Simple Markov-Perfect Industry Dynamics*

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October 14, 2010

Abstract

This paper develops a tractable model for the computational and empirical analysis of infinite-horizon oligopoly dynamics. It features aggregate demand uncertainty, sunk entry costs, stochastic idiosyncratic technological progress, and irreversible exit. We develop an algorithm for computing a symmetric Markov-perfect equilibrium quickly by finding the fixed points to a finite sequence of low-dimensional contraction mappings. If all firms are identical, the result is the unique such equilibrium. If at most two heterogenous firms serve the industry, it is the unique "natural" equilibrium in which a high profitability firm never exits leaving behind a low profitability competitor. With more than two firms, the algorithm always finds a natural equilibrium. We present a simple rule for checking ex post whether the calculated equilibrium is unique, and we illustrate the model's application by assessing how price collusion impacts consumer and total surplus in a market for a new product that requires costly development. The results confirm Fershtman and Pakes' (2000) finding that collusive pricing can increase consumer surplus by stimulating product development. A distinguishing feature of our analysis is that we are able to assess their robustness for *hundreds* of parameter values in only a few minutes on an off-the-shelf laptop computer.

*We thank R. Andrew Butters for his expert research assistance.

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JEL Code: L13

Keywords: Sunk costs, Demand uncertainty, Markov-perfect equilibrium, Learning-by-doing, Technology innovation

1 Introduction

This paper supplies fast, effective, and simple computational methods for important special cases of Ericson and Pakes' (1995) model of dynamic oligopoly. These cases feature aggregate uncertainty, sunk entry costs, and stochastic firm-specific technological progress; but they exclude investment decisions other than entry and exit. This simplification facilitates a range of equilibrium characterization, existence, and uniqueness results that are not available for the more general framework. Moreover, it enables the development of algorithms that calculate equilibria by finding the fixed points of a finite sequence of low-dimensional contraction mappings. These results can be used to explore some key aspects of Ericson and Pakes' model with very low computational cost. This is often useful in itself, and can serve as a first stage of a richer analysis with a more complex specification.

Substantial methodological progress in the computation of Markov-perfect equilibria followed Ericson and Pakes' original presentation of their framework. Nevertheless, Doraszelski and Pakes (2007) note that these methodological developments are only in their infancy and applications remain rare. This paper contributes to this literature by developing relatively rich analytical results and effective computational methods for a comparatively simple model. It shares this approach with Abbring and Campbell's (2010) analysis of last-in first-out oligopoly dynamics. They consider a dynamic extension of Bresnahan and Reiss' (1990) static entry model that can naturally be applied to the empirical analysis of market level entry and exit data (Abbring, Campbell, and Yang, 2010). Timing and expectational assumptions simplify its equilibrium analysis: Otherwise homogeneous firms move sequentially, oldest first; and older firms never exit expecting to leave a younger firm behind. The present paper contributes more directly to the analysis of Ericson and Pakes' framework and its potential applications, because it allows for idiosyncratic technological progress in a model with simultaneously moving incumbent firms.

Our results leverage one key insight into the structure of payoffs in a symmetric Markov-perfect equilibrium: If any firm chooses to exit with positive probability, then all identically situated firms must have an expected continuation value of zero. This allows us to calculate firms' expected continuation values at some nodes of the game tree without knowing everything about how the game will proceed thereafter. Our results demonstrate how to use these initial calculations to recover all equilibrium payoffs and actions. For this task, it is very helpful to know beforehand that adding an active firm to an industry weakly reduces all other firms' continuation values. We prove that this intuitive property must hold if there is no idiosyncratic technological progress (so all active firms are the same) or if at most two firms can serve the industry at one time. For the more general case, we show that if a Markov-perfect equilibrium with this monotonicity exists, then it is essentially unique. In this case, the algorithm we propose always computes it. If no such equilibrium exists, then our algorithm can be easily adapted to find all equilibria satisfying a desirable property we call "one-shot renegotiation proofness".

The remainder of this paper proceeds as follows. The next section presents the model's primitives. It also discusses the equilibrium concept used, natural Markov-perfect equilibrium. As in Cabral (1993), the restriction to "natural" equilibrium requires no firm with high flow profits to exit leaving a lower-profitability rival in the market.

Section 3 analyzes the case with homogeneous firms. It begins with the key ideas in a simple duopoly example, and proceeds to proofs of equilibrium existence and uniqueness in the general case with two or more homogeneous firms. These proofs are constructive, and so they naturally generate an algorithm for equilibrium computation. Its central steps find the fixed points of a finite number of low-dimensional contraction mappings. Section 3 finishes with an illustrative application that compares the model's industry dynamics with those from Abbring and Campbell's model. Because incumbent firms decide simultaneously on continuation in our model, it may feature coordination failures in the exit decisions that Abbring and Campbell's assumption of sequential moves excludes. We numerically illustrate the way these coordination failures generate short-run hysteresis in the market structure.

Section 4 provides a parallel analysis for the duopoly case with general idiosyncratic technological progress. The central results from the homogeneous case; equilibrium existence, uniqueness, and computation with a finite number of contraction mappings; all extend to this setting. We implement the algorithm for this case and apply it to a numerical analysis of the effects of relaxing short-term price competition on welfare-enhancing product development, earlier explored by Fershtman and Pakes (2000).

Section 5 begins with extending the algorithm for the duopoly model to accommodate three or more potentially heterogeneous firms. We then show that *if* a natural equilibrium in which adding incumbent firms weakly lowers continuation values exists, then it is essentially unique and the algorithm computes it. Next, we illustrate with an example that it is possible for entry to *increase* an incumbent's expected discounted payoff. This counterintuitive effect of entry arises from the entry deterring effects of competition. Our analysis identifies two sources of equilibrium multiplicity, both of which require entry to raise an incumbent's equilibrium payoff at some point. One arises from the failure of incumbent firms to coordinate on survival when this is mutually beneficial. We propose to exclude such coordination failures by requiring equilibria to be "one-shot renegotiation proof". The other occurs when multiple mixed strategies leave incumbents indifferent between exit and continuation. The section concludes by extending the application in Section 4 to the case with three or more firms.

2 The Model

In Ericson and Pakes (1995), a countable number of firms with heterogeneous productivity levels serve a single industry. Entry requires the payment of a sunk cost, and exit allows firms to avoid per-period fixed costs of production. Surviving incumbent firms choose investments that stochastically improve their technologies. Exogenous stochastic increases in the knowledge stock outside the industry increase the quality of an outside good and, this way, decrease all incumbent firms' profits simultaneously. These outside knowledge shocks are embodied in potential entrants to the industry, and therefore do not affect their profits.

Two main changes to Ericson and Pakes' primitive assumptions facilitate our demonstration of Markov-perfect equilibrium uniqueness and our algorithm for its rapid computation. First, we assume that productivity evolves exogenously, instead of allowing firms to make costly investments in accelerating technological progress. Second, we replace the common negative shocks to the incumbents' profits by general aggregate demand shocks that equally affect the profits of incumbent firms and potential entrants.

2.1 Primitive Assumptions

The model consists of a single oligopolistic market in discrete time $t \in \mathbb{Z}_\star \equiv \{0, 1, \ldots\}$. A countable number of firms potentially serve the market. These are indexed by $f \in \mathbb{Z}_\star \times \mathbb{N}$. Below we refer to f as the firm's *name*. At a given time t, some of the firms are *active*, and the remaining producers are *inactive*. Each active firm f has an idiosyncratic productivity type K_t^f that takes values in $\mathbb{K} \equiv \{1, \ldots, \check{k}\}$. Stack the numbers of active firms with each productivity level at time t into the $\check{k} \times 1$ vector N_t, the *market structure*. Initially, no firms serve the market: N_0 equals a vector of zeros. Subsequently; entry, stochastic productivity improvement, and exit determine its evolution.

Figure 1 illustrates the sequence of events and actions within a period t. It begins with the inherited values of two state variables, N_t and the number of consumers in the market, C_t. With these in place, the active participants receive their profits from serving the market. For a type k firm facing the market structure N_t, these equal $C_t\pi_k(N_t) - \kappa$. Here, $\pi_k(N_t)$ is the average producer surplus earned by the firm from each of the C_t consumers.[1] The term $\kappa > 0$ represents fixed costs of production.

We assume that a firm's producer surplus decreases with the number and productivity of its competitors and increases with its own productivity. For this assumption's formal statement, we use ι_k to denote a $\check{k} \times 1$ vector with a one in its kth position and zeros elsewhere, and set $\iota_0 \equiv 0$. This allows us to denote a market structure with at least one type k firm with $n + \iota_k$.

[1] We leave this undefined if the k'th element of N_t equals zero.

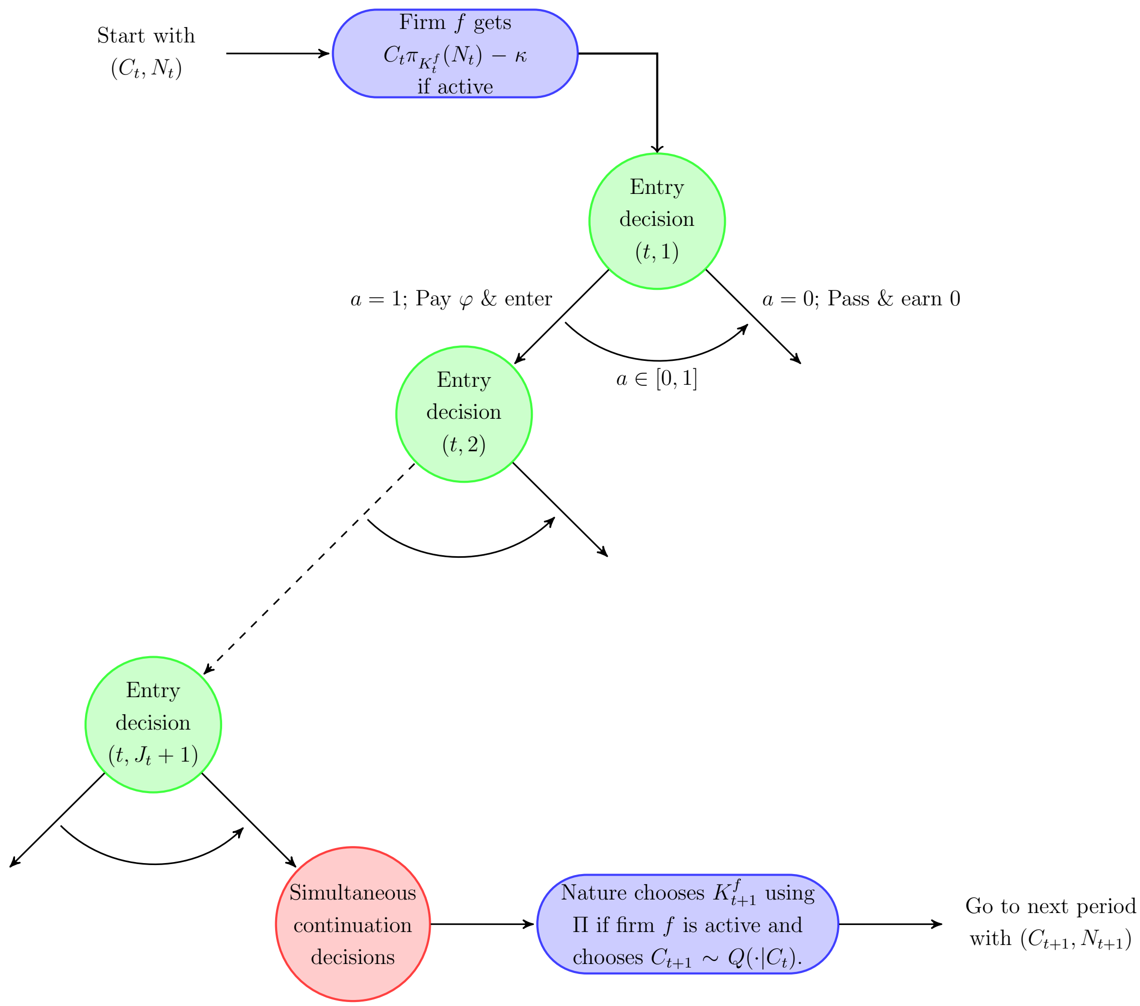


Figure 1: The Sequence of Events and Actions within a Period

Assumption 1 (Monotone Producer Surplus)**.** For all productivity types $k \in \mathbb{K}$ and market structures $n \in \mathbb{Z}_\star^{\check{k}}$:

1. $\pi_k(\iota_k + n + \iota_l) < \pi_k(\iota_k + n + \iota_{l-1})$ for all $l \in \mathbb{K}$;
2. $\pi_k(\iota_k + n) \to 0$ as the number of firms in n goes to infinity; and
3. $\pi_k(\iota_k + \iota_l + n) < \pi_l(\iota_k + \iota_l + n)$ for all $k, l \in \mathbb{K}$ such that $k < l$.

After production, firms with names in $\{t\} \times \mathbb{N}$ make entry decisions sequentially in the order of their names, starting with $(t, 1)$. These continue until a firm chooses to remain out of the industry. We denote the number of entrants in period t with J_t, so the name of the first potential entrant choosing to stay out of the market and thereby ending its entry stage is $(t, J_t + 1)$. The cost of entry is $\varphi > 0$, and after paying this cost the entrant immediately joins the set of active firms, with productivity type 1.[2] A firm with an entry opportunity cannot delay its choice, so the payoff to staying out of the industry is zero.

After the entry decisions, all active firms— including those that just entered the market— decide simultaneously between survival and exit. Exit is irreversible but otherwise costless. It allows firms to avoid future periods' fixed production costs. Firms' entry and exit decisions maximize their expected profit streams discounted with a factor $\beta < 1$.

In the period's final stage, C_t and the firms' productivity types evolve. The number of consumers in the market evolves exogenously according to a nonnegative first-order Markov process bounded between $\hat{c}$ and $\check{c} < \infty$. We denote the conditional distribution of C_{t+1} with $Q\left(c \,|\, C_t\right) \equiv \Pr\left(C_{t+1} \leq c \,|\, C_t\right)$, and the corresponding probability density function with $q(\cdot \mid C_t)$. Each firm's idiosyncratic productivity type follows an independent Markov chain with a common $(\check{k} \times \check{k})$ transition matrix Π. Its typical element is $\Pi_{k,k'} \equiv \Pr\left(K^f_{t+1} = k' \,\middle|\, K^f_t = k\right)$. Following Ericson and Pakes (1995), we assume that the idiosyncratic productivity types never regress:

Assumption 2 (No Productivity Regress)**.** Π is upper diagonal.

We further assume that K^f_{t+1} (weakly) stochastically increases with K^f_t.

Assumption 3 (Monotone Productivity Dynamics)**.** For all $k', k, l \in \mathbb{K}$ such that $k < l$,

$$\Pr\left(K^f_{t+1} \geq k' \,\middle|\, K^f_t = k\right) \leq \Pr\left(K^f_{t+1} \geq k' \,\middle|\, K^f_t = l\right).$$

This assumption gives high technology firms no worse opportunities than low technology firms have to advance to any given technological level.

2.2 Markov-Perfect Equilibrium

A *Markov-perfect equilibrium* is a subgame-perfect equilibrium in strategies that are only contingent on *payoff-relevant* variables. For a potential participant $f = (t, j)$ contemplating entry these are C_t and the market structure M^f_t just after f's possible entry. The latter is period t's initial market structure N_t augmented with j type 1 entrants: $M^f_t \equiv N_t + j\iota_1$. Denote the market structure after the period's final entry with $M_{E,t} \equiv N_t + \iota_1 J_t$. If firm f is contemplating survival in period t, the payoff-relevant variables are this market structure, the current demand state (C_t), and its productivity type (K^f_t).

A *Markov strategy* for firm f is a pair (a^f_E, a^f_S) of functions

$$a^f_E : \mathbb{Z}^{\check{k}}_{\star} \times [\hat{c}, \check{c}] \longrightarrow [0, 1] \qquad \text{and} \qquad a^f_S : \mathbb{Z}^{\check{k}}_{\star} \times [\hat{c}, \check{c}] \times \mathbb{K} \longrightarrow [0, 1].$$

[2] Since entrants' productivity types evolve before their first period of production, we can use the distribution of K^f_{t+1} given $K^f_t = 1$ to distribute new firms' types nontrivially. That is, the assumption that all entrants have $K^f_t = 1$ is not overly restrictive.

This strategy's *entry rule* a_E^f assigns a probability of becoming active given an entry opportunity to each possible value of (M_t^f, C_t). Similarly, its *exit rule* a_S^f assigns a probability of being active in the next period given that the firm is currently active to each possible value of its payoff-relevant state $(M_{E,t}, C_t, K_t^f)$. Since calendar time is not payoff-relevant, we hereafter drop the t subscript from all variables. A symmetric equilibrium is an equilibrium in which all firms follow the same strategy (a_E, a_S). In the remainder of the paper, we focus on symmetric equilibria and drop the superscript f from the firms' common strategy.

Throughout the paper, we will focus on equilibria in which a high productivity firm never exits when a low productivity competitor survives. Such equilibria are natural, because a high productivity firm earns strictly higher flow profit in each period than a low productivity firm. Formally, we define a natural Markov-perfect equilibrium as follows:

Definition 1. *A* natural *Markov-perfect equilibrium is a symmetric Markov-perfect equilibrium in a strategy* (a_E, a_S) *such that for all* $k, l \in \mathbb{K}$ *such that* $k < l$*;* $m_k \geq 1$*,* $m_l \geq 1$*, and* $a_S(m, c, k) > 0$ *together imply that* $a_S(m, c, l) = 1$*.*

Cabral (1993) restricts attention to similar natural equilibria in a model with deterministic productivity progression.

Firms' expected discounted profits at each node of the game depend on that node's payoff-relevant state variables when they all use Markov strategies. The payoffs in two of each period's nodes are of particular interest, the *post-entry* value and the *post-survival* value. The post-entry value $v_E(M_E, C, K)$ equals the expected discounted profits of a type K firm facing C consumers in a market with structure M_E just after all entry decisions have been sequentially realized. Since it gives the payoffs to a potential producer from entering in each possible market structure that could arise from other players subsequent entry decisions, it determines optimal entry choices. The post-survival value $v_S(M_S, C, K)$ equals the expected discounted profits of a type K firm facing C consumers in a market with structure M_S just after all survival decisions have been realized. It gives the payoffs to a surviving firm in each possible market structure following firms' simultaneous continuation decisions, so it is central to the analysis of exit.

The value functions v_E and v_S satisfy

$$v_E(m_E, c, k) = a_S(m_E, c, k)\mathbb{E}\left[v_S(M_S, c, k) \,|\, M_E = m_E\right] \tag{1}$$

and

$$v_S(m_S, c, k) = \beta\mathbb{E}\left[C'\pi_{K'}(N') - \kappa + v_E(M'_E, C', K') \,|\, M_S = m_S, C = c, K = k\right]. \tag{2}$$

Here and throughout, we denote the variable corresponding to X in the next period with X'. The conditional expectation in (1) is computed given that the firm of interest continues, and embodies the use of a_S by all other active firms. In fact, the only nontrivial randomness it embodies is that from firms' possible use of mixed strategies. The conditional expectation in (2) accounts for the use of a_E by all potential participants with entry opportunities in the next period as well as the exogenous evolutions of C and the firms' productivity types.[3]

For (a_E, a_S) to form a symmetric Markov-perfect equilibrium, it is necessary and sufficient that no firm can gain from a one-shot deviation from (a_E, a_S) (e.g. Fudenberg and Tirole, 1991, Theorem 4.2):

$$a_E(m,c) \in \arg\max_{a\in[0,1]} a\left\{\mathbb{E}\left[v_E\left(M_E,c,1\right) \mid M=m\right]-\varphi\right\} \text{ and} \tag{3}$$

$$a_S(m_E,c,k) \in \arg\max_{a\in[0,1]} a\,\mathbb{E}\left[v_S(M_S,c,k) \mid M_E=m_E\right]. \tag{4}$$

The conditional expectations in (3) and (4) are computed like those in (1) and (2). For example, $\mathbb{E}\left[v_E\left(M_E,c,1\right) \mid M=m\right]$ is the payoff, gross of the entry cost φ, that a potential participant in state (m,c) expects from entering if all firms with entry opportunities later in the period use the entry rule a_E and the value of ending up as a type 1 firm in a market with structure m_E and c consumers equals $v_E\left(m_E,c,1\right)$.

Together, conditions (1)–(4) are sufficient and necessary for a strategy (a_E, a_S) to form a symmetric Markov-perfect equilibrium with payoffs v_E and v_S.

Before proceeding to examine the set of natural Markov-perfect equilibria, consider one uninteresting source of equilibrium multiplicity. If immediately following a sequence of moves in a game, we can change the action of one player without an identically situated rival and give that player the same equilibrium payoff; then we can construct multiple subgame perfect equilibria by simply varying that player's choice. To eliminate this possibility, we require any incumbent firm that is the only active firm of its type and all potential entrants to choose inactivity whenever it gives the same payoff as continuation or entry, respectively.

Definition 2. *A Markov strategy* (a_S, a_E) *with corresponding payoff* v_E defaults to inactivity *if*

- $a_S(m,c,k)=0$ *if* $v_S(m,c,k)=0$ *and* $m_k=1$,
- $a_E(m,c)=0$ *if* $v_E(m,c,1)=\varphi$,

for all $k\in\mathbb{K}$ *and all* c.

The remainder of the paper restricts attention to equilibria with strategies that default to inactivity, unless otherwise mentioned.[4]

[3] This paper's computational appendix presents the two conditional distributions underlying the conditional expectations in (1) and (2) in detail.

[4] It should stressed here that we do *not* restrict the game's strategy space to include only strategies that default to inactivity.

3 Homogeneous Oligopoly

It is instructive to first consider the special case without productivity heterogeneity ($\check{k} = 1$), which we call the *homogeneous-firm* model. In this case; the market structure variables N, M, M_E, and M_S simply count the numbers of active firms in different nodes of the game; all active firms receive the flow payoff $C\pi(N) - \kappa$; the state for active firms contemplating survival reduces to the $\mathbb{N} \times [\hat{c}, \check{c}]$-valued pair (M_E, C); and $N' = M_S$. Note that the restriction to natural equilibria has no bite here beyond its requirement of symmetry.

3.1 Preliminary Results

We begin with establishing two results that are central to this section's equilibrium analysis. First, there exists a finite upper bound on the number of firms that will ever be active in equilibrium. Denote $\check{m} \equiv \max\{n \in \mathbb{N} : \check{c}\pi(n) - \kappa \geq 0\}$. Monotone producer surplus (Assumption 1) and $\check{c} < \infty$ ensure that $\check{m} < \infty$ exists. For all $n > \check{m}$, flow profits $c\pi(n) - \kappa$ are strictly negative, even in the most favorable demand state. The following result bounds the number of active firms with $\check{m}$.

Lemma 1 (Bounded Number of Firms in the Homogeneous-Firm Model)**.** *In a symmetric Markov-perfect equilibrium, $v_E(m, c) = 0$ for all $m > \check{m}$ and all $c \in [\hat{c}, \check{c}]$.*

Proof. See Appendix A. □

Lemma 1 implies that no firm will enter if the resulting number of active firms, including incumbents surviving from the previous period and the current period's earlier entrants, would be larger than $\check{m}$: $a_E(m, \cdot) = 0$ for all $m > \check{m}$. With $N_0 = 0$, this implies that M_E is bounded from above by $\check{m}$. In addition, because $N \leq M_E$ and $M_S \leq M_E$, both N and M_S are bounded by $\check{m}$ as well. Consequently, there is no need to specify $a_S(m, \cdot)$ for $m > \check{m}$ and we can analyze the equilibrium on a bounded state space $\{1, 2, \ldots, \check{m}\} \times [\hat{c}, \check{c}]$.

The second preliminary result demonstrates that an active firm's expected discounted profits decrease with the number of firms in the market. Specifically, monotone flow profits (Assumption 1) ensure that v_E and v_S are monotone in equilibrium.

Lemma 2 (Monotone Equilibrium Payoffs in the Homogeneous-Firm Model)**.** *In a symmetric Markov-perfect equilibrium, for all $c \in [\hat{c}, \check{c}]$, $v_E(m, c)$ and $v_S(m, c)$ weakly decrease with m.*

Proof. See Appendix A. □

This is the intuitive and desirable monotonicity property of equilibrium payoffs we mentioned in the introduction.

Figure 2: Reduced-form Representation of the Duopoly Continuation Game

	Survive	Exit
Survive	$v_S(2,c)$, $v_S(2,c)$	$v_S(1,c)$, 0
Exit	0, $v_S(1,c)$	0, 0

Note: In each cell, the upper-left expression gives the row player's payoff. Please see the text for further details.

With Lemmas 1 and 2 in hand, we proceed by first illustrating some key ideas with a simple duopoly example. Section 3.3 shows that the homogeneous-firm model generally admits a unique natural Markov-perfect equilibrium, and provides an algorithm for its computation. Finally, Section 3.4 applies this algorithm to a numerical comparison of the model's industry dynamics with those from Abbring and Campbell's (2010) model of last-in first-out oligopoly dynamics.

3.2 A Duopoly Example

If $\check{c}\pi(3) - \kappa < 0$, then no more than two firms will ever produce. Limiting the number of firms with this assumption allows us to illustrate the homogeneous-firm model's moving parts without undue notational burden. We do so by constructing a Markov-perfect equilibrium for it in three steps. We then illustrate the constructed equilibrium with a presentation of its strategies and continuation values when C follows a particular simple stochastic process.

Step 1: Calculation of $v_E(2,\cdot)$, $v_S(2,\cdot)$, and $a_E(2,\cdot)$ The equilibrium construction begins with a characterization of the duopoly payoffs $v_E(2,\cdot)$ and $v_S(2,\cdot)$. In a Markov-perfect equilibrium, the survival rule $a_S(2,c)$ satisfies (4): Given c, it is a Nash equilibrium of the static simultaneous-move game with payoffs given by the expected continuation values. Figure 2 gives the reduced-form representation of this game with the two pure strategies "Survive" and "Exit". The upper-left expression in each cell is the row player's payoff. Both firms receive the duopoly post-survival payoff $v_S(2,c)$ if they both choose to survive. This payoff consists of the discounted expected value of, in the next period, first collecting the duopoly flow payoff $C'\pi(2) - \kappa$ and then receiving the duopoly post-entry payoff $v_E(2,C')$. Consequently, it satisfies a special case of Equation (2):

$$v_S(2,c) = \beta\mathbb{E}\left[C'\pi(2) - \kappa + v_E(2,C') \,|\, C = c\right].$$

A firm that survives while its rival exits earns the monopoly post-survival value $v_S(1, c)$. Suppose that $v_S(2, c) > 0$. Lemma 2 guarantees that $v_S(1, C) > 0$, so in this case "Survive" is a dominant strategy.

