# Inference with an Incomplete Model of English Auctions*

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**Abstract**

Standard models of English auctions abstract from actual practice by assuming that bidders continuously affirm their willingness to pay as the price rises exogenously. We show that one need not rely on these models to make useful inferences on the latent demand structure at private value English auctions. Weak implications of rational bidding provide sufficient structure to nonparametrically bound the distribution of bidder valuations and the optimal reserve price, based on observed bids. If auctions and/or bidders differ in observable characteristics, our approach also yields bounds on parameters characterizing the effects of observables on valuations. Whenever observed bids are consistent with the standard model, the identified bounds collapse to the true distribution (or parameters) of interest. Throughout, we propose estimators that consistently estimate the identified features. We conduct a number of Monte Carlo experiments and apply our methods to data from U.S. Forest Service timber auctions in order to assess reserve price policy.

Keywords: English auctions, incomplete models, optimal reserve prices, nonparametric identification and testing, bounds analysis, timber auctions

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# 1 Introduction

By far the most common type of auction in practice is the "English" or "oral ascending bid" auction, in which bidders offer progressively higher prices until only one bidder remains. In the dominant theoretical model of English auctions (Milgrom and Weber (1982)), each bidder continuously affirms her participation by holding down a button while the price rises exogenously. If bidders know their valuations, each drops out (in equilibrium) by releasing her button when either the price reaches her valuation or all her opponents have exited. At most real English auctions, however, there are no buttons. Bidders call out prices—or type them in, in the case of an online auction—whenever they wish and need not indicate whether they are "in" or "out" as the auction proceeds. Active bidding by a player's opponents may eliminate any incentives for her to make a bid close to her valuation, or even to bid at all. Observed bids may therefore be poor proxies for those envisioned in the theory. This mismatch threatens hopes of obtaining estimates of the latent demand structure at auctions that could be used to guide market design policy.

One possible remedy is a structural model providing a more accurate mapping between bidders' private information and observed bids. Such a model must account for bidders' choices of prices to call out and whether to call out a price at all. These choices seem likely to depend on factors that are difficult to capture in a tractable model. An alternative is to rely on standard models but look only at winning bids, which may be good approximations of the second highest valuation at each auction.[1] This is a useful approach if minimum bid increments are small and there is little "jump bidding," so that the price rises gradually and stops as soon as the second-highest valuation is passed. One drawback of this approach is that it ignores information contained in the losing bids—typically information about the lower end of the distribution of valuations, which can be important for policy simulations or calculation of an optimal reserve price. A potentially more serious concern with this approach is the approximation of the second highest valuation by the winning bid: there are many auctions with fairly large bidding increments, and in many auctions we see a great deal of jump bidding. Hence, in many auctions there may be no bid with a precise analog in existing theoretical models.

We propose an approach to inference for English auctions with independent private values[2] that neither relies on approximate interpretations of the data nor requires a complete model of bidding. We rely on only two assumptions about the behavioral process generating the observed bids:

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[1] See Paarsch (1992b), Hendricks and Paarsch (1995), Baldwin, Marshall and Richard (1997), Haile (2000), or Athey and Haile (2000).

[2] Extensions to other models of bidder demand are discussed in Haile and Tamer (2000). See the surveys of McAfee and McMillan (1987), Milgrom (1985), or Wilson (1992) for a discussion of standard auction models.

**Assumption 1** *Bidders do not bid more than they are willing to pay.*

**Assumption 2** *Bidders do not allow an opponent to win at a price they are willing to beat.*

While these assumptions permit bidding as in the standard "button auction" model, they also allow bids that are inconsistent with that model. For example, bids need not be close to the corresponding bidders' valuations nor even monotonic in valuations. Because assumptions 1 and 2 do not imply a unique distribution of observed bids given a distribution of bidder valuations, they constitute an *incomplete* model of bidding. Our objective is to investigate the empirical content of this model—to see what can be learned about the underlying demand structure based only on observed bids and these two restrictions on their interpretation.

While Assumptions 1 and 2 impose only limited structure on the data, we show that this is sufficient to provide informative bounds on the distribution characterizing the latent demand structure. These bounds can be very tight. In fact, when the button auction model captures the true data generating process, our bounds collapse to the true distribution function. When our upper and lower bounds do not coincide, no existing empirical methods based on the button auction model generally yields estimates lying within our bounds.

Our bounds on the distribution of bidder valuations can be used to construct bounds on the optimal reserve price, which (under a regularity condition) fully characterizes the optimal selling mechanism. Constructing these bounds is nontrivial because the optimal reserve price can decline with an increase in demand. When auctions and/or bidders differ in observable characteristics, bounds on parameters characterizing the dependence of valuations on these covariates can also be estimated. These results are of use, for example, in estimating hedonic models of valuations, testing for premia for certain products or sellers, testing for complementarities, testing for bidder asymmetry, or testing overidentifying restrictions of the private values assumption. Finally, we show that Assumptions 1 and 2 provide sufficient structure to enable nonparametric testing of the IPV assumption. We conduct a number of Monte Carlo experiments to evaluate our approaches and also present an application to U.S. Forest Service timber auctions, focusing on reserve prices. The results enable evaluation of the trade-off between revenue and the probability of successful sale, which is central to the policy debate regarding this issue.

Our main contribution is to a growing literature on methods for structural estimation of auction models, begun by Paarsch (1992a).[3] Much of this work has focused on parametric specifications of the distributions of bidder valuations. Nonparametric methods for first-price auctions are developed by Guerre, Perrigne, and Vuong (2000) and Li, Perrigne, and Vuong (1999a,b). Athey and

[3] See Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999), and Hendricks and Porter (2000) for surveys of the recent empirical literature on auctions.

Haile (2000) address nonparametric identification and testing of standard auction models, including English auctions that closely match the button auction model. We are unaware of any empirical work using nonparametric methods to study English auctions. Prior work addressing incomplete structural models in other contexts includes Jovanovic (1989) and Tamer (1999). Our work also builds on approaches to inference developed for applications with incomplete data (e.g., Horrowitz and Manski (1998), Manski and Tamer (2000)). More broadly, other work considering models that identify bounds on parameters or distributions of interest includes Manski (1990, 1995) and Horrowitz and Manski (1995).

The following section sets up the model and notation. In section 3 we show how Assumptions 1 and 2 identify bounds on the distribution of bidder valuations. Section 4 addresses construction of bounds on the optimal reserve price. Section 5 then discusses incorporation of covariates accounting for heterogeneity in distributions of valuations across auctions and/or across bidders. Section 6 presents the results of our Monte Carlo experiments. We present the empirical application in Section 7 and conclude in section 8. Proofs are collected in an appendix.

## 2 Model and Notation

We focus initially on a symmetric independent private values environment in which, for each auction $t = 1, \ldots, T$, each of $n_t \geq 2$ bidders draws her valuation $V_{it}$ independently from a common continuous distribution $F(\cdot)$ with support $[\underline{v}, \bar{v}]$. To focus on the implications of Assumptions 1 and 2 for inference, we assume that any reserve price lies below $\underline{v}$.[4] All bidders participate in an ascending bid (English) auction of a single object. The reserve price (or zero, if there is no reserve) is initially designated the standing high bid. Bidders are then permitted to offer progressively higher prices in free form, using some minimum bid increment $\Delta \geq 0$, until no bidder is willing to beat the standing high bid by $\Delta$.

We leave the remaining details of the true underlying model unspecified, including what motivates individual players to bid at any particular point in the auction, whether bidders know $n_t$ and/or $F(\cdot)$, and how the seller ends the auction. We assume only that bidding satisfies Assumptions 1 and 2.

The highest price called out by each bidder is recorded as his "bid." A bidder who does not call

[4] This is a reasonable assumption for the many auctions with a reserve price (often implicit) of zero. In the Forest Service auctions we consider below, the positive reserve prices are widely viewed as nonbinding. When a binding reserve price $p_t$ is set, our approach to inference yields bounds on the *conditional* (on participation) distribution $\frac{F(\cdot)-F(p_t)}{1-F(p_t)}$.

out a bid has the reserve price (or zero) listed as his bid.[5] Let $B_{jt}$ denote $j$'s bid at auction $t$ and $B_{1:n_t}, \ldots, B_{n_t:n_t}$ the order statistics of the bids, with $b_{i:n_t}$ denoting the $i$th lowest of the $n_t$ realized bids. Let $G_{i:n_t}(\cdot)$ denote the distribution of $B_{i:n_t}$. Similarly, let $V_{1:n_t}, \ldots, V_{n_t:n_t}$ denote the ordered set of valuations, with each $V_{i:n_t}$ distributed according to $F_{i:n_t}(\cdot)$.

The key structural feature of interest is the distribution $F(\cdot)$, which is the primitive needed for most policy experiments of interest, including calculation of an optimal reserve price. Our objective is to determine what can be learned about $F(\cdot)$ and the optimal reserve price from bidding data based only on Assumptions 1 and 2. The asymptotics of interest are $T \to \infty$ with $n_t$ finite for all $t$. We assume that there is a finite integer $N \geq 2$ such that for all $n \leq N$, $T_n \equiv \sum_{t=1}^{T} 1[n_t = n] \to \infty$ as $T \to \infty$. Henceforth, we consider only $n_t \leq N$. To simplify notation, when it will not cause confusion we will suppress the subscript $t$ on $n_t$.

## 3 Bounds on the Distribution of Valuations

Assumption 1 implies that each bidder's own bid is a lower bound on her valuation, while Assumption 2 implies that the winning bid (plus one bid increment) at each auction is an upper bound on the valuations of the losing bidders. Hence, assumptions 1 and 2 have the following implications.

**Lemma 1** $b_{i:n} \leq v_{i:n}\, \forall i, n.$

**Lemma 2** $v_{i:n} \leq b_{n:n} + \Delta\ \forall n,\ i < n.$

Assumptions 1 and 2 do not impose sufficient structure on the data to identify $F(\cdot)$.[6] However, these assumptions do identify bounds on $F(\cdot)$. We propose two methods for constructing these bounds. The first exploits the ordered nature of the data from each auction to give tight bounds. A second method leads to wider bounds but is particularly simple to construct and can be useful in some applications.

[5] This is a common method of recording data at English auctions and corresponds exactly to that for the Forest Service auctions we consider below. For our analysis, the critical assumption is that we observe the number of bidders with valuations above the reserve price. This assumption is also essential to all existing approaches to structural inference from bids at English auctions.

[6] Note in particular that even if $F(\cdot)$ belonged to a known parametric family, one could not use maximum likelihood methods to obtain consistent estimates of the unknown parameters. This might seem surprising, since *given* a vector of bids one could easily write down the probability that the valuations lie within the bounds these bids imply. However, this probability is *not* the likelihood of the data (the bids). Indeed, our assumptions do not imply a unique distribution of bids given a vector of valuations.

