Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions*

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Abstract

This paper establishes the nonparametric identification of the bidders' utility function in first-price sealed-bid auctions under exclusion restrictions within a private value framework. Our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values independent of the number of bidders. The basic idea is to exploit that the bid distribution varies with the number of bidders while the private value distribution does not. We also characterize all the theoretical restrictions imposed by such an exclusion restriction on observables to rationalize the model. Though derived for a benchmark model, our results extend to more general cases such as a binding reserve price, affiliated private values and asymmetric bidders. Our theoretical results are extended to include observed and unobserved heterogeneity. In particular, we consider endogenous bidders' participation with some exclusion restrictions or the availability of instruments that do not affect the bidders' private value distribution.

Key words: Risk Aversion, Private Value, Nonparametric Identification, Exclusion Restrictions, Unobserved Heterogeneity

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1 Introduction

The empirical and experimental literature has shown that risk aversion may be relevant in auctions. See Baldwin (1995), Athey and Levin (2001), Perrigne (2003) for the former and Cox, Smith and Walker (1988), Goeree, Holt and Palfrey (2002) and Bajari and Hortacsu (2005) for the latter to name a few.[1] The problem of identifying and estimating risk aversion in first-price sealed-bid auctions has been first studied by Campo, Guerre, Perrigne and Vuong (2005).[2] These authors show that risk aversion is not identified in general and that identification can be achieved under parameterization of the bidder's utility function and a conditional quantile restriction of the bidders' private value distribution. In practice, the choice of a parametric family of utility functions displaying risk aversion may affect the estimated results, while various concepts of risk aversion have different implications on economic agents' behavior. There is, however, no general agreement on which concept of risk aversion is the most appropriate to explain observed phenomena such as in finance through the diversification of portfolios, insurance when low risk car drivers tend to buy more insurance than needed, or in auctions through overbidding. A

[1]In particular, relying on recent structural econometric methods, Bajari and Hortacsu (2005) find that risk aversion provides the best fit to some experimental data within a large set of competing models including learning ones.

[2]Since risk aversion does not affect bidding in ascending auctions within the private value paradigm, identification of risk aversion cannot achived in such auctions. Moreover, first-price sealed-bid auctions generate more revenue than ascending ones under risk aversion, thereby contributing to the use of the former for some goods. Under such circumstances, identifying risk aversion is an important issue.

typical example of such controversies is whether a measure of risk aversion should be absolute or relative to economic agent's wealth. As a matter of fact, little is known on the shape of agents' utility functions. Several families of utility functions have been developed to embody some economic properties related to risk aversion. See Gollier (2001) for an extensive survey on risk aversion.

Given the importance of risk aversion in auctions and our ignorance about bidders' utility functions, the nonparametric identification of the latter is a crucial question that we address in this paper. In particular, we show that bidders' utility function is nonparametrically identified under some exclusion restrictions. Our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values that is independent of the number of bidders. Exclusion restrictions are widely used in econometrics. A typical example is the use of instrumental variables in labor economics to solve for the endogeneity problem of education in the estimation of the wage equation. Exclusion restrictions have also been used in the structural auction literature. Athey and Haile (2002) and Haile, Hong and Shum (2003) exploit some exclusion restrictions to test for common values in first-price sealed-bid auctions. In particular, both papers assume exogenous participation to detect for the winner's curse. While considering unobserved heterogeneity affecting both bidders' participation and the latent distribution, Haile, Hong and Shum (2003) also introduce some exogenous variables or instruments independent of the latent distribution but affecting bidders' participation. In a different framework, Bajari and Hortacsu (2005) use exogenous participation to estimate an auction model with constant relative risk aversion from experimental data.[3] In the spirit of this literature, we consider observed and unobserved heterogeneity. In particular, we extend our results to a model with endogenous bidders' participation and some exclusion restrictions or the availability of instruments that do not affect the bidders' private value distribution.

Our nonparametric identification result exploits variations of the bid distribution in the number of bidders when the number of bidders does not affect the latent private value distribution. We also characterize all the theoretical restictions on observables implied by such an exclusion restriction. In particular, we show that the rationalization

[3]Exogenous participation is not necessary to estimate the model in their paper. Such a restriction avoids the use of a conditional quantile restriction as in Campo and al. (2005).

of the observed bid distributions involves additional restrictions that the data must satisfy. Though we consider first a benchmark model with symmetric bidders, independent private values and no reserve price, our results extend to a binding reserve price, affiliated private values and asymmetric bidders, whether asymmetry arises from private values and/or heterogenous preferences. As such, our results apply to a large class of auction models.

The paper is organized as follows. A second section presents the benchmark model and shows that the equilibrium strategy is increasing in the number of bidders. This result is important for the rest of the paper as nonparametric identification relies on such a property. A third section presents the nonparametric identification of the bidders' utility function and characterizes the theoretical restrictions that observed bids need to satisfy. A fourth section extends our results to a binding reserve price, affiliated private values and asymmetric bidders. A last section considers observed and unobserved heterogeneity and presents a general approach for dealing with the latter. An appendix collects the proofs.

2 The Model and Property of $s(\cdot)$

As a benchmark model, we consider the risk aversion model in Campo, Guerre, Perrigne and Vuong (2005) with symmetric bidders, independent private values and no reserve price defined by the structure $[U(\cdot), F(\cdot)]$, where $U(\cdot)$ and $F(\cdot)$ do not depend on the number I of (potential) bidders. As such, bidders' participation is exogenous. The functions $U(\cdot)$ and $F(\cdot)$ satisfy the regularity conditions of Definitions 1 and 2 in that paper with smoothness $R \geq 1$. In particular, $U(0) = 0$, while $F(\cdot)$ is strictly increasing and continuously differentiable with density $f(v) > 0$ for $v \in [\underline{v}, \overline{v}]$, which is a compact subset with nonempty interior of $[0, +\infty)$. The Bayesian Nash equilibrium strategy $s(\cdot)$ is defined as the solution of the differential equation

$$s'(v_i) = (I-1)\frac{F(v_i)}{f(v_i)}\lambda(v_i - b_i), i = 1, \ldots, I, \tag{1}$$

where v_i and b_i express the private value and the bid, respectively, $\lambda(\cdot) = U(\cdot)/U'(\cdot)$, with boundary condition $U(\underline{v} - s(\underline{v})) = 0$, i.e. $s(\underline{v}) = \underline{v}$ because $U(0) = 0$. Note that following Definition 1 in Campo and al. (2005), $\lambda(\cdot)$ is strictly increasing and continuously differentiable with $\lambda'(\cdot) \geq 1$. Moreover, $s(\cdot)$ is strictly increasing on $[\underline{v}, \overline{v}]$

Following Guerre, Perrigne and Vuong (2000), $G(b) = F(s^{-1}(v))$. Thus, (1) can be written as follows using the bid distribution $G(\cdot)$ and its corresponding density $g(\cdot)$

$$v_i = s^{-1}(b_i) = \xi(b_i) = b_i + \lambda^{-1}\left(\frac{1}{I-1}\frac{G(b_i)}{g(b_i)}\right), i = 1, \ldots, I, \tag{2}$$

where $\xi(\cdot)$ is strictly increasing and differentiable on $[\underline{b}, \overline{b}]$ with $\underline{b} = \underline{v}$ and $\overline{b} = s(\overline{v})$. Note that $G(\cdot)$ is strictly increasing and continuously differentiable on $[\underline{b}, \overline{b}]$ with density $g(\cdot) > 0$. The basic idea of our nonparametric identification result is to exploit variations of the quantiles of the bid distribution in the number of bidders, while the corresponding quantiles of the private value distribution remain the same. Our result relies on a property of the equilibrium strategy, namely $s(\cdot)$ is increasing in bidders' participation. In simple terms, increased competition renders bidding more aggressive.