If instead $v_S(2, c) < 0$, then a symmetric equilibrium strategy must put positive probability on "Exit". That pure strategy's payoff always equals zero. Since $v_E(2, c)$ equals the symmetric equilibrium payoff to this game, these facts together yield the following special case of Equation (1):

$$\begin{aligned} v_E(2, c) &= \max\left\{0, v_S(2, c)\right\} \\ &= \max\Big\{0, \beta\mathbb{E}\left[C'\pi(2) - \kappa + v_E(2, C') \,|\, C = c\right]\Big\}. \end{aligned}$$

The right-hand side defines a contraction mapping, so this necessary condition uniquely determines $v_E(2, \cdot)$ and, using (2), $v_S(2, .)$. This is the key technical insight that makes the calculation of the model's Markov-perfect industry dynamics simple. Although duopoly is *not* an absorbing state for the industry, we can calculate the equilibrium duopoly payoffs without knowledge of the firms' payoffs in possible future market structures. This is because firms' common post-entry value in a symmetric equilibrium equals zero unless joint continuation is individually profitable.

With the duopoly post-entry value in hand, we can proceed to the problem of a potential entrant facing a single incumbent. By Equation (3), this firm enters if $v_E(2, c) > \varphi$ and stays out of the market if $v_E(2, c) \leq \varphi$. For all c,

$$a_E(2, c) = I\left\{v_E(2, c) > \varphi\right\}.$$

Note that this specification embodies the restriction of default to inactivity. When C has an atomless distribution, this strategy almost surely prescribes the same action as any other entry strategy that does not default to inactivity but nevertheless consistent with profit maximization. For this reason, our requirement that the potential entrant default to inactivity has no substantial economic content.

Step 2: Calculation of $v_E(1, \cdot)$, $v_S(1, \cdot)$, $a_E(1, \cdot)$, and $a_S(1, \cdot)$ We proceed to consider the monopoly payoffs, a potential entrant's decision to enter an empty market, and an incumbent monopolist's survival decision. Because an incumbent monopolist choosing to survive will earn $v_S(1, c)$, the post-entry value to a monopolist in (1) reduces to

$$\begin{aligned} v_E(1, c) &= \max\left\{0, v_S(1, c)\right\} \\ &= \Big\{0, \beta\mathbb{E}\Big[C'\pi(1) - \kappa + a_E(2, C')v_E(2, C') + \left\{1 - a_E(2, C')\right\} v_E(1, C') \;\Big|\; C = c\Big]\Big\}. \end{aligned}$$

Given $v_E(2, \cdot)$ and $a_E(2, \cdot)$ from Step 1, the right-hand side defines a contraction mapping that uniquely determines $v_E(1, \cdot)$ and, using Equation (2), $v_S(1, \cdot)$. It is not difficult to demonstrate that the $v_E(1,c)$ and $v_S(1,c)$ so constructed always weakly exceed, respectively, $v_E(2,c)$ and $v_S(2,c)$ from Step 1; so that the constructed value functions are consistent with the requirements of Lemma 2.

Just as with a potential duopolist, we select the unique entry rule for a potential monopolist that defaults to inactivity. Since $v_E(2,c) \leq v_E(1,c)$, this is

$$a_E(1,c) = I\left\{v_E(1,c) > \varphi\right\}.$$

By (4), a monopolist chooses survival in demand states c such that $v_S(1,c) > 0$ and exit if $v_S(1,c) < 0$. Our equilibrium construction uses the unique monopoly survival rule that defaults to inactivity:

$$a_S(1,c) = I\left\{v_S(1,c) > 0\right\}.$$

Step 3: Calculate $a_S(2,\cdot)$ The first two steps have determined the only possible post-entry and post-survival values, as well as an entry rule and a monopoly survival rule that are consistent with them. This last step completes the equilibrium strategy's construction by determining a duopoly survival rule that satisfies (4).

As we noted above in Step 1, equilibrium requires $a_S(2,c) = 1$ if $v_S(2,c) > 0$. It only remains to determine the survival rule in demand states c such that $v_S(2,c) \leq 0$. If profit maximization would require even a monopolist to exit (i.e. $v_S(1,c) \leq 0$), then both duopolists exit for sure and $a_S(2,c) = 0$. If instead $v_S(1,c) > 0$, then the reduced-form continuation game above has no pure strategy equilibrium. In its unique mixed-strategy equilibrium, each firm's survival probability leaves its rival indifferent between exiting (and getting a payoff of zero for sure) and surviving. That is, in demand states c such that $v_S(2,c) \leq 0$ and $v_S(1,c) > 0$, the indifference condition

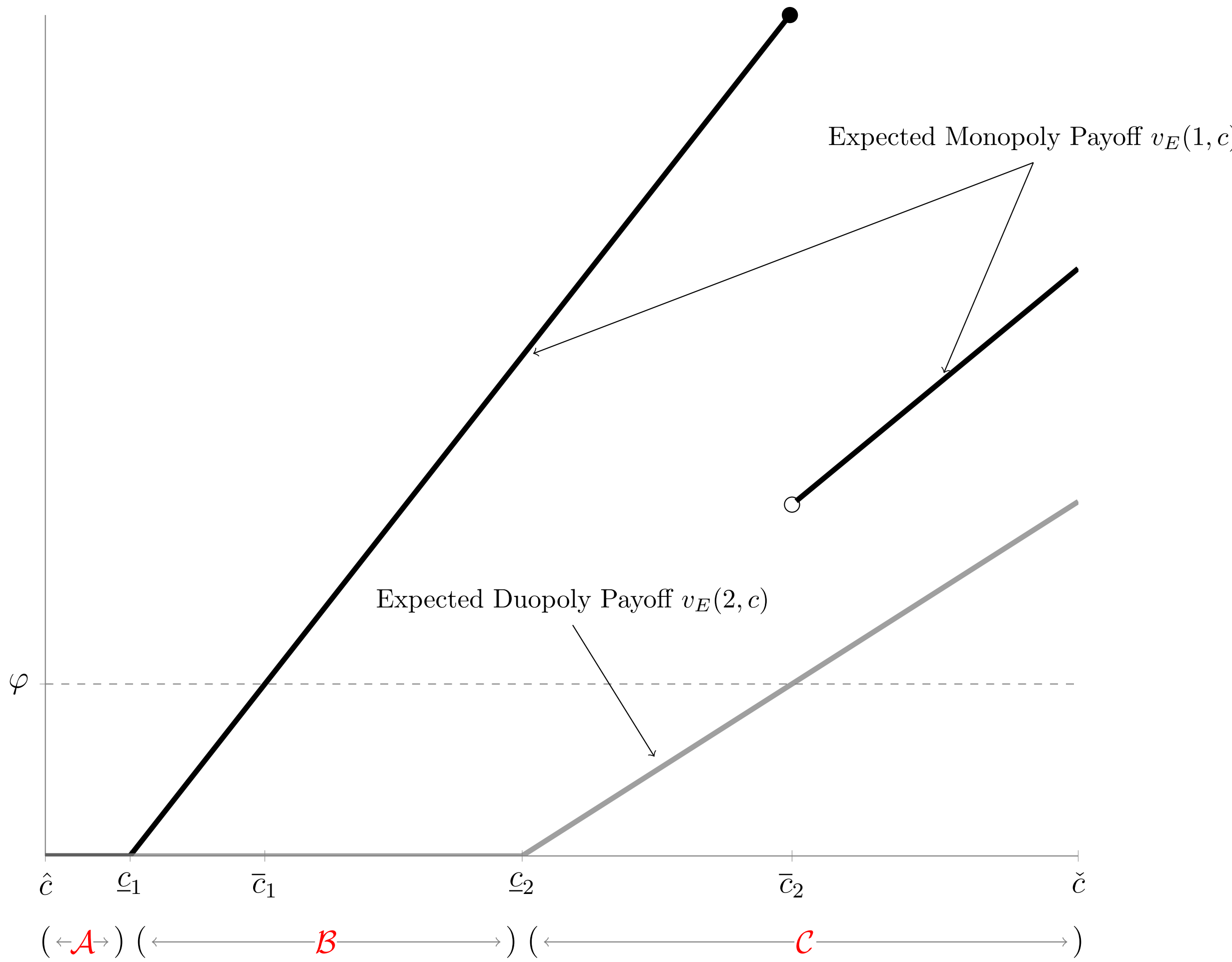
$$a_S(2,c)v_S(2,c) + \left\{1 - a_S(2,c)\right\} v_S(1,c) = 0$$

uniquely determines $a_S(2,c)$.

Illustration of the Constructed Equilibrium The entry and survival rules so calculated form our equilibrium. Figure 3 plots the payoffs for a particular numerical example. In it, $C' = c$ with probability $1-\lambda$ and equals a draw from a uniform distribution over $[\hat{c}, \check{c}]$ with the complementary probability. With this process the current value of C has no influence on the equilibrium probability that a firm will enter or exit in a future period. In turn, this guarantees that $v_E(1,c)$ and $v_E(2,c)$ are piecewise linear in c.

The lower (continuous) function in grey gives the duopoly post-entry value, $v_E(2,c)$. By construction, this is identical to the expected discounted profits of a duopolist facing a rival

Figure 3: Equilibrium Payoffs in the Homogeneous Duopoly Example



that will never exit first. It equals zero for $c \leq \underline{c}_2$. Thereafter it rises $\pi(2)/(1-\beta(1-\lambda))$ for each extra consumer. For $c > \overline{c}_2$, entry into a market with one incumbent is optimal.

The monopoly post-entry value $v_E(1,c)$ equals zero for demand levels $c \leq \underline{c}_1$, and it increases $\pi(1)/(1-\beta(1-\lambda))$ with each extra consumer until reaching $\overline{c}_2$. . When $c > \overline{c}_2$, the period's entry stage always ends with two firms, so the equilibrium payoff to a firm that began the period as a monopolist drops to the grey expected duopoly payoff. The disconnected line segment above this gives the expected payoff to a firm that finds itself as a monopolist after the period's entry stage is complete.[5] Given this value function, the equilibrium strategy for a potential entrant facing an empty market is $a_E(1,c) = I\{v_E(1,c) > \varphi\} \equiv I\{c > \overline{c}_1\}$, and the analogous continuation rule for an incumbent monopolist is $a_S(1,c) = I\{v_E(1,c) > 0\} = I\{v_S(1,c) > 0\} \equiv I\{c > \underline{c}_1\}$.

Duopolists' common continuation strategy corresponds to the unique Nash equilibrium

[5] This would require some potential entrant to deviate from the equilibrium strategy.

of the game in Figure 2. Exit is a dominant strategy when $c \in \mathcal{A}$, and survival is dominant when $c \in \mathcal{C}$. When $c \in \mathcal{B}$ the firms mix over survival and exit.

The specific three-step procedure we followed to compute the duopoly equilibrium employed two contraction mappings to calculate its continuation values, so these are obviously uniquely determined. Furthermore, after eliminating actions of entry and continuation that essentially result in the same equilibrium payoff as inactivity on at most three values of c, we construct the essentially unique equilibrium.

3.3 Equilibrium Existence, Uniqueness, and Computation

We now extend the analysis of the duopoly example to markets with arbitrary numbers of active firms. Lemma 1 connects this general case to the duopoly example, by providing an upper bound $\check{m}$ on the equilibrium number of active firms. This allows us to develop a finitely recursive argument along the lines of the duopoly example. As in the duopoly example, we will rely on the result that payoffs decrease in the number of active firms (Lemma 2) to ensure equilibrium uniqueness.

We first present an algorithm that computes a candidate equilibrium with strategy (α_E, α_S) and payoffs w_E and w_S. We then verify that (α_E, α_S) indeed forms a natural Markov-perfect equilibrium and establish that this is the only one with a strategy that defaults to inactivity.

The algorithm sequences two procedures: Procedure 1 computes w_E and α_E, beginning with the calculation of $\check{m}$. As shown by Lemma 1, no firm will enter a market that is already served by the maximum number of firms, so we can compute $w_E(\check{m}, \cdot)$ as the fixed point of the contraction mapping defined by the right-hand side of

$$w_E(\check{m}, c) = \max\Big\{0, \beta\mathbb{E}\left[C'\pi(\check{m}) - \kappa + w_E\left(\check{m}, C'\right) \mid C = c\right]\Big\} \tag{5}$$

Like the equation that yields $v_E(2, \cdot)$ in the duopoly example, this expression assumes that all $\check{m}$ incumbents will continue whenever this yields positive payoffs to each one of them. Lemma 2 again ensures that this is so in any natural equilibrium. With $w_E(\check{m}, \cdot)$ in hand, the corresponding entry rule for a potential entrant facing $\check{m} - 1$ incumbents immediately follows.

Procedure 1 continues by computing $w_E(\check{m} - 1, \cdot)$ using $\alpha_E(\check{m}, \cdot)$ to form expectations about the entry of an additional firm. That is, it solves

$$\begin{aligned} w_E(\check{m} - 1, c) = \max\Big\{0, \beta\mathbb{E}\Big[&C'\pi(\check{m} - 1) - \kappa + \alpha_E(\check{m}, C')w_E\left(\check{m}, C'\right) \\ &+ \{1 - \alpha_E(\check{m}, C')\}w_E(\check{m} - 1, C') \Big| C = c\Big]\Big\} \end{aligned}$$

for $w_E(\check{m}-1, \cdot)$. Again, this Bellman equation uses Lemma 2 to express a firm's continuation value as the maximum of zero and the value of continuing with all rivals.

With $w_E(\check{m}-1,\cdot)$ in place, $\alpha_E(\check{m}-1,\cdot)$ can be constructed. The recursion continues until $w_E(m,\cdot)$ and $\alpha_E(m,\cdot)$ are determined for all $m \in \{1,\ldots,\check{m}\}$. Procedure 1's flow chart lays out this recursion explicitly. In it, we use

$$\mu(m,c) \equiv m + \sum_{i=1}^{\check{m}-m} \alpha_E(m+i,c),$$

to denote the number of firms that will be active after all potential entrants' choices.

Using μ and w_E from Procedure 1, Procedure 2 constructs the candidate post-survival value $w_S(\cdot,c)$ and exit rule $\alpha_S(\cdot,c)$ for a given c. As in the duopoly example, $\alpha_S(m,c)=1$ if survival with m active firms is profitable, and to 0 if even survival as a monopolist would not be profitable. In all other cases, the m incumbents' survival probability equates each individual's expected payoff from continuation to zero. The following result ensures that the calculation of this mixed strategy always has a unique solution.

Lemma 3 (Monotone Candidate Equilibrium Payoffs in the Homogeneous-Firm Model)**.** *For all $c \in [\hat{c},\check{c}]$, $w_E(m,c)$ and $w_S(m,c)$ decrease with m.*

Proof. See Appendix A. □

Lemma 3 establishes that the candidate equilibrium payoffs w_E and w_S constructed by Procedures 1 and 2 have the monotonicity property that, by Lemma 2, equilibrium requires.

Procedures 1 and 2 can be combined in an algorithm to compute a candidate equilibrium.

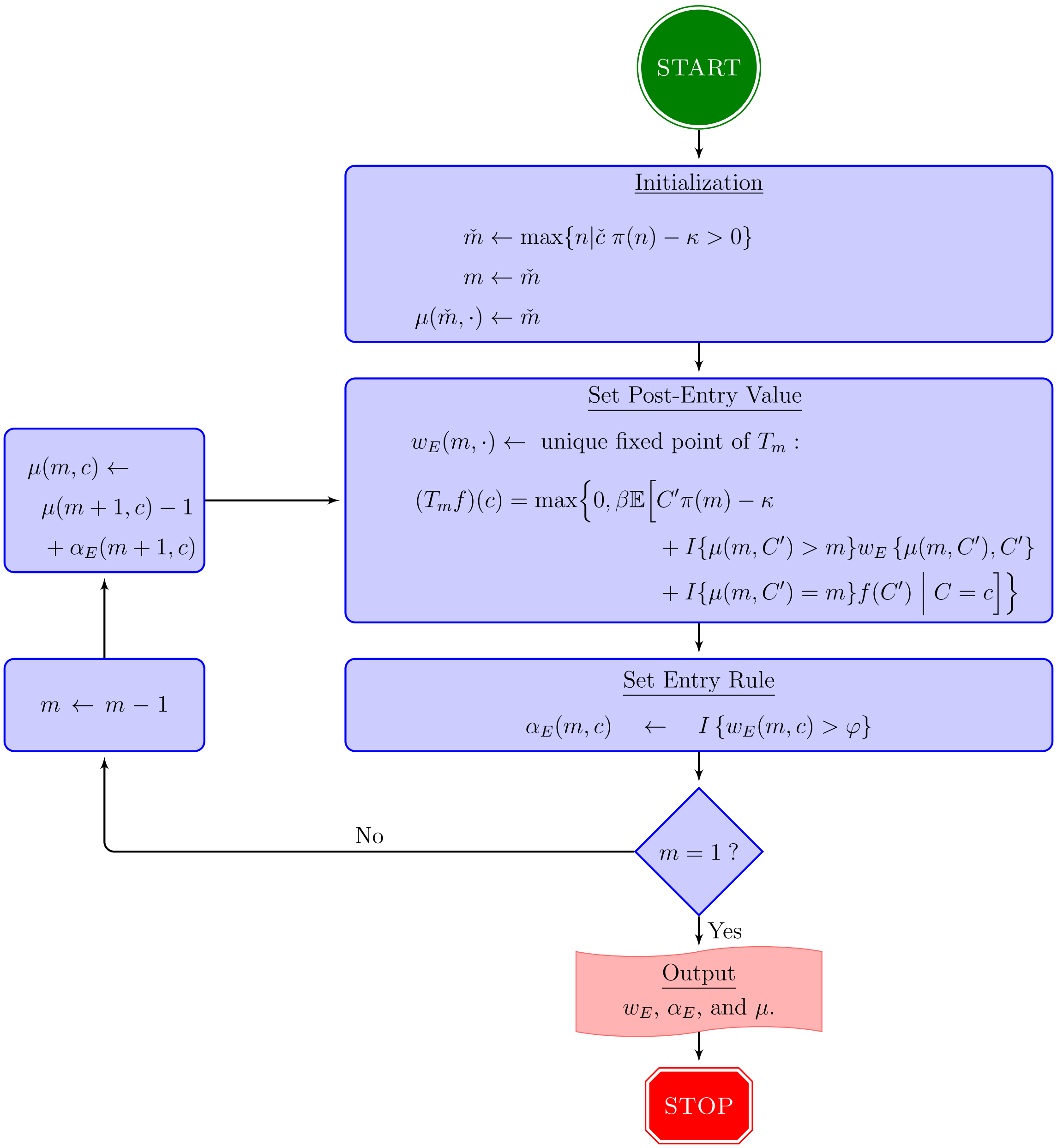
Algorithm 1 (Candidate Equilibrium in the Homogeneous-Firm Model)**.** *Compute a candidate equilibrium strategy (α_S,α_E) and payoffs w_S and w_E in two steps:*

1. *Use Procedure 1 to compute $\check{m}$; and μ, w_E and α_E on $\{1,\ldots,\check{m}\}\times[\hat{c},\check{c}]$.*

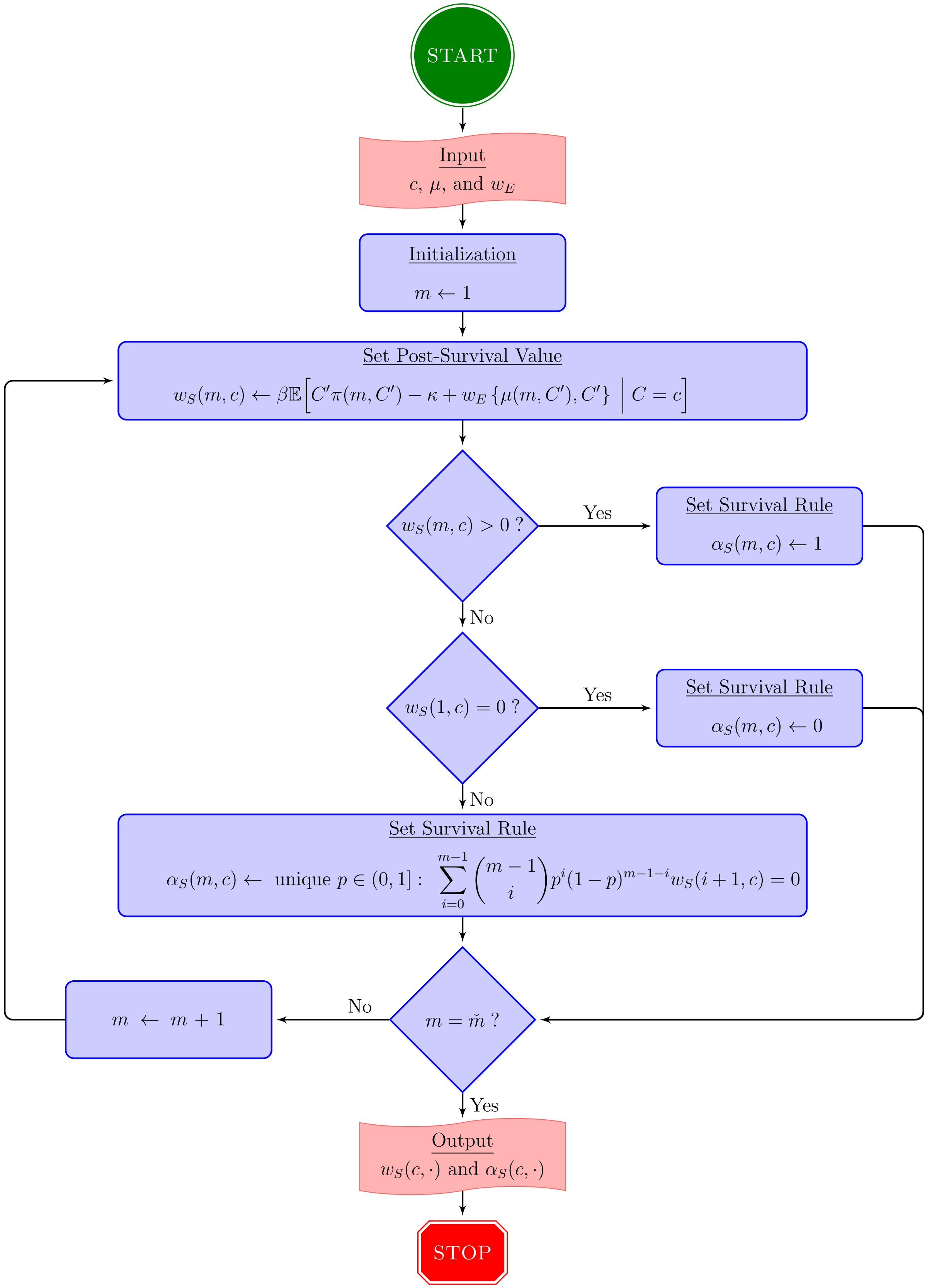
2. *For all $c \in [\hat{c},\check{c}]$, use Procedure 2 to compute $w_S(\cdot,c)$ and $\alpha_S(\cdot,c)$.*

Verifying that the candidate equilibrium constructed by Algorithm 1 is indeed a natural Markov-perfect equilibrium is straightforward.[6] Furthermore, with Lemma 2 we can show that the computed equilibrium is the only one.

[6] Formally, the algorithm only computes a candidate equilibrium strategy and value function on a subset $\{1,\ldots,\check{m}\}\times[\hat{c},\check{c}]$ of the state space. Lemma 1 implies that w_E and α_E, if they are indeed an equilibrium post-entry value function and survival rule, can be uniquely extended to the entire state space by setting $w_E(m,\cdot)=0$ and $\alpha_E(m,\cdot)=0$ for all $m>\check{m}$. Given an initially empty market, the equilibrium path's realization can never have more than $\check{m}$ firms serving the market. Nevertheless, Algorithm 1 can straightforwardly be adapted to extend the post-survival value w_S and the survival rule α_S to such states off the equilibrium path if the need would arise. Henceforth, we will not make the possibility or potential need to uniquely extend the candidate equilibrium to the full state space explicit.



Procedure 1: Calculation of w_E and α_E for the Homogeneous-Firm Model



Procedure 2: Calculation of Candidate w_S and α_S for the Homogeneous-Firm Model

Proposition 1 (Equilibrium in the Homogeneous-Firm Model)**.** *There exists a unique natural Markov-perfect equilibrium. Algorithm 1 computes its payoffs and strategy. The equilibrium payoffs* $v_S = w_S$ *and* $v_E = w_E$*. The equilibrium strategy* $(a_S, a_E) = (\alpha_S, \alpha_E)$*.*

Proof. See Appendix A. □

3.4 Comparison with LIFO Dynamics

This section's homogeneous-firm model differs from Abbring and Campbell's (2010) LIFO model in one substantial respect: Incumbent firms make their survival decisions simultaneously rather than sequentially. This leads to a noticeable difference in their dynamics following demand contractions. In the LIFO model, the youngest firms exit when all incumbents cannot profitably remain active. In this section's model with symmetric survival decisions, the only equilibrium often resembles a war of attrition, in which incumbent firms mix between the pure strategies "continue" and "exit". As a result, the adjustment to a negative demand shock generally takes longer. Moreover, the realized adjustment may be too large, and followed by entry even when demand does not increase.