### 3.1 Bounds Based on Order Statistics of the Bid Data

#### 3.1.1 Main Results

For $F \in [0,1], n \in \{2,\ldots,N\}$, and $i \in \{1,\ldots,n\}$ define $\phi(F;i,n)$ implicitly as the solution to

$$F = \frac{n!}{(n-i)!(i-1)!} \int_0^{\phi} s^{i-1}(1-s)^{n-i} ds.$$

Note that $\phi(\cdot;i,n)$ is a strictly increasing, differentiable function mapping $[0,1] \to [0,1]$. The following lemma describes a well known and useful relationship between the distribution $F_{i:n}(\cdot)$ of the $i$th order statistic from and i.i.d. sample and the parent distribution $F(\cdot)$.[7]

**Lemma 3** *Given the i.i.d. random variables $\{V_i\}_{i=1}^n$, the distribution $F_{i:n}(\cdot)$ of the ith order statistic $V_{i:n}$ is related to the parent distribution $F(\cdot)$ by*

$$F_{i:n}(v) = \frac{n!}{(n-i)!(i-1)!} \int_0^{F(v)} s^{i-1}(1-s)^{n-i} ds \quad \forall v. \tag{1}$$

Hence,

$$F(v) = \phi(F_{i:n}(v);i,n). \tag{2}$$

We don't observe the realizations of $V_{i:n}$ nor, therefore, the distribution $F_{i:n}(\cdot)$ that would enable us to infer $F(\cdot)$; however, we can infer bounds on this distribution. In particular, Lemma 1 implies

$$F_{i:n}(v) \leq G_{i:n}(v) \quad \forall i,n,v. \tag{3}$$

Applying the monotone transformation $\phi(\cdot;i,n)$ to each side of (3) and exploiting (2) gives the following result.[8]

**Proposition 1** *For all $v \in [\underline{v}, \overline{v}]$, $F(v) \leq F_U(v) \equiv \min_{n\in\{2,\ldots,N\},i\in\{1,\ldots,n\}} \phi(G_{i:n}(v);i,n)$.*

Proposition 1 provides an upper bound on the distribution $F(v)$ at every $v$ based on observed bids. For each $v$, (3) implies $\sum_{n=2}^N n$ bounds (one from $F_{i:n}(v)$ for each $i$ and $n$) on the distributions of various order statistics of the latent valuations. Through (2) these bounds imply $\sum_{n=2}^N n$ different bounds on $F(v)$. We obtain the tightest bound at $v$ by taking the minimum. Hence, the bound on any one of the distributions $F_{i:n}(\cdot)$ obtained from the data could provide the bound on $F(\cdot)$ at a given value of $v$, depending on for which $i$ and $n$ the $i$th lowest bidder is most likely to be bidding close to his valuation when making a bid near $v$. We often observe some bidders who

[7] See, for example, Arnold et al. (1992). An equivalent representation is $F_{i:n}(v) = \sum_{r=i}^n \binom{n}{r} F(v)^r (1-F(v))^{n-r}$.

[8] Note that Assumptions 1 and 2 imply that the distribution of observed bids must have support covering $(\underline{v}, \overline{v})$.

apparently like to bid frequently, while others appear to bid only if the auction is about to end at a price they are willing to beat. Such differences would imply that the bids of some bidders would yield tighter bounds on their valuations than those of others. Furthermore, this may vary across the support $[\underline{v}, \overline{v}]$. By taking the minimum (over $i$ and $n$) at each evaluation point $v$, we obtain the tightest bound possible on each $F(v)$ without having to make a precise assumption about the process generating the bids.[9]

An estimate of this upper bound for each $v$ is easily obtained from a sample analog. Using the empirical distribution functions $\hat{G}_{i:n}(\cdot)$, the following result characterizes the asymptotic behavior of each $\phi(\hat{G}_{i:n}(v); i, n))$.

**Proposition 2** *For $v \in [\underline{v}, \bar{v}]$, $n \in \{2, ..., N\}$, and $i \in \{1, ..., n\}$ define*

$$\hat{G}_{i:n}(v) = \frac{1}{T_n} \sum_{t=1}^{T} \mathbb{1}[n_t = n, b_{i:n_t} \leq v].$$

*Then $\hat{F}_U(v) \equiv \min_{n\in\{2,...,N\}, i\in\{1,...,n\}} \phi(\hat{G}_{i:n}(v); i, n)$ is a consistent estimator of $F_U(v)$.*

We now turn to the lower bound on $F(\cdot)$ using Lemma 2, which is equivalent to

$$v_{n-1:n} \leq b_{n:n} + \Delta \quad \forall n. \tag{4}$$

Let $G^{\Delta}_{n:n}(\cdot)$ denote the distribution of $B_{n:n} + \Delta$. From (4) we know

$$F_{n-1:n}(v) \geq G^{\Delta}_{n:n}(v) \quad \forall n, v. \tag{5}$$

Applying the monotone transformation $\phi(\cdot; n-1, n)$ to each side of (5) gives the following result.

**Proposition 3** *For all $v \in [\underline{v}, \overline{v}]$, $F(v) \geq F_L(v) \equiv \max_n \phi(G^{\Delta}_{n:n}(v); n-1, n)$.*

Just as for the upper bound, a consistent estimate of the lower bound $F_L(v)$ for each $v$ is easily obtained by using transformations of the empirical distributions of the order statistics $B_{n:n} + \Delta$ (we omit the proof).

**Proposition 4** *For $v \in [\underline{v}, \bar{v}]$ and $n \in \{2, ..., N\}$ define $\hat{G}^{\Delta}_{n:n}(v) = \frac{1}{T_n} \sum_{t=1}^{T} \mathbb{1}[n_t = n, b_{n_t:n_t} + \Delta \leq v]$. Then $\hat{F}_L(v) \equiv \max_n \phi(G^{\Delta}_{n:n}(v); n-1, n)$ is a consistent estimator of $F_L(v)$.*

[9] Note that the validity of taking the minimum when $n$ takes on different values relies on the assumption that the distribution of valuations is fixed across auctions with different numbers of bidders. In some applications one may wish to condition on the number of bidders (see, e.g., Li, Perrigne and Vuong (1999a, 1999b), Hendricks, Pinkse and Porter (1999)), in which case the minimum would be taken over $i$ only.

### 3.1.2 Discussion

Note that equilibrium strategies in Milgrom and Weber's (1982) button auction model give one example of bidding consistent with Assumptions 1 and 2. In this special case, the top losing bidder exits at his valuation, followed immediately by the winning bidder. Hence the upper and lower bounds on $v_{n-1:n}$ are always identical, and a test of equality of $F_L(\cdot)$ and $F_U(\cdot)$ would provide a test of the button auction model.[10]

**Corollary 1** *In the dominant strategy equilibrium of the button auction,* $b_{n-1:n} = v_{n-1:n} = b_{n:n}$. *Hence,* $F_L(v) = F_U(v)\ \forall v$.

In practice, we may seldom find these distributions to be equal, even if an adjustment is made to account for discrete bidding increments (so that $b_{n:n} - b_{n-1:n} = \Delta$). However, this result illustrates a sense in which our approach to inference nests another fully nonparametric approach that assumes the full structure of the button auction model. Whenever observed bids are consistent with that model, our bounds collapse to the true distribution, and our estimates to consistent point estimates of each $F(v)$.

Our bounds could also be used to test the joint hypothesis that (a) the (symmetric) IPV model applies, and (b) auctions are independent of each other. Both of these are maintained assumptions of most prior empirical work on English auctions, and testing has been given little attention. However, the following corollary to Propositions 1 and 3 suggests a simple testing principle.

**Corollary 2** *In the symmetric IPV model with independent auctions,* $F_L(v) \leq F_U(v)\ \forall v$.

This stochastic dominance relation could be tested using the approach proposed by McFadden (1989), giving a nonparametric test relying only on Assumptions 1 and 2 as maintained hypotheses.[11] The power of the test arises from the fact that the relation (1) holds only for i.i.d. random variables. Unobserved heterogeneity in the objects sold at each auction, common value components in bidder valuations, dependence of valuations on prior auction outcomes, or other sources of correlation in values within or between auctions can cause this relation to fail. Examples based on both simulations and field data will be given below.

[10] This is equivalent to Athey and Haile's (2000) aproach to testing the IPV model at second-price sealed bid auctions.

[11] Alternatively, with (a) and (b) as maintained assumptions, this test could be used to detect systematic violations of Assumptions 1 and 2. Note that a full treatment of testing based on Corollaries 1 and 2 would first require analysis of the distributions of estimators based on maxima and minima of estimated values. While we have shown consistency and asymptotic normality, little more is currently known about these distributions. Addressing this problem—which arises in other contexts as well (e.g., Manski and Pepper (2000))—is beyond the scope of this paper.

## 3.2 Bounds Based on Pooled Bid Data

We briefly discuss an alternative approach to inference on $F(\cdot)$. Lemmas 1 and 2 imply that for all $n_t$ and $i < n_t$

$$b_{i:n_t} \leq v_{i:n_t} \leq b_{n_t:n_t} + \Delta \tag{6}$$

and

$$b_{n_t:n_t} \leq v_{n_t:n_t} \leq \overline{v}. \tag{7}$$

Hence, we have an upper bound and a lower bound on the valuation of each bidder in the sample.[12] Let $G_U(v)$ be the distribution of the lower bounds in (6) and (7), and $G_L(v)$ that for the upper bounds. This immediately gives the following result.

**Proposition 5** *$G_L(v) \leq F(v) \leq G_U(v)$.*

Consistent and asymptotically normal estimates of the pointwise upper and lower bounds in Proposition 5 can be obtained by taking the idealized sample analogs of these endpoints. This is a standard case of nonparametric estimation of a CDF, and we state the following result without proof.

**Proposition 6** *Let $TN = \sum_{t=1}^{T} n_t$ denote the number of observed bids and define*

$$\hat{G}_U(v) = \frac{1}{TN} \sum_{i,t} \mathbb{1}[b_{i:n_t} \leq v].$$

*$\hat{G}_U(v)$ converges at the parametric rate $\sqrt{TN}$ (as $T \to \infty$) to the identified upper bound, $G_U(v)$. Furthermore, $\hat{G}_U(v)$ is asymptotically normal. Similar results hold for $\hat{G}_L(v)$.*

The bounds $G_L(\cdot)$ and $G_U(\cdot)$ are necessarily wider than those obtained using the method based on order statistics. The upper bound $G_U(\cdot)$ does not exploit the fact that observed bids may imply bounds of differing tightness on the valuations of differently ranked bidders. Meanwhile, the lower bound $G_L(v)$ cannot exceed $\frac{N-1}{N}$ for any $v$. However, the use of all bids in estimation of each bound is a virtue of this approach. Another is the simplicity of constructing these bounds, which could serve as a simple test of whether stronger identifying assumptions yield estimates consistent with Assumptions 1 and 2. As we will see below, this approach to inference on $F(\cdot)$ can also be useful when considering semiparametric models of bidder and/or auction heterogeneity.

[12] In practice, if $\overline{v}$ is not known, we use the consistent estimator (under Assumptions 1 and 2) $\max_t b_{n_t:n_t}$.

## 4 Bounds on the Optimal Reserve Price

A primary motivation for interest in the distribution of bidder valuations is the potential for evaluating market design policy. In many cases the key policy instrument is the reserve price. Here we address derivation of bounds on the optimal reserve price based on the bounds on $F(\cdot)$ above.[13] This is non-trivial because the optimal reserve price can decline with a rightward shift in demand. Hence, the optimal reserve price need not lie between the prices that would be optimal if $F_L(\cdot)$ or $F_U(\cdot)$ were the true distribution. Note that this same complication arises in constructing confidence bands around an estimate of optimal reserve price when one has estimated the distribution $F(\cdot)$ using additional identifying assumptions. The solution we derive here could be applied in that case as well, treating the upper and lower confidence bands around $\hat{F}(\cdot)$ as the bounds.

Since we do not have a complete model of bidding at English auctions, strictly speaking we address the determination of bounds on the optimal reserve price for a sealed bid auction.[14] Myerson (1981) has shown that with a regularity condition on $F(\cdot)$, a standard auction with an optimal reserve price is optimal among all selling mechanisms.[15] Hence, when this regularity condition holds, we address bounds on the single unknown parameter of an optimal selling mechanism. However, note that if the seller sets $\Delta = 0$, Assumptions 1 and 2 imply that the English auction allocates the good efficiently among the bidders with valuations above the reserve price and ensures a payoff of zero to a bidder with valuation $\underline{v}$. Therefore, as long as bidding in an English auction can be characterized by incentive compatible behavior in *some* well-defined selling mechanism in which each bidder's private information consists of his valuation, the optimality derived by Myerson (and optimality of the reserve price we bound below) carries over to the English auction (see Myerson (1981)).