Let $I_2 > I_1 \geq 2$ be two different numbers of bidders. We use the index $j = 1, 2$ to refer to the level of competition. Because the equilibrium strategy defined in (1) varies with the number of bidders, the bid distribution will also vary with the number of bidders giving $s_j(\cdot)$ and $G_j(\cdot)$. Though the lower bound of the bid distribution remains the same because of the boundary condition, the upper bound $\overline{b}_j$ varies with competition. The next lemma gives some lower and upper bounds for each equilibrium strategy in terms of the other equilibrium strategy. In particular, it establishes that the equilibrium strategy strictly increases in the number of bidders. As far as we know, the latter property was obtained for the risk neutral case and the constant relative risk aversion (CRRA) case, but not when risk aversion can take the general form $U(\cdot)$.[4]

Lemma: *Under the previous assumptions, we have*

$$\frac{I_1 - 1}{I_2 - 1}s_2(v) + \frac{I_2 - I_1}{I_2 - 1}\underline{v} < s_1(v) < s_2(v) < \frac{I_2 - 1}{I_1 - 1}s_1(v) + \frac{I_1 - I_2}{I_1 - 1}\underline{v}$$

for any $v \in (\underline{v}, \overline{v}]$.

The preceding lemma provides some testable implications in terms of stochastic dominance between the observed equilibrium bid distributions as well as their quantiles. Let $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$ denote that the distribution $G_1(\cdot)$ is strictly stochastically dominated by $G_2(\cdot)$ except at the common lower bound $\underline{b}$ of their supports. That is, $G_1(b) > G_2(b)$ for

[4] Only the middle inequality will be used for establishing the nonparametric identification of $[U(\cdot), F(\cdot)]$.

any $b \in (\underline{b}, \overline{b}_1]$, where the support of $G_j(\cdot)$ is $[\underline{b}, \overline{b}_j]$, which is a compact subset with non-empty interior of $[0, \infty)$. For $j = 1, 2$, let $b_j(\alpha)$ denote the α-quantile of the equilibrium bid distribution $G_j(\cdot)$, i.e. $G_j[b_j(\alpha)] = \alpha$ for any $\alpha \in [0, 1]$. Because $b_j = s_j(v)$ where $s_j(\cdot)$ is strictly increasing on $[\underline{v}, \overline{v}]$, $b_j(\alpha) = s_j[v(\alpha)]$, where $v(\alpha)$ is the α-quantile of $F(\cdot)$. Hence, from the Lemma the quantiles of $G_1(\cdot)$ and $G_2(\cdot)$ satisfy

$$\frac{I_1 - 1}{I_2 - 1} b_2(\alpha) + \frac{I_2 - I_1}{I_2 - 1} \underline{b} < b_1(\alpha) < b_2(\alpha) < \frac{I_2 - 1}{I_1 - 1} b_1(\alpha) + \frac{I_1 - I_2}{I_1 - 1} \underline{b} \qquad (3)$$

for any $\alpha \in [0, 1]$. Equivalently, let $G_{jk}(\cdot)$ denote the distribution of $[(I_k - 1)b_j + (I_j - I_k)\underline{b}]/[I_j - 1]$, where $j, k = 1, 2$, and $b_j = s_j(v)$.[5] Thus, the lower bound of the support of $G_{jk}(\cdot)$ is $\underline{b}$ and we have $G_{21}(\cdot) \prec_{\underline{b}} G_2(\cdot) \prec_{\underline{b}} G_1(\cdot) \prec_{\underline{b}} G_{12}(\cdot)$.

When the number I of bidders can take more than two values, the previous results imply many testable stochastic dominance relations among the observed bid distributions associated with the different values for I. Many of them are actually redundant. For instance, suppose that $I \in [\underline{I}, \overline{I}]$ with $2 \leq \underline{I} < \overline{I} < \infty$. The above implies that there are $4[1 + 2 + \ldots + (\overline{I} - \underline{I})] = 2(\overline{I} - \underline{I})(\overline{I} - \underline{I} + 1)$ stochastic dominance relations. The next corollary shows that there are at most $2(\overline{I} - \underline{I} + 1)$ relevant relations.

Corollary: *Suppose that $I \in [\underline{I}, \overline{I}]$ with $2 \leq \underline{I} < \overline{I}$. Under the previous assumptions, the quantiles of the equilibrium bid distributions $G_I(\cdot)$ satisfy*

$$\max\left\{b_{I-1}(\alpha), \frac{I-1}{I} b_{I+1}(\alpha) + \frac{1}{I}\underline{b}\right\} < b_I(\alpha) < \min\left\{b_{I+1}(\alpha), \frac{I-1}{I-2} b_{I-1}(\alpha) - \frac{1}{I-2}\underline{b}\right\}$$

for any $\alpha \in (0, 1]$ and any $I \in [\underline{I}, \overline{I}]$.[6] *Equivalently, let $\underline{G}_I(\cdot)$ denote the distribution of the maximum of b_{I-1} and $[(I-1)b_{I+1} + \underline{b}]/I$ and $\overline{G}_I(\cdot)$ denote the distribution of the minimum of b_{I+1} and $[(I-1)b_{I-1} - \underline{b}]/(I-2)$. Hence, $\underline{G}_I(\cdot) \prec_{\underline{b}} G_I(\cdot) \prec_{\underline{b}} \overline{G}_I(\cdot)$, for any $I \in [\underline{I}, \overline{I}]$.*

3 Nonparametric Identification and Restrictions

Given two different numbers of bidders $I_2 > I_1$, we now turn to the nonparametric identification of $[U(\cdot), F(\cdot)]$ or equivalently $[\lambda(\cdot), F(\cdot)]$ as $U(x) = \exp \int_1^x 1/\lambda(t) dt$ using the

[5] When $j = k$, $G_{jk} = G_j(\cdot)$.

