Dynamic Price Competition with Network Effects

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preliminary and incomplete

Abstract

I consider a dynamic model of competition between two proprietary networks. Consumers die with a constant hazard rate and are replaced by new consumers. Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference shock for each of the networks. Upon joining a network, in each period consumers enjoy a benefit which is increasing in network size. Firms receive revenues from new consumers joining the network (possibly negative revenues) as well as from consumers already belonging to its network.

I discuss various properties of the equilibrium, including the pricing function, the system's expected motion, the stationary distribution of market shares, and how welfare depends on market structure. Finally, I use the model to estimate to what extent network externalities create entry barriers. I do this both in terms of the value loss for an entrant and the expected time for an entrant to reach a certain market share.

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1 Introduction

Many industries exhibit some form of network externalities, the situation whereby a consumer's valuation is increasing in the number of other consumers in the same network. There are several possible sources of network effects. The most obvious one is direct network effects. Take the example of operating systems. If I use the Windows OS then, when I travel, it is more likely I will find a computer that I can use (both in terms of knowing how to use and in terms of being able to run files and programs I carry with me).[1]

A second source of network effects is the availability of complementary products. For example, it seems reasonable to assume that the variety and quality of the software available for the Palm system is greater the more users buy PDAs that us the Palm system. A similar argument applies for complementary services. For example, the greater the number of Cannon photocopiers sold, the more likely it is I will be able to find good post-sale service providers.

Finally, a third source of network effects is the pricing of network services.[2] Take the example of wireless telecommunications. To the extent that operators set different on-net and off-net prices, the utility from being connected to a given network is increasing in the number of other users on the same network.

In this paper, I consider a dynamic model of competition between two proprietary networks. Consumers die with a constant hazard rate and are replaced by new consumers.[3] Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference shock for each of the networks. Upon joining a network, in each period the consumers enjoy a benefit which is increasing in network size. Firms receive revenues from new consumers joining the network (possibly negative revenues) as well as from consumers already belonging to its network.

I develop a general model with the above features. I derive the firms' value functions as well as the consumer value functions of joining each of the networks. I show an equilibrium exists. In some particular cases, I also show uniqueness. The model does not have, in general, a closed-form analytical solution. Whenever needed, I compute the equilibrium numerically.

[1]Another source of direct network effects would be file sharing. While this is frequently proposed as the main source of direct network effects, in the example at hand I think this is relatively less important.

[2]Laffont, Rey and Tirole (1998a) refer to this case as "tariff-mediated network externalities."

[3]In macroeconomics, this is known as an overlapping families model.

I discuss various properties of the equilibrium, including the pricing function, the system's expected motion, the stationary distribution of market shares, and how welfare depends on market structure. Finally, I use the model to estimate to what extent network externalities create entry barriers. I do this both in terms of the value loss for an entrant and the expected time for an entrant to reach a certain market share.

To be completed.

■ **Related literature.** To be completed.

2 Model

Consider a set of N consumers who are divided between two networks, which I denote by i and j. Although I will be working with a discrete time model, the underlying reality I have in mind is one of continuous time. Suppose that consumers "die" according to a Poisson process with rate λ. Whenever a consumer dies a new consumer is born as a replacement. The new consumer chooses one of the existing networks and sticks to that network until death. An alternative interpretation of the model is that consumers normally have very high costs of switching between networks. At specific moments in time, which arrive according to a Poisson process, a consumer's switching cost drops to zero, allowing for a reassessment of the choice between networks.

Figure 1 shows the model's structure in continuous time. I will consider a reduced form of this continuous time model. Essentially, I consider the time between two consecutive deaths as a period in my discrete time model. By assuming risk-neutral agents, I can summarize the Poisson arrival process in a discount factor δ that reflects the average length of a discrete period: $\delta = \exp(-r/\lambda)$, where r is the continuous time discount rate and λ the Poisson arrival rate of a consumer death.

I consider an infinite period of price competition between two proprietary networks, owned by firms A and B. Since I am looking for symmetric equilibria, I will denote each firm by the size of their network, i or j. The timing of moves in each period is summarized in Table 1. Initially, a total of $N-1$ consumers are distributed between the two firms, so $i+j = N-1$. A new consumer is born and firms simultaneously set prices $p(i)$, $p(j)$ for the consumer to join their network. After the new consumer makes his choice, there are a total of N consumers distributed across the networks.

