Equilibrium Cartel Pricing in the Presence of an Antitrust Authority*

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September 2002
JEL Classification: L1, L4
Key Words: Collusion, Cartel Detection, Antitrust
Preliminary and Incomplete

Abstract

Price-fixing is characterized when firms are concerned about creating suspicions about collusion and bringing forth antitrust penalties. Antitrust laws have a complex effect on pricing as they influence the conditions determining the internal stability of the cartel. Dynamics are driven by two forces - the sensitivity of detection to price movements causes a cartel to gradually raise price while the sensitivity of penalties to the price level induces the cartel to lower price over time in order to maintain the stability of the cartel. While antitrust laws can lower collusive prices, they can also raise them by making it easier for firms to collude.

*I appreciate the comments of seminar participants at Cal Tech, UC-Berkeley, Stanford, and the Duke/UNC Theory Workshop. This research is supported by the National Science Foundation under grant SES-0209486.

1 Introduction

As evidenced by recent cases in lysine, graphite electrodes, vitamins, and auction houses, price-fixing remains a perennial problem. Though there is a voluminous theoretical literature on collusive pricing, an important dimension to price-fixing cartels has received very little attention. In light of its illegality, a critical goal faced by a cartel is to avoid the appearance that there is a cartel. Firms want to raise prices but not suspicions that they are coordinating their behavior. If high prices or rapidly increasing prices or, more generally, anomalous price movements may make customers and the antitrust authorities suspicious that a cartel is operating, one would expect this to have implications for how the cartel prices.

This paper is part of a research project whose objective is to explore cartel pricing dynamics in the presence of detection considerations. Some of the questions to be addressed include: What are the dynamics of the collusive price path when a cartel seeks to avoid detection? How does the decision to form a cartel and the properties of the collusive price path respond to various instruments of antitrust policy? What types of industry traits make detection more difficult and what are the implications of those traits for cartel pricing?

In an earlier paper (Harrington, 2002a), the joint profit maximizing price path was characterized under the constraint of possible detection and antitrust penalties. Previous static models assumed that detection is more likely when the price is higher (Block, Nold, and Sidak, 1981). Examining the dynamical extension of those models, the counterfactual result was derived that, after initially raising price, the cartel price path is decreasing over time. When instead the probability of detection is specified to be increasing in the extent of price *changes* rather than the price level, the cartel gradually raises price and price converges. Comparative statics on the steady-state price reveal that it is decreasing in the damage multiple and the probability of detection but is independent of the level of fixed fines. Furthermore, if fines are the only penalty, the cartel's steady-state price is the same as in the absence of antitrust laws, though fines do affect the path to the steady-state. Another intriguing result is that a more stringent standard for calculating damages increases the steady-state price.

That analysis presumed the incentive compatibility constraints ensuring the internal stability of the cartel were not binding. In the current paper, these constraints are explicitly introduced and allowed to bind. The optimal cartel price path is characterized and three considerations come into play - a desire to set high prices to realize high profit, a desire to gradually raise price so as to make detection less likely, and a need to adjust price so as to maintain the internal stability of the cartel. After laying out the model in Section 2 and defining an optimal collusive price path in Section 3, properties of that price path are described in Section 4. At play are two distinct sources of dynamics as the price path influences both the probability of detection and the penalties in the event of detection. I consider each of these dynamics in turn. When penalties are fixed but the probability of detection is endogenous, the cartel price path is shown to maintain the property that it is increasing over time even when incentive compatibility constraints are taken into account. When instead the probability of detection is fixed but penalties are endogenous, the cartel price path is decreasing over time (after price is raised in the first period of collusion). Numerical work is in progress to allow for both of these dynamics to operate. I also explore the impact of antitrust laws by comparing the cartel price path with that which would occur if cartels were legal (though not enforceable by the courts). While antitrust laws can have their desired effect of lowering price, they can also raise price because the prospect of detection and possible penalties can enhance cartel stability and thereby allow the cartel to set higher prices.

In concluding, let me briefly mention related work that takes account of detection considerations in the context of cartel pricing. Block, Nold, and Sidak (1981) initially considered a static cartel model in which the probability of detection depends on the price-cost margin and the penalty is a multiple of above-normal profits. They show that the optimal cartel price is below the monopoly price and that the cartel price is decreasing in the penalty multiple and the level of enforcement expenditures (higher levels of which raise the probability of detection). Three papers consider dynamic models with antitrust laws. Cyrenne (1999) modifies Green and Porter (1984) by assuming that a price war, and the ensuing raising of price after the war, results in detection for sure and with it a fixed fine. Spagnolo (2000) and Motta and Polo (2001) consider the effects of leniency programs on the incentives to collude in a repeated game of perfect monitoring.[1] Though considering collusive behavior in a dynamical setting with antitrust laws, these papers exclude the sources of dynamics that are the foci of the current analysis; specifically, that the probability of detection and penalties are sensitive to firms' pricing behavior. It is that sensitivity that will generate predictions about cartel pricing dynamics.

2 Model

Consider an industry with n symmetric firms. $\pi(P_i, P_{-i})$ denotes firm i's profit when its price is P_i and all other firms charge P_{-i}. Define $\pi(P) \equiv \pi(P, P)$ to be a firm's profit when all firms charge P. The space of feasible prices is Ω which is assumed to be a non-empty, compact, convex subset of $\Re_+$.

A1 $\pi(P_i, P_{-i}) : \Omega^2 \rightarrow \Re$ is continuously differentiable in P_i and P_{-i} and quasi-concave in P_i.

By A1, $\psi(P_{-i}) \in \arg\max \pi(P_i, P_{-i})$ exists.

A2 $\exists$ unique $\widehat{P}$ such that $P \gtreqless \psi(P)$ as $P \lesseqgtr \widehat{P}$.

$\widehat{P}$ is a symmetric Nash equilibrium for the static game and let $\widehat{\pi} \equiv \pi\left(\widehat{P}\right)$ denotes the associated profit. A3 defines the joint profit-maximizing price.

[1] Rey (2001) offers a nice review of some of this work along with other theoretical analyses pertinent to optimal antitrust policy.

A3 $\pi(P)$ is quasi-concave in P, if $\pi(P) > 0$ then it is strictly quasi-concave in P, and $\exists P^m > \widehat{P}$ such that $\pi(P^m) > \pi(P) \, \forall P \neq P^m$.

Firms engage in this price game for an infinite number of periods. The setting is one of perfect monitoring so that firms' prices over the preceding $t-1$ periods are common knowledge in period t. Assume a firm's payoff is the expected discounted sum of its income stream where the common discount factor is $\delta \in (0,1)$. To make the analysis interesting, it is assumed that, in the absence of antitrust policy, incentive compatibility constraints (ICCs) are binding. Specifically, A4 specifies that firms cannot support the simple monopoly price if they use a grim trigger strategy (so that deviation from a collusive price results in infinite repetition of the static Nash equilibrium). $\widetilde{P}$ represents the benchmark collusive price in the absence of antitrust laws.

A4 $\exists \widetilde{P} \in \left(\widehat{P}, P^m\right)$ such that $\pi(P)/(1-\delta) \gtreqless \pi(\psi(P), P) + \delta(\widehat{\pi}/(1-\delta))$ as $P \lesseqgtr \widetilde{P}$, $\forall P \in \left[\widehat{P}, P^m\right]$.

There are two reasons why I focus on reversion to the static Nash equilibrium as the punishment for deviation, as opposed to some other form like optimal penal codes. As is described below, our game is not repeated as antitrust laws result in elements of the history altering firms' payoff functions. To make the analysis more manageable, the punishment will be a Markov Perfect Equilibrium (MPE). As, in the repeated game setting, infinite repetition of the static Nash equilibrium is the unique MPE, it is then only appropriate to make $\widetilde{P}$ the benchmark collusive price. The second reason (and, in my mind, the more important one) is that my own reading of the evidence is that when there is clear deviation from a price-fixing agreement, the cartel simply breaks down. I don't mean they go to some "cooperative" punishment but rather that they simply stop cooperating. In many cases then, I think static Nash equilibrium is the most compelling description of post-deviation behavior. Assuming that firms never collude again is unreasonable but I don't think results would change if the reversion was finite.

If firms form a cartel, they meet to determine price. Assume these meetings, and any associated documentation, provides the "smoking gun" if an investigation is pursued.[2] The cartel is detected with some probability and incurs some level of penalties in that event. Assume, for simplicity, that detection results in the discontinuance of collusion forever. Detection then generates a terminal payoff of $[\widehat{\pi}/(1-\delta)] - X^t - F$ for a firm where X^t is its damages in the event the cartel is detected during period t and F is any (fixed) fines (which may include the monetary equivalent of prison sentences).[3] If not detected, collusion continues on to the next period.

[2] Though it is assumed that an investigation leads to conviction with probability one, all results would go through if the probability of conviction is only required to be positive.

[3] In that our focus will be on symmetric cartel solutions, firms will always have the same level of damages. In this model, damages refers to any penalty that is sensitive to the price charged and the length of time the cartel has been in place while fines refer to penalties that are fixed with respect to the endogenous variables.

Damages are assumed to evolve in the following manner:

$$X^t = \beta X^{t-1} + \gamma x\left(P^t\right) \text{ where } \beta \in [0,1), \gamma \geq 0.$$

As time progresses, damages incurred in previous periods become increasingly difficult to document and $1-\beta$ measures the rate of the deterioration of the evidence. $x\left(P^t\right)$ is the level of damages incurred by each firm in the current period where γ is the damage multiple applied. While U.S. antitrust law specifies treble damages, γ could be less than three because a case is settled out-of-court. Single damages is not unusual for an out-of-court settlement. Current U.S. antitrust practice is $x\left(P^t\right) = \left(P^t - \widehat{P}\right) D\left(P^t\right)$ where $D\left(P^t\right)$ is the number of units sold by each firm. $\widehat{P}$ is referred to as the "but for price" and is the price that would have occurred but for collusion.

A5 $x : \Omega \rightarrow \Re_+$ is bounded, continuous, and non-decreasing.

To ensure that there are indeed antitrust penalties, it is assumed throughout the paper that: (i) $\gamma > 0$ and if $P > \widehat{P}$ then $x(P) > 0$; and/or (ii) $F > 0$.

Detection of a cartel can occur from many sources; some of which are related to price - such as customer complaints - and some of which are unrelated to price - such as internal whistleblowers and incidental detection via an unrelated legal case. Hay and Kelley (1974) find that detection was attributed to a complaint by a customer or a local, state, or federal agency in 13 of 49 price-fixing cases. In the recent graphite electrodes case, it was reported that the investigation began with a complaint from a steel manufacturer which is a purchaser of graphite electrodes (Levenstein and Suslow, 2001). High prices or anomalous price movements (not easily explained by a non-collusive model) may cause customers to become suspicious and pursue legal action or share their suspicions with the antitrust authorities.[4] Though it isn't important for my model, I do imagine that buyers (in many price-fixing cases, they are industrial buyers) are the ones who are becoming suspicious about collusion.[5]

To capture these ideas in a tractable manner, I specify an exogenous probability of detection function which depends at most on the current and previous periods' price vectors. $\phi\left(\underline{P}^t, \underline{P}^{t-1}\right)$ is the probability of detection during the time in which firms are colluding where $\underline{P}^t \equiv \left(P_1^t, \ldots, P_n^t\right)$ is the vector of firms' prices.[6] When firms charge a common price, the vector will be replaced with that common scalar. This specification can capture how high prices and big price changes can create suspicions

[4] The Nasdaq case is one in which truly anomalous pricing resulted in suspicions about collusion. It was scholars rather than market participants who observed that dealers avoided odd-eighth quotes and ultimately explained it as a form of collusive behavior (Christie and Schultz, 1994). Though the market-makers did not admit guilt, they did pay an out-of-court settlement of around $1 billion.

[5] "As a general rule, the [Antitrust] Division follows leads generated by disgruntled employees, unhappy customers, or witnesses from ongoing investigations. As such, it is very much a reactive agency with respect to the search for criminal antitrust violations. ... Customers, especially federal, state, and local procurement agencies, play a role in identifying suspicious pricing, bid, or shipment patterns." [McAnney, 1991, pp. 529, 530]

[6] The specification of the probability of detection after the cartel ceases to operate will be discussed later.

among buyers that firms may not be competing.[7] The impact of the properties of this detection technology on the joint profit-maximizing price path was explored in Harrington (2002a).[8] There I found that cartel pricing dynamics are much more plausible when detection is driven by price changes rather than price levels. As a result, in this paper I will largely focus on when detection is sensitive to price changes. While additional structure will be imposed later, one common assumption to the ensuing analysis is that the probability of detection is minimized when prices don't change and is weakly higher with respect to price increases. In A6, note that $\phi(P,P)$ is allowed to vary with P in which case price levels can matter. For example, if $\phi(P,P)$ is increasing in P then, when price is stable, detection is more likely when price is higher.

A6 $\phi : \Omega^{2n} \rightarrow [0,1]$ is continuous and: i) $\phi(P^o, P^o) \leq \phi(P', P^o)$ and $\phi(P^o, P^o) \leq \phi(P^o, P'), \forall P', P^o \in \Omega$; ii) if $\underline{P}'' \geq \underline{P}' \geq \underline{P}^o$ (component-wide) then $\phi(\underline{P}'', \underline{P}^o) \geq \phi(\underline{P}', \underline{P}^o)$.

As further motivation for the emphasis on price movements, it is worth noting that while it may be difficult for individuals external to a firm, such as buyers, to have reasonably-informed beliefs about the level of cost and demand, it may be quite reasonable for them to receive noisy signals of changes in cost or demand and be able to make assessments about what is a reasonable change in price. This view has previously been articulated:

> Simultaneous price increases and output reductions unexplained by an increase in cost may therefore be good evidence of the initiation of a price-fixing scheme, while changes in the opposite direction might be used as evidence that a cartel has just collapsed. Notice that it is not necessary to determine what the firms' marginal costs are or what the competitive price and output would be. One simply observes price and output changes and asks whether changes in costs or in demand explain them, or whether it is necessary to posit cartelization. [Posner (1976), pp. 66-67.]

Rather than explicitly model these cost and demand shocks, I have instead postulated that bigger price changes are more likely to trigger suspicions. In the context of a stationary environment, greater price fluctuations should seem more puzzling to buyers. The virtue to this approach is that the cartel's price path is deterministic, which would not be the case with the presence of cost or demand shocks, and this will make it easier to derive clean results. Future research will explicitly consider encompassing such shocks.[9]

[7] While customers are implicitly assumed to be forgetful in that their likelihood of becoming suspicious depends only on recent prices, the inclusion of a more comprehensive price history would significantly complicate the analysis, by greatly expanding the state space, without any apparent gain in insight.