To illustrate these coordination problems, we specified particular parameter values for the models' common parameters, calculated their equilibria, and simulated them for a particular realization of the demand process lasting 200 periods/years.[7] Figure 4 gives the results. Its middle panel presents the logarithm of C_t, which displays considerable persistence by design. The top and bottom panels plot the equilibrium number of firms in the LIFO model and this paper's Simple model. For both models, the market has two incumbent firms at the start of the simulation and the initial realization of C_t, drawn from its stationary distribution, is slightly below its mean. The simulation begins with a sequence of demand contractions. In the LIFO equilibrium these induce the exit of one firm after the simulation's third period. In contrast, two firms remain active through the simulation's ninth period in the Simple equilibrium. After that, their mixed strategies produced *simultaneous* exit for both of them. In this example, a potential entrant always chooses to enter an empty industry, so the next period's entry corrects this "excess" exit from coordination failure. Later falls in demand cause similar delays in firm exit in the Simple model relative to the LIFO model, but without excess exits.

[7] The values of β, $\pi(N)$, κ, and ϕ; and the stochastic process for demand used are those underlying Tables I and II of Abbring and Campbell (2010). The demand approximates a random walk in the logarithm of C with innovation variance 0.3^2 reflected off of the state space's upper and lower boundaries, $\ln \hat{c} = -1.5$ and $\ln \check{c} = 1.5$.

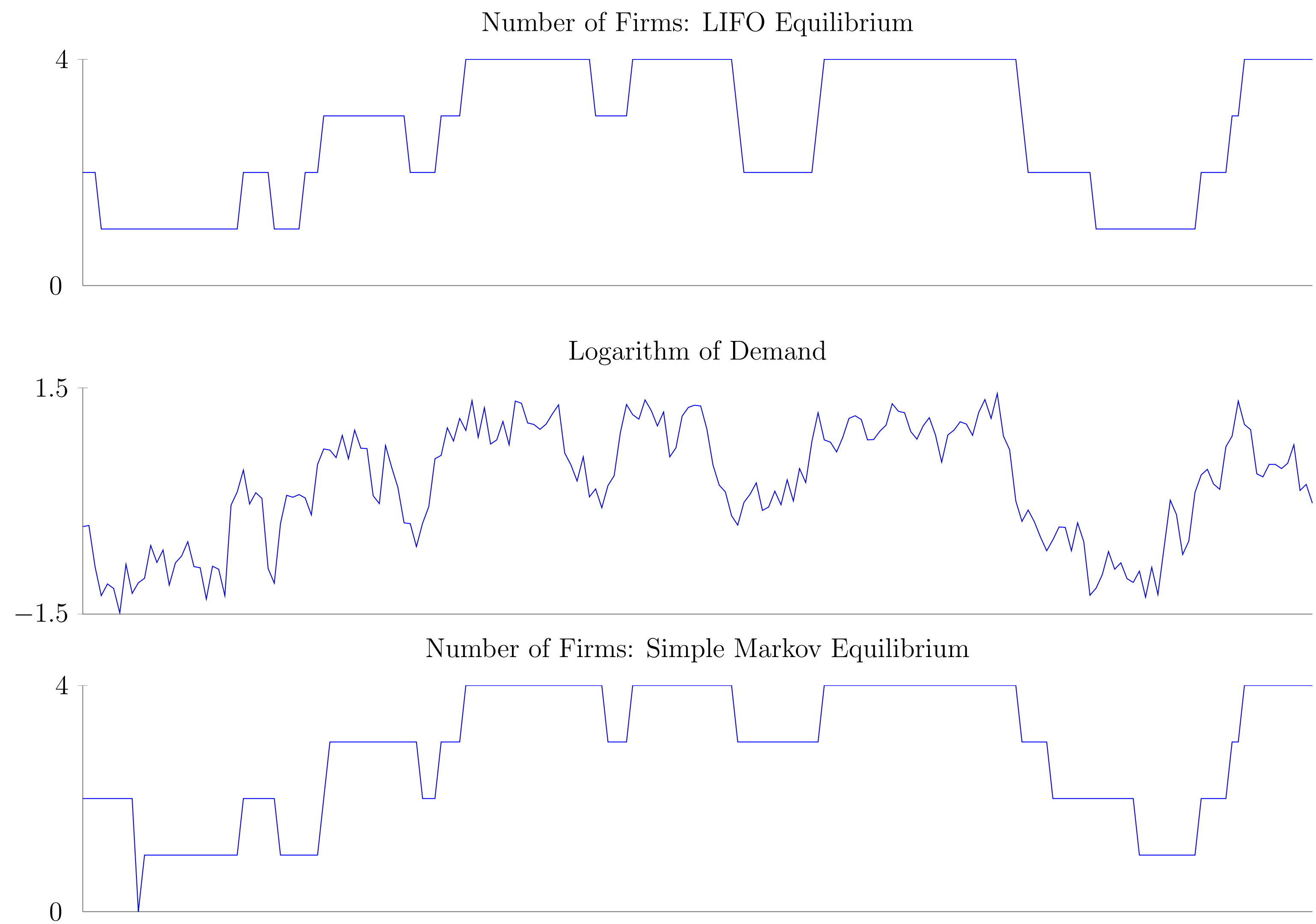


Figure 4: Simple Markov-Perfect and LIFO Industry Dynamics

4 Heterogeneous Duopoly

We now return to models with general productivity dispersion. In this section, we consider a special case with at most two firms serving the market, which we call the *heterogeneous-duopoly* model. In other words, we allow for any finite $\check{k}$ but require $\check{m} = 2$.

Throughout this section, we represent duopoly market structures with $\iota_k + \iota_l$; $k, l \in \mathbb{K} \cup \{0\}$. As in Section 3, our analysis relies on monotonicity of the post-entry and post-survival value functions. We have the following analogue to Lemma 2:

Lemma 4 (Monotone Payoffs in the Heterogenous-Duopoly Model)**.** *In a natural Markov-perfect equilibrium, for all $c \in [\hat{c}, \check{c}]$ and $k \in \mathbb{K}$, $v_E(2\iota_k, c, k) \leq v_E(\iota_k, c, k)$ and $v_S(2\iota_k, c, k) \leq v_S(\iota_k, c, k)$.*

Proof. See Appendix B. □

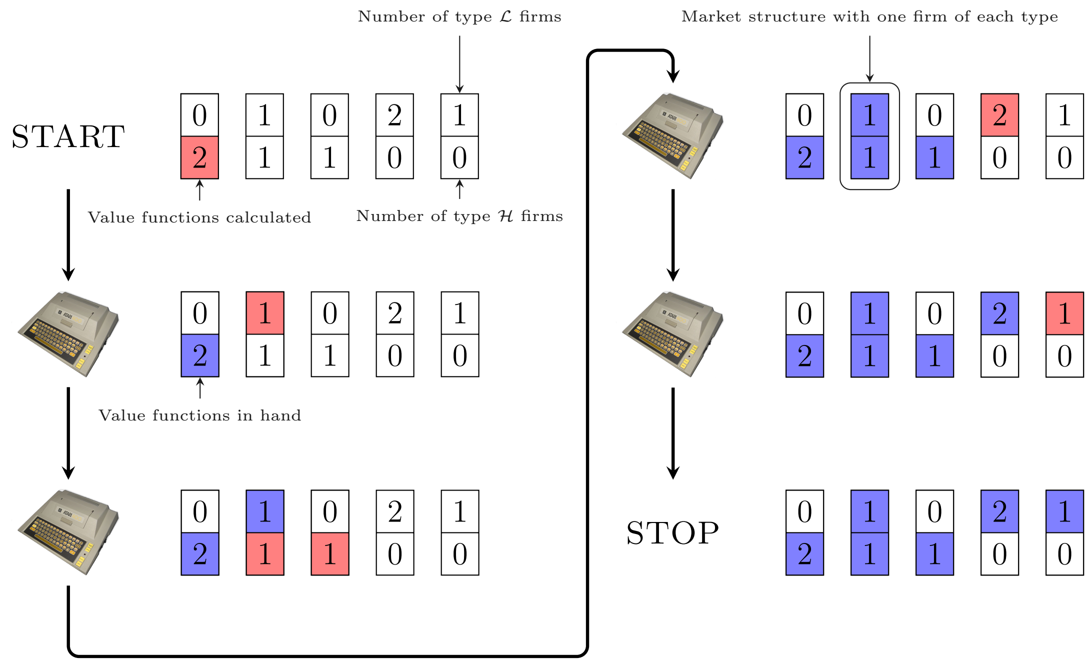
That is, a duopolist facing a rival of the same type always has a lower value than it would have without the rival present. This lemma renders the natural Markov-perfect equilibrium of this model essentially unique. With its help, we next analyze the special case with two productivity types. This illuminates a general procedure for computing a natural Markov-perfect equilibrium of the heterogenous-duopoly model. Section 4.2 formalizes this procedure into an algorithm and then establishes equilibrium existence and uniqueness results. Finally, Section 4.3 uses this algorithm for numerical analysis of the effects of technological progress and demand uncertainty on industry dynamics. This illustration demonstrates that the natural Markov-perfect equilibrium of this model can be easily computed.

4.1 Two Productivity Types

In the interest of expositional clarity, we denote the higher productivity type with the intuitive $\mathcal{H}$ (instead of 2) and the lower type with $\mathcal{L}$ (instead of 1). We construct this case's unique natural Markov-perfect equilibrium in six steps. As in Section 4.1's example without productivity heterogeneity, these steps take us through a finite partition of the state space. In each of the first five steps, we compute the equilibrium payoffs in the states considered by finding the unique fixed point of a contraction mapping. The results from the completed steps are used as inputs in the following steps. Figure 5 illustrates this sequence of computations. The construction ends by specifying the unique strategy that supports the equilibrium payoffs in the sixth step.

Step 1: Calculation of $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_S(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ As depicted by the upper-left panel in Figure 5, we start the equilibrium construction by considering a market populated

Figure 5: Equilibrium Computation for a Duopoly with Two Productivity Types



Note: There are five possible duopoly market structures. Each divided rectangle represents one of them, and each collection of five rectangles displays the value functions being calculated (in red) and the value functions already in hand (in blue) at one stage of the algorithm (which is Section 4.2's Algorithm 2 with $\check{k} = 2$).

by two type $\mathcal{H}$ firms. The analysis in this step is a carbon copy of the first step of Section 3.2's example. The simultaneous-move survival game between two type $\mathcal{H}$ firms is analogous to the one in Figure 2, and Lemma 4 guarantees that "Survive" is the dominant strategy if joint continuation gives both firms positive payoffs. Therefore, finding the fixed point of a contraction mapping analogous to that in (1) yields $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$. The continuation payoff $v_S(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ immediately follows.

Step 2: Calculation of $v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$, $v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$, $a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot)$, and $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$. A type $\mathcal{L}$ firm that chooses to survive advances to $\mathcal{H}$ with probability $\Pi_{\mathcal{LH}}$ and remains unchanged with probability $\Pi_{\mathcal{LL}} \equiv 1 - \Pi_{\mathcal{LH}}$. In a natural MPE, the survival of the type $\mathcal{L}$ firm guarantees survival of any type $\mathcal{H}$ rival, so the continuation value $v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C, \mathcal{L})$ must satisfy

$$\begin{aligned} v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L}) =& \max\Big\{0, v_S(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, c, \mathcal{L})\Big\} \\ =& \beta \max\Big\{0, \Pi_{\mathcal{LL}}\mathbb{E}\left[C'\pi_L(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}) + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L}) | C = c\right] \\ & + \Pi_{\mathcal{LH}}\mathbb{E}\left[C'\pi_H(2\iota_{\mathcal{H}}) + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H}) | C = c\right]\Big\}. \end{aligned}$$

Since $v_E(2\iota_{\mathcal{H}}, \cdot, \iota_{\mathcal{H}})$ is in hand from Step 1, this defines a contraction mapping in the desired value function. With its fixed-point in hand, we can then easily compute $v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$ and

$$\begin{aligned} a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c) &= I\{v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L}) > \varphi\}, \\ a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L}) &= I\{v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L}) > 0\}. \end{aligned}$$

Step 3: Calculation of $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $a_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, $v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{H})$, and $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{H})$. A market with a monopolist incumbent with type $\mathcal{H}$ attracts an entrant next period if and only if $a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, C') = 1$, so $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_E(\iota_{\mathcal{H}}+\iota_{\mathcal{L}}, \cdot, \mathcal{H})$ together satisfy

$$\begin{aligned} v_E(\iota_{\mathcal{H}}, c, \mathcal{H}) =& \max\Big\{0, v_S(\iota_{\mathcal{H}}, c, \mathcal{H})\Big\} \\ =& \beta \max\Big\{0, \mathbb{E}[C'\pi_{\mathcal{H}}(\iota_{\mathcal{H}}) - \kappa + a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, C')v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, C', \mathcal{H}) \\ & + \{1 - a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, C')\}\, v_E(\iota_{\mathcal{H}}, C', \mathcal{H})|C = c]\Big\}. \end{aligned} \tag{6}$$

Step 2 determined $a_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot)$, so the only unknowns in (6) are the value functions. Since a type $\mathcal{H}$ duopolist facing a type $\mathcal{L}$ rival becomes a monopolist if and only if $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L}) = 0$, these value functions must also satisfy

$$\begin{aligned} & v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \iota_{\mathcal{H}}) \\ &= a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L})v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \iota_{\mathcal{H}}) + \{1 - a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L})\}\, v_E(\iota_{\mathcal{H}}, c, \mathcal{H}) \\ &= a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L})\beta\Big\{\Pi_{\mathcal{L}\mathcal{L}}\mathbb{E}[C'\pi_{\mathcal{H}}(\iota_{\mathcal{H}}+\iota_{\mathcal{L}}) - \kappa + v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, C', \mathcal{H})|C = c] \\ &\qquad + \Pi_{\mathcal{L}\mathcal{H}}\mathbb{E}[C'\pi_{\mathcal{H}}(2\iota_{\mathcal{H}}) - \kappa + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H})|C = c]\Big\} \\ &\quad + \{1 - a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{L})\}\, v_E(\iota_{\mathcal{H}}, c, \mathcal{H}). \end{aligned} \tag{7}$$

We have $v_E(2\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ from Step 1 and $a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{L})$ from Step 2, so together, (6) and (7) determine $v_E(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_E(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{H})$. Obtaining $v_S(\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ and $v_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, \cdot, \mathcal{H})$ from these is straightforward. Since we seek a natural equilibrium, the survival strategies of interest must satisfy

$$a_S(\iota_{\mathcal{H}}, c, \mathcal{H}) = a_S(\iota_{\mathcal{L}}+\iota_{\mathcal{H}}, c, \mathcal{H}) = I\{v_S(\iota_{\mathcal{H}}, c, \mathcal{H}) > 0\}.$$

Step 4: Calculation of $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $a_E(2\iota_{\mathcal{L}}, \cdot)$, and $v_S(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$. Next, we consider a duopoly market with two type $\mathcal{L}$ firms. If both firms choose survival, then their idiosyncratic shocks could change the market structure to either of the duopoly structures considered in Steps 1-3 or leave it unchanged. Lemma 4 guarantees that if the value of simultaneous survival to either incumbent is positive, then joint continuation is the only Nash equilibrium outcome of their survival game. Therefore, $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$ satisfies

$$\begin{aligned}
v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) =& \max\Big\{0, v_S(2\iota_{\mathcal{L}}, c, \mathcal{L})\Big\} \\
=&\beta \max\Big\{0, \Pi^2_{\mathcal{L}\mathcal{L}}\mathbb{E}\left[C'\pi_{\mathcal{L}}(2\iota_{\mathcal{L}}) - \kappa + v_E(2\iota_{\mathcal{L}}, C', \mathcal{L})|C = c\right] \\
&+ \Pi_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}\mathcal{H}}\mathbb{E}\left[C'\pi_{\mathcal{L}}(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}) - \kappa + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{L})|C = c\right] \\
&+ \Pi_{\mathcal{L}\mathcal{H}}\Pi_{\mathcal{L}\mathcal{L}}\mathbb{E}\left[C'\pi_{\mathcal{H}}(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}) - \kappa + v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H})|C = c\right] \\
&+ \Pi^2_{\mathcal{L}\mathcal{H}}\mathbb{E}\left[C'\pi_{\mathcal{H}}(2\iota_{\mathcal{H}}) - \kappa + v_E(2\iota_{\mathcal{H}}, C', \mathcal{H})|C = c\right]\Big\}.
\end{aligned}$$

The only unknown on its righthand side is $v_E(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, so we can use this Bellman equation to calculate it. With this in hand, we construct the rule for entry into a market with one type $\mathcal{L}$ incumbent as

$$a_E(2\iota_{\mathcal{L}}, c) = I\{v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) > \varphi\}.$$

Moreover, it is straightforward to determine $v_S(2\iota_{\mathcal{L}}, \cdot, \mathcal{L})$.

Step 5: Calculation of $v_E(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $v_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, $a_E(\iota_{\mathcal{L}}, \cdot)$, and $a_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$. If a type $\mathcal{L}$ monopolist chooses survival, then one of four market structures will prevail in the next period, depending on the incumbent's idosyncratic shock and on the decision of a potential entrant:

$$\begin{aligned}
v_E(\iota_{\mathcal{L}}, c, \mathcal{L}) =& \max\Big\{0, v_S(\iota_{\mathcal{L}}, c, \mathcal{L})\Big\} \\
=& \max\Big\{0, \Pi_{\mathcal{L}\mathcal{L}}\mathbb{E}\Big[C'\pi_{\mathcal{L}}(\iota_{\mathcal{L}}) - \kappa + a_E(2\iota_{\mathcal{L}}, C')v_E(2\iota_{\mathcal{L}}, C', \mathcal{L}) \\
&+ \{1 - a_E(2\iota_{\mathcal{L}}, C')\}\, v_E(\iota_{\mathcal{L}}, C', \mathcal{L})|C = c\Big] \\
&+ \Pi_{\mathcal{L}\mathcal{H}}\mathbb{E}\Big[C'\pi_{\mathcal{H}}(\iota_{\mathcal{H}}) - \kappa + a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C')v_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C', \mathcal{H}) \\
&+ \{1 - a_E(\iota_{\mathcal{L}} + \iota_{\mathcal{H}}, C')\}\, v_E(\iota_{\mathcal{H}}, C', \mathcal{H})|C = c\Big]\Big\}.
\end{aligned} \tag{8}$$

Given the quantities calculated in Steps 1–4, the righthand side of (8) defines a contraction mapping with $v_E(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$ as its fixed point. With this, it is straightforward to compute $v_S(\iota_{\mathcal{L}}, \cdot, \mathcal{L})$, which gives the survival rule

$$a_S(\iota_{\mathcal{L}}, c, \mathcal{L}) = I\{v_S(\iota_{\mathcal{L}}, c, \mathcal{L}) > 0\}.$$

Since $v_E(2\iota_{\mathcal{L}}, c, \mathcal{L}) \leq v_E(\iota_{\mathcal{L}}, c, \mathcal{L})$, the entry rule for a potential monopolist can be written as

$$a_E(\iota_{\mathcal{L}}, c) = I\left\{v_E(\iota_{\mathcal{L}}, c) > \varphi\right\}$$

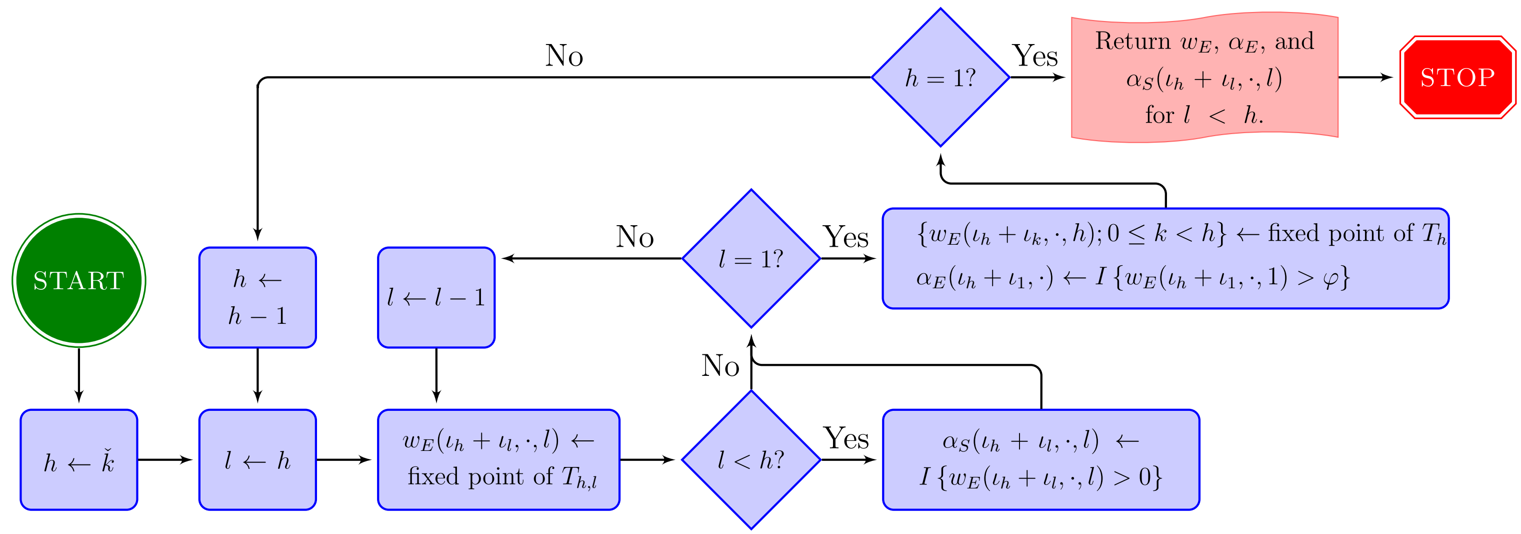
Step 6: Calculation of $a_S(2\iota_{\mathcal{H}},\cdot,\mathcal{H})$ and $a_S(2\iota_{\mathcal{L}},\cdot,\mathcal{L})$. Steps 1–5 have determined all equilibrium continuation values, entry strategies, and survival strategies for firms facing no identical rival. All that remains is to determine the exit strategies for duopolies of identical firms. Their construction parallels that from the case with homogeneous firms: Unless either survival or exit is a dominant strategy, both firms mix between the two pure actions to leave each other indifferent between them.

$$a_S(2\iota_{\mathcal{L}},c,\mathcal{L}) = \begin{cases} 1 & \text{if } v_S(2\iota_{\mathcal{L}},c,\mathcal{L}) > 0, \\ \frac{v_S(\iota_{\mathcal{L}},c,\mathcal{L})}{v_S(\iota_{\mathcal{L}},c,\mathcal{L})-v_S(2\iota_{\mathcal{L}},c,\mathcal{L})} & \text{if } v_S(2\iota_{\mathcal{L}},c,\mathcal{L}) \leq 0 \text{ and } v_S(\iota_{\mathcal{L}},c,\mathcal{L}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$a_S(2\iota_{\mathcal{H}},c,\mathcal{H}) = \begin{cases} 1 & \text{if } v_S(2\iota_{\mathcal{H}},c,\mathcal{H}) > 0, \\ \frac{v_S(\iota_{\mathcal{H}},c,\mathcal{H})}{v_S(\iota_{\mathcal{H}},c,\mathcal{H})-v_S(2\iota_{\mathcal{H}},c,\mathcal{H})} & \text{if } v_S(2\iota_{\mathcal{H}},c,\mathcal{H}) \leq 0 \text{ and } v_S(\iota_{\mathcal{H}},c,\mathcal{H}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Equilibrium Existence, Uniqueness, and Computation

We next extend the six-step procedure to the calculation of duopoly equilibrium with an arbitrary number of possible types. The resulting algorithm consists of two procedures. The first computes all payoffs, survival strategies for duopolists facing strictly higher productivity types, and strategy for a potential entrant facing an incumbent. The second procedure calculates the survival strategies for duopoly incumbents with weakly higher productivity types and the strategy for a potential entrant facing an empty market.



Required Functional Operators

$$T_{h,l}(f)(c) = \beta \max\left\{0, \mathbb{E}\left[\sum_{i,j}\Pi_{hi}\Pi_{lj}C'\pi_j(\iota_i+\iota_j) - \kappa + \sum_{(i,j)\neq(h,l)}\Pi_{hi}\Pi_{lj}w_E(\iota_i+\iota_j,C',j) + \Pi_{hh}\Pi_{ll}f(C') \,\middle|\, C=c\right]\right\}$$

$$\begin{aligned} T_h(f)(c,k) = \beta \max\Bigg\{0, \mathbb{E}\Bigg[&\alpha_S(\iota_h+\iota_k,c,k)\Bigg(\sum_{i,j}\Pi_{hi}\Pi_{kj}C'\pi_i(\iota_i+\iota_j) - \kappa + \sum_j \Pi_{hh}\Pi_{kj}f(C',j) + \sum_{i>h}\sum_j \Pi_{hi}\Pi_{kj}w_E(\iota_i+\iota_j,C',i)\Bigg) \\ &+ \{1-\alpha_S(\iota_h+\iota_k,c,k)\}\Bigg(\sum_i \Pi_{hi}C'\pi_i(\iota_i) - \kappa + \Pi_{hh}\{1-\alpha_E(\iota_h+\iota_1,C')\}f(C',0) \\ &\quad + \sum_{i>h}\Pi_{hi}\{1-\alpha_E(\iota_i+\iota_1,C')\}w_E(\iota_i,C',i) + \Pi_{hh}\alpha_E(\iota_h+\iota_1,C')f(C',1) \\ &\quad + \sum_{i>h}\Pi_{hi}\alpha_E(\iota_i+\iota_1,C')w_E(\iota_i+\iota_1,C',i)\Bigg) \Bigg| C=c\Bigg]\Bigg\}, \quad \alpha_S(\iota_h+\iota_0,\cdot,0)\equiv 0, \Pi_{00}\equiv 1 \end{aligned}$$

Procedure 3: Initial Equilibrium Calculations for the Heterogeneous Duopoly Model

In Procedure 3, h indexes the productivity type for the weakly better firm, and l for the weakly worse firm. In the course of this procedure, h decreases from $\check{k}$ to 1. For each level of h, l decreases from h to 1. For any pair of (h, l); the post-entry value w_E to the type l firm that faces a type h firm is computed as the fixed point of $T_{h,l}$. This functional operator is defined by the recursive condition for $w_E(\iota_l + \iota_h, \cdot, l)$. It is a contraction, because the type l firm has weakly lower productivity type, and rationally expects its rival to remain whenever it continues with positive probability. Hence, this firm's payoffs only depend on future states in which both firms survive for sure and possibly progress to higher productivity types. Therefore, $T_{h,l}$ only depends on $w_E(\iota_i + \iota_j, \cdot, j)$ for all $(i, j) \neq (h, l)$ such that $i \geq h$, $j \geq l$. Since Procedure 3 proceeds in descending order of (h, l), these post-entry values have been determined before computing $w_E(\iota_l + \iota_h, \cdot, l)$. This ensures that $T_{h,l}$ is a contraction mapping.