[6] The notations $G_I(\cdot)$ and $b_I(\cdot)$ are self-explanatory. Obviously, $b_{I-1}(\cdot)$ is dropped when $I = \underline{I}$, while $b_{I+1}(\cdot)$ is dropped when $I = \overline{I}$.

normalization $U(1) = 1$. More precisely, we show that the inverse function $\lambda^{-1}(\cdot)$, which exists because $\lambda(\cdot)$ is strictly increasing on $[0, +\infty)$, is nonparametrically identified on the range $\mathcal{R}_1$ of the function $R_1(\alpha)$, where $\alpha \in [0, 1]$ and

$$R_j(\alpha) = \frac{1}{I_j - 1} \frac{\alpha}{g_j[b_j(\alpha)]} \tag{4}$$

for $j = 1, 2$. Note that $\mathcal{R}_j$ is also the range of the function $[1/(I_j - 1)][G_j(b)/g_j(b)]$ where $b \in [\underline{b}, \overline{b}_j]$, and that $\mathcal{R}_j$ is of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ because $g_j(\cdot)$ is bounded away from zero and continuous on $[0, \overline{b}_j]$. Moreover, it follows from (2) that identifying nonparametrically $\lambda^{-1}(\cdot)$ on $\mathcal{R}_1$ is equivalent to identifying nonparametrically $\lambda(\cdot)$ on the range of the markdown $v - s_1(v)$, where $v \in [\underline{v}, \overline{v}]$. Because $\overline{r}_2 < \overline{r}_1$ so that $\mathcal{R}_2$ is strictly included in $\mathcal{R}_1$, as we show below, the risk aversion function $\lambda(\cdot)$ is identified nonparametrically on the largest set of possible markdowns $[0, \max_{v \in [\underline{v}, \overline{v}]} v - s_1(v)]$.[7] The next proposition gives our first result, while providing explicit expressions for $\lambda(\cdot)$ and $F(\cdot)$.

Proposition 1: *Under the previous assumptions, $\lambda^{-1}(\cdot)$ is identified nonparametrically on $\mathcal{R}_1$ and is given by*

$$\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} \Delta b(\alpha_t)$$

for any $u_0 \in \mathcal{R}_1$, where $\Delta b(\alpha) = b_2(\alpha) - b_1(\alpha)$ for $\alpha \in [0, 1]$, and the sequence $\{\alpha_t\}$ is strictly decreasing with $0 \le \alpha_t \le 1$ satisfying the nonlinear recursive relation $R_1(\alpha_t) = R_2(\alpha_{t-1})$ with initial condition $R_1(\alpha_0) = u_0$. Moreover, $F(\cdot)$ is identified nonparametrically on $[\underline{v}, \overline{v}]$ with $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$ for $j = 1, 2$.

The sequence $\{\alpha_t\}$ is not necessarily unique. Proposition 1 explains how to construct such a sequence recursively. The basic idea is to use the invariance of the quantile of the private value distribution $v(\alpha)$ for two different numbers of bidders I_1 and I_2. Specifically, using (2) leads to the *compatibility condition* $\lambda^{-1}[R_1(\alpha)] = \lambda^{-1}[R_2(\alpha)] + \Delta b(\alpha)$, where $\Delta b(\alpha) > 0$. The latter shows the need for equilibrium strategies increasing in the number

[7] In general, $\max_{v \in [\underline{v}, \overline{v}]} v - s_j(v) \neq \overline{v} - s_j(\overline{v})$, in which case $\overline{r}_j \neq R_j(1)$. On the other hand, if the markdown $v - s_j(v)$ is increasing in v, then $\max_{v \in [\underline{v}, \overline{v}]} v - s_j(v) = \overline{v} - s_j(\overline{v})$. Moreover, $R_j(\cdot)$ would be increasing in α and $\overline{r}_j = R_j(1)$.

of bidders as shown in the Lemma. By continuity of $R_2(\cdot)$, there exists a value $\tilde{\alpha}$ such that $\tilde{\alpha} < \alpha$ and $R_1(\tilde{\alpha}) = R_2(\alpha)$, which can be used to rewrite the previous equality. But the previous compatibility condition also holds at $\tilde{\alpha}$. Continuing the same exercise gives the sequence of values for α. As a matter of fact, we show that there exists at least one such sequence $\{\alpha_t\}$. When $R_1(\cdot)$ is strictly increasing, i.e. when the markdown or bidders' rent with I_1 bidders is strictly increasing in v, such a sequence is unique. When $R_1(\cdot)$ is not strictly increasing, the sequence $\{\alpha_t\}$ may not be unique, but all such sequences must lead to the same value for $\sum_{t=0}^{\infty} \Delta b(\alpha_t)$, which then defines $\lambda^{-1}(u_0)$ uniquely.

An important related question is to characterize all the restrictions on the observed equilibrium bid distributions that arise from the independence of the private value distribution $F(\cdot)$ on the number I of bidders. In particular, this is useful to assess whether the observed bid distributions, which typically vary with the number I of bidders, can be rationalized by a structure $[U(\cdot), F(\cdot)]$ that is independent of I. In other terms, these restrictions allow to test the validity of the model and its assumptions. Violation of one of these restrictions leads to reject the model for explaining the observed bids. In particular, it could mean that the exogeneity of bidders' participation is not justified. Our second result provides such restrictions when I can take two different values $I_2 > I_1$.

Proposition 2: *Let $\mathbf{G}_j(\cdot, \ldots, \cdot)$ be the joint distribution of $(b_1, \ldots, b_{I_j})$, $j = 1, 2$. The equilibrium bid distributions $\mathbf{G}_j(\cdot)$, $j = 1, 2$, are rationalized by a structure $[U(\cdot), F(\cdot)]$ independent of I if and only if*

(i) For each $j = 1, 2$, $\mathbf{G}_j(b_1, \ldots, b_{I_j}) = \prod_{i=1}^{I_j} G_j(b_i)$, with $G_j(\cdot) \in \mathcal{G}_R$ with support of the form $[\underline{b}, \overline{b}_j]$, where $\mathcal{G}_R$ is given by Definition 3 in Campo et al. (2005),

(ii) The α-quantiles of $G_1(\cdot)$ and $G_2(\cdot)$ satisfy $b_1(\alpha) < b_2(\alpha)$ for $\alpha \in (0, 1]$, i.e. $G_2(\cdot) \prec_{\underline{b}} G_1(\cdot)$,

(iii) $\exists \lambda(\cdot) : I\!\!R_+ \to I\!\!R_+$ with $R + 1$ continuous derivatives on $[0, +\infty)$, $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$ such that

(a) the compatibility condition is satisfied for any $\alpha \in [0, 1]$, namely,

$$b_2(\alpha) + \lambda^{-1}\left(\frac{1}{I_2 - 1}\frac{\alpha}{g_2(b_2(\alpha))}\right) = b_1(\alpha) + \lambda^{-1}\left(\frac{1}{I_1 - 1}\frac{\alpha}{g_1(b_1(\alpha))}\right), \tag{5}$$

(b) for $j = 1, 2$, $\xi_j'(\cdot) > 0$ on $[\underline{b}, \overline{b}_j]$, where $\xi_j(b) = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$.