During the period, network i receives a payoff $\pi(i)$ and consumers enjoy a

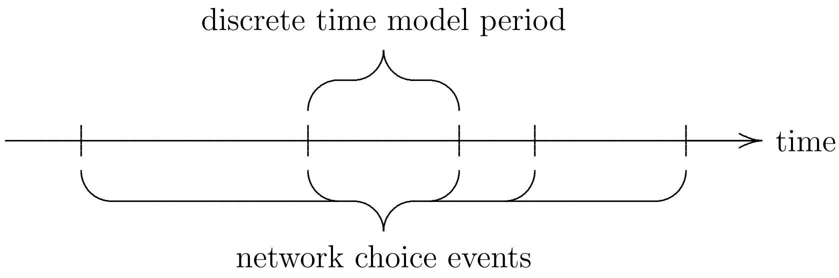

Figure 1: Continuous time and discrete time model.

Table 1: Timing of model: events occurring in each period t.

Event	Value functions	State of the game
Firms set network entry prices $p(i)$	$v(i)$	$i \in \{0, \ldots, N-1\}$
Nature chooses x_i, new consumer's preference for network i		
New consumer chooses network	$u(i)$	$i \in \{0, \ldots, N\}$
Stage competition takes place: period profits $\pi(i)$, consumer surplus $\lambda(i)$		
One consumer dies (probability $\frac{1}{N}$)		$i \in \{0, \ldots, N-1\}$

benefit $\lambda(i)$. (Note that firm profits and benefits do not include profits and consumer benefits from initially joining the network.) Finally, at the end of the period one consumer dies (each with equal probability).

The state is defined by i, the size of firm i's network at the beginning of the period. Note that, when consumers enjoy network benefits, there are a total of N consumers, divided between the two networks. However, at the time that prices are set there are only $N-1$ consumers, so $i \in \{0, \ldots, N-1\}$ at that moment. The state and the firm value functions are defined at this moment (beginning of period).

Each consumer's utility is given by two components: $\tilde{x}_i$ and $\lambda(i)$. The first component is the consumer's idiosyncratic preference for firm i. The second component is network benefit from a network with size i (*including* the consumer in question). I assume that consumers receive the $\tilde{x}_i$ component the moment they join a network, whereas $\lambda(i)$ is received each period that a consumer is still alive.

For simplicity, I measure network i idiosyncratic preference as a differential

Table 2: Notation.

N	Market size (number of consumers).
i	Firm i's network size.
$p(i)$	Price in state i (for new consumer).
$q(i)$	Probability of a sale in state i (to new consumer).
$u(i)$	Consumer's value in state i.
$v(i)$	Firm's value in state i.
x_i	Consumers's idiosyncratic preference for firm i.
$\lambda(i)$	Consumer's stage network benefit in state i.
$\pi(i)$	Firm's stage profit in state i.
ϕ	Network externality parameter.
σ	Product differentiation parameter.
α	Market share.
δ	Discount factor.

with respect to network j, so $\tilde{x}_i = -\tilde{x}_j$. I assume that $\tilde{x}_i$ is distributed according to $F(x)$, which satisfies the following properties:

Assumption 1 *(i) $F(x)$ is continuously differentiable; (ii) $f(x) = f(-x)$; (iii) $f(x) > 0$, $\forall x$; (iv) $F(x)/f(x)$ is strictly increasing.*

Many common distributions, including the Normal, satisfy Assumption 1. A summary of the model's notation is given in Table 2.

Let $u(i)$ be a consumer's "post-birth" value function, that is, excluding the $\tilde{x}$ component, which I assume is received at the time a consumer joins the network.[4] In other words, $u(i)$ measures, at the beginning of the period, the consumer's discounted expected stream of benefits $\lambda(i)$, where δ is the discount factor.

Consider a new consumer's decision. At state i, the indifferent consumer will have $\tilde{x} = x(i)$, where the latter is given by

$$x(i) - p(i) + u(i+1) = -p(j) + u(j+1), \tag{1}$$

or simply

$$x(i) = p(i) - p(j) - u(i+1) + u(j+1). \tag{2}$$

[4] The assumption that $\tilde{x}$ is received at birth is not important. I could have the consumer receive x each period of his or her lifetime. However, this way of accounting for consumer utility simplifies the calculations.