When l reaches 1, the next step is to compute simultaneously the monopoly payoff for a type h firm and the duopoly payoff for a type h firm facing a type k (for all $k < h$) rival as the fixed point of T_h. The operator T_h is again a contraction. When the type h firm is monopolizing the market, entry may happen and T_h depends on the entry rule of a potential entrant facing an incumbent with productivity type no worse than h, and all related post-entry values. When the h firm is facing a type k competitor, T_h depends on the survival strategy of this type k rival and all related post-survival values. Again, due to the descending order of h, l, all these relevant values have been determined before computing the fixed point of T_h.

Therefore, $T_{h,l}$ and T_h are two well-defined contraction mappings with unique fixed points. Consequently, Procedure 3 uniquely determines w_E.

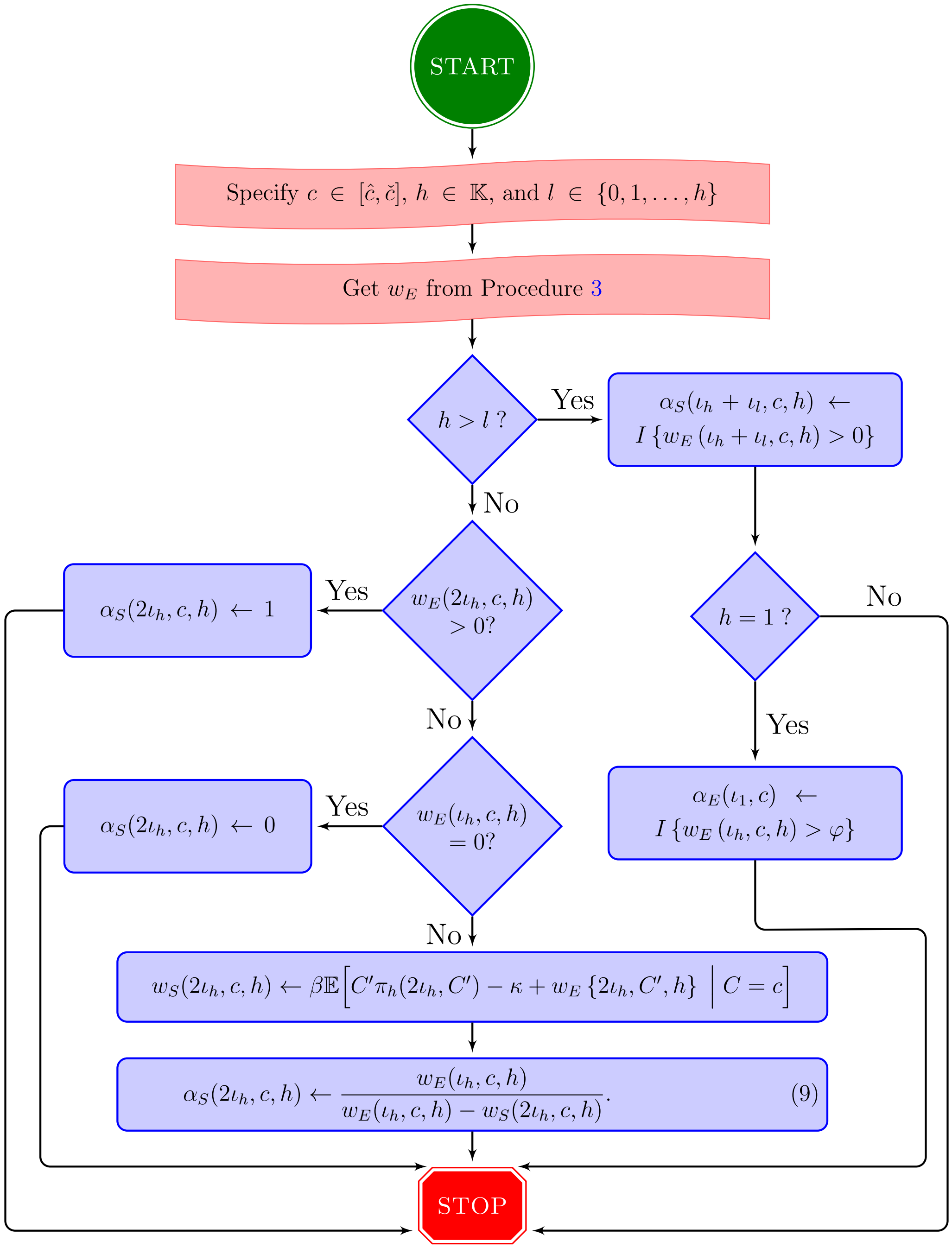
Procedure 4 complements Procedure 3 by determining; for any given c, h, l; the survival strategy for a firm with weakly better productivity type and the entry strategy for a potential monopolist.

Note that $w_S(2\iota_h, c, h) < 0$ and $w_E(\iota_h, c, h) > 0$ ensures that $\alpha_E(2\iota_h, c, h) \in (0, 1]$. Procedure 3 and 4 are combined in the following algorithm to compute a candidate natural Markov-perfect equilibrium.

Algorithm 2 (Heterogeneous Duopoly)**.** *Compute a candidate equilibrium strategy* (α_S, α_E) *and payoffs* w_S *and* w_E *in two steps:*

1. *Use Procedure 3 to compute* w_E, $\alpha_S(\iota_h + \iota_l, c, l)$ *and* $a_E(\iota_h + \iota_1, c)$ *for all* $h, l \in \mathbb{K}, l < h$ *and* $c \in [\hat{c}, \check{c}]$*.*

2. *For all* $h \in \mathbb{K}$*,* $l \in \{0, \dots, h\}$*, and* $c \in [\hat{c}, \check{c}]$*; use Procedure 4 to compute* $\alpha_S(\iota_h + \iota_l, c, h)$*. Also, for all* $c \in [\hat{c}, \check{c}]$*, compute* $\alpha_E(\iota_1, c)$*.*



Procedure 4: Calculation of Candidate Survival Rule for the Heterogeneous Duopoly Model

We can use Lemma 4 to prove that the constructed equilibrium is unique among all equilibria that default to inactivity.

Proposition 2 (Equilibrium in Heterogeneous Duopoly Model)**.** *There exists a unique natural Markov-perfect equilibrium. Algorithm 2 computes its payoffs and strategy. The equilibrium payoffs* $v_S = w_S$ *and* $v_E = w_E$*. The equilibrium strategy* $(a_S, a_E) = (\alpha_S, \alpha_E)$*.*

Proof. See Appendix B. □

4.3 Application

We apply our heterogenous-duopoly model to the welfare analysis of an R&D race game. Consider a market for some new good. In period t, C_t consumers populate the market. All of these consumers have the same utility function, which is quadratic in the quantity of the new good consumed. Consequently, total demand for the new good at time t and price p equals $C_t(a - p)/b$, for some parameters $a, b > 0$.

Firms must invent the good before they can supply it to the market. This requires that they enter the market, incurring an entry cost φ, and subsequently invest in R&D, at a fixed cost κ. There are several milestone stages in the invention process, marked by $1, 2, \ldots, \check{k}$. New entrants start in stage 1 and, as long as they stay in the market and pay the fixed cost κ, progress through the successive R& D stages according to a Markov chain with transition matrix Π. Once a firm reaches the final stage $\check{k}$, it has invented the product and can start selling it in the market. The fixed cost κ still needs to be paid to produce the good.[8] An active firm may exit the market in any stage of the R&D race to avoid paying future fixed costs.

We assume that at most two firms are active in the market at any give time. If only one firm is active in stage $\check{k}$, it sells the good at the monopoly price. If two firms are selling the good, they set symmetric quantities to maximize $q^o p + \lambda q^r p$, where q^o and q^r denote the quantities set by the firm and its rival, respectively, and λ indexes the level of collusion. If $\lambda = 0$, these two firms are Cournot competitors. At higher values of λ, they collude more. If $\lambda = 1$, then they operate as if they are branches of a monopoly firm that split their joint monopoly revenues evenly.

This game embodies Fershtman and Pakes' (2000) key "semi-collusion" assumption that firms may collude in setting quantities (or prices) but not when choosing R&D investment. Unlike Fershtman and Pakes, we take the level of collusion as exogenously given and ignore the intensive margin of the firms' strategic R&D investments. This focus on the (entry and exit) decisions to participate in the R&D race allow us to apply the heterogenous-duopoly model

[8] Alternatively, we can let the fixed cost decline with k. This only requires a minor adjustment of the model.

to analyze industry dynamics and welfare under different levels of collusion. We find that the model is sufficiently rich to replicate one of Fershtman and Pakes' main findings: Consumers may benefit from collusion, unlike in static models that take the industry structure as given. Intuitively, the direct negative effect of collusion on consumer welfare through weakened competition in the product market, well known from static models, is counteracted by a positive effect on R&D participation that increases product availability and product market competition.

To obtain this result, we first compute the model's unique natural Markov-perfect equilibrium for each value of λ between 0 and 1, with a 0.01 increment. Throughout, we specify $Q(\cdot|C)$ to approximate a random walk in the logarithm of C with innovation variance 0.3^2, reflected off of the state space's upper and lower boundaries, $\ln \hat{c} = -1.5$ and $\ln \check{c} = 1.5$ (see Footnote 7). Also, we specify $\check{k} = 4$, $\beta = 0.95$, $\kappa = 20$, $\varphi = 470$, demand parameters $a = 20$ and $b = 2$, and the Markov transition matrix Π for the R&D stages so that firms either progress one stage or remain put: $\Pi_{k,k} = \Pi_{k,k+1} = 0.5$ for all $k < \check{k}$ and $\Pi_{\check{k},\check{k}} = 1$.

Subsequently, for each value of λ, we use the equilibrium strategy to simulate the market's evolution over 100 periods, starting from a fixed $c_0 = 1.649$, drawn from the demand process's ergodic distribution, and an empty market. We repeat the simulation 10,000 times, drawing new demand and type transitions in each simulation, but using the same random draws across the different values of λ. To analyze the impact of collusion on welfare, we compute, for each level of collusion λ, the discounted sum $FP(\lambda)$ of all firms' revenues net of all firms' fixed costs and entry costs over the 100 periods, and the discounted sum $CS(\lambda)$ of the consumer surplus over the 100 periods, both averaged over the 10,000 simulation runs. We assume that consumers have the same discount factor as firms. The total surplus $TS(\lambda) \equiv FP(\lambda)+CS(\lambda)$.

The upper-left and upper-middle panels of Figure 6 show $CS(\lambda)$ and $FP(\lambda)$ for each value of λ, as a proportion of the competitive market's total surplus $TS(0)$. First, if λ increases from 0, $CS(\lambda)$ gradually increases and $FP(\lambda)$ gradually decreases. Then, $CS(\lambda)$ jumps up and $FP(\lambda)$ jumps down. At higher levels of collusion, increases in λ decrease the consumer surplus and increase firms' profits.

Clearly, for low values of λ, the positive effect of increasing collusion on R&D investment dominates its direct weakening effect on product market competition. Figure 6's bottom-left panel sheds further light on this. It plots the number of active firms for each λ, averaged over the 100 periods and all the simulation runs. We observe a gradual increase and then an upward jump in the number of firms, paralleling the increase and jump in the consumer surplus. If λ is low; that is, with little or no collusion; no entrant facing a monopoly market can recover the sunk cost of entry, even when demand is at its highest level. Therefore, markets with little collusion are often monopoly markets. If λ increases, firms expect higher payoffs from a duopoly product market, and are more willing to participate in the R&D race, even if one firm is already in this race. The value of λ at which the number of firms and welfare jump is the level of collusion above which two firms enter immediately, in the initial demand state c_0.

This increase in the number of firms improves the consumer surplus in two ways. First, it improves product availability. Specifically, in this example, on average the first product reaches the market faster with higher levels of collusion (see Figure 6's bottom-middle panel). Second, it mitigates the anticompetitive effects of collusion, by ensuring that consumers are more often charged the (collusive) duopoly price, which, for all $\lambda < 1$, is lower than the monopoly price. At low levels of λ, the consumer welfare enhancing effects dominate the direct negative effects of increased collusion.

In contrast, as is clear from Figure 6's bottom panels, at higher levels of collusion, the market is often served by the maximum number of two firms. Consequently, further increases in λ have only small effects on the number of firms serving the market and the speed at which the good becomes available. Therefore, at higher levels of collusion, the direct effects of collusion dominate, and the consumer surplus gradually falls if λ increases. Nevertheless, the benefits from earlier consumption under full collusion ($\lambda = 1$) ensure that $CS(1) > CS(0)$.

The variation of $FP(\lambda)$ with λ mirrors the variation of $CS(\lambda)$. If λ crosses the level at which two firms immediately enter the market, instead of one, the total fixed cost incurred

Figure 6: Welfare Analysis for Various Levels of Collusion

is doubled, but the total revenue is not. Consequently, $FP(\lambda)$ jumps down. For similar reasons, $FP(\lambda)$ falls gradually if λ increases at lower levels of collusion. In contrast, at higher levels of collusion, the market is usually a duopoly and the market structure does not change much with increases in λ. Consequently, the positive effects of such increases on the collusive duopoly price dominate, and $FP(\lambda)$ increases. Finally, $FP(1) < FP(0)$, because of scale savings: The monopoly price is usually charged at either collusive extreme, but two firms, instead of one, often incur fixed costs under full collusion.

Figure 6's upper-right panel plots the total surplus as a fraction of the competitive market's total surplus. At low levels of collusion, an increase in λ increases $TS(\lambda)$. In particular, the upward jump in the number of firms leads to an upward jump in the total surplus. At these levels of collusion, the positive effects of increased product market competition and earlier consumption on consumer welfare dominate its negative effects on firms through price decreases and fixed cost increases. At higher levels of collusion, the total surplus falls with increases in λ, because R&D activity is hardly affected and the negative welfare effects of collusion familiar from static models dominate.

In this specific example, as in static models that take the market structure as given, full collusion in the product markets lowers welfare below that in a competitive market: $TS(1) < TS(0)$. However, unlike in such static models, the competitive market is often served by only one firm and monopolistic pricing is common under both levels of collusion. Consequently, the result that full collusion lowers total welfare cannot be explained by the usual negative welfare effects of collusive pricing. Instead, it is due to the waste of fixed costs caused by excess entry of producers, which is not offset by the gains from earlier consumption.[9]

It is worth stressing that these results are obtained at a very low computational cost. For any particular value of λ, with 301 grid points for C and the parameter values of this section's experiment, we can solve the model within one second using Matlab on a PC. Even with $\beta = 0.995$ (monthly data) and $\check{k} = 10$, which implies a state space with over 33,000 points, we can solve the model in about 5–30 seconds on a PC.[10] This feature of our framework makes it a very useful complement to existing richer, but computationally more forbidding, frameworks for the analysis of industry dynamics. For example, Fershtman and Pakes' framework allows for a more detailed study of collusion dynamics by modeling, among other things, the intensive margin of R&D investment and endogenizing collusion. However,

[9] Obviously, this result can be reversed if consumers are impatient and/or have much larger weight in the total surplus than producers do.

[10] We use value function iteration to compute the fixed points of the contraction mappings, which simplifies our code, but results in a (slow) linear convergence rate in β. To cope with this issue, one can turn to more sophisticated approaches (see Judd, 1998, for a brief survey). For example, Ferris, Judd, and Schmedders's (2007) Newton-based method ensures global convergence with a quadratic convergence rate.

their framework's comparative richness comes at a substantial computational cost: It makes the replication of their results across different parameter values very hard. In contrast, our framework allows us to quickly examine the welfare implications of collusion for a wide range of parameter values.

5 The General Model

We now turn to the general model, with arbitrary finite $\check{m}$ and $\check{k}$. The central difficulty of the equilibrium analysis is that the equilibrium payoff function does not necessarily satisfy a monotonicity property analogous to the ones in Lemmas 2 and 4. In Section 5.1, we first analyze a type of equilibrium in which the payoffs still retain the monotonicity property: Adding an active firm into the industry weakly decreases other firms' payoffs. We can straightforwardly extend Algorithm 2 and use a sequence of contraction mappings to efficiently compute such an equilibrium, if it does exist. This monotonicity property is the key to establish the essential uniqueness of this type of equilibrium. However, since the monotonicity property does not always hold in the general model, this type of equilibrium may not exist. In Section 5.2, we discuss a simple example in which the monotonicity of equilibrium payoffs is violated and multiple equilibria emerge. In one class of those equilibria, if firms were allowed to renegotiate, they could strictly improve their payoffs by playing another equilibrium. We continue to focus on the type of equilibria that are renegotiation-proof and establish their existence. An extension of our algorithm can compute all such equilibria, if C has a discrete distribution.

5.1 Payoff-Monotone Equilibrium

We define an equilibrium to be *payoff-monotone* if the equilibrium payoffs satisfy conditions analogous to the ones in Lemma 2 and 4.

Definition 3. *A Markov-perfect equilibrium is payoff-monotone if its equilibrium payoff functions satisfies* $v_S(m, c, k) \geq v_S(m + \iota_k, c, k)$ *and* $v_E(m, c, k) \geq v_E(m + \iota_k, c, k)$ *for all* (m, c, k).

We showed in Section 4 that duopoly firms of the same type choose to continue if continuation guarantees positive payoff, because the heterogenous duopoly model's equilibrium payoffs satisfy Lemma 4. This property allows us to use conditions like (**??**) in Algorithm 2 to compute the unique natural Markov-perfect equilibrium. Similarly, in the general model, suppose that, for some parameter values, there exists a payoff-monotone natural Markov-perfect equilibrium. Then, if continuation renders the payoff to all firms of the same type positive, continuation is the dominant strategy for these firms. Following the argument leading to condition (**??**) in Section 4, we can establish similar necessary conditions on equilibrium payoffs. For instance, in the market with $\check{m}$ type $\check{k}$ firms, the payoff-monotone equilibrium payoff $v_E(\check{m}\iota_{\check{k}}, c, \check{k})$ necessarily satisfies

$$v_E(\check{m}\iota_{\check{k}}, c, \check{k}) = \max\{0, \beta\mathbb{E}[c'\pi_{\check{k}}(\check{m}\iota_{\check{K}}) - \kappa + v_E(\check{m}\iota_{\check{k}}, c', \check{k})|C = c]\}. \tag{10}$$

The right hand side of (10) defines a Bellman operator that uniquely determines $v_E(\check{m}\iota_{\check{k}}, \cdot, \check{k})$.

Note that the heterogeneous duopoly model and the general model only differ in the number of firms and share essentially the same dynamic specification. Therefore, since conditions like (**??**) extend to the general model, Algorithm 2 can also be extended to solve for the payoff-monotone natural equilibrium, by computed the fixed points of a sequence of contraction mappings. Similar to Algorithm 2, we partition the state space, order the parts, and compute the equilibrium in a corresponding sequence of steps, each one covering the computation on a single part of the state space. We order the steps so that all results that are needed in later steps can be passed on from earlier steps.

The partition and its order are defined using an oriental lexicographic order.

Definition 4. *Oriental lexicographical superiority (OLS) $\succ$ is a relation over $\mathbb{R}^n$. For any pair of vectors $x, y \in \mathbb{R}^n$, $x \succ y$ if $x_n > y_n$, or $(x_n = y_n$ and $x_{n-1} > y_{n-1})$, or $, \ldots,$ or $(x_n = y_n$ and $x_{n-1} = y_{n-1}$ and $\ldots$ and $x_1 > y_1)$.*

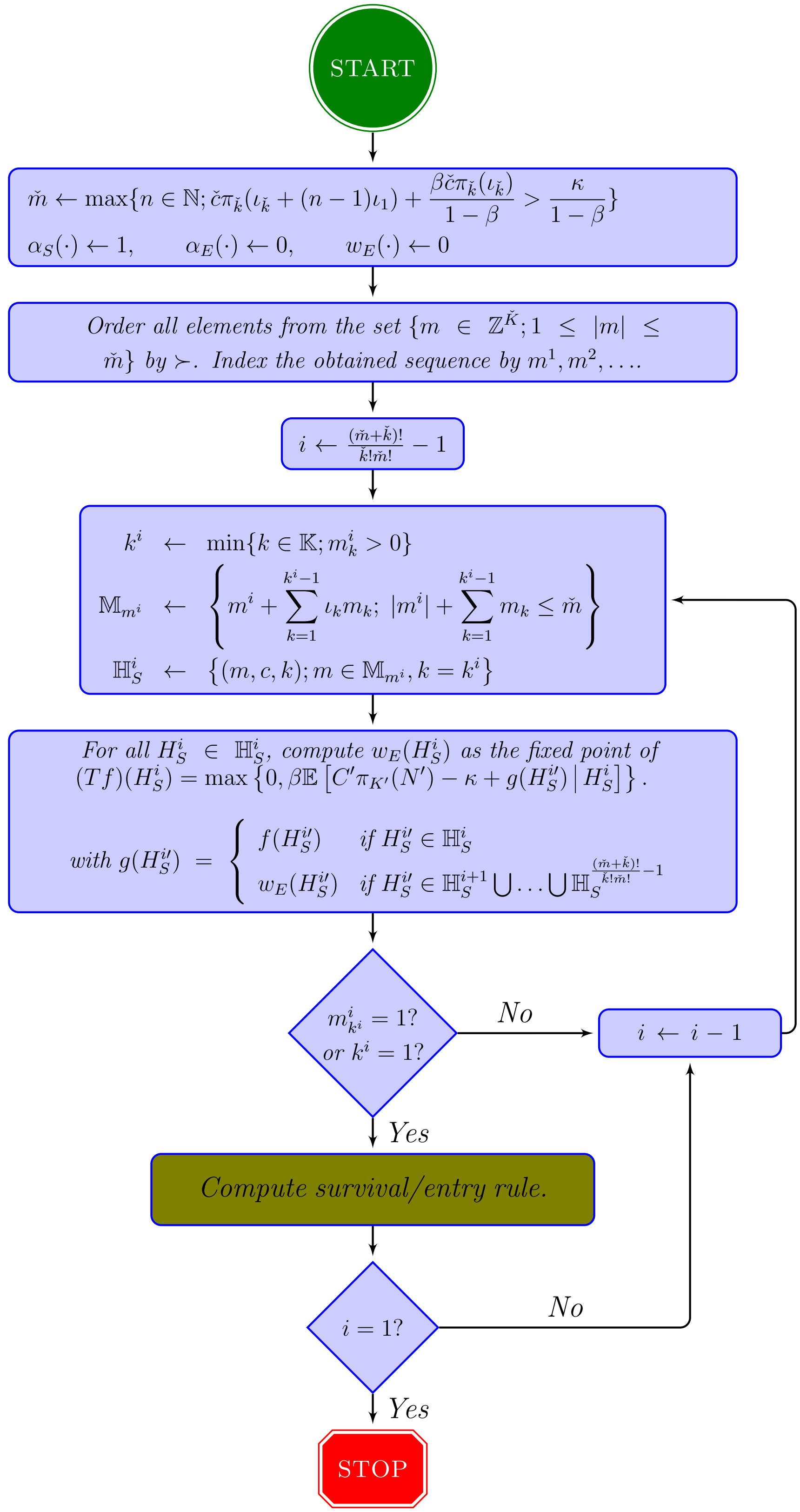
We use the phrase "oriental" because the vectors x, y are read from right to left when being compared, as in Arabic and Hebrew. In the previous sections, we have implicitly used an ordering based on OLS; the equilibrium payoff for an OLS market structure is always computed before the payoffs in any state it is superior to. For example, in Section 3's homogeneous-firm model, $\succ$ is equivalent to $>$ on $\mathbb{R}$ and the payoff to a duopolist is computed first, followed by the payoff to a monopolist. In Section 4.1, the sequence of market structures considered was $\{2\iota_{\mathcal{H}}, \iota_{\mathcal{H}} + \iota_{\mathcal{L}}, \iota_{\mathcal{H}}, 2\iota_{\mathcal{L}}, \iota_{\mathcal{L}}\}$. Thus, we partitioned the state space into five parts and ordered them in decreasing OLS to compute the equilibrium payoffs and strategy. Furthermore, this ordering extends to Algorithm 1 and 2 as well; the index number m in Procedure 1 and the index pair (h, l) in Procedure 3 are both decreasing in OLS.

Procedures 1 and 3 follow the ordering of OLS because this ensures that equilibrium payoffs and entry/survival rules necessary for computation in later steps are calculated in earlier steps. For example, in the homogeneous-firm model, computing firm's equilibrium payoff in a m-firm market requires entry rules for firms contemplating entry to all markets with at least m firms, as well as payoff to a firm in all markets with more than m firms. Markets with more than m firms rank higher than the m-firm market in the OLS sequence, so the payoff has been computed in earlier steps.

We construct the algorithm for the general model following the same ordering. For any $(\check{m}, \check{k})$ pair, there are $\binom{\check{m}+\check{k}}{\check{k}} - 1 = \frac{(\check{m}+\check{k})!}{\check{k}!\check{m}!} - 1$ possible non-empty markets. First, we partition the state space into $\frac{(\check{m}+\check{k})!}{\check{k}!\check{m}!} - 1$ parts, with each step of the algorithm computing the payoff on one of these parts. In step i, to see what the states in this part are, suppose the i-th ranked market structure in the OLS sequence is m^i. Let $k^i = \min\{k \in \mathbb{K}; m^i_k > 0\}$ be the lowest type of active firm in m^i and let a set $\mathbb{M}_{m^i}$ collect all the market structures that share the same number of type-$k^i, k^i + 1, \ldots, \check{k}$ firms as m^i. The part of the state space considered in this step is then $\{(m, c, k^i); m \in \mathbb{M}_{m^i}, c \in [\hat{c}, \check{c}]\}$. In other words, in the i-th step, we compute a type-k^i firm's payoff in every market structure in $\mathbb{M}_{m^i}$ for all c. Since this part of the state space are constructed from m^i, we say that it is *indexed* by m^i, and hence name m^i as the *indexing market structure*.