Unlike the corollary, which only provides some (testable) implications, Proposition

2 characterizes all the theoretical restrictions imposed by the model with an exogenous bidders' participation. Proposition 2 can be extended straightforwardly to the case where $I \in [\underline{I}, \overline{I}]$. Relative to the general case studied by Campo et al. (2005, Lemma 1), in which $F(\cdot)$ can vary across I, the set of bid distributions that can be rationalized is much reduced because of the restrictions (ii) and (iii)(a). Indeed, Proposition 1 in Campo et al. (2005) implies that any distribution $G_j(\cdot) \in \mathcal{G}_R$ can be rationalized by a structure $[U, F_j]$, where $U(\cdot)$ is not identified. As a matter of fact, these additional restrictions help in identifying nonparametrically the structure $[U, F]$.[8]

4 Extensions

Binding Reserve Price

A binding reserve price, i.e. $p_0 > \underline{v}$, introduces a truncation in the observed bid distribution as only the I^* bidders who have a value above p_0 will bid at the auction. Let $G_j^*(\cdot)$ be the truncated bid distribution on $[p_0, \overline{b}_j]$. We observe I_j^* the number of active bidders, $I_j^* \leq I_j$ and $I_1 < I_2$. Note that the Lemma still holds with a binding reserve price as its introduction just reduces the shading relative to the case with no reserve price. We maintain the previous assumption regarding the exogeneity of I. Because $G_j^*(b^*) = [F(v) - F(p_0)]/[1 - F(p_0)]$ for $b^* \in [p_0, \overline{b}_j]$, elementary algebra gives the following inverse equilibrium strategies

$$v = s_j^{-1}(b^*) = b^* + \lambda^{-1}\left(\frac{1}{I_j - 1}\frac{G_j^*(b^*)}{g_j^*(b)} + \frac{1}{I_j - 1}\frac{1}{g_j^*(b^*)}\frac{F(p_0)}{1 - F(p_0)}\right),$$

for $j = 1, 2$. Thus, the function $R_j(\alpha)$ becomes

$$R_j(\alpha) = \frac{1}{I_j - 1}\frac{1}{g_j^*[b_j^*(\alpha)]}\left(\alpha + \frac{F(p_0)}{1 - F(p_0)}\right),$$

for $j = 1, 2$. Note that $R_j(\alpha)$ differs from (4) through the additional term $F(p_0)/(1 - F(p_0))$. Following Guerre, Perrigne and Vuong (2000, Theorem 4), the number of potential bidders I_j and $F(p_0)$ are identified from the distribution of I_j^*. The problem reduces to identify $\lambda^{-1}(\cdot)$ and $F(\cdot)$ on $[0, \overline{r}_1]$ and $[p_0, \overline{v}]$, respectively. A simple extension of

[8] Note that the above compatibility condition is similar in spirit to the ones used to identify the model with risk aversion and bidders with heteregenous preferences. See Section 7.3 in Campo et al. (2005).

Proposition 1 shows that $[\lambda^{-1}(\cdot), F(\cdot)]$ is nonparametrically identified on these intervals using the quantiles $b_j^*(\alpha)$ of $G_j^*(\cdot)$. Similarly, Proposition 2 can be readily adapted.

Affiliated Private Values

The vector $(v_1, \ldots, v_{I_j})$ is distributed as $F_j(\cdot, \ldots, \cdot)$, which is exchangeable in its I_j arguments, affiliated and defined on $[\underline{v}, \overline{v}]^{I_j}$, where $F_1(v_1, \ldots, v_{I_1}) = \int_{\underline{v}}^{\overline{v}} \ldots \int_{\underline{v}}^{\overline{v}} F_2(v_1, \ldots, v_{I_1}, v_{I_1+1}, \ldots, v_{I_2}) dv_{I_1+1} \ldots dv_{I_2}$. We follow the notations of Li, Perrigne and Vuong (2002). Let $G^j_{B_i|b_i}(b_i|b_i)$ express the probability that i has a bid larger than all his opponents conditional on his bid b_i with $B_i = \max_{k\neq i} b_k$ and $b_i = s_j(v_i)$, $j = 1, 2$. Without loss of generality, we can consider $G^j_{B_1|b_1}(\cdot|\cdot)$ as all bidders are symmetric. To simplify the notations, we omit the index 1. The inverse equilibrium strategy becomes

$$v = s_j^{-1}(b) = b + \lambda^{-1}\left(\frac{G^j_{B|b}(b|b)}{g^j_{B|b}(b|b)}\right),$$

for $j = 1, 2$ and $b \in [\underline{b}, \overline{b}_j]$. Note that $G_{B|b}(\cdot|\cdot)/g_{B|b}(\cdot|\cdot) = G_{B\times b}(\cdot,\cdot)/g_{B,b}(\cdot,\cdot)$, where $G_{B\times b}(\cdot,\cdot) = \partial G_{B,b}(\cdot,\cdot)/\partial b$ and $g_{B,b}(\cdot,\cdot)$ is the joint density. The competition effect is unclear with affiliated private values. As a matter of fact, some distributions $F(\cdot, \ldots, \cdot)$ may lead to decreasing bidding strategies in the number of bidders as shown by Pinkse and Tan (2005). We assume that the structure satisfies $s_1(v) < s_2(v)$ for the pair (I_1, I_2), i.e. for this pair competition renders bidding more aggressive. Here again, the exogeneity of the number of bidders allows us to identify $\lambda^{-1}(\cdot)$ on $[0, \overline{r}_1]$ by exploiting the variation in the number of bidders, where $R_j(\cdot) = G^j_{B\times b}(b_j(\alpha), b_j(\alpha))/g^j_{B,b}(b_j(\alpha), b_j(\alpha))$ with $b_j(\alpha)$ the α-quantile of the marginal bid density $g_j(\cdot)$ associated with I_j bidders. Proposition 2 is similarly adapted to this case.

Asymmetric Bidders

Asymmetry among bidders, which is assumed to be known ex ante to all participants, can arise from two different sources, namely from (i) different distributions of private values and/or (ii) different utility functions. We consider these cases separately.[9] While Campo et al. (2005) show that asymmetry in private values is a straightforward extension of the symmetric case, asymmetry in preferences provides additional restrictions leading

[9] Different utility functions among bidders may involve different attitudes towards risk and/or different wealth levels. If the wealth levels are not known ex ante to all bidders, the game becomes a multi-signal game, which is beyond the scope of this paper.

to a semiparametric identification result which requires the parameterization of the utility functions only. In contrast, we assume here the exogeneity of bidders' participation, i.e. the private value distribution of every bidder is independent of the level of competition. As before, bidders' preferences do not depend on I.

Let $I_2 > I_1 \geq 2$. Since our results exploit the difference in bidding behavior under two competitive environments, it is crucial that at least one bidder participates in both auctions.[10] For instance, $I_1 = 2$ with two different bidders and $I_2 = 3$ with a third bidder possibly different from the two others. In the case of asymmetry, because of the complexity of the system of differential equations defining the equilibrium strategies, a property stating that the equilibrium strategies are increasing in the number of bidders is difficult if not impossible to prove. Nevertheless, because of the independence of private values, it is reasonable to postulate that equilibrium strategies are increasing in the number of bidders due to the competition effect. Let $s_{ij}(\cdot)$ denote the equilibrium strategy for bidder $i = 1, \ldots, I_j$, when the number of bidders is I_j, $j = 1, 2$. When $I_1 = 2$ and $I_2 = 3$, the equilibrium strategies must satisfy the boundary conditions $s_{11}(\underline{v}) = s_{12}(\underline{v}) = s_{21}(\underline{v}) = s_{22}(\underline{v}) = s_{32}(\underline{v}) = \underline{v}$ and $s_{11}(\overline{v}_1) = s_{12}(\overline{v}_1) = s_{21}(\overline{v}_2) = s_{22}(\overline{v}_2) = s_{32}(\overline{v}_3)$. Thus, we assume that $s_{11}(v) < s_{12}(v)$ and $s_{21}(v) < s_{22}(v)$ so that the lemma applies.