[8] That paper ignored incentive compatibility constraints; the inclusion of which is the focus of the current paper.

[9] For one final motivation for focusing on price movements, note that current methods for estimating market conduct in I.O. use information embedded in price changes rather than price levels.

This modelling of detection warrants further discussion since it does not model those agents who might engage in detection. The first point to make concerns tractability. With two distinct sources of structural dynamics - detection and antitrust penalties - in addition to the usual (repeated game-style) behavioral dynamics, this model is rich enough to provide new insight into cartel pricing dynamics even with a simple modelling of the detection process and its complexity already pushes the boundaries of formal analysis. Tractability issues aside, there is another motivation for our approach. The objective of this paper is not to develop insight and testable hypotheses about detection but rather about cartel pricing. A good model of the detection process is then one that is a plausible description of how cartel members *perceive* the detection process. To my knowledge, there is little evidence from past cases that cartels hold a sophisticated view of buyers (which is implied if one were to model buyers as strategic agents and derive an equilibrium). It strikes me as quite reasonable that firms might simply postulate that higher prices or bigger price changes result in a greater likelihood of creating suspicions without having derived that relationship from first principles about buyers. Thus, even if this modelling of the detection process is wrong, the resulting statements about cartel pricing may be accurate if that model is a reasonable representation of firms' perceptions.[10]

In period 1, firms have the choice of forming a cartel, and risking detection and penalties, or earning non-collusive profit of $\widehat{\pi}$. If they choose the former, they can, at any time, choose to discontinue colluding. However, a finitely-lived cartel will cause collusion to unravel so that, in equilibrium, firms either collude forever or not at all (subject to the cartel being exogenously terminated because of detection). Firms are then not allowed to form and dissolve a cartel more than once. While the possibility of temporarily shutting down the cartel is not unreasonable (firms may want to "lay low" for a bit of time), the analysis is complicated enough without allowing for such. Exploration of that strategic option is left for future research.

3 Optimal Symmetric Subgame Perfect Equilibrium

The cartel's problem is to choose an infinite price path so as to maximize the expected sum of discounted income subject to the price path being incentive compatible (IC). In determining the set of IC price paths, the assumption is made that deviation from the collusive path results in the cartel being dissolved and firms behaving according to a Markov Perfect Equilibrium (MPE). As an initial step in characterizing the cartel's problem, I define a MPE for the post-cartel game.[11]

That is, market conduct is inferred by measuring how price changes in response to exogenous events rather than trying to directly estimate the price-cost margin.

[10] Nor do I believe it is inconsistent to model firms as choosing prices optimally - as that is a statement about what one thinks is best for one's self - and, at the same time, suppose that firms do not derive buyers' optimal detection behavior - as that is a statement about what one thinks is best for others. An agent may know what is best for themselves without having a clue as to what is best for someone else.

[11] For what I believe is the only other paper to characterizes cooperative equilibria among multiple infinitely-lived players when there are state variables, see Roth (1996).

3.1 Markov Perfect (Punishment) Equilibrium

Suppose a firm deviates and collusion dissolves. Since cartel meetings are no longer taking place, the damage variable simply depreciates at the exogenous rate of $1 - \beta$: $X^t = \beta X^{t-1}$.[12] This is still a dynamic problem, however, in that price movements can create suspicions and, while firms are no longer colluding, an investigation could reveal evidence of past collusion. The state variables at t are the vector of lagged prices, $\underline{P}^{t-1}$, and damages, X^{t-1}. Let firm i's payoff at a MPE is denoted $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$. When there is a symmetric state and a symmetric MPE, the payoff is denoted $V^{mpe}\left(P^{t-1}, X^{t-1}\right)$.

In this paper, a common property imposed on $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ is:

$$\widehat{\pi}/\left(1-\delta\right) \geq V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right), \forall \underline{P}^{t-1} \in \Omega^n, \forall X^{t-1} \geq 0, \tag{1}$$

so MPE results in a payoff weakly lower than the static Nash equilibrium payoff. In some cases, this property is assumed and in other cases it occurs for free; being implied by other assumptions. Though this property need not always hold (for an example where it doesn't, see Harrington 2002b), it is useful to limit our attention to when it does so as to be able to provide a coherent set of results. A fuller examination of cartel pricing for other MPE is left to future research.

As long as expected penalties are positive, note that (1) holds when infinite repetition of the static Nash equilibrium is a MPE. Let us offer sufficient conditions for that to be the case and thereby establish that (1) is not vacuous.

The first case assumes homogeneous products and constant marginal cost of c. The static Nash equilibrium then has firms price at cost: $\widehat{P} = c$. Also assume that the probability of detection is independent of the prices of firms with zero demand. Hence, the probability of detection depends only on the minimum price among firms. Initially, suppose $P_i^{t-1} \geq c \forall i$ so that, in the preceding period, firms are pricing at or above cost. Suppose strategies are such that when $P_i^{t-1} \geq c \forall i$ that a firm prices at c. An individual firm that instead prices above c doesn't raise its current profit and doesn't impact the current or next period's probability of detection (since its demand is zero). Also, since all firms price at c in the ensuing periods, it doesn't influence the future payoff. Hence, it is not better off by pricing above c relative to pricing at c. Now suppose this firm instead prices below c. Relative to pricing at c, it lowers its current profit and weakly raises the probability of detection (by A6). [INCOMPLETE: I still need to argue that the future payoff is weakly less than $\widehat{\pi}/\left(1-\delta\right)$. I believe this to be the case but it's matter of characterizing equilibrium.]

The second case is when the probability of detection (when the cartel is inactive) is independent of an individual firm's price when that price is different from a common price charged by other firms. For example, this occurs when $n \geq 3$ and the probability of detection depends on the change in the median price or on the change in the average transaction price after discarding the greatest outlier. It is also the case

[12] Implicit in this specification is that damages stop accumulating once the cartel is dismantled. This is a useful approximation though need not be descriptively accurate. If the post-cartel price exceeds $\widehat{P}$, it is because of past collusion so one could argue that additional damages should be assessed. Whether they are assessed, in practice, is another matter.

when the probability of detection is fixed and independent of firms' prices. Suppose this assumption holds and consider a symmetric MPE path. Given all other firms price at some common level P, an individual firm's price cannot influence the current probability of detection nor the next period's probability of detection. Thus, the future path and payoff are unaffected by its price. All this argues to a firm's optimal price being that which maximizes current profit. By symmetry, this implies that all firms charge a price of $\widehat{P}$.

3.2 Collusive Equilibrium

It is natural to assume that, at the start of the cartel, damages are zero and firms are charging the non-collusive price: $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$. While many of the ensuing results are robust to these initial conditions, they will be assumed throughout the paper so as to simplify some of the proofs. Before providing the conditions defining the cartel solution, the assumption is made that damages are assessed only in those periods for which the cartel has been functioning properly and, more specifically, are not assessed when a firm deviates from the cartel price. Thus, when a firm considers cheating on the agreement, it assumes the act of deviation negates damages for that period. In practice, it is not clear when damages are no longer assessed and this assumption is probably as good as any other. Furthermore, it has a nice property which is useful for both analytical and numerical work. If damages were assigned in the period that a firm deviated then, entering the post-deviation phase, firms would have different levels of damages and this would complicate the characterization of a MPE path. With my assumption, firms have a common damage variable in the post-deviation phase which means that the state space is $\Omega^n \times \Re_+$ rather than $\Omega^n \times \Re_+^n$. Let me add that all results have been derived when it is instead assumed that damages are assessed in the period of deviation based on the price that the cartel set (which also serves to preserve common damages). I conjecture, but have not shown, that results are also true when damages are assessed for the deviating firm based on the price and quantity that it sets. In any case, I feel it is a second-order effect as to how damages are determined in the period of deviation and I have no reason to believe that results are sensitive to this assumption.

As the focus is on symmetric collusive solutions, it is sufficient to define the state variables as $\left(P^{t-1}, X^{t-1}\right) \in \Omega \times \Re_+$ for the cartel's problem where P^{t-1} is the common lagged price and X^{t-1} is the common accumulated damages. One way in which to describe the problem is with dynamic programming. Letting $V\left(P^{t-1}, X^{t-1}\right)$ denote the value associated with firms colluding, it is defined by a constrained Bellman equation where the constraints ensure the stability of the cartel.

$$\begin{aligned} V\left(P^{t-1}, X^{t-1}\right) &= \max_{P \in \Omega} \pi(P) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta X^{t-1} - \gamma x(P) - F\right] \\ &\quad + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right]\max\{V\left(P, \beta X^{t-1} + \gamma x(P)\right), \\ &\quad V^{mpe}\left(P, \beta X^{t-1} + \gamma x(P)\right)\} \end{aligned}$$

subject to

$$\begin{aligned} & \pi(P)+\delta\phi\left(P,P^{t-1}\right)\left[(\widehat{\pi}/(1-\delta))-\beta X^{t-1}-\gamma x(P)-F\right] \\ & +\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\left\{V\left(P,\beta X^{t-1}+\gamma x(P)\right),V^{mpe}\left(P,\beta X^{t-1}+\gamma x(P)\right)\right\} \\ \geq\ & \max_{P_i\in\Omega}\pi(P_i,P)+\delta\phi\left((P,\ldots,P_i,\ldots,P),P^{t-1}\right)\left[(\widehat{\pi}/(1-\delta))-\beta X^{t-1}-F\right] \\ & +\delta\left[1-\phi\left((P,\ldots,P_i,\ldots,P),P^{t-1}\right)\right]V_i^{mpe}\left((P,\ldots,P_i,\ldots,P),\beta X^{t-1}\right). \end{aligned}$$

Note that we give the cartel the option of shutting down and receiving a MPE payoff. Of course, in equilibrium, that option will not be exercised as a cartel that has a known finite terminal date is unstable from its initiation. This then implies $V\left(P^{t-1},X^{t-1}\right)\geq V^{mpe}\left(P^{t-1},X^{t-1}\right)$.[13] Though we may use this notational representation of the problem, our proofs will not use dynamic programming. Unfortunately, this constrained Bellman equation does not necessarily yield a mapping which is a contraction and, therefore, there need not be a unique fixed point. Intuitively, if the k^{th} iteration of the value function is increased by some arbitrary amount $\varepsilon>0$, this could sufficiently loosen ICCs (which serves to expand the feasible set of prices) so that the $k+1^{st}$ iteration is greater by more than ε, in spite of discounting. In addition, it has already been shown that the value function is not concave when ICCs are not binding (Harrington, 2002a) which limits the usefulness of this approach.

By the boundedness of $x(\cdot)$, it follows that damages are bounded by $\overline{X}\equiv\overline{x}/(1-\beta)$ where $\overline{x}\geq x(P)\,\forall P\in\Omega$. We then have that $X^{t-1}\in\left[0,\overline{X}\right]$. The firms' problem is either to not form a cartel - and price at $\widehat{P}$ in every period with each firm receiving a payoff of $\widehat{\pi}/(1-\delta)$ - or form a cartel and choose a price path such that:

$$\begin{aligned} & \max_{\{P^t\}_{t=1}^{\infty}\in\Gamma}\sum_{t=1}^{\infty}\delta^{t-1}\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\pi\left(P^t\right) \qquad (2) \\ & +\sum_{t=1}^{\infty}\delta^t\phi\left(P^t,P^{t-1}\right)\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\left[(\widehat{\pi}/(1-\delta))-\sum_{j=1}^{t}\beta^{t-j}\gamma x\left(P^j\right)-F\right] \end{aligned}$$

where

$$\begin{aligned} \Gamma\ \equiv\ & \{\left\{P^t\right\}_{t=1}^{\infty}\in\Omega^{\infty}:\sum_{\tau=t}^{\infty}\delta^{t-\tau}\Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\pi\left(P^{\tau}\right) \\ & +\sum_{\tau=t}^{\infty}\delta^{\tau-t+1}\phi\left(P^{\tau},P^{\tau-1}\right)\Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\left[(\widehat{\pi}/(1-\delta))-\sum_{j=1}^{\tau}\beta^{\tau-j}\gamma x\left(P^j\right)-F\right] \end{aligned}$$

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[13] To reduce the length of equations, and hopefully without confusion, I will from hereon replace

$$\max\left\{V\left(P,\beta X^{t-1}+\gamma x(P)\right),V^{mpe}\left(P,\beta X^{t-1}+\gamma x(P)\right)\right\}$$

with $V\left(P,\beta X^{t-1}+\gamma x(P)\right)$.

$$\geq \max_{P_i} \pi\left(P_i, P^t\right) + \delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right) - F\right]$$

$$+\delta\left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}((P^t, \ldots, P_i, \ldots, P^t), \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right)),$$

$$\forall t \geq 1\}$$

Γ is the set of price paths that satisfy the incentive compatibility constraints (ICCs). A solution to (2) is referred to as an Optimal Symmetric Subgame Perfect Equilibrium (OSSPE) price path.

As I do not have a general proof of existence for a pure-strategy MPE, to provide a general proof of the existence of an OSSPE price path, it is necessary to assume A7. By our earlier analysis, note that A7 holds when products are homogeneous.

A7 $\forall\left(\underline{P}^{t-1}, X^{t-1}\right) \in \Omega^n \times \left[0, \overline{X}\right]$, $\exists$ a Markov Perfect Equilibrium and, furthermore, $\exists$ a continuous function $V_i^{mpe} : \Omega^n \times \left[0, \overline{X}\right] \rightarrow \Re$ such that $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ is the payoff associated with a Markov Perfect Equilibrium.

Define the firms' choice set as $\{No\ Cartel\} \cup \Gamma$ where it is understood that choosing an element from Γ implies forming a cartel while choosing $No\ Cartel$ implies all firms price at $\widehat{P}$ in all periods. An OSSPE price path is a selection from $\{No\ Cartel\} \cup \Gamma$ that maximizes each firm's payoff.