[13] Paarsch (1997) addresses estimation of optimal reserve prices in a parametric model of English auctions, interpreting the losing bids as those envisioned in the button auction model. Li, Perrigne, and Vuong (1997) consider semiparametric estimation of the optimal reserve price based on bids at first-price auctions. McAfee and Vincent (1992) and McAfee, Quan, and Vincent (1995) construct a lower bound on the optimal reserve price based on the discounted expected price at which an item attracting no bids would sell in a later auction. Carter and Newman (1998) estimate a reduced form model to assess the effect of marginal changes in reserve prices on revenues from USFS timber auctions in North Carolina.

[14] Note that we implicitly assume that the seller can commit to the reserve price; in particular, the seller will not reoffer an item failing to attract a bid above the reserve.

[15] Myerson's regularity condition—that $v - \frac{1-F(v)}{f(v)}$ increases monotonically in $v$, where $f(\cdot)$ is a density associated with $F(\cdot)$—is satisfied by many known distribution functions.

## 4.1 Derivation

Setting an optimal reserve price is equivalent to setting an optimal monopoly price with demand curve $q = 1 - F(p)$ and marginal cost $v_0$, equal to the seller's opportunity cost of selling the object.[16] Hence we consider the problem of identifying bounds on

$$p^* \in \arg\max R(p)$$

where

$$R(p) = (p - v_0)(1 - F(p)).$$

For simplicity we assume $v_0 = 0$; the generalization is straightforward and is spelled out in section 7. Let $F_U(\cdot)$ and $F_L(\cdot)$ denote upper and lower bounds on $F(\cdot)$, with $F_U(p) \geq F_L(p)$ for all $p$. Let

$$R_1(p) = p(1 - F_U(p)) \qquad \text{and} \qquad R_2(p) = p(1 - F_L(p))$$

so that

$$R_1(p) \leq R(p) \leq R_2(p) \quad \forall p.$$

Let

$$p_1^* \in \arg\max R_1(p) \qquad \text{and} \qquad p_2^* \in \arg\max R_2(p).$$

Now define $p_L$ and $p_U$ to be the highest and lowest prices at which the upper revenue function $R_2(\cdot)$ gives revenues at least as high as the maximal revenues on $R_1(\cdot)$; i.e.,

$$p_L = \inf\{p \in [0, \overline{v}] : R_2(p) \geq R_1(p_1^*)\}$$

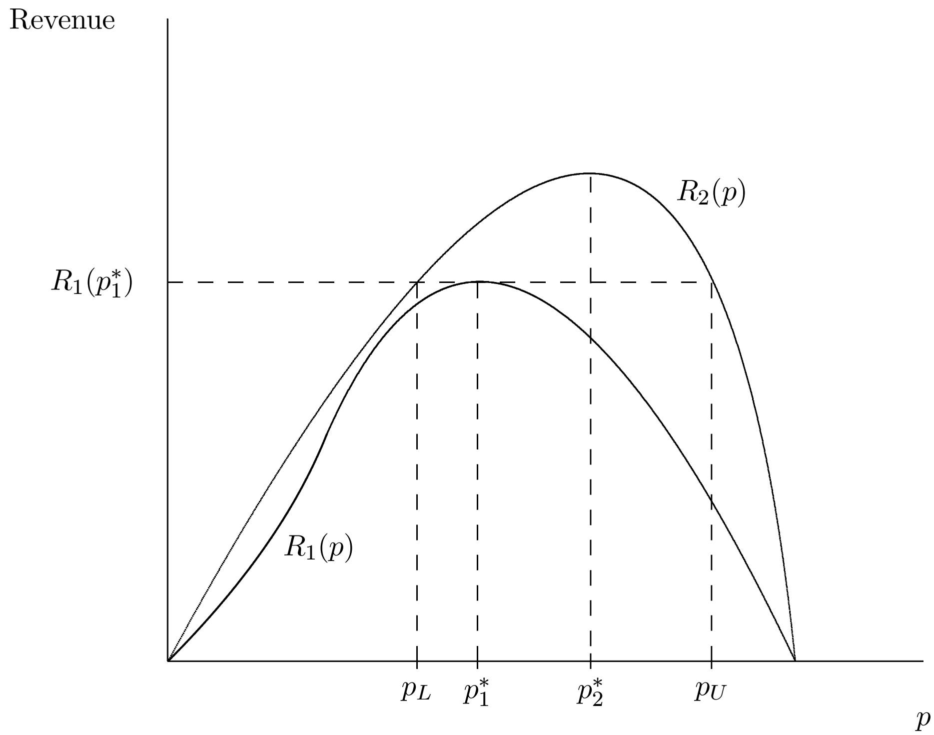
and

$$p_U = \sup\{p \in [0, \overline{v}] : R_2(p) \geq R_1(p_1^*)\}.$$

Since $R_2(p_1^*) \geq R_1(p_1^*)$, both $p_L$ and $p_U$ are well defined. Since $R_2(\cdot)$ is continuous and $R_1(p) = R_2(p)$ for $p \in \{0, \underline{v}, \overline{v}\}$, $R_2(p_L) = R_2(p_U) = R_1(p_1^*)$. Figure 1 illustrates.

[16] See Riley and Samuelson (1981), Myerson (1981), or Bulow and Roberts (1989).

Figure 1



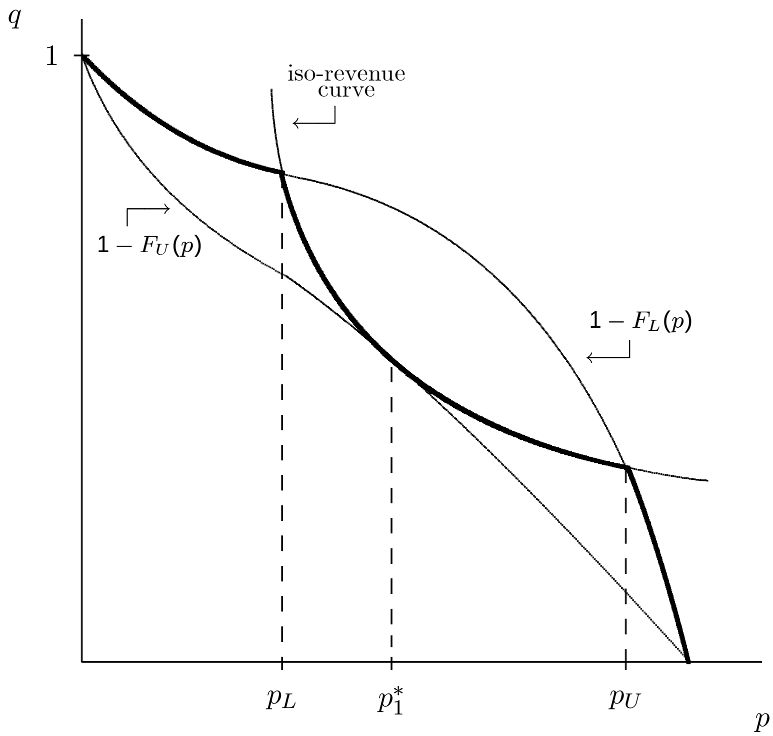

The following results show that $p_L$ and $p_U$ enclose the optimal reserve price and that these are the tightest bounds one can obtain from the bounds $F_U(\cdot)$ and $F_L(\cdot)$.

**Lemma 4** *$p_U \geq \max\{p_1^*, p_2^*\}$ and $p_L \leq \min\{p_1^*, p_2^*\}$.*

**Proposition 7** *$p^* \in [p_L, p_U]$. These bounds are sharp; i.e., there exist sequences of distributions $\{\tilde{F}_s(\cdot)\}$ and $\{\hat{F}_s(\cdot)\}$ with $\tilde{F}_s(p), \hat{F}_s(p) \in [F_L(p), F_U(p)]$ $\forall s, p$, such that the implied unique optimal reserve prices $\{\tilde{p}_s\}$ and $\{\hat{p}_s\}$ satisfy $\lim_{s\to\infty} \tilde{p}_s = p_L$ and $\lim_{s\to\infty} \hat{p}_s = p_U$.*

Some intuition can be gained from Figure 2, which shows the "demand curves" $q_j = 1 - F_j(p)$, $j \in \{U, L\}$, along with an iso-revenue curve tangent to the point $(p_1^*, 1 - F_U(p_1^*))$. Note that intersections of this iso-revenue curve and the upper demand curve define $p_L$ and $p_U$. Any downward sloping demand curve lying between the two original demand curves is consistent with the upper and lower bounds. The bold curve illustrates one possibility, tracing the outer demand curve for $p \notin [p_L, p_U]$ and tracing the iso-revenue curve for $p \in [p_L, p_U]$. With this demand curve (i.e., with the corresponding distribution function), any price in $[p_L, p_U]$ maximizes revenue. The proof verifies that for any $\epsilon > 0$ one can always construct a similar distribution function that makes $p_L + \epsilon$ (or $p_U - \epsilon$) the unique optimum.

Figure 2




If one is willing to assume that the true revenue function $R(p) \equiv p(1 - F(p))$ is quasiconcave, then tighter bounds can sometimes be obtained.[17] While $R_2(p)$ may exceed $R_1(p_1^*)$ for a non-convex set of prices $p$, the true revenue function cannot if it is quasiconcave. Hence we look for the boundaries of a convex set of prices around $p_1^*$:

$$\begin{aligned} p_L^q &= \inf\{p : R_2(\hat{p}) \geq R_1(p_1^*) \ \forall \hat{p} \in (p, p_1^*]\} \\ p_U^q &= \sup\{p : R_2(\hat{p}) \geq R_1(p_1^*) \ \forall \hat{p} \in [p_1^*, p)\}. \end{aligned}$$

[17] Myerson's (1981) regularity condition is another restriction one might consider imposing. Regularity requires that there be no flat point in the revenue function $R(\cdot)$ except at its maximum. Small perturbations of the revenue function constructed in the proof of Proposition 8 satisfy this condition (as well as differentiability of $F(\cdot)$).

Note that under the assumption of quasiconcave revenues, $p_L^q$ and $p_U^q$ are well defined, with $[p_L^q, p_U^q] \subseteq [p_L, p_U]$; if $R_2(\cdot)$ is quasiconcave, $(p_L^q, p_U^q) = (p_L, p_U)$. When $(p_L^q, p_U^q) \neq (p_L, p_U)$, imposing quasiconcavity will yield tighter bounds.

**Proposition 8** *If $R(\cdot)$ is quasiconcave, then $p^* \in [p_L^q, p_U^q]$. These bounds are sharp.*

### 4.2 Estimation

We can use either of the methods discussed in section 3 to obtain estimates of the revenue functions $R_1(p) = p(1 - F_U(p))$ and $R_2(p) = p(1 - F_L(p))$. We simply replace $F_U(\cdot)$ and $F_L(\cdot)$ by appropriate nonparametric estimators to obtain $\hat{R}_1(p) = p(1 - \hat{F}_U(p))$ and $\hat{R}_2(p) = p(1 - \hat{F}_L(p))$. The following lemma shows that

$$\hat{p}_1^* \equiv \arg\max_p \ \hat{R}_1(p)$$

is a consistent estimator of $p_1^*$.

**Lemma 5** $\hat{p}_1^* \xrightarrow{a.s.} p_1^*$.

We then obtain estimates of $p_U$ and $p_L$ by using $\hat{R}_2(\cdot)$ to solve for the largest and smallest values of $p$ giving $\hat{R}_2^s(p) \geq \hat{R}_1^s(\hat{p}_1^*)$.[18]

**Proposition 9** *Let $\hat{p}_U$ and $\hat{p}_L$ denote, respectively, the* sup *and* inf *of the set $\{p : \hat{R}_2^s(p) \geq \hat{R}_1^s(\hat{p}_1^*)\}$. Then $\hat{p}_U$ and $\hat{p}_L$ converge, respectively, to $p_U$ and $p_L$ as $T \to \infty$.*

## 5 Auction and Bidder Heterogeneity

Let $Z_{it} = (X_t, W_{it})$, where $X_t$ is a vector of observable characteristics of auction $t$ and $W_{it}$ is a vector of observable characteristics of bidder $i$ at auction $t$. We develop techniques that condition on these observables, enabling us to capture shifts in valuations that are auction- and/or bidder-specific. Conditioning on auction-specific heterogeneity is often important in practice, since one rarely observes multiple auctions of identical objects. Furthermore, one is often interested directly in the ways that observables affect valuations. For instance, letting $X_t = n_t$ one could ask whether valuations depend on the number of bidders, providing an overidentifying test of the private values assumption. One may also be interested in how characteristics of the object for sale, the identity or characteristics of the seller (for example, his eBay rating[19]), or other exogenous factors affect

[18] In practice one can smooth the revenue functions (using a kernels, for example), since the selected bounds may be sensitive to irregularities arising from the step functions $\hat{F}_U(\cdot)$ and $\hat{F}_L(\cdot)$.

