Because data are only observed at discrete times, we do not have full information on the trajectory of the process. Therefore we may have modeling errors due to the discreteness of the observations, but this problem can be significantly mitigated with high frequency data which is now available with technological progress.

High frequency data, however, generate their own challenges: We are not sure that the data generating process is continuous; there may be jumps. Since continuous diffusion models do not capture jumps, researchers must know whether the data contain jumps or not. Furthermore, many datasets (such as when returns are measured over short intervals - say 1 to 5 minutes) contain some contamination commonly called "market microstructure noise". Our aim in this paper is to propose an optimal test for the null hypothesis of continuous diffusion models against an alternative hypothesis of jump diffusion models while allowing for the presence of market microstructure noise in the data. Though several tests (Barndorff-Nielsen and Shephard (2006), hereafter BNS, Ait-Sahalia and Jacod (2009), hereafter AJ) were already introduced, their power properties were not explicitly considered. In this paper, we derive a rate-optimal test valid under fairly general assumptions about the data generating process. Furthermore we compare power of our test and that of other competing tests.

2 Local Power Bound

As the null, we consider the usual (i.e., purely continuous) diffusion model:

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where $W = (W_t : t \in [0, 1))$ is a Wiener process and μ_t and σ_t are non-anticipating random processes fulfilling the usual requirements of Ito-calculus. Later on we will impose additional assumptions on μ and σ. (For example, we will assume them to be smooth to a certain extent, so that the process X_t specified in equation(1) has certain "nice" properties.)

As the alternative, we want to consider jumps present in the diffusion model :

$$dX_t = \mu_t dt + \sigma_t dW_t + J_t d\kappa_t,$$

where J_t is a non-zero random variable whose absolute value specifies the jump size and κ_t is a counting process governing whether there is a jump or not at time t.

The problem, however, is that in empirical practice, we cannot observe the entire process. Instead, it is observed only at discrete times $t = i/n$, where n is a positive integer denoting the "sample size" and

$$0 \leq i \leq n.$$

The first test for this testing problem - and still the "gold standard" for all tests was developed by Barndorff-Nielsen and Shephard (2002). Since the problem of detecting jumps is of enormous practical importance, further research has occurred in this field. An alternative test was developed by Ait-Sahalia and Jacod (2009), and an informal "testing procedure" was provided by Lee and Mykland (2008).

To the best of our knowledge, none of these contributions, however, discussed power of the tests. In this paper, we establish the following two results.

1. Clearly, when n, the number of observations increases, we should expect our test to have "better" power. In particular, we want to consider the power against local alternatives for the jumping processes. So we consider for each n the alternative

$$J_t^{(n)} = c_n$$

where we assume that c_n is a sequence converging to zero, and assume the process κ_t remains (uniformly) bounded. (So we assume there is only a maximum number of jumps). Let $\varepsilon > 0$ be arbitrary. Then we show that - even if we know that $\sigma_t = \sigma$ - it is impossible to construct tests that have nontrivial power against alternatives with

$$c_n = \sigma \frac{\sqrt{2(1-\varepsilon)\ln n}}{\sqrt{n}} \tag{2}$$

2. As a main result, we show that one can construct a test so that (even in the general case)

$$c_n = \sigma_t \frac{\sqrt{2(1+\varepsilon)\ln n}}{\sqrt{n}}$$

the power of the test converges to one. So in a certain way, our tests attain the "optimal rate". This is an advantage over the classical BNS or AJ tests: their local alternatives shrink with the order $n^{-1/4}$ (or - in the case of AJ - with the order of $n^{-(1/2+1/p)}$, where p is a positive number determining the test statistic. Moreover, our test is able to deal with simple mis-specification due to microstructure).

Let us first deal with our first assertion. Let assume we even deal with the simplest case, namely $\mu_t = 0$ and $\sigma_t = 1$, so our underlying process X_t is a Wiener process. Then let us assume that - under the alternative - we only have one jump, and the time of the jump is distributed uniformly in the interval $[0, 1]$. We first will show that even under this rather ideal conditions we will be unable to construct tests with nontrivial power if the c_n are following (2).

Theorem 1 *We want to test the null of X_t being a Wiener process W_t ,of known variance, against the alternative of*

$$X_t = W_t + c_n I\{\tau \geq t\},$$

where τ is an independent random variable following an uniform distribution. Suppose we observe the process X_t only at the time points $0, 1/n, 2/n, ...1$. Suppose c_n follows (2) (or is smaller then this bound). Then it is impossible to construct nontrivial tests.

Proof. We did assume the variance of the Wiener process W to be known. Without loss of generality, we can assume this variance to be 1. Let P_n be the probability measure of $(X_0, X_{1/n}, X_{2/n}, ..., X_1)$ under the null, and Q_n be the measure under the alternative. Let the z_i be defined as

$$z_i = \left(X_{i/n} - X_{(i-1)/n}\right)\sqrt{n}$$

Then we can easily see that the z_i are i.i.d. standard normal, and that

$$\frac{dQ_n}{dP_n} = \frac{1}{n}\sum_{i=1}^{n} \exp\left\{(c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2\right\}$$

Since each of the z_i is standard normal, the expectation of each $\exp\left\{(c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2\right\}$ equals one. Moreover, $E\left[\exp\left\{(c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2\right\}^2\right] = E\left[\exp\left\{2(c_n\sqrt{n})z_i - (c_n\sqrt{n})^2\right\}\right] = \exp\left\{(c_n\sqrt{n})^2\right\}$. Hence the variance of $\frac{dQ_n}{dP_n}$ is smaller than $\exp\left\{(c_n\sqrt{n})^2\right\}/n$, which converges to zero if c_n follows (2).

Since we are interested in small $\varepsilon > 0$ (the smaller we choose ε, the bigger are our jumps), we can maintain the assumption that

$$\varepsilon < \frac{2}{3}. \tag{3}$$

We are convinced that our result holds for larger ε, too. Assuming (3), however, greatly simplifies one part of the proof.

Let us first show that

$$\frac{dQ_n}{dP_n} \to 1. \tag{4}$$

in probability. The density $\frac{dQ_n}{dP_n}$ is a nonnegative random variable. Hence we can show (4) by showing that its Laplace transforms obeys

$$E\left[\exp\left(-s\frac{dQ_n}{dP_n}\right)\right] \to \exp(-s)$$

for all positive s, which is equivalent to

$$\ln E\left[\exp\left(-s\frac{dQ_n}{dP_n}\right)\right] \to -s.$$

With (2), we can simplify the expression for the density. One can easily see that

$$\frac{dQ_n}{dP_n} = \frac{1}{n^{2-\varepsilon}}\sum_{i=1}^{n}\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right).$$

Since each of the z_i is standard normal, and the z_i are independent and identically distributed, we can simplify the Laplace transforms

$$\ln E\left[\exp\left(-s\frac{dQ_n}{dP_n}\right)\right] = n\ln\left[-\frac{s}{n^{2-\varepsilon}}E\left\{\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right)\right\}\right].$$

Hence we have to show that

$$n\ln\left[-\frac{s}{n^{2-\varepsilon}}E\left\{\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right)\right\}\right] \to -s. \tag{5}$$

It is well known that

$$\lim_{x\to 1}\frac{\ln x}{x-1} = 1.$$

Using that result, it can easily be established that (5) is equivalent to

$$\frac{n}{s}E\left[1-\frac{s}{n^{2-\varepsilon}}\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right)\right] \to 1,$$

which we will prove to be correct. Let $\Phi(.)$ be the cumulative distribution function (for short cdf) of the standard normal. Then the cdf of

$$\exp\left\{z_i\sqrt{2(1-\varepsilon)\ln n}\right\}$$

equals

$$\Phi\left(\frac{\ln x}{\sqrt{2(1-\varepsilon)\ln n}}\right).$$

Hence

$$\begin{aligned}
& E\left[1-\exp\left\{-\frac{s}{n^{2-\varepsilon}}\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right)\right\}\right] \\
= & \int\left\{1-\exp\left(-\frac{s}{n^{2-\varepsilon}}x\right)\right\}d\Phi\left(\frac{\ln x}{\sqrt{2(1-\varepsilon)\ln n}}\right) \\
= & \frac{1}{\sqrt{2\pi}}\int\left\{1-\exp\left(-\frac{s}{n^{2-\varepsilon}}x\right)\right\}\exp\left(-\frac{(\ln x)^2}{4(1-\varepsilon)\ln n}\right)\frac{1}{x}\frac{1}{\sqrt{2(1-\varepsilon)\ln n}}dx.
\end{aligned}$$

Let us re-scale this expression and define S_n by

$$\begin{aligned}
S_n &= \frac{n}{s}E\left[1-\exp\left\{-\frac{s}{n^{2-\varepsilon}}\exp\left(z_i\sqrt{2(1-\varepsilon)\ln n}\right)\right\}\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{n^{-1+\varepsilon}}{\sqrt{2(1-\varepsilon)\ln n}}\int_0^\infty\frac{1-\exp\left(-\frac{s}{n^{2-\varepsilon}}x\right)}{\frac{s}{n^{2-\varepsilon}}x}\exp\left(-\frac{(\ln x)^2}{4(1-\varepsilon)\ln n}\right)dx.
\end{aligned}$$

Then we have to show that

$$S_n \to 1. \tag{6}$$

Substituting

$$y = \frac{s}{n^{2-\varepsilon}}x$$

yields

$$\begin{aligned}
S_n &= \frac{1}{\sqrt{2\pi}}\frac{n^{-(1-\varepsilon)}}{\sqrt{2(1-\varepsilon)\ln n}}\int_0^\infty\frac{1-\exp(-y)}{y}\exp\left[-\frac{\{\ln y+(2-\varepsilon)\ln n-\ln s\}^2}{4(1-\varepsilon)\ln n}\right]\frac{n^{2-\varepsilon}}{s}dy \\
&= \frac{1}{\sqrt{2\pi}}\frac{n}{\sqrt{2(1-\varepsilon)\ln n}}\frac{1}{s}\int_0^\infty\frac{1-\exp(-y)}{y}\exp\left[-\frac{\{\ln y+(2-\varepsilon)\ln n-\ln s\}^2}{4(1-\varepsilon)\ln n}\right]dy.
\end{aligned}$$