Theorem 1 *If A1 and A5-A7 hold then an OSSPE price path exists.*

Proof: Suppose Γ is empty. As the choice set is the singleton $\{No\ Cartel\}$, the OSSPE price path is $\widehat{P}$ forever. For the remainder of the proof, suppose Γ is non-empty. Consider the payoff function in (2). Since π and x are bounded functions and $\delta, \beta \in (0,1)$, the payoff function is defined for all price paths. The payoff function is continuous in $\left\{P^t\right\}_{t=1}^{\infty}$ by the continuity of π, x, and ϕ. To show that Γ is a compact set, first note that it is a subset of Ω^{∞} which, by the compactness of Ω and Tychonoff's Product Theorem, is itself compact. The lhs expression of the ICC is continuous in $\left\{P^t\right\}_{t=1}^{\infty}$ as is the rhs expression (using A7). It follows that Γ is a closed set. Since Γ is a closed subset of a compact set, Γ is compact. There is then a solution to (2) as it involves maximizing a continuous function over a non-empty compact set. If the associated payoff exceeds $\widehat{\pi}/\left(1-\delta\right)$ then such a solution is an OSSPE price path. If it does not exceed $\widehat{\pi}/\left(1-\delta\right)$ then an OSSPE price path is $\widehat{P}$ forever. ♦

$V\left(P^{t-1}, X^{t-1}\right)$ will denote the payoff that is associated with an OSSPE path. Finally, let me note that when something is stated to be a property of an OSSPE path, it is meant to refer to an OSSPE path which involves cartel formation.

4 Results

In order to simplify some of the ensuing the proofs, the assumption is made that $\Omega = [0, P^m]$. By A3, this implies that cartel profit is non-decreasing in price over the price space. While I don't believe this assumption is essential for any result, one can construct examples (which, while extreme, are not pathological) in which an OSSPE path does have price exceed the simple monopoly price in some periods. An explanation as to how this can occur is provided later. However, I also conjecture that the most relevant part of the parameter space is where an OSSPE path lies below P^m. If all of this creates doubt for the reader, a sufficient condition for this assumption to be made without loss of generality is for demand to be perfectly inelastic up to P^m. Prices in excess of P^m then generate zero demand and can be shown never to be part of an OSSPE path.

4.1 Dynamical Properties of the Collusive Price Path

When ICCs are not binding and the probability of detection depends only on movements in price, an OSSPE price path is increasing over time (Harrington, 2002a). When those constraints are binding, the analysis is sufficiently more complex that it prevents any such general result. Both the probability of detection and the associated antitrust penalties are evolving over time and how they impact the price path depends on whether ICCs are tightening or loosening. If those constraints are loosening over time then one would expect the price path to be increasing as if it is IC to initially raise price then it is becoming increasingly easy to do so. If instead those constraints are tightening then one might imagine that price may initially rise but then decline as the constraints bind; forcing the cartel to set a lower price in order to stabilize it. Of course, there is nothing assuring us that the ICCs are monotonically changing; they could become looser then tighter than looser and so forth. Indeed, the combination of two dynamics - the sensitivity of detection to price changes and the dependence of penalties to past prices - means the model is sufficiently rich that I do not think general analytical results are to be had. My approach will then be a blend of analytical and numerical results. First, I isolate the implications of each one of these two dynamics. In Section 4.1.1, penalties are fixed (so there are fines but no damages) but the probability of detection remains endogenous and thus evolves over time. In that case, an OSSPE price path is increasing over time. It is shown that if it is IC to raise price to some level then it is IC to keep price at that level. Thus, cartel stability never requires that price be lowered. In Section 4.1.2, the probability of detection is fixed but the penalty is increasing with past prices. On an OSSPE path, it is established that the penalty rises over time which lowers the payoff from colluding and from deviating. However, the rate of decline in the collusive payoff is shown to be faster than that for the deviator's payoff which causes ICCs to tighten. To ensure cartel stability, an OSSPE price path must decline over time. The second step in my research plan is to use numerical analysis to explore what happens when both of these dynamics are operative. That work is in progress and will be reported in a future version of this paper.

4.1.1 Cartel Price Path is Increasing

Assume there are only fines: $\gamma = 0$ and $F > 0$. The lone state variable for the cartel is lagged price and, in the event of a deviation, the vector of lagged prices. Though penalties are fixed, the probability of detection is sensitive to how the cartel prices as specified in A6. However, further structure is required to establish our main result.

B1 If $P' \geq P$ and $P' > P^o$ then $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is non-increasing in P.

To interpret B1, suppose that the lagged cartel price is P and the cartel is to raise price to P'. If an individual firm deviates to a price of P^o, $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is the associated change in the probability of detection between colluding and deviating. B1 says that this change is weakly lower when the increase in the collusive price is smaller (that is, P is higher and closer to P'). Suppose $P' \geq P > P^o$. As P is increased, the extent of the rise in price from P to P' diminishes but the extent of the fall in price from P to P^o rises. Thus, if the probability of detection is increasing in individual firms' price changes then $\phi(P', P)$ should fall while $\phi((P', \ldots, P^o, \ldots, P'), P)$ should rise which implies $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is smaller as a result of P being higher. B1 is then quite compelling when $P' \geq P > P^o$. The problematic case is $P' > P^o > P$ in which $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ measures the additional increase in the probability of detection by having all firms raise price from P to P' rather than having $n-1$ firms do so and one firm having a more modest price increase from P to P^o. That this additional increase is required to be greater when P is lower - so that the price rises are greater - is a form of convexity in that the rise in the probability from a bigger price change is greater when the initial price change is larger. Some examples of ϕ satisfying B1 are provided in Appendix A.

As mentioned previously, our focus is on when the punishment payoff is no greater than that which occurs in the absence of antitrust laws. To recall, a sufficient condition for B2 to hold is that products are homogeneous.

B2 $\widehat{\pi}/(1-\delta) \geq V_i^{mpe}(\underline{P}) \geq (\widehat{\pi}/(1-\delta)) - F, \forall \underline{P} \in \Omega^n$.

Theorem 2 shows that when only detection, and not penalties, are sensitive to cartel pricing, the cartel price path is non-decreasing over time.

Theorem 2 *Assume A1-A3, A6-A7, B1-B2, and* $\gamma = 0$. *If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE path then* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is non-decreasing over time.*

Proof: The proof is comprised of two steps. Suppose $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an OSSPE path. First, it is shown that if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$ then it is IC to keep price constant at $\overline{P}^{t'}$ in $t'+1$. Second, if, contrary to the theorem, this price path has a decreasing subsequence then one can construct a non-decreasing IC price path that yields a strictly higher payoff and produces the desired contradiction. Given this OSSPE, let $V\left(\overline{P}^{t}\right)$ denote the associated payoff starting with period $t+1$.

In performing the first step, let us initially show that if $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then it is IC to keep price at $\overline{P}^{t'}$. There are two cases to consider: i) $V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$; and ii) $V\left(\overline{P}^{t'-1}\right) \leq V\left(\overline{P}^{t'}\right)$. Starting with case (i) and recognizing that the lhs of (3) is $V\left(\overline{P}^{t'-1}\right)$, we have

$$\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]V\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'}\right). \tag{3}$$

This implies

$$\frac{\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]}{1-\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]} > V\left(\overline{P}^{t'}\right). \tag{4}$$

Substituting the lhs of (4) for $V\left(\overline{P}^{t'}\right)$ in (3), it follows that

$$\begin{aligned} V\left(\overline{P}^{t'-1}\right) &< \pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\ &\quad +\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]\left[\frac{\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]}{1-\delta\left(1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right)}\right]. \end{aligned}$$

Re-arranging yields

$$V\left(\overline{P}^{t'-1}\right) < \frac{\pi\left(\overline{P}^{t'}\right)+\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]}{1-\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]} \tag{5}$$

which gives us an upper bound on $V\left(\overline{P}^{t'-1}\right)$.

Now consider a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$. The payoff, denoted $W\left(\overline{P}^{t'}\right)$, is defined by:

$$W\left(\overline{P}^{t'}\right)=\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]W\left(\overline{P}^{t'}\right),$$

and solving for $W\left(\overline{P}^{t'}\right)$:

$$W\left(\overline{P}^{t'}\right)=\left\{\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\Big/\left\{1-\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\right\}. \tag{6}$$

$W\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'-1}\right)$ follows from (5) and (6) since $\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right) \leq \phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)$ by A6. Given that, by supposition, $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then $W\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$.

The next step is to show that this constant price path is IC. The ICC for period $t'+1$ for the original OSSPE path is:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]V\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\pi\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{7}$$

As $W\left(\overline{P}^{t'}\right)>V\left(\overline{P}^{t'-1}\right)>V\left(\overline{P}^{t'}\right)$ then (7) continues to hold if $W\left(\overline{P}^{t'}\right)$ replaces $V\left(\overline{P}^{t'}\right)$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]W\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\pi\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{8}$$

Now consider the ICC for a constant price path of $\overline{P}^{t'}$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]W\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\pi\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{9}$$

Note that we only need to be concerned with $P_i<\overline{P}^{t'}$ as deviating with a price in excess of $\overline{P}^{t'}$ cannot yield a higher payoff as current profit is weakly lower, the probability of detection is weakly higher, and, by B2, $W\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ implies $W\left(\overline{P}^{t'}\right)\geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)$ so that the future collusive payoff is at least as high as the MPE payoff. Re-arranging (8) and (9), we need to show that:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)-\pi\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right] \\
\geq\ &\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\times \\
&\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}
\end{aligned}\tag{10}$$

implies

$$\begin{aligned}
& \pi\left(\overline{P}^{t'}\right)-\pi\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right] \quad (11)\\
\geq\ & \delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
& -\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\times\\
& \left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\\
\forall P_i\ <\ & \overline{P}^{t'}.
\end{aligned}$$

As the lhs of (10) and (11) are identical, (10) implies (11) if:

$$\begin{aligned}
& \delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
& -\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
\geq\ & \delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
& -\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}.
\end{aligned}$$

Re-arranging this inequality,

$$\begin{aligned}
& \left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \geq \quad (12)\\
& \left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times\\
& \left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}.
\end{aligned}$$

Since

$$W\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)\geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-F$$

and $\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)$ then (12) holds if:

$$\begin{aligned}
& \phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\text{-}\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right) \geq\\
& \phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
& \phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\text{-}\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right) \geq \quad (13)\\
& \phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right).
\end{aligned}$$

Since $\overline{P}^{t'}\geq P_i,\overline{P}^{t'-1}$ then (13) holds by B1. Having shown that a constant price path of $\overline{P}^{t'}$ starting from $t'+1$ is IC and yields a payoff strictly greater than $V\left(\overline{P}^{t'}\right)$, we have a contradiction that the original price path is an OSSPE path. Therefore, if $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ and $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$, it cannot be true that $V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$.

Let us now consider case (ii): $V\left(\overline{P}^{t'-1}\right) \leq V\left(\overline{P}^{t'}\right)$. Consider keeping price constant at $\overline{P}^{t'}$ in period $t'+1$ but then continuing with the original path. The ICC is:

$$\begin{aligned} & \pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'}\right) \\ \geq & \max_{P_i\in\Omega}\pi\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right)-F\right] \\ & +\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right). \end{aligned} \tag{14}$$

Using the same series of steps as with case (i), (7) implies (14) iff

$$\begin{aligned} & \left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/(1-\delta)\right)-F\right]\right\} \geq \\ & \left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/(1-\delta)\right)-F\right]\right\}. \end{aligned}$$

The same argument is used to show that this inequality also holds. The important point to note is that $V\left(\overline{P}^{t'}\right) \geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)$ because $V\left(\overline{P}^{t'}\right) \geq V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$. Hence, if $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ then it is IC to keep price at $\overline{P}^{t'}$ and, in addition, $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/(1-\delta)$.

To summarize, it has been shown that if $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$ and $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ then: i) it is IC to keep price at $\overline{P}^{t'}$; ii) $V\left(\overline{P}^{t'}\right) \geq V\left(\overline{P}^{t'-1}\right)$; and iii) $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/(1-\delta)$. Arguing by strong induction, we will show that if the price path is non-decreasing, it is IC to keep price constant in the future. First note that, by the conditions of an OSSPE, $V\left(P^0\right) \geq \widehat{\pi}/(1-\delta)$. Hence, if $P^0 \leq \overline{P}^1$ then, by our previous argument, it is IC to keep price at $\overline{P}^1$ and $V\left(\overline{P}^1\right) \geq \widehat{\pi}/(1-\delta)$. Now suppose $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$. By the previous argument, it follows from $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1}$ that $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$. Therefore, it is IC to keep price at $\overline{P}^{t'}$. Also note that as long as an OSSPE price path is non-decreasing then so is the value to colluding: if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$ then $V\left(P^0\right) \leq \cdots \leq V\left(\overline{P}^{t'}\right)$.

The second step is to suppose that $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is not non-decreasing and show that there exists another IC path which yields a strictly higher payoff. This contradicts it being supported by an OSSPE. Suppose the price path declines at some time and let $t'+1$ be the first period in which it does so,

$$P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'} > \overline{P}^{t'+1}.$$

Define $t''+1$ as the first period after t' for which price is at least as great as in t': $\overline{P}^{t} \leq \overline{P}^{t'}$ $\forall t \in \{t'+1,\ldots,t''\}$ and $\overline{P}^{t''+1} > \overline{P}^{t'}$. t'' might be ∞. Now consider an alternative price path in which price equals $\overline{P}^{t'}$ for periods $t'+1,\ldots,t''$ and is identical to the original path starting at $t''+1$. First note that this alternative path yields a strictly higher payoff than the original path since it generates strictly higher profit in periods $t'+1,\ldots,t''$ and the same profit thereafter. Furthermore, by A6, it results in a weakly lower probability of detection in periods $t'+1,\ldots,t''+1$ because, with this alternative path, price doesn't change over $t'+1,\ldots,t''$. With respect to $t''+1$, the price rise is $\overline{P}^{t''+1}-\overline{P}^{t'}$ with the alternative path as opposed to a higher price rise of $\overline{P}^{t''+1}-\overline{P}^{t''}$ with the original path which means a weakly lower probability of detection by A6.