[19] See Bryan et al. (2000) for a reduced form analysis of the effects of this and other auction/seller characteristics at eBay auctions.

bidder valuations. The approaches discussed above are easily adapted to allow for this type of heterogeneity, and we show how our bounds on the *conditional* distributions of valuations can be used to assess the effects of auction-specific observables on valuations following the identification and estimation strategies in Manski and Tamer (2000).

## 5.1 Heterogeneous Auctions

We first address the case in which bidders are symmetric at each auction but auctions differ in observable dimensions. This is the case considered in the majority of empirical studies of auctions, where symmetry at each auction is assumed but object heterogeneity is accounted for with covariates. We make the following assumption regarding the effect of auction characteristics $X_t$ on valuations, although one could also use functionals such as the conditional median rather than the conditional mean.

**Assumption 3** *$E[V_{it}|X_t = x_t] = l(x_t\beta_0)$, with $\beta \in \mathcal{B}$, where $\mathcal{B}$ is a compact subset of $\mathsf{R}^d$ and $l(\cdot)$ is a known function mapping $\mathsf{R} \to \mathsf{R}$.*

The results from the preceding sections carry through when the distributions $F_{i:n}(\cdot)$ and $G_{i:n}(\cdot)$ are replaced with $F_{i:n}(\cdot|X_t)$ and $G_{i:n}(\cdot|X_t)$.[20] Hence, one can easily construct bounds $F_U(\cdot|X)$ and $F_L(\cdot|X)$ on the distribution of valuations, conditional on a given value of $X$. Given these bounds, let $\overline{Z}_x$ be a random variable with distribution $F_L(\cdot|x)$ and let $\underline{Z}_x$ be a random variable with distribution $F_U(\cdot|x)$. Assumption 3, Lemmas 1 and 2, and the definition of first-order stochastic dominance then imply

$$E[\underline{Z}_x] \leq l(x\beta_0) \leq E[\overline{Z}_x]$$

This is an incomplete econometric model that provides inequality restrictions on regressions. For $b \in \mathcal{B}$ define

$$V(b) = \left\{x : l(x_t b) < E[\underline{Z}_x] \quad \bigcup \quad E[\overline{Z}_x] < l(x_t b)\right\}$$

Then, given Assumptions 1, 2, and 3, a parameter $\beta_0$ is *identified* relative to $b$ if and only if $\Pr[V(b)] > 0$. Hence, the identified set of parameters is

$$\Sigma = \{b \in \mathcal{B} : \Pr[V(b)] = 0\}.$$

The parameter $\beta_0$ is *point identified* (i.e., $\Sigma = \beta_0$) iff for all $b \in \mathcal{B}_0$, $\Pr[V(b)] > 0$. In general the identified set will be a non-empty set[21] in $\mathsf{R}^d$, where $d$ is the dimension of $\beta_0$. However, sufficient

[20] Nonparametric estimators of the conditional distribution functions $G_{i:n}(\cdot|X_t)$ must replace the simple empirical distribution functions suggested in the previous section. These estimators will not achieve the parametric rates of convergence that the unconditional empirical CDFs do.

[21] If $l(\cdot)$ is monotone, the set will be convex.

conditions for point identification can be derived, even when the bounds $F_U(\cdot|X)$ and $F_L(\cdot|X)$ are not coincident. For example, if and $l(a) = a$ and both $E[\underline{Z}_x]$ and $E[\overline{Z}_x]$ are linear in $X$, all parameters on regressors with unbounded support are point identified.[22] In the next lemma, we define the modified minimum distance estimator that we will use to estimate the identified features of the model. This estimator was introduced in Manski and Tamer (2000).

**Lemma 6** *Let $g_1(x,c) = 1[E[\underline{Z}_x] > l(xb)]$ and $g_2(x,c) = 1[E[\overline{Z}_x] < l(xb)]$. Let*

$$Q(b) = \int \left[(E[\underline{Z}_x] - l(xb))^2 g_1(x,b) + (l(xb) - E[\overline{Z}_x])^2 g_2(x,b)\right] dP_X$$

*and $P_X$ is the distribution of the conditioning variable(s) $X$. Then $Q(b) \geq 0$ for all $b \in \mathcal{B}$. $Q(b) = 0$ if and only if $b \in \Sigma$.*

Estimation of the identified set is based on a sample analog, using sampling from the estimated distributions $\hat{F}_U(\cdot|X)$ and $\hat{F}_L(\cdot|X)$. Let $E_T\left[\underline{Z}_x\right]$ and $E_T\left[\overline{Z}_x\right]$ denote the means of a sample draws from $\hat{F}_U(\cdot|X)$ and $\hat{F}_L(\cdot|X)$, respectively. Define the objective function

$$\begin{aligned} Q_T(b) \quad = \quad & \frac{1}{T}\sum_{t=1}^{T}\Bigg[\left(E_T\left[\underline{Z}_x\right] - l(X_t\beta)\right)^2 1\left[E_T\left[\underline{Z}_x\right] > l(X_t\beta)\right] \\ & + \left(l(X_t\beta) - E_T\left[\overline{Z}_x\right]\right)^2 1\left[l(X_t\beta) > E_T\left[\overline{Z}_{,x}\right]\right]\Bigg] \end{aligned}$$

and the distance function

$$\rho(\Sigma_T, \Sigma) = \sup_{b\in\Sigma_T} \inf_{b^*\in\Sigma} |b - b^*|$$

The following result describes the properties of the modified minimum distance estimator based on minimization of $Q_T(b)$.

**Proposition 10** *Let assumptions 1,2 and 3 hold. Let $E_T\left[\underline{Z}_x\right]$ and $E_T\left[\overline{Z}_x\right]$ be nonparametric estimates of the corresponding conditional expectations such that $E_T\left[\underline{Z}_x\right] \xrightarrow{a.s.} E\left[\underline{Z}_x\right]$ and $E_T\left[\overline{Z}_x\right] \xrightarrow{a.s.} E\left[\overline{Z}_x\right]$. Assume that there exists an integrable function $\psi : \mathsf{R}^d \to \mathsf{R}$ that dominates $(E\left[\underline{Z}_x\right] - l(xb))^2 g_1(X,b) + (E\left[\overline{Z}_x\right] - l(xb))^2 g_2(X,b)$. Let $\epsilon_T \to 0$, then*

$$\Sigma_T \equiv \{b \in \mathcal{B} : Q_T(b) \leq \min_{c\in\mathcal{B}} Q_T(c) + \epsilon_T\} \tag{8}$$

*is such that*

$$\rho(\Sigma_T, \Sigma) \xrightarrow{a.s.} 0.$$

*If $sup_{b\in\mathcal{B}}\{|Q_T(b) - Q(b)|\}/\epsilon_T \to 0$, then $\rho(\Sigma, \Sigma_T) \to 0$.*

[22]This is easy to see since for any $b \neq \beta$, one can drive $x$ far enough on the support that the implied mean crosses the upper bound.

The proposition above provides a result in terms of set convergence. In the presence of set-valued estimates, we must ensure that the estimates derived from (8) above do not converge to a proper subset of the identified set $\Sigma$. This is ensured by having $\Sigma_T$ include not just parameter values that minimize the sample objective function, but also parameters that come $\epsilon_T$ away from minimizing it. Letting $\epsilon_T$ go to zero at an appropriate rate ensures that $\Sigma_T$ converges in the sense above to the set $\Sigma$. In particular, $\epsilon_T$ must converge to zero at a slower rate than that of $sup_b\,|Q_T(b) - Q(b)|$. For additional discussion of this type of estimator, see Manski and Tamer (2000).

## 5.2 Heterogeneous Bidders

The more general case, allowing bidders to differ in observable characteristics, is similar.[23] With asymmetric bidders, our order-statistic based approach cannot be applied by simply conditioning on the observables, since equation (1) relies on symmetry between bidders at a given auction. Our approach based on pooled bid data, however, can be applied in this case.[24] For conditioning variables $Z_{it} \in \mathsf{R}^d$, let $F_L(\cdot|Z_{it})$ and $F_U(\cdot|Z_{it})$ denote the conditional distributions of the upper and lower bounds in equations (6) and (7). These equations immediately imply that for all $v$,

$$F_L(v|Z_{it}) \leq F(v|Z_{it}) \leq F_U(v|Z_{it}). \tag{9}$$

One is often interested in measuring the sign and/or magnitude of the effects of $Z_{it}$ on the distribution of valuations. For example, tests for the presence of inventory constraints or complementarities rely on estimates of such effects. Here we show how an assumption regarding the median of the valuations enables inference on these effects.

**Assumption 4** *Let $Med[v_{it}|Z_{it}] = l(x_t\beta_0 + w_{it}\delta_0)$, with $(\beta_0, \gamma_0) \in \mathcal{B} \subset \mathsf{R}^d$ and $l(\cdot)$ a known function defined on* $\mathsf{R}$.

Since in practice we often will not know the value of $\bar{v}$, we must assume that it is an arbitrary large number. This makes a conditional mean assumption unattractive. The median is a more robust statistic in cases like this. As in the preceding section, one can entertain a number of specifications for the "link" function $l(\cdot)$. For example one specification could be the linear model where $l(a) = a$. Since bids are nonnegative in practice, another natural specification is the exponential link function $l(x_t\beta_0 + w_{it}\delta_0) = exp(x_t\beta_0 + w_{it}\delta_0)$.

[23] Empirical work considering asymmetric first-price sealed bid auction includes Bajari and Ye (2000) and Pesendorfer and Jofre-Bonet (1999). We are unaware of any empirical analysis of English auctions allowing bidder asymmetry.

[24] The approaches suggested in Haile and Tamer (2000), which addresses a more general model of bidder asymmetry, could also be applied.