Hence we will have to evaluate

$$\int_0^\infty\frac{1-\exp(-y)}{y}\exp\left[-\frac{\{\ln y+(2-\varepsilon)\ln n-\ln s\}^2}{4(1-\varepsilon)\ln n}\right]dy. \tag{7}$$

Let us first deal with the second factor. We can split it up into three factors:

$$\begin{aligned} & \exp\left[-\frac{\{\ln y + (2-\varepsilon)\ln n - \ln s\}^2}{4(1-\varepsilon)\ln n}\right] \\ = & \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} \exp\left\{\frac{-(\ln y)((2-\varepsilon)\ln n - \ln s)}{2(1-\varepsilon)\ln n}\right\} \\ & \times \exp\left[-\frac{\{(2-\varepsilon)\ln n - \ln s\}^2}{4(1-\varepsilon)\ln n}\right]. \end{aligned}$$

The last factor is a constant, so when evaluating (7) we can take it outside the integral. Moreover, we can easily see that with

$$C_n = \exp\left[-\frac{\{(2-\varepsilon)\ln n - \ln s\}^2}{4(1-\varepsilon)\ln n}\right]$$

we can simplify the expression to

$$C_n = n^{-\frac{(2-\varepsilon)^2}{4(1-\varepsilon)}} s^{\frac{(2-\varepsilon)}{2(1-\varepsilon)}} \exp\left\{-\frac{(\ln s)^2}{4(1-\varepsilon)\ln n}\right\}.$$

For the second factor, we can see that it equals to

$$y^{F_n}$$

where

$$F_n = -\frac{2-\varepsilon}{2(1-\varepsilon)} + \frac{\ln s}{2(1-\varepsilon)\ln n}.$$

So

$$S_n = \frac{1}{\sqrt{2\pi}} \frac{n}{\sqrt{2(1-\varepsilon)\ln n}} \frac{1}{s} C_n \int_0^\infty \frac{1-\exp(-y)}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy. \tag{8}$$

Hence we have to evaluate

$$\begin{aligned} & \int_0^\infty \frac{1-\exp(-y)}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy & (9) \\ = & \int_0^\infty \frac{1-\exp(-y)-y}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy & (10) \\ & + \int_0^\infty y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy & (11) \end{aligned}$$

The second part can be evaluated rather easily: Substitute

$$z = \frac{1}{\sqrt{2(1-\varepsilon)\ln n}} \ln y$$

and therefore

$$dz = \frac{1}{\sqrt{2(1-\varepsilon)\ln n}} \frac{dy}{y}.$$

$$\begin{aligned} & \int_0^\infty y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy \\ = & \sqrt{2(1-\varepsilon)\ln n} \int_{-\infty}^\infty \exp\left\{z\left(F_n+1\right)\sqrt{2(1-\varepsilon)\ln n}\right\} \exp\left(-\frac{z^2}{2}\right) dz. \end{aligned} \tag{12}$$

Then we have

$$\begin{aligned} & \int_{-\infty}^\infty \exp\left\{z(F_n+1)\sqrt{2(1-\varepsilon)\ln n}\right\} \exp\left(-\frac{z^2}{2}\right) dz \\ = & \exp\left[\left\{(F_n+1)\sqrt{2(1-\varepsilon)\ln n}\right\}^2/2\right] \int_{-\infty}^\infty \exp\left[-\left\{z - F_n\sqrt{2(1-\varepsilon)\ln n}\right\}^2/2\right] dz \\ = & \sqrt{2\pi}\exp\left\{(F_n+1)^2(1-\varepsilon)\ln n\right\}. \end{aligned}$$

Since

$$\begin{aligned} & (F_n+1)^2(1-\varepsilon) \\ = & \left\{1 - \frac{2-\varepsilon}{2(1-\varepsilon)} + \frac{\ln s}{2(1-\varepsilon)\ln n}\right\}^2 (1-\varepsilon) \\ = & \left\{-\frac{\varepsilon}{2(1-\varepsilon)} + \frac{\ln s}{2(1-\varepsilon)\ln n}\right\}^2 (1-\varepsilon) \\ = & \frac{\varepsilon^2}{4(1-\varepsilon)} - \frac{\ln s}{\ln n}\left(\frac{\varepsilon}{2(1-\varepsilon)}\right) + \frac{(\ln s)^2}{(\ln n)^2}\frac{1}{4(1-\varepsilon)}, \end{aligned}$$

we have

$$\begin{aligned} & \exp\left\{(F_n+1)^2(1-\varepsilon)\ln n\right\} \\ = & \exp\left[\left\{\frac{\varepsilon^2}{4(1-\varepsilon)} - \frac{\ln s}{\ln n}\left(\frac{\varepsilon}{2(1-\varepsilon)}\right) + \frac{(\ln s)^2}{(\ln n)^2}\frac{1}{4(1-\varepsilon)}\right\}\ln n\right] \\ = & n^{\frac{\varepsilon^2}{4(1-\varepsilon)}} s^{-\frac{\varepsilon}{2(1-\varepsilon)}} \exp\left(\frac{1}{\ln n}\frac{(\ln s)^2}{4(1-\varepsilon)}\right). \end{aligned}$$

Hence (using (12)) we can conclude that for all $s > 0$,

$$\lim_{n\to\infty} \frac{\int_0^\infty y^{F_n}\exp(-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n})dy}{n^{\frac{\varepsilon^2}{4(1-\varepsilon)}} s^{-\frac{\varepsilon}{2(1-\varepsilon)}}\sqrt{2\pi}\sqrt{2(1-\varepsilon)\ln n}} = 1. \tag{13}$$

We now return to our original task, namely the analysis of (9), which is the sum of (10) and (11). We can easily see from (13) that for all fixed $s > 0$, (11) diverges to infinity as $n \to \infty$.

We now will show that (10) remains $O(1)$. First of all let us observe that

$$F_n \to -\frac{2-\varepsilon}{2(1-\varepsilon)}. \tag{14}$$

As $\varepsilon > 0$,

$$-\frac{2-\varepsilon}{2(1-\varepsilon)} < -1.$$

Our assumption (3), namely that $\varepsilon > 2/3$, implies that

$$-\frac{2-\varepsilon}{2(1-\varepsilon)} > -2.$$

So the limit of the F_n lies between -1 and -2. Hence we can find constants α, β with

$$-2 < \alpha < \beta < -1.$$

So that for all but finitely many n,

$$\alpha < F_n < \beta. \tag{15}$$

Without loss of generality we can assume that (15) holds for all n. Let us now analyze the integrand in (10):

$$\frac{1-\exp(-y)-y}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} \tag{16}$$

The first factor,

$$\psi(y) = \frac{1-\exp(-y)-y}{y}$$

is easily seen to be uniformly bounded for all nonnegative real y. So there exists an M that

$$|\psi(y)| \leq M$$

for all nonnegative y. Using the power series representation for $\exp(\cdot)$, one can easily see that $\psi(.)$ is an analytic function for all y and

$$\psi(0) = 0.$$

Since any analytic function has derivatives bounded on any compact set, we may conclude that for $0 \leq y \leq 1$,

$$|\psi(y)| \leq Cy$$

for some universal constant C. The third factor is easily seen to have absolute value smaller than 1.

Let us now distinguish two cases: For $y \leq 1$,

$$\begin{aligned} & \left| \frac{1-\exp(-y)-y}{y} y^{F_n} \exp(-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}) \right| \\ \leq \ & C y^{1+\alpha}. \end{aligned}$$

Since $\alpha > -2$,

$$\begin{aligned} & \left| \int_0^1 \frac{1-\exp(-y)-y}{y} y^{F_n} \exp(-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}) dy \right| \\ \leq \ & C \int_0^1 y^{1+\alpha} dy = C \frac{1}{2+\alpha}. \end{aligned} \tag{17}$$

For $y \geq 1$,

$$\begin{aligned} & \left| \frac{1-\exp(-y)-y}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} \right| \\ \leq \ & M y^{\beta} \end{aligned}$$

and since $\beta < -1$,

$$\begin{aligned} & \left| \int_1^\infty \frac{1-\exp(-y)-y}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy \right| \\ \leq \ & M \int_1^\infty y^{\beta} dy = -M \frac{1}{1+\beta}. \end{aligned}$$

Hence we may conclude that

$$\left| \int_0^\infty \frac{1-\exp(-y)-y}{y} y^{F_n} \exp\left\{-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}\right\} dy \right|$$

remains uniformly bounded in n. Therefore for all s,

$$\lim_{n\to 0} \frac{\int_0^\infty \frac{1-\exp(-y)-y}{y} y^{F_n} \exp(-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}) dy}{\sqrt{2(1-\varepsilon)\ln n}\sqrt{2\pi} n^{\frac{\varepsilon^2}{4(1-\varepsilon)}} s^{-\frac{\varepsilon}{2(1-\varepsilon)}}} = 0.$$

Now we can combine this result with (13) and (9) and conclude that

$$\lim_{n\to 0} \frac{\int_0^\infty \frac{1-\exp(-y)}{y} y^{F_n} \exp(-\frac{(\ln y)^2}{4(1-\varepsilon)\ln n}) dy}{\sqrt{2(1-\varepsilon)\ln n}\sqrt{2\pi} n^{\frac{\varepsilon^2}{4(1-\varepsilon)}} s^{-\frac{\varepsilon}{2(1-\varepsilon)}}} = 1.$$