We consider first the case with asymmetry in private values only with different private value distributions $F_i(\cdot)$ defined on $[\underline{v}, \overline{v}_i]$ but with the same utility function $U(\cdot)$. As discussed above, we assume that bidders $1, \ldots, I_0$ participate to both auctions, where $I_0 \geq 1$. Let $v_i(\alpha)$ and $b_{ij}(\alpha)$ be the α-quantiles of $F_i(\cdot)$ and $G_{ij}(\cdot)$, respectively. Following Campo et al. (2005, Section 7.3), instead of (2), we now have at the α-quantile

$$v_i(\alpha) = b_{ij}(\alpha) + \lambda^{-1}\left(R_{ij}(\alpha)\right), i = 1, \ldots, I_j, j = 1, 2 \tag{6}$$

for $\alpha \in [0, 1]$, where $R_{ij}(\alpha) = 1/H_{ij}(b_{ij}(\alpha))$ takes values in the range $\mathcal{R}_{ij} = [0, \overline{r}_{ij}]$ with $H_{ij}(\cdot) = \sum_{k \neq i} g_{kj}(\cdot)/G_{kj}(\cdot)$. A straightforward extension of Proposition 1 shows that $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{i=1,\ldots,I_0} \overline{r}_{i1}]$ as well as $[F_1, \cdots, F_{I_0}]$ on their respective supports. Since $\lambda^{-1}(\cdot)$ is identified, (6) helps to identify $[F_{I_0+1}, \ldots, F_{I_2}]$. Because $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{i=1,\ldots,I_0} \overline{r}_{i1}]$, which may be a strict subset of $[0, \max_{i=1,\ldots,I_2} \overline{r}_{i1}]$, the

[10] In practice, a few "types" are often entertained in empirical studies involving asymmetric bidders. See for instance Campo, Perrigne and Vuong (2003), Athey, Levin and Seira (2004) and Flambard and Perrigne (2005). In this case, it is important that we observe at least one bidder's type in both auctions.

latter distributions may not be identified on the upper range of their supports. Proposition 2 also extends where the compatibility conditions (5) now hold for each of the I_0 bidders.

The second case involves asymmetry in preferences only with the $\lambda_i(\cdot)$ varying across bidders and with the same private value distribution $F(\cdot)$. Again, we assume that bidders $1, \ldots, I_0$ participate to both auctions, where $I_0 \geq 1$. Equation (6) takes a similar form with $v(\alpha)$ and $\lambda_i^{-1}(\cdot)$ replacing $v_i(\alpha)$ and $\lambda^{-1}(\cdot)$, respectively. A similar argument as in Proposition 1 applies for identifying nonparametrically $\lambda_i^{-1}(\cdot)$ on $[0, \overline{r}_{i1}]$ for $i = 1, \ldots, I_0$ from which we can identify $F(\cdot)$ on its full support. Since $F(\cdot)$ is identified, following a similar argument as in Lu and Perrigne (2004), $\lambda_{I_0+1}^{-1}(\cdot)$, $\ldots$, $\lambda_{I_2}^{-1}(\cdot)$ are identified from (6). Specifically, because the $v(\alpha)$s are identified, we can recover the latter $\lambda_i(\cdot)$ on $[0, \overline{r}_{i1}]$. Again Proposition 2 extends with the compatibility conditions (5) holding for each of the I_0 bidders and $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$.

The third case involves asymmetry in both private value distributions and preferences. Equation (6) takes a similar form with $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$. Despite the complexity of this case, which has not been considered to our knowledge, we can show that the structure $[\lambda_i, F_i]$ is nonparametrically identified for the $I_0 \geq 1$ bidders who participate to both auctions. Specifically, we can apply Proposition 1 to each common bidder to identify nonparametrically $\lambda_i^{-1}(\cdot)$ and $F_i(\cdot)$ on $[0, \overline{r}_{i1}]$ and $[\underline{v}, \overline{v}_i]$, respectively. On the other hand, we cannot identify the pair $[\lambda_i^{-1}(\cdot), F_i(\cdot)]$ for the other bidders. Again Proposition 2 extends with the compatibility conditions (5) holding for each of the common bidders and $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$.

5 Observed and Unobserved Heterogeneity

The previous sections have shown that exogenous variations in the number of bidders can be exploited to identify nonparametrically the bidders' utility functions and their private value distributions with a binding or nonbinding reserve price, affiliated private values and asymmetric bidders. In practice, some additional variables and possible unobserved heterogeneity can explain auctioned objects heterogeneity and bidders' participation. This section discusses how our theoretical results can be interpreted in this context. In particular, bidders' participation may be endogenous and some of these additional variables can play the role of instrumental variables through exclusion restrictions.

We first consider the case of an exogenous number of bidders. Let W be a vector of observed variables characterizing the heterogeneity across auctioned objects. These variables are assumed to affect both bidders' private value distribution and bidders' participation.[11] Bidders' participation is modeled as $I = I(W, \epsilon)$, where ϵ can be interpreted as a term of unobserved heterogeneity or as a traditional error term. We assume $v \perp \epsilon | W$, namely bidders' private values are independent of ϵ given the auction characteristics W. This assumption implies $F(v|W, \epsilon) = F(v|W)$, which translates into an exclusion restriction, while the observed bid distribution is $G(b|I, W)$ since $b = s(v, I, U, F)$. This model is similar to the one in Section 3 since the latent private value distribution does not depend on the number of bidders or equivalently bidders' private values are independent of I given W leading to the exogeneity of I. The only difference between the two models is the introduction of the vector of conditioning variables W. This exclusion restriction allows us to exploit the variation in bidding behavior under two competitive environments, i.e. $I_2 > I_1$, at W given. Proposition 1 applies and the pair $[U(\cdot), F(\cdot|\cdot)]$ is nonparametrically identified, while the quantile becomes $b_j(\alpha, W)$. Regarding Proposition 2, the bids are now conditionally independent given W in (i), while the rest extends straightforwardly.

The term of unobserved heterogeneity ϵ may, however, affect the private value distribution as ϵ can capture some unobserved characteristics affecting private values as well as bidders' participation. This leads to a model of endogenous participation. The introduction of additional variables or instruments combined with appropriate exclusion restrictions solves this problem. Specifically, bidders' participation is modeled as $I = I(W, Z, \epsilon)$, while we assume $v \perp Z | (W, \epsilon)$, namely the bidders' private values are independent of the variables Z given (W, ϵ). This implies $F(v|W, Z, \epsilon) = F(v|W, \epsilon)$, which translates into an exclusion restriction. Hence, the variables Z can be viewed as instruments. Moreover, this model corresponds to an endogenous number of bidders as the unobserved heterogeneity ϵ affects both bidders' private values and bidders' participation. This model is reminiscent of Bajari and Hortacsu (2003) modeling of bidders' entry in online coin auctions within a common value framework. Their empirical results show that the variables explaining bidder's entry are the estimated value of the auctioned object (W), the reserve price for the auctioned object (Z_1) and the seller's reputation (Z_2), while the bidder's private sig-

[11] For instance, W contains variables affecting the value of auctioned objects, and explaining bidders' participation. See Athey, Levin and Seira (2004) for example of such variables.

nal distribution is conditional on the estimated value of the auctioned object only. Haile, Hong and Shum (2003) also adopt such a framework to test for common value in first-price sealed-bid auctions when I is endogenous.

