Econometrics of Share Auctions*

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Abstract

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a common value model in which a single and perfectly divisible good is sold to a group of symmetric and risk-neutral buyers. The parameters in the distribution function of the value of the good and the signals received by the buyers are estimated using a two-step estimation procedure. The methods are applied to French Treasury securities auctions held in 1995. A counterfactual comparison shows that the Treasury's revenue in the discriminatory share auction (the mechanism adopted by the French Treasury) is 5% higher than in the uniform share auction.

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1 Introduction

Over the past forty years the theoretical literature on auctions has considerably expanded (see Klemperer, 1999, for a recent survey). Auctions are relatively simple and well-defined economic environments, and the theoretical models often lead to clear-cut predictions. For this reason and because auction data are readily available, the theory has triggered a vast amount of empirical articles (see Hendricks and Paarsch, 1995, for a survey on empirical work). Two types of approaches can be distinguished in the empirical literature. There is first of all the so-called reduced form approach. In reduced form econometrics little or no restrictions are imposed on the data generating process. The objective is typically to provide a descriptive analysis of the auction data, or to test certain predictions of the theory (such as the well-known Revenue Equivalence theorem). The second approach is known as the structural econometric form approach. Structural econometrics consists in methods to estimate the parameters of a particular auction model incorporating all restrictions and constraints of the theory. This approach is of particular interest when one desires to measure the effects of a change in auction design on the behavior of bidders and the revenue of sellers.

Until now the structural econometric literature has mainly focused its attention on the standard first-price auction of a single indivisible good.[1] Various structural estimation techniques have been developed for this auction model under different bidding environments. Donald and Paarsch (1993, 1996), Elyakime, Laffont, Loisel and Vuong (1994), Laffont, Ossard and Vuong (1995), and Guerre, Perrigne and Vuong (2000) consider the Independent Private Value (IPV) paradigm so that the structural element to be estimated is the distribution of private values of the buyers. Li, Perrigne and Vuong (2002) consider an affiliated private value framework so that their objective is to estimate the joint distribution of private values. Finally, Paarsch (1992), Li, Perrigne and Vuong (2000) and Hong and Shum (2003) consider the common value paradigm and propose methods to estimate the joint distribution function of the value of the good and the signals received by the buyers.

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a

[1] Some recent papers, however, consider the estimation of multi-unit auctions. See for example Donald, Paarsch and Robert (1999), and Cantillon and Pesendorfer (2001).

common value model in which a single and perfectly *divisible* good is sold to a group of symmetric and risk-neutral buyers. As in the standard common value auction model for an indivisible good, the value of the object is unknown at the time of bidding, and prior to the auction the buyers independently receive signals that are informative about the value. Unlike the standard model, each potential buyer not only submits a price but also a share (or fraction) he requests at that price. Each bidder can submit as many price/share combinations as he wants, thereby constituting a demand curve for the good. By aggregating the individual demand curves, the auctioneer can determine the equilibrium price that clears the market. Given the announced (prior to the auction) allocation mechanism and pricing rule, the shares are then awarded to the winning bidders, who make their payments to the seller. Wilson restricts his analysis primarily to two auction formats: the uniform price auction and the discriminatory auction. In both auction formats the allocation devise is the same, and consists in awarding to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction mechanisms. Roughly speaking, in the uniform auction, all winners (bidders with positive demand at the equilibrium price) pay the equilibrium price, whereas in the discriminatory auction, for each marginal share they receive, winners pay the price at which the bid was submitted.

Although basically our estimation procedure can be applied to both the discriminatory and uniform share auction model, we only present the methods for the former auction format. The structural elements of interest are the marginal distribution function of the value of the good, and the conditional distribution function of the signal given the value. The two distribution functions are specified parametrically so that the goal is to estimate a vector of unknown parameters. We do this via a two-step procedure which is similar in spirit as the two-step procedure proposed by Elyakime et al. (1994) and Guerre et al. (2000) (see Jofre-Bonet and Pesendorfer (2003) and Hortaçsu (2002) for other applications and extensions of the two-stage approach). As in their first step, we exploit the relationship between the distribution of observables (shares) and the structural distribution of the model, estimate the distribution of observables, and insert the estimate into the first-order condition. More specifically, our first step consists in estimating nonparametrically the distribution function of the demand for the good using the observed price/share bids. The estimated inverse demand function is then replaced into the Euler condition resulting from the bidder's maximization program. The second step of their method of statistical inference is however different in nature from our's. Their first-order condition allows them to simulate for each bidder a private value, which in turn allows them to estimate the distribution of private values using the generated pseudo sample. We shall instead exploit the fact that our Euler condition can be seen as a set of moment restrictions that depend on the unknown parameters of interest. Minimization of the empirical counterpart of the moment restrictions then leads to our estimator.

As Elyakime et al. (1994) and Guerre et al. (2002) have pointed out, an attractive feature of the two-step method is that it only relies on the first-order condition implied by the bidder's maximization problem. Although it must be assumed that there exists an equilibrium strategy and that all bidders behave according to this strategy, the explicit form of the equilibrium does not need to be known. In our context this is of particular importance since recent work has shown that, unlike the uniform share auction, it appears difficult to find explicit optimal strategies in the case of the discriminatory share auction (Viswanathan, Wang and Witelski (2000) and Hortaçsu (2001) have obtained explicit solutions for the very specific two-bidder case only).

This paper also studies the identification of the share auction model. As in the standard common value auction model (see Athey and Haile (2002)), it turns out that the structural distribution functions are not nonparametrically identified. We show however that the model is parametrically identified. That is, given our parametric choice of the distribution functions, the true value of the parameter is uniquely determined from the moment restrictions.

The asymptotic properties of our estimator are derived by drawing on Newey and McFadden (1994, Chapter 8). Our estimator belongs to their class of semiparametric two-step estimators. By verifying the regularity conditions of Newey and McFadden for this class of models, we show that our estimator is consistent and asymptotically normally distributed, and indicate how the asymptotic variance matrix can be consistently estimated.

The methods are applied to data from Treasury securities auctions. Treasury securities auctions are often cited as a good example of a share auction (see for example the survey on Treasury auction theory by Das and Sundaram, 1996). Treasury securities auctions are auctions in which huge quantities of strictly identical goods—i.e. securities—are sold to a group of buyers, generally large investment institutions. Since these institutions typically require different amounts of securities at different prices, the Treasury allows them to submit price/quantity combinations, that is a demand curve for the securities. Normalizing the total amount of securities offered to one, and dividing the quantity bids by the total volume, a Treasury auction is indeed a good approximation of a share auction.

Although Wilson did not mention Treasury auctions in his article (may be because at that time the practice of selling securities at auction was much less widespread than nowadays; see Bartolini and Cottarelli, 1997), his model and its underlying assumptions are often regarded as well adapted to the context of Treasury auctions. For instance, in their discussion about the theoretical literature on divisible-good auctions, Back and Zender (1993) refer to Wilson's share auction model as the most relevant model for Treasury securities. Bikhchandani and Huang (1993) state that in Treasury securities auctions the common value assumption is appropriate because the value for each bidder is a common and unknown resale price. Each investor is likely to have some private information about the resale price of the security, on which to base a bid function. Finally, Das and Sundaram (1996) argue that bidders in Treasury securities auctions can be considered as symmetric and risk-neutral.

Our empirical analysis is based on very detailed bidder-level data from French Treasury securities auctions held in 1995. In France, the Treasury sells the securities via discriminatory auctions. In the first part of the empirical analysis our purpose is therefore to estimate the parameters of the discriminatory auction model using the two-step estimation procedure. The second part is more policy-oriented. Given an explicit optimal strategy in the uniform auction model (derived under the same distributional assumptions as in the discriminatory model), and using the parameter estimates of the discriminatory auction model, we can approximate the hypothetical equilibrium price that would emerge in the uniform auction, and thereby also the corresponding revenue. Comparing, for each auction held in 1995, the observed revenue (from the discriminatory auction) with the hypothetical revenue generated by the uniform auction, we can evaluate if the discriminatory auction is revenue-superior to the uniform auction, or not.

We hereby contribute to a debate that has been going on at least since Friedman (1960). He supported the uniform auction format, claiming that collusion is less likely in this auction system and that for this reason the uniform auction would be better from the Treasury's viewpoint than a discriminatory auction. His article initiated a vast literature on the revenue generating properties of the uniform and discriminatory auction formats. Although most of the papers in this literature are theoretical ***Conflicting theoretical results here (RP)***, there are some empirical contributions. The empirical studies on the optimal revenue debate are either based on experimental data (see for example Smith, 1967, and Abbink, Brandts and Pezanis-Christou, 2001), or on natural experiment-type data (see for example Umlauf, 1993, who examines the revenue issue by exploiting the fact that the Mexican treasury switched from uniform pricing to discriminatory pricing, and Berg, Boukai and Landsberger, 1998, who exploit the fact that the Norwegian central Bank simultaneously applied both auction rules to similar types of securities). Outside a laboratory setting, and in the absence of natural experiments—France has never experimented with the uniform auction format—it is necessary to adopt a structural form approach in order to determine the best auction mechanism for the French Treasury.

Castellanos and Oviedo (2002) and Hortaçsu (2002) are the only papers we are aware of that also address the revenue-issue in a structural way. Castellanos and Oviedo apply our method of statistical inference to Mexican Treasury (discriminatory) auctions. They find that the uniform auction produces more revenue than the discriminatory auction, which, interestingly, is in line with the conclusion that Umlauf (1993) draws from his analysis of the Mexican natural experiment. Hortaçsu proposes a model of strategic bidding behavior in the discriminatory share auction. His model differs from Wilson's model because the IPV paradigm is adopted instead of the common value paradigm, and also because the prices and shares submitted by the bidders are discrete instead of continuous variables. As in Elyakime et al. (1994) and Guerre et al. (2000), the Euler condition is used to estimate the marginal valuations of the bidders. The estimated marginal valuations enable him to reconstruct the seller's revenue in the perfectly competitive outcome, which constitutes an upper bound to revenue from the uniform auction. Using data from the Turkish Treasury auctions, he finds that the revenue generated by the discriminatory auctions exceeds the competitive revenue, implying that in Turkey the discriminatory auction is revenue-superior to the uniform auction format.

The main advantage of Hortaçsu's approach over our's is that his method is distribution-free. In his setup the distribution function of private values does not need to be specified, whereas we rely on a parametric framework to evaluate and compare the auction performances. The major disadvantage of Hortaçsu's approach relatively to our's is that his methodology does not always allow to rank the auction institutions in terms of revenue. Indeed, in case the competitive outcome exceeds the revenue from the discriminatory auction, then the only conclusion that can be drawn is that both the uniform and discriminatory auctions are inferior to the competitive outcome, but nothing whatsoever can be concluded about the relative performance of the 2 auction formats. In contrast, our methodology does not suffer from this drawback since simulations under the uniform auction rule can be performed under all circumstances, and hence revenue-comparisons can always be made.

The paper proceeds as follows. Section 2 presents Wilson's (1979) share auction model, the estimation method, and the asymptotic properties of the two-step estimator. Section 3 describes the institutional background of the French Treasury securities auctions, and contains a descriptive analysis of the auction data. Section 4 presents the results, and section 5 concludes.

2 Share auction theory and the estimation method

2.1 The share auction model

This subsection presents the theory of share auctions developed by Wilson (1979). A divisible good is auctioned. There are $n \geq 2$ risk-neutral bidders. The value of the good is the same for all bidders but unknown at the start of the auction. It is assumed that the value is a realization of a random variable V, which has the distribution function $F_V(v) = \Pr(V \leq v)$.[2] Prior to the auction, each bidder $i = 1, ..., n$ receives a private signal about the value of the good. The signal received by individual i is assumed to be a realization of the random variable S_i. As usual in common value-type auction models, the bidder's signals $S_1, ..., S_n$ are i.i.d. given V. The distribution function of S_i given V is thus the same for all bidders i, and is denoted $F_{S|V}(s|v) = \Pr(S_i \leq s|V = v)$. It is assumed that signal S_i is only observed by bidder i, and not by the seller or the other potential buyers. Furthermore, the number of bidders n, and the distribution functions $F_V(\cdot)$ and $F_{S|V}(\cdot|\cdot)$ are common knowledge.

Each bidder i is required to submit to the auctioneer a tender (sealed, and written) stating, for each price of the good, the desired share of the good. The price-share combinations submitted by bidder i constitute bidder's i demand

[2]Throughout the paper random variables are distinguished from their realizations by denoting the former by upper case letters and the latter by lower case letters.

function for the good. By aggregating the individual demand functions, the auctioneer can determine the equilibrium price that clears the market, i.e. the price for which aggregate demand equals one. Given the pre-announced allocation devise and pricing rule, the shares are then awarded to the winners, who pay the owner of the good. In the two auction formats mentioned in the introduction, the uniform price auction and the discriminatory auction, the allocation mechanism is the same, and consists in allocating to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction formats. In the uniform price auction each winner simply pays the equilibrium price multiplied by his requested share. In the discriminatory auction each winner has to pay the area under his inverse demand curve between zero and his requested share, so that here the payment is bidder-specific.

Let $x_i(\cdot,\cdot)$ be a strategy of bidder i. A strategy is a function of the price of the good p and the signal s_i, such that when bidder i receives signal $S_i = s_i$, he submits a demand schedule specifying that at each price p he demands a share $x_i(p, s_i)$ of the good. In characterizing an optimal strategy, attention is restricted to symmetric strategies, so that $x_i(\cdot,\cdot) = x(\cdot,\cdot)$ for all i. By *optimal* strategy is meant a symmetric Bayesian Nash equilibrium of the auction game. A strategy is thus optimal if no player can deviate in a profitable way from equilibrium behavior if the other players adopt the optimal strategy.

The form of an optimal strategy depends, among other things, on the auction mechanism that is used to sell the good. Consider first the characterization of an optimal strategy in the case of a uniform auction. Let $x(\cdot,\cdot)$ now designate the optimal strategy. Suppose that all bidders except i use the strategy $x(\cdot,\cdot)$, and that i uses the strategy $y(\cdot,\cdot)$. Let p^0 denote the equilibrium price, i.e. p^0 is the price such that

$$\sum_{j \neq i} x(p^0, s_j) + y(p^0, s_i) = 1. \tag{1}$$

The above market clearing equation shows that the equilibrium price depends on the signals received by the competitors of bidder i. Since these signals are unknown, the equilibrium price is also unknown to bidder i. But since bidder i knows the distribution function from which the signals S_j, $j \neq i$, are drawn, and also the function $x(\cdot,\cdot)$, he can determine the (conditional) distribution function of the random variable P^0. That is, he can determine

$$\begin{aligned} H(p;v,y) &\equiv \Pr\left(P^0 \le p | V = v, y(p,s_i) = y, S_i = s_i\right) \\ &= \Pr\left(\sum_{j\neq i} x(p,S_j) \le 1 - y | V = v, S_i = s_i\right) \\ &= \Pr\left(\sum_{j\neq i} x(p,S_j) \le 1 - y | V = v\right). \end{aligned}$$

When bidder i uses the strategy $y(\cdot,\cdot)$, and if the value of the good and equilibrium price are respectively v and p^0, his profit is $(v-p^0)y(p^0,s_i)$. Bidder's i expected profit in a uniform auction is therefore

$$E\left\{\int_0^\infty (V-p)y(p,s_i)dH(p;V,y(p,s_i))|S_i = s_i\right\} \tag{2}$$

where, as the notation suggests, the expectation is with respect to V given $S_i = s_i$. The strategy $x(\cdot,\cdot)$ is indeed optimal if the maximum of (2) is attained at $y(\cdot,\cdot) = x(\cdot,\cdot)$. A solution to this maximization problem can be characterized by applying the principles of calculus of variations. Wilson has shown that the necessary condition for optimization (the Euler condition) is that for all $p \in [0,\infty)$

$$0 = E\left\{(V-p)\partial H(p;V,y)/\partial p + x(p,s_i)\partial H(p;V,y)/\partial y | S_i = s_i\right\} \tag{3}$$

where the partial derivatives of H with respect to p and y are evaluated at $y = x(p,s_i)$.

Next consider the discriminatory auction. Using the same notation as above, bidder's i profit in a discriminatory auction, when he adopts the strategy $y(\cdot,\cdot)$, and when the value of the good and equilibrium price are v and p^0, equals $(v-p^0)y(p^0,s_i) - \int_{p^0}^{p^{\max}} y(u,s_i)du$, where $p^{\max}$ is the largest price for which demand $y(\cdot,s_i)$ is non-negative. Bidder's i expected profit is therefore

$$E\left\{\int_0^\infty \left[(V-p)y(p,s_i) - \int_p^{p^{\max}} y(u,s_i)du\right] dH(p;V,y(p,s_i))|S_i = s_i\right\}. \tag{4}$$

Wilson (1979) does not derive the Euler condition for the discriminatory share auction. We show in appendix A that the Euler condition in this case is

$$0 = E\left\{(V-p)\partial H(p;V,y)/\partial p - H(p;V,y)|S_i = s_i\right\} \tag{5}$$

where the distribution H and the derivative of H are evaluated at $y = x(p, s_i)$, and the condition must hold for all $p \in [0, \infty)$. Note that this Euler condition differs from the uniform-auction Euler condition (3) only in the second term of the expectation.