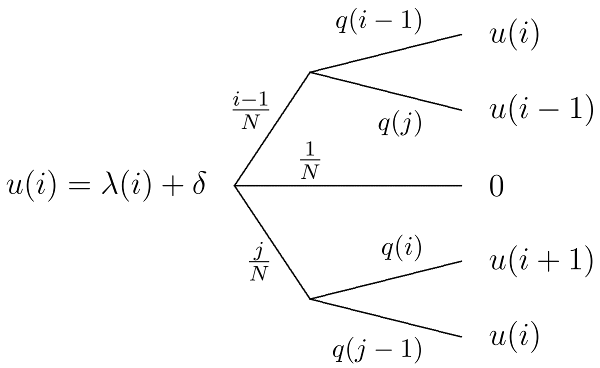

Figure 2: Consumer's value function.

where $p(i)$ is firm i's price. Firm i's demand is the probability of attracting the new consumer to its network. It is given by

$$q(i) = 1 - F\left(x(i)\right) = F\left(-x(i)\right). \tag{3}$$

The consumer value functions, introduced above, are illustrated in Figure 2. The corresponding formula is given by

$$u(i) = \lambda(i) + \delta \left(\frac{j}{N}\, q(i)\, u(i+1) + \left(\frac{j}{N}\, q(j-1) + \frac{i-1}{N}\, q(i-1) \right) u(i) + \right.$$
$$\left. + \frac{i-1}{N}\, q(j)\, u(i-1) \right), \tag{4}$$

where $q(i)$ is given by (3), $i = 1, \ldots, N$, and $j = N - i$. Recall that the argument of u includes the network adopter to whom the value function applies, thus i must be strictly positive in order for the value function to apply.[5]

The firm's value functions are given by

$$v(i) = q(i) \left(p(i) + \pi(i+1) + \delta\, \frac{j}{N}\, v(i+1) + \delta\, \frac{i+1}{N}\, v(i) \right)$$
$$+ \left(1 - q(i)\right) \left(\pi(i) + \delta\, \frac{j+1}{N}\, v(i) + \delta\, \frac{i}{N}\, v(i-1) \right), \tag{5}$$

where $i = 0, \ldots, N-1$ and $j = N - 1 - i$.[6]

[5]Notice that, for the extreme values $i = 1$ and $i = N$, (4) calls for values of $q(\cdot)$ and $u(\cdot)$ that are not defined. However, these values are multiplied by zero.

[6]Again, notice that, for the extreme case $i = 0$, (11) calls for values of $v(\cdot)$ which are not defined. However, these values are multiplied by zero.

Equation (11) leads to the following first-order conditions for firm value maximization:

$$\begin{aligned} q(i) + \frac{\partial\, q(i)}{\partial\, p(i)} \Big(p(i) + \pi(i+1) - \pi(i) \\ + \delta\, \frac{j}{N}\, v(i+1) + \delta\, \frac{i+1}{N}\, v(i) \\ - \delta\, \frac{j+1}{N}\, v(i) - \delta\, \frac{i}{N}\, v(i-1) \Big) = 0, \end{aligned}$$

or simply

$$p(i) = h(i) - w(i), \tag{6}$$

where

$$h(i) \equiv -\frac{q(i)}{q'(i)} = \frac{F\big(-x(i)\big)}{f\big(-x(i)\big)} \tag{7}$$

$$w(i) \equiv \big(\pi(i+1) - \pi(i)\big) + \delta \left(\frac{j}{N}\, v(i+1) + \frac{i-j}{N}\, v(i) - \frac{i}{N}\, v(i-1) \right) \tag{8}$$

Finally, substituting (6) into (11) and simplifying, we get

$$v(i) = r(i) + \pi(i) + \delta \left(\frac{j+1}{N}\, v(i) + \frac{i}{N}\, v(i-1) \right), \tag{9}$$

where

$$r(i) \equiv F\big(-x(i)\big)\; h(i). \tag{10}$$

This is a recursive system, the solution of which is given by

$$v(i) = \left(1 - \delta\, \frac{N-i}{N} \right)^{-1} \left(r(i) + \pi(i) + \delta\, \frac{i}{N}\, v(i-1) \right), \tag{11}$$

$i = 0, \ldots, N-1$.

3 Equilibrium

In this section, I deal with issues of existence and uniqueness. The proofs of all results are presented in the Appendix. Notice that some of the results below are still missing complete proofs. I am also considering some generalizations of the $N = 2$ and $N = 3$ results.

Proposition 1 *If $N = 2$, then there exists a unique equilibrium, which is symmetric.*

Proposition 2 *If $N = 3$, then there exists a unique symmetric equilibrium.*

Now for general N results.