Assumption 4, the inequality (9), and the definition of first-order stochastic dominance imply

$$Med[B_{it}|Z_{it}] \leq l(x_t\beta_0 + w_{it}\gamma_0) \leq Med[B_{n:n}|Z_{it}].$$

So for $(b,c) \in \mathcal{B}$, define

$$V(b,c) = \left\{(x,w) : l(x_t b + w_{it}c) < Med[B_{it}|z_{it}] \quad \bigcup \quad Med[B_{n:n}|z_{it}] < l(x_t b + w_{it}c)\right\}.$$

Then a parameter pair $(\beta_0, \gamma_0)$ is identified relative to $(b,c)$ if and only if $\Pr[V(b,c)] > 0$. Point identification is equivalent to $(\beta_0, \gamma_0)$ being identified relative to all $(b,c) \in \mathcal{B}\backslash(\beta_0,\gamma_0)$. A sufficient condition for point identification is that for some $Z$ such that $\Pr(Z) > 0$, $Med[B_{it}|Z] = Med[B_{n:n}|Z]$. In general we would not expect this condition to hold. However, if $Med[B_{it}|Z]$ and $Med[B_{n:n}|Z]$ are very close for values of $Z$ occurring with positive probability, we should obtain tight bounds on $(\beta_0, \gamma_0)$. In general the set of observationally equivalent parameters is

$$\Sigma = \{(b,c)|\Pr[V(b,c)] = 0\}$$

**Lemma 7** *Let $\eta_1(z) \equiv Med[B_{it}|z]$ and $\eta_2(z) = Med[B_{n:n}|z]$. Let $g_1(z,b,c) = \mathbb{1}[Med[B_{it}|z] > l(xb + wc)]$ and $g_2(z,b,c) = \mathbb{1}[Med[B_{n:n}|z] < l(xb+wc)]$. Define*

$$Q(b,c,\eta_1,\eta_2) = \int \left[(\eta_1(z) - l(xb+wc))^2 g_1(z,b,c) + (\eta_2(z) - l(xb+wc))^2 g_2(z,b,c)\right] dP_Z$$

*and $P_Z$ is the distribution of the conditioning variables $Z$. Then $Q(b,c) \geq 0$ for all $(b,c) \in \mathcal{B}$. Moreover, $Q(b,c) = 0$ if and only if $(b,c) \in \Sigma$.*

The lemma implies that every $(b,c) \in \Sigma$ solves the problem

$$min_{(b,c)\in\mathcal{B}}\ Q(b,c)$$

This suggests the following analog estimator of $Q(b,c)$:

$$Q_T(b,c,\eta_1,\eta_2) = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{n_t}\sum_{i=1}^{n_t}\left[(\eta_1(z_{it}) - l(x_t b - w_{it}c))^2 g_1(z_{it},b,c) + (\eta_2(z_{it}) - l(x_t b - w_{it}c))^2 g_2(z_{it},b,c)\right].$$

Since we do not observe the functions $\eta_1$ and $\eta_2$, we replace them by nonparametric estimators $\eta_{1T}$ and $\eta_{2T}$ based on local approximations of the conditional median functions. Then, as above, we consider sets of parameter values that approximately minimize $Q_T(b,c,\eta_{1T},\eta_{2T})$. For a specified $\epsilon_T > 0$ with $\lim_{T\to\infty}\epsilon_T = 0$, this set is

$$\Sigma_T = \{(b,c) : Q_T(b,c,\eta_{1T},\eta_{2T}) \leq \arg\min_{(b,c)\in\mathcal{B}} Q_T(b,c,\eta_{1T},\eta_{2T}) + \epsilon_T\}.$$

The following proposition gives conditions guaranteeing almost sure convergence of $\Sigma_T$ to $\Sigma$.

**Proposition 11** *Let assumptions 1, 2 and 4 hold. Assume there exists an integrable function* $\psi : R^d \to R$ *that dominates* $(\eta_1(z) - l(xb - wc))^2 g_1(z, b, c) + (\eta_2(z) - l(xb - wc))^2 g_2(z, b, c)$. *Let* $\eta_{1T}(\cdot) \overset{a.s.}{\to} \eta_1(\cdot)$ *and* $\eta_{2T}(\cdot) \overset{a.s.}{\to} \eta_2(\cdot)$ *as* $T$ *gets large. If* $\epsilon_T \to 0$, *then*

$$\rho(\Sigma_T, \Sigma) \overset{a.s.}{\to} 0.$$

*If* $sup_{(b,c)}\{|Q_T(b, c, \eta_{1T}, \eta_{2T}) - Q(b, c, \eta_1, \eta_2)|\}/\epsilon_T \to 0$, *then* $\rho(\Sigma, \Sigma_T) \to 0$.

# 6 Monte Carlo Experiments

To examine the small sample performance of the proposed estimation approaches and compare our bounds to the estimates obtained using alternative approaches considered previously, we have performed a number of Monte Carlo experiments. To generate artificial bidding data for each experiment, $T$ samples of $n_t$ valuations were first drawn from a known distribution $\Phi(\cdot)$, the lognormal with location parameter 3 and scale parameter 0.5. Bids were then generated based on the following selling mechanism: All bidders are initially identified as active. As long as at least two bidders are active, the seller picks one of the active bidders at random. This bidder must either raise the current standing bid by at least one bid increment ($\Delta$) or decline to bid. If the bidder declines, he becomes inactive and the process iterates. If the bidder accepts, the standing bid is raised by this increment and the process iterates. A bidder who accepts in one iteration is prevented from being picked again on the next iteration so that bidders are not asked to raise their own bids. A bidder is declared the winner when his is the standing bid and no other bidders are active. Hence, it is optimal for a bidder to agree to bid whenever his valuation exceeds the standing bid by at least $\Delta$.

Up to the discrete bidding increment (except where otherwise noted, we use a small increment of 5% of the standing high bid) this procedure replicates the outcomes of the button auction. In a variant, with probability $\lambda$ a bidder who agrees to bid jumps to a uniform draw between the standing bid (plus $\Delta$) and his valuation. This binomial draw on whether to "jump bid" is made independently at each bidding opportunity. This is, of course, only an approximation of the true data generating process at English auctions.[25] Nonetheless, the data generated in this way capture deviations from the button auction model that are common in field data. In particular, gaps between the top two bids may exist; bids may be poor approximations of valuations; and

[25] We have considered a number of further variations, including asymmetric bidding propensities (asymmetric probabilities of being selected to bid), chosen at random or as a function of the realized valuations. The results we present below are representative of those obtained with these variants.

the ordering of bidders according to their bids may be different from the ordering according to valuations.

## 6.1 Order Statistic Based Approach

Figures 3a and 3b illustrate the results for two experiments with $n_t = 3$ and $T = 200$, using 100 replications. In figure 2a, $\lambda = 0$, while in figure 2b $\lambda = \frac{1}{4}$. We report the (pointwise) means of our estimated upper and lower bounds, as well as a 90% confidence interval, constructed by taking the (pointwise) 5th percentile of the lower bounds and 95th percentile of the upper bounds.

Figures 4a and 4b present the results when we consider auctions with 3, 5, and 7 bidders—200 of each. These figures illustrate two important phenomena. First, because we take a maximum of the individual lower bounds (from different $n$) and a minimum of the individual lower bounds (from the different order statistics of the different auction sizes), the bounds are necessarily tighter than when we had only $n = 3$. Second, there is severe bias in the lower tails, where the upper and lower bounds often cross. This occurs because, it is likely (particularly when $n$ is large) that at least one of the empirical distributions $\hat{G}_{j:n}(v)$ is equal to zero for small values of $v$, causing the upper bound to go to zero when we take the minimum. This type of problem is inherent in estimators that involve taking maxima or minima of estimated values. Intuitively, by taking a maximum (minimum) we sometimes pick up the estimated value with the highest positive (negative) estimation error. This problem disappears in sufficiently large samples, but appears to be a serious problem in sample sizes typical of those available in practice. Fortunately, this problem can be easily overcome with a minor modification to the approach proposed above.

### 6.1.1 Averaging Bounds in Finite Samples

Under the order-statistic based approach, for each $v$ and $n$ we obtain a lower bound by solving the equation

$$F_{n-1:n}(v) = \frac{n!}{(n-2)!} \int_0^{F(v)} s^{n-2}(1-s)\, ds \tag{10}$$

for $F(v)$ while substituting for each $F_{n-1:n}(v)$ a value $(G^{\Delta}_{n:n}(v))$ that is known to be smaller; we then take the largest implied bound on $F(v)$ as $F_L(v)$. A natural alternative to taking the maximum bound is to take an average. While several ways of doing this are possible, we propose a simple weighted averaging approach.

Let $\{\mathcal{N}_L^{\rho}\}$ denote a set of disjoint subsets of $\{2, \ldots, N\}$ indexed by $\rho$. Let $\{\gamma_n\}$ be a vector of weights, one for each $n \in \{2, 3, \ldots, N\}$. Then for each $v$ let

$$F_L^\rho(v) = \arg\min_{F_L} \sum_{n \in \mathcal{N}_L^\rho} \gamma_n \left[ G_{n:n}^\Delta(v) - \frac{n!}{(n-2)!} \int_0^{F_L} s^{n-2}(1-s)\, ds \right]^2 .$$

When $\gamma_n = \gamma$ for all $n$, $F_L^\rho(v)$ is just an unweighted mean of $\phi(G_{n:n}^\Delta(v); n-1, n)$ for $n \in \mathcal{N}_L^\rho$. Since each $\phi(G_{n:n}^\Delta(v); n-1, n)$ is a valid lower bound on $F(v)$, so is each average $F_L^\rho(v)$. Hence, $F_L(v) \equiv \max_\rho F_L^\rho(v)$ is also a valid bound. A consistent estimator of this bound can again be obtained from the sample analog.

A similar procedure can be followed for the upper bound. Let $\{\mathcal{N}_U^\rho\}$ be a set of disjoint subsets of $\{(i,n)\}_{n=2,\ldots,N; i=1,\ldots,n-1}$ and define weights $\gamma_{i,n}$ for each pair $(i,n)$. Then for each $v$ and $\rho$ let

$$F_U^\rho(v) = \arg\min_{F_U} \sum_{(i,n) \in \mathcal{N}_U^\rho} \gamma_{i,n} \left[ G_{i:n}(v) - \frac{n!}{(n-i)!(i-1)!} \int_0^{F_U} s^{i-1}(1-s)^{n-i}\, ds \right]^2 .$$

Then $F_U(v) \equiv \min_\rho F_U^\rho(v)$ gives an upper bound on $F(v)$ that is consistently estimated by sample analog.

Other natural weighting schemes could be constructed based on the standard "inverse variance rule," e.g., $\gamma_n = var(\phi(G_{n:n}^\Delta(v); n-1, n)^{-\frac{1}{2}}$, using either the asymptotic variance derived above or bootstrapped finite sample variances. This weighting can then account for the facts that there often will be more data for certain auction sizes than others, and that the precision of the estimate $\phi(\hat{G}_{i:n}(v); i, n)$ will vary with the evaluation point $v$ in different ways depending on the values of $i$ and $n$.

Note that while this approach will reduce the finite sample bias by averaging over more data before taking a min or max, the estimated bounds will necessarily be wider than those proposed above, unless $\{\mathcal{N}_L^\rho\} = \{\mathcal{N}_U^\rho\} = \{\{2\}, \{3\}, \ldots, \{N\}\}$.[26] An attractive feature of this approach, however, is that by varying the fineness of the partitions $\{\mathcal{N}_L^\rho\}$ and $\{\mathcal{N}_U^\rho\}$, trade-offs can be made between the bias of the estimates and the tightness of the bounds being estimated.

For simplicity we will partition bids into two sets, with the marginal bids $B_{n-1:n}$ in one set and all other bids in the second set. This choice of partition is ad hoc, but is based on our expectation that in practice the top losing bid is most likely to be close to the true valuation of the corresponding bidder. We use equal weights in each cell of the partition to construct the averages. For the lower bound we do not split the data, instead just averaging the individual lower bounds using equal weights. Figures 5a and 5b show the results when the experiments underlying figures 4a and 4b

[26] By letting $\{\mathcal{N}_L^\rho\}$ and $\{\mathcal{N}_U^\rho\}$ converge to $\{\{2\}, \{3\}, \ldots, \{N\}\}$ as $T \to \infty$, the probability limits of the averaging-based estimators will be the bounds derived in section 3.1.

are re-run. The behavior of these estimators is quite good and, in fact, remains so in (unreported) experiments with even smaller samples. While averaging necessarily results in wider bounds, it addresses the finite sample bias problem effectively at what appears to be a small cost in terms of the tightness of the bounds.

Figure 6 illustrates the important point made in Remark 1. Here we modify our simulation procedure by setting the bid increment $\Delta$ to zero, i.e., by generating bids according to the equilibrium of the button auction model. The upper and lower bounds are coincident on the true distribution.