We can now use this limit to characterize S_n defined in (8):

$$\lim_{n\to\infty} \frac{S_n}{\frac{1}{\sqrt{2\pi}}\frac{n}{\sqrt{2(1-\varepsilon)\ln n}}\frac{1}{s}\cdot C_n\left\{\sqrt{2(1-\varepsilon)\ln n}\sqrt{2\pi}n^{\frac{\varepsilon^2}{4(1-\varepsilon)}}s^{-\frac{\varepsilon}{2(1-\varepsilon)}}\right\}} \tag{18}$$

$$= \lim_{n\to\infty} \frac{S_n}{n^{(2-\varepsilon)^2/4(1-\varepsilon)}\left\{s^{-\frac{\varepsilon}{2(1-\varepsilon)}-1}\right\}C_n} = 1.$$

Since the C_n were defined as

$$C_n = n^{-\frac{(2-\varepsilon)^2}{4(1-\varepsilon)}}s^{\frac{(2-\varepsilon)}{2(1-\varepsilon)}}\exp\left(-\frac{(\ln s)^2}{4(1-\varepsilon)\ln n}\right),$$

we can easily see that the denominator of the expression in (18) converges to one. But this implies that

$$\lim_{n\to\infty} S_n = 1,$$

which is exactly (6), the result we wanted to prove. Hence the Laplace transforms

$$E\left[\exp\left\{-s\frac{dQ_n}{dP_n}\right\}\right]$$

of the density ratios converge to $\exp(-s)$, which is the Laplace transform of a measure concentrated in 1.

We therefore have shown that the density ratios

$$\frac{dQ_n}{dP_n}$$

converge in distribution to a constant, namely 1 (Theorem 2, p. 431, Feller (1971)). Therefore they converge in probability to this constant, too. Hence for an arbitrary $\eta > 0$,

$$P_n\left[\left\{\left|\frac{dQ_n}{dP_n}-1\right|>\eta\right\}\right]\to 0.$$

Now let A_n be a sequence of events. Then we have

$$\begin{aligned}(1-\eta)P_n(A_n) - P_n\left[\left\{\left|\frac{dQ_n}{dP_n}-1\right|>\eta\right\}\right] &< Q_n(A_n)\\ &< (1+\eta)P_n(A_n) + P_n\left[\left\{\left|\frac{dQ_n}{dP_n}-1\right|>\eta\right\}\right].\end{aligned}$$

Since η was arbitrary, we can conclude that

$$P_n(A_n) - Q_n(A_n) \to 0.$$

Since A_n is an arbitrary sequence of events, we can conclude that the total variation between P_n and Q_n converges to zero, hence for all measurable functions φ_n with $0 \leq \varphi_n \leq 1$ we have

$$\int \varphi_n dP_n - \int \varphi_n dQ_n \to 0.$$

But this is exactly what we wanted to show: For every sequence of tests, the power under the null (P_n) is the same as under the alternative (Q_n). ■

Now we want to present a test statistic, for which we will show that the local power of the test statistic attains this bound. The preceding result indicates that for fixed c_n our best statistic is an exponential sum of the z_i. We should, however, keep in mind that our z_i are increments over shorter and shorter time intervals. In order to consider relevant alternatives, we might be interested in alternatives where c becomes "large". In this case, the test statistic gives more and more increasing influence to bigger values. Continuing with this line of reasoning, it may be a good idea to look mainly at the "largest" absolute values of the increments of X_t: The standard theory of diffusion processes guarantees that, when divided by σ_t, these increments are approximately normal. Next, because we do not know σ_t, we have to estimate it. As σ_t is time-varying, a moving average of the squares of the increments seems to be a natural candidate estimator. This leads us to propose the following test statistic: Define (for an arbitrary n) the return $r_i = r_{i,n}$ by

$$r_i = r_{i,n} = \left(X_{i/n} - X_{(i-1)/n}\right)$$

Then choose an integer ℓ (the "length of the window") and reject the null whenever

$$\tau_n = \sup_i \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-\ell}^2)/\ell}$$

is "too large". This is quite analogous to the test statistics of Lee and Mykland (2008): We standardize the returns by an estimator for $\sigma^2(t, X_t)$. We use the usual quadratic estimator instead of the bipower estimator. One might argue that jumps could unduly affect the properties of our estimator. We think, however, that the much simpler form is justified, essentially in part for the following two reasons:

1. We assume that the jumps are separated events: *Before* the first jump occurs, our estimator for $\sigma^2(t, X_t)$ is not influenced by it.

2. We only use a window of length ℓ to estimate $\sigma^2(t, X_t)$. So a jump will only influence a small number of estimated values of $\sigma^2(t, X_t)$. Our test would get only distorted if we had two (or more) jumps within an interval of length ℓ/n, an event whose probability we assume converges to zero.

The main reason, however, for using this specific estimator is convenience. Specifically, only Lemma 6 is essential to prove the rate-optimality of our proposed test. We think analogous results will hold for more general classes of estimators.

3 Critical Values and Local Power of Our Test

Let $z_i, i = 1, .., n$ be a sequence of independent, identically distributed standard normal random variables. Assume that for each n we have given an $\ell = \ell(n)$, and let us denote by $\mathcal{F}_i$ the $\sigma-$algebra generated by $z_i, z_{i-1}, ..$ Then let us define

$$w_i = \sum_{j=1}^{\ell} z_{i-j}^2,$$

$$\widehat{\sigma}_i^2 = w_i/\ell$$

and

$$\tau_i = z_i^2/\widehat{\sigma}_i^2.$$

For the computation of the critical values the following lemma is very helpful.

Lemma 2 *Suppose that*

$$\ell = o(n),$$

but also

$$\ell \geq 2\log n.$$

Define for each $c > 0$, *a* $K_n^* = K_n^*(c)$ *such that*

$$2E\left[\exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2}\right] = c/n \tag{19}$$

Then,

$$P\left(\max_{i=\ell+1,..n}\tau_i > K_n^*\right) \to 1-\exp\left(-c\right) \ \text{ as } n\to\infty.$$

Proof. First of all we observe that $P\left(\max_{i=\ell+1,..n}\tau_i > K_n^*\right) = 1 - P\left(\max_{i=\ell+1,..n}\tau_i \le K_n^*\right)$ and

$$P\left(\max_{i=\ell+1,..n}\tau_i \le K_n^*\right) = E\left[\prod_{i=\ell+1,..n} I\left\{\tau_i \le K_n^*\right\}\right].$$

It can immediately be seen that the τ_i are $\mathcal{F}_i$−measurable. We will repeatedly apply the optional sampling theorem for various stopping times. Let $\varepsilon > 0$ be arbitrary, and let $M(\varepsilon)$ be defined as in the Appendix A (29), (30), (31).

Let us define the stopping time ν as follows: Define ν to be the first index $m \le n-1$ so that

$$\begin{aligned}\sum_{j=\ell+1}^{m+1}\log E\left[I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right] &< -c(1+\varepsilon)^3 \ \text{ or } \widehat{\sigma}^2_{m+1} < M\left(\varepsilon\right)^2/K_n \text{ or}\\ \sum_{j=\ell+1}^{m+1}\log E\left[I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right] &> -c(1-\varepsilon)\end{aligned}$$

and

$$n \text{ if no such } m \text{ exists.}$$

Observe that ν is indeed a stopping time adapted to $\mathcal{F}_i$: Since for $i \le m+1$ $E\left(\left(\tau_i \le K_n^*\right)/\mathcal{F}_{i-1}\right)$ as well as $\widehat{\sigma}^2_{m+1}$ are $\mathcal{F}_m$-measurable, the event

$$\{\nu = n\} \in \mathcal{F}_m.$$

We contend that

$$\lim_{n\to\infty} P\left[\{\nu = n\}\right] = 1. \tag{20}$$

To demonstrate (20), it is sufficient to first show that

$$P\left[\left\{\inf \widehat{\sigma}^2_i > M\left(\varepsilon\right)^2/K_n\right\}\right] \to 1 \tag{21}$$

and then, because $\log E\left(I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right) \le 0$, then that

$$P\left[\left\{\sum_{j=\ell+1}^{n}\log E\left(I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right) \ge -c(1+\varepsilon)^3\right\} \cap \left\{\inf \widehat{\sigma}^2_i > M\left(\varepsilon\right)^2/K_n\right\}\right] \to 1. \tag{22}$$

Equation (21) is an immediate consequence of Lemma 6, which shows that

$$P\left[\left\{\inf \widehat{\sigma}^2_i \le M\left(\varepsilon\right)^2/K_n\right\}\right] \le nP\left[\left\{\widehat{\sigma}^2_i \le M\left(\varepsilon\right)^2/K_n\right\}\right] \to 0.$$

To prove that (22) is valid, first observe that

$$E\left[I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right] = 2\Phi\left(\sqrt{K_n^*\widehat{\sigma}_i^2}\right) - 1.$$

If $\widehat{\sigma}_i^2 > M(\varepsilon)^2/K_n$, we can use inequality (31) to conclude that

$$\log\left(2\Phi\left(\sqrt{K_n^*\widehat{\sigma}_i^2}\right) - 1\right) \ge -2(1+\varepsilon)^2 \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2}$$

Hence

$$\left\{\sum_{j=\ell+1}^{n} \log E\left[I\left\{\tau_i \le K_n^*\right\}|\mathcal{F}_{i-1}\right] \ge -c(1+\varepsilon)^3\right\} \cap \left\{\inf \widehat{\sigma}_i^2 > M(\varepsilon)^2/K_n\right\}$$

$$\subseteq \left\{-2(1+\varepsilon)^2 \sum_{j=\ell+1}^{n} \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2} \ge -c(1+\varepsilon)^3\right\} \cap \left\{\inf \widehat{\sigma}_i^2 > M(\varepsilon)^2/K_n\right\}$$

Since we already know that $P\left[\left\{\inf \widehat{\sigma}_i^2 > M(\varepsilon)^2/K_n\right\}\right] \to 1$, it is sufficient to show that

$$P\left[\left\{2\sum_{j=\ell+1}^{n} \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2} \le c(1+\varepsilon)\right\}\right] \to 1 \tag{23}$$

Let us now introduce the term Y_j by

$$Y_j = 2\exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2}$$

Then we can easily see that (23) is fulfilled if

$$\sum_{j=\ell+1}^{n} Y_j \to c \tag{24}$$

in probability. By our definition of K_n^*, $EY_j = c/n$. Moreover, we know that $\widehat{\sigma}_i^2$ is distributed according to a scaled χ^2 distribution with ℓ degrees of freedom. Hence it is an elementary exercise to show that $EY_j^2 = O(1/n^2)$, and that Y_j and Y_k are independent if

$$|j-k| > \ell + 1.$$

As $\ell/n \to 0$, we can easily see that the variance of $\sum Y_j$ converges to zero.