The first step of the algorithm is indexed by the most superior market structure, $\check{m}\iota_{\check{k}}$. Then, we proceed the algorithm and sequentially determine the payoffs and strategy for market structures in the order of decreasing OLS.

Algorithm 3 (Calculation of a Candidate Equilibrium for the General Model).



In Algorithm 3, when computing the candidate post-entry payoff w_E as the fixed point of the Bellman equation, the expectation relies on the relevant parts of other firms strategy. Because of the algorithm's OLS ordering, for the firms with lower productivity types than the firm of interest and the potential entrants, these values have been computed in previous steps. The survival rules for firms with productivity types at least as good as the firm of interest are set to continuation. Also, the OLS ordering of the algorithm helps to ensure that the current states that the firm of interest is facing only evolves to states that have been covered in previous computation, providing that this firm continues. Hence, all relevant values of future post-entry payoff have been computed in previous steps. Then, T is always a contraction mapping with unique fixed point $w_E(H_S)$.

Procedure 5 is devoted to computing the survival/entry strategy. In this procedure, when firms are randomizing between survival and exit, the mixing probability is chosen to be one of possible probabilities that solves the indifference condition (11). If it is not profitable to unilaterally deviate from exit to survival given all other firms of the same type opt for exit, the mixing probability can also be set to 0.

Note that Algorithm 3 does not require w_E to be monotone as in Definition 3. After computing w_E with Algorithm 3, we can check whether it satisfies this monotonicity condition. If it does, we can show that the candidate equilibrium strategy (α_S, α_E) is unique. We can also verify that (α_S, α_E) forms a natural Markov-perfect equilibrium. Since the Bellman equation for w_E defines the necessary condition for any payoff-monotone natural Markov-perfect equilibrium payoff, if one such equilibrium exists, not only we are able to compute it using Algorithm 3, but also we can prove its essential uniqueness.

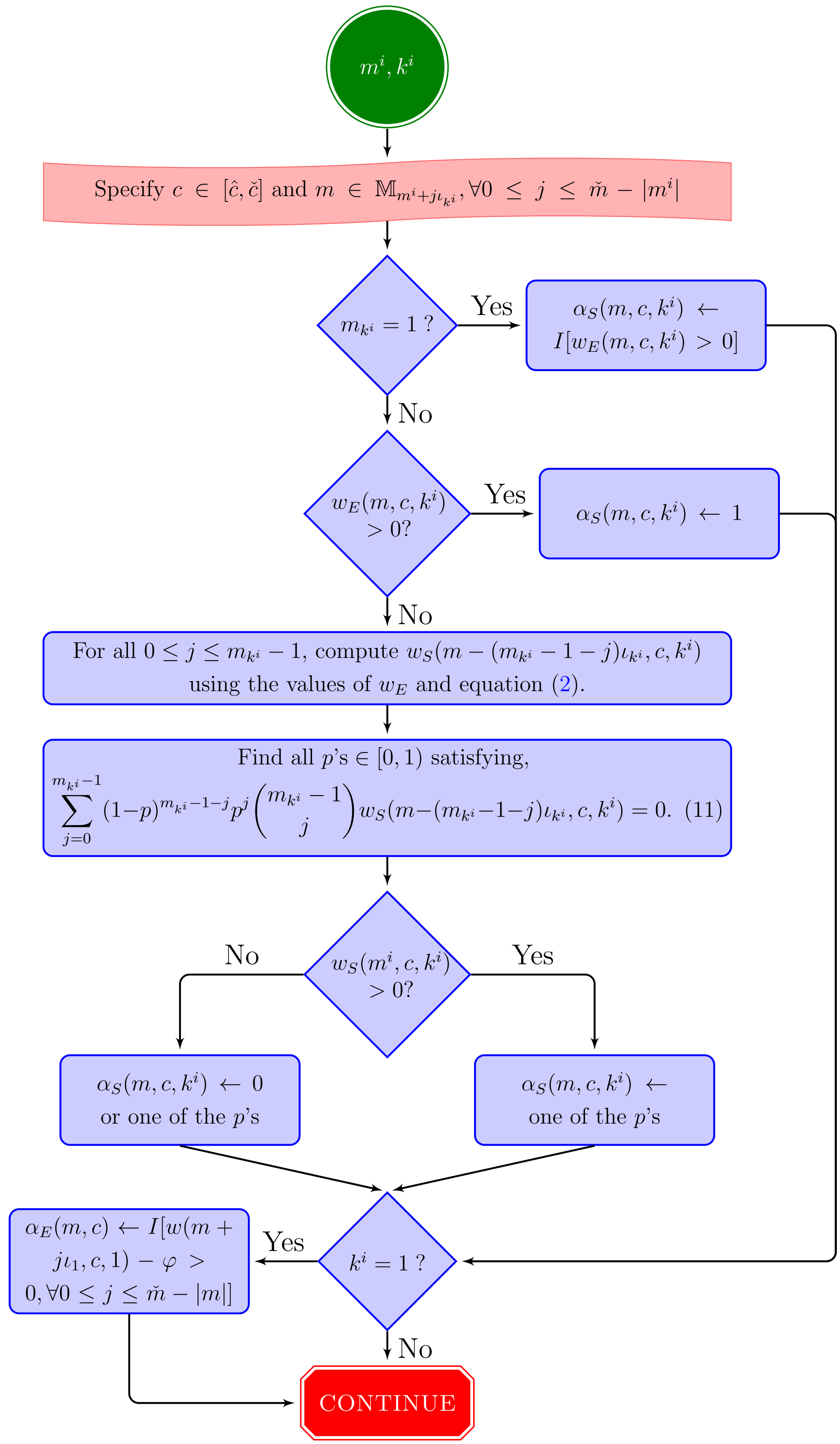
Proposition 3 (Payoff-Monotone Equilibrium in the General Model)**.** *If there exists a payoff-monotone natural Markov-perfect equilibrium, it is the unique such equilibrium and Algorithm 3 computes it. The post-entry equilibrium payoff function is w_E and the equilibrium strategy is (α_S, α_E).*

Proof. See Appendix C. □

5.2 Renegotiation-Proof Equilibrium

Proposition 3 implies that a payoff-monotone natural Markov-perfect equilibrium does not exist if w_E is not monotone as in Definition 3.

In Appendix D, we present a simple example in which equilibrium payoffs are not monotone and there are multiple natural Markov-perfect equilibria. In this example, we consider an industry with at most three active firms. We assume that firms can be type $\mathcal{H}$ or type $\mathcal{L}$. For two type $\mathcal{H}$ duopoly firms contemplating survival, we create a situation that if these two firms jointly continue to next period, any type $\mathcal{L}$ potential entrant will never find it



Procedure 5: Calculation of Candidate Entry/Survival Rule for the General Model

profitable to enter this market. This way, the type $\mathcal{H}$ duopolists deter any future entry by joint survival and enjoy a high duopoly surplus forever. Otherwise, if one of the firms exits, then two type-$\mathcal{L}$ firms will enter the market and remain active onwards. The survived type-$\mathcal{H}$ firm will only receive a low triopoly surplus thereafter. Connecting this example to the static survival game depicted in Figure 2, we construct the payoff matrix such that, for some c, the post-survival value satisfies $v_S(2\iota_{\mathcal{H}}, c, \mathcal{H}) > 0 > v_S(\iota_{\mathcal{H}}, c, \mathcal{H})$. Therefore, although "Survive, Survive" remains an equilibrium in this static game, "Exit, Exit" emerges as another equilibrium. Also, there could be equilibria involving mixed strategies. Indeed, we show in Appendix D that we do have three possible equilibrium actions at this particular point of the game tree. Namely, to survive for sure, to exit for sure, and to survive with some probability. We further demonstrate that when three firms are randomizing between survival and exit because joint survival is not profitable, the mixing probability can be multiple.

We distinguish two sources of equilibrium multiplicity using this example. One comes from the incumbents' failure to jointly continue if this is profitable. If the two type $\mathcal{H}$ firms can coordinate on continuation, they can strictly improve their equilibrium payoffs. Since these two firms repeatedly interact, it seems reasonable to assume that they are able to "renegotiate" to joint continuation whenever this is profitable. Henceforth, we restrict attention to equilibria with the desirable property that firms cannot improve their payoffs by one-shot change of action. We call this property *renegotiation-proofness.*

Definition 5. *A natural Markov-perfect equilibrium is (one-shot) renegotiation-proof if, for any* (m, c) *pair, no one-shot agreement satisfying the following properties can be negotiated:*

- *all firms in the agreement change their survival actions once;*
- *the agreement is self-enforcing, so no firm in the agreement has incentive to unilaterally change the agreed action;*
- *if one type* k *firm is in the agreement, all type* k *firms are; and*
- *the payoffs to all firms* in the agreement *are strictly improved.*

In any equilibrium, a firm earns positive payoffs only when continuing for sure. Therefore, if all firms of a certain type can strictly improve their payoff by changing their actions, it must be the case that (i) the actions must be changed from exiting with non-negative probability to surviving with probability one (ii) the actions of joint continuation must give all firms in the agreement positive payoff. Therefore, this refinement has bite only when all incumbents of certain type(s) could coordinate on sure joint continuation and earn positive payoffs, but will not unilaterally continue if others do not. Note that Lemmas 2 and 4 both ensure that all incumbents of the same type continue for sure if joint continuation renders payoff positive.

Therefore, no further improvement is possible via renegotiation in the homogenous-firm model and the heterogenous-duopoly model. Consequently, the symmetric equilibrium in Section 3 and the natural equilibrium in Section 4 are renegotiation-proof. Since the monotonicity in Definition 3 essentially functions in the same way as the monotonicity in Lemmas 2 and 4, the payoff-monotone equilibrium in the general model is also renegotiation-proof.

Recall that in Algorithm 3, $\alpha_S(m, c, k)$ is set to its initial value of one when computing $w_E(m, c, k)$. This implies that all type k firms are "forced" to jointly continue if positive payoff is expected. Therefore, the Bellman equation for w_E is a necessary condition on w_E for a renegotiation-proof natural Markov-perfect equilibrium. When verifying that (α_S, α_E) forms a natural equilibrium, we also verify that it is a renegotiation-proof one. We can further show that Algorithm 3 always delivers some (α_S, α_E) as its outcome, which proves the existence of a renegotiation-proof natural Markov-perfect equilibrium.

Proposition 4 (Renegotiation-proof Equilibrium in the General Model)**.** *Algorithm 3 always computes some* (α_S, α_E) *and this strategy* (α_S, α_E) *forms a renegotiation-proof natural Markov-perfect equilibrium. So, a renegotiation-proof natural Markov-perfect equilibrium always exists.*

Proof. See Appendix C. □

The renegotiation-proof property helps to eliminate the equilibria involving exit and mixing actions when joint continuation is profitable. However, the other source of the multiplicity persists. As we have illustrated in the example in Appendix D, when joint survival is not profitable and more than two firms are randomizing between survival and exit, there can be multiple equilibrium mixing probabilities. The property of renegotiation-proofness is silent on which probability to select. Therefore, each distinct equilibrium mixing probability leads to a different equilibrium survival rule. Different combinations of the equilibrium survival rules result in different renegotiation-proof natural Markov-perfect equilibria. Since the Bellman equation for w_E defines the necessary condition for renegotiation-proof natural Markov-perfect equilibrium payoff, any such equilibrium should be an outcome of Algorithm 3, if the mixing probabilities that correspond to this equilibrium are used in the computation.

Recall that we do have proven in Proposition 3 that if a payoff-monotone equilibrium exists, the mixing probability is always unique, and such equilibrium is the unique outcome of Algorithm 3. This implies the following corollary to Proposition 3.

Corollary 1. *If there exists a payoff-monotone natural Markov-perfect equilibrium, it is also the unique renegotiation-proof natural Markov-perfect equilibrium.*

If there is no payoff-monotone equilibrium, the possible multiplicity of mixing probabilities may challenge our equilibrium computation. After all, each step of Algorithm 3 requires the unique input of payoffs and rules computed in the previous steps. (In Section 5.1, Algorithm 3 simply selects an arbitrary mixing probability to continue when the multiplicity arises.) In the computational appendix, we prove that the number of renegotiation-proof natural Markov-perfect equilibria is finite if C is discrete. We also extend Algorithm 3 so that it computes all renegotiation-proof natural Markov-perfect equilibria, by creating parallel branches of Algorithm 3 every time the multiplicity arises, with each branch corresponding to a distinct choice of mixing probability.

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Appendices

A Proofs for the Homogeneous Oligopoly Case

Proof of Lemma 1. First, consider v_E. Equations (1) and (4) imply that for any (m, c),

$$v_E(m, c) = \max_{a\in[0,1]} a\mathbb{E}\left[v_S(M_S, c) \,|\, M_E = m\right] \geq 0.$$

Moreover, $v_E(m, c) > 0$ only if $a_S(m, c) = 1$.

We need to show that $v_E(m, \cdot) = 0$ for all $m > \check{m}$. To this end, suppose that instead $v(m, c) > 0$ for some c and some $m > \check{m}$. Then, from Equations (1) and (2)

$$v_E(m, c) = \beta\mathbb{E}\Big[C'\pi(m) - \kappa + I\left\{a_S(M', C') = 1\right\} v_E(M', C') \,\Big|\, M = m, C = c\Big];$$

because $a_S(m, c) = 1$, and $v(M', C') > 0$ only if $a_S(M', C') = 1$. Moreover, $M' \geq m > \check{m}$ (M' cannot be smaller than m because there is no exit ($a_S(m, c) = 1$); it will be larger if there is entry). Iterating this argument for $v_E(M', C')$, etcetera; this implies that $v_E(m, c)$ equals a discounted sum of strictly negative expected flow payoffs from next period up to and including the first period in which the active firms fail to continue for sure. This implies that $v_E(m, c) < 0$, which contradicts the supposition that $v_E(m, c) > 0$. Consequently, $v_E(m, \cdot) = 0$ for all $m > \check{m}$. □

Proof of Lemma 2. By Lemma 1, there exist an upper bound $\check{m} \in \mathbb{N}$ on the number M of active firms in equilibrium. Denote the space of all functions

$$f : \{1, \ldots, \check{m}\} \times [\hat{c}, \check{c}] \to \left[0, \frac{\beta\pi(1)\check{c}}{1-\beta}\right]$$

with $\mathcal{W}$. Pick an arbitrary equilibrium strategy $a \equiv (a_S, a_E)$. Define $T_E^a : \mathcal{W} \to \mathcal{W}$ and $T_S^a : \mathcal{W} \to \mathcal{W}$ such that for any (m, c)

$$\begin{aligned}(T_E^a f)(m, c) = \\ \max\left\{0, \sum_{i=0}^{m-1} \binom{m-1}{i} a_S(m, c)^{m-i-1} \left[1 - a_S(m, c)\right]^i (T_S^a f)(m - i, c)\right\}\end{aligned} \tag{12}$$

and

$$\begin{aligned}(T_S^a f)(n, c) = \\ \beta\mathbb{E}\Big[C'\pi(n) - \kappa + \{1 - a_E(n + 1, C')\} f(n, C') \\ + \sum_{l=1}^{\infty} \{1 - a_E(n + l + 1, C')\} \Pi_{j=1}^{l} a_E(n + j, C') f(n + l, C') \,\Big|\, C = c\Big].\end{aligned} \tag{13}$$

The function $(T_S^a f)(n,\cdot)$ gives the expected discounted payoffs to a firm just after it has survived in the industry with $n-1$ rivals, in the case that its rivals use the strategy a_S and its expected discounted payoffs when deciding on survival are f. The sum in the right hand side of (12) is the expected discounted payoff to a continuing firm in this same case.

The metric space $\mathcal{W}$ is complete with respect to the supremum norm and T_E^a satisfies Blackwell's sufficient conditions for a contraction mapping. Consequently T_E^a has a unique fixed point in $\mathcal{W}$: The payoff v_E^a in the equilibrium with strategy a.[11] The corresponding unique $T_S^a v_E^a$ is the post-exit value v_S^a corresponding to v^a.

Now consider $\mathcal{W}^a$, the space of all functions $f \in \mathcal{W}$ such that $f \leq v^a$ and $f(m,\cdot)$ weakly decreases with m. We will prove that $v_E^a(m,\cdot)$ and $v_S^a(m,\cdot)$ weakly decrease with m by first establishing that

(A) $(T_S^a f)(n,\cdot)$ decreases with n for all $f \in \mathcal{W}^a$; and then showing that

(B) $v_E^a \in \mathcal{W}^a$.

Note that (B) directly implies that $v_E^a(m,\cdot)$ weakly decreases with m and, with (A), implies that $v_S^a(m,\cdot) = (T_S^a v_E^a)(m,\cdot)$ weakly decreases with m. Because a is arbitrary, this establishes the stated result. We finish by proving (A) and (B).

(A) Using (13), for $N' > 1$,

$$\begin{aligned}
&(T_S^a f)(n,c) - (T_S^a f)(n-1,c) \\
&\quad = \beta \mathbb{E}\Big[C' \{\pi(n) - \pi(n-1)\} + \{1 - a_E(n,C')\}\{f(n,C') - f(n-1,C')\} + \\
&\qquad\qquad \{1 - a_E(n,C')\} \sum_{l=1}^{\infty} \{f(n+l,C') - f(n+l-1,C')\} \prod_{j=1}^{l} a_E(n+j,C') \,\Big|\, C = c\Big].
\end{aligned}$$

By Assumption 1, the first term in the right hand side is negative; for all $f \in \mathcal{W}^a$, the remaining terms in the right hand side are weakly negative. Consequently, $(T_S^a f)(n,\cdot)$ decreases with n for all $f \in \mathcal{W}^a$.

(B) Because $\mathcal{W}^a$ is complete, it suffices to show that $T_E^a : \mathcal{W}^a \to \mathcal{W}^a$:

1. Because T_E^a is monotone and v_E^a is its unique fixed point, $T_E^a f \leq v_E^a$ for all $f \in \mathcal{W}^a$.

2. Remains to show that $(T_E^a f)(m,\cdot)$ weakly decreases with m for all $f \in \mathcal{W}^a$. Take any $f \in \mathcal{W}^a$, $m > 1$ and c.

[11] More precisely, this fixed point equals the restriction of v^a to $\{1,\ldots,\check{m}\} \times [\hat{c},\check{c}]$ and, by Lemma 1, v^a itself uniquely follows by setting $v_E^a(m,\cdot) = 0$ for all $m > \check{m}$. Because there is little risk of confusion, we will keep this qualification implicit here and below.

(a) If $a_S(m,c) < 1$, then $v_E^a(m,c) = 0$ by (4). Because $f \leq v_E^a$ and T_E^a is monotone, $(T_E^a f)(m,c) \leq (T_E^a v_E^a)(m,c) = v_E^a(m,c) = 0$. Because, by (12), $(T_E^a f)(m-1,c) \geq 0$; this implies that $(T_E^a f)(m,c) \leq (T_E^a f)(m-1,c)$.

(b) If $a_S(m,c) = 1$; then, by (12) and (A),

$$(T_E^a f)(m,c) = \max\{0, (T_S^a f)(m,c)\} \leq \max\{0, (T_S^a f)(m-1,c)\} \\ \leq (T_E^a f)(m-1,c).$$

This establishes that $(T^a f)(m,\cdot)$ weakly decreases with m for all $f \in \mathcal{W}^a$.

Taken together, this shows that $T^a : \mathcal{W}^a \to \mathcal{W}^a$, so that $v_E^a \in \mathcal{W}^a$. □

Proof of Lemma 3. First, because $\mu(\check{m}-1,c) \leq \check{m}$, let $g(c) = w_E(\check{m},c), \forall c$ and we have that

$$\begin{aligned}(T_{\check{m}-1}g)(c) &= \max\{0, \beta\mathbb{E}[C'\pi(\check{m}-1) - \kappa + I\{\mu(\check{m}-1,C') > \check{m}-1\}g(C') \\ &\quad + I\{\mu(\check{m}-1,C') = \check{m}-1\}g(C')|C=c]\} \\ &\geq \max\{0, \beta\mathbb{E}[C'\pi(\check{m}) - \kappa + g(C')|C=c]\} \\ &= g(c)\end{aligned}$$

Because $T_{\check{m}-1}$ is monotone, this implies that $w_E(\check{m}-1,c) \geq g(c) = w_E(\check{m},c), \forall c$.

Next, iterate the following argument for $m = \check{m}-2,\ldots,1$. Suppose that $w_E(m+1,c) \geq \ldots \geq w_E(\check{m},c), \forall c$, because $\mu(m+1,c) = m+1$ if $\mu(m,c) = m$ and $\mu(m+1,c) = \mu(m,c)$ if $\mu(m,C) > m$, let $g(C) = w_E(m+1,c), \forall c$ and we have that

$$\begin{aligned}(T_m g)(c) &= \max\{0, \beta\mathbb{E}[C'\pi(m) - \kappa + I\{\mu(m,C') > m\}w(\mu(m,C'),C') \\ &\quad + I\{\mu(m,C') = m\}g(C')|C=c]\} \\ &\geq \max\{0, \beta\mathbb{E}[C'\pi(m+1) - \kappa + I\{\mu(m+1,C') > m+1\}w(\mu(m+1,C'),C') \\ &\quad + I\{\mu(m+1,C') = m+1\}g(C')|C=c]\} \\ &= g(c)\end{aligned}$$

Because T_m is monotone, this implies that $w_E(m,c) \geq \ldots \geq w_E(\check{m},c), \forall c$. □

Proof of Proposition 1. First, we establish a lemma that the strategy (α_S, α_E) forms a symmetric Markov-perfect equilibrium that defaults to inactivity, with payoff w_E and w_S. Then, we check that it is the unique such equilibrium.

Lemma 5 (Verification of the Homogeneous-Firm Model's Candidate Equilibrium)**.** *The strategy (α_E, α_S) constructed by Algorithm 1 forms a symmetric Markov-perfect equilibrium, with payoffs w_E and w_S.*

Proof. To prove this lemma, we need to verify that no firm can gain from a one-shot deviation if all firms follow (α_E, α_S). This requires that for any (m, c) equation (3) and (4) are satisfied

$$\begin{aligned} \alpha_S(m,c) &\in \arg\max_{a\in[0,1]} a\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] \text{ and} \\ \alpha_E(m,c) &\in \arg\max_{a\in[0,1]} a(\mathbb{E}\left[w_E\left(M_E,c\right)|\; M=m\right]-\varphi), \end{aligned}$$

where

$$\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = \sum_{i=0}^{m-1}\binom{m-1}{i}\alpha_S(m,c)^{m-i-1}\left\{1-\alpha_S(m,c)\right\}^i w_S(i+1,c) \text{ and}$$

$$\mathbb{E}\left[w_E\left(M_E,c\right)|\; M=m\right] = w_E\left\{\mu(m,c),c\right\}.$$

First, consider (14):

- For (m,c) such that $w_E(m,c) > 0$, $\alpha_S(m,c) = 1$ and $\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = w_E(m,c) > 0$, so that

$$\arg\max_{a\in[0,1]} a\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = \{1\} \ni \alpha_S(m,c) = 1.$$

- For (m,c) such that $w_E(m,c) = 0$ and $w_E(1,c) > 0$, $\alpha_S(m,c)$ is such that $\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = 0$, so that

$$\arg\max_{a\in[0,1]} a\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = [0,1] \ni \alpha_S(m,c).$$

- For (m,c) such that $w_E(m,c) = w_E(1,C) = 0$, $\alpha_S(m,c) = 0$ and $\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] = w_S(1,C) \leq w_E(1,C) = 0$, so that

$$\arg\max_{a\in[0,1]} a\mathbb{E}\left[w_S(M_S,c)\,|\,M_E=m_E\right] \supseteq \{0\} \ni \alpha_S(m,c) = 0.$$

Next, we verify (14):

- For (m,c) such that $w(m,c) \leq \varphi$, $\alpha_E(m,c) = 0$. Moreover, by Lemma 3 and because $\mu(m,c) \geq m$, $\mathbb{E}\left[w_E\left(M_E,c\right)|\; M=m\right] = w_E\{\mu(m,c),C\} \leq w_E(m,c) \leq \varphi$, so that

$$\arg\max_{a\in[0,1]} a(\mathbb{E}\left[w_E\left(M_E,c\right)|\; M=m\right]-\varphi) \supseteq \{0\} \ni \alpha_E(m,c) = 0.$$

- For (m,c) such that $w_E(m,c) > \varphi$, $\alpha_E(m,c) = 1$. Moreover, it follows from the construction of $\mu(m,c)$ that $w_E\left\{\mu(m,c),c\right\} > \varphi$, so that

$$\arg\max_{a\in[0,1]} a(\mathbb{E}\left[w_E\left(M_E,c\right)|\; M=m\right]-\varphi) = \{1\} \ni \alpha_E(m,c) = 1.$$

Taken together, this establishes that (α_S, α_E) forms a symmetric Markov-perfect equilibrium. Then it is straightforward that w_E and w_S are associated equilibrium payoffs. □

The proof for the uniqueness of v_E and v_S proceeds recursively, from a market served by $\check{m}$ firms down to a monopoly market. Throughout, we use that, by Assumption 1 and Lemma 2, $v_E(m, \cdot)$ weakly decreases with m.

First, consider $v_E(\check{m}, \cdot)$, $a_S(\check{m}, \cdot)$, and $a_E(\check{m}, \cdot)$. Clearly, in a symmetric equilibrium, if $v_S(\check{m}, c) = 0$, then $v_E(\check{m}, c) = 0$.