Table 1: Monte Carlo Simulations: Optimal Reserve Price

| Lognormal Parameters: | $\mu = 3, \sigma = 0.5$ | $\mu = 3, \sigma = 1.0$ | $\mu = 3, \sigma = 1.5$ |
|---|---|---|---|
| true $p^*$ | 15.5 | 27.2 | 45.1 |
| $F(p^*)$ | 0.30 | 0.62 | 0.71 |
| $\underline{T = 200}$ | | | |
| mean bounds $[\hat{p}_L, \hat{p}_U]$ | [11.7,21.7] | [19.4,56.0] | [45.1,109.6] |
| 90% confidence interval | [10.5,24.0] | [13.8,83.8] | [21.3,222.0] |
| true expected revenue at: | | | |
| $p_t = 0$ | 21.2 | 25.1 | 28.5 |
| $p_t = \hat{p}_L$ | 21.3 | 26.3 | 32.0 |
| $p_t = p^*$ | 21.4 | 26.6 | 32.0 |
| $p_t = \hat{p}_U$ | 20.4 | 18.3 | 27.3 |
| $\underline{T = 800}$ | | | |
| mean bounds $[\hat{p}_L, \hat{p}_U]$ | [11.5,22.3] | [16.4,48.6] | [27.0,91.1] |
| 90% confidence interval | [10.7,23.2] | [13.8,58.8] | [21.3,135.0] |
| true expected revenue at: | | | |
| $p_t = \hat{p}_L$ | 21.3 | 26.0 | 31.2 |
| $p_t = \hat{p}_U$ | 20.3 | 24.6 | 29.1 |

Table 1 summarizes the bounds on the optimal reserve price obtained using variations on the lognormal experiments above. Several observations can be made from these results. First, although in each case we set $\lambda = 0$, so that our bounds on $F(\cdot)$ are tight, the bounds on the optimal reserve are fairly wide. In all but one of the experiments, setting a reserve equal to our estimated upper bound would yield lower revenues under the true distribution than setting a nonbinding reserve price of zero. In general the tightness of the bounds on the optimal reserve depend on the shapes of the bounding upper bounding revenue function in the neighborhood of the price that maximizes the lower bounding revenue function. Second, if the upper or lower bounds lie in the tails of the true distribution, the finite sample estimates of the upper and lower bounds can be quite noisy. This problem is sufficiently severe when $\sigma = 1.5$, for example, that our estimated lower bound in samples of 200 often lies above the true optimal reserve. Meanwhile, the upper bound on the optimal reserve is virtually uninformative. This problem is clearly reduced in larger samples, although the bounds are still quite wide, and 800 auctions is a fairly large sample. Since our bounds on the optimal reserve are sharp given the (tight) bounds on $F(\cdot)$, these results suggest fundamental sensitivity of policy implications to seemingly small variations in the primitives. A virtue of the inferential approach we propose here is that we can explicitly account for this in evaluating policy implications—something we will do in the application below.

## 6.2 Comparison to Alternative Methods

We next consider experiments using the same data generating processes but two estimation approaches considered previously in the literature, based on the button auction model. The first of these approaches ignores the winning bids and treats all losing bids as if they were equilibrium bids in the button auction model. This approach was proposed by Donald and Paarsch (1996) and implemented by Paarsch (1997) and Hong and Shum (1999). We follow this literature by using a maximum likelihood implementation of this model. Letting $\tilde{F}(\cdot;\theta)$ denote a parametric family of distribution functions with density $\tilde{f}(\cdot;\theta)$, the likelihood function is obtained from the joint density of the lowest $n_t - 1$ order statistics of the $n_t$ valuations at each auction; i.e.,[27]

$$\mathcal{L}(\theta) = \prod_{t=1}^{T} \left\{ n_t \left[ 1 - \tilde{F}\left(b_{n_t-1:n_t};\theta\right) \right] \prod_{i=1}^{n_t-1} \tilde{f}(b_{i:n_t};\theta) \right\}.$$

We refer to this as "Model 1." While in practice one would not typically know the true parametric family, we will let $\tilde{F}(\cdot;\theta)$ be the true (lognormal) family. Figures 7a and 7b illustrate the results for $n = 3$, $T = 200$, and $\lambda \in \{0, \frac{1}{4}\}$, where we compare the true lognormal distribution to that at the mean parameter estimates. Even when $\lambda = 0$, the performance of this model is extremely poor. These estimates would lie within our bounds over little of the support. Evidently the deviations from the button auction data generating process created by the 5% bid increments is sufficient to lead to large bias. Figure 7c confirms that this is not a small sample problem by

[27] Note that because $b_{n_t:n_t} = b_{n_t-1:n_t}$ in the equilibrium of the button auction, one could substitute $1-\tilde{F}(b_{n_t:n_t};\theta)$ for $1-\tilde{F}(b_{n_t-1:n_t};\theta)$, as Donald and Paarsch (1996) do.

applying the same empirical model to data generated according to the button auction model. The MLE performs extremely well in this case.

An alternative, "Model 2," avoids misinterpretation of the losing bids by ignoring them altogether and assuming only that the winning bid $b_{n_t:n_t}$ is equal to the second highest valuation $v_{n_t-1:n_t}$. This approach was proposed by Paarsch (1992b) and has also been used by Baldwin, Marshall, and Richard (1997) and Haile (2000). We take the fully nonparametric implementation of this model suggested by Athey and Haile (2000). Up to the bid increment $\Delta$, which could be accounted for in different ways, this estimator is identical to the mean of the lower bounds $\phi(G^{\Delta}_{n:n}(v); n-1, n)$. Hence, up to the treatment of $\Delta$, this method will always give estimates lying outside our bounds—strictly so except in the case in which our lower bound is constructed by averaging $\phi(G^{\Delta}_{n:n}(v); n-1, n)$ across all $n$. However, when the gap between each $b_{n_t:n_t}$ and $v_{n_t-1:n_t}$ is small, Model 2 will still give estimates that are close to the truth. Figure 8a illustrates, showing the results for $n = 3$, $T_n = 200$, $\lambda = 0$. The mean estimates and the true CDF are nearly indistinguishable, and the sampling error is fairly small. However, figure 8b shows the results when $\lambda = \frac{1}{4}$. Here, the bias resulting from misinterpretation of the winning bid is substantial, with the true distribution lying outside the 90 percent confidence bands over most of the support.

## 6.3 Pooled Bid Data Approach

For comparison, we report a few results using the approach based on pooled bid data. Figure 9 shows the 90 percent Monte Carlo confidence bands with the true CDF for the case $n = 3, T_n = 200, \lambda = 0$. While the true distribution of course lies within the bounds, the bounds are considerably wider than those obtained above. This is expected, as discussed above, since the derivation of these bounds does not take advantage of the ordered nature of the bid data. Indeed, this is nearly the worst possible data configuration for this estimator, since the lower bound cannot exceed $\frac{n-1}{n}$.

Despite the width of these bounds, however, they are sufficient to provide fairly tight inference on parameters characterizing heterogeneity across auctions. To illustrate, we consider a simple model in which

$$v_{it} = \exp(x_t\beta + \epsilon_{it})$$

with each $\epsilon_{it}$ normally distributed (with mean zero and variance $\sigma_1^2$) and each $x_t$ distributed $\mathrm{N}(\mu, \sigma_2^2)$. Table 2 summarizes the results of two Monte Carlo experiments, each with $n = 3$ and $T = 100$. We use $\beta = 1$ in each experiment, varying the relative standard deviations of the covariates and the "error term" $\epsilon_{it}$.

Table 2: Monte Carlo Simulations
Auction Heterogeneity Model
Estimation Using Pooled Bid Data

| | mean lower bound | mean upper bound | 90% Confidence Interval |
|---|---|---|---|
| $\mu = 3, \sigma_1 = .1, \sigma_2 = 1$ | 1.01 | 1.07 | (.74,1.27) |
| $\mu = 3, \sigma_1 = 1, \sigma_2 = 1$ | .86 | 1.03 | (.67,1.37) |

# 7 Application to Forest Service Timber Auctions

## 7.1 Background and Data

We apply our estimation approaches to data from oral ascending (English) auctions of timber harvesting contracts held by the U.S. Forest Service. We focus on sales in Washington and Oregon held between 1982 and 1990. We discuss only a few key aspects of these auctions, referring readers to Baldwin, Marshall, and Richard (1997), Athey and Levin (2000), or Haile (1996, 2000) for more detailed background.

As in Baldwin, Marshall, and Richard (1997) and Haile (2000), we focus on a subset of sales for which our independent private values (IPV) assumption is most compelling.[28] By focusing on "scaled sales"[29] with contract lengths of one year or less, most bidder uncertainty regarding timber volumes and prices is eliminated or at least insured by the Forest Service.[30] We also exclude from our sample salvage sales, sales set aside for small bidders, and sales of contracts with road construction requirements. We assume bidding in each auction in our sample is competitive and

[28] Other studies assuming the IPV model for timber auctions include Johnson (1979), Hansen (1985, 1986), Paarsch (1992, 1995), Cummins (1994), Elyakime et al. (1994), Carter and Newman (1998), and Campo, Perrigne, and Vuong (2000). See Baldwin, Marshall, and Richard (1997), and Haile (2000) for discussion of this choice.

[29] In a scaled sale, bids are species-specific prices per unit harvested, so the Forest Service effectively insures away uncertainty over the volume of timber actually on the tract. Athey and Levin (2000) point out that "skew bidding" can arise if bidders have private information about errors in the Forest Services estimates of the distribution of volume across species, implying a common values model. Following Baldwin, Marshall, and Richard (1997) and Haile (2000), we assume there is little information of this sort in the auctions we consider and focus on the total bid made by each bidder.

[30] This, along with our restriction to sales after 1981, also minimizes the likelihood that opportunities for subcontracting introduce a common value element as in Haile (2000).

can be treated independently of the others.[31]

Before each sale in our sample, the Forest Service published a "cruise report," which provided estimates of timber volume, harvesting costs, costs of manufacturing end products, and revenues from end product sales. Forest Service officials also used these estimates to construct a reserve price for the sale. Bidders were required to submit sealed bids of at least the reserve price to qualify for the sale. Hence, sales records indicate the registration of all bidders, including any who do not actually call out a bid at the auction. Bids at the auction are made on a per unit (thousand board-feet, denoted by "MBF") basis. Bidding opens at the highest sealed bid and then proceeds orally, with a minimum bid increment of 5 cents per thousand board-feet.[32] As Table 3 suggests, jump bidding is common, with the gap between the highest and second-highest bids of several hundred dollars in the majority of auctions.[33] Since the cost of jump bidding (the risk that one wins with the jump bid and pays too much) is highest at the end of the auction, jump bidding may be even more significant early in the auctions. Nonetheless, the gaps are generally small relative to the total bid, suggesting that we may be able to obtain tight bounds.

Table 3: Gaps Between First- and Second-Highest Bids

| | High Bid | Gap | Minimum Increment | Gap ÷ Increment |
|---|---|---|---|---|
| Quantiles: | | | | |
| 10% | 9,151 | 30 | 4.1 | 1.2 |
| 25% | 22,041 | 92 | 10.1 | 6.9 |
| 50% | 55,623 | 309 | 23.4 | 14.8 |
| 75% | 127,475 | 858 | 52.1 | 20.0 |
| 90% | 292,846 | 2,048 | 110.5 | 76.4 |

[31] Baldwin, Marshall, and Richard (1997) consider the possibility of collusion in auctions in this region in an earlier time period.

[32] Forest Service rules require only that total bids rise as the auction proceeds, although local officials often specified discrete increments. In the time period we consider, the 5 cent increment was one common practice in this region. Sometimes increments of 1 cent per MBF were used, although many sales used no minimum increment. We use the 5 cent increment since this results in a more conservative bound, although variations of this magnitude actually have very little effect on the results.

[33] Forest Service officials report that jump bidding is a common practice.