Our method of statistical inference is highly facilitated by rewriting the above Euler equation. The following proposition states how (5) can conveniently be reformulated.

Proposition 1. *The Euler condition (5) can be rewritten as*

$$\begin{aligned} 0 &= E\left\{(n-1) \cdot [E\{V|S_1 = s_1, ..., S_n = s_n\} - p] \cdot 1(P^0 \leq p)\right\} \\ &\quad -E\left\{(p - P^0) \cdot 1(P^0 \leq p)\right\} \end{aligned} \tag{6}$$

In the above equation $1(\cdot)$ represents the indicator function, the first expectation is with respect to $S_1, ..., S_n$ (the random variable P^0 only depends on $S_1, ..., S_n$), and, as is clear from the notation, the second expectation is with respect to V given $S_1, ..., S_n$, and the third is with respect to P^0. The condition (6) must hold for all $p \in [0, \infty)$. The proof of Proposition 1 is given in appendix B. As the proof shows, the step from equation (5) to equation (6) depends very crucially on the "symmetry assumption" (i.e. the assumption that the conditional distribution of S_i is independent of i, plus the fact that attention is restricted to symmetric equilibria).

The Euler condition forms the basis of our estimation method for the discriminatory share auction model. As explained in the next subsection, our estimator is defined as the minimum of an empirical counterpart of the Euler restriction. The crucial advantage of the reformulated Euler condition (6) over the Euler condition (5) is that it is much easier to obtain an empirical counterpart of the former. This comes from the fact that (6) no longer depends on $H(\cdot; \cdot, \cdot)$, the distribution function of the market clearing price. Indeed this function (and its derivative with respect to p) is difficult to compute and evaluate because of its implicit dependence on the equilibrium strategy $x(\cdot, \cdot)$.

2.2 Estimation

In Section 2.2.1 we rewrite the Euler conditions (6) once more and show that they can be seen as moment conditions that depend on the unknown parameters of the distribution functions. Section 2.2.2 is devoted to the issue of identification of the model. Section 2.2.3 presents our two-step estimator and Section 2.2.4 presents the asymptotic properties of the estimator.

2.2.1 Theoretical moments

The estimation procedure exploits the fact that the results from several different auctions are available. Suppose there are L auctions and let l index the l-th auction. In many applications the goods sold in the different auctions are not completely identical. Also, the number of bidders typically varies from auction to auction. To capture this between-auction heterogeneity we introduce the commonly known vector of variables z_l characterizing the good sold at the l-th auction, and the number of bidders, n_l.

It is assumed that the random variables (N_l, Z_l), $l = 1, ..., L$, are independently and identically distributed. The value of the good in the l-th auction, V_l, is assumed to depend on Z_l but not on N_l. Similarly, the signal received by individual i in auction l, S_{il}, depends on Z_l (and on V_l) but not on N_l. Conditionally on $Z_1, ..., Z_L$, the values $V_1, ..., V_L$ are assumed to be independently and identically distributed. Furthermore, $S_{1l}, ..., S_{n_l l}$ are independent conditionally on (V_l, Z_l), and the signals S_{il} and $S_{i'l'}$ are independent conditionally on Z_l and $Z_{l'}$ for all $l \neq l'$. We adopt a parametric framework. The distribution functions are specified parametrically and are thus known up to a vector of parameters. The conditional distribution of V_l given $Z_l = z$ is denoted $F_{V|Z}(\cdot|z; \theta_1)$, where θ_1 is a vector of parameters. The conditional distribution function of S_{il} given $V_l = v$ and $Z_l = z$ is denoted $F_{S|V,Z}(\cdot|v, z; \theta_2)$, where θ_2 is a vector of parameters. Given the distribution functions $F_{V|Z}(\cdot|\cdot; \cdot)$ and $F_{S|V,Z}(\cdot|\cdot, \cdot; \cdot)$, we can determine the distribution function of S_{il} given $Z_l = z$, denoted $F_{S|Z}(\cdot|z; \theta)$, where $\theta = (\theta_1', \theta_2')'$. The true value of θ is denoted θ^0.

The objective of this subsection is to rewrite the Euler condition (6) once again and show how the reformulated theoretical moments depend on the true parameter value. To proceed, let us from now on explicitly write the optimal strategy as a function of not only the price and the signal, but also the number of bidders, the vector of characteristics of the good, and the parameter. That is, $x(p, s, n, z; \theta^0)$ is the optimal demand for the good at price p for an individual with signal s, when the auction is attended by n bidders and the good on sale has characteristics z, and when the true value of the parameter equals θ^0. Furthermore, for any given $p \in [0, \infty)$, let $G(\cdot|n, z; p)$ denote the distribution function of $x(p, S_{il}, N_l, Z_l; \theta^0)$ conditionally on $N_l = n$ and

$Z_l = z$. We have

$$\begin{aligned} G(x|n,z;p) &= \Pr(x(p,S_{il},N_l,Z_l;\theta^0) \leq x|N_l = n, Z_l = z) \\ &= \Pr(x(p,S_{il},n,z;\theta^0) \leq x|N_l = n, Z_l = z) \\ &= \Pr(S_{il} \geq x^{-1}(x,p,n,z;\theta^0)|N_l = n, Z_l = z) \\ &= \Pr(S_{il} \geq x^{-1}(x,p,n,z;\theta^0)|Z_l = z) \\ &= 1 - F_{S|Z}(x^{-1}(x,p,n,z;\theta^0)|z;\theta^0) \end{aligned} \tag{7}$$

where[3] the fourth equation follows from the assumption that S_{il} and N_l are conditionally independent. The third equation holds under the additional hypothesis that the optimal strategy $x(\cdot, s, \cdot, \cdot; \cdot)$ is a strictly decreasing function in s.[4] Note that from (7) we immediately obtain the inverse demand function:

$$x^{-1}(x,p,n,z;\theta^0) = F^{-1}_{S|Z}(1 - G(x|n,z;p)|z;\theta^0). \tag{8}$$

Let us now rewrite the Euler condition (6), first by incorporating the auction-specific notation and variables. For auction l with characteristics z_l and with n_l bidders, the condition becomes

$$\begin{aligned} 0 = E\Big\{(n_l - 1) \cdot \Big[E\Big\{V_l|S_{1l} = s_{1l}, ..., S_{n_l l} = s_{n_l l}, N_l = n_l, Z_l = z_l\Big\} - p\Big] \\ \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} - E\Big\{(p - P_l^0) \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} \end{aligned} \tag{9}$$

where the random variable P_l^0 represents the equilibrium price in auction l, and the first expectation is with respect to $S_{1l}, ..., S_{n_l l}$ given $N_l = n_l, Z_l = z_l$. The condition must hold for all $p \in [0, \infty)$, all possible values of n_l and z_l, and all $l = 1, ..., L$. Letting x_{ilp} represent the observed optimal demand by bidder i in auction l at the price p, i.e., $x_{ilp} = x(p, s_{il}, n_l, z_l; \theta^0)$, and since $s_{il} = x^{-1}(x_{ilp}, p, n_l, z_l; \theta^0)$, and using (8), the condition (9) can be rewritten as

[3]Note that there is the function $x(\cdot,\cdot,\cdot,\cdot;\cdot)$, the inverse function $x^{-1}(\cdot,\cdot,\cdot,\cdot;\cdot)$, and the scalar x.

[4]Given specific parametric specifications of the distributions functions $F_{V|Z}(\cdot|\cdot;\cdot)$ and $F_{S|V,Z}(\cdot|\cdot,\cdot;\cdot)$, section 4.2 derives an explicit optimal strategy in the uniform auction model. This strategy (shown to be the unique equilibrium in a large class of demand functions) turns out to be strictly decreasing in s. It is quite natural therefore to assume that the optimal strategy in the discriminatory auction also satisfies this property.

$$0 = E\Big\{(n_l - 1) \cdot \Big[E\Big\{V_l|S_{1l} = F^{-1}_{S|Z}(1 - G(x_{1lp}|n_l, z_l; p)|z_l; \theta^0),$$
$$..., S_{n_l l} = F^{-1}_{S|Z}(1 - G(x_{n_l lp}|n_l, z_l; p)|z_l; \theta^0), N_l = n_l, Z_l = z_l\Big\} - p\Big]$$
$$\cdot 1(P^0_l \leq p)|N_l = n_l, Z_l = z_l\Big\} - E\Big\{(p - P^0_l) \cdot 1(P^0_l \leq p)|N_l = n_l, Z_l = z_l\Big\} \quad (10)$$

where now the first expectation is with respect to the random variables $X_{1lp}, ..., X_{n_l lp}$, given $N_l = n_l, Z_l = z_l$.

Since the above moment condition is actually a conditional moment condition (the first and third expectations are conditional on N_l, Z_l), there is an infinity of possible unconditional moments for each p. More precisely, the condition (10) implies that

$$0 = E\Big\{w(N_l, Z_l) \cdot (N_l - 1) \cdot \Big[E\Big\{V_l|S_{1l} = F^{-1}_{S|Z}(1 - G(x_{1lp}|n_l, z_l; p)|z_l; \theta^0),$$
$$..., S_{n_l l} = F^{-1}_{S|Z}(1 - G(x_{n_l lp}|n_l, z_l; p)|z_l; \theta^0), N_l = n_l, Z_l = z_l\Big\} - p\Big]$$
$$\cdot 1(P^0_l \leq p)\Big\} - E\Big\{w(N_l, Z_l) \cdot (p - P^0_l) \cdot 1(P^0_l \leq p)\Big\} \quad (11)$$

for any function $w(\cdot, \cdot)$.[5] The first expectation is an unconditional expectation with respect to $N_l, Z_l, X_{1lp}, ..., X_{n_l lp}$, the third is an unconditional expectation with respect to N_l, Z_l, P^0_l. The moment condition (11) must hold for all $p \in [0, \infty)$ and forms the basis of our estimation method. Section 2.2.3 proposes empirical counterparts for the theoretical moments and shows how the former can used in our method of statistical inference. But first we turn to the issue of identification.

2.2.2 Identification

An important issue in the structural estimation of auction models is whether the fundamental elements of the model are identified. In the share auction model the fundamental elements are the conditional distribution functions of $V|Z$ and $S|V, Z$. Although we have adopted a parametric framework in this paper (i.e. the distribution functions are assumed to be known up to a vector

[5]A function $w(\cdot, \cdot)$ is a scalar function of the "instrumental" variables n_l, z_l.

of parameters), we shall investigate both the parametric and nonparametric identification of the model.

In our setting, the nonparametric identification problem consists in establishing if the distribution functions of $V|Z$ and $S|V,Z$ are uniquely determined from the Euler condition and knowledge of the distribution function of the observable variables. More precisely, the question that should be answered is: given the distribution function of share bids $G(\cdot|\cdot,\cdot;\cdot)$ defined in (7), and given the "nonparametric version" of the moment condition (11)[6] for all $p \in [0,\infty)$ and all functions $w(\cdot,\cdot)$, are the nonparametric distribution functions $F_{V|Z}(\cdot|\cdot)$ and $F_{S|V,Z}(\cdot|\cdot,\cdot)$ uniquely determined? The answer to this question is: no. The proof is similar to the proof of the non-identifiability of a related auction model, namely the standard common value first-price auction model (see Laffont and Vuong (1996) and Athey and Haile (2002)). Given a normalization of the signals similar to the normalization considered by Athey and Haile (2002, page 2125), it might be possible to identify the distribution of $S|V,Z$ but one cannot recover the distribution of $V|Z$. It is of particular interest to understand the precise sources of nonidentification, and to study what are the possible identifying restrictions (as is done by Li, Perrigne and Vuong (2000) in the case of the common value first-price auction model). However, this topic is beyond the scope of this paper and will therefore be left for future research.

Instead we turn now to the parametric identifiability of our model. The question that has to be answered now is whether the true parameter θ^0 is uniquely determined given the distribution function of share bids, the theoretical moment conditions and the parametric specifications of the distribution functions. Of course the answer to this question is dependent on which specific parametric specifications are chosen. The next proposition establishes the parametric identification of the share auction model for the case where the distribution functions of $V|Z$ and $S|V,Z$ are given by (15) and (16) respectively. These specifications are the ones that are used in the empirical part of the paper (Section 4.1 and Section 4.2), and they can be seen as a generalization of the specifications chosen by Wilson (1979, example 1).

Proposition 2. *Given i) the distribution function $G(\cdot|\cdot,\cdot;\cdot)$, ii) the moment condition (11) for all $p \in [0,\infty)$ and all functions $w(\cdot,\cdot)$, and iii) the parametric specifications of $F_{V|Z}(\cdot|z;\theta_1^0)$ and $F_{S|V,Z}(\cdot|v,z;\theta_2^0)$ defined in respectively*

[6] The same condition as condition (11) except that the parametric distribution functions are replaced by nonparametric ones.

(15) and (16), the true value $\theta^0 = (\theta_1^{0\prime}, \theta_2^{0\prime})'$ is uniquely determined.

The proof of Proposition 2 is in appendix E. As the proof shows, it is not necessary to assume that (11) holds for all values of p. Indeed, identification follows under the weaker assumption that (11) holds only for a certain interval of values for p.

2.2.3 Definition of the estimator

Our estimator of the parameter of interest θ^0 is a two-step estimator. The first step consists in estimating the distribution function of bids $G(\cdot|\cdot,\cdot;\cdot)$. This distribution function is not parameterized in any way but is is instead left completely unspecified. The advantage of not imposing any a priori restrictions on the distribution function of bids is that potential misspecification problems are avoided.

For any given $p \in [0, \infty)$ the distribution function $G(\cdot|\cdot,\cdot;p)$ can be estimated nonparametrically from the observed share bids x_{ilp}, $i = 1, ..., n_l$, $l = 1, ..., L$, and the variables n_l, z_l, $l = 1, ..., L$, using kernel estimation methods. The kernel estimate of $G(\cdot|\cdot,\cdot;p)$ is

$$\hat{G}(x|n, z; p) = \frac{\sum_{l=1}^{L} \frac{1}{n_l} \sum_{i=1}^{n_l} 1(x_{ilp} \leq x) K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)}{\sum_{l=1}^{L} K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)} \tag{12}$$

where $K(\cdot,\cdot)$ is a kernel and h_N and h_Z are bandwidth parameters (h_Z is actually a vector of bandwidth parameters with the same dimension as z).

The second step of the estimation method consists in minimizing, over θ, an appropriate criterion function involving the empirical counterparts of our theoretical moments. To find the empirical moments, consider again the theoretical moment restrictions (11). Replacing θ^0 by an arbitrary value θ, and given a specific function $w(\cdot,\cdot)$, the only unknown component in (11) is the distribution function $G(\cdot|\cdot,\cdot;p)$. Replacing this distribution function by its consistent estimate $\hat{G}(\cdot|\cdot,\cdot;p)$, a natural empirical counterpart of the righthand side of (11) is

$$
\begin{aligned}
&m(x_{11p},...,x_{n_L Lp},n_1,...,n_L,p_1^0,...,p_L^0,z_1,...,z_L,p;\theta) \\
&\equiv \frac{1}{L}\sum_{l=1}^{L} w(n_l,z_l)\cdot(n_l-1)\cdot 1(p_l^0 \le p)\cdot\Big[E\Big\{V_l|S_{1l}=F_{S|Z}^{-1}(1-\hat{G}(x_{1lp}|n_l,z_l;p)|z_l;\theta), \\
&\qquad ...,S_{n_l l}=F_{S|Z}^{-1}(1-\hat{G}(x_{n_l lp}|n_l,z_l;p)|z_l;\theta),N_l=n_l,Z_l=z_l\Big\}-p\Big] \\
&\qquad -\frac{1}{L}\sum_{l=1}^{L} w(n_l,z_l)\cdot(p-p_l^0)\cdot 1(p_l^0\le p). \qquad (13)
\end{aligned}
$$

Since (11) must also hold for all $p \in [0,\infty)$ and any weighting function $w(\cdot,\cdot)$, there is an infinity of moment conditions of the form (11), and for each of these theoretical moments there exists an empirical counterpart of the form (13). The question now arises which of the theoretical moments should be exploited in the method of statistical inference. One option is to somehow use *all* possible moments, but, for a practical reason given in the next subsection, this is not the course that we shall follow. Instead, our estimation method exploits a fixed number of the theoretical moments. That is, we impose the restriction (13) only for a finite number of different weighting functions and prices.

The restriction (13) is imposed for a single function $w(\cdot,\cdot)$. Its functional form is chosen in a pragmatic way, that is we experiment with several specifications and select the one that performs best. This somewhat ad hoc way of choosing the weighting function is quite common in empirical applications of the GMM. Even when an optimal weighting function $w(\cdot,\cdot)$ (or an optimal vector of weighting functions) is available in theory, empirical researchers generally opt for less than optimal instruments since it is typically hard if not impossible to implement the optimal weighting functions (see for example Davidson and MacKinnon (1993, section 17.4)). Of course it is possible to impose the restriction for several weighting functions simultaneously. But as it turns out in the empirical application, imposing additional weighting functions (beyond the one that we select) does not fundamentally alter the estimation results. Because of this and also for notational simplicity, the estimator presented below corresponds to the case where a single weighting function is used, but one should bear in mind that the statistical method is easily generalized to the case of multiple weighting functions.