Suppose that $v_S(\check{m}, c) > 0$ and $v_E(\check{m}, C) = 0$. By Lemma 2, the monotonicity of π, and $\mu(M', C') \leq \check{m}$, $\beta \mathbb{E}_C[C'\pi(M') - \kappa + v_E(\mu(M', C'), C')|C = c] \geq v_S(\check{m}, c) > 0$ for any M', a_E, where $\mathbb{E}_C$ is the expectation over C. By (2) and (4), this means that $a_S(\check{m}, c) = 1$ and hence $v_E(\check{m}, c) = v_S(\check{m}, c) > 0$, which contradicts the original supposition. This argument justifies (5) so $v_E(\check{m}, c) = w_E(\check{m}, c)$. Then it follows naturally that $v_S(\check{m}, c) = w_S(\check{m}, c)$. Moreover, the contraction mapping in equation (5) uniquely determines the post-entry payoff for a potential entrant facing $\check{m} - 1$ incumbent firms, so the strategy's action (enter if and only if this payoff is positive) is the only one consistent with the maximization of this payoff and a default to inactivity $(a_E(\check{m}, c) = \alpha_E(\check{m}, c))$.

Next, iterate the following argument for $m = \check{m} - 1, \ldots, 1$. Suppose that we have demonstrated that $v_E(m+1, \cdot), \ldots, v_E(\check{m}, \cdot)$ have been uniquely determined as $w_E(m + 1, \cdot), \ldots, w_E(\check{m}, \cdot)$ by a sequence of contraction mappings and $a_E(m + 1, \cdot), \ldots, a_E(\check{m}, \cdot)$ are also uniquely computed as $\alpha_E(m + 1, \cdot), \ldots, \alpha_E(\check{m}, \cdot)$. By Lemma 2, $a_E(m + 1, c) \geq \ldots \geq a_E(\check{m}, c)$ for all c. Due to equilibrium symmetry, if $v_E(m, c) > 0$, then it must be true that $a_S(m, c) = 1$ and $v_S(m, c) > 0$. Now suppose $v_E(m, c) = 0$ and $v_S(m, c) > 0$. By Lemma 2 and the monotonicity of π, a_E, $\beta \mathbb{E}_C[C'\pi(n) - \kappa + v_E(\mu(n, C'), C')|C = c] \geq v_S(m, c) > 0$ for any $n \leq m$. By (2) and (4), this means that $a_S(m, c) = 1$ and hence $v_E(m, c) = v_S(m, c) > 0$, which contradicts the original supposition. Therefore, $v_E(m, c) > 0$ if and only if $v_S(m, c) > 0$. This allows us to write

$$v_E(m, c) = \max\{0, v_S(m, c)\} = \max\{0, \beta \mathbb{E}[C'\pi(m) - \kappa + v_E(\mu(m, C'), C') \,|\, C = c]\}.$$

So $v_E(m, c)$ is uniquely determined as the fixed point of T_m in Algorithm 1 $(v_E(m, c) = w_E(m, c))$. Then, $v_S(m, c)$ is uniquely computed as $w_S(m, c)$ and $a_E(m, c)$ by $\alpha_E(m, c)$.

We next check the uniqueness of a_S. Given a unique v_E, from the argument verifying that α_S is an equilibrium strategy, we know that the optimal strategy sets when $v_S(m, c) > 0$ and when $v_S(1, c) \leq 0$ are singletons. Remains is to show that when $v_S(m, c) < 0$ and $v_S(1, c) > 0$, a unique $p(m, c)$ solves

$$\binom{m-1}{i} p^i (1-p)^{m-1-i} w_S(i + 1, c) = 0.$$

Since v_E is the equilibrium payoff, according to Lemma 2, it is non-increasing in m. In addition, $\mu(m,c)$ is non-decreasing in m. Thus, on the left hand side of the equation, $w_S(i,c) = \beta\mathbb{E}[C'\pi(i) - \kappa + v_E(\mu(i,C'),C'),C')|C=c]$ is non-increasing in i. This means that the left hand side changes monotonically and continuously from $w_S(1,c) > 0$ to $w_S(m,c) \le 0$, when $p(m,c)$ changes from 0 to 1. Therefore, there is a unique $p(m,c) \in (0,1]$ which solves the equation. Hence, the constructed α_S is the unique equilibrium strategy a_S. □

B Proofs for the Heterogeneous Duopoly Case

Proof of Lemma 4. A sufficient condition for the results in Lemma 4 is the following. For all k and all x, $1 \le k \le \check{k}$ and $0 \le x \le \check{k}$, and for all natural equilibrium strategy (a_S, a_E),

Condition 1. $v^a(\iota_k + \iota_x, \cdot, k)$ is weakly decreasing in x.

In order to prove this sufficient condition, we need to verify it along with the other two conditions,

Condition 2. $v^a(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x.

Condition 3. $v^a(\iota_k, \cdot, k)$ is weakly increasing in k.

First, we prove Conditions 1 and 2 for $k = \check{k}$ and Condition 3 for $v^a(\iota_{\check{k}}, \cdot, \check{k}) \ge v^a(\iota_{\check{k}-1}, \cdot, \check{k}-1)$. Then, for any $k < \check{k}$, suppose that those conditions hold good for $k+1, k+2, \dots, \check{k}$, we prove that they also hold for k. In this way, we prove that these conditions hold for all $k \in \mathbb{K}$. Eventually, we proof Lemma 4 using Condition 1.

Now suppose (a_S, a_E) forms a natural equilibrium with equilibrium payoff v_S^a, v_E^a. Define $\mathcal{F}$ to be the space of all functions

$$f_E : \left\{ (n_1, \dots, n_{\check{k}}) : \sum_{i=1}^{\check{k}} n_i \le 2 \right\} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \left[0, \frac{\beta \pi_{\check{k}}(\iota_{\check{k}})\check{c}}{1-\beta} \right],$$

and $T^a : \mathcal{F} \to \mathcal{F}$ with

For any $k^1, k^2 \in \{0, 1, \dots, \check{k}\}$, if $k^1 \ge k^2$,

$$\begin{aligned}
(T^a f_E)(\iota_{k^1} + \iota_{k^2}, c, k^1) &= \max\Bigg\{ 0, a_S(\iota_{k^1} + \iota_{k^2}, c, k^2) \sum_{i=k^1}^{\check{k}} \sum_{j=k^2}^{\check{k}} \Pi_{k^1,i} \Pi_{k^2,j} f_S(\iota_i + \iota_j, c, i) \\
&\quad + (1 - a_S(\iota_{k^1} + \iota_{k^2}, c, k^2)) \sum_{i=k^1}^{\check{k}} \Pi_{k^1,i} f_S(\iota_i, c, i) \Bigg\} \\
&= \max\big\{ 0, a_S(\iota_{k^1} + \iota_{k^2}, c, k^2) \mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1'}} + \iota_{K^{2'}}, c, K^{1'}) | K^2 = k^2] | K^1 = k^1] \\
&\quad + (1 - a_S(\iota_{k^1} + \iota_{k^2}, c, k^2)) \mathbb{E}[f_S(\iota_{K^{1'}}, c, K^{1'}) | K^1 = k^1] \big\}. \qquad (14)
\end{aligned}$$

If $k^1 < k^2$,

$$
\begin{aligned}
(T^a f_E)(\iota_{k^1}+\iota_{k^2},c,k^1) &= \max\left\{0, \sum_{i=k^1}^{\check{k}}\sum_{j=k^2}^{\check{k}} \Pi_{k^1,i}\Pi_{k^2,j} f_S(\iota_i+\iota_j,c,i)\right\} \\
&= \max\left\{0, \mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}}+\iota_{K^{2\prime}},c,K^{1\prime})|K^2=k^2]|K^1=k^1]\right\} \quad (15)
\end{aligned}
$$

With f_S defined analogously to v_S in Section 2 as

$$
\begin{aligned}
f_S(\iota_{k^1}+\iota_{k^2},c,i) \equiv \beta\mathbb{E}[C'\pi_{k^1}(\iota_{k^1}+\iota_{k^2}) - \kappa + a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C') f_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C',i) \\
+ [1-a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,C')]\, f_E(\iota_{k^1}+\iota_{k^2},C',i)|C=c]
\end{aligned}
$$

where, for definiteness, $a_E(\iota_{k^1}+\iota_{k^2}+\iota_1,c)\equiv 0$ if $k^1,k^2>0$.

The metric space $\mathcal{F}$ is complete with respect to the supremum norm. T^a satisfies Blackwell's sufficient conditions for a contraction mapping. The equilibrium payoff v^a is the unique fixed point of T^a. We prove that v^a satisfies Conditions 1–3 by showing that the fixed point of T^a lies in the space in which all functions satisfy these conditions. We introduce a Lemma which we will use repeatedly to achieve this result. Recall that firm types' evolution has the first-order stochastic dominance property, as stated in Assumption 3. The following Lemma exploits this property

Lemma 6. *X, Y are random variables with densities F and G respectively. If X first-order stochastically dominates Y, then $\mathbb{E}[h(X)] \geq \mathbb{E}[h(Y)]$ for all weakly increasing function h.*

First, denote a subspace of $\mathcal{F}$ in which any functions f_E satisfy $f_E \geq v_E^a$ as $\mathcal{F}^0$. Because $\mathcal{F}^0$ is also a complete metric space and $v_E^a \in \mathcal{F}^0$, $T^a: \mathcal{F}^0 \to \mathcal{F}^0$. Then, for any natural Markov-perfect equilibrium with strategy (a_S, a_E) and payoff v_S^a, v_E^a, we gradually narrow down the space that v_S^a, v_E^a lies in.

Prove Condition 2 for $k=\check{k}$. We take two steps to prove Condition 2 for $k=\check{k}$.

1. Consider $\mathcal{F}_{\check{k}}^1$, a subspace of $\mathcal{F}^0$ in which any function f_E satisfies $(T^a f_E)(2\iota_{\check{k}},\cdot,\check{k}) = v_E^a(2\iota_{\check{k}},\cdot,\check{k})$ and $f_E(2\iota_{\check{k}},\cdot,\check{k}) \leq f_E(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ for $0\leq x\leq \check{k}$. This is also a complete metric space. Since function f_E^* which satisfies $f_E^*(2\iota_{\check{k}},\cdot,\check{k}) = v_E^a(2\iota_{\check{k}},\cdot,\check{k})$ and $f_E^*(2\iota_{\check{k}},\cdot,\check{k}) = f_E^*(\iota_{\check{k}}+\iota_x,\cdot,\check{k})$ for $0\leq x\leq\check{k}$ is in $\mathcal{F}_{\check{k}}^1$, this subspace is nonempty. We aim to prove that $T^a: \mathcal{F}_{\check{k}}^1 \to \mathcal{F}_{\check{k}}^1$. Since v_E^a is the unique fixed point of T^a, this result ensures that $v_E^a \in \mathcal{F}_{\check{k}}^1$ and in particular $v_E^a(2\iota_{\check{k}},\cdot,\check{k}) \leq v_E^a(\iota_{\check{k}},\cdot,\check{k})$.

   In a symmetric equilibrium, any rival's exit implies that both firms expect non-positive payoffs from continuing. Therefore, firms earn positive expected payoffs only when both firms continue with probability 1. So

   $$v_E^a(2\iota_{\check{k}},c,\check{k}) \leq \max\{0, \beta\mathbb{E}[C'\pi_{\check{k}}(2\iota_{\check{k}}) - \kappa + v_E^a(2\iota_{\check{k}},C',\check{k})|C=c]\} \leq \max\{0, v_S^a(2\iota_{\check{k}},c,\check{k})\}.$$

Note that for all $f_E \in \mathcal{F}^1_{\check{k}}$, we have for any (a_S, a_E), $f_S(\iota_{\check{k}} + \iota_x, \cdot, \check{k}) \geq v^a_S(2\iota_{\check{k}}, \cdot, \check{k})$ for $0 \leq x \leq \check{k}$. Therefore, $\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{X'}, \cdot, \check{k})|X = x] \geq v^a_S(2\iota_{\check{k}}, \cdot, \check{k})$. Then, for any $0 \leq x \leq \check{k}$, we have

$$\begin{aligned}
& (T^a f_E)(\iota_{\check{k}} + \iota_x, c, \check{k}) \\
= \; & \max\left\{0, a_S(\iota_{\check{k}} + \iota_x, c, x)\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{X'}, c, \check{k})|X = x] + (1 - a_S(\iota_{\check{k}} + \iota_x, c, x))f_S(\iota_{\check{k}}, c, \check{k})\right\} \\
\geq \; & \max\{0, v^a_S(2\iota_{\check{k}}, c, \check{k})\} \\
\geq \; & v^a_E(2\iota_{\check{k}}, \cdot, \check{k}) \\
= \; & (T^a f_E)(2\iota_{\check{k}}, \cdot, \check{k})
\end{aligned}$$

Therefore, $T^a : \mathcal{F}^1_{\check{k}} \to \mathcal{F}^1_{\check{k}}$ and $v^a_E(2\iota_{\check{k}}, \cdot, \check{k}) \leq v^a_E(\iota_{\check{k}}, \cdot, \check{k})$. Then, an analogous argument which motivates equation (**??**) leads to

$$v^a_E(2\iota_{\check{k}}, c, \check{k}) = \max\left\{0, \beta\mathbb{E}\left[C'\pi_{\check{k}}(2\iota_{\check{k}}) - \kappa + v^a_E\left(2\iota_{\check{k}}, C', \check{k}\right) \,\middle|\, C = c\right]\right\}. \tag{16}$$

(16) defines a contraction mapping with a unique fixed point $v^a_E(2\iota_{\check{k}}, \cdot, \check{k})$. So

$$(T^a f_E)(2\iota_{\check{k}}, c, \check{k}) = \max\{0, f_S(2\iota_{\check{k}}, c, \check{k})\}.$$

2. We move on to a subspace of $\mathcal{F}^1_{\check{k}}$, which we denote by $\mathcal{F}^2_{\check{k}}$. In this subspace, any function f_E satisfies that $f_E(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x for $0 \leq x \leq \check{k}$. We will further show that $T^a : \mathcal{F}^2_{\check{k}} \to \mathcal{F}^2_{\check{k}}$. Note that for $f_E \in \mathcal{F}^2_{\check{k}}$, $f_S(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x as well. For any k^1, k^2 such that $1 \leq k^1 \leq k^2 \leq \check{k}$, we use $K^{1'}, K^{2'}$ to denote the random variables for the type succeeding $K^1 = k^1, K^2 = k^2$ respectively. $K^{2'}$ stochastically dominates $K^{1'}$. According to Lemma 6, $\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{1'}}, \cdot, K^{1'})|K^1 = k^1] \leq \mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{2'}}, \cdot, K^{2'})|K^2 = k^2]$. Therefore,

$$\begin{aligned}
(T^a f_E)(\iota_{\check{k}} + \iota_{k^1}, c, k^1) \; &= \; \max\left\{0, \mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{1'}}, c, K^{1'})|K^1 = k^1]\right\} \\
&\leq \; \max\left\{0, \mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{2'}}, c, K^{2'})|K^2 = k^2]\right\} \\
&= \; (T^a f_E)(\iota_{\check{k}} + \iota_{k^2}, c, k^2)
\end{aligned}$$

This result guarantees that $T^a : \mathcal{F}^2_{\check{k}} \to \mathcal{F}^2_{\check{k}}$. Therefore, $v^a(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x and Condition 2 is satisfied. This result also implies that any equilibrium strategy must satisfy that $a_S(\iota_{\check{k}} + \iota_x, \cdot, x)$ is weakly increasing in x.

Prove Condition 1 for $k = \check{k}$. Next, we focus on a subspace of $\mathcal{F}^2_{\check{k}}$, denoted by $\mathcal{F}^3_{\check{k}}$. In this subspace any function f_E must satisfy that $f_E(\iota_{\check{k}} + \iota_x, \cdot, \check{k})$ is weakly decreasing in x for any $0 \le x \le \check{k}$. Note that for $f_E \in \mathcal{F}^3_{\check{k}}$, $f_S(\iota_{\check{k}} + \iota_x, \cdot, \check{k})$ is weakly decreasing in x as well. Also for any $0 \le k^1 \le k^2 \le \check{k}$, $\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{2\prime}}, \cdot, \check{k})|K^2 = k^2] \le \mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{1\prime}}, \cdot, \check{k})|K^1 = k^1] \le f_S(\iota_{\check{k}}, \cdot, \check{k})$. Then,

$$
\begin{aligned}
& (T^a f_E)(\iota_{\check{k}} + \iota_{k^2}, c, \check{k}) \\
= \; & \max\left\{0, a_S(\iota_{\check{k}} + \iota_{k^2}, c, k^2)\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{2\prime}}, c, \check{k})|K^2 = k^2] + (1 - a_S(\iota_{\check{k}} + \iota_{k^2}, c, k^2))f_S(\iota_{\check{k}}, c, \check{k})\right\} \\
\le \; & \max\left\{0, a_S(\iota_{\check{k}} + \iota_{k^1}, c, k^1)\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{2\prime}}, c, \check{k})|K^2 = k^2] + (1 - a_S(\iota_{\check{k}} + \iota_{k^1}, c, k^1))f_S(\iota_{\check{k}}, c, \check{k})\right\} \\
\le \; & \max\left\{0, a_S(\iota_{\check{k}} + \iota_{k^1}, c, k^1)\mathbb{E}[f_S(\iota_{\check{k}} + \iota_{K^{1\prime}}, c, \check{k})|K^1 = k^1] + (1 - a_S(\iota_{\check{k}} + \iota_{k^1}, c, k^1))f_S(\iota_{\check{k}}, c, \check{k})\right\} \\
= \; & (T^a f_E)(\iota_{\check{k}} + \iota_{k^1}, c, \check{k})
\end{aligned}
$$

Therefore, $T^a : \mathcal{F}^3_{\check{k}} \to \mathcal{F}^3_{\check{k}}$ and $v^a_E(\iota_{\check{k}} + \iota_x, \cdot, \check{k})$ is weakly decreasing in x, so Condition 1 is satisfied.

Prove Condition 3: $v^a_E(\iota_{\check{k}}, \cdot, \check{k}) \ge v^a_E(\iota_{\check{k}-1}, \cdot, \check{k} - 1)$. Before proving this condition, we need to prove Conditions 2 and 1 for $k = \check{k} - 1$. We skip the details because later we will demonstrate how to do so for any k. Given that these two conditions are verified for $k = \check{k}-1$, we know that $v^a_E \in \mathcal{F}^3_{\check{k}-1}$ where $\mathcal{F}^3_{\check{k}-1}$ is defined analogously to $\mathcal{F}^3_{\check{k}}$. Hence, for $0 \le x \le \check{k}$, $v^a_E(\iota_{\check{k}-1} + \iota_x, \cdot, \check{k} - 1)$ is weakly decreasing in x and $v^a_E(\iota_{\check{k}-1} + \iota_x, \cdot, x)$ is weakly increasing in x. Then we take two steps to achieve Condition 3.

1. First, denote the subspace of $\mathcal{F}^3_{\check{k}-1}$ in which any f_E functions such that $f_E(\iota_{\check{k}} + \iota_x, \cdot, x) \le f_E(\iota_{\check{k}-1} + \iota_x, \cdot, x)$ for all x by $\mathcal{F}^4_{\check{k}-1}$.

   Then, $f_S(\iota_{\check{k}} + \iota_x, \cdot, x) \le f_S(\iota_{\check{k}-1} + \iota_x, \cdot, x)$ holds for $f_E \in \mathcal{F}^4_{\check{k}-1}$, by Lemma 6, we have

$$
\begin{aligned}
(T^a f_E)(\iota_{\check{k}} + \iota_x, c, x) &= \max\{0, \mathbb{E}[f_S(\iota_{\check{k}} + \iota_{X'}, c, X')|X = x]\} \\
&\le \max\left\{0, \mathbb{E}[\mathbb{E}[f_S(\iota_{K'} + \iota_{X'}, c, X')|X = x]|\check{k} - 1]\right\} \\
&= (T^a f_E)(\iota_{\check{k}-1} + \iota_x, c, x)
\end{aligned}
$$

   Therefore, $T^a : \mathcal{F}^4_{\check{k}-1} \to \mathcal{F}^4_{\check{k}-1}$ and $v^a_E(\iota_{\check{k}} + \iota_x, \cdot, x) \le v^a_E(\iota_{\check{k}-1} + \iota_x, \cdot, x)$ for all x. In particular, this result ensures that $a_E(\iota_{\check{k}} + \iota_1, \cdot) \le a_E(\iota_{\check{k}-1} + \iota_1, \cdot)$.

2. Define a subspace of $\mathcal{F}^4_{\check{k}-1}$ in which any function f_E satisfies $f_E(\iota_{\check{k}-1}, \cdot, \check{k} - 1) \le f_E(\iota_{\check{k}}, \cdot, \check{k})$, as $\mathcal{F}^5_{\check{k}-1}$. Because $a_E(\iota_{\check{k}} + \iota_1, \cdot) \le a_E(\iota_{\check{k}-1} + \iota_1, \cdot)$, we have $f_S(\iota_{\check{k}-1}, \cdot, \check{k} - 1) \le f_S(\iota_{\check{k}}, \cdot, \check{k})$ and then $(T^a f_E)(\iota_{\check{k}-1}, \cdot, \check{k} - 1) \le (T^a f_E)(\iota_{\check{k}}, \cdot, \check{k})$. Therefore, $T^a : \mathcal{F}^5_{\check{k}-1} \to \mathcal{F}^5_{\check{k}-1}$ and hence Condition 3 is verified.

Now suppose that for any $k \le \check{k}$, we have established the following results

Result 1. $v_E^a(\iota_{k+1}+\iota_x,\cdot,x), v_E^a(\iota_{k+2}+\iota_x,\cdot,x),\ldots,v_E^a(\iota_{\check{k}}+\iota_x,\cdot,x)$ are all weakly increasing in x, $1\le x\le\check{k}$.

Result 2. $v_E^a(\iota_x+\iota_{k+1},\cdot,k+1), v_E^a(\iota_x+\iota_{k+2},\cdot,k+2),\ldots,v_E^a(\iota_x+\iota_{\check{k}},\cdot,\check{k})$ are all weakly decreasing in x, $0\le x\le\check{k}$.

Result 3. $v_E^a(\iota_{k+1}+\iota_x,\cdot,x)\ge v_E^a(\iota_{k+2}+\iota_x,\cdot,x)\ge\ldots\ge v_E^a(\iota_{\check{k}}+\iota_x,\cdot,x)$ for all $1\le x\le k$.

Result 4. $v_E^a(\iota_{k+1},\cdot,k+1)\le v_E^a(\iota_{k+2},\cdot,k+2)\le\ldots\le v_E^a(\iota_{\check{k}},\cdot,\check{k})$.

We then need to prove that

Condition 4. $v_E^a(\iota_k+\iota_x,\cdot,x)$ is weakly increasing in x, $1\le x\le\check{k}$.

Condition 5. $v_E^a(\iota_x+\iota_k,\cdot,k)$ is weakly decreasing in x, for all $0\le x\le\check{k}$.

Condition 6. $v_E^a(\iota_k+\iota_x,\cdot,x)\ge v_E^a(\iota_{k+1}+\iota_x,\cdot,x)$, for all $1\le x\le k-1$.

Condition 7. $v_E^a(\iota_k,\cdot,k)\le v_E^a(\iota_{k+1},\cdot,k+1)$.

Note that Result 2 and Condition 4 together implies Condition 2, Result 1 and Condition 5 implies Condition 1, and Result 4 and Condition 7 implies Condition 3 for any general k. Result 3 and Condition 6 are intermediate steps which are necessary to achieve Condition 3. Next, we prove Conditions 4–7.

Prove Condition 4. We follow three steps to achieve this end.