## 7.2 Reserve Price Policy

The Forest Service's mandated objective in setting a reserve price is to ensure that timber is sold at a "fair market value," which is defined as the value to an "average operator, rather than that of the most or least efficient" (USFS 1995). This loose guideline can be interpreted as an attempt to give local Forest Service officials sufficient flexibility to pursue the various objectives of the timber sales program. Many observers have argued, however, that Forest Service reserve prices fall short of the "average operator" criterion and are essentially nonbinding floors.[34] Bidders, for example, claim that Forest Service reserve prices never prevent them from bidding on a tract (Baldwin, Marshall, and Richard (1997)). We assume that this is correct.[35]

Recently, there has been controversy over so-called "below cost sales." Because the Forest Service harvests are part of a forest management program with multiple objectives, including a range of "forest stewardship" objectives, setting reserve prices sufficiently high to cover all costs could mean pricing above market value, as determined by private timber sales. This creates a potential conflict with another Forest Service objective: providing a supply of timber to meet U.S. demand for wood and other timber products. While the Forest Service has identified a range of cost estimates that might be weighed against timber sale revenues, determination of an appropriate reserve price policy has been hindered in part by an inability to assess the impact of alternative reserve price policies on auction outcomes:

> "Studies indicate it is nearly impossible to use sale records to determine if marginal sales made in the past would have been purchased under a different [reserve price] structure." (U.S. Forest Service (1995))

Estimates of the distribution of bidder valuations, however, would enable exactly this. With such estimates in hand, simulations could be used to assess the effects of alternative reserve price rules

[34] See, for example, Baldwin, Marshall, and Richard (1997), Mead, et al. (1981, 1983), and Haile (1996). Among the sales meeting our selection criteria, there was no advertised sale that failed to attract a qualifying bid.

[35] As noted previously, if the actual reserve price $r$ is such that $F(r) > 0$, our bounds will correspond to bounds on the conditional distribution $F(\cdot|r) = \frac{F(\cdot)-F(r)}{1-F(r)}$, which would give the common distribution from which participating bidders draw their valuations. It is easy to verify that as long as the solution to

$$\max_p \ p(1 - F(p))$$

exceeds $r$, this solution also solves

$$\max_p \ p(1 - F(p|r)).$$

Hence, our bounds on the optimal reserve price remain valid under this weaker assumption.

on sales prices and on the likelihood of tracts going unsold. To do this, an estimate of the cost of allowing the harvest of the tract is needed. The costs of conducting the cruise and the auction are sunk given the decision to hold the auction, so relevant costs include costs of administering the contract and opportunity costs of the forest harvest. In a recent review of reserve price policies, the Forest Service estimated that costs fall between \$25 and \$65 per MBF, depending on which timber sales program costs are included (USFS (1995)). Some of the sunk costs of preparing the sale are include in these figures, suggesting that they may be too high. On the other hand, only costs of the timber sale program itself are included—no opportunity costs and, in particular, no environmental costs are accounted for. Hence we take this range as a conservative estimate of a range of appropriate cost figures.[36]

### 7.3 Results

While our sample contains over 1,200 auctions, when stratified by the number bidders, each subsample has fewer than 200 observations. The Monte Carlo experiments suggest that this is too few for the pure order statistic approach, particularly since tract heterogeneity creates the need to condition on observables. To illustrate, we consider the sample of 177 3-bidder auctions, assuming tract homogeneity up to a normalization by the Forest Service's estimate of the timber volume. Figure 10a shows the results, with the estimated upper bound lying below the estimated lower bound. This violation of the ordering required by Corollary 2 suggests the presence of tract heterogeneity, which makes the valuations correlated within each auction, invalidating equation (1). Figure 10b shows similar results obtained from simulated data, where we have let valuations at each auction be shifted by a uniform random draw between zero and 50 but ignored this in the estimation.

This leads us to focus on the averaging approach, which still utilizes the ordered nature of the data but enables us to condition on tract observables while avoiding severe small sample bias. The vector of conditioning variables $X_t$ consists of a constant and the following: year of the auction, an index of species concentration,[37] estimated manufacturing costs, estimated selling value, estimated harvesting costs, and a 6-month inventory of timber sold in the same region. Given prior results suggesting correlation between the number of potential bidders and unobserved tract characteristics (Haile (2000)), we also condition on $n_t$. Table 4 present summary statistics.

Figure 11 shows our estimates of the upper and lower bounds on the distribution of valuations,

[36] Given tract characteristics $X$ and a value $v_0$ representing the cost of selling the timber, bounds on the optimal reserve price are obtained by substituting $\hat{R}_1(p|X) = (p - v_0)(1 - \hat{F}_U(p|X))$ and $\hat{R}_2(p|X) = (p - v_0)(1 - \hat{F}_L(p|X))$ for the expressions $\hat{R}_1(p)$ and $\hat{R}_2(p)$ discussed in section 4.

[37] Letting $q_j$ denote the estimated volume of species $j$ timber on the tract, the index is equal to $\sum_j q_j^2$. Because bidders are typically specialized sawmills, a tract may be more attractive if it consists primarily of a single species.

Table 4: Summary Statistics

| | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|
| Number of Bidders | 5.7 | 3.0 | 2 | 12 |
| Year | 1985 | 2.6 | 1982 | 1990 |
| Species Concentration | .672 | .23 | .24 | 1.0 |
| Manufacturing Costs | 199.3 | 43.0 | 56.7 | 286.5 |
| Selling Value | 415.4 | 61.4 | 202.2 | 746.8 |
| Harvesting Cost | 120.2 | 34.1 | 51.1 | 283.1 |
| 6-Month Inventory* | 1364.4 | 376.5 | 286.4 | 2084.3 |
| Bids | 98.27 | 76.65 | 3.09 | 1977.01 |

* In millions of board-feet.

along with bootstrap[38] confidence bands (based on 100 replications), evaluated at the mean of the $X_t$ vector.[39] These bounds are remarkably tight and obey the dominance relation required by our assumptions. The shape of the true distribution is nailed down almost completely by these bounds and resembles the lognormal distribution, which has been used successfully in several prior studies.

Table 5 shows the results of some simulations used to evaluate the trade-offs between revenues and the probability that a tract goes unsold with alternative reserve prices. Values of $v_0 \in \{\$25, \$45, \$65\}$ are considered and the implied bounds on the optimal reserve prices calculated. The table reports simulated gains in revenues per MBF (relative to those using the true average reserve prices) and the simulated frequency of tracts going unsold. These calculations are made for reserve prices equal to our upper and lower bounds on the optimal reserve, as well the mean of the two. We use the empirical distribution of $n_t$ to select the number of bidders at each simulated auction. To simulate bidder behavior, we consider the extreme cases in which $F(\cdot)$ is equal to either $F_L(\cdot)$ or $F_U(\cdot)$ and assume equilibrium bidding in a sealed bid auction.

[38] We use a simple block bootstrap procedure, drawing entire auctions of bids with replacement from the sample of all auctions until a sample (of bids) of the same size as the true sample is obtained.

[39] We use a uniform kernel with a bandwidth of 1.33 standard deviations for each component of $X_t$. This results in inclusion of 2080 of the 7117 bids in the sample. This gives an average of 47 data points per empirical distribution function $G_{i:n}(\cdot|X)$ being estimated—roughly one-fourth the number used in the Monte Carlo simulations above. Narrower bandwidths yield similar results but require more smoothing of the revenue functions, due to sparse data in the lower tail.

Table 5: Simulated Outcomes with Alternative Reserve Prices

| *reserve price:* | $p_L$ | | $\frac{p_L+p_U}{2}$ | | $p_U$ | |
|---|---|---|---|---|---|---|
| *distribution of valuations:* | $F_L$ | $F_U$ | $F_L$ | $F_U$ | $F_L$ | $F_U$ |
| Optimal Reserve when $v_0 = \$25$ : | 60.40 | | 74.00 | | 87.60 | |
| $\Delta$ revenue | 0.00 | 0.08 | -0.04 | 0.02 | -0.19 | -0.19 |
| Prob(no bids) | 0.01 | 0.01 | 0.04 | 0.04 | 0.08 | 0.08 |
| Optimal Reserve when $v_0 = \$45$ : | 73.52 | | 84.08 | | 94.64 | |
| $\Delta$ revenue | 0.22 | 0.41 | 0.65 | 0.66 | 0.12 | 0.39 |
| Prob(no bids) | 0.04 | 0.04 | 0.06 | 0.07 | 0.11 | 0.12 |
| Optimal Reserve when $v_0 = \$65$ : | 83.44 | | 92.88 | | 102.32 | |
| $\Delta$ revenue | 1.33 | 1.38 | 1.29 | 1.61 | 1.16 | 1.23 |
| Prob(no bids) | 0.06 | 0.07 | 0.11 | 0.11 | 0.15 | 0.16 |

Results based on 1,000,000 simulations evaluated at the mean value of $X_t$ and using the empirical distribution of the number of bidders. Revenue figures are per MBF and are shown in differences relative to simulated revenues using actual mean reserve prices (stratified by $n$).

These simulations indicate that reserve prices could be raised considerably without causing many tracts to go unsold—indeed a reserve price of about \$100/MBF (almost twice the average actual reserve) would be required to drive the probability of a tract going unsold past 15 percent—a key threshold given a Forest Service policy of setting appraisal and reserve price practices to ensure that at least 85 percent of all offered timber volume is actually sold (USFS (1992)). In terms of revenues, our lowest estimated lower bound on the optimal reserve still exceeds the actual average of \$55, although our 90 percent confidence interval on this lowest lower bound is $[\$54, \$69]$. Our estimates suggest that implementing our lower bound on the optimal reserve could not reduce revenues[40] and could increase revenues (unconditional on a sale occurring) by as much as \$1.61 per MBF. However, the revenue gains to increasing the reserve clearly depend on the appropriate measure of costs. With $v_0 = \$25$, these simulations suggest small potential gains from increasing reserve prices, with small losses possible if our estimated upper bound were used. However, if the \$65 cost figure is more appropriate, substantial gains may exist, primarily from withholding a small number of contracts that attract bids below $v_0$. If, for example, if gains of \$1.25 per MBF were achieved on all sales, this would imply approximately \$5 million in additional net revenue annually from the timber sales program.

The estimation approach based on pooled bid data is also useful in this application—in particular, for estimating the effects of the covariates $X_t$ on bidder valuations. While this can also be done using the order statistic method (with averaging), doing so requires using relatively large bandwidths in smoothing the conditional distributions $G_{i:n}(\cdot|X)$; with narrow bandwidths, for some values of $X$ and $v$ there is sufficiently little nearby data that sampling error in estimates of each $G_{i:n}(v|X)$ result in upper and lower bounds that cross. When the estimated bounds cross, our modified minimum distance estimator breaks down. With the pooling method, the bounds being estimated are sufficiently wide the estimates do not cross, even with narrow bandwidths.

Using the pooling method, we estimate the model

$$v_{it} = X_t\beta + \epsilon_{it}$$

assuming $Med[\epsilon_{it} \mid X_{it}] = 0$. Table 6 summarizes the results obtained using the modified minimum distance estimator described above. We report the estimates and a bootstrapped 95% confidence intervals. For every coefficient, zero lies outside the 95 percent confidence interval, meaning that we can reject the hypothesis that any one of these conditioning variables has no effect on valuations. The implied signs are as expected: larger inventories, higher harvesting costs, or higher

[40] This will always be true if the true revenue function is quasiconcave and actual reserve prices are below the lower bound on the optimal reserve.

manufacturing costs reduce valuations. Greater species concentration and higher selling value of the timber lead to higher valuations. Moreover, the bounds are fairly tight and the magnitudes are reasonable. For example, if the Forest Service estimates of selling value, manufacturing cost, and harvesting costs were exact, these covariates should have coefficients of +1, -1, and -1, respectively, which are close to or within the estimated intervals.