This establishes (20). Now it is rather easy to establish the claim stated in our lemma: We have to show that

$$E\left[\prod_{i=\ell+1,..n} I\left\{\tau_i \le K_n^*\right\}\right] \to \exp(-c)$$

Using (20), it is sufficient to show

$$E\left[\prod_{i\leq\nu} I\left\{\tau_i \leq K_n^*\right\}\right] \to \exp(-c)$$

Trivially,

$$E\left[\frac{I\left\{\tau_i \leq K_n^*\right\}}{E\left(I\left\{\tau_i \leq K_n^*\right\}|\mathcal{F}_{i-1}\right)}|\mathcal{F}_{i-1}\right] = 1.$$

A straightforward argument, perfectly analogous to the optional sampling theorem, yields

$$E\left[\frac{E\left[\prod_{i\leq\nu} I\left\{\tau_i \leq K_n^*\right\}\right]}{\prod_{i\leq\nu} E\left[(I\left\{\tau_i \leq K_n^*\right\}|\mathcal{F}_{i-1}\right]}\right] = 1. \tag{25}$$

According to the definition of ν,

$$-(1+\varepsilon)^2 \sum_{j=\ell+1}^{\nu} Y_j \leq \log \prod_{i\leq\nu} E\left[I\left\{\tau_i \leq K_n^*\right\}|\mathcal{F}_{i-1}\right] \leq -(1-\varepsilon)^2 \sum_{j=\ell+1}^{\nu} Y_j \tag{26}$$

and

$$\log \prod_{i\leq\nu} E\left[I\left\{\tau_i \leq K_n^*\right\}|\mathcal{F}_{i-1}\right] \geq -c(1+\varepsilon)^3 \tag{27}$$

Moreover, (20) implies that $P\left[\left\{\sum_{j=\ell+1}^{\nu} Y_j = \sum_{j=\ell+1}^{n} Y_j\right\}\right] \to 1$. Therefore $\sum_{j=\ell+1}^{\nu} Y_j \to c$ as well. Hence it can be seen that (27) and (26) allow us to deduce from (25) that

$$\begin{aligned}\exp(-(1+\varepsilon)^2 c) &\leq \liminf E \prod_{i\leq\nu} I\left\{\tau_i \leq K_n^*\right\} \\ &\leq \limsup E \prod_{i\leq\nu} I\left\{\tau_i \leq K_n^*\right\} \leq \exp(-(1-\varepsilon)^2 c).\end{aligned}$$

Now one can easily see that (20) allows us to replace ν with n in the preceding inequalities, which proves our lemma. ■

So far, we have computed the distribution of our test statistic for a very specific case, namely when $\mu_t = 0$ and $\sigma_t = 1$. We now have to show that the general case given by (1) can be reduced to the specific case discussed above. To achieve this goal, we have to impose some stronger assumptions on μ_t and σ_t.

Theorem 3 *Suppose μ_t and $\ln \sigma_t$ are diffusion processes with a.s. uniformly bounded diffusion coefficients. Then - provided that the conditions of Lemma 2 are satisfied and that $\ell_n / \ln n$ converges to a constant different from 0 - the difference between the test statistic applied to X_t and W_t converges to zero in probability.*

Proof. The proof is rather technical and is provided in the Appendix B. ∎

Lemma 2 and Theorem 3 establish that our construct - rejecting when the max τ_i are larger than K_n^* - is indeed a test. Moreover, it is an easy, but tedious exercise to establish the order of magnitude of K_n^*. Because the distribution of $\widehat{\sigma}_i^2$ is a scaled χ^2 distribution, the left-hand side of equation (19) can be evaluated using the gamma function.[1] Then it is then an easy task to show that

$$K_n^* / (2 \ln n) = 1.$$

Finally, it is elementary to establish our assertion that the test is consistent against jumps of the order

$$\sigma_t \frac{\sqrt{2(1+\varepsilon)\ln n}}{\sqrt{n}}.$$

4 Power of the Some Competing tests

As mentioned in the introduction, several tests for this testing problem already been published. Two of the best-known ones are the tests of BNS and of AJ. These tests are based on the following test statistics:

[1] Given an $\alpha-$level of significance, we can plug-in $c = -\log(1-\alpha)$. Then we can solve for the critical value K, which by definition satisfies

$$\left(\frac{K}{\ell}+1\right)^{-\frac{(\ell-1)}{2}} \left(\frac{K}{\ell}\right)^{-1/2} \frac{\Gamma(\ell/2-1/2)}{\Gamma(1/2)\Gamma(\ell/2)} + \frac{\log(1-\alpha)}{n} = 0 \tag{28}$$

Definition 4 *BNS test statistic(Barndorff-Nielsen and Shephard (2006))*

$$\widehat{\tau}_{BNS}^{LIN} = \frac{\sqrt{n}\left(RV - \frac{\pi}{2}BPV\right)}{\sqrt{\int_0^1 \sigma_u^4 du}}, \widehat{\tau}_{BNS}^{ADJ} = \frac{\sqrt{n}\left(1 - \frac{\pi BPV}{2RV}\right)}{\sqrt{\max\left[1, \int_0^1 \sigma_u^4 du / \left\{\int_0^1 \sigma_u^2 du\right\}^2\right]}} \text{ where}$$

$$RV = \sum_{j=1}^{1/\Delta} r_{t+j\Delta}^2 \text{ and } BPV = \sum_{j=2}^{1/\Delta} \left|r_{t+j\Delta}\right| \left|r_{t+(j-1)\Delta}\right|$$

Another alternative test has recently been proposed by Ait-Sahalia and Jacod. This test is based on the $p-$th power variation of the process X_t and it compares the estimates of this variation for different time scales.

Definition 5 *AJ test statistic(Ait-Sahalia and Jacod (2009))*[2]

$$\begin{aligned}
\widehat{\tau}_{AJ}^{p,k} &= \left(k^{p/2-1} - \widehat{S}(p,k,\Delta)\right) / \sqrt{\widehat{V}_{p,k}} \text{ where } p > 3, k \geq 2, \\
\widehat{S}(p,k,\Delta) &= \widehat{B}(p,k\Delta)/\widehat{B}(p,\Delta),\ \widehat{B}(p,k\Delta)_t = \sum_{i=1}^{n/k} \left|r_{t+ik\Delta}\right|^p \text{ and} \\
& \widehat{V}_{p,k} \text{ is the variance of } \widehat{S}(p,k,\Delta) \text{ under the null.}
\end{aligned}$$

The behavior of both sets of test statistics under our alternatives can easily be analyzed. We just add a specific jump to one of returns. One can easily see that if the jump is of $o(n^{-1/4})$, the difference between the BNS test statistic under the null and the alternative converges to 0. Hence their test will be much less powerful against the alternatives we consider. The same is true for AJ tests: After some calculations, we can see that the corresponding bound is $o\left(n^{-1/2+1/p}\right)$. So - in a bit of contrast to the views of the authors of the test - we think that larger orders p deserve more attention (Our simulation results, however, indicate that the limiting distribution does not

[2] For $\widehat{V}_{p,k}$, they suggest two estimators:
$\widehat{V}_{p,k}^c = \frac{\Delta_n M(p,k)\widehat{A}(2p,\Delta)_t}{\widehat{A}(p,\Delta)_t^2}, \widetilde{V}_{p,k}^c = \frac{\Delta_n M(p,k)\widetilde{A}\left(\frac{p}{p+1},2p+2,\Delta\right)_t}{\widetilde{A}\left(\frac{p}{p+1},p+1,\Delta\right)_t^2}$ where
$M(p,k) = \frac{1}{m_p^2}\left(k^{p-2}(1+k)m_{2p} + k^{p-2}(k-1)m_p^2 - 2k^{p/2-1}m_{k,p}\right),$
$m_p = E(|Z_1|^p) = \pi^{-1/2}2^{p/2}\Gamma\left(\frac{p+1}{2}\right),$
$m_{k,p} = E\left[|Z_1|^p \left|Z_1 + \sqrt{k-1}Z_2\right|^p\right],$
$Z_i \sim^{iid} N(0,1),$
$\widehat{A}(p,\Delta_n)_t = \frac{\Delta_n^{1-p/2}}{m_p}\sum |\Delta_i^n X|^p 1\{|\Delta_i^n X| \leq \alpha\Delta_n^{\varpi}\}, \varpi \in (0,1/2),$
$\widetilde{A}(r,q,\Delta_n)_t = \frac{\Delta_n^{1-qr/2}}{m_r^q}\sum_{i=1}\Pi_{j=1}^q \left|\Delta_{i+j-1}^n X\right|^r$

well approximate the finite-sample distribution for higher order p). In any case, we think that this subject merits further research.