Lemma 1 *Given $u(i)$ and $v(i)$, there exist unique values $p(i)$ and $q(i)$ solving the sellers' optimality conditions.*

Corollary 1 *If δ is sufficiently small, then there exists a unique equilibrium.*

Proposition 3 *A Markov equilibrium exits.*

4 Solution and computation

The above dynamic system has no closed form solution (except for the cases $N = 2, 3$). In this section I describe the process of numerical computation of equilibrium values.

The exogenous parameters of the model correspond to the stage game: profit functions $\pi(i)$ and consumer surplus functions $\lambda(i)$. In addition, we have the parameter of network preference, σ, as well as the discount factor, δ. (I use Greek letters for exogenous parameters and functions.)

The endogenous variables are the price functions $p(i)$, the demand functions $q(i)$, the firm value functions $v(i)$, and the consumer value functions $u(i)$. (I use Roman letters for endogenous values.)

In order to obtain an equilibrium solution, I follow a Gaussian method similar to that proposed by Pakes and McGuire (1994):[7]

1. Start with "naïve" consumer and firm value functions: $u_0(i) = \lambda(i)/(1-\delta)$ and $v_0(i) = \pi(i)/(1-\delta)$.

2. At iteration $t = 1, \ldots$, use (6) to compute equilibrium prices and (3) to compute equilibrium demand given the latest estimate of u and v, that is, $u_{t-1}(i)$ and $v_{t-1}(i)$.

3. Use equilibrium prices and demands to compute the value functions at iteration t. The consumer value functions, $u_t(i)$ are obtained from (4). The firm value functions, $v_t(i)$, are obtained from (**??**).

[7]See also

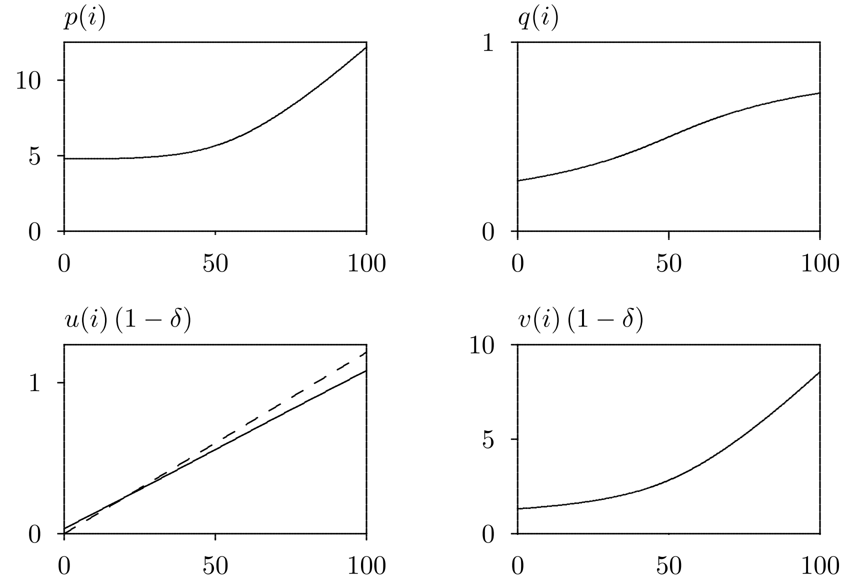

Figure 3: Equilibrium values as a function of state (horizontal axis) when F is normal with standard deviation σ and assuming $\pi(i) = 0,\ \lambda(i) = \phi\, i/N,\ \phi = 1.2,\ \sigma = 5,\ \delta = .9,\ N = 101$. Consumer and firm value functions are expressed in terms of per-period payoffs.

4. Compute the distance $\xi \equiv \sum_{i=1}^{N} \left| u_t(i) - u_{t-1}(i) \right| + \sum_{i=0}^{N-1} \left| v_t(i) - v_{t-1}(i) \right|$. Return to Step 2 until $\xi < \xi'$.[8]

From Proposition 1, (6) yields a unique solution $p(i)$ given the value functions $u(i), v(i)$; consequently, we also get a unique solution $q(i)$ given $u(i)$ and $v(i)$. Moreover, the systems determining $u(i), v(i)$ are linear, and so given $p(i), q(i)$ we get unique values of $u(i)$ and $v(i)$. It follows that the above process yields a unique solution at each iteration. Therefore, insofar as the sequence of p, q, u, v converges, we obtain a specific Nash equilibrium of the game.