Table 6: Forest Service Timber Auctions:
Pooling Method Estimation of the Auction Heterogeneity Model
Modified Minimum Distance Estimator

| | Mean Interval Estimate | 95% Bootstrapped CI |
|---|---|---|
| Constant | [0.00, 1.21] | [0.00, 1.34] |
| Species Concentration | [10.61, 18.46] | [8.91, 23.65] |
| Manufacturing Cost | [-1.19, -1.09] | [-1.35, -.91] |
| Selling Value | [.91, .96] | [.79, 1.03] |
| Harvesting Cost | [-.77, -.67] | [-.89, -.62] |
| 6-month Inventory | [-.017, -.0133] | [ -.0191, -.0091] |

## 8 Conclusion

Many theoretical models that serve well in capturing the essential elements of behavior in a market may nonetheless fall short of providing a mapping between primitives and observables that empirical researchers can view as exact. This need not preclude the use of theory to provide a structure for interpreting data. In some cases useful inferences can be made by relying on weaker conditions that, while insufficient to fully characterize the mapping from primitives to observables, provide a more robust structural framework. We have considered one example of this approach and have argued that while standard models of English auctions can imply unpalatable identifying assumptions for many applications, useful inferences on the primitives characterizing the demand and information structure can be made based on observed bids and weak implications of rational bidding.[41]

Our approach enables construction of bounds on the distributions of bidder valuations, on optimal reserve prices, and on the effects of auction and bidder characteristics on valuations. In fact, the case for focusing on bounds is particularly compelling in this application: our estimated bounds will be tight whenever the standard model is a good approximation of the true model, and collapse to consistent point estimates when the button auction model *is* the true model. When our bounds are wide, this provides a measure of the error that could be made by imposing the interpretation of the data implied by standard models. Our bounds may also serve in guiding the choice of additional identifying assumptions in such cases, since not all approaches will yield estimates lying within the bounds.

Finally, while we have focused on the case of independent private values, our interpretation of bids as bounds is natural in any English auction. Many of the ideas developed here can be extended to richer settings by exploiting properties of order statistics for dependent random variables. This is a topic of ongoing work (Haile and Tamer (2000)).

[41] Of course, while the structure we assume is weaker than that used in prior studies, even this structure cannot be imposed without some caution. Avery (1998) and Daniel and Hirshleifer (1999), for example, have constructed 2-bidder sequential bidding models with equilibria in which our Assumption 2 can be violated because jump bidding by one player reveals to the other that he has no chance of winning. Specification tests like those proposed above may be valuable for evaluating the structure we do impose.

# 9 Appendix

**Proof of Lemma 1:** Suppose $b_{i:n} > v_{i:n}$ for some $i \in \{1, \ldots, n\}$. Then there must be $n-i+1$ bids exceeding the $(n-i+1)$th highest valuation, contradicting Assumption 1. ¤

**Proof of Lemma 2:** Immediate from Assumption 2. ¤

**Proof of Proposition 2:** Because $\phi(\cdot; i, n)$ is strictly increasing and differentiable, the continuous mapping theorem, known asymptotic results for empirical distribution functions, the Lindberg-Levy CLT, and an application of the delta method imply that as $T_n \to \infty$

$$\sqrt{T_n}\left(\phi(\hat{G}_{i:n}(v); i, n) - \phi(G_{i:n}(v); i, n)\right) \xrightarrow{d} N\left(0, \phi'(G_{i:n}(v); i, n)^2 \left[G_{i:n}(v)(1 - G_{i:n}(v))\right]\right).$$

Continuity of the min function and the continuous mapping theorem then give the result. ¤

**Proof of Lemma 4:** Since $R_2(p_1^*) \geq R_1(p_1^*)$, the definitions of $p_U$ and $p_L$ imply $p_U \geq p_1^*$ and $p_L \leq p_1^*$. Similarly, since $R_2(p_2^*) \geq R_2(p_1^*) \geq R_1(p_1^*)$, we must have $p_U \geq p_2^*$ and $p_L \leq p_2^*$. ¤

**Proof of Proposition 7:** Since $R(p) \geq R_1(p) \forall p$, we must have $R(p^*) \geq R(p_1^*) \geq R_1(p_1^*)$. Take $p < p_L$. The definition of $p_L$ implies

$$R(p) \leq R_2(p) < R_1(p_1^*)$$

which precludes $p^* = p$. An analogous argument rules out the optimality of any $p > p_U$.

To see that the bounds are sharp, consider $p_L$. If $p_L = p_1^*$, $p_L$ is optimal given the distribution $F_L(\cdot)$ and the result is immediate. If $p_L < p_1^*$, then for $\epsilon > 0$ such that $p_L + \epsilon < p_1^*$ define

$$\tilde{F}_\epsilon(p) = \begin{cases} F_L(p) & p \leq p_L + \epsilon \\ \max\{F_L(p), \alpha(p)F_U(p_1^*) + (1-\alpha(p))F_L(p_L+\epsilon)\} & p \in (p_L+\epsilon, p_1^*) \\ F_U(p) & p \geq p_1^* \end{cases}$$

where

$$\alpha(p) \equiv \frac{p_1^* p - p_1^*(p_L+\epsilon)}{p_1^* p - p(p_L+\epsilon)}.$$

Note that $\alpha(p)$ lies in $(0,1)$ for $p \in (p_L+\epsilon, p_1^*)$ and is strictly increasing. Since $p_L < p_1^*$ and

$$p_L(1 - F_L(p_L)) = p_1^*(1 - F_U(p_1^*))$$

we know that for sufficiently small $\epsilon$ we must have $F_L(p_L+\epsilon) < F_U(p_1^*)$, implying that $\tilde{F}_\epsilon(\cdot)$ is a strictly increasing distribution. Finally, the "max" ensures that $\tilde{F}_\epsilon(\cdot)$ always lies within the bounds. The corresponding revenue function $\tilde{R}_\epsilon(p) = p(1 - \tilde{F}_\epsilon(p))$ can then be written

$$\tilde{R}_\epsilon(p) = \begin{cases} R_2(p) & p \leq p_L + \epsilon \\ \min\{R_2(p), R_2(p_L+\epsilon) + \frac{p-(p_L+\epsilon)}{p_1^*-(p_L+\epsilon)}(R_1(p_1^*) - R_2(p_L+\epsilon)\} & p \in (p_L+\epsilon, p_1^*) \\ R_1(p) & p \geq p_1^*. \end{cases}$$

For all sufficiently small $\epsilon$, $\tilde{R}_\epsilon(p)$ has a unique maximum at $p_L+\epsilon$. Taking a sequence $\{\epsilon_s\}$ such that $\lim_{s\to\infty}\epsilon_s=0$ then gives a sequence of distributions $\{\tilde{F}_{\epsilon_s}(\cdot)\}$ that imply unique optimal reserve prices $\{\tilde{p}_{\epsilon_s}\}$ such that $\lim_{s\to\infty}\tilde{p}_{\epsilon_s}=p_L$. A similar argument applies to $p_U$. ¤

Proof of Proposition 8: Given the result in Proposition 7, to prove the first part we need to show that no $p\in[p_L,p_U]\backslash[p_L^q,p_U^q]$ can be strictly optimal. Suppose $p_U^q<p_U$ and take $p\in(p_U^q,p_U]$. Because $R(\cdot)$ is quasiconcave by assumption and $R_1(p)\le R(p)\le R_2(p)$, the definition of $p_U^q$ implies $R(p)\le R_1(p_1^*)$, giving the result. A similar argument rules out $p\in[p_L,p_L^q)$ when $p_L<p_L^q$.

To see that the bounds are sharp, let $R_2^q(\cdot)$ be a revenue function equal to $R_2(p)$ for $p\in[p_L^q,p_U^q]$ and equal to a lower quasiconcave envelope of $R_2(\cdot)$ otherwise. Then define $F_L^q(p)=\frac{R_2^q(p)}{p}$. Similarly, let $R_1^q(\cdot)$ be a revenue function equal to $R_1(p)$ for $p\in[p_L^q,p_U^q]$, and equal to an upper quasiconcave envelope of $R_1(\cdot)$ otherwise. Define $F_U^q(p)=\frac{R_1^q(p)}{p}$. One can confirm that, given the implications of quasiconcavity of $R(\cdot)$ for the bounding distributions, $F_U^q(\cdot)$ and $F_L^q(\cdot)$ must be proper distribution functions lying within the original bounds. Substituting $R_1^q(\cdot), R_2^q(\cdot), F_L^q(\cdot), F_U^q(\cdot), p_L^q$, and $p_U^q$ for $R_1(\cdot), R_2(\cdot), F_L(\cdot), F_U(\cdot), p_L$, and $p_U$ in the proof of Proposition 7 shows that there is a sequence of distribution functions lying within the bounds $F_L^q(\cdot)$ and $F_U^q(\cdot)$ that induce quasiconcave revenue functions and yield optimal reserve prices converging to $p_L^q$. A similar argument proves the sharpness of the upper bound $p_U^q$. ¤

Proof of Lemma 5: Uniform convergence of $\hat{R}_1(p)$ to $R_1(p)$ is guaranteed by convergence of empirical distribution functions and the continuous mapping theorem. Since the function $R_1(p)$ is maximized uniquely at $p_1^*$, consistency of $\hat{p}_1^*$ follows from uniform convergence of $\hat{R}_1^s(p)$ to $R_1(p)$, which holds due to the convergence of $\hat{R}_1^s(p)$ to $\hat{R}_1(p)$ as $T\to\infty$. ¤

Proof of Proposition 9: We know that $\hat{R}_1(p)$ converges uniformly to $R_1(p)$, $\hat{R}_2(p)$ converges uniformly to $R_2(p)$, and $\hat{p}_1^*$ converges in probability to $p_1^*$. Since $p_L=\inf\{p:R_2(p)=R_1(p_1^*)\}$ and $\hat{p}_L=\inf\{p:\hat{R}_2(p)\ge\hat{R}_1(\hat{p}_1^*)\}$, we have $p_L\xrightarrow{p}p_L$. A similar argument applies to $\hat{p}_U$. ¤

Proof of Lemma 6: Let $b\in\Sigma$. Then $g_1(x,b)=g_2(x,b)=0$ and the so the objective function is zero. Let $b\notin\Sigma$. Then

$$(E[\underline{Z}_x]-l(xb))^2g_1(x,b)+(\eta_2-E[\overline{Z}_x])^2g_2(x,b)>0$$

for all $x\in V(b)$, and $\Pr[V(b)]>0$ for all $b\notin\Sigma$. ¤

Proof of Proposition 10: Given estimates of $\hat{F}_L(\cdot|X)$ and $\hat{F}_U(\cdot|X)$, one can estimate the conditional means by numerically integrating the estimated density, or by simulations . We follow the latter method. Replace $E[\underline{Z}_x]$ and $E[\overline{Z}_x]$ by corresponding estimates $E_T[\underline{Z}_x]$ and $E_T[\overline{Z}_x]$ based on simulated draws from $\hat{F}_L(\cdot|X)$ and $\hat{F}_U(\cdot|X)$. These simulation estimators converge almost surely to their population counterparts as long as $\hat{F}_L(\cdot|X)$ and $\hat{F}_U(\cdot|X)$ converge almost surely to $F_L(\cdot|X)$ and $F_U(\cdot|X)$ (see for example Stern (1997) and references therein). Given the above assumptions, the argument in the proof of proposition 5 in Manski and Tamer (2000) completes this proof. ¤

Proof of Lemma 7: Follows the proof of Lemma 6. ¤

Proof of Proposition 11: We first replace the conditional median functions with sample analogs based on local linear approximations. Define

$$(\eta_{1T}(z), \partial\eta_{1T}(z)) = (\hat{a}, \hat{b}) = \underset{a,b}{argmin} \sum_{i,t} |b_{it} - a - bz| \, K\left(\frac{z_{it} - z}{h_T}\right)$$

where $K(\cdot)$ is a kernel function, and $h_T > 0$, $h_T \to 0$ *as* $T \to \infty$. Almost sure convergence of local linear approximations of conditional median estimators follows from well known results (Loader (1999)). Given the assumptions of the theorem, we can then use the proof of Proposition 5 of Manski and Tamer (2000) to complete the argument. ¤

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