Despite the fact that both tests have "low" power against "our" alternatives, it should be noted that there are situations where the BNS and AJ tests have large advantages over our test. For instance, the relevant alternative could be the occurrence of many jumps in the sample. Assume there is not only one jump, but many. So let us assume that we have L jumps of size J (rather evenly distributed), and let us assume that the number of intervals between successive jumps is always greater than 1. Then it is easily seen from the definition of BNS statistics that the tests are consistent (i.e., its power converges to 1) if

$$\sqrt{n}LJ^2 \to \infty.$$

An analogous result holds for AJ tests. These results are easily explainable by taking into account that both test statistics are constructed from sums: The contributions of several small jumps can cumulate, whereas our test does not allow for this. So one should employ their tests not as tests against the alternative of the occurrence of a single jump, but as tests against Levy-type alternatives. It might be worthwhile to investigate their power of the test against alternatives of this specific type.

5 Simulations

We first consider a very simple ideal process for returns:

$$r_{i/n} = \int_{(i-1)/n}^{i/n} dp_t = \int_{(i-1)/n}^{i/n} \sigma dW_t + \int_{(i-1)/n}^{i/n} J d\kappa_t \qquad \text{(Model 1)}$$

Second, we considered the model devised by Barndorff-Nielsen and Shephard (2002).

$$\begin{aligned} dp\left(t\right) &= \sigma\left(s\right)W\left(ds\right) + Jd\kappa\left(t\right) \text{ with} \qquad \text{(Model 2)} \\ \sigma^2\left(t\right) &= \sigma_1^2\left(t\right) + \sigma_2^2\left(t\right), \\ \sigma_k^2\left(t\right) &= -\int_0^t \lambda_k\left\{\sigma_k^2\left(s\right) - \xi_k\right\}ds + \int_0^t \sigma_k\left(s\right)\omega_k\sqrt{\lambda_k}dB\left(s\right), \\ \xi_k &= p_k\xi_0,\ \omega_k^2 = 2\omega_0^2/\xi_0,\ k = 1, 2, 3. \end{aligned}$$

We track the performance of four test statistics : Lee-Ploberger (LP), Barndorff-Nielsen & Shephard(BNS), Ait-Sahalia & Jacod(AJ), and Lee-Mykland (LM). We assume $J \sim N(0, \sigma_c^2)$ and consider three cases for jump sizes : no jump($\sigma_c^2 = 0$), 20% jump ($\sigma_c^2 = 0.2\xi(s)$), and $\ln(n)/n$ jump($\sigma_c^2 = \ln(n)/n * \xi(s)$). The sample sizes considered are 72, 288, 1440, 2880, 8640; these sample sizes correspond to sampling interval lengths of 20 minutes, 5 minutes, 1 minute, 30 seconds, and 10 seconds, respectively, over a 24-hour trading day. We repeat this simulation 5,000 times. The parameters are calibrated by Barndorff-Nielsen and Shephard (2002) as follows : $\sigma^2 = 0.513$ (Model 1), $\xi_0 = 0.513$, $\omega_0^2 = 0.374$, $\lambda_1 = 1.44$, $p_1 = 1$ (Model 2, 1 factor), $\xi_0 = 0.509$, $\omega_0^2 = 0.461$, $\lambda_1 = 0.0429$, $\lambda_2 = 3.74$, $p_1 = 0.218$, $p_2 = 1 - p_1$ (Model 2, 2 factor), $\xi_0 = 0.508$, $\omega_0^2 = 4.79$, $\lambda_1 = 0.0331$, $\lambda_2 = 0.973$, $\lambda_3 = 268$, $p_1 = 0.0183$, $p_2 = 0.0180$, $p_3 = 1 - p_1 - p_3$ (Model 2, 3 factor).

<To be completed>

6 Empirical Applications

<To be completed>

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A Normal and χ^2 distributions for small and large values

It is well known that for the standard distribution function $\Phi(x)$

$$\lim_{x\to\infty} (1-\Phi(x))\sqrt{2\pi}x\exp\left(x^2/2\right) = 1$$

or equivalently

$$\lim_{x\to\infty}\left(\log(2\Phi\left(x\right)-1)\right)\sqrt{2\pi}x\exp\left(x^2/2\right)/2=-1$$

we can define $M\left(\varepsilon\right)$ as the smallest value so that for all

$$x>M\left(\varepsilon\right) \tag{29}$$

$$\left(1-\varepsilon\right)\leq\left|\sqrt{2\pi}x\exp\left(x^2/2\right)\left(1-\Phi\left(x\right)\right)\right|\leq\left(1+\varepsilon\right). \tag{30}$$

and

$$-2\left(1+\varepsilon\right)^2\leq\left(\log(2\Phi\left(x\right)-1)\right)\sqrt{2\pi}x\exp\left(x^2/2\right)\leq-2\left(1-\varepsilon\right)^2 \tag{31}$$

Lemma 6 *So let us now choose an arbitrary $\varepsilon>0$, and let ℓ, n, w_i be the integers defined in the main section of the paper. If*

$$\ell\geq 2\ln n$$

and

$$K\to\infty,$$

then

$$p_n=P\left[w_i\leq\ell M\left(\varepsilon\right)^2/K\right]=o\left(n^{-1}\right).$$

.

Proof. Since w_i is distributed according to a χ^2distribution with ℓ degrees of freedom, we have

$$p_n=\frac{1}{\Gamma\left(\ell/2\right)}\int_0^{\ell M(\varepsilon)^2/(K)}x^{\ell/2-1}\exp\left(-x/2\right)dx.$$

Since $\exp\left(-x/2\right)\leq 1$, we have with

$$C=\frac{M\left(\varepsilon\right)^2}{K}$$

$$p_n\leq\frac{1}{\Gamma\left(\ell/2\right)}\frac{1}{\ell/2}C^{\ell/2}\ell^{\ell/2},$$

and therefore

$$\log p_n\leq-\log\Gamma\left(\ell/2\right)-\log\left(\ell/2\right)+\frac{\ell}{2}\log C+\ell/2\log\ell.$$

The well known formula of Stirling implies that for $\ell\to\infty$ (with $m=\ell/2-1$)

$$\log\Gamma\left(\ell/2\right)-\left\{m\log\left(m\right)-m+\log\left(\sqrt{2\pi m}\right)\right\}\to 0.$$

Therefore

$$\log p_n \leq \{(\ell/2)\log\ell - m\,(\log(m)\} + (m + \frac{\ell}{2}\log C) + O(\log\ell)$$

One can easily see that $\{(\ell/2)\log\ell - m\,(\log(m)\} = (\ell/2)\log(\ell/m) + O(\log m)$. So the terms linear in ℓ dominate the right hand side of the inequality, Moreover, as $M(\varepsilon)$ is fixed and $K \to \infty$, we may conclude that $C \to 0$. Therefore, $\frac{\ell}{2}\log C$ will become negative. Therefore, $\frac{\ell}{2}\log C$ will become negative and dominate other parts. Therefore it can immediately be seen that lim sup

$$\frac{-\log p_n}{\ell/2\,(-\log C)} \geq 1,$$

which implies $p_n \leq \exp\left(-\ell/2\right)\frac{M(\varepsilon)^2}{K}$. ■

B The proof of Theorem 3

Our proof of Theorem 3 is based on the following lemma.

Lemma 7 *Suppose we have given a standard Wiener process W, an adapted process f and a constant α so that*

$$\int_z^b f^2 dt \leq B.$$

Then, where $\int_a^b f dW$ is the usual Ito-integral,

$$P\left[\left\{\left|\int_a^b f dW\right| \geq C\right\}\right] \leq 2\exp\left(-\frac{C^2}{2B}\right).$$

Proof. Novikov's theorem guarantees that for all u

$$E\left[\exp\left(u\int_a^b f dW - \frac{u^2}{2}\int_a^b f^2 dt\right)\right] = 1$$

Hence

$$E\left[\exp\left(u\int_a^b f dW - \frac{u^2}{2}B\right)\right] \leq 1$$

and therefore

$$\exp\left(uC - \frac{u^2}{2}B\right) P\left[\left\{\int_a^b f dW > C\right\}\right] \leq 1.$$

Setting

$$u = \frac{C}{B}$$

and repeating the same idea with $-\int_a^b f dW$ proves our proposition. ■

We are now prove Theorem 3 applying Lemma 7.

Proof of Theorem 3. We have

$$d\mu_t = A_t dt + B_t dV_t^{(1)},$$

$$d(\log \sigma_t) = C_t dt + D_t dV_t^{(2)},$$

where A_t, B_t, C_t, D_t are continuous processes and $V_t^{(1)}$,$V_t^{(2)}$ are (standard) Wiener processes. First of all let us demonstrate that without limitation of generality we can assume that A_t, B_t, C_t, D_t and $\mu_t, \log \sigma_t$ as well are uniformly bounded.