■ **Example.** A solution consists of four primary outputs: the mappings $p(i)$ and $q(i)$, which give equilibrium price and probability of a sale at each state $i = 0, \ldots, N-1$; the firm value functions $v(i),\ i = 0, \ldots, N-1$; and the consumer value functions $u(i),\ i = i, \ldots, N$. Figure 3 shows these four mappings for the particular case when $\pi(i) = 0,\ \lambda(i) = \phi\, i/N,\ \phi = 1.2,\ \sigma = 5$, and $\delta = .9$.

■ **Transition matrix.** Given the equilibrium values of $q(i)$, I can compute

[8] I have also experimented with computing a moving average of values of u and v from previous iterations. This alternative procedure improved the speed of convergence for some parameter values.

a Markov transition matrix $M = m(i,j)$ where $m(i,j)$ is the probability of moving from state i to state j. For $0 < i < N-1$, we have

$$\begin{aligned} m(i,i-1) &= \frac{i}{N}\Big(1-q(i)\Big) \\ m(i,i) &= \frac{i+1}{N}\,q(i) + \frac{N-i}{N}\Big(1-q(i)\Big) \\ m(i,i+1) &= \frac{N-1-i}{N}\,q(i) \end{aligned}$$

Moreover, $m(i,j) = 0$ if $j < i-1$ or $j > i+1$. Finally, the boundary values are obtained as follows. For $i = 0$, I apply the general equations and add the value obtained for $m(0,-1)$ to the value of $m(0,0)$. For $i = N-1$, again I apply the general equations and add the value obtained for $m(N-1,N)$ to $m(N-1,N-1)$. As a result, I get

$$\begin{aligned} m(0,0) &= 1 - \frac{N-1}{N}\,q(0) \\ m(0,1) &= \frac{N-1}{N}\,q(0) \\ m(N-1,N-2) &= \frac{N-1}{N}\Big(1-q(N-1)\Big) \\ m(N-1,N-1) &= 1 - \frac{N-1}{N}\Big(1-q(N-1)\Big) \end{aligned}$$

■ **Steady state.** Given the assumption that $F(\cdot)$ has full support (Assumption 1), $q(i) \in (0,1)\ \forall i$, that is, there are no corner solutions in the pricing stage. It follows that the Markov process is ergodic and I can compute the stationary distribution over states, that is, over market shares. This is given by the (transposed) vector s that solves $s\,M = s$. This vector s can also be computed by repeatedly multiplying M by itself. That is, $\lim_{k\to\infty} M^k$ is a matrix with s in every row. Figure 6 shows the stationary distribution for the same equilibrium as that considered in Figure 3.

5 Discussion

This section is still work in progress. I am interested in studying the properties of the equilibrium (policy function, value functions, stationary distribution over states) for different values of the network externality parameter, for different functional forms of the π and λ functions, for different values of the

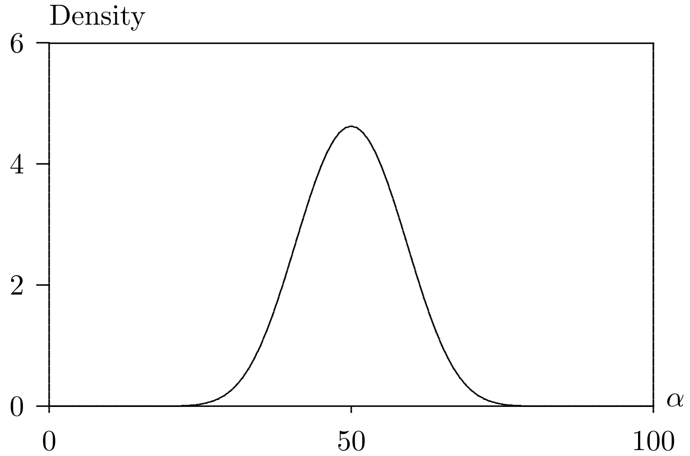

Figure 4: Stationary distribution of market share, α (assuming $\pi(i) = 0$, $\lambda(i) = \phi\, i/N$, $\phi = 1.2$, $\sigma = 10$, $\delta = .9$, $N = 101$).

discount factor. I am also interested in looking at the dynamic properties of the equilibrium, e.g., average time to transit from state i to state j.

To be completed.

■ **Equilibrium prices.** In the example I considered above, the pricing function is non-monotonic in i. A firm with a small, positive market share sets a price higher than a firm with zero market share. But a firm with a high market share sets a price higher than a firm with low market share. The intuition is that there are three main effects determining optimal price: current profits and future profits. In terms of current profits, a larger firm has a a greater market power (consumers are willing to pay more), and this is reflected in a higher price. In terms of future prices, the value of winning a same is roughly proportional to the slope of the value function, which is increasing in market share.