Having established that this alternative price path yields a strictly higher payoff, let me argue that it is IC. Consider incentive compatibility over $t'+1,\ldots,t''$. If $t'=0$ then, since $P^{0}=\widehat{P}$, a constant price path of $P^{t'}$ over $t'+1,\ldots,t''$ is certainly IC. If $t'\geq 1$ then $\overline{P}^{t'-1}\leq\overline{P}^{t'}$ and, by our previous analysis, a constant price of $\overline{P}^{t'}$ starting with period $t'+1$ is IC. It is also IC for periods after $t''+1$ since the previous period's price and the current period's price are the same as with the original path which, by supposition, is IC. The only remaining ICC is for period $t''+1$. The period $t''+1$ price is the same for both paths but with the original path the lagged price is $\overline{P}^{t''}$ and with the alternative path it is $\overline{P}^{t'}$ where $\overline{P}^{t'}\geq\overline{P}^{t''}$. There is no problem if $\overline{P}^{t'}=\overline{P}^{t''}$ so let us suppose $\overline{P}^{t'}>\overline{P}^{t''}$. The ICC for $t''+1$ for the original path is:

$$\begin{aligned}
& \pi\left(\overline{P}^{t''+1}\right)+\delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\right]V\left(\overline{P}^{t''+1}\right) \\
\geq\; & \pi\left(P_i,\overline{P}^{t''+1}\right)+\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
& +\delta\left[1-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)\right]V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right), \\
\forall P_i \;\leq\; & P^{t''+1}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
& \pi\left(\overline{P}^{t''+1}\right)-\pi\left(P_i,\overline{P}^{t''+1}\right) \\
& +\delta\left[V\left(\overline{P}^{t''+1}\right)-V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)\right] \quad \geq \\
& \delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)+F\right]\right\} \\
& -\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)\times \\
& \left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\forall P_i\leq\overline{P}^{t''+1}.
\end{aligned} \tag{15}$$

The ICC for the alternative path at $t''+1$ is:

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right)+\delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t''+1}\right)\\
\geq\ &\pi\left(P_i,\overline{P}^{t''+1}\right)+\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\\
\forall P_i\ \leq\ &P^{t''+1}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right)-\pi\left(P_i,\overline{P}^{t''+1}\right) \qquad (16)\\
&+\delta\left[V\left(\overline{P}^{t''+1}\right)-V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)\right]\quad\geq\\
&\delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
&-\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\times\\
&\left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\forall P_i\leq\overline{P}^{t''+1}.
\end{aligned}$$

We then need to show that the rhs of (15) is at least as great as the rhs of (16):

$$\begin{aligned}
&\left[\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \qquad (17)\\
\geq\ &\left[\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\right]\times\\
&\left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\ \forall P_i\leq\overline{P}^{t''+1}.
\end{aligned}$$

Let me argue that $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. As the OSSPE price path is non-decreasing over $1,\ldots,t'$ then, by our earlier argument, $V\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)$. Next note that since a constant price path of $\overline{P}^{t'}$ is IC - and recalling that $W\left(\overline{P}^{t'}\right)$ denotes the associated payoff - then the conditions of an OSSPE imply $V\left(\overline{P}^{t'}\right)\geq W\left(\overline{P}^{t'}\right)$. Since the payoff stream from the OSSPE path is less than that from the constant price path over $t'+1,\ldots,t''$ (recall that the former generates strictly lower profit and a weakly higher probability of detection in those periods), it must deliver a higher payoff stream after t''. Since $V\left(\overline{P}^{t''}\right)$ is the payoff associated with the stream after t'', it follows that $V\left(\overline{P}^{t''}\right)>V\left(\overline{P}^{t'}\right)$. We then have that $V\left(\overline{P}^{t''}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ and $\overline{P}^{t''+1}\geq\overline{P}^{t''}$ which, we've previously showed, implies $V\left(\overline{P}^{t''+1}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ and therefore $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. Since $\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\geq$

$\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)$, a sufficient condition for (17) to hold is:

$$\begin{aligned}&\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\\ \geq\ &\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)\\ \geq\ &\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right).\end{aligned}$$

Since $\overline{P}^{t''+1}>\overline{P}^{t'}>\overline{P}^{t''}$ and $\overline{P}^{t''+1}>P_i$, this condition follows from B1. ♦

When ICCs are not binding, the optimal price path is non-decreasing over time (Harrington, 2002a). Since bigger price movements are more likely to trigger suspicions about a cartel having been formed (by A6), the cartel gradually raises price so as to balance profit and the probability of detection. Thus, if, when ICCs bind, the price path is decreasing, it is because incentive compatibility requires it. The issue then is under what circumstances does the cartel find themselves charging a price that they can't maintain over time. In the proof of Theorem 2, it is established that if, on an OSSPE path, the cartel raises price to some level then it is IC to keep price at that level. Therefore, it is never necessary to reduce price in order to maintain the stability of the cartel which implies that the price path is non-decreasing over time.

There are two key assumptions used in showing that if it is IC to raise price to some level then it is IC to keep price at that level. First, "convexity" of the probability of detection function as characterized in B1. Consider the ICC associated with the cartel currently pricing at P''. If a firm deviates and prices at $P^o<P''$, it changes its current profit by an amount $\pi\left(P^o,P'\right)-\pi\left(P'\right)$ and alters the future payoff, in the event the cartel is not detected, by an amount $V_i^{mpe}\left(P'',\ldots,P^o,\ldots,P''\right)-V\left(P''\right)$. Those components to the ICC are the same regardless of whether the cartel is raising price to P'' or keeping it there. What differs, however, is how cheating influences the current likelihood of detection. Suppose the cartel is raising price from P' to P''. If a firm goes along with that price increase, detection occurs with probability $\phi\left(P'',P'\right)$ while if a firm deviates by pricing at P^o then the probability of detection is $\phi\left(\left(P'',\ldots,P^o,\ldots,P''\right),P'\right)$. Thus, cheating changes the probability of detection by $\phi\left(P'',P'\right)-\phi\left(\left(P'',\ldots,P^o,\ldots,P''\right),P'\right)$. If instead the cartel is maintaining price at P'' then cheating alters the probability of detection by $\phi\left(P'',P''\right)-\phi\left(\left(P'',\ldots,P^o,\ldots,P''\right),P''\right)$. B1 ensures us that

$$\phi\left(P'',P'\right)-\phi\left(\left(P'',\ldots,P^o,\ldots,P''\right),P'\right)\geq\phi\left(P'',P''\right)-\phi\left(\left(P'',\ldots,P^o,\ldots,P''\right),P''\right)$$

so that cheating has a more favorable effect on the likelihood of detection when the cartel is raising price than when it is keeping price constant. Given the other components of the ICCs are identical, if it is IC to raise price to P'' then it is IC to keep price at P''. The second key assumption used in proving Theorem 2 is that penalties are fixed. If penalties are endogenous then, depending on how they are changing and how they influence ICCs, it is possible that the cartel could find it IC to raise price to some level but find it is not IC to keep price there because of how the penalty is evolving. For example, suppose higher penalties reduce the collusive payoff more than they reduce the payoff to deviating. If the penalty is rising over time then the ICCs are tightening, in which case the cartel may ultimately be forced to lower price. This possibility is explored in the next sub-section.

In conclusion, when the dynamics are solely due to how the price path influences the likelihood of detection and the probability of detection is, roughly speaking, convex in price changes then the cartel price path is non-decreasing over time.

4.1.2 Cartel Price Path is Decreasing

While the previous section allowed detection to be endogenous and penalties to be fixed, let us now assume the contrary. Suppose detection is independent of prices - being exclusively driven by such factors as internal whistleblowers - and $\gamma > 0$ so that penalties are sensitive to the prices the cartel set.

C1 $\exists \phi^o \in (0,1)$ such that $\phi(\underline{P}', \underline{P}^o) = \phi^o \ \forall \underline{P}', \underline{P}^o \in \Omega^n$.

As the probability of detection is fixed, the problem simplifies considerably. First, as argued in Section 3.1, the unique MPE is the infinite repetition of the static Nash equilibrium. Second, the optimal deviation price is that which maximizes current profit, $\psi\left(P^t\right)$. Since the probability of detection is fixed and the price at which a firm deviates doesn't influence penalties (where recall it is assumed that damages are not assessed when the cartel is not functioning), a deviating firm's price only affects current profit. Using these properties, the cartel's problem can be stated as:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Omega^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} (1-\phi^o)^{t-1} \left[\pi\left(P^t\right) - \Delta\gamma x\left(P^t\right)\right] + [\delta\phi^o / (1-\delta(1-\phi^o))] \left[(\widehat{\pi}/(1-\delta)) - F\right] \tag{18}$$

subject to:

$$\begin{aligned} &\sum_{\tau=t}^{\infty} \delta^{\tau-t} (1-\phi^o)^{\tau-t} \left[\pi(P^\tau) - \Delta\gamma x(P^\tau)\right] \\ &+ [\delta\phi^o / (1-\delta(1-\phi^o))] \left[(\widehat{\pi}/(1-\delta)) - F\right] - \Delta\beta X^{t-1} \quad \geq \\ &\pi\left(\psi\left(P^t\right); P^t\right) + \delta\left(\widehat{\pi}/(1-\delta)\right) - \theta\beta X^{t-1} - \kappa F, \ \forall t \geq 1, \end{aligned} \tag{19}$$

where $\Delta \equiv \delta\phi^o / [1-\delta\beta(1-\phi^o)]$. $\pi\left(P^t\right) - \Delta\gamma x\left(P^t\right)$ represents the net income from collusion in period t. A firm receives profit of $\pi\left(P^t\right)$ by colluding but incurs a liability in the form of $\Delta\gamma x\left(P^t\right)$ which is the expected present value of damages.[14] This expression is multiplied by $(1-\phi^o)^{t-1}$ which is the probability that

[14] More specifically, the expected present value of damages is $\gamma x\left(P^t\right) \sum_{\tau=0}^{\infty} \delta \left[\delta\beta(1-\phi^o)\right]^{\tau} \phi^o$ where $(1-\phi^o)^{\tau} \phi^o$ is the probability of detection in τ periods and $\beta^{\tau}\gamma x\left(P^t\right)$ is the value of damages at that time.

collusion has not yet been detected. Turning to the payoff to deviating in (19), θ and κ measure the marginal effect of damages and fines, respectively, on the punishment payoff and depend on the likelihood of detection after the cartel has become inactive. Suppose $\rho(\tau)$ is the probability of detection τ periods after the last cartel meeting (which was in the period during which a firm cheated) so that $\theta = \sum_{\tau=0}^{\infty} \delta(\delta\beta)^{\tau} \Pi_{h=0}^{\tau-1}(1-\rho(h))\rho(\tau)$ and $\kappa = \sum_{\tau=0}^{\infty} \delta^{\tau+1} \Pi_{h=0}^{\tau-1}(1-\rho(h))\rho(\tau)$. Compelling restrictions are: $\rho(0) = \phi^o$ and $\rho(\tau)$ is non-increasing in τ. In that case, $\theta \leq \delta\phi^o/(1-\delta\beta(1-\phi^o))$ and $\kappa \leq \delta\phi^o/(1-\delta(1-\phi^o))$ which is stated as assumption C2.

C2 $\theta \in [0, \delta\phi^o/(1-\delta\beta(1-\phi^o))]$ and $\kappa \in [0, \delta\phi^o/(1-\delta(1-\phi^o))]$.

The key implication of C2 is that if, starting from period t, some price path is IC given $X^{t-1} = X'$ then it is also IC if $X^{t-1} < X'$ as the collusive payoff is decreasing with respect to damages at a faster rate then the deviation payoff. The next assumption says that the difference between the maximal current profit and the collusive profit is increasing in the collusive price. It'll imply that if a price path is IC then so is a price path which is identical except that the period t price is lower, holding X^t fixed.

C3 $\pi(\psi(P), P) - \pi(P)$ is increasing in P $\forall P \geq \widehat{P}$.

C4 strengthens A5 by assuming the damage function is strictly monotonic.

C4 $x(\cdot)$ is differentiable and non-decreasing, $x\left(\widehat{P}\right) = 0$, and x is strictly increasing over $\left[\widehat{P}, P^m\right]$.

Define $\xi(x)$ as the price that generates current damage penalties of $x : x = \gamma x(\xi(x))$. Since the damage function is monotonic over $\left[\widehat{P}, P^m\right]$, ξ is well-defined $\forall x \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

C5 $\pi(\xi(x))$ is concave $\forall x \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

It is shown in Appendix B that C5 holds when demand is weakly concave, marginal cost is constant, damages take the standard form, and the but-for price weakly exceeds the competitive price. Note that C4 is also implied by these conditions.

The next result shows that damages are non-decreasing over time so that the penalty from detection is weakly increasing.

Lemma 3 *Assume A1-A3 and C1-C5. If* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is consistent with an OSSPE then* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is non-decreasing.*

Proof: A critical property that will be used is that if, on an OSSPE path, the cartel prices at P' and the damage state variable at the end of the period is X' then pricing at P with end-of-period damages of X is also IC if $P \leq P'$ and $X \leq X'$. To see this, consider the ICC for $\left(P^t, X^t\right) = (P', X')$:

$$\pi\left(P'\right) + \sum_{\tau=t+1}^{\infty} \delta^{\tau-t}\left(1-\phi^o\right)^{\tau-t}\left[\pi\left(P^t\right) - \Delta\gamma x\left(P^\tau\right)\right] + \left[\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right]\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]$$
$$-\Delta\beta X' \geq \pi\left(\psi\left(P'\right), P'\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) - \theta X' + \theta\gamma x\left(P'\right) - \kappa F.$$

Since, by deviating rather than colluding, a firm avoids current period damages of $\gamma x(P')$, if the end-of-period damages are X' when a firm colludes then they are $[X' - \gamma x(P')]$ when it deviates. By C2, the lhs decreases at a faster rater with respect to X' than the rhs. Hence, if X' is replaced with a lower value for the damage variable, this condition still holds. By C3 and C4, $\pi(\psi(P), P) + \theta\gamma x(P) - \pi(P)$ is increasing in P. Hence, this ICC holds if P' is replaced with a lower price. I conclude that, on an OSSPE path, if $\left(P^t, X^t\right)$ is replaced with a lower price and/or lower damage variable then the ICC at t still holds.

Since $X^0 = 0$, if $\overline{X}^1 = 0$ then, by the stationarity of the policy function, $\overline{X}^t = 0 \forall t$ and thus, trivially, damages are non-decreasing.[15] Next suppose that $X^0 < \overline{X}^1$. If Lemma 3 is not true then $\exists t' \geq 1$ such that $X^0 < \overline{X}^1 < \cdots < \overline{X}^{t'} > \overline{X}^{t'+1}$. (Note that if damages are constant from one period to the next then they are constant in all future periods by the stationarity of the optimal pricing function.) Given the path of damages on an OSSPE path, the associated prices in t' and $t'+1$ are defined by $\overline{P}^{t'} = \xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ and $\overline{P}^{t'+1} = \xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)$. That is, $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ is the price that results in damages of $\overline{X}^{t'}$ given inherited damages of $\overline{X}^{t'-1}$.