Since the processes A_t, B_t, C_t, D_t and $\mu_t, \ln \sigma_t$ are continuous, for every $\varepsilon > 0$ there exists a $M = M(\varepsilon)$ so that

$$P\left[\{\sup|A_t|, \sup|B_t|, \sup|C_t|, \sup|D_t|, \sup|\mu_t|, \sup|\ln \sigma_t| < M(\varepsilon)\}\right] > 1 - \varepsilon.$$

Let us now define the stopping time $\tau^{(\varepsilon)}$ be defined as the first time one of the absolute values of A_t, B_t, C_t, D_t and $\mu_t, \ln \sigma_t$ becomes larger than $M(\varepsilon)$, or 1 if the absolute values of the processes remain below $M(\varepsilon)$ all the time. Then

$$P\left[\left\{\tau^{(\varepsilon)} = 1\right\}\right] > 1 - \varepsilon. \tag{32}$$

Let $r_{i,n} = \left(X_{i/n} - X_{(i-1)/n}\right)$ and $s_{i,n} = \left(W_{i/n} - W_{(i-1)/n}\right)$. Then let

$$\rho_n = \sup_i \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-\ell}^2)/\ell},$$

$$\rho_n^{(\varepsilon)} = \sup_{i \leq \tau^{(\varepsilon)}} \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-\ell}^2)/\ell},$$

$$\xi_n = \sup_i \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-\ell}^2)/\ell},$$

and

$$\xi_n^{(\varepsilon)} = \sup_{i \le \tau^{(\varepsilon)}} \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-\ell}^2)/\ell}.$$

Then - by definition, ρ_n and ξ_n are our test statistics applied to $X_{i/n}$ and $W_{i/n}$, respectively. Moreover, (32) guarantees that

$$P\left[\{\rho_n = \rho_n^{(\varepsilon)}\}\right] > 1 - \varepsilon$$

and

$$P\left[\left\{\xi_n = \xi_n^{(\varepsilon)}\right\}\right] > 1 - \varepsilon,$$

too. Hence it is sufficient to show that for all $\varepsilon > 0$ the difference between converges to zero. For showing this, let us first observe that

$$\min\left(\frac{\sigma^2_{(i-k)/n}}{\sigma^2_{i/n}}\right) \le \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-\ell}^2)/\ell} \Big/ \frac{\sigma^2_{i/n} s_i^2}{(\sigma^2_{(i-1)/n} s_{i-1}^2 + \sigma^2_{(i-2)/n} s_{i-2}^2 + ..\sigma^2_{(i-\ell)} s_{i-\ell}^2)/\ell} \le \max\left(\frac{\sigma^2_{(i-k)/n}}{\sigma^2_{i/n}}\right).$$

For analyzing the difference of the left and right side of the above inequality and one, it is sufficient to consider

$$\sup_{k \le \ell} \left| \ln\left(\frac{\sigma^2_{(i-k)/n}}{\sigma^2_{i/n}}\right) \right|.$$

Now observe that $\ln(\sigma^2_{i/n}) - \ln(\sigma^2_{(i-k)/n}) = \int_{(i-k)/n}^{i/n} C_t dt + D_t dV_t^{(2)}$. For $i < \tau^{(\varepsilon)}$ $\left|\int_{(i-k)/n}^{i/n} C_t dt\right| \le kM/n$.Moreover, we have due to Lemma 7

$$P\left[\left\{\left|\int_{(i-k)/n}^{i/n} D_t dV_t^{(2)}\right| > 2\sqrt{M\ell}\sqrt{\frac{\ln n}{n}}\right\}\right] \le \frac{1}{n^2}$$

and hence

$$P\left[\left\{\sup_{i \le \tau^{(\varepsilon)}, k \le \ell} \left|\int_{(i-k)/n}^{i/n} D_t dV_t^{(2)}\right| > 2\sqrt{M\ell}\sqrt{\frac{\ln n}{n}}\right\}\right] \le \frac{\ell}{n} \to 0.$$

Hence we can conclude that

$$P\left[\left\{\sup_{k \le \ell} \left|\ln\left(\frac{\sigma^2_{(i-k)/n}}{\sigma^2_{i/n}}\right)\right| > 4\sqrt{M\ell}\sqrt{\frac{\ln n}{n}}\right\}\right] \to 0.$$

Since

$$\sup \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-\ell}^2)/\ell} = O(\ln n),$$

we can conclude that the difference between

$$\sup \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-\ell}^2)/\ell}$$

and

$$\sup \frac{\sigma_{i/n}^2 s_i^2}{(\sigma_{(i-1)/n}^2 s_{i-1}^2 + \sigma_{(i-2)/n}^2 s_{i-2}^2 + ..\sigma_{(i-\ell)}^2 s_{i-\ell}^2)/\ell}$$

converges to zero.

It now remains to show that the differences

$$\left| r_{i,n} - \sigma_{(i-1)/n} s_{i,n} \right|$$

remain small. Now observe that

$$\begin{aligned} \left| r_{i,n} - \sigma_{(i-1)/n} s_{i,n} \right| &= \left| \int_{(i-1)/n}^{i/n} \left(\mu_t dt + \sigma_t dW_t - \sigma_{(i-1)/n} dW_t \right) \right| \\ &= \left| \int_{(i-1)/n}^{i/n} \mu_t dt \right| + \left| \int_{(i-1)/n}^{i/n} \left(\sigma_t - \sigma_{(i-1)/n} \right) dW_t \right| \\ &\leq \max |\mu_t| \frac{1}{n} + \left| \int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right) dW_u \right|. \end{aligned}$$

For the analysis of

$$\left| \int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right) dW_u \right|$$

we will apply Lemma 7. Since σ_u is a diffusion process, where drift and diffusion coefficients were assumed to be bounded, we can conclude that for all $\alpha > 0$ there exists a M so that

$$P\left[\left\{ \text{for all } i \text{ and } (i-1)/n \leq u \leq i/n \ \left| \sigma_u - \sigma_{(i-1)/n} \right| \leq M \left| u - (i-1)/n \right|^{1/2-\alpha} \right\} \right] \to 1.$$

Hence

$$P\left[\left\{ \int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right)^2 du \leq 2Mn^{-2+\alpha} \right\} \right] \to 1.$$

To apply Lemma 7, however, we need to guarantee an uniform bound on the integral $\int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right)^2 du$. This can easily be achieved by using a stopping time.

We stop the process at time S, where

$$i/n \geq S \geq (i-1)/n,$$

if for the first time

$$\int_{(i-1)/n}^{S} \left(\sigma_u - \sigma_{(i-1)/n}\right)^2 du = 2Mn^{-2+\alpha},$$

otherwise we set

$$S = 1.$$

Obviously the definition of M guarantees that

$$P\left(S = 1\right) \geq 1 - \varepsilon$$

Hence if we define

$$\sigma_u^* = \begin{cases} \sigma_u \text{ for } u \leq S \\ \sigma_S \text{ otherwise,} \end{cases}$$

we have

$$P\left[\{\sigma_u^* = \sigma_u \text{ for all } u\}\right] \geq 1 - \varepsilon.$$

Hence it is sufficient to give estimates for $\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u$. For this task, however, we can apply Lemma 7 and conclude that

$$P\left[\left|\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u\right| > \sqrt{8Mn^{-2+\alpha}\ln n}\right] \leq \frac{2}{n^2}$$

Since $\alpha > 0$ was arbitrary, we can conclude that for arbitrary $\beta > 0$

$$P\left[\sup\left|\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u\right| > n^{-1+\beta}\right] \to 0,$$

which demonstrates that these terms are negligible. ■

C Data

We use the FOREX historical database which has intra-daily transactions data of stock indices and foreign exchange rates. We use the intra-daily data of US Dollar Index(DXA0), USD/JPY(Japanese yen, JPYA0), EUR/USD(EURA0), GBP/USD(British pound, GBPA0), and USD/CAN(Canadian dollar, CADA0). Since the European and US foreign exchange markets are larger than other markets, we first consider the data from 2:00 to 16:00 eastern time. We exclude the first and last 30 minutes data because we observe relatively infrequent trading. We also consider the whole 24 hour data including data from the Asian and pacific market.

For stock indices, we analyze Dow Jones Industrial Average (DJIA), NASDAQ Composite Index (COMPQ), NASDAQ-100 Index(NDX)[3], S&P 500 Index(SPX), S&P 100 Index(OEX), and Russell 2000 Index (RUT). Contrary to other market value weighted indices, the Dow Jones index is a price weighted index and represents well-established blue-chip stocks. The NASDAQ Composite is the index of all of the common stocks and similar securities listed on the NASDAQ stock market, so it measures the performance of technology stocks. The NASDAQ-100 is the index of 100 of the largest non-financial companies listed on the NASDAQ. The S&P 500 is a large-cap stock market index of 500 of largest common stocks actively traded in the US stock market. The S&P 100 chooses 100 largest companies in the S&P 500 considering sector balance. The Russell 2000 Index is a small-cap stock market index of the bottom 2,000 stocks in the Russell 3000 Index which measures the performance of the small-cap segment of the US stock market. Since US stock market opens at 9:30 and closes at 16:00 eastern time, we consider that time span. But we exclude the first and last 30 minutes because of infrequent transactions.

We also use the New York Stock Exchange (NYSE) Trade and Quote (TAQ) database which covers intra-daily transactions data for securities listed on the major stock exchanges. Because of limited accessibility, we mainly consider data of 2005 year. Dow Jones 30 stocks are chosen as main subjects because they are generally leading blue-chip stocks represent their industry and constitute a popular stock market indicator, Dow Jones Industrial Average (DJIA). We use the transaction data from the New York Stock Exchange (NYSE), American Stock Exchange

[3]Note that NASDAQ100(NDX) starts from 2.24. 1998 in FOREX database. So number of sample is smaller.

(AMEX), and NASDAQ National Market System (NMS). (We choose data whose Ex field is "N","A", or "T") Furthermore, only regular way sales are selected. We exclude special sales like Bunched sales (B), Automatic Executed sales(E), and Burst Basket Executed sales(F). TAQ database deals with intra-daily data which may have trading error or canceled transactions. By choosing trades whose CORR field is equal to either zero or one, we exclude erroneous data like cancelled trades and obvious error records. About 99.71 percentage data have the proper CORR field. When we see multiple trades with different prices at the same time, we choose a volume-weighted average of the trade price. Since we consider jumps, we did not filter data based on the size of price change.[4] During the opening and closing hours, we observe volatile movement of prices. We consider rather a clean time horizon excluding near opening and closing hours : from 10:30 to 15:30. We also exclude holidays and some trading days which had few transactions, say Labor day, Thanksgiving day, Black Friday, Christmas, and so on. On December 1th, AT&T substitutes SBC. We used the return of SBC until Nov. 30th and used that of AT&T after December 1st.