The restriction (13) is furthermore imposed for different values of the price p. We choose T values and they are denoted $p_1, ..., p_T$. The second step of our estimation procedure consists in minimizing over θ the sum of the T squared empirical moments. More precisely, the second-stage estimate of θ^0 is

$$\hat{\theta} = Arg \min_{\theta} \sum_{t=1}^{T} m^2(x_{11p_t}, ..., x_{n_L L p_t}, n_1, ..., n_L, p_1^0, ..., p_L^0, z_1, ..., z_L, p_t; \theta). \tag{14}$$

Before turning to the asymptotic properties of the two-step estimator, we want to make 2 remarks. The first remark concerns the actual choice of the T price values. Although in principle any set of values $p_1, ..., p_T$ can be chosen, it may be unwise to select values that are too "small". Indeed, it may be that the demand functions are filled in with relatively less precision for low values of p as bidders may anticipate that demands expressed far below the expected equilibrium price are not executed and thus without real consequence. On the other hand, it may also be unwise to pick price values that are too "large". Indeed, as the equilibrium demand functions may be relatively flat for large values of p, the variation in the estimates $G(\cdot|\cdot, \cdot; \cdot)$ may be too small, which might in turn lead to numerical problems in the second step of the estimation procedure. A reasonable compromise seems therefore to choose price values that are close to the equilibrium prices observed in the sample. Our second remark concerns, once again, the identification of the model. In section 2.2.2 it was shown that the share auction is parametrically identified if (11) holds for all p (or an interval of values for p) and any function $w(\cdot, \cdot)$. Since in our estimation method the restriction (13) is imposed only for a finite number of price values and weighting functions, it is necessary to assume that the identification is not lost by considering a subset of moments.

2.2.4 Asymptotic properties of the estimator

Newey and McFadden (1994, section 8) consider a class of estimators that are defined as the solution of a set of equations involving the observables, an unknown vector of parameters of interest, and a "first-step" estimate of a function. Since the first-step estimator is a nonparametric estimator of a function rather than an estimator of a finite-dimensional parameter, this class is referred to as the class of semiparametric two-step estimators. Newey and McFadden show that these estimators can converge at a rate equal to the root of the number of observations, even though the first-step estimator converges at a slower rate. They also give regularity conditions for asymptotic normality of the second-step estimator.

By rewriting the criterion function (14) we show in appendix D that our estimator of θ^0 belongs to the class of semiparametric two-step estimators (the nonparametric first-step estimate being $\hat{G}$). By verifying in the appendix that the regularity conditions of Newey and McFadden hold, we show that our estimator is $\sqrt{L}$-consistent, and that it is asymptotically normally distributed. For the form of the asymptotic variance of the estimator we refer to appendix D, as it involves a lot of (too much) new notation.

We restrict ourselves to a finite number of moments T to fit into the framework of Newey and McFadden. The extension to a setting where an infinity of moment conditions is exploited is beyond the scope of this paper (see however Carrasco and Florens, 1999, for an extension of the GMM to a continuum of moments).

3 Data

3.1 The institutional setting of the French government securities auctions

Since 1985 the French government securities are sold through auctions.[7] The Treasury securities are auctioned via discriminatory auctions.[8] The auctions are organized by the Bank of France, but all decisions regarding the dates the auctions are held, the characteristics of the securities, the quantities offered, etc., are taken by the Treasury. The three main types of French Treasury securities are:

- The *Bons du Trésor à taux Fixe et à intérêts précomptés* (the BTFs); these are tradable fixed-rate short-term discount Treasury bills with maturities of up to one year.

[7]This section draws on the French Government Securities annual report (we use the 1995 version; see also the web site of the French Treasury: http://www.oat.finances.gouv.fr), and on personal discussions with Sébastien Moynot and Benoît Cœré of the French Treasury.

[8]Bartolini and Cottarelli (1997) find that 42 out of 77 countries in their sample (including all G-7 countries) used auctions in 1993 to sell government bills. More than 90% of the 42 countries that relied on auctions adopted the discriminatory auction format.

- The *Bons du Trésor à taux Fixe et à intérêts ANnuels* (the BTANs); these are tradable fixed-rate medium-term Treasury notes with interest paid annually and with maturities of two or five years.

- The *Obligations Assimilables du Trésor* (the OATs); these are fungible Treasury bonds with maturities ranging between 7 and 30 years.

Since our empirical analysis is based on the auctions for OATs and BTANs that were held in 1995, we describe the auction environment for these particular securities only, and present the auction rules as they prevailed at that time, even though they may have slightly changed by now.

The Treasury auctions for OATs and BTANs are held once a month—OATs on the first Thursday of the month and BTANs on the third Thursday. The scheduling of the auctions is as follows:

- Two business days prior to the auction, the Treasury announces the line that is to be auctioned, i.e. the Treasury describes the characteristics of the securities (the nominal yield, the maturity, etc.), and also determines the amount of securities it plans to sell.

- A bid consists of a price/quantity pair. The price is expressed as a percentage (of the nominal value of the security), and the quantity is the amount (in FFr) of the security the bidder wants at the corresponding price.[9] The minimal amount that bidders may submit is FFr1 million for the BTANs and FFr50 million for the OATs. Bidders are allowed to submit as many bids as they wish. Bids can be submitted until 10 minutes before the start of the auction. Most bids are submitted via TELSAT, a computerized bidding system, but bidders can also submit their proposals in sealed envelops directly to the Bank of France. The start of the auction is at 11 am.

- The Bank of France ranks bids in descending order of price, and immediately transmits the list of ranked bids to the Treasury without revealing the identity of the bidders.

[9]For example, a bidder submitting (90%;FFr100 million) states that at the price of 90% he requests FFr100 million worth of securities. Re-formulated in a more familiar way, assuming the nominal value equals FFr2000, this bidder demands 50000 securities (FFr100 million divided by FFr2000) when the price per security is FFr1800 (90% of FFr2000).

- The Treasury calculates the stop-out price (the equilibrium price), i.e. the price for which aggregate demand equals the amount of securities offered at the auction. The highest prices are served first, and the allocation of securities stops when the amount (sold at auction) is reached. Any ties are settled on a pro rata basis.

- The auction results are announced via TELSAT (and also via some other information networks). The delay between the deadline for submission and the release of the results was less than 20 minutes in 1995.

- Delivery and settlement of the OATs take place on the 25th of the month the auction is held. In the case of 2-year (resp. 5-year) BTANs this occurs on the 5th (resp. 12th) of the month following the auction.

The majority of bidders are so-called *Spécialistes en Valeurs du Trésor* (SVTs).[10] The SVTs are primary dealers selected by the Treasury among the most active players on the government securities market. They belong to large French or international banks. Generally operating within these banks as independent entities, the SVTs are traders who buy and sell the securities and try to maximize profits. The SVTs sell the securities not only on the secondary market, but also directly to their clients (pension funds and insurance companies) and their own bank. On average the SVTs account for 90% of the securities bought at auctions, the remaining 10% being purchased by other banks or financial institutions.

France belongs to a small group of countries[11] in which bidders have the possibility to submit *Offres Non Compétitives* (ONCs). These non-competitive bids (the competitive bids being the auction bids described until now) can actually only be submitted by the SVTs. There are two kinds of non-competitive bids: the ONC1s, which must be submitted at the same time as the competitive bids, and the ONC2s, which may be submitted once the auction is over and until one day after the auction. The SVTs are not obliged to engage in non-competitive bidding. Each SVT can not submit more than one ONC1 bid and one ONC2 bid. Unlike a competitive bid, a non-competitive bid consists only in an amount (in FFr) of the security the bidder wants. The amount submitted by a bidder is sealed and may

[10] On 1 April 1995 there were 19 SVTs.

[11] According to Bartolini and Cottarelli (1997), only 38% of 40 countries that relied on auctions accepted non-competitive bids.

not exceed a certain bidder-specific limit, but except for this restriction each bidder is guaranteed the quantity he bids for. The limit may differ for the two types of ONCs, and its height is determined by the participation of the bidder in the three last auctions. The price at which the non-competitive bids must be paid is identical for all bidders and corresponds to the quantity-weighted average price of the awarded competitive bids. Since ONC1s are submitted *before* the main auction and ONC2s *after* the auction results are revealed, the former are submitted under *price-uncertainty* and the latter under *price-certainty*.

French Treasury securities can not only be bought at auction but also on two other markets: the so-called when-issued market and the secondary market (see Bikhchandani and Huang, 1993, for a detailed description of the US-version of these two market forms). The when-issued market for a security is a forward market that starts the day when the Treasury announces an auction for that security, and ends on the settlement date. Once the announcement has been made by the Treasury, dealers can trade forward contracts on the Treasury securities that are to be auctioned. Contract sellers and contract buyers commit themselves to respectively deliver and take delivery of certain specified amounts of securities at the forward prices. The forward contracts are delivered on the issue date (=settlement date) of the security, which explains why the market is called a when-issued market. The BTANs and OATs are also traded on the secondary market. The secondary market is a permanently active market where the smaller financial institutions and individual investors can trade in securities and where the competitive bidders can resale their securities obtained at auction.

3.2 The link between theory and practice

The purpose of this part of the paper is to motivate some of the model assumptions, in light of our description of the institutional setting of the French Treasury auctions. We will also comment on some of the apparent deviations between theory and the real-life auctions.

Perhaps the most crucial assumption underlying the share auction model is that bidding behavior can be adequately modeled within the common value paradigm. An argument in favor of the common value assumption is that in France the bidders' objective is to resell the securities purchased at auction. Indeed, according to officials at the French Treasury, the SVTs eventually resell all the securities they have won. It is well known that such resale opportunities give rise to a common value component in the bidders' valuation (see for example Haile, 2001). Most of the securities are resold well after the day of the auction.[12] At the time of bidding, the future common resale value of the securities is unknown to the SVTs. An argument in favor of the assumption that the SVTs receive different signals about the future value is that they typically have different forecasts on interest movements and different anticipations of the market demand for securities.

So there are strong arguments for using a common value environment. However, there may also be private value aspects to the Treasury auctions we are analyzing. As mentioned in section (3.1), the SVTs resell part of the securities directly to their clients. It is likely that these clients place orders (stipulating the amount of securities they wish to buy from the SVTs at given transaction prices) *before* the opening of the auction. This means that the SVTs know, in advance, that certain quantities of T-bills can be disposed of at the predetermined transaction prices. This in turn means that the SVTs may not only participate in the Treasury auctions for speculative purposes (as in the common value paradigm), but also to fill customer orders. It may thus be possible that the valuation of each SVT is determined not only by a common value component but also by a private value component, the latter corresponding to the transaction price proposed by the client of the SVT (which is unknown to other SVTs).[13]

Two other model assumptions that need to be motivated are the risk-neutrality assumption and the symmetry assumption (which was used in the proof of Proposition 1. The risk-neutrality hypothesis seems natural given that the bidders in our sample are large financial institutions. These institutions are wealthy agents and are therefore unlikely to be averse to risk.

The symmetry assumption seems less natural as the SVTs that partic-

[12]Although some SVTs resell part of their securities immediately, i.e. during the afternoon of the auction day (in which case it can be argued that the resale value at the time of bidding is known from the secondary market), many T-bills are resold days or even weeks thereafter. The fact that resale is not immediate is not surprising since there is an important time lag between the moment the auction is organized and the moment the securities are actually delivered and paid (see section (3.1)).

[13]Hortaçsu (2002) argues that there are private value aspects in his data because bidders in Turkey (mostly banks) primarily buy securities to fulfill their liquid asset reserve requirements. The securities won at auction are therefore for the bidder's personal use only, and the private values are the privately known liquidity needs of the banks. Hortaçsu's argument does not apply in our context as bidders in France do not enter the auctions to meet liquid asset reserve requirements.

ipated in the auctions of 1995 differed considerably in size and financial importance. In 1995 there were for example financial heavyweights such as *Crédit Lyonnais* and *Société Générale*, but also smaller institutions such as *Banque d'Escompte* and *Jouis Dreyfuss Finance*.[14] This seems like evidence against our hypothesis of symmetry across bidders. However, the auxiliary data that we have on the transaction behavior of the SVTs (kindly given to us by the French Treasury) are rather in support of the symmetry assumption. These data are summarized in Table 1. The table reports for each SVT the total volume of T-bill transactions in 1995 (quantity of T-bills bought by the SVT in 1995 plus the quantity sold in that year).[15] The transaction data are given separately for the BTANs and OATs.

Table 1. Annual volume (FFr millions) of T-bill transactions

SVT	BTAN	OAT
1	526 541 (11%)	564 430 (10%)
2	343 722 (7%)	515 588 (9%)
3	359 467 (8%)	465 576 (8%)
4	363 048 (8%)	432 786 (8%)
5	305 332 (6%)	433 722 (8%)
6	352 273 (7%)	351 596 (6%)
7	309 322 (6%)	318 588 (5%)
8	218 745 (5%)	260 500 (4%)
9	270 869 (6%)	379 563 (7%)
10	241 348 (5%)	318 876 (6%)
11	258 056 (5%)	324 900 (6%)
12	238 761 (5%)	247 326 (4%)
13	218 296 (5%)	261 707 (5%)
14	225 613 (5%)	195 862 (3%)
15	200 944 (4%)	208 932 (4%)
16	190 780 (4%)	221 927 (4%)
17	158 296 (3%)	194 929 (3%)

[14]The above mentioned web site of the French Treasury can be consulted for the list of SVTs that are currently active in the French Treasury auctions. There is much overlap between this list (2003 version) and the one of 1995.

[15]The original dataset contains information on traded volumes for each quarter of 1995 and a total of 20 different SVTs. We aggregated the quarterly data into annual data, and only kept the SVTs whose transaction data are available in all quarters (17 out of 20 SVTs).

Although the traded quantities of T-bills are certainly not identical, the disparity between the SVTs is not that important. Most SVTs have transaction volumes varying between 4% and 8% of the total trade activity. The SVTs are thus quite similar in terms of their T-bill transaction activities. Admittedly, this is by no means a definite or overwhelming proof of the symmetry assumption. But unfortunately we do not have other relevant information on the SVTs that would allow us to perform additional tests of the symmetry hypothesis. Also, given the strictly anonymous nature of our bidding data, it would anyhow be hard if not impossible to estimate a generalized asymmetric auction model. Finally, we do not have any identification results in this case.[16]For these reasons we shall maintain the symmetry assumption, acknowledging that it is a limitation of the empirical part of the paper and that it is potentially restrictive. The theory however can be extended to the asymmetric case (see Appendix F).

Next we comment on the deviations between theory and practice. As mentioned in the introduction, Wilson's share auction model is often regarded as a realistic model of Treasury auctions. From the description of the institutional setting it is clear, however, that the theory deviates in several ways from practice. The first deviation is that, unlike the share auction model, part of the bidders in France—the SVTs—have the possibility to submit ONC1s and ONC2s. The fact that bidders have this opportunity implies that their maximization problem differs from the maximization problem (4). If non-competitive bidding is allowed then agents maximize expected earnings derived from competitive *and* non-competitive bidding, and consequently the actual optimal bidding strategies may differ from those derived in subsection (2.1). The reason why we have not attempted to extend Wilson's model to account for non-competitive bidding is that this phenomenon is not important in France. Indeed, the descriptive statistics in the next subsection (Table 1) indicate that on average ONC1s (ONC2s) only amount to 1% (8%) of the total amount of securities issued by the Treasury (that is the amount sold at auction plus the ONC1s plus the ONC2s). Apparently bidders in France do not make much use of non-competitive bidding, and the effect of neglecting non-competitive bidding in our analysis is therefore expected to

[16]Identification in the case of normal distributions (see section on Robustness) requires variations in the total number of bidders and in the number of bidders in each category that we do not have in our data. Armantier and Sbai (2003), in a recent papet, use normal distributions to estimate a model with asymmetric bidders. They do not study identification but rely on non-linear forms for strategies that give them "numerical" identification.

be negligible.

The second deviation is that unlike most Treasury auctions in the world (including the BTF auctions in France), the French Treasury does not announce the precise amount of OATs or BTANs it plans to sell. Instead it announces an issue interval, wherein the quantity of securities eventually sold at auction necessarily lies. Ex-ante this induces some uncertainty about the total amount of securities sold at auction. However, in practice the intervals announced by the Treasury are quite tight, so that we have not attempted to allow for supply uncertainty in the model.[17]

The third deviation is that Wilson's model describes the main auction as an isolated market, unaffected by what happens on the when-issued market and the secondary market. The interdependencies that exist between the three market forms might affect bidder's behavior at the auction. However, we feel that it is beyond the scope of this paper to construct (and estimate) a unified model that integrates the three markets for Treasury securities.

3.3 Descriptive analysis

Our empirical analysis is based on all French-franc denominated OAT and BTAN auctions that were held in 1995.[18] Table A1 in the appendix gives the auction dates, the lines auctioned (this column in the table gives for each auction the nominal yield of the security and the year of maturity), the settlement dates, and the exact maturity dates. As mentioned in subsection (3.1), OATs and BTANs were auctioned once per month—OATs on the first Thursday of the month and BTANs on the third Thursday of the month. There are therefore 24 different auction dates. As table A1 indicates, the Treasury did not necessarily sell just one line on a given auction date, but often sold two or even three lines on the same day. If several lines were offered on a given day, they were sold simultaneously but via strictly separate auctions.[19] Six different lines of OATs and five different lines of BTANs were

[17]Note that Wang and Zender (2002) have generalized Wilson's model by allowing for supply uncertainty.