The observed bid distribution $G(b|W, Z, \epsilon)$ is equal to $G(b|I, W, \epsilon)$ using $G(b|\cdot) = F[s^{-1}(b)|\cdot]$, while the exclusion restriction $v \perp Z|(W, \epsilon)$ implies $v \perp I|(W, \epsilon)$ as $I = I(W, Z, \epsilon)$. The parallel with Proposition 1 appears as we can exploit the variation in bidding behavior under two competitive environments while the latent distribution remains the same at (W, ϵ) given. The term of heterogeneity ϵ is, however, unobserved. To solve this problem, we assume as usual an additive error term, i.e. $I(W, Z, \epsilon) = I(W, Z) + \epsilon$, where $\mathrm{E}(\epsilon|W, Z) = 0$. Because $I(W, Z) = \mathrm{E}(I|W, Z)$, it follows that $\mathrm{E}(I|\cdot, \cdot)$ is the regression of I on (W, Z). Hence, $\mathrm{E}(I|\cdot, \cdot)$ is nonparametrically identified so that ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|\cdot, \cdot)$. Proposition 1 applies and $[U, F(\cdot|\cdot, \cdot)]$ is nonparametrically identified, while the quantile becomes $b_j(\alpha, W, \epsilon)$. Regarding Proposition 2, the bids are now conditionally independent given (W, ϵ) in (i), while the rest extends straightforwardly.

It should be noted that endogenous entry with an additive error term is a general method that can be used to solve for the problem of unobserved heterogeneity. The exclusion restriction or existence of instrument Z is not needed in general and is used here because the model is not identified otherwise. For instance, consider the risk neutral model with endogenous entry $I = I(W) + \epsilon$, where the bidders' private value distribution is $F(v|W, \epsilon)$. Hence, unobserved heterogeneity affects both bidders' participation and private values. As above, ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|W, \epsilon)$. Thus, $F(\cdot|W, \epsilon)$ is nonparametrically identified following Guerre, Perrigne and Vuong (2000, Theorem 1). This method differs from Krasnokutskaya (2004) who identifies the term of unobserved heterogeneity in a private value model $F(v|W, \epsilon)$ with exogenous participation $I = I(W)$ using a multiplicative decomposition of private values. Her result then relies on a measurement error model with multiple indicators studied in Li and Vuong (1998).

This paper addresses the problem of identification of bidders' utility function. The corresponding nonparametric estimation method clearly needs to be developed. Recent results on the nonparametric estimation of quantiles can be used. The difficulty relies, however, in the infinite series of differences in quantiles that indentify $\lambda^{-1}(\cdot)$ in Proposition 1 as the sequence of estimated $\{\hat{\alpha}_t\}$ and hence the estimated $\Delta\hat{b}(\hat{\alpha}_t)$ are serially correlated. This complicates greatly the derivation of the asymptotic properties of the resulting estimator and its rate of convergence. If one is willing to impose some parameterization, one can use e.g. the semiparametric estimator proposed in Campo et al. (2005) as the model is nonparametrically and hence semiparametrically identified.

Appendix

Proof of Lemma: (i) We first prove that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$. We have $s_2(\underline{v}) = s_1(\underline{v}) = \underline{v}$. Moreover, from Theorem 1-(ii) in Campo and al. (2005) we have

$$0 < s_j'(\underline{v}) = \frac{(I_j - 1)\lambda'(0)}{(I_j - 1)\lambda'(0) + 1} = 1 - \frac{1}{(I_j - 1)\lambda'(0) + 1} < 1 \tag{A.1}$$

where $\lambda'(0) \geq 1$. Thus $0 < s_1'(\underline{v}) < s_2'(\underline{v}) < 1$. In particular, by continuity of $s_j(\cdot)$ it follows that $\underline{v} < s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$ for some ϵ satisfying $\underline{v} < \epsilon \leq \overline{v}$.

The proof is now by contradiction. Suppose that $s_1(v) \geq s_2(v)$ for some $v \in [\epsilon, \overline{v}]$. Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$, the continuity of $s_j(\cdot)$ would imply the existence of some $v_0 \in [\epsilon, \overline{v}]$ such that $s_1(v_0) = s_2(v_0)$. Moreover, for (at least) one of such v_0 denoted v_0^*, the strategy $s_1(\cdot)$ must intersect the strategy $s_2(\cdot)$ from below, i.e. $s_1'(v_o^*) \geq s_2'(v_o^*)$. From (1), we have

$$s_j'(v_0^*) = (I_j - 1)\frac{f(v_0^*)}{F(v_0^*)}\lambda\left(v_0^* - s_j(v_0^*)\right)$$

for $j = 1, 2$, where $f(v_0^*) > 0$ and $F(v_0^*) > 0$ since $v_0^* \in (\underline{v}, \overline{v}]$, while $\lambda(v_0^* - s_j(v_0^*)) > 0$ since $v_0^* > s_j(v_0^*)$ by Theorem 1-(i) in Campo and al. (2005) and $\lambda(\cdot) > 0$ on $(0, +\infty)$. By construction, $s_1(v_0^*) = s_2(v_0^*)$, the previous equation then implies $s_1'(v_0^*) < s_2'(v_0^*)$, contradicting $s_1'(v_0^*) \geq s_2'(v_0^*)$.

(ii) Next, we prove the first inequality, which implies the third inequality after immediate algebra. For each $j = 1, 2$, (1) gives

$$\frac{s_j'(v)}{I_j - 1} = \frac{f(v)}{F(v)}\lambda\left(v - s_j(v)\right) \tag{A.2}$$

for any $v \in [\underline{v}, \overline{v}]$. From (i), $v - s_2(v) < v - s_1(v)$ for any $v \in (\underline{v}, \overline{v}]$. Because $0 < v - s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ by Theorem 1-(i) in Campo et al. (2005), and $\lambda(\cdot)$ is strictly increasing with $\lambda(\cdot) > 0$ on $(0, +\infty)$, then $0 < \lambda(v - s_2(v)) < \lambda(v - s_1(v))$ for any $v \in (\underline{v}, \overline{v}]$. Hence, because $f(\cdot) > 0$ and $F(\cdot) > 0$ on $(\underline{v}, \overline{v}]$, it follows from (A.2) that

$$s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1) \tag{A.3}$$

for any $v \in (\underline{v}, \overline{v}]$.[12] Integrating (A.3) from $\underline{v}$ to $v > \underline{v}$ and using $s_j(\underline{v}) = \underline{b}$ give

$$\frac{s_2(v) - \underline{b}}{I_2 - 1} < \frac{s_1(v) - \underline{b}}{I_1 - 1} \tag{A.4}$$

for any $v \in (\underline{v}, \overline{v}]$. The desired result follows after immediate algebra. □