■ **Consumer value function.** In Figure 3, in addition to $(1-\delta)\, u(\alpha)$, I plot short-run consumer payoff, $\lambda(\alpha)$ (dashed line). Notice that $(1-\delta)\, u$ is flatter than λ. This results from the consumers being forward looking. When $\alpha = 0$, consumers expect the firm will increase in size, and so $(1-\delta)\, u$ is greater than λ. If $\alpha = 1$, however, then consumers expect the network to decrease in size, and so $(1-\delta)\, u$ is lower than λ.

■ **Social welfare.** This section is work in progress. Figure 3 suggests that joint profits are increasing in market share asymmetry (that is, the value

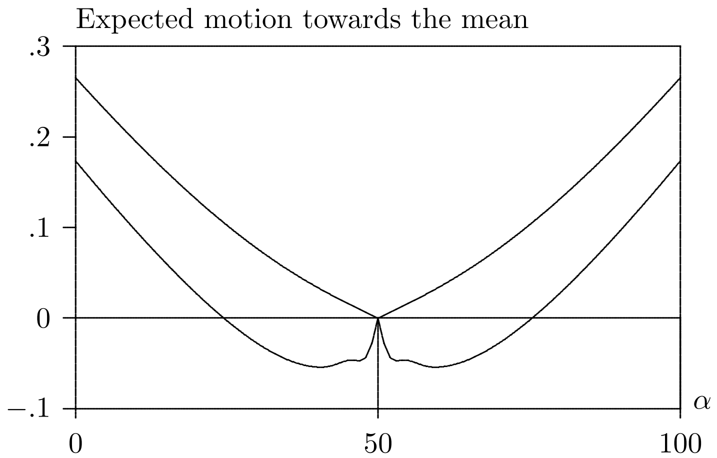

Figure 5: Expected motion towards the mean for $\phi = 1.2$ (higher mapping) and $\phi = 2.4$ (lower mapping). (As before, I assume $\pi(i) = 0$, $\lambda(i) = \phi\, i/N$, $\sigma = 5$, $\delta = .9$, $N = 101$.) Negative values indicate motion away from the mean.

function is convex). An increasing u function also implies that post entry consumer welfare is increasing in market share asymmetry. However, for the cases I have considered the net welfare of a new consumer is greater the closer together the firms are. This is because the price function is so convex. In fact, price is typically lowest for equal market shares.

■ **Increasing dominance.** Figure 3 shows that the firm with a greater market share is more likely to attract the next new consumer. In fact, that probability is increasing in market share, a property that Cabral and Riordan (1994) refer to as increasing increasing dominance. Notice however that a larger network also has a greater death rate. What is the net effect? Figure 5 shows the system's expected motion. A positive value indicates a movement towards the mean market share (50%). A negative value denotes a movement away from the mean. For a relatively lower value of ϕ (weak network effects), we always have reversion to the mean. For higher values of ϕ, however, the system moves away from the mean for value of α closer to $\frac{1}{2}$ and towards the mean for extreme value of α (that is, close to 0 or close to 1).

■ **Symmetric equilibria and asymmetric outcomes.** The fact that I consider a unique, symmetric equilibrium, does not preclude outcomes from being rather asymmetric. Figure 6 shows the stationary distribution of market shares for a series of values of ϕ, a parameter which reflects the size of network effects. The stationary distribution over states, while symmetric around 50%,

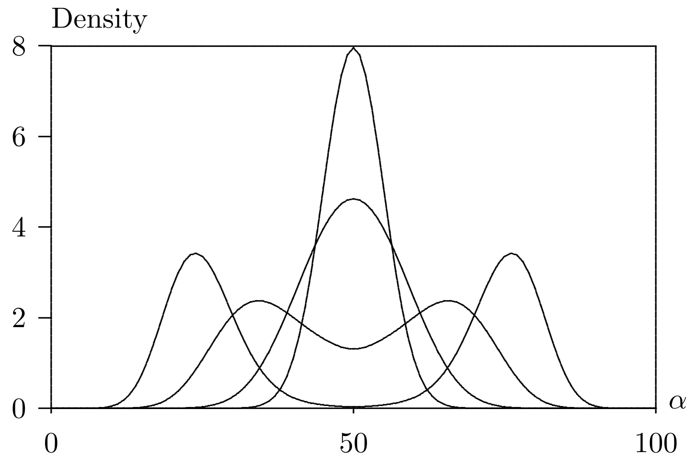

Figure 6: Network externalities and market shares: stationary distribution for values of $\phi = 0,\ 1.2,\ 1.8,\ 2.4$. (assuming $\pi(i) = 0,\ \lambda(i) = \phi\, i/N,\ \sigma = 5,\ \delta = .9,\ N = 101$).

may have a lot of variation. In fact, for sufficiently high ϕ, the distribution is bi-modal: although the average market share is 50%, the system spends most of the time at relatively asymmetric market shares.