[18]Three other types of French government securities were issued through auctions by the Treasury in 1995: BTFs, ECU-denominated OATs/BTANs, and OATs for private investors. We have excluded the data from these auctions for homogeneity reasons. Indeed, the auction rules under which they were conducted differed from the rules under which the auctions in our sample were held.

[19]For example, the investors interested in both lines issued on May 1995—"7.75% April 2005" OATs and "8.5% October 2008" OATs—had to submit their bids separately for the

issued in 1995.

Apart from the information already given in table A1, we observe for each auction all competitive bids submitted by all bidders (the non-competitive bids are not observed in our data), the stop-out price, the amount of securities sold at auction, and the total amounts of ONC1s and ONC2s awarded by the Treasury. The auction participants are unidentified in our data, i.e. we cannot tell from the data which bidder in a given auction corresponds to which bidder in some other auction.

Table 2 gives some overall information about the auctions. In 1995 a total of 45 auctions took place: 25 OAT auctions and 20 BTAN auctions. The total quantity issued by the Treasury in the 45 auctions is FFr464.579 billion, so the mean amount of securities sold is FFr10.324 billion per auction. ***REFER TO TABLE 1*** The total quantity can be split up into the total amount of awarded competitive bids (FFr 423.72 billion), total amount of awarded ONC1s (FFr4.831 billion), and total of awarded ONC2s (FFr36.028 billion). A total of 937 "different " bidders have participated in the 45 auctions, and the total number of competitive bids submitted by these bidders is 2677. About 38% of these 2677 bids were served by the Treasury, 16% were only partially served (because they were at the stop-out price), and almost half of the bids (46%) were loosing bids.

Table 2. Overall information about the auctions

Number of auctions	45
OAT	25 (56%)
BTAN	20 (44%)
Number of bidders	937
Number of bids	2677
Totally served	1 016 (38%)
Partially served	423 (16%)
Not served	1 238 (46%)
Total amount issued by the Treasury (FFr millions)	464 579
competitive bids (FFr millions)	423 720 (91%)
ONC1 (FFr millions)	4 831 (1%)
ONC2 (FFr millions)	36 028 (8%)

Table 3 presents summary statistics per auction. The average number of

two auctions, according to the procedures described in subsection (3.1), and at 11 am the two lines were auctioned independently from each other.

bidders per auction is 20.82. The number of bidders is quite stable across auctions and is close to the total number of SVTs in 1995. The number of bids per auction ranges between 28 and 102 bids, and the mean is about 60 bids. The mean of the auction coverage—which is the ratio of the sum of all submitted competitive bids and ONC1s to the total amount served (the awarded competitive bids and ONC1s)—is equal to 2.25.

Table 3. Summary statistics per auction

Variable	Mean	Std. dev.	Min	Max	Obs
Number of bidders	20.82	1.71	15	23	45
Number of bids	59.49	17.41	28	102	45
Amount issued by Treasury (FFr millions)	10 324	5 922	2 052	21 849	45
Winning competitive bids (FFr millions)	9 416	5 335	1 800	19 125	45
ONC1 (FFr millions)	107	121	0	496	45
ONC2 (FFr millions)	801	820	0	2 553	45
Auction coverage	2.25	0.75	1.29	5.18	45
Maturity of security (in days)	3 749	3 227	586	11 231	45
Nominal yield (%)	7.31	0.80	5.75	8.50	45
Secondary market price	98.07	9.29	71.33	108.50	45
Stop-out price	97.94	9.40	70.88	108.16	45
Highest price bid - lowest price bid	0.32	0.13	0.10	0.68	45
Auction scatter (average price - stop-out price)	0.03	0.02	0.00	0.16	45

The mean of the maturity of the security—defined as the number of days between the settlement date and the date of maturity—is 3 749 days. The average nominal yield in the sample is 7.31%. The average of the secondary market price—defined as the opening secondary market price of the security on the day of the auction—equals 98.07. The stop-out price varies between 70.88 and 108.16, and has a mean equal to 97.94. The dispersion of the submitted prices—defined as the highest price minus the lowest price— is on average 0.32. Finally, the mean of the auction scatter—still another measure of the dispersion of auction prices, and defined as the difference between the weighted average price of the winning bids and the stop-out price—equals 0.03.

Table 4 gives summary statistics per bidder or per bid. The number of submissions per bidder ranges from 1 to 9, and the mean equals 2.86. This is slightly lower than the mean of 3.2 bids found by Gordy (1999) (based on data from Portuguese Treasury auctions), but substantially lower than the mean of 6.9 bids found by Hortaçsu (2000) (Turkish Treasury auctions).

Table 4. Summary statistics per bidder or per bid

Variable	Mean	Std. dev.	Min	Max	Obs
Number of bids	2.86	1.58	1	9	937
Demanded quantity per bid (FFr millions)	326	328	10	2500	2677
Price bid	98.54	7.93	70.54	108.26	2677
Highest price bid - lowest price bid	0.07	0.07	0	0.54	937

In our sample the submitted quantities range from FFr10 million to FFr2.5 billion, and prices from 70.54 to 108.26. The price dispersion per bidder ranges between 0 (if the bidder has submitted only one bid) and 0.54.

4 Results

4.1 Estimation of the parameters of the discriminatory model

The secondary market price, the nominal yield and the maturity of the security (divided by 1000) sold at the l-th auction are the variables included in the vector z_l. The dimension of z_l is thus equal to 3, and we denote $z_l = (z_{1l}, z_{2l}, z_{3l})$. In the first step of our estimation procedure we nonparametrically estimate the distribution function $G(x|n, z; p)$ using the Epanechnikov kernel. To avoid any confusion with the vector z_l, we denote the 3-dimensional vector of explanatory variables at which $G(\cdot|\cdot, \cdot; \cdot)$ is evaluated, as $z = (z^1, z^2, z^3)$. In expression (12) we thus have $K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right) = K\left(\frac{n-n_l}{h_N}\right) K\left(\frac{z^1-z_{1l}}{h_{1Z}}\right) K\left(\frac{z^2-z_{2l}}{h_{2Z}}\right) K\left(\frac{z^3-z_{3l}}{h_{3Z}}\right)$ where $K(u) = 0.75(1-u^2)1\{|u| \leq 1\}$, and h_N, h_{1Z}, h_{2Z} and h_{3Z} are the bandwidth parameters. The choice of all bandwidth parameters were chosen according to the rule of thumb defining each bandwidth as 2.214 multiplied by the standard error of the variable multiplied the number of observations (L) to the power $-\frac{1}{7}$.[20] We find $h_N = 2.2$,

[20]Newey and McFadden (1994, pp. 2203-2210) impose conditions on the choice of the kernel and the convergence rate of the bandwidth parameters. To satisfy these conditions,

$h_{1Z} = 11.9$ (bandwidth of the secondary market price), $h_{2Z} = 1.0$ (nominal yield), and $h_{3Z} = 4.1$ (maturity of the security divided by 1000).

To proceed with the second step, we have to choose parametric specifications for the distribution functions of the signal and the value. For the revenue comparison in the next subsection, the specifications should be chosen such that explicit optimal strategies can be obtained in the uniform share auction model. Bearing this in mind, we assume that the value V_l given $Z_l = z_l$ has the distribution function

$$F_{V|Z}(v|z_l;\theta_1) = \int_0^v \gamma u^{\gamma-1}\frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)}u^{\gamma(\alpha_l-1)}\exp\left[-\beta_l u^\gamma\right]du \tag{15}$$

where

$$\alpha_l = (1, z_l)\cdot\alpha$$
$$\beta_l = (1, z_l)\cdot\beta$$

and $\Gamma(\cdot)$ is the gamma function, α and β are vectors (of dimension 4 by 1) of parameters, and γ is a scalar parameter. Note that if $\gamma = 1$ then the above distribution function corresponds to the gamma distribution function with parameters α_l and β_l, i.e. in this case V_l follows a gamma distribution with conditional mean α_l/β_l and conditional variance α_l/β_l^2; if $\gamma \neq 1$ then V_l^γ is distributed as a gamma distribution with parameters α_l and β_l; if $\alpha_l = 1$ then V_l follows a weibull distribution with parameters γ and β_l. Note also that $\theta_1' = (\alpha', \beta', \gamma)$.

We furthermore assume that the signal S_{il} given $V_l = v_l$ and $Z_l = z_l$ follows an exponential distribution:

$$F_{S|V,Z}(s|v_l, z_l;\theta_2) = 1 - \exp\left[-s v_l^\gamma\right] \tag{16}$$

where γ is the scalar parameter that also appears in the conditional distribution function of V_l. Note that the conditional expectation and variance of S_{il} are assumed to be independent of z_l. Note also that $\theta_2 = \gamma$, so the complete vector of parameters is therefore $\theta' = (\alpha', \beta', \gamma)$.

In the second step of the estimation procedure we estimate θ^0, the true value of θ. The estimate of this parameter is defined by (14), where, given

we chose the Epanechnikov kernel ($m = 2$ in the notation of Newey and McFadden's lemma 8.10, p. 2206), and the convergence rate $L^{-\frac{1}{7}}$. Furthermore in this formula, the factor 2.214 is the constant associated with the Epanechnikov kernel.

the specifications (15) and (16), the conditional expectation of V_l appearing in the empirical moment $m(.)$ is:

$$E\left(V_l | S_{1l} = \tilde{x}^{-1}(x_{1lp}, p, n_l, z_l; \theta), ..., S_{n_l l} = \tilde{x}^{-1}(x_{n_l lp}, p, n_l, z_l; \theta), N_l = n_l, Z_l = z_l\right)$$

$$= \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{\left(\beta_l + \sum_{i=1}^{n_l} \beta_l \left[\frac{1}{\widehat{G}^{1/\alpha_l}(x_{ilp}|n_l, z_l; p)} - 1\right]\right)^{1/\gamma}}. \tag{17}$$

We choose $T = 45$, i.e. the number of moments equals the number of auctions in the sample, and the prices $p_1, ..., p_T$ are equal to the observed stop-out prices.

The second-step estimates are presented in Table 5, together with the estimated standard errors (using the estimated asymptotic variance matrix of $\hat{\theta}$ given in the appendix).

Table 5. Second-step estimate of θ (est. standard error)

Estimate of α :	
Constant	-15848.93 (320.72)**
Secondary market price	66.67 (3.48)**
Nominal yield	1617.29 (8.76)**
Maturity of security (in days/1000)	142.24 (6.96)**
Estimate of β :	
Constant	8596.67 (114.87)**
Secondary market price	-104.30 (1.24)**
Nominal yield	340.41 (6.77)**
Maturity of security (in days/1000)	-2.04 (1.57)
γ	12.28 (0.0085)**

Note: Actual estimates and standard errors in β are very small, and those reported in the table are multiplied by 10^{24}.

All parameters are significantly different from zero at the 5% level, except the parameter corresponding to the maturity of the security in β. Given the estimate $\hat{\theta}$ and using (15), we can calculate $E(V_l|Z_l = z_l)$, and the derivative of this expectation with respect to each variable in z_l. Evaluated at the empirical mean of the explanatory variables (i.e. we replace z_l by the averages reported in Table 3), we find that the conditional mean of the value equals 99.79, which is above the average secondary market price (98.07) and the average stop-out-price (97.94). The derivative with respect to the nominal yield, the secondary market price and the maturity are 1.06, 1.18 and 0.40 respectively. All these variables thus have a positive effect on the expected value of the security, as intuition suggests. Table A2 in the appendix gives for all auctions $l = 1, ..., 45$, the estimated expectations $E[V_l|S_{1l} = \hat{s}_{1l}, ..., S_{n_l l} = \hat{s}_{n_l l}, Z_l = z_l]$ (with the estimated signal $\hat{s}_{il} = F^{-1}_{S|Z}(1 - \hat{G}(x_{ilp_l^0}|n_l, z_l; p_l^0)|z_l; \hat{\theta})$, and z_l the value of the characteristics of the security at the l-th auction) and $E[V_l|Z_l = z_l]$, the stop-out-price, and the secondary market price.

4.2 Revenue comparison

In this subsection we compare the actual income of the French Treasury with the hypothetical income the Treasury would have earned had it adopted the uniform share auction mechanism. The actual revenue from a given discriminatory auction simply equals the sum, over all winning bids in the auction, of awarded quantities multiplied by the associated prices. The total actual income earned by the Treasury is then the sum, over the 45 auctions held in 1995, of these actual revenues. In 1995 the total actual income thus calculated amounts to FFr421.543 billion.

The calculation of the hypothetical total income under the uniform auction format is less straightforward. First we need to determine an explicit optimal bidding strategy in the uniform auction format. Given our parametric specifications of the distribution functions (15) and (16), we show in appendix C that an optimal strategy in the uniform auction is (i.e. a solution of (3))[21]

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - \left\{\frac{\beta_l}{n_l} + s_{il}\right\}\left\{\frac{\Gamma(n_l + \alpha_l)}{\Gamma(n_l + \alpha_l + 1/\gamma)}\frac{1+\gamma}{\gamma}p\right\}^{\gamma}\right] / (n_l - 1)\,. \tag{18}$$

As mentioned in subsection 2.2.3, this optimal bidding function is decreasing in the signal s_{il} (it is also decreasing in p). In appendix C we

[21]Of course, as with the Euler condition of the discriminatory model, the auction-specific notation and variables should be incorporated in the uniform Euler condition (3).

also show that given our distributional assumptions, (18) is actually the unique equilibrium strategy in the class of demand functions of the form $x(p, s_{il}, n_l, z_l; \theta) = a(p, n_l, z_l; \theta) + b(p, n_l, z_l; \theta)s_i$. There are no restrictions on the functions $a(\cdot, \cdot, \cdot; \cdot)$ and $b(\cdot, \cdot, \cdot; \cdot)$ except that they must be such that $x(\cdot, \cdot, \cdot, \cdot; \cdot)$ is decreasing in p and s_{il}.

The strategy (18) is a generalization of the equilibrium strategy derived in Wilson (1979, example 1). Wilson assumes that V_l follows a gamma distribution with parameters α_l and β_l, and that S_{il} is exponentially distributed with parameter v_l. His setup thus corresponds to the special case where the parameter γ appearing in the distribution functions (15) and (16) is equal to 1. When $\gamma = 1$, (18) reduces to

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - 2p\frac{\beta_l + n_l s_{il}}{n_l(n_l + \alpha_l)}\right] / \, (n_l - 1) \tag{19}$$

which is the optimal bidding function given by Wilson.[22]

Now that an optimal bidding strategy has been determined, a possible way to calculate the hypothetical income under the uniform auction format is as follows. Given the two-step estimate $\hat{\theta}$ and using (**??**), define, for each bidder i and auction l, the estimated signal $\hat{s}_{il} = F^{-1}_{S|Z}(1 - \hat{G}(x_{ilp}|n_l, z_l; p)|z_l; \hat{\theta})$. Replacing, in (18), θ by $\hat{\theta}$, and for all i s_{il} by $\hat{s}_{il}$, gives the estimated uniform demand functions for all bidders i in auctions l. The estimated stop-out price in the l-th uniform auction can then be determined by equating estimated aggregate demand and total supply (i.e. the amount of securities sold at the l-th discriminatory auction). The hypothetical revenue from auction l is then the product of total supply and the estimated stop-out price, and the total hypothetical income under the uniform auction follows from summation over all 45 auctions in the sample.

We will however calculate the hypothetical total income in a different and more direct way by exploiting the fact that for our uniform share auction model there is an explicit solution for the stop-out price. Under the assumption that agents bid according to the optimal demand function (18),

[22] In Wilson (1979) the numerator is $\beta_l - n_l s_{il}$ instead of $\beta_l + n_l s_{il}$. But this simply reflects a difference in the choice of the conditional distribution function of the signal. In Wilson the conditional distribution function is $F_{S|V,Z}(s_{il}|v_l, z_l; \theta_2) = e^{v_l s_{il}}$ for $s_{il} \leq 0$.

it can be shown that the stop-out price in the l-th uniform auction satisfies

$$\begin{aligned} p_l^0 &= \frac{1}{1+1/\gamma} E\left(V_l | S_{1l} = s_{1l}, ..., S_{n_l l} = s_{n_l l}, Z_l = z_l\right) \qquad (20) \\ &= \frac{1}{1+1/\gamma} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\beta_l + \sum_{i=1}^{n_l} s_{il})^{1/\gamma}}. \end{aligned}$$

The equation for the stop-out price given by Wilson (1979, page 682) corresponds to (20) with $\gamma = 1$.[23]

In order to calculate the hypothetical uniform revenue we construct, as above, the estimated signals $\hat{s}_{il}$. The estimated stop-out price in auction l can be obtained directly on replacing θ and s_{il} by their estimates in (20). The hypothetical income in auction l and the total hypothetical income are then calculated as above.[24]

The hypothetical uniform revenue we calculate in this manner equals FFr400.421 billion. Therefore, had the French Treasury adopted the uniform auction format instead of the discriminatory auction format, it would have earned FFr21.122 billion less (5% of total income in the discriminatory auctions). In calculating the variance of the above estimate, we use the fact that the hypothetical revenue is simply the sum over all auctions l of p_l^0 times the amount of securities sold in the l-th auction. Hence, the estimated hypothetical revenue is some function of $\hat{\theta}$ and (via the estimated signals $\hat{s}_{il}$) $\hat{G}(.)$, so its variance can be calculated by applying the delta method. In applying the delta method we ignore the variance in $\hat{G}(.)$, i.e. we consider it as fixed. The estimated standard error we find is FFr1.11 billion, and the 95% confidence interval for the hypothetical revenue is [FFr398.210 billion; FFr402.632 billion], implying that the difference in revenue between the discriminatory and uniform auction is significant at the 5% level.