Since, by supposition, $\overline{X}^{t'} > \overline{X}^{t'+1}$ and furthermore $\overline{X}^{t'+1} \geq \beta\overline{X}^{t'} > \beta\overline{X}^{t'-1}$ then $\overline{X}^{t'+1} \in \left[\beta\overline{X}^{t'-1}, \overline{X}^{t'}\right]$. Hence, it was feasible to set price in t' so that damages equalled $\overline{X}^{t'+1}$ at t' and the price that would have done this is $\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)$. Since $\overline{X}^{t'} - \beta\overline{X}^{t'-1} > \overline{X}^{t'+1} - \beta\overline{X}^{t'-1}$ and ξ is increasing then $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right) > \xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)$. Given that, by supposition, charging a price of $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ with resulting total damages of $\overline{X}^{t'}$ is IC (as it is part of an OSSPE) then the price-damage pair $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$ is also IC (assuming the future price path is the same) as it involves a lower collusive price and lower damages. Since $\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'}\right)$ was selected in t' and, as just argued, the cartel could have chosen $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$, I conclude that the former yields at least as high a payoff. Letting $V(X)$ denote the payoff associated with the OSSPE when

[15] The assumption $X^0 = 0$ could be replaced with the condition that, on the optimal path, $X^0 < \overline{X}^1$.

damages are X, the previous statement is then represented as:

$$\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'}\right)\geq$$
$$\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'+1}\right)\quad\Leftrightarrow$$

$$\begin{aligned}&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)\geq\\&\delta\left(1-\phi^{o}\right)\left[V\left(\overline{X}^{t'+1}\right)-V\left(\overline{X}^{t'}\right)\right]+\delta\phi^{o}\left(\overline{X}^{t'}-\overline{X}^{t'+1}\right).\end{aligned}\tag{20}$$

Next note that $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right),\overline{X}^{t'}\right)$ being IC implies $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right),\overline{X}^{t'}\right)$ is as well since $\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)>\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)$. Given that $\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right),\overline{X}^{t'+1}\right)$ was chosen in $t'+1$, it follows that $\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right),\overline{X}^{t'+1}\right)$ yields at least as high a payoff as $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right),\overline{X}^{t'}\right)$:

$$\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'+1}\right)\geq$$
$$\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'}\right)\quad\Leftrightarrow$$

$$\begin{aligned}&\delta\left(1-\phi^{o}\right)\left[V\left(\overline{X}^{t'+1}\right)-V\left(\overline{X}^{t'}\right)\right]+\delta\phi^{o}\left(\overline{X}^{t'}-\overline{X}^{t'+1}\right)\geq\\&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right).\end{aligned}\tag{21}$$

(20)-(21) imply:

$$\begin{aligned}&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)\quad\geq\\&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right).\end{aligned}\tag{22}$$

Note that the difference in the arguments on the lhs of (22) is

$$\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)-\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)=\overline{X}^{t'}-\overline{X}^{t'+1},$$

and on the rhs is:

$$\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)-\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)=\overline{X}^{t'}-\overline{X}^{t'+1}.$$

By the concavity of $\pi\left(\xi\left(\cdot\right)\right)$, it then follows from (22) that:

$$\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\leq\overline{X}^{t'+1}-\beta\overline{X}^{t'}\Leftrightarrow\overline{X}^{t'}\leq\overline{X}^{t'-1},$$

which is a contradiction. This proves that $\overline{X}^{t}$ is non-decreasing on an OSSPE path. ♦

C6 imposes quasi-concavity of net income - profit less the expected present value of damages. Sufficient conditions for C6 are strict concavity of the profit and damage functions.

C6 $\exists P^* \in \left(\widehat{P}, P^m\right]$ such that $\pi'(P) - \Delta\gamma x'(P) \gtreqless 0$ as $P \lesseqgtr P^*$ $\forall P \in \left[\widehat{P}, P^m\right]$.

Theorem 4 *Assume A1-A3 and C1-C6. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then it is non-increasing* $\forall t \geq 1$.

Proof: Let $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ denote the path of damages associated with $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$. Recall that $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$.[16] If $\overline{P}^1 \leq \widehat{P}$ then, since $X^0 = 0$, $\overline{X}^1 = 0$ (by C4). Hence, by stationarity, an OSSPE price path then involves pricing at $\overline{P}^1$ in period 2 and every period thereafter. As this contradicts the optimality of colluding, it is inferred that $\overline{P}^1 > \widehat{P}$ and, therefore, $\overline{P}^1 > P^0$.[17]

If Theorem 4 is not true then $\exists t' \geq 1$ such that $P^0 < \overline{P}^1 \geq \cdots \geq \overline{P}^{t'} < \overline{P}^{t'+1}$. For the OSSPE price path under consideration, let $\overline{P}^{t'} = P'$ and $\overline{P}^{t'+1} = P''$ where $P' < P''$. The analysis will involve comparing the original price path - $\left\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P', P'', \overline{P}^{t'+2}, \ldots\right\}$ - with an alternative price path - $\left\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P'', P', \overline{P}^{t'+2}, \ldots\right\}$ - which has the prices in t' and $t'+1$ switched. It'll be shown that if a price path has price rise from one period to the next then an alternative price path in which those two prices are switched yields a strictly higher collusive payoff and if the original price path was IC then so is this one. This contradicts the original price path being induced by an OSSPE and thus contradicts the supposition that an OSSPE price path is not non-increasing.

The first step is to show that an OSSPE price path is bounded from above by P^* (which is defined in C6). Suppose not so that in some periods price exceeds P^*. Consider an alternative price path which is identical except that it has a price of P^* in those periods for which price exceeded P^*. By C6, the collusive payoff, which is expressed in (18), is strictly higher since $\pi(P^*) - \Delta\gamma x(P^*)$ exceeds the comparable expression when price exceeded P^*. By C4, accumulated damages are lower. As ICCs are loosened when damages are reduced, if the original price path was IC then so is this one. In that a price path has been constructed which generates a higher payoff and is IC, it contradicts the supposition that the original path was generated by an OSSPE. I conclude that an OSSPE price path is bounded from above by P^*.

Given $P' < P'' \leq P^*$, it follows from C6 that $\pi(P'') - \Delta\gamma x(P'') > \pi(P') - \Delta\gamma x(P')$. Inspection of (18) then reveals that, due to discounting, the alternative price path yields a strictly higher payoff as it has the cartel receive $\pi(P'') - \Delta\gamma x(P'')$ in period t' and $\pi(P') - \Delta\gamma x(P')$ in $t'+1$; which is the reverse for the original path. The remainder of the proof involves showing that if the original price path is IC then so is the alternative price path.

I begin with the supposition that the original path is IC in all periods. With the alternative path, the ICCs over periods $1, \ldots, t'-1$ are still satisfied since the collusive payoff is higher and the deviation payoff is unchanged. Next consider the

[16] The assumption $P^0 = \widehat{P}$ can be replaced with $\overline{P}^1 > P^0$ on the optimal path.

[17] If $F > 0$ then colluding and pricing at or below $\widehat{P}$ is clearly inferior to not colluding. If $F = 0$ then it could be optimal to collude and price at $\widehat{P}$ though that is a non-generic result.

period t constraint where $t \geq t' + 2$. As the current and future price path is the same as with the original path, the only difference in the constraint is lagged damages. Note that accumulated damages at t, where $t \geq t'+2$, under the alternative path and under the original path are identical in all terms except for the damages incurred in periods t' and $t'+1$. The difference between the accumulated damages at t, where $t \geq t'+2$, under the alternative path and under the original path then equals:

$$\begin{aligned} & \left[\beta^{t-t'}\gamma x\left(P''\right)+\beta^{t-t'-1}\gamma x\left(P'\right)\right]-\left[\beta^{t-t'}\gamma x\left(P'\right)+\beta^{t-t'-1}\gamma x\left(P''\right)\right] \\ = & \ -\beta^{t-t'-1}\left(1-\beta\right)\gamma\left[x\left(P''\right)-x\left(P'\right)\right]<0. \end{aligned}$$

Since, compared to the original path, the alternative path substitutes higher damages in t' for lower damages in $t'+1$, accumulated damages are lower after $t'+1$ since the higher damages have had one more period to depreciate. Given that damages are lower under the alternative price path, the path is IC for $t \geq t'+2$.

Next consider the ICC at $t'+1$. With the original price path, price is P'' and damages are $\beta^2\overline{X}^{t'-1}+\beta\gamma x\left(P'\right)+\gamma x\left(P''\right)$ at $t'+1$. With the alternative price path, price is P' and damages are $\beta^2\overline{X}^{t'-1}+\beta\gamma x\left(P''\right)+\gamma x\left(P'\right)$. As price is lower then, by C3, this loosens the ICC. As damages are lower, this also serves to loosen the ICC. I conclude that the ICC is satisfied at $t'+1$ for the alternative price path.

Finally, consider the ICC at t'. Using Lemma 3, it'll be shown that if the original price path is IC at $t'+1$ then the alternative path is IC at t'. As an initial step, compare the damages at $t'+1$ for the original path with those at t' for the alternative path. The latter is weakly smaller iff $\beta\overline{X}^{t'}+\gamma x\left(P''\right) \geq \beta\overline{X}^{t'-1}+\gamma x\left(P''\right)$. As $\overline{X}^{t'-1} \leq \overline{X}^{t'}$ by Lemma 3, this is then indeed true. Since then damages at t' for the alternative path are weakly lower than damages at $t'+1$ for the original path, *ceteris paribus*, if the original path is IC at $t'+1$ then the alternative path is IC at t'. For the next step, recall that the collusive payoff at t' for the alternative path exceeds the collusive payoff at t' for the original path. Since $\overline{X}^{t'} \leq \overline{X}^{t'+1}$, it must then be true, for the original path, that $V\left(\overline{X}^{t'}\right) \geq V\left(\overline{X}^{t'+1}\right)$. The reason is that, at t', the cartel can use the price path starting at $t'+1$ and, since damages are weakly lower in t', the collusive payoff must be weakly higher. Holding fixed the level of accumulated damages, it follows that the collusive payoff at t' for the alternative path exceeds the collusive payoff at $t'+1$ for the original path. Still holding fixed the level of accumulated damages, since the price at t' for the alternative path is the same as the price at $t'+1$ for the original path, the deviation payoffs are the same. Finally, since the accumulated damages at t' for the alternative path are weakly lower than the accumulated damages at $t'+1$ for the original path, the ICC being satisfied at $t'+1$ for the original path then implies it holds at t' for the alternative path.

It has then been shown that the incentive compatibility of the original price path implies the incentive compatibility of the alternative price path. As the latter yields a strictly higher payoff, this contradicts the original path being generated by an OSSPE and thereby establishes that a price path which is not non-increasing is inconsistent with it being supported by an OSSPE. ⧫

As the probability of detection is independent of lagged prices, all dynamics are generated by the evolution of damages. Since detection is more likely when the cartel is active, the collusive payoff is then more sensitive than the deviation payoff to damages. Given that damages grow over an OSSPE path (Lemma 3), the collusive payoff is then declining faster than the deviation payoff over time. This tightens ICCs and, in order to ensure they are satisfied, the cartel must lower price (by C3).

The idea of a cartel raising price for one period to then monotonically lower it is, I believe, counterfactual. However, if one appends this analysis with that of the preceding subsection (thereby allowing the probability of detection to be increasing in the extent of the price change), I conjecture the following price path will emerge. Initially, the cartel gradually raises price over time so as to balance earning higher profit with avoiding creating suspicions about collusion. When ICCs are never binding, the price path would continue to increase and eventually asymptote some steady-state price (Harrington, 2002a). When these constraints bind, however, I conjecture that price peaks in finite time after which it declines and asymptotes some steady-state price. In that damages are growing, the equilibrium constraints are tightening which forces the cartel to reduce price for the purposes of internal stability rather than considerations related to detection (indeed, reducing price could raise the likelihood of detection relative to keeping price fixed). Numerical analysis is in progress to assess the validity of this conjecture.

4.2 Effect of Antitrust Law on Price Levels

Having identified some properties of cartel pricing dynamics, let us next explore how antitrust laws impact the level of cartel prices. Do antitrust laws have their desired effect of constraining a cartel's price increases? The first result considers the preceding model in which the probability of detection is fixed and independent of price movements. The intuitive result is derived that the introduction of antitrust laws induces a cartel to price lower in all periods. I then consider the other extreme by assuming that detection depends only on price and, more specifically, only on price movements. This is a special case of the model of Section 4.1.1 which, by Theorem 2, we know the cartel price path is rising. While antitrust laws are shown to initially lower prices, they eventually cause the cartel to price *higher* than would have occurred in the absence of such laws. The risk of detection and the prospect of penalties serve to loosen ICCs and thereby allow the cartel to support higher prices.

4.2.1 Detection is Independent of Price Movements

Suppose detection is independent of prices; being exclusively driven by such factors as internal whistleblowers. Recall that, in the absence of antitrust laws, a price of $\widetilde{P}$ is charged in all periods. The next result shows that antitrust laws lower prices in all periods.

Theorem 5 *Assume A1-A5 and C1-C2. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\sup\left\{\overline{P}^t\,|t=1,2,\ldots\right\}<\widetilde{P}$.

Proof: The ICCs can be re-arranged to be

$$\begin{aligned} &\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left[(1-\phi^o)^{\tau-t} \left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-(1-\phi^o)^{\tau-t}\right)\widehat{\pi}\right] \\ &- \left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right) - \kappa\right] F - (\Delta-\theta)\,\beta\overline{X}^{t-1} \\ \geq\ & \pi\left(\psi\left(\overline{P}^{t}\right); \overline{P}^{t}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right), \forall t \geq 1, \end{aligned}$$

Contrary to the theorem, suppose an OSSPE price path is not bounded below $\widetilde{P}$. First, suppose the price path reaches a maximum in finite time: $\exists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ and $\overline{P}^{t'} \geq \widetilde{P}$. We then have

$$\begin{aligned} &\pi\left(\psi\left(\overline{P}^{t'}\right); \overline{P}^{t'}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \geq \pi\left(\overline{P}^{t'}\right)/(1-\delta) \geq \textstyle\sum_{\tau=t'}^{\infty} \delta^{\tau-t'}\pi\left(\overline{P}^{\tau}\right) \quad > \\ &\sum_{\tau=t'}^{\infty} \delta^{\tau-t'} \left[(1-\phi^o)^{\tau-t'} \left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-(1-\phi^o)^{\tau-t'}\right)\widehat{\pi}\right] \\ &- \left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right) - \kappa\right] F - (\Delta-\theta)\,\beta\overline{X}^{t-1}. \end{aligned}$$

The first inequality follows from A4 and $\overline{P}^{t'} \geq \widetilde{P}$. The second inequality is because $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ and thus $\pi\left(\overline{P}^{t'}\right) \geq \pi\left(\overline{P}^{t}\right) \forall t$. The third inequality follows from C2, $\phi^o > 0$, and $\pi\left(\overline{P}^{t'}\right) > \widehat{\pi}$. We conclude that the ICC for t' is violated. This contradiction establishes that if $\exists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ then $\overline{P}^{t'} < \widetilde{P}$ and thus $\sup\left\{\overline{P}^{t} \,|t=1,2,\dots\right\} < \widetilde{P}$.