6 Network effects as a barriers to entry

Among the many possible applications of my framework, I am particularly interested in addressing questions that can only be answered with a dynamic model like the one I consider here. One such question is the size of the entry barrier implied by network effects. There are several ways by which to measure barriers to entry. Table 3 suggests a few measures. For a series of values of ϕ, I first computed firm values for 0, 50, and 100% market shares. I then consider the following measures: the percent value difference between an entrant and a firm with 50% market share; the percent value difference between an entrant and the incumbent (that is, a firm with 100% market share; and the percent value difference between an entrant with $\phi \geq 0$ and an entrant with $\phi = 0$, that is, with no network effects. (The latter measure probably best captures the idea of network effects as a barrier to entry.)

An alternative way of measuring the difficulty to enter is to compute the average time it takes for an entrant to achieve a market share of α. Figure 7 shows expected time as a function of α, for various values of ϕ. The qualitative conclusion from this figure is that network effects don't affect much the time it

Table 3: Network effects as a barrier to entry.

ϕ	0	1.2	1.8	2.4
$v(0)$	1.566	1.320	1.061	0.920
$v(50)$	1.566	2.834	2.642	2.384
$v(100)$	1.566	8.575	12.790	17.416
$\frac{v(50)-v(0)}{v(50)}$ (%)	0	53.42	59.84	61.41
$\frac{v(100)-v(0)}{v(100)}$ (%)	0	84.61	91.70	94.72
$\frac{v(0)\|_{\phi=0}-v(0)}{v(0)\|_{\phi=0}}$ (%)	0	15.71	38.26	60.89

takes for an entrant to reach a low market share (say, 10%), but do significantly affect the time required for an entrant to achieve a 50% market share.

To see this, consider the extreme cases of network effects I have been considering, $\phi = 0$ and $\phi = 2.4$. It takes an average of 25 periods for an entrant to achieve a 10% market share if there are no effects. With strong network effects, $\phi = 2.4$, it takes three times as long for the entrant to achieve the same market share (3.32, to be more precise). If however we consider how long it takes to get to a 50% market share, then the different between $\phi = 0$ and $\phi = 2.4$ is a factor of 123 (the ratio of 40,116 and 324)!

To be completed. . . .

7 Conclusion

There are many other potential applications of my framework. I am particularly interested in answering important questions that can only be answered with a dynamic model like mine. Examples include

- Look at the extent to which network sizes are socially optimal. Compare with Mitchell and Skrzypacz (2006).

- Introductory pricing. I have assumed that prices can be negative. However, imposing the constraint that $p(i) \geq 0$, we may find equilibria where equilibrium prices are $p(i) = 0$ for $i < i'$. (This is just a conjecture.) Compare with Radner and Sundararanjan (2006).

- Bertrand supertraps — how $v(N/2)$ changes with ϕ. Compare with Cabral and Riordan (1994), Cabral and Villas-Boas (2005).

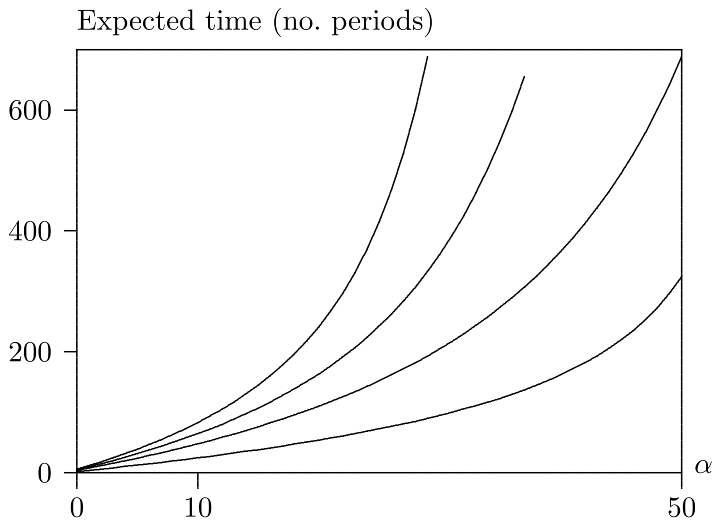

Figure 7: Expected time for an entrant to reach market share α, for values of $\phi = 0,\ 1.2,\ 1.8,\ 2.4$. (assuming $\pi(i) = 0,\ \lambda(i) = \phi\, i/N,\ \sigma = 5,\ \delta = .9,\ N = 101$).