When we restrict $\gamma = 1$, the results are very different: the estimated revenue under the uniform auction drops to FFr 215.462 billion, implying a substantial loss in income of FFr206.081 billion (almost 49%). However, since the hypothesis $\gamma = 1$ is rejected by the data (see Table 4), these last

[23]Except that the denominator in the last term equals $\beta_l - \sum_{i=1}^{n_l} s_{il}$ instead of $\beta_l + \sum_{i=1}^{n_l} s_{il}$.

[24]The total hypothetical income has been calculated in two ways. First by estimating the signals as $\hat{s}_{il} = F_{S|Z}^{-1}(1 - \hat{G}(x_{ilp_l^0} | n_l, z_l; p_l^0) | z_l; \hat{\theta})$, and second by estimating the signals as an average $\hat{s}_{il} = \frac{1}{L} \sum_{l'=1}^{L} F_{S|Z}^{-1}(1 - \hat{G}(x_{ilp_{l'}^0} | n_l, z_l; p_{l'}^0) | z_l; \hat{\theta})$. The estimated hypothetical income is not affected by the manner in which the signals are estimated.

figures have no statistical justification, and cannot therefore be treated without much suspicion. They merely show that Wilson's model is too restrictive for the analysis of our data, and leads to overly negative conclusions regarding the performance of the uniform auction format.

As in Hortaçsu (2000), we find that the discriminatory auction is revenue-superior to the uniform auction. Our estimated revenue loss of 5% is smaller though than the revenue loss obtained by Hortaçsu using Turkish Treasury auctions. He reports counterfactual revenue comparisons for each of the 25 auctions in his sample. His *ex-ante* revenue differences vary between 0.12% and 27%, and on average the uniform auction generated 14% less than the discriminatory auction.

4.3 Robustness analysis

In Section 2.2.2 we showed that the share auction model is not identified nonparametrically. This means that the economic question being answered in this paper may be very sensitive to the parametric specification being utilized. It is therefore important to estimate alternative specifications and to check if our previous empirical results are robust.

The set of possible alternative models is unfortunately quite small since we need to confine ourselves to specifications that imply an analytical solution of the equilibrium strategy in the corresponding uniform auction. One possible alternative model that satisfies this criterion is the model considered by Kyle (1989). TO BE COMPLETED.

5 Conclusion

This paper has proposed structural econometric methods for the empirical study of Wilson's share auction model. We have shown how the parameters of this model, i.e. the joint distribution function of the value of the good and the signals received by the bidders, can be estimated via a two-step estimation procedure. Using the estimation theory for semiparametric two-step estimators developed by Newey and McFadden (1994), we have established the asymptotic properties of our estimator. A crucial feature of the method of statistical inference is that it only relies on the first-order condition of the bidder's maximization problem. Our estimation method seems therefore potentially of interest whenever one wants to estimate game theoretic models for which no explicit strategies can be found.

The methods have been applied to Treasury auctions held in France. Our results suggest that the Treasury's revenue in the discriminatory share auction is 5% higher than in the uniform share auction, which is a relatively high figure given the enormous amounts of money at stake. This result can be seen as an ex-post justification for the fact that the majority of countries rely on the discriminatory auctions to sell their Treasury securities.

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APPENDIX

A The Euler condition for the discriminatory auction model

First we rewrite the expected profit (4). We have

$$
\begin{aligned}
& E\left\{\int_0^\infty \left[(V-p)y(p,s_i) - \int_p^{p^{\max}} y(u,s_i)du\right] dH(p;V,y(p,s_i))|S_i = s_i\right\} \\
= \;& E\left\{-\int_0^\infty \left[-y(p,s_i) + (V-p)y_p(p,s_i) + y(p,s_i)\right] H(p;V,y(p,s_i))dp|S_i = s_i\right\} \\
= \;& E\left\{\int_0^\infty \left[-(V-p)y_p(p,s_i)H(p;V,y(p,s_i))\right] dp|S_i = s_i\right\}
\end{aligned}
$$

where the first equality follows from an integration by parts, and y_p is the derivative of y with respect to p. The expression in brackets can be written as a function $g(p, y(p, s_i), y_p(p, s_i), V)$. We want to stress that there is no connection at all between this function g and the distribution function G defined in the main text. Using this notation, the expected profit can be rewritten as

$$
E\left\{\int_0^\infty \left[g(p,y(p,s_i),y_p(p,s_i),V)\right] dp|S_i = s_i\right\}. \tag{21}
$$

The necessary condition for $y(\cdot, s_i)$ to maximize (21) is that for all $p \in [0, \infty)$

$$
0 = E\left\{\frac{\partial g}{\partial y} - \frac{d}{dp}\frac{\partial g}{\partial y_p}|S_i = s_i\right\} \tag{22}
$$

(see Chiang, 1992, p. 46). Given the specific form of the function g, we have

$$
\frac{\partial g}{\partial y} = -(V-p)y_p(p,s_i)\frac{\partial H(p;V,y(p,s_i))}{\partial y}
$$

and

$$
\frac{d}{dp}\frac{\partial g}{\partial y_p} = H(p;V,y(p,s_i)) - (V-p)\left\{\frac{\partial H(p;V,y(p,s_i))}{\partial p} + y_p(p,s_i)\frac{\partial H(p;V,y(p,s_i))}{\partial y}\right\}
$$

so that (22) can be rewritten as

$$0 = E\left\{-H(p;V,y(p,s_i)) + (V-p)\frac{\partial H(p;V,y(p,s_i))}{\partial p}|S_i = s_i\right\}.$$

The strategy $x(\cdot,\cdot)$ is optimal if the above condition is satisfied for $y(\cdot,s_i) = x(\cdot,s_i)$, which gives the Euler condition (5).

B Proof of Proposition 1

Here we show that the Euler condition (5) can be rewritten as (6). We have

$$H(p;v,y) = \int ... \int_{s_j;j\neq i} \mathbf{1}\left\{\sum_{j\neq i} x(p,s_j) \leq 1-y\right\} \prod_{j\neq i} f_{S|V}(s_j|v)ds_j$$

where $f_{S|V}(\cdot|\cdot)$ is the density associated with $F_{S|V}(\cdot|\cdot)$. Defining $x_{jp} = x(p,s_j)$, the above expression can be written as

$$H(p;v,y) = \int ... \int_{x_{jp};j\neq i} \mathbf{1}\left\{\sum_{j\neq i} x_{jp} \leq 1-y\right\} \prod_{j\neq i} \frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}$$

so that (because the integrand is symmetric in all the x_{jp})

$$\begin{aligned}
\frac{\partial H(p;v,y)}{\partial p} &= (n-1)\int ... \int_{x_{jp};j\neq i} \mathbf{1}\left\{\sum_{j\neq i} x_{jp} \leq 1-y\right\} \\
&\quad \times \frac{\partial}{\partial p}\left[\frac{\partial x^{-1}(p,x_{1p})}{\partial x_{1p}} f_{S|V}(x^{-1}(p,x_{1p})|v)\right] dx_{1p} \\
&\quad \times \prod_{\substack{j\neq i \\ j\neq 1}} \frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}.
\end{aligned}$$

Now define $H(p;v) = \Pr(P^0 \leq p|V=v)$ and $H(p) = \Pr(P^0 \leq p)$. We have

$$\begin{aligned}
H(p;v) &= \int ... \int \mathbf{1}\left\{\sum_{j=1}^{n} x(p,s_j) \leq 1\right\} \prod_{j=1}^{n} f_{S|V}(s_j|v)ds_j \\
&= \int ... \int \mathbf{1}\left\{\sum_{j=1}^{n} x_{jp} \leq 1\right\} \prod_{j=1}^{n} \frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}.
\end{aligned}$$

So, again by symmetry, we have

$$\begin{aligned}\frac{dH(p;v)}{dp} &= n\int ...\int \mathbf{1}\left\{\sum_{j=1}^{n} x_{jp}\leq 1\right\}\frac{\partial}{\partial p}\left[\frac{\partial x^{-1}(p,x_{1p})}{\partial x_{1p}}f_{S|V}(x^{-1}(p,x_{1p})|v)\right]dx_{1p}\\ &\quad \times\prod_{j=2}^{n}\frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}}f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}.\end{aligned}$$

After some straightforward calculations it follows that

$$\begin{aligned}E\left\{\frac{\partial H(p;v,x(p,S_i))}{\partial p}|V=v\right\} &= \int \frac{\partial H(p;v,x(p,s_i))}{\partial p}f_{S|V}(s_i|v)ds_i\\ &= \frac{(n-1)}{n}\frac{dH(p;v)}{dp}\end{aligned}$$

and therefore

$$E\left\{\frac{\partial H(p;V,x(p,S_i))}{\partial p}\right\}=\frac{(n-1)}{n}\frac{dH(p)}{dp}.$$

We also have

$$\begin{aligned}E\left\{V\frac{\partial H(p;V,x(p,S_i))}{\partial p}\right\} &= \frac{(n-1)}{n}\int vf_V(v)\frac{dH(p;v)}{dp}dv\\ &= \frac{(n-1)}{n}\frac{d}{dp}\left[\int vf_V(v)\int ...\int \mathbf{1}\left\{P^0\leq p\right\}\prod_{j=1}^{n}f_{S|V}(s_j|v)ds_jdv\right]\\ &= \frac{(n-1)}{n}\frac{d}{dp}\left[E\left(E\left(V|S_1=s_1,...,S_n=s_n\right)\cdot\mathbf{1}\left\{P^0\leq p\right\}\right)\right]\end{aligned}$$

where $f_V(\cdot)$ is the density associated with $F_V(\cdot)$. Finally we have $E\left\{H(p;V,x(p,S_i))\right\}=H(p)$.

Therefore, taking the expectation with respect to V, S_i, the Euler condition (5) can be rewritten as

$$\begin{aligned}0 &= E\left\{V\frac{\partial H(p;V,x(p,S_i))}{\partial p}\right\}-pE\left\{\frac{\partial H(p;V,x(p,S_i))}{\partial p}\right\}-E\left\{H(p;V,x(p,S_i))\right\}\\ &= \frac{(n-1)}{n}\frac{d}{dp}\left[E\left(E\left(V|S_1=s_1,...,S_n=s_n\right)\cdot\mathbf{1}\left\{P^0\leq p\right\}\right)\right]\\ &\quad -p\frac{(n-1)}{n}\frac{dH(p)}{dp}-H(p).\end{aligned}$$

Integrating over p gives

$$\begin{aligned} C &= E\left\{(n-1)\cdot\left(E\left(V|S_1=s_1,...,S_n=s_n\right)-p\right)\cdot\mathbf{1}\left\{P^0\leq p\right\}\right\} \\ &\quad -E\left\{\left(p-P^0\right)\cdot\mathbf{1}\left\{P^0\leq p\right\}\right\} \end{aligned}$$

where C is the integration constant. We now assume that $\Pr(P^0 \leq 0) = 0$. Replacing $p = 0$ in the above expression, it follows then that $C = 0$, which gives the condition (6).

C An optimal strategy for the uniform auction model

In this appendix we prove the claim that (18) is the unique equilibrium strategy belonging to the class of demand functions of the form $x(p, s_{il}, n_l, z_l; \theta) = a(p, n_l, z_l; \theta) + b(p, n_l, z_l; \theta)s_{il}$.

First we derive the conditional expectation $E(V_l|\tilde{S}_l = \tilde{s}_l, S_{il} = s_{il})$ where $\tilde{S}_l \equiv \sum_{j\neq i} S_{jl}$. In this section of the appendix a density function (marginal, joint, or conditional) is represented by the letter g. Again, there is no connection between this function g and the distribution function G defined in the main text. We have

$$
\begin{aligned}
E(V_l|\tilde{S}_l &= \tilde{s}_l, S_{il} = s_{il}) \\
&= \int_0^\infty v_l g(v_l|\tilde{s}_l, s_{il}) dv_l \\
&= \frac{\int_0^\infty v_l g(\tilde{s}_l|v_l) g(s_{il}|v_l) g(v_l) dv_l}{\int_0^\infty g(\tilde{s}_l|v_l) g(s_{il}|v_l) g(v_l) dv_l} \\
&= \frac{\int_0^\infty v_l \frac{v_l^{\gamma(n_l-1)}}{\Gamma(n_l-1)} \tilde{s}_l^{n_l-2} e^{-v_l^\gamma \tilde{s}_l} v_l^\gamma e^{-v_l^\gamma s_{il}} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} v_l^{\gamma(\alpha_l-1)} e^{-\beta_l v_l^\gamma} \gamma v_l^{\gamma-1} dv_l}{\int_0^\infty \frac{v_l^{\gamma(n_l-1)}}{\Gamma(n_l-1)} \tilde{s}^{n_l-2} e^{-v_l^\gamma \tilde{s}_l} v_l^\gamma e^{-v_l^\gamma s_{il}} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} v_l^{\gamma(\alpha_l-1)} e^{-\beta_l v_l^\gamma} \gamma v_l^{\gamma-1} dv_l} \\
&= \frac{\int_0^\infty v_l^{1+\gamma(n_l+\alpha_l-1)} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l^\gamma} \gamma v_l^{\gamma-1} dv_l}{\int_0^\infty v_l^{\gamma(n_l+\alpha_l-1)} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l^\gamma} \gamma v_l^{\gamma-1} dv_l} \\
&= \frac{\int_0^\infty v_l^{n_l+\alpha_l-1+1/\gamma} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l} dv_l}{\int_0^\infty v_l^{n_l+\alpha_l-1} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l} dv_l} \\
&= \frac{\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l+1/\gamma}} \int_0^\infty \frac{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l+1/\gamma}}{\Gamma(n_l+\alpha_l+1/\gamma)} v_l^{n_l+\alpha_l-1+1/\gamma} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l} dv_l}{\frac{\Gamma(n_l+\alpha_l)}{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l}} \int_0^\infty \frac{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l}}{\Gamma(n_l+\alpha_l)} v_l^{n_l+\alpha_l-1} e^{-(\tilde{s}_l+\beta_l+s_{il})v_l} dv_l} \\
&= \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l+1/\gamma}} \frac{(\tilde{s}_l+\beta_l+s_{il})^{n_l+\alpha_l}}{\Gamma(n_l+\alpha_l)} \\
&= \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \frac{1}{(\tilde{s}_l+\beta_l+s_{il})^{1/\gamma}}.
\end{aligned}
$$

The second equality follows from the assumption that the signals $S_{1l}, ..., S_{nl}$ are independent given $V_l = v_l$. The third equality follows from the assumption that V_l^γ follows a gamma distribution with parameters α_l and β_l, and $S_{il}|V_l = v_l$ an exponential distribution with parameter v_l^γ, and the well-known fact that the sum of $n_l - 1$ i.i.d. exponential variables (with mean $1/v_l^\gamma$) follows a gamma distribution with parameters v_l^γ and $n_l - 1$.