Proof of Corollary: As $I = \underline{I}$ and $I = \overline{I}$ follow from the general case, we fix $I \in (\underline{I}, \overline{I})$. Let $I = I_1 < I_2$ in (3). The first inequality in (3) gives

$$(I-1)\frac{b_2(\alpha) - \underline{b}}{I_2 - 1} + \underline{b} < b_I(\alpha),$$

for any $\alpha \in (0, 1]$ and any $I_2 > I$. Equation (A.4) applied to an arbitrary pair (I_2, I_2') with $I_2 < I_2'$ shows that the LHS in the above inequality is strictly decreasing in I_2. Hence, the most stringent inequality is obtained when I_2 is the smallest, i.e. when $I_2 = I + 1$. Similarly, let $I = I_2 > I_1$ in (3). The third inequality in (3) gives

$$b_I(\alpha) < (I-1)\frac{b_1(\alpha) - \underline{b}}{I_1 - 1} + \underline{b}$$

for any $\alpha \in (0, 1]$ and any $I_1 < I$. The RHS in the above inequality is strictly decreasing in I_1 from (A.4). Hence, the most stringent inequality is obtained when I_1 is the largest, i.e. when $I_1 = I - 1$. Combining these two results gives

$$\frac{I-1}{I} b_{I+1}(\alpha) + \frac{1}{I}\underline{b} < b_I(\alpha) < \frac{I-1}{I-2} b_{I-1}(\alpha) - \frac{1}{I-2}\underline{b}, \tag{A.5}$$

for any $\alpha \in (0, 1]$. On the other hand, the middle inequality of (3) gives

$$b_{I-1}(\alpha) < b_I(\alpha) < b_{I+1}(\alpha), \tag{A.6}$$

for any $\alpha \in (0, 1]$ and any $I \in [\underline{I}, \overline{I}]$. The desired result follows by combining (A.5) and (A.6).

The second part of the corollary follows by noting that b_I is a strictly increasing function of v, namely $b_I = s_I(v)$ fo each I. Hence, the random variables $\max\{b_{I-1}, [(I-1)b_{I+1} + \underline{b}]/I\}$ and $\min\{b_{I+1}, [(I-1)b_{I-1} - \underline{b}]/(I-2)\}$ are also strictly increasing functions of v. It follows that the α-quantiles of their corresponding distributions $\underline{G}_I(\cdot)$ and $\overline{G}_I(\cdot)$ are equal to these functions evaluated at $v(\alpha)$. Thus, they are equal to the first term and third term of the two inequalities displayed in the Corollary, respectively since $b_I(\alpha) = s_I[v(\alpha)]$. The stochastic dominance assertion then follows from these two inequalities. □

[12] Equation (A.1) shows that $s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1)$ also holds at $v = \underline{v}$.

Proof of Proposition 1: From $b_j(\alpha) = s_j[v(\alpha)]$ and (2) evaluated at $v = v(\alpha)$, we obtain the crucial relation

$$v(\alpha) = b_j(\alpha) + \lambda^{-1}\left(\frac{1}{I_j - 1}\frac{\alpha}{g_j[b_j(\alpha)]}\right) \tag{A.7}$$

for $j = 1, 2$ and any $\alpha \in [0, 1]$. Hence, using (4) we obtain the nonlinear relation

$$\lambda^{-1}[R_1(\alpha)] = \lambda^{-1}[R_2(\alpha)] + \Delta b(\alpha) \tag{A.8}$$

for any $\alpha \in [0, 1]$.

For future use, we note that $\Delta b(0) = 0$ as $s_1(\underline{v}) = s_2(\underline{v}) = \underline{b}$. Moreover, from the Lemma, we know that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$, which implies $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0, 1]$. Hence, $\Delta b(\alpha) > 0$ for any $\alpha \in (0, 1]$. Because $\lambda^{-1}(\cdot)$ is strictly increasing and $R_j(\alpha) > 0$ for any $\alpha \in (0, 1]$, it follows from (A.8) that $R_1(\alpha) > R_2(\alpha) > 0$ for any $\alpha \in (0, 1]$. In particular, because $R_j(\cdot)$ is continuous on $[0, 1]$ and $R_j(0) = 0$ for $j = 1, 2$, the range $\mathcal{R}_j$ of $R_j(\cdot)$ must be of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ and $\overline{r}_1 > \overline{r}_2$, as claimed in the text.

Now, by assumption u_0 belongs to $\mathcal{R}_1 = [0, \overline{r}_1]$. We consider two cases. Suppose first that $u_0 = 0$. Then, there exists a unique $\alpha_0 \in [0, 1]$, namely $\alpha_0 = 0$ such that $u_0 = R_1(\alpha_0)$, which implies that the sequence $\{\alpha_t\}$ is unique, namely $\alpha_t = 0$ for $t = 0, 1, \ldots$. It follows that $\lambda^{-1}(u_0) = \sum_{t=0}^{\infty} \Delta b(\alpha_t)$ holds trivially since $\lambda^{-1}(0) = 0$ and $\Delta b(0) = 0$.

Next, consider the general case $u_0 \in (0, \overline{r}_1]$. Thus, there exists some $\alpha_0 \in (0, 1]$ such that $u_0 = R_1(\alpha_0)$. In particular, we have $u_0 = R_1(\alpha_0) > R_2(\alpha_0) > 0 = R_1(0)$ because $R_1(\cdot) > R_2(\cdot) > 0$ on $(0, 1]$. Moreover, because $R_1(\cdot)$ is continuous on $[0, 1]$, there exists some α_1 satisfying $\alpha_0 > \alpha_1 > 0$ and $R_1(\alpha_1) = R_2(\alpha_0)$. Continuing such a construction, we have $R_1(\alpha_1) > R_2(\alpha_1) > 0 = R_1(0)$, which implies that there exists some α_2 satisfying $\alpha_1 > \alpha_2 > 0$ and $R_1(\alpha_2) = R_2(\alpha_1)$. Thus, we have constructed a sequence, which is not necessarily unique such that $1 \geq \alpha_0 > \alpha_1 > \ldots > \alpha_t > \ldots > 0$ with $u_0 = R_1(\alpha_0) > R_2(\alpha_0) = R_1(\alpha_1) > R_2(\alpha_1) = R_1(\alpha_2) > \ldots > R_2(\alpha_{t-1}) = R_1(\alpha_t) > \ldots > 0$, as indicated in the text.[13] Because the sequence $\{\alpha_t\}$ is strictly decreasing and is in $(0, 1]$, it must converge to some finite limit $\alpha_\infty \in [0, 1]$. Because $R_j(\cdot)$ is continuous on $[0, 1]$, then $\lim_{t\to+\infty} R_j(\alpha_t) = R_j(\alpha_\infty)$ for $j = 1, 2$. But $R_2(\alpha_{t-1}) = R_1(\alpha_t)$ by construction, implying that $R_2(\alpha_\infty) = R_1(\alpha_\infty)$. Because $R_2(\alpha) = R_1(\alpha)$ only for $\alpha = 0$, this implies that $\alpha_\infty = 0$, and consequently $\lim_{t\to+\infty} R_j(\alpha_t) = 0$ for $j = 1, 2$.

[13] When $R_1(\cdot)$ is strictly increasing, or equivalently by (2) when the bidder's rent is strictly increasing in v, then $\alpha_0 = R_1^{-1}(u_0)$ and $\alpha_t = [R_1^{-1} \circ R_2]^t(\alpha_0)$, for $t = 1, 2, \ldots$, where $\circ$ denotes the composition of two functions and $[R_1^{-1} \circ R_2]^t$ denotes the t-composition of $R_1^{-1} \circ R_2$. In particular, the sequence $\{\alpha_t\}$ is unique.