1. First, consider $\mathcal{F}_k^1$, the subspace of $\mathcal{F}_{k+1}^5$ in which any function f satisfies that $(T^a f_E)(2\iota_k,\cdot,k)=v_E^a(2\iota_k,\cdot,k)$, $f_E(\iota_x+\iota_k,\cdot,k)$ is weakly decreasing in x, $k\le x\le\check{k}$, and $f_E(2\iota_k,\cdot,k)\le f_E(\iota_k+\iota_x,\cdot,k)$, for all $0\le x\le k$. Note that at least a function f_E^* with $f_E^*(2\iota_k,\cdot,k)=v_E^a(2\iota_k,\cdot,k)=f_E(\iota_k+\iota_x,\cdot,k)$ for all $\mathcal{F}_k^1$ is in $\mathcal{F}_k^1$, so $\mathcal{F}_k^1$ is nonempty. For any $f_E\in\mathcal{F}_k^1$, f_S shares the properties with f_E and hence also has the properties stated in Results 1–4. To prove that $(T^a f_E)(\iota_x+\iota_k,\cdot,k)$ is weakly decreasing in x, $k\le x\le\check{k}$, consider the following cases for any k^1,k^2 such that $k\le k^1<k^2\le\check{k}$.
    (a) If $k<k^1$, according to Lemma 6, $\mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}}+\iota_{K'},\cdot,K')|K=k]|K^1=k^1]\ge\mathbb{E}[\mathbb{E}[f_S(\iota_{K^{2\prime}}+\iota_{K'},\cdot,K')|K=k]|K^2=k^2]$. Thus, from equation (15), $(T^a f_E)(\iota_{k^2}+\iota_k,c,k)\le(T^a f_E)(\iota_{k^1}+\iota_k,c,k)$.
    (b) If $k=k^1$, because $f_S(2\iota_k,\cdot,k)\le f_S(\iota_k+\iota_x,\cdot,k)$, for all $0\le x\le k$, and because Result 2 ensures that $f_S(\iota_k+\iota_d,\cdot,d)\le f_S(\iota_x+\iota_d,\cdot,d)$, for all $0\le x\le k$ and $k<d\le\check{k}$. Thus, $\mathbb{E}[f_S(\iota_k+\iota_{K'},\cdot,K')|K=k]\le\mathbb{E}[f_S(\iota_x+\iota_{K'},\cdot,K')|K=k]$ for all $0\le x\le k$. In addition, because $f(\iota_x+\iota_k,\cdot,k)$ is weakly decreasing in x,

$k \leq x \leq \check{k}$, according to Lemma 6, $\mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}} + \iota_{K'}, \cdot, K')|K = k]|K^1 = k] \leq \mathbb{E}[f_S(\iota_{K'}, \cdot, K')|K = k]$. Then,

$$\begin{aligned}
(T^a f_E)(\iota_{k^2} + \iota_k, c, k) &\leq \max\left\{0, \mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\} \\
&\leq \max\{0, a_S(2\iota_k, c, k)\mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k] \\
&\quad +(1 - a_S(2\iota_k, c, k))\mathbb{E}[f_S(\iota_{K'}, c, K')|K = k]\} \\
&= (T^a f_E)(2\iota_k, c, k) = (T^a f_E)(\iota_{k^1} + \iota_k, c, k)
\end{aligned}$$

To prove that $(T^a f_E)(2\iota_k, \cdot, k) \leq (T^a f_E)(\iota_k + \iota_x, \cdot, k)$, for all $0 \leq x \leq k$, note that,

$$\begin{aligned}
(T^a f_E)(2\iota_k, c, k) &= v_E^a(2\iota_k, c, k) \\
&\leq \max\left\{0, \mathbb{E}[\mathbb{E}[v_S^a(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\} \\
&\leq \max\left\{0, \mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\} \\
&\leq \max\{0, a_S(\iota_k + \iota_x, c, x)\mathbb{E}[\mathbb{E}[f_S(\iota_{X'} + \iota_{K'}, c, K')|K = k]|X = x] \\
&\quad +(1 - a_S(\iota_k + \iota_x, c, x))\mathbb{E}[f_S(\iota_{K'}, c, K')|K = k]\} \\
&= (T^a f_E)(\iota_k + \iota_x, c, k)
\end{aligned}$$

The first inequality is due to equilibrium symmetric; for two type-k firm, either firm's equilibrium payoff is bounded by payoff from joint continuation. The second inequality is because $f_E \in \mathcal{F}^0$ so $f_E \geq v_E^a$. These results show that $T^a : \mathcal{F}_k^1 \to \mathcal{F}_k^1$. An analogous argument which motivates equation (**??**) leads to

$$v_E^a(2\iota_k, c, k) = \max\left\{0, \mathbb{E}[\mathbb{E}[v_S^a(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\}. \tag{17}$$

(17) defines a contraction mapping with a unique fixed point $v^a(2\iota_k, \cdot, k)$,

$$(T^a f_E)(2\iota_k, c, k) = \max\left\{0, \mathbb{E}[\mathbb{E}[f_S(\iota_{K^{1\prime}} + \iota_{K'}, c, K')|K = k]|K^1 = k]\right\}.$$

2. We move on to a subspace of $\mathcal{F}_k^1$, which we denote by $\mathcal{F}_k^2$. Any function f_E in this subspace satisfy that $f_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing with x. Note that for $f_E \in \mathcal{F}_k^2$, $f_S(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x as well. Combine it with Result 1, and then we have that $f_S(\iota_l + \iota_x, \cdot, x)$ is weakly increasing in x for all l such that $k \leq l \leq \check{k}$. Therefore, $\mathbb{E}[f_S(\iota_{K'} + \iota_x, \cdot, x)|K = k]$ is weakly increasing in x. For any k^1, k^2 such that $1 \leq k^1 \leq k^2 \leq \check{k}$, according to Lemma 6,

$$[\mathbb{E}[f_S(\iota_{K'} + \iota_{K^{1\prime}}, \cdot, K^{1'})|K = k]|K^1 = k^1] \leq [\mathbb{E}[f_S(\iota_{K'} + \iota_{K^{2\prime}}, \cdot, K^{2'})|K = k]|K^2 = k^2]. \tag{18}$$

We then consider the following cases.

(a) For $1 \le k^1 \le k^2 \le k$, from equation (15), we can observe that equation (18) leads to $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_{\check{k}} + \iota_{k^2}, c, k^2)$.

(b) For $k < k^1 \le k^2 \le \check{k}$, from Result 3, we have $a_S(\iota_k + \iota_{k^1}, \cdot, k) \ge a_S(\iota_k + \iota_{k^2}, \cdot, k)$. Also, from Result 2 and Lemma 6,

$$[\mathbb{E}[f_S(\iota_{K'} + \iota_{K^{i'}}, \cdot, K^{i'})|K = k]|K^i = k^i] \le \mathbb{E}[f_S(\iota_{K^{i'}}, \cdot, K^{i'})|K^i = k^i], i = 1, 2.$$

From Result 4 and Lemma 6,

$$\mathbb{E}[f_S(\iota_{K^{2'}}, \cdot, K^{2'})|K^2 = k^2] \le \mathbb{E}[f_S(\iota_{K^{1'}}, \cdot, K^{1'})|K^1 = k^1].$$

Using equation (14), we can show $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_k + \iota_{k^2}, c, k^2)$ by exploiting the inequalities.

(c) For $k^1 \le k \le k^2 \le \check{k}$, similarly we can show $(T^a f_E)(\iota_k + \iota_{k^1}, c, k^1) \le (T^a f_E)(\iota_k + \iota_{k^2}, c, k^2)$ with the above results.

This result guarantees that $T^a : \mathcal{F}^2_{\check{k}} \to \mathcal{F}^3_{\check{k}}$. Therefore, any equilibrium payoff must satisfy that $v^a_E(\iota_k + \iota_x, \cdot, x)$ is weakly increasing in x for all $1 \le x \le \check{k}$, which implies the same monotonicity for any equilibrium strategy.

Prove Condition 5. Next, we focus on a subspace of $\mathcal{F}^3_{\check{k}}$, denoted by $\mathcal{F}^3_{\check{k}}$. In this subspace, any function f_E satisfies that $f_E(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $0 \le x \le k$. Note that for $f_E \in \mathcal{F}^3_{\check{k}}$, $f_S(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $0 \le x \le k$. Combine it with Result 2, and then we have that $f_S(\iota_x + \iota_l, \cdot, l)$ is weakly decreasing in x, $0 \le x \le \check{k}$ and for all l such that $k \le l \le \check{k}$. Therefore, $\mathbb{E}[f_S(\iota_{K'} + \iota_x, \cdot, K')|K = k]$ is weakly decreasing in x. For any k^1, k^2 such that $0 \le k^1 \le k^2 \le \check{k}$, Lemma 6 implies that

$$[\mathbb{E}[f_S(\iota_{K'} + \iota_{K^{1'}}, \cdot, K')|K = k]|K^1 = k^1] \ge [\mathbb{E}[f_S(\iota_{K'} + \iota_{K^{2'}}, \cdot, K')|K = k]|K^2 = k^2].$$

Also, we have $a_S(\iota_k + \iota_{k^1}, \cdot, k^1) \le a_S(\iota_k + \iota_{k^2}, \cdot, k^2)$. Therefore, it must be true that $(T^a f_E)(\iota_k + \iota_{k^2}, c, k) \le (T^a f_E)(\iota_k + \iota_{k^1}, c, k)$. So $T^a : \mathcal{F}^3_{\check{k}} \to \mathcal{F}^3_{\check{k}}$ and the equilibrium payoff $v^a_E(\iota_x + \iota_k, \cdot, k)$ is weakly decreasing in x, $0 \le x \le \check{k}$.

Prove Condition 6. Next, we further look into a subspace of $\mathcal{F}^3_{\check{k}}$, denoted by $\mathcal{F}^4_{\check{k}}$, in which any function f_E satisfies that $f_E(\iota_k + \iota_x, \cdot, x) \le f_E(\iota_{k+1} + \iota_x, \cdot, x)$ for all $x < k$. Note that Result 2 and Condition 5 ensure that $f_E(\iota_k + \iota_x, \cdot, x) \le f_E(\iota_{k+1} + \iota_x, \cdot, x)$ for all $x \ge k$, so for any $f_E \in \mathcal{F}^3_{\check{k}}$, $f_S(\iota_{k+1} + \iota_x, \cdot, x) \le f_S(\iota_k + \iota_x, \cdot, x)$ for all x. Combine it with Result 3, and then we have $\mathbb{E}[f_S(\iota_l + \iota_{X'}, \cdot, X')|X = x]$ is weakly decreasing in l for $k \le l \le \check{k}$. According to Lemma 6,

$$\mathbb{E}[\mathbb{E}[f_S(\iota_{L'} + \iota_{X'}, \cdot, X')|X = x]|L = l] \le \mathbb{E}[\mathbb{E}[f_S(\iota_{K'} + \iota_{X'}, \cdot, X')|X = x]|K = k].$$

Then using equation (15) we can show that $(T^a f_E)(\iota_{k+1} + \iota_x, c, x) \leq (T^a f_E)(\iota_k + \iota_x, c, x)$ for any $x < k$. Therefore, $T^a : \mathcal{F}_k^4 \to \mathcal{F}_k^4$ and $v_E^a(\iota_{k+1} + \iota_x, c, x) \leq v_E^a(\iota_k + \iota_x, c, x)$ for all x. In particular, this result ensures that $a_E(\iota_{k+1} + \iota_1, \cdot) \leq a_E(\iota_k + \iota_1, \cdot)$.

Prove Condition 7. Finally, define a subspace of $\mathcal{F}_k^4$, in which any function f_E satisfies $f_E(\iota_k, \cdot, k) \leq f_E(\iota_{k+1}, \cdot, k+1)$, as $\mathcal{F}_k^5$. Because $a_E(\iota_{k+1} + \iota_1, \cdot) \leq a_E(\iota_k + \iota_1, \cdot)$, we have $f_S(\iota_k, \cdot, k) \leq f_S(\iota_{k+1}, \cdot, k+1)$ as well. Combine it with Result 4, and then we have that $f_S(\iota_l, \cdot, l)$ is weakly increasing in l for $k \leq l \leq \check{k}$. Using Lemma 6, we have $(T^a f_E)(\iota_k, c, k) \leq (T^a f_E)(\iota_{k+1}, c, k+1)$ so $T^a : \mathcal{F}_k^5 \to \mathcal{F}_k^5$ and Condition 7 is verified.

This completes the verification for the sufficient conditions for any arbitrary k. Since (a_S, a_E) is also arbitrarily chosen, any natural equilibrium payoff function v must satisfy Conditions 1, 2, and 3. Then we can prove Lemma 4.

Prove Lemma 4. For any strategy (a_S, a_E), as a special case of Condition 1, $v_E^a(2\iota_k, c, k) \leq v_E^a(\iota_k, c, k)$ for any $k \leq \check{k}$. To prove $v_S^a(2\iota_k, c, k) \leq v_S^a(\iota_k, c, k)$, note that

$$v_S^a(2\iota_k, c, k) = \mathbb{E}[\mathbb{E}[v_S^a(\iota_{K'} + \iota_{K^{1\prime}}, c, K')|K = k]|K^1 = k]$$

$$v_S^a(\iota_k, c, k) = \mathbb{E}[v_S^a(\iota_{K'}, c, K')|K = k]$$

For any $k^1, k^2 \geq k$, Condition 1 ensures that $v_S^a(\iota_{k^1} + \iota_{k^2}, c, k^1) \leq \tilde{v}_S(\iota_{k^1}, c, k^1)$, so $\mathbb{E}[\tilde{v}_S(\iota_{k^1} + \iota_{K^{2\prime}}, c, k^1)|K^2 = k^2] \leq \tilde{v}_S(\iota_{k^1}, c, k^1)$, which implies that $\mathbb{E}[\mathbb{E}[v_S^a(\iota_{K'} + \iota_{K^{1\prime}}, c, K')|K = k]|K^1 = k] \leq \mathbb{E}[v_S^a(\iota_{K'}, c, K')|K]$. □

Proof of Proposition 2. We prove the proposition in two steps. First, we establish a lemma verifying that the candidate equilibrium computed by Algorithm 2 is indeed a natural Markov-perfect equilibrium. Then, we use Lemma 4 to prove that the constructed equilibrium is essentially unique.

Lemma 7. *The strategy (α_S, α_E) and payoff function w_E constructed by Algorithm 2 form a natural Markov-perfect equilibrium.*

Proof. To prove Lemma 7, we proceed in two steps. First, note that Algorithm 2 already embodies the requirement in Definition 1, i.e., for $k^1 > k^2$, holding $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$ when computing the payoff and strategy for k^2 firm. We then need to verify that the candidate equilibrium payoff function w_E supports this heuristic, i.e., $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) > 0$ whenever $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) > 0$. To this end, we prove a sufficient condition, $w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^1) \geq w_E(\iota_{k^1} + \iota_{k^2}, \cdot, k^2)$. Second, we show that (α_S, α_E) forms a natural equilibrium by proving that it is one-shot-deviation proof.

Prove $w_E(\iota_{k^1}+\iota_{k^2},\cdot,k^1) \geq w_E(\iota_{k^1}+\iota_{k^2},\cdot,k^2)$. Note that

$$(T^\alpha g)(\iota_{k^1}+\iota_{k^2},c,k^1) = \left\{ \begin{array}{ll} (T_{k^1,k^2}f)(C), f(C) \equiv g(\iota_{k^1}+\iota_{k^2},c,k^2) & \text{if } k^1 \leq k^2 \\ (T_{k^1}f)(C,k^2), f(C,k^2) \equiv g(\iota_{k^1}+\iota_{k^2},c,k^2) & \text{if } k^1 > k^2 \end{array} \right\}.$$

Thus, T^α is exactly assembled by T_{k^1,k^2} and T_{k^1} and w_E computed by Algorithm 2 is the unique fixed point of T^α. Now consider a subspace of $\mathcal{F}$, which we denote as $\mathcal{F}^N$. In this space, any function f_E satisfies that $f \leq w_E$, $f_E(\iota_{k^1}+\iota_{k^2},\cdot,k^1) \geq f_E(\iota_{k^1}+\iota_{k^2},\cdot,k^2)$ for all $k^1 > k^2$.

We aim to prove $T^\alpha : \mathcal{F}^N \to \mathcal{F}^N$. For all $f_E \in \mathcal{F}^N$, $f_S \in \mathcal{F}^N$ as well. Consider the following cases

1. If Algorithm 2 computes $\alpha_S(\iota_{k^1}+\iota_{k^2},c,k^2) = 1$, then Algorithm 2 also prescribes $\alpha_S(\iota_{k^1}+\iota_{k^2},c,k^1) = 1$. Since both type-$k^1,k^2$ firms survive with probability one, according to Lemma 6 and Equation (15), $(T^\alpha f_E)(\iota_{k^1}+\iota_{k^2},c,k^1) \geq (T^\alpha f_E)(\iota_{k^1}+\iota_{k^2},c,k^2)$.

2. If Algorithm 2 computes $\alpha_S(\iota_{k^1}+\iota_{k^2},c,k^2) = 0$, then it must be the case that $w_E(\iota_{k^1}+\iota_{k^2},c,k^2) = 0$. For any $f_E \in \mathcal{F}$, $f_E(\iota_{k^1}+\iota_{k^2},c,k^2) \leq w(\iota_{k^1}+\iota_{k^2},c,k^2) = 0$. Since $w_E(\iota_{k^1}+\iota_{k^2},c,k^2) = (T^{\alpha\infty} f_E)(\iota_{k^1}+\iota_{k^2},c,k^2)$ and T^α is a monotone operator, $(T^\alpha f_E)(\iota_{k^1}+\iota_{k^2},c,k^2) \leq w_E(\iota_{k^1}+\iota_{k^2},c,k^2) = 0$. Thus, $(T^\alpha f_E)(\iota_{k^1}+\iota_{k^2},c,k^1) \geq 0 \geq (T^\alpha f_E)(\iota_{k^1}+\iota_{k^2},c,k^2)$.

By point-wise comparison, we conclude that $T^\alpha : \mathcal{F}^N \to \mathcal{F}^N$ hence $w_E(\iota_{k^1}+\iota_{k^2},\cdot,k^1) \geq w_E(\iota_{k^1}+\iota_{k^2},\cdot,k^2)$ for all $k^1 > k^2$. This means that whenever $w_E(\iota_{k^1}+\iota_{k^2},\cdot,k^2) > 0$, $w_E(\iota_{k^1}+\iota_{k^2},c,k^1) > 0$ as well.

Verify one-shot deviation proofness. To verify one-shot deviation proofness for α_S, we need to show that for any k^1,k^2,C, $1 \leq k^1 \leq \check{k}$ and $0 \leq k^2 \leq \check{k}$,

$$\alpha_S(\iota_{k^1}+\iota_{k^2},c,k^1) \in \arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M',c,k^1)|M=\iota_{k^1}+\iota_{k^2}] \tag{19}$$

where

$$\mathbb{E}[w_S(M',c,k^1)|M=\iota_{k^1}+\iota_{k^2}] = \begin{cases} a_S(\iota_{k^1}+\iota_{k^2},c,k^2)w_S(\iota_{k^1}+\iota_{k^2},c,k^1) & \\ +(1-a_S(\iota_{k^1}+\iota_{k^2},c,k^2))w_S(\iota_{k^1},c,k^1) & \text{if } k^1 \geq k^2 \\ w_S(\iota_{k^1}+\iota_{k^2},c,k^1) & \text{if } k^1 < k^2 \end{cases}$$

and w_S is defined analogously as v_S. To verify (19), consider the following cases

1. For all c such that $\alpha_S(\iota_{k^1}+\iota_{k^2},c,k^1) = 1$, we show that it must be the case that $\mathbb{E}[w_S(M',c,k^1)|M=\iota_{k^1}+\iota_{k^2}] > 0$.

(a) If $k^1 \leq k^2$, then $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = w_S(\iota_{k^1} + \iota_{k^2}, c, k^1)$. Because $w_E(\iota_{k^1}+\iota_{k^2}, c, k^1) = \max\{0, w_S(\iota_{k^1} + \iota_{k^2}, c, k^1)\} > 0$, then it must be that $w_S(\iota_{k^1}+\iota_{k^2}, c, k^1) > 0$ and hence $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] > 0$.

(b) If $k^1 > k^2$, then $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, \mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}]\} > 0$, so $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] > 0$.

So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = \{1\} \ni \alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 1$.

2. For all c such that $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 0$, we show that it must be the case that $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = 0$.

(a) If $k^1 < k^2$, $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = w_S(\iota_{k^1} + \iota_{k^2}, c, k^1)$. Because $w_E(\iota_{k^1} + \iota_{k^2}, c, k^1) = \max\{0, w_S(\iota_{k^1}+\iota_{k^2}, c, k^1)\} = 0$, $w_S(\iota_{k^1}+\iota_{k^2}, c, k^1) = \mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] \leq 0$.

(b) If $k^1 \geq k^2$, then in natural equilibrium it must be that $\alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^2) = 0$, and hence $\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = w_S(\iota_{k^1}, c, k^1)$. Because $w_E(\iota_{k^1}, c, k^1) = \max\{0, w_S(\iota_{k^1}, c, k^1)\} = 0$, $w_E(\iota_{k^1}+\iota_{k^2}, c, k^1) = \mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1}+\iota_{k^2}] \leq 0$.

So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] \ni \alpha_S(\iota_{k^1} + \iota_{k^2}, c, k^1) = 0$. Note that if we restrict attention to the case in which firm chooses to exit when continuation also gives zero expected payoff, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = \{0\}$.

3. For all c such that $\alpha_S(2\iota_{k^1}, c, k^1)$ is determined by (9), then $k^1 = k^2$ and

$$\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1}+\iota_{k^2}] = \alpha_S(2\iota_{k^1}, c, k^1)w_S(2\iota_{k^1}, c, k^1)+(1-\alpha_S(2\iota_{k^1}, c, k^1))w_S(\iota_{k^1}, c, k^1) = 0.$$

The last equality is due to equation (9). So, $\arg\max_{a\in[0,1]} a\mathbb{E}[w_S(M', c, k^1)|M = \iota_{k^1} + \iota_{k^2}] = [0, 1] \ni \alpha_S(2\iota_{k^1}, c, k^1)$.

To verify one-shot deviation proofness for α_E, we need to show that for any k, C, $0 \leq k \leq \check{k}$,

$$\alpha_E(\iota_k + \iota_1, C) \in \arg\max_{a\in[0,1]} a(\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] - \varphi) \tag{20}$$

where

$$\mathbb{E}[w_E(M', c, 1)|M = \iota_k+\iota_1] = \begin{cases} w_E(\iota_k + \iota_1, c, 1) & \text{if } k > 0 \\ (1 - \alpha_E(2\iota_1, c))w_E(\iota_1, c, 1) + \alpha_E(2\iota_1, c)w_E(2\iota_1, c, 1) & \text{if } k = 0 \end{cases}$$

By construction, $\alpha_E(\iota_k + \iota_1, C)$ satisfies (20) except for $k = 0$. Thus, at this moment, we can assert that $\alpha_E(\iota_k + \iota_1, C)$ is a natural Markov-perfect equilibrium strategy for $k > 0$. Since T^α does not depend on $\alpha_E(\iota_1, C)$, this is sufficient to ensure that w_E is the natural Markov-perfect equilibrium payoff corresponding to (α_S, α_E). Lemma 4 then guarantees that w_E also exhibits $w_E(2\iota_1, c, 1) \leq w_E(\iota_1, c, 1)$. Thus,

1. when $\alpha_E(2\iota_1, C) = 1$, it must be the case that $w_E(\iota_1, c, 1) \geq w_E(2\iota_1, c, 1) > \varphi$ so $\alpha_E(\iota_1, C) = 1$. The right-hand-side of (20) is

$$\arg\max_{a\in[0,1]} a(\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] - \varphi) = \arg\max_{a\in[0,1]} a(w_E(2\iota_1, c, 1) - \varphi) = \{1\} \ni \alpha_E(\iota_1, C).$$

2. when $\alpha_E(2\iota_1, C) = 0$, $\mathbb{E}[w_E(M', c, 1)|M = \iota_k + \iota_1] = w_E(\iota_1, c, 1)$. So $\alpha_E(\iota_1, C) = I(w_E(\iota_1, c, 1) > \varphi$ satisfies (20).

So we can conclude that (α_S, α_E) forms a natural Markov-perfect equilibrium with w the equilibrium payoff when firms deciding on survival. □

With Lemma 7 in hand, we can prove the uniqueness of natural Markov-perfect equilibrium following the order of Procedure 3 in Algorithm 2. When $k_1^1 = \check{k}$. Note that (16) defines the same contraction mapping as $T_{\check{k},\check{k}}$ in Algorithm 2, which means the unique $v_E^a(2\iota_{\check{k}}, \cdot, \check{k})$ for any possible (a_S, a_E) is $w_E(2\iota_{\check{k}}, \cdot, \check{k})$. Consequently, for $k < \check{k}$, $w_E(\iota_{\check{k}} + \iota_k, \cdot, k)$ is determined as the unique natural Markov-perfect equilibrium payoff. Recall that in the proof for Lemma 7, the optimal strategy sets for type-k firm are singletons if strategy defaults to inactivity. Therefore, $a_S(\iota_{\check{k}} + \iota_K, \cdot, K)$ and $a_E(\iota_{\check{k}} + \iota_1, \cdot)$ is the unique natural Markov-perfect equilibrium strategy that defaults to inactivity, which guarantees that $w_E(\iota_{\check{k}} + \iota_K, \cdot, \check{k})$ is the unique natural Markov-perfect equilibrium payoff. The uniqueness of $a_S(2\iota_{\check{k}}, \cdot, \check{k})$ is ensured by equation (9).

In the i-th steps in Algorithm 2, by equation (17), $w_E(2\iota_{k_1^i}, \cdot, k_1^i)$ is the unique natural Markov-perfect equilibrium payoff. For any $k_2^i < k_1^i$, since we have argued that $T_{k_1^i, k_2^i}$ does not depend on any strategy, $w_E(\iota_{k_1^i} + \iota_{k_2^i}, \cdot, k_2^i)$ is determined as the unique natural Markov-perfect equilibrium payoff and $a_S(\iota_{k_1^i} + \iota_{k_2^i}, \cdot, k_2^i)$, $a_E(\iota_{k_1^i} + \iota_1, \cdot)$ as the unique natural Markov-perfect equilibrium strategy that defaults to inactivity. Since $T_{k_1^i}$ only depends on natural Markov-perfect equilibrium strategy that has been verified to be unique, $w_E(\iota_{k_1^i} + \iota_{k_2^i}, \cdot, k_1^i)$ is then uniquely determined. The uniqueness of $a_S(2\iota_{k_1^i}, \cdot, k_1^i)$ is ensured by equation (9). □

C Proofs for the General Case

To prove Propositions 4 and 3, we prove a useful lemma first.

Lemma 8. *Algorithm 3 always delivers some (α_S, α_E) as outcome. Furthermore, (α_E, α_S) forms a natural Markov-perfect equilibrium, with payoffs w_E and w_S.*

Proof. We follow four steps to prove this lemma. We first show that Algorithm 3 computes w_E, α_S, and α_E for all (m, c, k). Second, we prove that Algorithm 3 always delivers some well-defined (α_S, α_E) as outcome. This is a nontrivial step. Because we need to show that

w_E as the fixed point of T always exist, and α_S is well defined when Procedure 5 assign p from equation (11) to it.

After proving the first part of the lemma, we verify in the third step that α_S satisfies the requirement in Definition 1. Eventually, we prove that in each step of Algorithm 3, w_E is constructed as an equilibrium post-entry value, and the corresponding w_S gives the equilibrium post-survival payoff. Along the way, we also show that (α_S, α_E) is an natural equilibrium strategy.

First, note that the set of all indexing market structures is $\mathbb{M} \equiv \{m \in \mathbb{Z}^{\check{k}}; 1 \leq |m| \leq \check{m}\}$, which is also the set of all payoff-relevant market structures. Consider any (m, k) pair such that $m \in \mathbb{M}$ and $m_k > 0$, $w_E(m, \cdot, k)$ is computed in the step with indexing market structure $(0, \ldots, 0, m_k, m_{k+1}, \ldots, m_{\check{k}})$. For any (m, k) pair such that $m \in \mathbb{M}$, $\alpha_S(m, \cdot, k)$ is computed in the step with indexing market structure $(0, \ldots, 0, 1, m_{k+1}, \ldots, m_{\check{k}})$. For any m such that $m \in \mathbb{M}$ and $m_1 > 0$, $\alpha_E(m, k)$ is computed in the step with indexing market structure m. Therefore, w_E, α_S, α_E for all payoff-relevant (m, c, k) are computed in Algorithm 3.