Let us now simplify somewhat the notation by suppressing n_l, z_l and θ in $x(\cdot)$, $a(\cdot)$ and $b(\cdot)$. Since we are looking for equilibria of the form $x(p, s_{il}) =$

$a(p) + b(p)s_{il}$, we have

$$\begin{aligned} H(p; v_l, y) &= \Pr\left\{\sum_{j \neq i} x(p, S_{jl}) \leq 1 - y | V_l = v_l, S_{il} = s_{il}\right\} \\ &= \Pr\left\{\sum_{j \neq i} [a(p) + b(p)S_{jl}] \leq 1 - y | V_l = v_l, S_{il} = s_{il}\right\} \\ &= \Pr\left\{\tilde{S}_l \geq \frac{1 - y - (n_l - 1)a(p)}{b(p)} | V_l = v_l, S_{il} = s_{il}\right\} \\ &= 1 - \Pr\left\{\tilde{S}_l \leq \frac{1 - y - (n_l - 1)a(p)}{b(p)} | V_l = v_l, S_{il} = s_{il}\right\} \end{aligned}$$

where the last equation follows from the fact that the functions $a(\cdot)$ and $b(\cdot)$ are such that $x(\cdot, \cdot)$ is decreasing in p and s_{il}. Letting $\tilde{s}_l = [1 - y - (n_l - 1)a(p)]/b(p)$ we have

$$H_p(p; v_l, y) = -g(\tilde{s}_l | v_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial p}$$

$$H_y(p; v_l, y) = -g(\tilde{s}_l | v_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial y}$$

and therefore

$$\begin{aligned} & E\left\{(V_l - p)H_p(p; V_l, y) + yH_y(p; V_l, y) | S_{il} = s_{il}\right\} \\ = \; & E\left\{-(V_l - p)g(\tilde{s}_l | V_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial p} - yg(\tilde{s}_l | V_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial y} | S_{il} = s_{il}\right\} \\ = \; & E\left\{-(V_l - p) | \tilde{S}_l = \tilde{s}_l, S_{il} = s_{il}\right\} g(\tilde{s}_l | s_{il}) \frac{\partial \tilde{s}_l}{\partial p} - yg(\tilde{s}_l | s_{il}) \frac{\partial \tilde{s}_l}{\partial y} \\ = \; & -\left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\tilde{s}_l + \beta_l + s_{il})^{1/\gamma}} - p\right) \\ & \times g(\tilde{s}_l | s_{il}) \frac{-(n_l - 1)a'(p)b(p) - b'(p)[1 - y - (n_l - 1)a(p)]}{b^2(p)} - yg(\tilde{s}_l | s_{il}) \frac{-1}{b(p)}. \end{aligned}$$

The Euler condition (3) (including the auction-specific notation and variables) amounts to evaluating the above expression at $y = x(p, s_{il}) = a(p) +$

$b(p)s_{il}$ and then equating the expression to zero:

$$\begin{aligned} 0 &= E\left\{(V_l-p)H_p(p;V_l,y)+x(p,s_{il})H_y(p;V_l,y)|S_{il}=s_{il}\right\} \\ &= \left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{1}{(\beta_l+\frac{1-n_la(p)}{b(p)})^{1/\gamma}}-p\right)\left(-\frac{(n_l-1)a'(p)}{b(p)}-\frac{b'(p)}{b(p)}\tilde{s}_l\right) \\ &\quad -\frac{1-(n_l-1)a(p)-b(p)\tilde{s}_l}{b(p)} \end{aligned}$$

where we have used that $\tilde{s}_l=[1-n_la(p)-b(p)s_{il}]/b(p)$ so that $\tilde{s}_l+s_{il}=(1-n_la(p))/b(p)$. The above equality must hold for all values of s_{il} and hence for all values of $\tilde{s}_l$. Therefore it must be that

$$\left\{\begin{array}{l} \left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{1}{(\beta_l+\frac{1-n_la(p)}{b(p)})^{1/\gamma}}-p\right)\left(-\frac{(n_l-1)a'(p)}{b(p)}\right)-\frac{1-(n_l-1)a(p)}{b(p)}=0 \\ -\left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{1}{(\beta_l+\frac{1-n_la(p)}{b(p)})^{1/\gamma}}-p\right)\frac{b'(p)}{b(p)}+1=0 \end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l} \left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{(-b(p))^{1/\gamma}}{(-\beta_lb(p)-1+n_la(p))^{1/\gamma}}-p\right)((n_l-1)a'(p))+1-(n_l-1)a(p)=0 \\ \left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{(-b(p))^{1/\gamma}}{(-\beta_lb(p)-1+n_la(p))^{1/\gamma}}-p\right)b'(p)-b(p)=0. \end{array}\right. \tag{23}$$

The two equalities imply that

$$-\frac{b(p)}{b'(p)}=\frac{1-(n_l-1)a(p)}{(n_l-1)a'(p)}$$

$$\Leftrightarrow -(n_l-1)a'(p)b(p)+(n_l-1)a(p)b'(p)=b'(p)$$

$$\Leftrightarrow (n_l-1)\left(\frac{a(p)b'(p)-a'(p)b(p)}{b^2(p)}\right)=\frac{b'(p)}{b^2(p)}$$

$$\Leftrightarrow (n_l-1)\frac{a(p)}{b(p)}=\frac{1}{b(p)}+C_1$$

$$\Leftrightarrow a(p)=(1+C_1b(p))/(n-1).$$

where C_1 is the integration constant. Inserting this expression for $a(p)$ into the second equality of (23) gives

$$\left(\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}\frac{(-b(p))^{1/\gamma}}{\left(-\beta_l b(p)-1+\frac{n_l}{n_l-1}+\frac{n_l C_1 b(p)}{n_l-1}\right)^{1/\gamma}}-p\right)b'(p)-b(p)=0$$

$$\Leftrightarrow b'(p)\left[\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}(-b(p))^{1/\gamma}-\left(\left(\frac{C_1 n}{n_l-1}-\beta_l\right)b(p)+\frac{1}{n_l-1}\right)^{1/\gamma}p\right]$$
$$-\left(\left(\frac{C_1 n_l}{n_l-1}-\beta_l\right)b(p)+\frac{1}{n_l-1}\right)^{1/\gamma}b(p)=0.$$

Defining $C_2=\frac{C_1 n_l}{n_l-1}-\beta_l$ the last equation becomes

$$b'(p)\left[\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}(-b(p))^{1/\gamma}-\left(C_2 b(p)+\frac{1}{n_l-1}\right)^{1/\gamma}p\right]-\left(C_2 b(p)+\frac{1}{n_l-1}\right)^{1/\gamma}b(p)=0$$

which can be rewritten as (using the transformation $u=b(p)$, so that $p(u)=b^{-1}(p)$)

$$\frac{1}{p'(u)}\left[\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}(-u)^{1/\gamma}-\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}p(u)\right]-\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}u=0$$

or as

$$p'(u)\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}u+p(u)\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}=\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}(-u)^{1/\gamma}. \quad (24)$$

The solution of the differential equation

$$p'(u)\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}u+p(u)\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}=0$$

is $p(u)=C_3/u$ where C_3 is an integration constant. The solution of the differential equation (24) is therefore necessarily of the form $p(u)=C(u)/u$ where $C(\cdot)$ is some function of u. Inserting this solution into (24) gives

$$\frac{C'(u)}{u}\left(C_2 u+\frac{1}{n_l-1}\right)^{1/\gamma}u=\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)}(-u)^{1/\gamma}$$

so that

$$C'(u) = \begin{cases} \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \left[\frac{-u}{C_2 u + \frac{1}{n_l-1}} \right]^{1/\gamma} \text{ if } C_2 \neq 0 \\ \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \left[-(n_l-1)u \ \right]^{1/\gamma} \text{ if } C_2 = 0 \end{cases}$$

so that

$$C(u) = \begin{cases} \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \int_u^0 \left[\frac{-t}{C_2 t + \frac{1}{n_l-1}} \right]^{1/\gamma} dt + C_4 \text{ if } C_2 \neq 0 \\ -\frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \left[(n_l-1) \ \right]^{1/\gamma} \frac{\gamma}{1+\gamma} (-u)^{(1+\gamma)/\gamma} + C_5 \text{ if } C_2 = 0 \end{cases}$$

where C_4 and C_5 are integration constants. Since $p(u) = C(u)/u$ the solutions are therefore

$$p(u) = \begin{cases} \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \frac{\int_u^0 \left[\frac{-t}{C_2 t + \frac{1}{n_l-1}} \right]^{1/\gamma} dt}{u} + \frac{C_4}{u} \text{ if } C_2 \neq 0 \\ \frac{\Gamma(n_l+\alpha_l+1/\gamma)}{\Gamma(n_l+\alpha_l)} \left[(n_l-1) \ \right]^{1/\gamma} \frac{\gamma}{1+\gamma} (-u)^{1/\gamma} + \frac{C_5}{u} \text{ if } C_2 = 0. \end{cases} \tag{25}$$

Now recall from the main text that the functions $a(p)$ and $b(p)$ must be such that $x(p, s_{il}) = a(p) + b(p)s_{il}$ is decreasing in p and s_{il}. This implies in particular that $b(p) < 0$ and $b'(p) < 0$, which in turn implies that $p(u) \geq 0$ for all $u < 0$ and $p'(u) < 0$ for all $u < 0$. Now consider the case $C_2 = 0$, i.e. $C_1 = \frac{n_l-1}{n_l}\beta_l$. The condition $p(u) \geq 0$ for all $u < 0$ then implies $C_5 \leq 0$, and the condition $p'(u) < 0$ for all $u < 0$ implies $C_5 \geq 0$. Thus $C_5 = 0$, which from (25) implies that

$$b(p) = -\left[\frac{\Gamma(n_l+\alpha_l)}{\Gamma(n_l+\alpha_l+1/\gamma)} \frac{1+\gamma}{\gamma} \right]^{\gamma} \frac{p^{\gamma}}{n_l-1}$$

and

$$a(p) = \frac{(1 + C_1 b(p))}{n_l - 1} = \frac{1}{n_l-1} - \frac{\beta_l}{n_l(n_l-1)} \left[\frac{\Gamma(n_l+\beta_l)}{\Gamma(n_l+\alpha_l+1/\gamma)} \frac{1+\gamma}{\gamma} \right]^{\gamma} p^{\gamma}.$$

Inserting these expressions into $x(p, s_{il}) = a(p) + b(p)s_{il}$ gives (18). One can show that there does not exist a solution that verifies the conditions for the case $C_2 \neq 0$.

D Asymptotic properties of the estimator

This appendix extensively uses chapter 8 of Newey and McFadden (1994, pp. 2194-2215). The results in this appendix cannot be read independently without consulting Newey and McFadden.

In deriving the asymptotic properties of our estimator (14), we suppose, for simplicity, that $n_l = n$ for all l, and $T = 1$. The proofs are similar but more involved when the number of bidders is allowed to vary from auction to auction. The proof for $T = 1$ corresponds to what has to be done component by component for $T > 1$.

Some notations are specific to this appendix, and do not necessarily refer to the notations in the main body of the paper. We do this to facilitate the apprehension of this appendix, which heavily relies on Newey and McFadden and their notations. To avoid any possible confusion, all notations that have different meanings in the appendix and the main text, are redefined.

We first show that our estimator (14) belongs to the class of semiparametric two-step estimators considered by Newey and McFadden (1994), with the first-step estimator being a kernel estimator (section 8.3). Our estimator (14) solves the equation:

$$\frac{1}{L}\sum_{l=1}^{L} m(p_l^0, z_l; \theta; \widehat{\gamma}(x_{.lp}, z_l); p) = 0 \tag{26}$$

where $x_{.lp} = (x_{1lp}, ..., x_{nlp})$. As we restrict ourselves to the case $T = 1$, p corresponds in fact to p_1 in the main text. $\widehat{\gamma}(.,.)$, evaluated at the vector $x_p = (x_{1p}, ..., x_{np})$ and the vector z, is a vector of dimension $n+1$, where the i-th component, with $i = 1, ..., n$, is given by:

$$\widehat{\gamma}_i(x_p, z) \quad = \quad \frac{1}{L}\sum_{l}\frac{1}{n}\sum_{j} 1(x_{jlp} \leq x_{ip}) K_{h_Z}(z - z_l)$$

and the $(n+1)$-th component is given by:

$$\widehat{\gamma}_{n+1}(x_p, z) \quad = \quad \frac{1}{L}\sum_{l} K_{h_Z}(z - z_l)$$

where $K_{h_Z}(z - z_l) = K(\frac{z-z_l}{h_Z})$.

Finally, the function $m(.)$ is defined by:

$$m(p^0,z;\theta;\widehat{\gamma}(x_p,z);p)$$
$$= \left\{ \begin{array}{c} (n-1)(E[V|S_1 = F^{-1}_{S|Z}(1-\frac{\widehat{\gamma}_1(x_p,z)}{\widehat{\gamma}_{n+1}(x_p,z)}|z;\theta),...,S_n = F^{-1}_{S|Z}(1-\frac{\widehat{\gamma}_n(x_p,z)}{\widehat{\gamma}_{n+1}(x_p,z)}|z;\theta),z]-p) \\ -(p-p^0) \end{array} \right\}$$
$$\times 1(p^0 \leq p).$$

Note that this function $m(.)$ is not exactly the function (13) defined in the main text.[25]

Thus, the equation that defines our estimator, (26), can be viewed as a special case of equations (8.1) and (8.7) in Newey and McFadden. To apply their theorems 8.11, 8.12, and 8.13, we introduce the following three additional functions.

First of all, $\nu(p^0,z,x_p;\theta^0;p)$ which is a vector of dimension $(n+1)$, where the i-th component, with $i=1,...,n$, is given by

$$\nu_i(p^0,z,x_p;\theta^0;p)$$
$$= -\left\{\frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i}(F^{-1}_{S|Z}(1-\frac{\gamma^0_1(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0),...,F^{-1}_{S|Z}(1-\frac{\gamma^0_n(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0),z)\right\}$$
$$\times(n-1)F^{-1\prime}_{S|Z}(1-\frac{\gamma^0_i(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0)\frac{1(p^0\leq p)}{\gamma^0_{n+1}(x_p,z)}g^0(x_p,z)$$

and where the $(n+1)$-th component is given by

$$\nu_{(n+1)}(p^0,z,x_p;\theta^0;p)$$
$$= \sum_{i=1}^{n}\left\{\frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i}(F^{-1}_{S|Z}(1-\frac{\gamma^0_1(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0),...,F^{-1}_{S|Z}(1-\frac{\gamma^0_n(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0),z)\right\}$$
$$\times(n-1)F^{-1\prime}_{S|Z}(1-\frac{\gamma^0_i(x_p,z)}{\gamma^0_{n+1}(x_p,z)}|z;\theta^0)\frac{\gamma^0_i(x_p,z)1(p^0\leq p)}{\left(\gamma^0_{n+1}(x_p,z)\right)^2}g^0(x_p,z).$$

Second, $\mu(z,x_p;\theta^0;p)$ which is a vector of dimension n where the i-th term is given by:

$$\mu_i(z,x_p;\theta^0;p) = \int \nu_i(p^0,z,(y_{1p},...,y_{(i-1)p},x_{ip},y_{(i+1)p},...,y_{np});\theta^0;p)dy_{1p}...dy_{(i-1)p}dy_{(i+1)p}...dy_{np}$$

[25] Note also that the function $G(.)$ in the main text corresponds in the appendix to ratios of the form $\frac{\widehat{\gamma}_i(x_p,z)}{\widehat{\gamma}_{1+n}(x_p,z)}$.

Third, $\lambda(z;\theta^0;p)$ which is a scalar given by

$$\lambda(z;\theta^0;p) = \int \nu_{(n+1)}(p^0, z, (y_{1p}, ..., y_{np}); \theta^0; p) dy_{1p} ... dy_{np}.$$

In our context, under their hypotheses, Theorem 8.12 of Newey and McFadden can written as:

Theorem 1.

$$\sqrt{L}(\widehat{\theta} - \theta^0) \rightarrow N(0, (M'_\theta M_\theta)^{-1} M'_\theta \Omega M_\theta (M'_\theta M_\theta)^{-1})$$

where

$$\begin{aligned} M_\theta &= E[\nabla_\theta m(P^0, Z; \theta^0; \gamma^0(X_p, Z); p)] \\ &\quad \textit{and} \\ \Omega &= Var[m(P^0, Z; \theta^0; \gamma^0(X_p, Z); p) + \delta(Z, X_p; \theta^0; p)] \end{aligned}$$

and where $\delta(z, x_p; \theta^0; p)$ is a scalar given by:

$$\delta(z, x_p; \theta^0; p) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{x_{jp}}^{+\infty} \mu_i(z, y_p; \theta^0; p) dy_{ip} + \lambda(z; \theta^0; p).$$

Proof. Newey and McFadden have introduced the function δ in their theorems 8.11 and 8.12. What has to be proved is that this function δ takes in our context the specific form given in our theorem. For this we follow, step by step, the reasoning of Newey and McFadden (pp. 2207-2208).

The first step consists in linearizing the function $m(p^0, z; \theta^0; \gamma(x_p, z); p)$ around the true value $\gamma^0(x_p, z)$. This development allows us to find $M(p^0, z; \theta^0; \gamma(x_p, z); p)$, the linearization of $m(.)$. Here $M(.)$ is a function given by :

$$
\begin{aligned}
& M(p^0,z;\theta^0;\gamma(x_p,z);p) \\
= & \\
& -\sum_{i=1}^{n}\left\{\frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i}(F_{S|Z}^{-1}(1-\frac{\gamma_1^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0),...,F_{S|Z}^{-1}(1-\frac{\gamma_n^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0),z)\right\} \\
& \times(n-1)F_{S|Z}^{-1'}(1-\frac{\gamma_i^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0)\frac{1(p^0\leq p)}{\gamma_{n+1}^0(x_p,z)}\gamma_i(x_p,z) \\
& +\sum_{i=1}^{n}\left\{\frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i}(F_{S|Z}^{-1}(1-\frac{\gamma_1^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0),...,F_{S|Z}^{-1}(1-\frac{\gamma_n^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0),z)\right\} \\
& \times(n-1)F_{S|Z}^{-1'}(1-\frac{\gamma_i^0(x_p,z)}{\gamma_{n+1}^0(x_p,z)}|z;\theta^0)\frac{\gamma_i^0(x_p,z)1(p^0\leq p)}{\left(\gamma_{n+1}^0(x_p,z)\right)^2}\gamma_{n+1}(x_p,z).
\end{aligned}
$$

The next step is to calculate the function $\nu(p^0,z;\theta^0;p)$ defined by

$$\int_z\int_{x_p} M(p^0,z;\theta^0;\gamma(x_p,z);p)g^0(x_p,z)dx_pdz = \int_z\int_{x_p}\nu(p^0,z,x_p;\theta^0;p)\gamma(x_p,z)dx_pdz$$

where $g^0(x_p,z)$ is the joint distribution of $x_{1p},...,x_{np}$ and z. This is equation 8.12 of Newey and McFadden. The function $\nu(.)$ that solves this equation is the function $\nu(.)$ defined previously.