[4]Standard filtering rules exlcude trades which are less than 50% or greater 150% of the previous prices. (Boehmer, Saar, and Yu (2005))

C.1 Tables and Figures

Following tables and figures summarize rejection probabilities under 5% size.

Model1-1 : Pure Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	3.04	3.74	11.14	8.26	6.42	3.24	3.70	48.64	17.72	48.64
288	4.82	4.56	8.40	6.98	6.18	3.84	3.68	43.38	28.52	66.40
1440	4.76	4.90	5.84	5.30	5.16	4.28	4.30	28.76	39.44	85.96
2880	5.06	5.64	6.02	5.56	5.30	5.10	4.74	22.82	45.60	92.62
8640	5.12	5.18	4.92	4.64	4.58	5.06	4.68	16.46	52.90	96.54
Model1-2 : Pure Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	33.24	25.44	32.74	28.50	26.36	6.78	9.98	68.32	49.60	68.32
288	59.32	52.58	48.90	46.96	46.06	20.46	36.26	78.62	72.92	87.60
1440	78.80	75.92	65.28	64.56	64.44	34.76	68.26	86.66	88.94	97.74
2880	84.38	81.54	71.26	70.94	70.84	39.62	76.96	88.94	92.32	98.84
8640	90.88	88.78	78.36	78.14	78.08	43.32	85.62	93.12	96.12	99.78
Model1-3 : Pure Diffusion W LN(2N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	22.22	16.44	24.82	20.92	18.58	5.36	7.70	62.32	39.28	62.32
288	25.84	19.14	18.62	16.72	15.68	8.22	12.68	60.38	49.24	76.54
1440	26.80	20.52	12.60	11.64	11.36	9.14	13.76	49.96	57.78	90.68
2880	27.26	21.76	11.30	10.66	10.40	9.52	14.38	46.98	62.76	94.80
8640	27.74	21.94	7.74	7.44	7.32	9.02	13.40	41.70	67.12	97.58

Table 1: Simulated rejection probability of Model 1 : Pure Diffusion

Model2-1-1 : CIR 1 Factor SV-Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	25.30	16.86	14.84	10.96	9.64	3.30	6.02	71.24	47.94	71.24
288	32.24	20.38	9.60	7.84	7.70	4.18	4.90	72.64	65.44	83.80
1440	33.02	19.64	6.82	6.12	6.12	5.02	5.20	65.28	70.06	92.78
2880	30.64	17.66	6.60	6.04	6.04	5.52	5.10	61.68	72.98	96.00
8640	28.22	16.38	6.04	5.72	5.72	5.22	5.12	56.02	74.00	97.86
Model2-1-2 : CIR 1Factor SV-Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	55.18	45.76	41.92	37.24	35.96	3.48	9.52	83.96	70.96	83.96
288	72.46	64.86	52.58	50.46	50.38	13.34	25.28	89.60	86.84	93.76
1440	85.48	80.68	66.64	65.94	65.92	31.90	63.98	93.02	93.88	98.82
2880	88.94	85.12	72.42	72.10	72.10	36.80	74.16	94.00	95.92	99.36
8640	93.36	91.42	78.80	78.52	78.52	41.38	83.82	95.78	97.42	99.80
Model2-1-3 : CIR 1Factor SV-Diffusion W LN(2N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	40.58	32.46	27.00	22.38	20.68	3.28	7.60	78.60	60.60	78.60
288	46.08	35.20	18.66	16.22	16.08	4.72	7.64	78.70	72.88	87.42
1440	47.64	35.18	12.60	11.60	11.60	6.52	8.94	73.22	77.24	94.64
2880	44.20	32.92	10.36	9.78	9.78	6.98	9.14	69.18	78.48	96.72
8640	43.68	33.24	8.18	7.94	7.94	6.56	8.24	65.22	79.28	98.36

Table 2: Simulated rejection probability of Model 2-1 : 1 Factor SV

Model2-2-1 : CIR 2 Factor SV-Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	33.34	23.68	17.56	13.50	12.52	3.50	8.44	78.56	58.36	78.56
288	46.04	29.46	10.98	8.94	8.86	4.36	5.98	82.10	78.10	88.72
1440	44.96	28.26	7.94	6.96	6.96	4.98	4.96	76.40	78.76	95.12
2880	40.78	25.36	7.44	6.76	6.76	5.54	5.52	73.42	79.72	97.50
8640	37.28	24.06	6.04	5.72	5.72	5.64	5.28	66.42	78.82	98.56
Model2-2-2 : CIR 2Factor SV-Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	57.70	48.20	41.48	36.70	35.32	3.74	12.32	87.46	76.00	87.46
288	77.40	67.92	50.68	48.12	48.10	10.88	22.82	92.90	91.26	95.76
1440	86.86	81.34	64.06	63.10	63.10	28.78	57.76	94.92	95.46	98.94
2880	89.72	86.18	70.16	69.70	69.70	34.18	69.92	95.42	96.92	99.68
8640	93.38	91.12	76.56	76.28	76.28	38.94	80.38	96.28	97.96	99.86
Model2-2-3 : CIR 2Factor SV-Diffusion W LN(2N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	44.54	35.36	27.54	23.12	21.68	3.30	10.10	83.36	67.30	83.36
288	56.52	42.12	18.60	16.18	16.16	4.86	8.06	86.14	82.82	91.42
1440	55.48	41.04	12.14	11.22	11.22	5.82	7.66	81.46	83.24	96.12
2880	52.04	38.80	10.70	9.78	9.78	6.88	8.12	79.18	84.48	98.16
8640	51.20	39.22	7.94	7.40	7.40	6.50	7.40	73.94	84.00	98.88

Table 3: Simulated rejection probability of Model 2-2 : 2 Factor SV

Model2-3-1 : CIR 3 Factor SV-Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	54.48	50.52	61.80	56.20	53.06	2.80	21.12	94.62	81.70	94.62
288	97.24	95.02	69.18	65.16	65.14	3.02	31.66	99.86	99.62	99.90
1440	99.82	99.64	43.94	40.80	40.80	2.32	14.82	99.96	99.94	100.00
2880	99.86	99.76	33.34	30.98	30.98	2.16	10.32	99.96	99.98	100.00
8640	99.84	99.74	19.70	18.62	18.62	1.66	6.58	99.94	99.94	100.00
Model2-3-2 : CIR 3Factor SV-Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	58.44	53.54	64.96	59.80	56.96	2.66	21.48	95.40	85.60	95.40
288	97.84	96.24	75.72	72.06	72.06	2.60	34.42	99.90	99.70	99.90
1440	99.84	99.72	64.24	62.02	62.02	3.30	24.64	99.98	99.98	100.00
2880	99.90	99.82	61.80	59.96	59.96	5.76	27.98	99.98	100.00	100.00
8640	99.92	99.88	60.42	59.52	59.52	16.32	43.36	99.96	99.96	100.00
Model2-3-3 : CIR 3Factor SV-Diffusion W LN(2N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	54.92	50.58	62.34	56.38	53.48	2.82	20.94	94.64	82.56	94.64
288	97.14	95.18	69.78	65.64	65.64	3.04	31.82	99.86	99.64	99.90
1440	99.82	99.64	44.58	41.46	41.46	2.36	14.82	99.96	99.94	100.00
2880	99.86	99.76	33.92	31.48	31.48	2.14	10.32	99.96	99.98	100.00
8640	99.84	99.74	20.02	18.88	18.88	1.66	6.62	99.94	99.94	100.00