Now suppose $\nexists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ and, furthermore, $\sup\left\{\overline{P}^{t} \,|t=1,2,\dots\right\} \equiv P^z \geq \widetilde{P}$. Construct an infinite subsequence of prices comprised of the maximal price set up to that period: $\left\{\overline{P}^{t_1}, \overline{P}^{t_2}, \dots\right\}$ where $t_1 < t_2 < \cdots$, $\overline{P}^{t_{i-1}} < \overline{P}^{t_i}$, and $\overline{P}^{t} \leq \overline{P}^{t_{i-1}} \forall t \in \{t_{i-1}, t_{i-1}+1, \dots, t_i - 1\}$. This sequence is obviously increasing and, since price is bounded from above by P^m, its limit then exists and, in fact, equals P^z. First note that:

$$\begin{aligned} \pi\left(P^z\right)/\left(1-\delta\right) \quad > \quad & \lim_{i\to\infty} \sum_{\tau=t'}^{\infty} \delta^{\tau-t_i} \left[(1-\phi^o)^{\tau-t_i} \left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-(1-\phi^o)^{\tau-t_i}\right)\widehat{\pi}\right] \\ & - \left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right) - \kappa\right] F - (\Delta-\theta)\,\beta\overline{X}^{t_i-1}. \end{aligned}$$

Next note that:

$$\lim_{i\to\infty} \pi\left(\psi\left(\overline{P}^{t_i}\right); \overline{P}^{t_i}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) = \pi\left(\psi\left(P^z\right); P^z\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right).$$

Since $P^z \geq \widetilde{P}$ and thus

$$\pi\left(\psi\left(P^z\right); P^z\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \geq \pi\left(P^z\right)/\left(1-\delta\right),$$

it follows from the preceding two equations that

$$\begin{aligned}
& \lim_{i\to\infty} \pi\left(\psi\left(\overline{P}^{t_i}\right);\overline{P}^{t_i}\right)+\delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \\
> \; & \lim_{i\to\infty}\sum_{\tau=t'}^{\infty}\delta^{\tau-t_i}\left[\left(1-\phi^o\right)^{\tau-t_i}\left(\pi\left(\overline{P}^{\tau}\right)-\Delta\gamma x\left(\overline{P}^{\tau}\right)\right)+\left(1-\left(1-\phi^o\right)^{\tau-t_i}\right)\widehat{\pi}\right] \\
& -\left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right)-\kappa\right]F-\left(\Delta-\theta\right)\beta\overline{X}^{t_i-1}
\end{aligned}$$

which means the ICC is violated. We conclude that if $\nexists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ then $\sup\left\{\overline{P}^{t} \,|t=1,2,\ldots\right\} < \widetilde{P}$. ♦

This result is quite general and is due to two effects. First, introducing detection results in the cartel being finitely-lived almost surely. This reduces the collusive payoff and tightens ICCs. While we have assumed that detection results in the permanent cessation of collusion, a temporary cessation is sufficient for this force to be operative. Lest the reader thinks that this straightforward effect is exclusively driving our conclusions, a finitely-lived cartel in expectation is neither necessary nor sufficient for antitrust laws to cause the collusive price to be lower.[18] The second effect is due to detection depending only on whether firms are colluding, and not on the prices they set or the way in which prices change. In that case, a firm that cheats reduces the likelihood of detection by causing the cartel to dissolve. While antitrust penalties reduce both the collusive payoff and the payoff to a deviator (as detection may still occur during the period of deviation or afterwards), it has a bigger impact on the collusive payoff since detection is more likely when the cartel is active. As a result, antitrust penalties depress the collusive payoff more than the payoff to deviating. This tightens ICCs and forces the cartel to set lower prices.

4.2.2 Detection Depends on Price Movements

Now consider the other extreme - detection depends only on price and, furthermore, it depends only on price movements. This is embodied by assuming the baseline probability of detection, which is that associated with the price vector not changing, is zero.

D1 $\phi:\Omega^{2n}\rightarrow\Re_{+}$ is continuously differentiable.

D2 $\phi\left(P,P\right)=0\ \forall P\in\Omega$.

D3 If $\underline{P}'\geq\underline{P}^o$ and $\underline{P}'\geq\underline{P}''$ (component-wise) then

$$\phi\left(\underline{P}',\underline{P}^o\right)+\left[1-\phi\left(\underline{P}',\underline{P}^o\right)\right]\phi\left(\underline{P}'',\underline{P}'\right)\geq\phi\left(\underline{P}'',\underline{P}^o\right).$$

[18] Theorem 6 shows that the collusive price is higher in the long-run even though with positive probability the cartel is finitely-lived. Thus, a finitely-lived cartel is not sufficient for antitrust laws to have their intended effect. An example is provided in Harrington (2002b) for which the collusive price is lower in all periods even though the cartel is infinitely-lived with probability one. Hence, it is then not a necessary condition either.

D2 says that if prices don't change then the cartel is not detected for sure. I believe results are robust to minor variations in this assumption and this will be discussed after they are presented. It follows from D1 and D2 that

$$\partial\phi\left(P,P\right)/\partial P^{t}=0=\partial\phi\left(P,P\right)/\partial P^{t-1}\ \forall P. \tag{23}$$

In interpreting D3, the lhs is the probability of detection over two periods where prices are initially raised from $\underline{P}^{o}$ to $\underline{P}'$ and then lowered from $\underline{P}'$ to $\underline{P}''$. D3 specifies that this probability exceeds the probability of directly changing the price vector from $\underline{P}^{o}$ to $\underline{P}''$. To see why this condition is compelling, consider it at the level of an individual firm. Compare firm i: i) raising price from P_i^o to P_i' and then lowering it from P_i' to P_i''; with ii) changing price from P_i^o to P_i''. Suppose $P_i' > P_i'' > P_i^o$. Since raising price from P_i^o to P_i'' should be less likely to trigger suspicions than raising price from P_i^o to P_i', the probability of detection for (i) ought to be at least as great as that for (ii). If instead $P_i' > P_i^o > P_i''$ then lowering price from P_i' to P_i'' should be worse than lowering price from P_i^o to P_i'' so that, once again, the probability of detection for (i) ought to be at least as great as that for (ii).

A7 will be assumed so that a MPE exists. The following additional property is imposed which holds, for example, when products are homogeneous.

D4 $V_i^{mpe}\left(\underline{P},X\right)$ is non-increasing in X and if $\underline{P}\neq\left(\widehat{P},\ldots,\widehat{P}\right)$ and $X>0$ then

$$\widehat{\pi}/\left(1-\delta\right)>V_i^{mpe}\left(\underline{P},X\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta X-F.$$

While D1-D3 do not imply the probability of detection is ever positive, such is implicit in D4. Define $\overline{\Lambda}\left(P\right)$ to be the maximal payoff from deviating when the cartel is in a steady-state of charging a price of P. This means that P was charged last period and this period and damages are at their steady-state level of $\gamma x\left(P\right)/\left(1-\beta\right)$.

$$\begin{aligned}\overline{\Lambda}\left(P\right) \equiv\ & \max_{P_i\in\Omega}\pi\left(P_i,P\right)+\delta\phi\left(\left(P,\ldots,P_i,\ldots,P\right),P\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\left(\gamma x\left(P\right)/\left(1-\beta\right)\right)-F\right]\\ & +\delta\left[1-\phi\left(\left(P,\ldots,P_i,\ldots,P\right),P\right)\right]V_i^{mpe}\left(\left(P,\ldots,P_i,\ldots,P\right),\beta\gamma x\left(P\right)/\left(1-\beta\right)\right).\end{aligned}$$

Since damages of $\gamma x\left(P\right)/\left(1-\beta\right)$ are carried over from the previous period, damages in the period of deviation are $\beta\left(\gamma x\left(P\right)/\left(1-\beta\right)\right)$.

Note that A1-A7 imply that $\overline{\Lambda}\left(P\right)$ is defined. In D5, P^{*} is defined to be the highest steady-state price path that is IC. By D2, the steady-state collusive payoff is $\pi\left(P\right)/\left(1-\delta\right)$.

D5 P^{*} exists and is unique where, if

$$\pi\left(P\right)/\left(1-\delta\right)\geq\overline{\Lambda}\left(P\right)\forall P\in\left[\widehat{P},P^{m}\right]$$

then $P^{*}=P^{m}$ and, otherwise, P^{*} is defined by

$$\pi\left(P\right)/\left(1-\delta\right)\gtreqless\overline{\Lambda}\left(P\right)\ \text{ as } P\lesseqgtr P^{*},\forall P\in\left[\widehat{P},P^{m}\right].$$

Furthermore, it is straightforward to show that $P^* > \widetilde{P}$ where recall that $\widetilde{P}$ is the highest equilibrium price in the absence of antitrust laws. By the definition of $\widetilde{P}$,

$$\pi\left(\widetilde{P}\right)/\left(1-\delta\right) = \max_{P_i\in\Omega} \pi\left(P_i, \widetilde{P}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right).$$

It follows from D4 that

$$\begin{aligned} \pi\left(P_i, P\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \quad &> \\ \pi\left(P_i, P\right) + \delta\phi\left(\left(P, \ldots, P_i, \ldots, P\right), P\right)&\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta\left(\gamma x\left(P\right)/\left(1-\beta\right)\right) - F\right] \\ +\delta\left[1-\phi\left(\left(P, \ldots, P_i, \ldots, P\right), P\right)\right] &V_i^{mpe}\left(\left(P, \ldots, P_i, \ldots, P\right), \beta\gamma x\left(P\right)/\left(1-\beta\right)\right), \forall P_i \end{aligned}$$

when $P > \widehat{P}$. It is then true that $\pi\left(\widetilde{P}\right)/\left(1-\delta\right) > \overline{\Lambda}\left(\widetilde{P}\right)$ which implies $P^* > \widetilde{P}$.

The cartel's problem is:

$$\begin{aligned} &\max_{\{P^t\}_{t=1}^{\infty}\in\Omega^\infty} \sum_{t=1}^{\infty} \delta^{t-1}\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\pi\left(P^t\right) \\ &+\sum_{t=1}^{\infty}\delta^t\phi\left(P^t, P^{t-1}\right)\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^t X^0 - \sum_{j=1}^{t}\beta^{t-j}\gamma x\left(P^j\right) - F\right] \end{aligned} \tag{24}$$

subject to

$$\Psi\left(\left\{P^\tau\right\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \geq \Lambda\left(\left\{P^\tau\right\}_{\tau=t-1}^{\infty}, X^{t-1}\right),\ \forall t \geq 1; \tag{25}$$

where

$$\begin{aligned} \Psi\left(\left\{P^\tau\right\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \quad \equiv \quad &\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\pi\left(P^\tau\right) \\ &+\sum_{\tau=t}^{\infty}\delta^{\tau-t+1}\Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\phi\left(P^\tau, P^{\tau-1}\right)\times \\ &\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^{\tau-t+1}X^{t-1} - \sum_{j=t}^{\tau}\beta^{\tau-j}\gamma x\left(P^j\right) - F\right], \end{aligned}$$

$$\begin{aligned} \Lambda\left(\left\{P^\tau\right\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \quad \equiv \quad &\max_{P_i}\pi\left(P_i, P^t\right) \\ &+\delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X^{t-1} - F\right] \\ &+\delta\left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right]\times \\ &V_i^{mpe}(\left(P^t, \ldots, P_i, \ldots, P^t\right), \beta X^{t-1}). \end{aligned}$$

Theorem 6 states that the price path is bounded below P^* and converges to it. Hence, the introduction of antitrust laws and an authority to enforce them results in the cartel eventually pricing higher.

Theorem 6 *Assume A1-A7 and D1-D5. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\overline{P}^t \leq P^* \forall t$ *and* $\lim_{t\to\infty}\overline{P}^t = P^*$.

Proof: Define $\mathcal{P}^t \equiv \max\left\{\overline{P}^0, \overline{P}^1, \ldots, \overline{P}^t\right\}$ to be the maximum price set over the first t periods. As an initial step, let us shown that, on an OSSPE path, if the current period's price is at least as great as all past prices, $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$, then $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on the value in that period: $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\overline{P}^{t'}\right)/(1-\delta)$; where $\overline{X}^t$ is the value of the state variable on the OSSPE path. Intuitively, if it was IC to change price to $\overline{P}^{t'}$ then it is IC to keep price at $\overline{P}^{t'}$ since the probability of detection is zero from doing so (by D2).

Assume $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ in which case $\overline{P}^{t'} \geq \overline{P}^{t'-1}$. The ICC for period t' is

$$\begin{aligned} &\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right] \\ &\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) \quad \geq \\ &\max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\ &\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right). \end{aligned} \tag{26}$$

We want to make two substitutions in (26). First, replace $\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right]$ on the lhs with $\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right]$. Second, suppose, contrary to the claim, $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) < \pi\left(\overline{P}^{t'}\right)/(1-\delta)$ and replace $V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right)$ with $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ on the lhs. If (26) holds then it follows that the inequality is still true after these substitutions:

$$\begin{aligned} &\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\ &\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] \pi\left(\overline{P}^{t'}\right)/(1-\delta) \quad \geq \\ &\max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\ &\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right). \end{aligned} \tag{27}$$

The objective is to show that pricing at $\overline{P}^{t'}$ from $t'+1$ onward is IC and thus $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right)$ which gives us the desired contradiction.

As an alternative price path, consider the firm maintaining price at the t' level; that is, pricing at $\overline{P}^{t'}$ in period $t, \forall t \geq t'+1$. The ICC for period $t'+1$ is

$$\begin{aligned} &\pi\left(\overline{P}^{t'}\right) + \delta\left(\pi\left(\overline{P}^{t'}\right)/(1-\delta)\right) \quad \geq \\ &\max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'} - F\right] \\ &\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'}\right), \end{aligned} \tag{28}$$

where $\overline{X}^{t'} = \beta \overline{X}^{t'-1} + \gamma x \left(\overline{P}^{t'} \right)$. Note that damages are no longer present in the collusive payoff since, by D2, $\phi \left(\overline{P}^{t'}, \overline{P}^{t'} \right) = 0$, and $\overline{X}^{t'} \geq \overline{X}^{t'-1}$ since $\overline{P}^{t'}$ is the highest price charged thus far. For both (27) and (28), the ICC holds when $P_i > \overline{P}^{t'}$ as pricing above $\overline{P}^{t'}$ weakly lowers current profit (by A1-A2), weakly raises the probability of detection (by A6), and it'll be shown that the MPE payoff does not exceed the collusive payoff.