Finally, we have to calculate the function $\delta(.)$. For this purpose, we rely on equation 8.14 of Newey and McFadden, and define $\delta(.)$ as the solution of the following equation:

$$\int_z\int_{x_p} M(p^0,z;\theta^0;\widehat{\gamma}(x_p,z)-\gamma^0(x_p,z);p)g^0(x_p,z)dx_pdz = \int_z\int_{x_p}\delta(z,x_p;\theta^0;p)\widehat{g}(x_p,z)dx_pdz \tag{27}$$

where, as in Newey and McFadden, the integral of a function $a(z,x_p)$ over $d\widehat{g}$ is equal to $\frac{1}{L}\sum_{l=1}^{L}\int a(z,x_{.lp})K_{h_Z}(z-z_l)dz$. Given the functions $\mu(.)$ and $\lambda(.)$ introduced previously, we obtain:

$$
\begin{aligned}
\int_z\int_{x_p}\nu(p^0,z,x_p;\theta^0;p)\widehat{\gamma}(x_p,z)dx_pdz &= \frac{1}{L}\sum_l\frac{1}{n}\sum_j\sum_i\int dz\int_{x_{jlp}}^{+\infty}\mu_i(z,x_p;\theta^0;p)K_{h_Z}(z-z_l)d \\
&+\frac{1}{L}\sum_l\int\lambda(z;\theta^0;p)K_{h_Z}(z-z_l)dz
\end{aligned}
$$

and

$$\int_z \int_{x_p} \nu(p^0, z, x_p; \theta^0; p)\gamma^0(x_p, z) dx_p dz = \int_z \int_{x_p} \left\{ \begin{array}{c} \sum_i \nu_i(p^0, z, x_p; \theta^0; p)\gamma_i^0(x_p, z) \\ +\nu_{(n+1)}(p^0, z, x_p; \theta^0; p)\gamma_{n+1}^0(x_p, z) \end{array} \right\}$$

$$= 0.$$

The function δ that solves (27) is thus defined by

$$\delta(z, x_p; \theta^0; p) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{x_{jp}}^{+\infty} \mu_i(z, y_p; \theta^0; p) dy_{ip} + \lambda(z; \theta^0; p)$$

as stated in our theorem.

This allows us to obtain the analogue of Newey and McFadden's equation 8.15, that is:

$$\int_z \int_{x_p} \delta(z, x_p; \theta^0; p)\widehat{g}(x_p, z) dx_p dz - \int_z \int_{x_p} \delta(z, x_p; \theta^0; p)\widetilde{g}(x_p, z) dx_p dz$$
$$= \frac{1}{L} \sum_l \frac{1}{n} \sum_j \sum_i \int_{x_{jlp}}^{+\infty} \left\{ \int \mu_i(z, x_p; \theta^0; p) K_{h_Z}(z - z_l) dz - \mu_i(z_l, x_p; \theta^0; p) \right\} dx_{ip}$$
$$+ \frac{1}{L} \sum_l \left\{ \int \lambda(z; \theta^0; p) K_{h_Z}(z - z_l) dz - \lambda(z_l; \theta^0; p) \right\}$$

where $\widetilde{g}(x_p, z)$ is the empirical distribution of (x_p, z), and $\widehat{g}(x_p, z)$, defined previously, corresponds to the distribution of (x_p, z) where the empirical distribution of z has been replaced by a smoothed version $\frac{1}{L} \sum_l K_{h_z}(z - z_l)$, exactly as in Newey and McFadden. This equation is important to verify the condition (iv) of theorem 1 (page 2196) in Newey and McFadden, which is a necessary condition to apply theorems 8.11 and 8.12.

□

It has been shown that our problem fits exactly in the framework of Newey and McFadden, and we have defined in our context all the functions introduced in their general case.

Newey and McFadden also study the estimation of the asymptotic variance-covariance matrix. By introducing the three following functions:

$$\widehat{M_\theta} = \frac{1}{L} \sum_{l=1}^{L} \nabla_\theta m(p_l^0, z_l; \widehat{\theta}; \widehat{\gamma}(x_{.lp}, z_l); p)$$

$$\widehat{\lambda}(z;\widehat{\theta};\widehat{\gamma};p) = \begin{array}{l} \frac{1}{L}\sum_l \sum_{i=1}^n \left\{ \frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i} (F_{S|Z}^{-1}(1-\frac{\widehat{\gamma}_1(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta}),...,F_{S|Z}^{-1}(1-\frac{\widehat{\gamma}_n(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta}) \right. \\ \times(n-1)F_{S|Z}^{-1\prime}(1-\frac{\widehat{\gamma}_i(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta})\frac{\widehat{\gamma}_i(x_{.lp},z_l)1(p_l^0\leq p)}{\left(\widehat{\gamma}_{n+1}(x_{.lp},z_l)\right)^2}K_{h_Z}(z-z_l) \end{array}$$

$$\widehat{\rho}_{ij}(z,x_p;\widehat{\theta};\widehat{\gamma};p) = \begin{array}{l} \frac{1}{L}\sum_l \left\{ \frac{\partial E[V|S_1,...,S_n,Z]}{\partial S_i} (F_{S|Z}^{-1}(1-\frac{\widehat{\gamma}_1(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta}),...,F_{S|Z}^{-1}(1-\frac{\widehat{\gamma}_n(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta}), \right. \\ \times(n-1)F_{S|Z}^{-1\prime}(1-\frac{\widehat{\gamma}_i(x_{.lp},z_l)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}|z_l;\widehat{\theta})\frac{1(p_l^0\leq p)}{\widehat{\gamma}_{n+1}(x_{.lp},z_l)}1(x_{jp}\leq x_{ilp})K_{h_Z}(z-z_l) \end{array}$$

we obtain, under the hypotheses of Newey and McFadden, the analogue of their theorem 8.13:

Theorem 2.

$$(\widehat{M_\theta'}\widehat{M_\theta})^{-1}\widehat{M_\theta'}\widehat{\Omega}\widehat{M_\theta}(\widehat{M_\theta'}\widehat{M_\theta})^{-1} \to (M_\theta' M_\theta)^{-1}M_\theta'\Omega M_\theta(M_\theta' M_\theta)^{-1}$$

where

$$\widehat{\Omega} = \frac{1}{L}\sum_{l=1}^{L}[m(p_l^0,z_l;\widehat{\theta};\widehat{\gamma}(x_{.lp},z_l);p)+\widehat{\delta}(z_l,x_{.lp};\widehat{\theta};\widehat{\gamma};p)][m(p_l^0,z_l;\widehat{\theta};\widehat{\gamma}(x_{.lp},z_l);p)+\widehat{\delta}(z_l,x_{.lp};\widehat{\theta};\widehat{\gamma};p)]'$$

and

$$\widehat{\delta}(z,x_p;\widehat{\theta};\widehat{\gamma};p) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n}\widehat{\rho}_{ij}(z,x_p;\widehat{\theta};\widehat{\gamma};p)+\widehat{\lambda}(z;\widehat{\theta};\widehat{\gamma};p).$$

Proof. The function $\widehat{\delta}(z,x_p;\theta;\gamma;p)$ itself has to be estimated (whereas in Newey and McFadden the function $\delta(.)$ is assumed to be known; see their equation 8.17). This is not a problem as shown by the following decomposition:

$$\parallel \widehat{\delta}(z,x_p;\widehat{\theta};\widehat{\gamma};p)-\delta(z,x_p;\theta^0;\gamma^0;p)\parallel \leq \begin{array}{l} \parallel \widehat{\delta}(z,x_p;\widehat{\theta};\widehat{\gamma};p)-\widehat{\delta}(z,x_p;\theta^0;\gamma^0;p)\parallel \\ +\parallel \widehat{\delta}(z,x_p;\theta^0;\gamma^0;p)-\delta(z,x_p;\theta^0;\gamma^0;p)\parallel . \end{array}$$

The first term of the decomposition is of the type considered by Newey and McFadden, and converges to zero under their hypotheses. The second term is a standard one, and converges to zero as expectations are estimated by empirical means.

□

Finally, for the previous asymptotic results to be valid, the distribution functions $F_{V|Z}(.)$ and $F_{S|V,Z}(.)$ should be chosen such that the conditions of our theorems 1 and 2 (i.e. the hypotheses of Newey and McFaddens' theorems) are verified. It can be shown that for our particular choices of the distribution functions (see the specifications 15 and 16), this is indeed the case. The calculations that these verifications entail are lengthy and fastidious and are therefore omitted. They are based on first -and second order developments, and can be obtained from the authors upon request.

E Identification

In the case of gamma extended distributions, the expectation, that can be evaluate from the first order condition, take the following form :

$$E\{E(V_l|S_{1l} = \widetilde{x}^{-1}(x_{1lp}), ..., S_{n_l l} = \widetilde{x}^{-1}(x_{n_l lp}), N_l = n_l, Z_l = z_l)1\{P_l^0 \leq p\}|N_l = n_l, Z_l = z_l\}$$

$$= \int_{x_{1lp}} \cdots \int_{x_{n_l lp}} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{g_0(x_{1lp}, \ldots, x_{n_l lp})1\{\sum_{i=1}^{n_l} x_{ilp} \leq 1\} dx_{1lp} \ldots dx_{n_l lp}}{\left(\beta_l + \sum_{i=1}^{n_l} \beta_l \left[\frac{1}{G_0^{1/\alpha_l}(x_{ilp}|n_l, z_l; p)} - 1\right]\right)^{1/\gamma}}.$$

Since $G_0(x_{ilp}|n_l, z_l; p) = F_0(s_i) = \left(\frac{\beta_0}{\beta_0 + s_i}\right)^{\alpha_0}$, we can rewrite the previous expectation by changing the variables in the integrals and by using $f_0(s_1, ..., s_n) = \frac{\Gamma(n+\alpha_0)}{\Gamma(\alpha_0)} \frac{\beta_0^{\alpha_0}}{(\beta_0 + \sum_{i=1}^{n} s_i)^{n+\alpha_0}}$. We then obtain :

$$E\{E(V_l|S_{1l} = \widetilde{x}^{-1}(x_{1lp}), ..., S_{n_l l} = \widetilde{x}^{-1}(x_{n_l lp}), N_l = n_l, Z_l = z_l)1\{P_l^0 \leq p\}|N_l = n_l, Z_l = z_l\}$$

$$= \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{\Gamma(n + \alpha_0)}{\Gamma(\alpha_0)} \frac{\beta_0^{\alpha_0}}{\beta^{1/\gamma}}$$
$$\times \int_{s_1} \cdots \int_{s_n} \frac{1\{\sum_{i=1}^{n} x(p, s_i, n, z; \theta_0) \leq 1\} ds_1 \ldots ds_n}{(\beta_0 + \sum_{i=1}^{n} s_i)^{n+\alpha_0} \left(1 + \sum_{i=1}^{n_l} \left[\left(\frac{\beta_0 + s_i}{\beta_0}\right)^{\alpha_0/\alpha} - 1\right]\right)^{1/\gamma}}.$$

To obtain the identification, the idea is to study the multiple integral and his derivative in p for $p = p_{max}$. p_{max} is the highest equilibrium price. It is defined by $x(p_{max}, 0, n, z; \theta_0) = \frac{1}{n}$[26].

We therefore try to derive the previous multiple integral in p. We note

$$A(p) = \int_0^{+\infty} \cdots \int_0^{+\infty} h(s_1, ..., s_n) 1\{\sum_{i=1}^{n} x(p, s_i, n, z; \theta_0) \leq 1\} ds_1 \dots ds_n$$

.

It is important to note that for $p = p_{max}$, the function $1\{\sum_{i=1}^{n} x(p, s_i, n, z; \theta_0) \leq 1\}$ is always equal to 1. Let us now study $A(p)$ for p near p_{max}. $A(p)$ can be rewritten as :

$$A(p) = \int_{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)}^{+\infty} ds_1 \int_0^{+\infty} ds_2 \cdots \int_0^{+\infty} ds_n h(s_1, \dots, s_n)$$

$$+$$

$$\int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1 \left[\int_0^{+\infty} \cdots \int_0^{+\infty} h(s_1, \dots, s_n) 1\{\sum_{i=2}^{n} x(p, s_i; \theta_0) \leq 1 - x(p, s_1; \theta_0\} ds_2 \dots \right.$$

$$\frac{dA}{dp}(p) = -\frac{d}{dp} \left[x^{-1}(p, 1-(n-1)x(p, 0; \theta_0); \theta_0) \right] \int_0^{+\infty} ds_2 \cdots$$

$$\int_0^{+\infty} ds_n h(x^{-1}(p, 1-(n-1)x(p, 0; \theta_0); \theta_0), s_2 \dots, s_n)$$

$$+$$

$$\frac{d}{dp} \left[x^{-1}(p, 1-(n-1)x(p, 0; \theta_0); \theta_0) \right] \int_0^{+\infty} ds_2 \cdots$$

$$\int_0^{+\infty} ds_n h(x^{-1}(p, 1-(n-1)x(p, 0; \theta_0); \theta_0), s_2 \dots, s_n) 1\{\sum_{i=2}^{n} x(p, s_i; \theta_0) \leq (n-1)x(p, 0; \theta_0\}$$

$$+$$

$$\int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1 \frac{d}{dp} \left[\int_0^{+\infty} \cdots \int_0^{+\infty} h(s_1, \dots, s_n) 1\{\sum_{i=2}^{n} x(p, s_i; \theta_0) \leq 1 - x(p, s_1; \theta_0)\} ds \right.$$

[26]In the uniform case for example, $p_{max} = \frac{1}{1+1/\gamma} \frac{\Gamma(n+\alpha+1/\gamma)}{\Gamma(n+\alpha)} \frac{1}{\beta^{1/\gamma}}$

As the indicate function in the second term is always equal to 1, we simply obtain :

$$\frac{dA}{dp}(p) = \int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1$$

$$\frac{d}{dp}\left[\int_0^{+\infty}\cdots\int_0^{+\infty} h(s_1,\ldots,s_n)1\{\sum_{i=2}^{n} x(p,s_i;\theta_0) \leq 1 - x(p,s_1;\theta_0\}ds_2\ldots ds_n\right]$$

By recursion, we finally have :

$$\frac{dA}{dp}(p) = \int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1\cdots\int_0^{x^{-1}(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0)} ds_{n-1}$$

$$\frac{d}{dp}\int_0^{+\infty} h(s_1,\ldots,s_n)1\{x(p,s_n;\theta_0) \leq 1 - \sum_{i=1}^{n-1} x(p,s_i;\theta_0)\}ds_n$$

which gives

$$\frac{dA}{dp}(p) = \int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1\cdots\int_0^{x^{-1}(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0)} ds_{n-1}$$

$$\frac{d}{dp}\left[x^{-1}(p,1-\sum_{i=1}^{n-1}x(p,s_i;\theta_0);\theta_0)\right] h(s_1,\ldots,s_{n-1},x^{-1}(p,1-\sum_{i=1}^{n-1}x(p,s_i;\theta_0);\theta_0))$$

Finally, one can rewrite

$$\frac{d}{dp}\left[x^{-1}(p,1-\sum_{i=1}^{n-1}x(p,s_i;\theta_0);\theta_0)\right] = \frac{\partial x^{-1}}{\partial p}(p,1-\sum_{i=1}^{n-1}x(p,s_i;\theta_0);\theta_0) - \left(\sum_{i=1}^{n-1}\frac{\partial x}{\partial p}(p,s_i;\theta_0)\right)$$

$$\frac{\partial x^{-1}}{\partial x}(p,1-\sum_{i=1}^{n-1}x(p,s_i;\theta_0);\theta_0)$$

We now search $\frac{d^2A}{dp^2}(p)$. The derivative of the previous equation gives n terms. The first $n-2$ terms correspond to the derivative of the $n-2$ first integral; the $n-1$th term to the derivative of the $n-1$th integral and the last term to the derivative of the function that is integrated.