- Aftermarkets. What is the difference between the situation where the network proprietor monopolizes the aftermarket (the market that leads to the payoff functions $\pi(i)$ and $\lambda(i)$) and the situation where the network proprietor invites competition in the after market.

Appendix

Proof of Lemma 1: Subtracting (6) from firm j's first-order conditions, I get

$$P(i) = H(i) - W(i) \tag{12}$$

where

$$\begin{aligned} P(i) &\equiv p(i) - p(j) && (13) \\ H(i) &\equiv h(i) - h(j) && (14) \\ W(i) &\equiv w(i) - w(j) && (15) \end{aligned}$$

and $j = N - 1 - i$.

Notice that $W(i)$ is only a function of $v(k)$ (various values of k). $H(i)$, in turn, is a function of $u(k)$ (various values of k); and of $x(i)$, which in turn is a function of $P(i)$. Specifically, from (2) we conclude that $x(i)$ is increasing in $P(i)$. From (7) and Assumption 1 we conclude that $h(i)$ is decreasing in $P(i)$. Since $h(j)$ is decreasing in $P(j)$ and $P(j) = -P(i)$, it follows that $h(j)$ is increasing in $P(i)$. Consequently, $H(i) \equiv h(i) - h(j)$ is decreasing in $P(i)$.

We can now rewrite (12) as $P(i) - H(i) = -W(i)$. The left-hand side is increasing in $P(i)$, ranging from $-\infty$ to $+\infty$. It follows that, for a given $W(i)$, there exists a unique $P(i)$ solving the first-order condition. Finally, given $P(i)$, the values of $p(i)$ are uniquely determined by (6). ■

Proof of Corollary 1: If $\delta = 0$, the value functions $u(i)$ and $v(i)$ are given by constants independent of $p(i), q(i)$. Proposition 1 implies a unique solution. By continuity, the same holds for δ arbitrarily close to zero. ■

Proof of Proposition 3: To be completed. The proof relies heavily on Lemma 1. ■

Proof of Proposition 1: When $N = 2$, the entire equilibrium is determined by the value of $x(1)$ (or equivalently $x(0) = -x(1)$). From (4),

$$u(1) \;=\; \lambda(1) + \delta\,\frac{1}{2}\Big(q(1)\,u(2) + q(0)\,u(1)\Big)$$

$$u(2) = \lambda(2) + \delta \frac{1}{2} \Big(q(1)\, u(2) + q(0)\, u(1) \Big)$$

It follows that

$$x(1) = p(1) - p(0) - u(2) + u(1) = P(1) - \lambda(2) + \lambda(1)$$

and

$$H(1) \equiv h(1) - h(0) = \frac{1}{f\Big(P(1) - c_1\Big)} \Big(F\Big(P(1) - c_1\Big) - F\Big(- P(1) + c_1\Big)\Big) \tag{16}$$

where $c_1 \equiv \lambda(2) - \lambda(1)$ is a constant.

From (8), I get

$$\begin{aligned} w(0) &= \pi(1) - \pi(0) + \delta \frac{1}{2} \Big(v(1) - v(0) \Big) \\ w(1) &= \pi(2) - \pi(1) + \delta \frac{1}{2} \Big(v(1) - v(0) \Big) \end{aligned}$$

It follows that

$$W(1) \equiv w(1) - w(0) = \pi(2) - 2\,\pi(1) + \pi(0). \tag{17}$$

Following the argument in the proof of Proposition 1, the equilibrium value of $P(1)$ is determined by

$$P(1) - H(1) = -W(1). \tag{18}$$

From (17), we conclude that $W(1)$ is a constant. From (16) and Assumption 1, we conclude that the left-hand side is continuous and strictly increasing in $P(1)$. Moreover, the left-hand side ranges from $-\infty$ to $+\infty$ as $P(1)$ varies from $-\infty$ to $+\infty$. By the intermediate value theorem, there exists a unique value of $P(1)$ that solves (18). Finally, a unique $P(1)$ determines $x(1)$, uniquely, which in turn determines $u(i)$ and $v(i)$ uniquely. ■

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This list is *very* incomplete. Please help.

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