I want to show that (27) implies (28) which will establish that if the original price path was IC at t' then so is a price of $\overline{P}^{t'}$ at $t'+1$. Since the rhs of (28) is non-increasing in damages (using D4), a sufficient condition for (28) to hold is

$$\begin{aligned} & \pi \left(\overline{P}^{t'} \right) + \delta \left(\pi \left(\overline{P}^{t'} \right) / (1-\delta) \right) \quad \geq \qquad (29) \\ & \max_{P_i \in \Omega} \pi \left(P_i, \overline{P}^{t'} \right) + \delta \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'-1} \right) \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \\ & + \delta \left[1 - \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'-1} \right) \right] V_i^{mpe} \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \beta \overline{X}^{t'-1} \right), \end{aligned}$$

where $\beta \overline{X}^{t'}$ has been replaced with $\beta \overline{X}^{t'-1}$. Let us then show that (27) implies (29). This is true if the rhs minus the lhs of (29) is at least as great as the rhs minus the lhs of (27):

$$\begin{aligned} & \pi \left(\overline{P}^{t'} \right) + \delta \left(\pi \left(\overline{P}^{t'} \right) / (1-\delta) \right) - \pi \left(P_i, \overline{P}^{t'} \right) \\ & - \delta \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'} \right) \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \\ & - \delta \left[1 - \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'} \right) \right] V_i^{mpe} \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \beta \overline{X}^{t'-1} \right) \\ \geq \; & \pi \left(\overline{P}^{t'} \right) + \delta \phi \left(\overline{P}^{t'}, \overline{P}^{t'-1} \right) \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \\ & + \delta \left[1 - \phi \left(\overline{P}^{t'}, \overline{P}^{t'-1} \right) \right] \left(\pi \left(\overline{P}^{t'} \right) / (1-\delta) \right) \\ & - \pi \left(P_i, \overline{P}^{t'} \right) - \delta \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'-1} \right) \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \\ & - \delta \left[1 - \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'-1} \right) \right] V_i^{mpe} \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \beta \overline{X}^{t'-1} \right), \\ \forall P_i \; < \; & \overline{P}^{t'}. \end{aligned}$$

Eliminating common terms on both sides and re-arranging:

$$\begin{aligned} & \phi \left(\overline{P}^{t'}, \overline{P}^{t'-1} \right) \left\{ \left(\pi \left(\overline{P}^{t'} \right) / (1-\delta) \right) - \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \right\} \qquad (30) \\ \geq \; & \left[\phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'-1} \right) - \phi \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \overline{P}^{t'} \right) \right] \times \\ & \left\{ V_i^{mpe} \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \beta \overline{X}^{t'-1} \right) - \left[(\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F \right] \right\}. \end{aligned}$$

As D4 implies

$$\pi \left(\overline{P}^{t'} \right) / (1-\delta) \geq V_i^{mpe} \left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'} \right), \beta \overline{X}^{t'-1} \right) \geq (\widehat{\pi} / (1-\delta)) - \beta \overline{X}^{t'-1} - F$$

then (30) holds if

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right). \tag{31}$$

Suppose $\overline{P}^{t'} > P_i \geq \overline{P}^{t'-1}$. By A6,

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)$$

so that (31) holds. If instead $\overline{P}^{t'} \geq \overline{P}^{t'-1} > P_i$ then, by D3,

$$\begin{array}{c}\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)+\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right) \geq \\ \phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\end{array}$$

which implies

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)+\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)$$

so that (31) is true. We conclude that a constant price path of $\overline{P}^{t'}$ is IC in period $t'+1$. As far as $t > t'+1$, the ICC is as specified in (28) except that $\overline{X}^{t'}$ is replaced with a weakly higher level of damages. Since the rhs of (28) is decreasing in damages and the lhs is independent of them, the ICC holds. In summary, if pricing at $\overline{P}^{t'}$ is IC in t', where $\overline{P}^{t'}$ exceeds all past prices, then a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$ is IC. This implies $V\left(\overline{P}^{t'},\overline{X}^{t'}\right)\geq\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ which gives us our desired contradiction. We have then show that if $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ then $V\left(\overline{P}^{t'},\overline{X}^{t'}\right)\geq\pi\left(\overline{P}^{t'}\right)/(1-\delta)$.

The next step is to show that, for all periods, a lower bound on the value function at the end of period t is $\pi\left(\mathcal{P}^t\right)/(1-\delta)$. The proof is by induction. Start with period t' and suppose that a lower bound on the value function is $\pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$. Note that t' exists since

$$V\left(P^0,X^0\right)\geq\pi\left(P^0\right)/(1-\delta)=\pi\left(\mathcal{P}^0\right)/(1-\delta).$$

Suppose $\overline{P}^t \leq \mathcal{P}^{t-1}\forall t\in\{t'+1,\ldots,t''-1\}$ and, by supposition,

$$V\left(\overline{P}^t,\overline{X}^t\right)\geq\pi\left(\mathcal{P}^t\right)/(1-\delta)\left(=\pi\left(\mathcal{P}^{t'}\right)/(1-\delta)\right),\forall t\in\left\{t'+1,\ldots,t''-1\right\}.$$

Let us show that $V\left(\overline{P}^{t''},\overline{X}^{t''}\right)\geq\pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$ or, equivalently, $V\left(\overline{P}^{t''},\overline{X}^{t''}\right)\geq\pi\left(\mathcal{P}^{t''}\right)/(1-\delta)$. If $\overline{P}^{t''} \geq \mathcal{P}^{t''-1}$ then it is immediate. Suppose $\overline{P}^{t''} < \mathcal{P}^{t''-1}$. By the definition of the value to colluding:

$$\begin{aligned}V\left(\overline{P}^{t''-1},\overline{X}^{t''-1}\right) &= \pi\left(\overline{P}^{t''}\right)+\delta\phi\left(\overline{P}^{t''},\overline{P}^{t''-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right)-\overline{X}^{t''}-F\right] \\ &\quad +\delta\left[1-\phi\left(\overline{P}^{t''},\overline{P}^{t''-1}\right)\right]V\left(\overline{P}^{t''},\overline{X}^{t''}\right).\end{aligned}$$

Since $V\left(\overline{P}^{t''-1}, \overline{X}^{t''-1}\right) \geq \pi\left(\mathcal{P}^{t''-1}\right)/(1-\delta)$ then

$$\begin{aligned} &\pi\left(\overline{P}^{t''}\right) + \delta\phi\left(\overline{P}^{t''}, \overline{P}^{t''-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \overline{X}^{t''} - F\right] \\ &+\delta\left[1-\phi\left(\overline{P}^{t''}, \overline{P}^{t''-1}\right)\right] V\left(\overline{P}^{t''}, \overline{X}^{t''}\right) \geq \pi\left(\mathcal{P}^{t''-1}\right)/(1-\delta). \end{aligned} \tag{32}$$

Given $\overline{P}^{t''} < \mathcal{P}^{t''-1}$ then $\pi\left(\overline{P}^{t''}\right) < \pi\left(\mathcal{P}^{t''-1}\right)$ which, using (32), implies

$$\begin{aligned} &\delta\phi\left(\overline{P}^{t''}, \overline{P}^{t''-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \overline{X}^{t''} - F\right] \\ &+\delta\left[1-\phi\left(\overline{P}^{t''}, \overline{P}^{t''-1}\right)\right] V\left(P^{t''}, X^{t''}\right) > \delta\pi\left(\mathcal{P}^{t''-1}\right)/(1-\delta). \end{aligned} \tag{33}$$

Given that

$$V\left(\overline{P}^{t''}, \overline{X}^{t''}\right) \geq (\widehat{\pi}/(1-\delta)) - \overline{X}^{t''} - F$$

then (33) implies

$$V\left(\overline{P}^{t''}, \overline{X}^{t''}\right) > \pi\left(\mathcal{P}^{t''-1}\right)/(1-\delta).$$

Since $\mathcal{P}^{t''} = \mathcal{P}^{t''-1}$ when $\overline{P}^{t''} < \mathcal{P}^{t''-1}$, we then have

$$V\left(\overline{P}^{t''}, \overline{X}^{t''}\right) > \pi\left(\mathcal{P}^{t''}\right)/(1-\delta)$$

which is the desired result.

For an OSSPE path, $\pi\left(\mathcal{P}^{t}\right)/(1-\delta)$ is then a lower bound for $V\left(\overline{P}^{t}, \overline{X}^{t}\right)$. Since π is increasing in price (here we use the fact that the upper bound on the price space is the simple monopoly price) and $\mathcal{P}^{t}$ is non-decreasing over time (being the maximum of all prices over the first t periods), this lower bound for the value function is a non-decreasing sequence. As it has an upper bound of $\pi(P^m)/(1-\delta)$, the sequence of lower bounds converges. Call $\overline{V}$ the value to which it converges.

Since $\mathcal{P}^{t}$ is non-decreasing and is bounded above by P^m, it converges and let $\mathcal{P}^{\infty} \equiv \lim_{t\to\infty}\mathcal{P}^{t}$. Thus, $\overline{V} = \pi(\mathcal{P}^{\infty})/(1-\delta)$. An OSSPE price path is bounded from above by $\mathcal{P}^{\infty}$. If it does not converge to $\mathcal{P}^{\infty}$ then V^t is bounded below $\pi\left(\mathcal{P}^{t}\right)/(1-\delta)$ as $t\to\infty$ but this contradicts $\pi\left(\mathcal{P}^{t}\right)/(1-\delta)$ being a lower bound on the value function. Therefore, an OSSPE price path must converge to $\mathcal{P}^{\infty}$. For incentive compatibility to hold, it must then be true that

$$\lim_{t\to\infty}\left[\left(\pi\left(\mathcal{P}^{t}\right)/(1-\delta)\right) - \Lambda\left(\mathcal{P}^{t}\right)\right] \geq 0. \tag{34}$$

By the definition of P^* being the highest constant price path that is IC in the steady-state (that is, with damages equal to their steady-state value of $\gamma d(P^*)/(1-\beta)$), it follows from (34) that $\mathcal{P}^{\infty} \leq P^*$. The final step is to show $\mathcal{P}^{\infty} = P^*$.

If $\mathcal{P}^{\infty} < P^*$ then

$$\lim_{t\to\infty}\left[\left(\pi\left(\overline{P}^{t}\right)/(1-\delta)\right) - \Lambda\left(\overline{P}^{t}\right)\right] > 0.$$

Recall that the cartel payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} \Pi_{j=1}^{t-1} \left[1 - \phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^t\right)$$
$$+\sum_{t=1}^{\infty} \delta^t \phi\left(P^t, P^{t-1}\right) \Pi_{j=1}^{t-1} \left[1 - \phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^t X^0 - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(P^j\right) - F\right].$$

Taking the derivative of it with respect to $P^{t'}$and evaluating it at $P^{t'} = \overline{P}^{t'}$, if the ICC is not binding at t' then optimality requires that:

$$\pi'\left(\overline{P}^{t'}\right) + \delta \left(\frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right) \left(\left(\frac{\widehat{\pi}}{1-\delta}\right) - \beta^{t'} X^0 - \sum_{j=1}^{t'} \beta^{t'-j} \gamma x\left(\overline{P}^j\right) - F\right) \tag{35}$$
$$+\delta^2 \left[\left(\frac{\partial \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\right) \left(1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right) - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right) \left(\frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right)\right] \times$$
$$\left[\left(\frac{\widehat{\pi}}{1-\delta}\right) - \beta^{t'+1} X^0 - \sum_{j=1}^{t'+1} \beta^{t'+1-j} \gamma x\left(\overline{P}^j\right) - F\right]$$

$$-\left[\left(\frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right) \left(1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right) + \left(1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right) \left(\frac{\partial \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\right)\right] \times$$
$$\sum_{t=t'+2}^{\infty} \delta^{t-t'+1} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \Pi_{j=t'+2}^{t-1} \left[1 - \phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right] \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^t X^0 - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(\overline{P}^j\right) - F\right]$$
$$-\sum_{t=t'}^{\infty} \delta^{t-t'+1} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \Pi_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right] \beta^{t-t'} \gamma x'\left(\overline{P}^{t'}\right) = 0.$$

As $t' \to \infty$, $\left(\overline{P}^{t'} - \overline{P}^{t'-1}\right) \to 0$ which implies, by D2 and (23), that

$$\begin{aligned} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) &\to 0, \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right) \to 0 \\ \frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t} &\to 0, \frac{\partial \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}} \to 0 \end{aligned}$$

Thus, (35) implies $\lim_{t' \to \infty} \pi'\left(\overline{P}^{t'}\right) = 0$. However, since, $P^* \leq P^m$ and, by supposition, $\lim_{t \to \infty} \overline{P}^t < P^*$ then $\lim_{t' \to \infty} \pi'\left(\overline{P}^{t'}\right) > 0$. This contradiction proves that our original claim that $\mathcal{P}^{\infty} < P^*$ is false. We conclude that $\lim_{t \to \infty} \overline{P}^t = P^*$. ⧫

Given the prospects of detection, the cartel will tend to gradually raise price so as to reduce the likelihood of triggering suspicions that a cartel has formed. This could cause the cartel price path to initially lie below $\widetilde{P}$, which is the cartel price in the absence of antitrust laws. Theorem 6 establishes, however, that eventually the cartel will price in excess of $\widetilde{P}$ *because* detection may occur and antitrust laws result in the levying of penalties. For example, suppose the MPE is infinite repetition of the static Nash equilibrium. The post-deviation period is then characterized by firms lowering their prices from some collusive level to $\widehat{P}$. This "price war" has associated with it some probability of triggering suspicions that firms may not be competing, leading to an investigation and the levying of costly antitrust penalties. These expected penalties represent an additional cost associated with deviation which serves to lower the payoff to deviating. Of course, detection can also occur with collusion so the collusive payoff is also reduced. However, since $\phi(P, P) = 0$ and the cartel price path eventually settles down, the probability of detection if firms continue colluding is approaching zero and, therefore, the collusive payoff is approaching that value which occurs without antitrust laws. In the long-run, antitrust laws then cause a loosening of ICCs which allows the cartel to support prices in excess of $\widetilde{P}$.[19]

As just argued, the assumption that $\phi(P, P) = 0$ means that antitrust penalties have no impact on the collusive payoff in the long-run because the probability of detection is converging to zero. However, they do have an impact on the payoff from deviating since deviation results in price discretely dropping which means a positive probability of detection. If instead $\phi(P, P) > 0$ then the presence of an antitrust authority depresses both the collusive payoff and the payoff from deviating so its effect on ICCs in the long-run is ambiguous. Still, by continuity, Theorem 6 would seem to be robust as long as, when $P >> \widehat{P}$, $\phi(P, P)$ is sufficiently small relative to $\phi\left(\widehat{P}, P\right)$ so that the prospect of penalties is significantly higher with a deviation-induced price war than with the stable prices associated with continued collusion. This condition and that the MPE is not sufficiently less competitive than the static Nash equilibrium (as implicit in D4) are the key assumptions driving this result. We then think that this perverse effect of antitrust policy on cartel pricing is actually quite general.[20]

Comparative statics on the long-run cartel price, P^*, with respect to the antitrust policy parameters are straightforward under some reasonable assumptions. If $P^* = P^m$ then, generically, P^* is independent of any marginal parameter changes. Thus,

[19] Here, I will elaborate on the comment made at the start of Section 4; that an OSSPE path could entail prices in excess of the simple monopoly price. The reason is that incentive compatibility might require it. By the preceding argument, higher collusive prices imply bigger price wars in response to a deviation and thus a higher chance of a deviator having to pay antitrust penalties.