We now iterate (A.8). Specifically, for any $u_0 \in \mathcal{R}_1$ and any corresponding sequence $\{\alpha_t\}$ as constructed above, we must have the nonlinear dynamic relation

$$\begin{aligned}
\lambda^{-1}(u_0) &= \lambda^{-1}[R_2(\alpha_0)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_1(\alpha_1)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_2(\alpha_1)] + \Delta b(\alpha_0) + \Delta b(\alpha_1) \\
&\vdots \\
&= \lambda^{-1}[R_2(\alpha_t)] + \Delta b(\alpha_0) + \ldots \Delta b(\alpha_t).
\end{aligned}$$

See the Figure for an illustration. Because $\lambda^{-1}(\cdot)$ is continuous on $[0, +\infty)$ with $\lambda^{-1}(0) = 0$ and $\lim_{t\to+\infty} R_2(\alpha_t) = 0$, as shown above, then $\lim_{t\to+\infty} \lambda^{-1}[R_2(\alpha_t)] = 0$. Because $\lambda^{-1}(u_0)$ is finite, it follows from the above equation that $\lim_{t\to+\infty} \sum_{\tau=0}^{t} \Delta b(\alpha_\tau)$ must exist and that it is equal to $\lambda^{-1}(u_0)$, i.e. $\lambda^{-1}(u_0) = \sum_{\tau=0}^{+\infty} \Delta b(\alpha_\tau)$ as desired. Note that this must be so irrespective of the sequence $\{\alpha_t\}$, whether such a sequence is unique. Moreover, because $\Delta b(\alpha_\tau)$ depends only on $b_j(\cdot)$ and $R_j(\cdot)$, which depend only on the distributions $G_j(\cdot)$, the latter equality shows that $\lambda^{-1}(\cdot)$ is identified nonparametrically on $\mathcal{R}_1$ from observed equilibrium bids.

The nonparametric identification of $F(\cdot)$ follows immediately from $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$. For, the nonparametric identification of $\lambda^{-1}(\cdot)$ on $\mathcal{R}_1$ and hence on $\mathcal{R}_2 \subset \mathcal{R}_1$ implies the nonparametric identification of $\xi_j(\cdot)$ on $[\underline{b}, \overline{b}_j]$ for $j = 1, 2$ by (2) and (4). The latter implies the nonparametric identification of $\xi_j^{-1}(\cdot) = s_j(\cdot)$ on $[\underline{v}, \overline{v}]$. Alternatively, pick an arbitrary $\alpha_0 \in [0, 1]$. From (4) and (A.7) for (say) $j = 1$, we have $v(\alpha_0) = b_1(\alpha_0) + \lambda^{-1}[R_1(\alpha_0)]$. Thus, the above explicit expression for $\lambda^{-1}(u_0)$ with $u_0 = R_1(\alpha_0)$ gives

$$v(\alpha_0) = b_1(\alpha_0) + \sum_{t=0}^{+\infty} \Delta b(\alpha_t) \tag{A.9}$$

showing that the α_0-quantile of $F(\cdot)$ is identified. Because α_0 is arbitrary in $[0, 1]$, it follows that $F(\cdot)$ is identified on $[\underline{v}, \overline{v}]$. $\square$

Proof of Proposition 2: First, we prove that (i), (ii) and (iii) are necessary. We use a double index (i, j) with i indexing bidder i among the I_j bidders and $j = 1, 2$ indicating the level of competition. Because $b_{ij} = s_j(v_i, U, F, I_j)$ and the v_{ij}s are i.i.d., the b_{ij}s are also i.i.d. given $j = 1, 2$. The fact that $G_j(\cdot) \in \mathcal{G}_R, j = 1, 2$ follows from Lemma 1 in Campo et al. (2005). This establishes (i). Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ from the Lemma and noting that $b_j = s_j(v)$ with $s_j(\cdot)$ strictly increasing, it follows that $b_j(\alpha) = s_j[v(\alpha)]$. Hence, $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0, 1]$ or equivalently $G_2(\cdot) \prec_{\underline{b}} G_1(\cdot)$. This establishes (ii). Lastly, because $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ and $U(\cdot)$ satisfies Definition 1 in Campo et al. (2005), then $\lambda(\cdot)$ is defined from $I\!R_+$ to $I\!R_+$ with $\lambda(0) = 0$, $\lambda'(\cdot) \geq 1$ and $\lambda(\cdot)$ is continuously differentiable on $[0, \infty)$. Because $F(\cdot)$ is invariant in I, its quantiles are also invariant in I. Thus, considering (2) for two values I_1 and I_2 at any α-quantile with $\alpha \in [0, 1]$ with $I_1 \neq I_2$ lead to (5). It remains to show that $\xi_j'(\cdot) > 0, j = 1, 2$. The equilibrium strategy $s_j(\cdot)$ must satisfy $\xi_j[s_j(v), U, G, I_j] = v$ for any $v \in [\underline{v}, \overline{v}]$ and $j = 1, 2$. We then obtain $\xi_j(b, U, G, I_j) = s_j^{-1}(b, U, F, I)$. This implies $\xi_j'(\cdot) = [s^{-1}(\cdot)]' > 0$. This establishes (iii).

Conversely, we show that (i), (ii) and (iii) are together sufficient. We construct a pair $[U, F]$ that satisfies Definitions 1 and 2 in Campo et al. (2005) and is independent of I. Let $U(\cdot)$ be such that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ or $1/\lambda(\cdot) = U'(\cdot)/U(\cdot)$. Integration of the latter with the normalization $U(1) = 1$ gives $U(x) = \exp \int_1^x 1/\lambda(t)dt$. We need to verify that $U(\cdot)$ satifies Definition 1 in Campo et al. (2005). This part of the proof can be found in Campo et al. (2005, Lemma 1). Let $F(\cdot)$ be the distribution of $X = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$, where $b \sim G_j(\cdot), j = 1, 2$. Note that X is independent of j because of (5). We need to verify that $F(\cdot)$ satifies Definition 2 in Campo et al. (2005). This part of the proof can be found in Campo et al. (2005, Lemma 1). Moreover, because the compatibility condition is satisfied for any $\alpha \in [0, 1]$, it implies that the corresponding α-quantile of $F(\cdot)$ does not depend on I. Hence $F(\cdot)$ does not depend on I. Lastly, we show that the pair $[U, F]$ can be rationalized by $G_1(\cdot)$ and $G_2(\cdot)$ with $I_2 > I_1$, i.e. that $G_j(\cdot) = F[s_j^{-1}(\cdot, U, F, I_j)], j = 1, 2$, where $s_j(\cdot, U, F, I_j)$ solves the first-order differential equation defining the equilibrium strategy with the boundary condition $s_j(\underline{v}, U, F, I_j) = \underline{v}$. By construction, $G_j(\cdot) = F[\xi_j(\cdot)]$. It suffices to show that $\xi_j^{-1}(\cdot), j = 1, 2$ solves the differential equation (2). This proof can be found in Campo et al. (2005, Lemma 1), which shows that $\xi_j^{-1}(\cdot), j =, 1, 2$ solves the differential equation with I_j under the boundary condition $\xi_j^{-1}(\underline{v}) = \underline{v}$. □

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