Table 4: Simulated rejection probability of Model 2-3 : 3 Factor SV

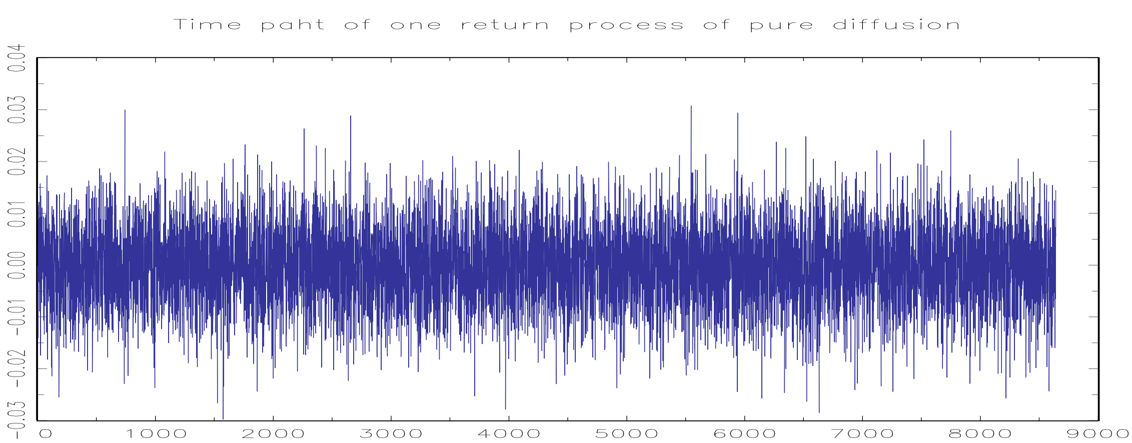

Figure 1: A time path of pure diffusion model under the null

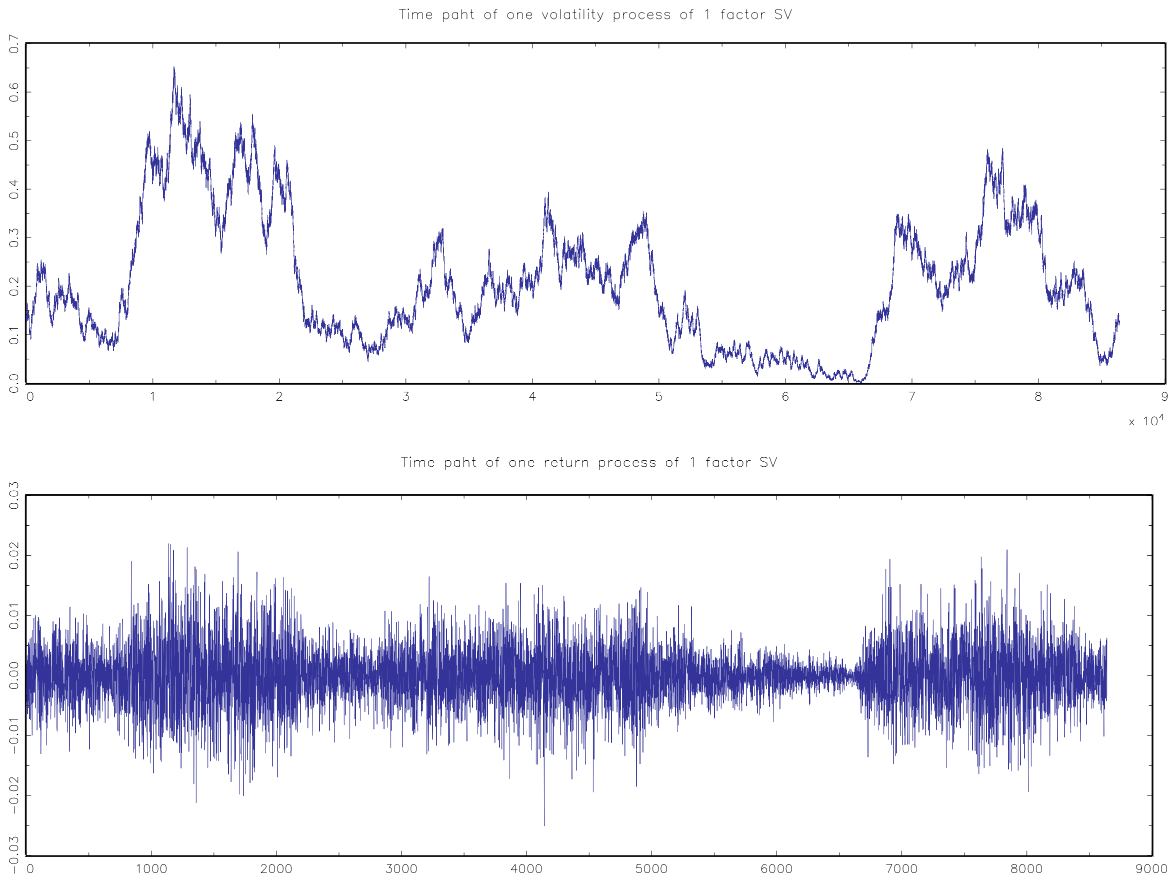

Figure 2: A time path of 1 factor SV model under the null

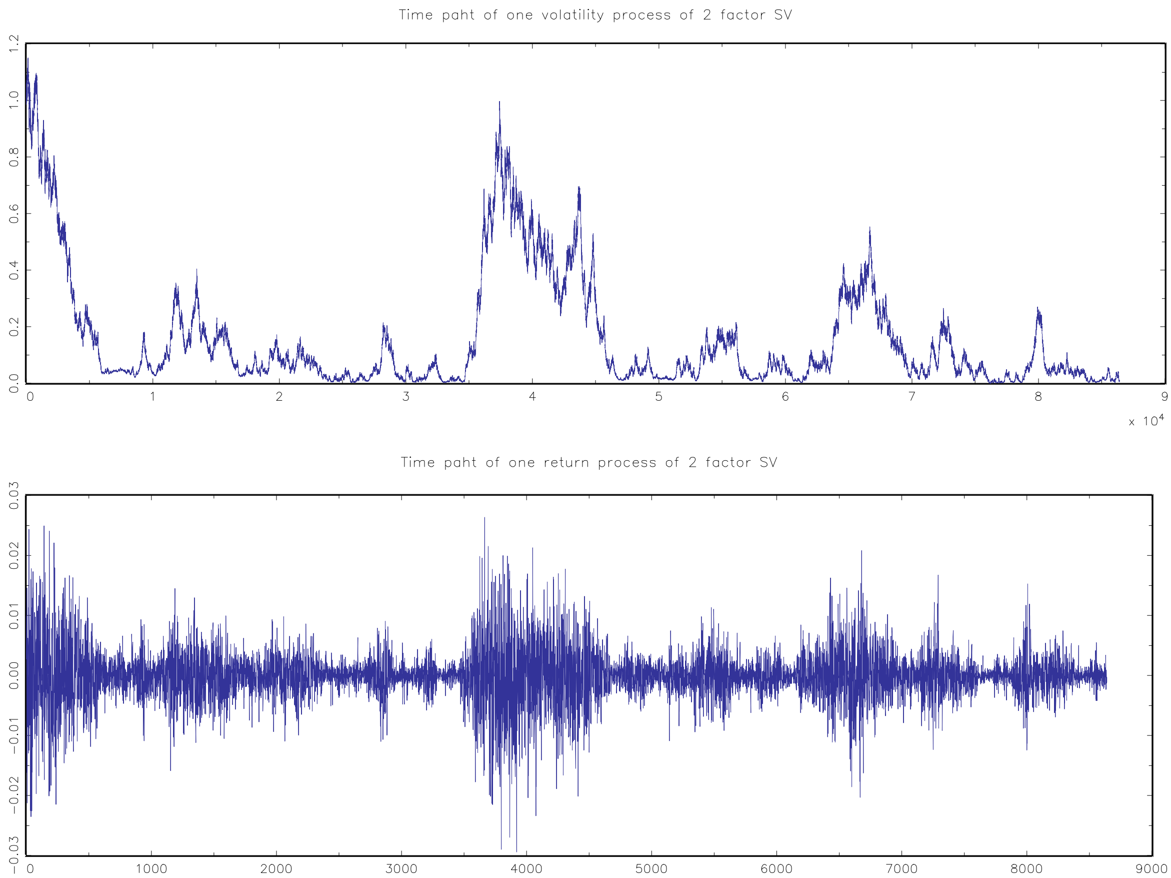

Figure 3: A time path of 2 factor SV model under the null

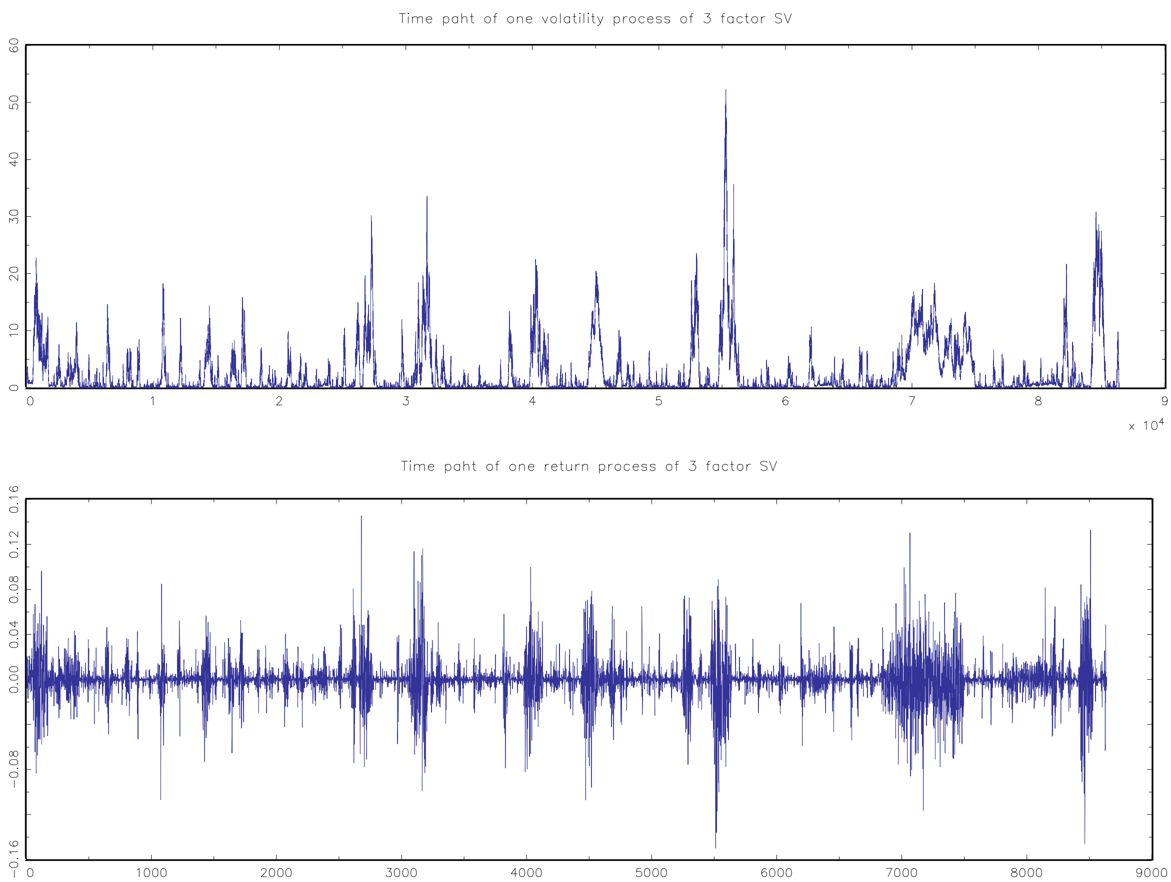

Figure 4: A time path of 3 factor SV model under the null