[20] For very different reasons, Athey and Bagwell (2001) also identify some perverse effects of antitrust policy on cartel pricing.

suppose $P^* < P^m$ so that it is defined by:

$$\begin{aligned}\pi(P)/(1-\delta) \quad &\gtreqless \quad \max_{P_i \in \Omega} \pi(P_i, P) + \delta\phi((P, \ldots, P_i, \ldots, P), P) \times \\ &[(\widehat{\pi}/(1-\delta)) - \beta(\gamma x(P)/(1-\beta)) - F] \\ &+\delta[1 - \phi((P, \ldots, P_i, \ldots, P), P)] \times \\ &V_i^{mpe}((P, \ldots, P_i, \ldots, P), \beta\gamma x(P)/(1-\beta)), \text{ as } P \lesseqgtr P^*.\end{aligned} \tag{36}$$

Assume $V_i^{mpe}((P, \ldots, P_i, \ldots, P), \beta\gamma x(P)/(1-\beta))$ is decreasing in γ, F, and $\overline{\phi}(\cdot)$ where the latter term is the probability of detection when the cartel is no longer operative. These properties hold, for example, if the MPE is infinite repetition of the static Nash equilibrium as then[21]

$$\begin{aligned}&V_i^{mpe}((P, \ldots, P_i, \ldots, P), \beta\gamma x(P)/(1-\beta)) \\ = \quad &(\widehat{\pi}/(1-\delta)) - \delta\overline{\phi}\left(\widehat{P}, (P, \ldots, P_i, \ldots, P)\right)[(\beta\gamma x(P)/(1-\beta)) + F].\end{aligned}$$

With that assumption, the rhs of (36) is decreasing in $\gamma, F, \overline{\phi}(\cdot)$, and $\phi(\cdot)$. As the lhs is independent of these variables then, when γ, F, $\phi(\cdot)$, or $\overline{\phi}(\cdot)$ are increased, the lhs strictly exceeds the rhs at the original value for P^*. By continuity, the new long-run cartel price must bring these two expressions back into equality which requires that it be higher.

In conclusion, it is important to note that Theorem 6 has an antecedent in Cyrenne (1999).[22] He modifies the imperfect monitoring model of Porter (1983) and Green and Porter (1984) by assuming that the transition into a punishment phase entails an additional cost which is interpreted as an antitrust fine. For the case of a quantity game, the optimal collusive price is independent of the fine though the length of the punishment is decreasing in the size of the fine. For the case of a price game (in which firms' prices are private information), the optimal collusive price is increasing in the fine. In both cases, a higher fine increases the average price set by the cartel. While this result is similar to the one here, there are two significant weaknesses to the analysis in Cyrenne (1999). First, the modelling of the detection process is nonsensical. As part of the standard Green-Porter mechanism, the cartel specifies a trigger price (for the quantity game) such that reversion to the static Nash equilibrium occurs when price falls below it. It is the process of price falling below the trigger price that brings forth cartel detection. No other element of the price series influences detection. If P' is the trigger price then the probability of detection equals 1 if firms are colluding in t and $P^t < P'$ and is zero otherwise. This has odd properties. For example, a small change in price can trigger detection - if price goes from being above P' to below P' - while a large change in price (up or down) can avoid detection as long as price remains above P'. Though Cyrenne (1999) motivates this specification by the notion that large price movements induce detection, his specification does not appear to capture that idea very well. Second, he characterized the steady-state

[21] By D2 and assuming the probability of detection is weakly lower when the cartel is inactive, the probability of detection is zero after price has settled down to $\widehat{P}$.

[22] I discovered this paper after developing the intuition for Theorem 6 but prior to proving it.

without describing how the cartel gets to it. The transitional path is of primary consideration in the case of a cartel trying to avoid detection as that is where the potentially revealing pricing dynamics lie. Furthermore, the ultimate steady-state that is reached can well be influenced by the transitional path. A proper analysis then requires modelling the entire cartel price path.

5 Concluding Remarks

This paper has enriched the classic repeated game model of collusion by taking account of how the manner in which the cartel prices can lead to its eventual detection and, in that event, the levying of penalties. Due to the complex way in which detection and penalties influence the conditions for the internal stability of the cartel, there is an array of implications. First, the introduction of antitrust laws can lower the prices set by the cartel but can also allow them to charge higher prices by loosening the incentive compatibility constraints associated with collusion. Second, while the optimal cartel price path is increasing when incentive compatibility constraints are not binding, when they bite the properties of the path depend on whether those constraints are loosening or tightening over time. When penalties are exogenously set, collusion becomes easier over time and this results in the price path being increasing. When penalties are endogenous but the probability of detection is fixed, collusion becomes more difficult over time as penalties accumulate. As long as detection is more likely when the cartel is active, the incentive to deviate becomes greater as penalties accumulate for the expected penalty is higher under continued collusion. As a result, the cartel price path is decreasing over time, after initially being raised right after cartel formation. Our conjecture is that when both penalties and detection are sensitive to cartel behavior that there are two possible paths: i) the cartel price path is monotonically increasing; and ii) the cartel gradually raises price but, after some point, lowers price so as to maintain the stability of the cartel. Numerical analysis is in progress that will assess the accuracy of that conjecture.

6 Appendices

6.1 Appendix A

The task is to provide some examples of functions for which B1 is holds.

B1 If $P' \geq P$ and $P' > P^o$ then $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is non-increasing in P.

Define $f : \Omega^n \to \Omega$ to be a summary statistic for a vector of prices and assume: i) $f(P, \ldots, P) = P$; and ii) if $P^o \leq P$ then $f(P, \ldots, P^o, \ldots, P) \leq P$. Further assume that the probability of detection is a non-decreasing affine or quadratic function of $\left| f\left(\underline{P}^t\right) - f\left(\underline{P}^{t-1}\right)\right|$:

$$\phi\left(\underline{P}^t, \underline{P}^{t-1}\right) = \alpha_0 \left| f\left(\underline{P}^t\right) - f\left(\underline{P}^{t-1}\right)\right| + \alpha_1 \left(f\left(\underline{P}^t\right) - f\left(\underline{P}^{t-1}\right)\right)^2$$

where $\alpha_0, \alpha_1 \geq 0$. Assume $P' \geq P, P^o$ and let $P'' \equiv f\ (P', \ldots, P^o, \ldots, P')$. By property (ii), $P' \geq P''$. We want to show that

$$\left[\alpha_0 \left|P' - P\right| + \alpha_1 \left(P' - P\right)^2\right] - \left[\alpha_0 \left|P'' - P\right| + \alpha_1 \left(P'' - P\right)^2\right] \tag{37}$$

is non-increasing in P. Suppose $P' \geq P'' > P$ then (37) equals

$$\left[\alpha_0 \left(P' - P\right) + \alpha_1 \left(P' - P\right)^2\right] - \left[\alpha_0 \left(P'' - P\right) + \alpha_1 \left(P'' - P\right)^2\right]$$

and taking the derivative with respect to P yields:

$$-\alpha_0 - 2\alpha_1 \left(P' - P\right) + \alpha_0 + 2\alpha_1 \left(P'' - P\right) = -2\alpha_1 \left(P' - P''\right) \leq 0.$$

Next suppose $P' > P > P''$ then (37) equals

$$\left[\alpha_0 \left(P' - P\right) + \alpha_1 \left(P' - P\right)^2\right] - \left[\alpha_0 \left(P - P''\right) + \alpha_1 \left(P'' - P\right)^2\right]$$

and taking the derivative with respect to P yields:

$$-\alpha_0 - 2\alpha_1 \left(P' - P\right) - \alpha_0 + 2\alpha_1 \left(P'' - P\right) = -2\left[\alpha_0 + \alpha_1 \left(P' - P''\right)\right] \leq 0.$$

Next, it is shown that B1 holds if the probability of detection depends only on the maximal price change. Suppose $\exists\widetilde{\phi} : \Re_+ \to [0, 1]$ such that

$$\phi\left(\left(P_1', \ldots, P_n'\right), \left(P_1^o, \ldots, P_n^o\right)\right) = \widetilde{\phi}\left(\max\left\{\left|P_1' - P_1^o\right|, \ldots, \left|P_n' - P_n^o\right|\right\}\right)$$

where $\widetilde{\phi}$ is non-decreasing. If $|P^o - P| > |P' - P|$ then

$$\phi\left(P', P\right) - \phi\left(\left(P', \ldots, P^o, \ldots, P'\right), P\right) = \widetilde{\phi}\left(P' - P\right) - \widetilde{\phi}\left(P - P^o\right).$$

Since

$$-\widetilde{\phi}'\left(P' - P\right) - \widetilde{\phi}'\left(P - P^o\right) \leq 0,$$

B1 holds. If instead $|P' - P| > |P^o - P|$ then

$$\phi\left(P', P\right) - \phi\left(\left(P', \ldots, P^o, \ldots, P'\right), P\right) = \widetilde{\phi}\left(P' - P\right) - \widetilde{\phi}\left(P' - P\right) = 0,$$

and, therefore, B1 holds trivially.

Now suppose the probability of detection depends only on the change in the minimal price. This case is natural when products are homogeneous and detection is based on the prices that customers pay. Thus,

$$\phi\left(\left(P_1', \ldots, P_n'\right), \left(P_1^o, \ldots, P_n^o\right)\right) = \phi\left(\min\left\{P_1', \ldots, P_n'\right\}, \min\left\{P_1^o, \ldots, P_n^o\right\}\right).$$

We then need to show that $\phi(P', P) - \phi(P^o, P)$ is non-increasing in P when $P' \geq P$ and $P' > P^o$. This is immediate if $P \geq P^o$ as raising P lowers $\phi(P', P)$ and raises $\phi(P^o, P)$. If $P^o > P$ then convexity delivers the required property.

6.2 Appendix B

$D(P)$ is a firm's demand when a common price of P is set.

Lemma 7 *Assume A1-A3; i) $D(\cdot)$ is twice continuously differentiable, $D'(P) \leq 0$ and if $D(P) > 0$ then $D'(P) < 0$, and $D''(P) \leq 0$; ii) $\pi(P) = (P - c) D(P)$; iii) $x(P) = \max\left\{\left(P - \widehat{P}\right) D(P)\right\}$; and iv) $\widehat{P} \geq c$. Then $\pi(\xi(x))$ is strictly concave in x, $\forall x \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right)$.*

Proof: First take the total derivative of $x = \gamma\left[\xi(x) - \widehat{P}\right] D(\xi(x))$ with respect to x:

$$\begin{aligned} 1 &= \xi'(x) \gamma D(\xi(x)) + \left[\xi(x) - \widehat{P}\right] \gamma D'(\xi(x)) \xi'(x) \Leftrightarrow \\ \xi'(x) &= 1/\gamma\left[D(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D'(\xi(x))\right] > 0. \end{aligned} \tag{38}$$

The second derivative is:

$$\begin{aligned} \xi''(x) &= -\frac{\left[2D'(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D''(\xi(x))\right] \xi'(x)}{\gamma\left[D(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D'(\xi(x))\right]^2} \\ &= -\gamma\left[2D'(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D''(\xi(x))\right] \left(\xi'(x)\right)^3 > 0. \end{aligned} \tag{39}$$

Taking the first two derivatives of $\pi(\xi(x)) = [\xi(x) - c] D(\xi(x))$ with respect to x:

$$\frac{d\pi(\xi(x))}{dx} = \left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] \xi'(x)$$

$$\begin{aligned} \frac{d^2\pi(\xi(x))}{dx^2} &= \left[2D'(\xi(x)) + (\xi(x) - c) D''(\xi(x))\right] \left(\xi'(x)\right)^2 \\ &\quad + \left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] \xi''(x). \end{aligned}$$

Substituting (38)-(39):

$$\begin{aligned}
\frac{d^2\pi(\xi(x))}{dx^2} &= \left[2D'(\xi(x)) + (\xi(x) - c) D''(\xi(x))\right] (\xi'(x))^2 \\
&\quad - \left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] \times \\
&\quad \gamma \left[2D'(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D''(\xi(x))\right] (\xi'(x))^3 \\
&= (\xi'(x))^2 \{\left[2D'(\xi(x)) + (\xi(x) - c) D''(\xi(x))\right] \\
&\quad - \left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] \left[2D'(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D''(\xi(x))\right] \times \\
&\quad \left[1/\left(D(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D'(\xi(x))\right)\right]\}.
\end{aligned}$$

Since $\xi(x) < P^m$ and $\widehat{P} \geq c$, A3 implies $D(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D'(\xi(x)) > 0$. The sign of $d^2\pi(\xi(x))/dx^2$ doesn't then change if we multiply through by $D(\xi(x)) + \left(\xi(x) - \widehat{P}\right) D'(\xi(x))$. Doing so, $d^2\pi(\xi(x))/dx^2 \leq 0$ iff $\chi\left(\widehat{P}\right) \geq 0$ where

$$\begin{aligned}
\chi(P) \equiv & \left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] \left[2D'(\xi(x)) + (\xi(x) - P) D''(\xi(x))\right] \\
& - \left[D(\xi(x)) + (\xi(x) - P) D'(\xi(x))\right] \left[2D'(\xi(x)) + (\xi(x) - c) D''(\xi(x))\right].
\end{aligned}$$

Next note that

$$\chi'(P) = \begin{array}{l} -\left[D(\xi(x)) + (\xi(x) - c) D'(\xi(x))\right] D''(\xi(x)) \\ +D'(\xi(x)) \left[2D'(\xi(x)) + (\xi(x) - c) D''(\xi(x))\right] \end{array} > 0.$$

Given that $\chi(c) = 0$ and $\widehat{P} \geq c$ then $\chi\left(\widehat{P}\right) \geq 0$. ♦

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