Second, because all α_S, α_E, w_E required to compute the fixed point of T in each step have been either initialized or determined in previous steps, T is always a well-defined contraction mapping with a unique fixed point. Then, w_E is always uniquely determined, as well as α_E. It remains to show that α_S is also always well-defined, in particular when it is determined from equation (11). Note that for any (m, c, k) such that $k = \underline{k}(m) \equiv \min\{j; m_j > 0\}$, when computing $w_E(m, c, k)$ using T, we always use the initialized value $\alpha_S(m, c, k^+) = 1$ for all $k^+ \geq k$, which leads to the condition that

$$w_E(m, c, k) = \max\{0, w_S(m, c, k)\}.$$

This implies that when $w_E(m, c, k) = 0$, $w_S(m, c, k) \leq 0$. In Procedure 5, when determining p using equation (11) in step i, it is the case that $w_E(m, c, k^i) = 0$ and $w_S(m, c, k^i) \leq 0$. If, in addition $w_S(m^i, c, k^i) = w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) > 0$, then the right hand side of equation (11) changes continuously from $w_S(m - (m_{k^i} - 1)\iota_{k^i}, c, k^i) > 0$ to $w_S(m, c, k) \leq 0$ when p changes from 0 to 1. This means that there exists at least one $p \in [0, 1)$ to satisfy equation (11). Considering that when $w_S(m^i, c, k^i) \leq 0$, 0 can be assigned if no p is found to satisfy equation (11), we conclude that α_S is always well-defined (although it can take multiple values if multiple p solve equation (11)).

Next, we show that α_S satisfies the requirement in Definition 1 by proving $w_E(m, c, k^1) \geq w_E(m, c, k^2)$ for all m, c and $k^1 \geq k^2$. To this end, for any computed w_E, define a functional space $\mathcal{G}^N$ containing all functions $g_E : \mathbb{M} \times [\hat{c}, \check{c}] \times \mathbb{K} \to \left[0, \frac{\beta \pi_{\check{k}}(\iota_{\check{k}})\check{c}}{1-\beta}\right]$ such that $g_E \leq w_E$, $g_E(m, c, k^1) \geq g_E(m, c, k^2)$ for all m, C and $k^1 = k^2 + 1$. We aim to prove $T : \mathcal{G}^N \to \mathcal{G}^N$.

Let g_S denote the analogous post-exit value computed by equation (2) using g_E. According to Lemma 6, for all $g_E \in \mathcal{G}^N$, $g_S \in \mathcal{G}^N$ as well. Furthermore, unless for all c such that

$Q(C' = 0|C = c) = 1$, $g_S(m, c, k^1) > g_S(m, c, k^2)$ for all m, c and $k^1 = k^2 + 1$. We assume away the existence of such degenerate distribution of C. Consider the following cases when $(Tg_E)(m, c, k^1)$ is being computed in Algorithm 3, noting that by the OLS ordering of the algorithm, at this moment, $\alpha_S(m, c, k^1)$ remains at its initial value 1 and $\alpha_S(m, c, k^2)$ has been determined in previous computation by Procedure 5.

1. If $\alpha_S(m, c, k^2) = 1$, since both type-k^1, k^2 firms survive with probability one, they expect same post-exit market structure, denoted by M_S.

$$(Tg_E)(m, c, k^1) = \mathbb{E}\left[g_S(M_S, c, k^1) \,\middle|\, M_E = m\right] > \mathbb{E}\left[g_S(M_S, c, k^2) \,\middle|\, M_E = m\right] = (Tg_E)(m, c, k^2).$$

2. If $0 < \alpha_S(m, c, k^2) < 1$, then $\alpha_S(m, c, k^2) = p$ with $p \in [0, 1)$ solving

$$\sum_{j=0}^{m_{k^2}-1} p^{m_{k^2}-1-j}(1-p)^j \binom{m_{k^2}-1}{j} g_S(m - j\iota_{k^2} - \sum_{i=1}^{k^2-1} m_i\iota_i, c, k^2) = 0.$$

   The right hand side is nothing but $(Tg_E)(m, c, k^2)$. Therefore, $(Tg_E)(m, c, k^2) = 0$. Since $g_S \in \mathcal{G}^N$, we have

$$(Tg_E)(m, c, k^1) = \max\left\{0, \sum_{j=0}^{m_{k^2}-1} p^{m_{k^2}-1-j}(1-p)^j \binom{m_{k^2}-1}{j} g_S(m - j\iota_{k^2} - \sum_{i=1}^{k^2-1} m_i\iota_i, c, k^1)\right\} > 0.$$

3. If $\alpha_S(m, c, k^2) = 0$, then $g_E(m, c, k^2) \le w_E(m, c, k^2) = 0$ for $g_E \in \mathcal{G}^N$. Since $w_E(m, c, k^2) = (T^\infty g_E)(m, c, k^2)$ and T is a monotone operator, $0 = w_E(m, c, k^2) \ge (Tg_E)(m, c, k^2)$ for all $g_E \in \mathcal{G}^N$. Thus, $(Tg_E)(m, c, k^1) \ge 0 \ge (Tg_E)(m, c, k^2)$.

By point-wise comparison, we conclude that $T : \mathcal{G}^N \to \mathcal{G}^N$, hence $w_E(m, c, k^1) \ge w_E(m, c, k^2)$ for all m, c and $k^1 = k^2 + 1$. The proof also verifies that $(Tg_E)(m, c, k^1) > 0$ whenever $\alpha_S(m, c, k^2) > 0$. Since T is a monotone operator, it means that $w_E(m, c, k^1) = (T^\infty g_E)(m, c, k^1) > 0$. Given that in Algorithm 3 α_S is set to be 1 if and only if $w_E(m, c, k^1) > 0$, $\alpha_S(m, c, k^1) = 1$ whenever $\alpha_S(m, c, k^2) > 0$. So α_S satisfies the requirement in Definition 1.

Finally, we prove that (α_S, α_E) forms an Markov-perfect equilibrium. To this end, we first show that w_E constructed by Algorithm 3 is the post-entry payoff under strategy (α_S, α_E). Then, we show that given w_E as payoff, (α_S, α_E) satisfies one-shot deviation proofness.

We begin with showing w_E is the post-entry payoff under (α_S, α_E) in the first step of Algorithm 3, where $m^1 = \check{m}\iota_{\check{k}}$. In this step, $\mathbb{H}_S^1 = \{(m^1, c, \check{k})|c \in [\hat{c}, \check{c}]\}$. When computing $w_E(H_S^1)$, we use $\alpha_S(H_S^1) = 1$ for all H_1^S, $\alpha_E(\cdot) = 0$, and $w_E(\cdot) = 0$. According to equation (1), if w_E is the post-entry payoff under strategy (α_S, α_E), then it satisfies

$$w_E(H_S^1) = \alpha_S(H_S^1)\mathbb{E}\left[w_S(M_S, c, \check{k}) \,\middle|\, M_E = m^1, C = c, K = \check{k}\right].$$

The construction of α_S in Procedure 5 implies that $\alpha_S(m^1, c, \check{k}) = 1$ if and only if $w_E(m^1, c, \check{k}) > 0$, and $\alpha_S(m^1, c, \check{k}) < 1$ if and only if $w_E(m^1, c, \check{k}) = 0$. Also, note that $\mathbb{E}\left[w_S(M_S, c, \check{k}) \,\middle|\, M_E = m^1, C = c, K = \check{k}\right] = w_S(m^1, c, \check{k})$ if $\alpha_S(m^1, c, \check{k}) = 1$. Then, the above condition for $w_E(H_S^1)$ under the constructed α_S is equivalent to

$$w_E(H_S^1) = \max\{0, I(w_E(H_S^1) > 0) w_S(H_S^1)\} = \max\{0, w_S(H_S^1)\}.$$

By setting $\alpha_S(H_S^1) = 1$, the right hand side of T is identical to this condition. Therefore, setting $\alpha_S(H_S^1) = 1$ is computationally equivalent to using $\alpha_S(H_S^1)$ determined by Procedure 5, i.e., gives the same $w_E(H_S^1)$. Also, under $\alpha_E(\cdot) = 0$, no firm will further enter. This means that $w_E(H_S^1)$ computed as the fixed point of T is the post-entry payoff under strategy (α_S, α_E).

Consequently, for all c, $w_S(m^1, c, \check{k})$ computed by equation (2) using $w_E(m^1, \cdot, \check{k})$ is the post-survival payoff under strategy (α_S, α_E).

Then suppose that the $1, \ldots, i-1$-step of Algorithm 3 have computed the $w_E(m, c, k)$ and $w_S(m, c, k)$ for all $(m, k) \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j} \times \{\underline{k}(m^j)\}$ and all c as the payoffs under (α_S, α_E). Then, Procedure 5 in the first $i-1$-step computes the following part of (α_S, α_E) for all c,

- $\alpha_S(m, c, k)$ for all $(m, k) \in \{(m, k); (m, k) \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j} \times \{\underline{k}(m^j)\}, m - n\iota_{k^i} \neq m^i, \forall n \in \mathbb{N}\}$.

- $\alpha_E(m, c)$ for all $m \in \{m; m \in \bigcup_{j=1}^{j=i-1} \mathbb{M}_{m^j}$ with $\underline{k}(m^j) = 1\}$.

Recall that $\underline{k}(m) \equiv \min\{j; m_j > 0\}$. Now, in the i-th step of the algorithm, $H_S^i \in \{(m, c, k); m \in \mathbb{M}_{m^i}, c \in [\hat{c}, \check{c}], k = k^i\}$. To make sure that $w_E(H_S^i)$ and $w_S(H_S^i)$ take their values under (α_S, α_E), we need to use in the construction of T the strategy $\alpha_S(m, \cdot, k)$ for all $m \in \mathbb{M}_{m^i}$ and k such that $m_k > 0$, $\alpha_E(n' + j\iota_1, \cdot)$ for all $j \in \mathbb{N}$ and all possible n', and $w_E(H_S^{i,\prime})$ for (m', k') such that $m' \notin \mathbb{M}_{m^i}$ and $k' \neq k^i$, conditional on type-k^i firms having positive payoff.

We check if the required values are in place.

1. From the argument for step-1 computation, the initialized value $\alpha_S(m, c, k^i) = 1$ leads to the same condition for $w_E(m, c, k^i)$ as the $\alpha_S(m, c, k^i)$ computed by Procedure 5. So, although $\alpha_S(m, c, k^i)$ has not been obtained, we can set it to 1.

2. For any (m, k^+) such that $m \in \mathbb{M}_{m^i}$ and $k^+ > k^i$, as we have shown, $\alpha_S(m, c, k^+) = 1$ conditional $w_E(m, c, k^i) > 0$, which is the same as the initialized value. For any (m, k^-) such that $m \in \mathbb{M}_{m^i}$ and $k^- < k^i$, note that $m \neq m^i$ because $m^i_{k^-} = 0$. By the definition of $\mathbb{M}_{m^i}$, for all $m \in \mathbb{M}_{m^i} \backslash \{m^i\}$, $m \succ m^i$ (so there is some $j < i$-step such that its indexing market structure $m^j = m$) and $m - b\iota_k^i \neq m^i, \forall b \in \mathbb{N}$. Therefore, $\alpha_S(m, c, k^-)$ for all $k^- < k^i$ have been computed.

3. Since according to α_S, all firms with type equal or better than k^i survive, which, together with non-regressive type evolution, implies that $n' \succeq m^i$ and $n' + b\iota_1 \succ m^i$ for all n' and all $b \in \mathbb{N}$. Therefore, for $|n' + b\iota_1| \leq \check{m}$, there is some $j < i$-th step with indexing market structure $m^j = n' + b\iota_1$. So, these $\alpha_E(n' + b\iota_1)$'s values have been computed in the j-th step by Procedure 5. For any n' such that $|n' + \iota_1| > \check{m}$, we use the initialized value $\alpha_E(n' + \iota_1) = 0$.

4. Based on the above argument, for any (m', k') following the transition governed by (α_S, α_E), $m' \succeq m^i$ and $k' \geq k^i$. If $m' \notin \mathbb{M}_{m^i}$ and $k' \neq k^i$, define $\underline{m'}(k') = (0, \ldots, 0, m'_{k'}, \ldots, m'_{\check{k}})$, the market structure which has exactly the same number of type-k' or better firms as m' does, but no type-$k'-1$ or worse firm. Then, $\underline{m'}(k') \succ m^i$, which means that there is some $j < i$-th step such that its indexing market structure $m^j = (0, \ldots, 0, m'_{k'}, \ldots, m'_{\check{k}})$. Since $m' \in \mathbb{M}_{m^j}$, $w_E(m', \cdot, k')$ is then computed in the j-th step. So, all necessary w_E's values have been computed.

Since all the required values of w_E, α_S, α_E have been obtained in earlier steps, $w_E(H_S^i)$ is computed as the payoff under (α_S, α_E), so as $w_S(H_S^i)$.

Then, we verify that (α_S, α_E) is an equilibrium strategy corresponding to w_E, w_S. To this end, we show that $\alpha_S(m, c, k)$ satisfies (4) for all (m, c, k), if all other firms follow α_S as well. For any (m, c, k), consider the following cases

1. If $w_E(m, c, k) > 0$, the algorithm sets $\alpha_S(m, c, k) = 1$. The right-hand-side of (4) is

$$\arg\max_{a \in [0,1]} a w_E(m, c, k) = \{1\} \ni \alpha_S(m, c, k).$$

2. If $w_E(m, c, k) = 0$, then the algorithm sets $\alpha_S \in [0, 1)$. Since any α_S computed by Algorithm 3 satisfies the requirement in Definition 1, it is implied that $\alpha_S(m, c, k^-) = 0$ for all $k^- < k$. Hence, $w_E(m, c, k) = w_E(m - \sum_{i=1}^{k-1} m_i, c, k) = \max\{0, w_S(m - \sum_{i=1}^{k-1} m_i, c, k)\}$ and $w_S(m - \sum_{i=1}^{k-1} m_i, c, k) \leq 0$. We look at three subcases,

   (a) If $w_S(m - (m_k - 1)\iota_k - \sum_{i=1}^{k-1} m_i \iota_i, c, k) > 0$, the algorithm sets $\alpha_S(m, c, k) = p \in [0, 1)$ to satisfy

$$\sum_{j=0}^{m_k - 1} p^{m_k - 1 - j}(1 - p)^j \binom{m_k - 1}{j} w_S(m - j\iota_k - \sum_{i=1}^{k-1} m_i, c, k) = 0,$$

   The right-hand-side of (4)

$$\arg\max_{a \in [0,1]} a \sum_{j=0}^{m_k - 1} p^{m_k - 1 - j}(1-p)^j \binom{m_k - 1}{j} w_S(m - j\iota_k - \sum_{i=1}^{k-1} m_i, c, k) = [0, 1] \ni \alpha_S(m, c, k).$$

(b) If $w_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i,c,k)>0$ and $\alpha_S(m,c,k)\in[0,1)$ solves the same polynomial as above, same result holds for $\alpha_S(m,c,k)$.

(c) If $w_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i,c,k)\le 0$ and $\alpha_S(m,c,k)=0$. All other type-k firms will exit from the market, so the right-hand-side of (4) is

$$\arg\max_{a\in[0,1]} aw_S(m-(m_k-1)\iota_k-\sum_{i=1}^{k-1} m_i,c,k)=\{0\}\ni\alpha_S(m,c,k).$$

For any (m,c,k) such that $m_k=1$, consider the following cases

1. If $w_E(m,c,k)>0$, then the algorithm sets $\alpha_S(m,c,k)=1$. The right-hand-side of (4) is $\arg\max_{a\in[0,1]} aw_S(m,c,k)=\{1\}\ni\alpha_S(m,c,k)$.

2. If $w_E(m,c,k)=0$, then α_S can not be 1. From the same argument as above, $w_E(m,c,k)=w_E(m-\sum_{i=1}^{k-1} m_i,c,k)=\max\{0,w_S(m-\sum_{i=1}^{k-1} m_i,c,k)$. So $w_S(m-\sum_{i=1}^{k-1} m_i,c,k)\le 0$ and the right-hand-side of (4) is $\arg\max_{a\in[0,1]} aw_S(m-\sum_{i=1}^{k-1} m_i,c,k)=\{0\}\ni\alpha_S(m,c,k)$.

Therefore, α_S satisfies (4). To show that α_E is an equilibrium strategy, first note that $\alpha_E(m,c)$ is determined in the step with indexing market structure m, while $w_E(m+b\iota_1,c,1)$ is computed in step with indexing market structure $m+b\iota_1$, which is (weakly) lexicographically superior than m. Therefore, $w_E(m+b\iota_1,c,1)$ has been determined as an equilibrium payoff. Then, if all potential entrants are using α_E, according to (1), post-entry payoff is $w_E(m+J\iota_1,c,1)$ where J is the largest possible number such that $w_E(m+J\iota_1,c,1)-\varphi>0$. Therefore, according to (3), α_E is the equilibrium strategy.

This completes the proof for Lemma 8. □

With Lemma 8 in hand, we proceed to prove Propositions 3 and 4.

Proof for Proposition 3. To prove this proposition, we again establish a lemma first.

Lemma 9. *If v_E is the post-entry payoff in a payoff-monotone natural Markov-perfect equilibrium, it necessarily satisfies that $v_E(m,c,k)>0$ if and only if $\mathbb{E}[v_S(M_S,c,k)|M_E=m]>0$, or*

$$v_E(m,c,k)=\max\{0,\mathbb{E}[v_S(M_S,c,k)|M_E=m]\},$$

where the expectation is computed given all equilibrium values $a_S(m,c,k^-)$ for all $k^-<k$, a tentative rule $a_S(m,c,k)=1$, and $a_S(m,c,k^+)=1$ for all $k^+>k$.

Proof. In any symmetric equilibrium, $v_E(m,c,k) > 0$ only if all firms with type k survive. In any natural equilibrium, this also implies that all firms with type k^+ survive as well. Therefore, the "only if" part is true.

The "if" part is true because (i) if $\mathbb{E}[v_S(M_S,c,k)|M_E = m] > 0$ and $a_S(m,c,k^-) > 0$, then it must be the case that in natural equilibrium $v_E(m,c,k^-) > 0$. Also, according to Definition 1, $a_S(m,c,k) = 1$ and $a_S(m,c,k^+) = 1$. Then, $v_E(m,c,k) \geq v_E(m,c,k^-) > 0$; (ii) if $\mathbb{E}[v_S(M_S,c,k)|M_E = m] > 0$ and $a_S(m,c,k^-) = 0$ for all $k^- < k$, then $\mathbb{E}[v_S(M_S,c,k)|M_E = m] = v_S(m - \sum_{i=1}^{k-1} m_i, c, k) > 0$. Recall that we have shown in the proof for Proposition 1 that in the homogenous oligopoly model, Lemma 2 ensures that $v_E(m,c) > 0$ if and only if $v_S(m,c) > 0$. Applying an analogous reasoning, we know that $v_E(m,c,k) > 0$ if $v_S(m - \sum_{i=1}^{k-1} m_i, c, k) > 0$. □

Lemma 9 gives a necessary condition for the post-entry payoff in a payoff-monotone natural Markov-perfect equilibrium. For $v_E(m^1,c,\check{k})$, where $m^1 = \check{m}\iota_{\check{k}}$ is the indexing market structure in the first step of Algorithm 3, this condition can be written as

$$v_E(m^1,c,\check{k}) = \max\{0, v_S(m^1,c,\check{k})\}.$$

In the first step of Algorithm 3, $w_E(m^1,c,\check{k})$ is uniquely computed by the contraction mapping generated by the above condition. Thus, it is the only payoff function satisfying the necessary condition for a payoff-monotone equilibrium. Providing that such equilibrium exists, its post-entry payoff $v_E(m^1,c,\check{k})$ has a unique value $w_E(m^1,c,\check{k})$ for all c. Also, $v_S(m^1,c,\check{k}) = w_S(m^1,c,\check{k})$ for all c.

In any succeeding step i of Algorithm 3, with α_S either properly initialized or computed in the previous steps as its equilibrium value (this is shown in Lemma 8), $w_E(m,c,k^i)$ is computed as the unique payoff *under* (α_S,α_E) that satisfies such necessary condition for all c and all $m \in \mathbb{M}_{m^i}$.

Moreover, the (α_S,α_E) constructed in Procedure 5 is also the unique equilibrium strategy given that the w_E, w_S computed in previous steps are unique equilibrium payoffs v_E, v_S. α_E's uniqueness trivially follows its construction. The uniqueness of α_S is due to the monotonicity of (the previously computed part of) w_S: When using (11) to compute the mixing probability p, because $w_S(m - (m_{k^i}-1)\iota_{k^i}, c, k^i) \geq w_S(m - (m_{k^i}-2)\iota_{k^i}, c, k^i) \geq \ldots \geq w_S(m,c,k^i)$, the right hand side (11) changes continuously and *monotonically* from $w_S(m-(m_{k^i}-1)\iota_{k^i}, c, k^i) > 0$ to $w_S(m,c,k^i) \leq 0$ when p changes from 0 to 1. Therefore, there is only one $p \in [0,1)$ that satisfies (11). So, α_S is single valued.

Therefore, if there exists a payoff-monotone equilibrium, (α_S,α_E) forms the unique equilibrium and w_E and w_S are the unique equilibrium payoffs. The equilibrium is subsequently unique. □

Proof for Proposition 4. First, note that from the definition of a renegotiation-proof natural Markov-perfect equilibrium, all firms with a same type survive for sure *if and only if* joint continuation gives them positive post-survival payoff. This implies that (i) any such equilibrium's post-entry equilibrium payoff must satisfy the condition in Lemma 9; (ii) if any natural Markov-perfect equilibrium's post-entry payoff satisfies the condition in Lemma 9, such equilibrium is renegotiation-proof.

Since we have shown in Lemma 8 that Algorithm 3 always gives some (α_S, α_E) to form a natural Markov-perfect equilibrium. We have also shown in the proof for Proposition 3 that w_E satisfies the necessary condition in Lemma 9. Therefore, (α_S, α_E) forms a renegotiation-proof natural Markov-perfect equilibrium. □

D An Example of Multiple Equilibria

We construct a three-firm two-type example where the equilibrium payoff is not weakly decreasing in the number of same-type competitors ($\check{m} = 3$ (by setting $\pi_k(n) = 0$ for any n with more than 3 firms) and $\check{k} = 2$).

Consider the following sequence of c_t: $c_1 = 1, c_2 = 1e^{-6}, c_t = 5$, for all $t \geq 3$. The number of consumers drops to nearly zero in the second period but is boosted to a high level in the third period, and stays high afterwards. We set $\beta = 0.5, \kappa = 4, \varphi = 1$, and $\Pi_{\mathcal{LH}} = \Pi_{\mathcal{LL}} = 0.5$. Some parts of the producer surplus π are summarized in the following table:

$\pi_{\mathcal{L}}/\pi_{\mathcal{H}}$	$1\iota_{\mathcal{H}}$	$2\iota_{\mathcal{H}}$	$3\iota_{\mathcal{H}}$
$0\iota_{\mathcal{L}}$	/102	/100	/1
$1\iota_{\mathcal{L}}$	99/101	0.89/1.24	
$2\iota_{\mathcal{L}}$	1.23/1.25		

One feature of this surplus structure is that a duopoly market promises much higher per consumer surplus than a triopoly market does. The duopoly-triopoly surplus difference overwhelmingly dominates the $\mathcal{L}, \mathcal{H}$-type difference in surplus. Since from period 3 onwards, the model is essentially an infinitely repeated game, we can use backward induction to compute the equilibrium payoffs[12]. Unlike the results stated in Lemma 2 and 4, v_S is not always monotonic in the number of firms,

$$v_S(\iota_{\mathcal{H}}, c_1, \mathcal{H}) = -1.1821, \quad v_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 246, \quad v_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) = -1.5$$

Not surprisingly, the low triopoly surplus implies a low payoff if continuing as one of three type-$\mathcal{H}$ firms. What is counter-intuitive is that the payoff to continuing as a duopolist is better than the payoff to continuing as a monopolist under c_1. This is because in period 2,

[12] We provide the details of such computation in an online appendix.

under a low c_2, a duopoly firm and a monopoly firm make similar flow profit and similar large losses. However, duopoly firms can, by jointly remaining active, preempt any further entrants and enjoy a high duopoly profit after demand increases to a high level in period 3. The future duopoly payoff compensates the loss in period 2, and make $v_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H})$ positive. In contrast, a monopoly market attracts two entrants for sure in period 2, which results in a triopoly market from period 3 onwards. Because demand increases to a high level in period 3, none of these firms will exit and, given the per consumer surplus structure, they will all earn a substantially lower payoff than duopoly firms, which can not compensate the loss in period 2. Consequently, $v_S(\iota_{\mathcal{H}}, c_1, \mathcal{H})$ is negative.

Given the computed non-monotone equilibrium payoff, (a_S, a_E) with $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 1$ is still an natural equilibrium. However, if one duopoly firm chooses to exit with probability 1, the rival firm receives -1.1821 if continuing alone and hence will choose to exit with probability 1 as well. Similarly, if one firm chooses to survive with probability $\frac{-1.1821}{-1.1821-246} = 4.782e^{-3}$, the other firm is indifferent between exiting and survival. Therefore, two other natural equilibria with $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ and $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 4.782e^{-3}$ exist.

Note that when both firms choose $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ or $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 4.782e^{-3}$, they receive zero payoffs. By "renegotiating" on jointly choosing $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 1$, they can strictly improve their equilibrium payoffs. Henceforth, we only restrict attention to equilibria in which there is no room for this type of one-shot joint improvement.

Unfortunately, this type of equilibria may not be unique. Since joint continuation and continuing as monopolist both render payoffs negative, in a one-shot renegotiation-proof equilibrium, a triopoly firm either chooses $a_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0$ or set $a_S(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) = p$, where p solves

$$p^2\bar{v}(3\iota_{\mathcal{H}}, c_1, \mathcal{H}) + \binom{2}{1}p(1-p)\bar{v}(2\iota_{\mathcal{H}}, c_1, \mathcal{H}) + (1-p)^2\bar{v}(\iota_{\mathcal{H}}, c_1, \mathcal{H}) = 0.$$

This quadratic equation has two roots, $p_1 = 0.0024$ and $p_2 = 0.997$, both between 0 and 1. Hence, there are three one-shot renegotiation-proof equilibria with $a_S(3\iota_{\mathcal{H}}, c_1, \mathcal{H})$ equal to 0, p_1 and p_2, respectively.