Let us first study the $n-2$ first term. They are equal to zero. Indeed, for example, the derivative of the first integral gives :

$$\frac{d}{dp}\left[x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)\right]\int_0^{x^{-1}(p,1-(n-2)x(p,0;\theta_0)-[1-(n-1)x(p,0;\theta_0)];\theta_0)} ds_2\cdots$$

and

$$x^{-1}(p,1-(n-2)x(p,0;\theta_0)-[1-(n-1)x(p,0;\theta_0)];\theta_0)=x^{-1}(p,x(p,0;\theta_0);\theta_0)=0$$

The $(n-1)$-th term gives a contribution to the derivative and "suppress" an integral. This term can be written as :

$$\int_0^{x^{-1}(p,1-(n-1)x(p,0;\theta_0);\theta_0)} ds_1\cdots\int_0^{x^{-1}(p,1-2x(p,0;\theta_0)-\sum_{i=1}^{n-3}x(p,s_i;\theta_0);\theta_0)} ds_{n-2}$$

$$\left[\frac{\partial x^{-1}}{\partial p}\left(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0\right)-\left(\frac{\partial x}{\partial p}(p,0;\theta_0)+\sum_{i=1}^{n-2}\frac{\partial x}{\partial p}(p,s_i;\theta_0)\right)\frac{\partial x^{-1}}{\partial x}\left(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0\right)\right]$$

$$\left[\frac{\partial x^{-1}}{\partial p}(p,x(p,0;\theta_0);\theta_0)-\left(\frac{\partial x}{\partial p}\left(p,x^{-1}(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0);\theta_0\right)+\sum_{i=1}^{n-2}\frac{\partial x}{\partial p}(p,s_i;\theta_0)\right)\frac{\partial x^{-1}}{\partial x}(p,x(p,0;\theta_0);\theta_0)\right]$$

$$h\left(s_1,\cdots,s_{n-2},x^{-1}\left(p,1-x(p,0;\theta_0)-\sum_{i=1}^{n-2}x(p,s_i;\theta_0);\theta_0\right),0\right)$$

The last term gives also a contribution to the derivative but do not suppress an integral. When the derivative will be taken in $p=p_{max}$, this term will give a contribution equal to 0 also as

$$x^{-1}(p_{max},1-(n-1)x(p_{max},0;\theta_0);\theta_0)=x^{-1}(p_{max},1/n;\theta_0)=x^{-1}(p_{max},x(p_{max},0;\theta_0);\theta_0)=0$$

The same reasoning can be used until the nth derivative and one obtain :

$$\frac{d^nA}{dp^n}(p_{max})=\left(\frac{\partial x^{-1}}{\partial p}(p_{max},x(p_{max},0;\theta_0);\theta_0)-\frac{\partial x^{-1}}{\partial x}(p_{max},x(p_{max},0;\theta_0);\theta_0)\left[(n-1)\frac{\partial x}{\partial p}(p_{max},x(p_{max},0;\theta_0);\theta_0)\right]\right)^n h(0,\ldots,0)$$

which can be rewritten as

$$\frac{d^n A}{dp^n}(p_{max}) = \left(-n\frac{\partial x^{-1}}{\partial p}(p_{max}, x(p_{max}, 0; \theta_0); \theta_0)\right)^n h(0, \ldots, 0)$$

and this expression has no reason to be zero [27]

Using this result, we can conclude that the first order condition allows us to identify

$$\frac{\Gamma(n+\alpha+1/\gamma)}{\Gamma(n+\alpha)}\frac{\Gamma(n+\alpha_0)}{\beta^{1/\gamma}} = \frac{\Gamma(n+\alpha_0+1/\gamma_0)}{\beta_0^{1/\gamma_0}}$$

By doing the ratio of this equation for n and $n+1$, we have :

$$\frac{n+\alpha+1/\gamma}{n+\alpha}(n+\alpha_0) = n+\alpha_0+1/\gamma_0$$

and some straightforward calculus prove that

$$n(1/\gamma - 1/\gamma_0) + \alpha_0/\gamma - \alpha/\gamma_0 = 0$$

for every n.

Thus γ and α are identified. Using the first equation, we obtain that β is also identified.

F Asymmetry

In this part, we show that the Euler condition given by equation (6) is still a necessary condition with asymmetric players. Let suppose that there is two types of bidders. There are k players of type A with signals $s_1^A, s_2^A, ..., s_k^A$. The density function of the signals conditionally on V is given by $f^A(.|V)$. Similarly, there are $n-k$ players of type B with signals $s_1^B, s_2^B, ..., s_{(n-k)}^B$.

[27] For example, in the uniform case, one obtain

$$\frac{\partial x^{-1}}{\partial p}(p_{max}, x(p_{max}, 0; \theta_0); \theta_0) = -\frac{n+\alpha}{2np_{max}^2}$$

Their density is denoted $f^B.|V$. We note $x^A(\cdot,\cdot)$ (resp. $x^B(\cdot,\cdot)$) the optimal strategy for type A (resp. type B) bidders.

Let first consider a player i of type A. By definition, we have

$$H^A(p;v,y) = \int_{s_j^A;j\neq i}\int_{s_j^B} \mathbf{1}\left\{\sum_{j=1;j\neq i}^{k} x^A(p,s_j^A) + \sum_{j=1}^{n-k} x^B(p,s_j^B) \leq 1-y\right\} \prod_{j=1;j\neq i}^{k} f_{S|V}(s_j^A|v)ds_j^A \prod_{j=1}^{n-k} f_{S|V}(s_j^B|v)ds_j^B$$

Defining $x_{jp}^X = x^X(p,s_j)$, the above expression can be written as

$$H^A(p;v,y) = \int_{x_{jp}^A;j\neq i}\int_{x_{jp}^B} \mathbf{1}\left\{\sum_{j=1;j\neq i}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \prod_{j=1;j\neq i}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v)dx_{jp}^A \prod_{j=1}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v)dx_{jp}^B$$

so that (because the integrand is symmetric in all the x_{jp}^A and the x_{jp}^B)

$$\begin{aligned}
\frac{\partial H^A(p;v,y)}{\partial p} = (k-1)\int_{x_{jp}^A;j\neq i}\int_{x_{jp}^B} & \mathbf{1}\left\{\sum_{j=1;j\neq i}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \\
& \frac{\partial}{\partial p}\left[\frac{\partial (x^A)^{-1}(p,x_{1p}^A)}{\partial x_{1p}^A} f_{S|V}^A((x^A)^{-1}(p,x_{1p}^A)|v)\right] dx_{1p}^A \\
& \prod_{j=2;j\neq i}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v)dx_{jp}^A \\
& \prod_{j=1}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v)dx_{jp}^B \\
+(n-k)\int_{x_{jp}^A;j\neq i}\int_{x_{jp}^B} & \mathbf{1}\left\{\sum_{j=1;j\neq i}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \\
& \frac{\partial}{\partial p}\left[\frac{\partial (x^B)^{-1}(p,x_{1p}^B)}{\partial x_{1p}^B} f_{S|V}^B((x^B)^{-1}(p,x_{1p}^B)|v)\right] dx_{1p}^B \\
& \prod_{j=1;j\neq i}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v)dx_{jp}^A \\
& \prod_{j=2}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v)dx_{jp}^B
\end{aligned}$$

A similar equation is obtained for $\frac{\partial H^B(p;v,y)}{\partial p}$ when we consider a player i of type B.

Now define $H(p;v) = \Pr(P^0 \leq p|V=v)$ and $H(p) = \Pr(P^0 \leq p)$. We have

$$\begin{aligned}
H(p;v) &= \int_{s_j^A}\int_{s_j^B} \mathbf{1}\left\{\sum_{j=1}^{k} x^A(p,s_j^A) + \sum_{j=1}^{n-k} x^B(p,s_j^B) \leq 1\right\} \prod_{j=1}^{k} f_{S|V}(s_j^A|v)ds_j^A \prod_{j=1}^{n-k} f_{S|V}(s_j^B|v)ds_j^B \\
&= \int_{x_{jp}^A}\int_{x_{jp}^B} \mathbf{1}\left\{\sum_{j=1}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \prod_{j=1}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v)dx_{jp}^A \\
&\quad \prod_{j=1}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v)dx_{jp}^B
\end{aligned}$$

So, again by symmetry, we have

$$\begin{aligned}
\frac{\partial H(p;v)}{\partial p} = k \int_{x_{jp}^A} \int_{x_{jp}^B} \quad & \mathbf{1}\left\{\sum_{j=1}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \\
& \frac{\partial}{\partial p}\left[\frac{\partial (x^A)^{-1}(p,x_{1p}^A)}{\partial x_{1p}^A} f_{S|V}^A((x^A)^{-1}(p,x_{1p}^A)|v)\right] dx_{1p}^A \\
& \prod_{j=2}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v) dx_{jp}^A \\
& \prod_{j=1}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v) dx_{jp}^B \\
+(n-k) \int_{x_{jp}^A} \int_{x_{jp}^B} \quad & \mathbf{1}\left\{\sum_{j=1}^{k} x_{jp}^A + \sum_{j=1}^{n-k} x_{jp}^B \leq 1-y\right\} \\
& \frac{\partial}{\partial p}\left[\frac{\partial (x^B)^{-1}(p,x_{1p}^B)}{\partial x_{1p}^B} f_{S|V}^B((x^B)^{-1}(p,x_{1p}^B)|v)\right] dx_{1p}^B \\
& \prod_{j=1}^{k} \frac{\partial (x^A)^{-1}(p,x_{jp}^A)}{\partial x_{jp}^A} f_{S|V}((x^A)^{-1}(p,x_{jp}^A)|v) dx_{jp}^A \\
& \prod_{j=2}^{n-k} \frac{\partial (x^B)^{-1}(p,x_{jp}^B)}{\partial x_{jp}^B} f_{S|V}((x^B)^{-1}(p,x_{jp}^B)|v) dx_{jp}^B
\end{aligned}$$

After some straightforward calculations it follows that

$$\begin{aligned}
kE_{S_i^A}\left\{\frac{\partial H^A(p;v,x^A(p,S_i^A))}{\partial p}|V=v\right\} \quad + \quad & (n-k)E_{S_i^B}\left\{\frac{\partial H^B(p;v,x^B(p,S_i))}{\partial p}|V=v\right\} \\
= \quad & (n-1)\frac{dH(p;v)}{dp}
\end{aligned}$$

and therefore

$$kE_{S_i^A,V}\left\{\frac{\partial H^A(p;v,x^A(p,S_i^A))}{\partial p}\right\} + (n-k)E_{S_i^B,V}\left\{\frac{\partial H^B(p;v,x^B(p,S_i))}{\partial p}\right\} = (n-1)\frac{dH(p)}{dp}$$

We also have

$$
\begin{aligned}
k \quad & E_{S_i^A,V} \quad \left\{ V \frac{\partial H^A(p;V,x^A(p,S_i^A))}{\partial p} \right\} + (n-k) E_{S_i^B,V} \left\{ V \frac{\partial H^B(p;V,x^B(p,S_i^B))}{\partial p} \right\} \\
= \quad & (n-1) \int v f_V(v) \frac{dH(p;v)}{dp} dv \\
= \quad & (n-1) \frac{d}{dp} \left[E \left(E \left(V | S_1^A = s_1^A, ..., S_k^A = s_k^A, S_1^B = s_1^B, ..., S_{(n-k)}^B = s_{(n-k)}^B \right) \cdot \mathbf{1} \left\{ P^0 \leq p \right\}
\end{aligned}
$$

Finally we have

$$
kE_{S_i^A,V} \left\{ H^A(p;V,x^A(p,S_i^A)) \right\} + (n-k) E_{S_i^B,V} \left\{ H^B(p;V,x^B(p,S_i^B)) \right\} = nH(p)
$$

Therefore, the sum over all players of the expectations of Euler conditions (5) (taking the expectation with respect to V, S_i^A for bidders of type A and with respect to V, S_i^B for bidders of type B) can be rewritten as

$$
\begin{aligned}
0 \quad = \quad & (n-1) \frac{d}{dp} \left[E \left(E \left(V | S_1^A = s_1^A, ..., S_k^A = s_k^A, S_1^B = s_1^B, ..., S_{(n-k)}^B = s_{(n-k)}^B \right) \cdot \mathbf{1} \left\{ P^0 \leq p \right\} \right) \right] \\
& -(n-1) p \frac{dH(p)}{dp} - nH(p).
\end{aligned}
$$

This equation was true in the symmetric case and led to the condition (6).

Table A1. The auctions

Type of auction	Auction date	Line auctioned	Settlement date	Maturity date
OAT	5 Jan 95	7.5% 2005	25 Jan 95	25 Apr 05
OAT	5 Jan 95	6% 2025	25 Jan 95	25 Oct 25
OAT	2 Feb 95	7.5% 2005	27 Feb 95	25 Apr 05
OAT	2 Feb 95	8.5% 2008	27 Feb 95	25 Oct 08
OAT	2 Feb 95	6% 2025	27 Feb 95	25 Oct 25
OAT	2 Mar 95	7.5% 2005	27 Mar 95	25 Apr 95
OAT	2 Mar 95	8.5% 2008	27 Mar 95	25 Oct 08
OAT	6 Apr 95	7.75% 2005	25 Apr 95	25 Apr 05
OAT	4 May 95	7.75% 2005	26 May 95	25 Apr 05
OAT	4 May 95	8.5% 2008	26 May 95	25 Oct 08
OAT	1 Jun 95	8.5% 2002	26 Jun 95	25 Nov 02
OAT	1 Jun 95	7.75% 2005	26 Jun 95	25 Apr 05
OAT	1 Jun 95	6% 2025	26 Jun 95	25 Apr 25
OAT	6 Jul 95	7.75% 2005	25 Jul 95	25 Oct 05
OAT	6 Jul 95	8.5% 2008	25 Jul 95	25 Oct 08
OAT	6 Jul 95	6% 2025	25 Jul 95	25 Oct 25
OAT	3 Aug 95	8.5% 2002	25 Aug 95	25 Nov 02
OAT	3 Aug 95	7.75% 2005	25 Aug 95	25 Oct 05
OAT	3 Aug 95	8.5%2008	25 Aug 95	25 Oct 08
OAT	7 Sep 95	7.75% 2005	25 Sep 95	25 Oct 05
OAT	7 Sep 95	6% 2025	25 Sep 95	25 Oct 25
OAT	5 Oct 95	7.25% 2006	25 Oct 95	25 Apr 06
OAT	2 Nov 95	7.25% 2006	27 Nov 95	25 Apr 06
OAT	2 Nov 95	6% 2025	27 Nov 95	25 Oct 25
OAT	7 Dec 95	7.25% 2006	26 Dec 95	25 Apr 06
BTAN	19 Jan 95	7.75% 2000	13 Feb 95	12 Apr 00
BTAN	16 Feb 95	6.5% 1996	6 Mar 95	12 Oct 96
BTAN	16 Feb 95	7.75% 2000	13 Mar 95	12 Apr 00
BTAN	16 Mar 95	7.25% 1997	5 Apr 95	12 Aug 97
BTAN	20 Apr 95	7.25% 1997	5 May 95	12 Aug 97
BTAN	20 Apr 95	7.75% 2000	12 May 95	12 Apr 00
BTAN	18 May 95	7.25% 1997	6 Jun 95	12 Aug 97
BTAN	18 May 95	7.75% 2000	12 Jun 95	12 Apr 00
BTAN	15 Jun 95	7.25% 1997	5 Jul 95	12 Aug 97
BTAN	15 Jun 95	7.75% 2000	12 Jul 95	12 Apr 00
BTAN	20 Jul 95	7% 2000	11 Aug 95	12 Oct 00
BTAN	17 Aug 95	7.25% 1997	5 Sep 95	12 Aug 97
BTAN	17 Aug 95	7% 2000	12 Sep 95	12 Oct 00
BTAN	21 Sep 95	7.25% 1997	5 Oct 95	12 Aug 97
BTAN	21 Sep 95	7% 2000	12 Oct 95	12 Oct 00
BTAN	19 Oct 95	7.25% 1997	6 Nov 95	12 Aug 97
BTAN	19 Oct 95	7% 2000	13 Nov 95	12 Oct 00
BTAN	16 Nov 95	5.75% 1998	5 Dec 95	12 Mar 98
BTAN	21 Dec 95	5.75% 1998	5 Jan 96	12 Mar 98
BTAN	21 Dec 95	7% 2000	12 Jan 96	12 Oct 00

Table A2. Expected value of V, stop-out price, and secondary market price

$E[V_l \mid S_{1l},...,S_{n_l l},Z_l]$	Stop-out price	$E[V_l \mid Z_l]$	Secondary market Price
96.28	93.96	96.40	94.17
71.75	70.88	72.09	71.33
97.75	95.72	97.80	95.86
104.37	102.26	104.40	102.39
75.15	72.94	75.41	73.35
98.72	96.56	98.77	96.91
105.65	103.28	105.67	103.45
101.98	99.68	101.99	99.65
102.36	99.74	102.39	99.75
107.63	104.70	107.65	104.80
111.79	106.74	111.80	106.80
105.84	101.76	105.82	102.37
80.70	77.46	80.55	78.07
103.97	101.20	103.94	101.21
110.49	106.46	110.51	106.25
79.27	76.18	79.22	76.80
115.56	107.96	115.57	107.91
107.03	102.80	106.99	103.00
118.80	108.16	118.80	108.50
106.71	102.72	106.68	102.85
79.92	77.22	79.84	77.50
99.87	97.78	99.88	98.16
102.01	99.62	102.02	99.75
80.20	77.80	80.05	77.80
108.32	102.60	108.27	102.80
101.07	99.41	101.10	99.50
97.69	99.21	97.81	99.01
101.87	100.19	101.86	100.14
99.90	99.21	99.99	99.24
101.02	99.97	101.02	99.95
102.83	100.90	102.81	100.88
104.03	101.47	104.02	101.54
106.77	103.27	106.74	103.21
103.25	101.30	103.23	101.24
106.08	102.80	106.04	102.87
101.59	99.90	101.61	100.06
104.90	101.96	104.85	102.02
102.98	100.78	102.95	100.75
105.29	102.13	105.24	102.19
104.76	101.47	104.72	101.53
103.94	101.61	103.90	101.60
102.56	100.60	102.54	100.57
99.58	100.36	99.31	100.42
105.56	100.73	105.29	100.78
115.78	103.91	115.72	104.08