# Collusion with Monitoring Based on Self-Reported Sales*

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March 10, 2008 (Preliminary)

**Abstract**

Motivated by some recent cartels, this paper characterizes a stable collusive mechanism when firms' prices and quantities are private information. An equilibrium is described in which firms truthfully report their sales and then make transfers based on these reports. Even though a higher sales report requires a higher transfer by a firm, the threat of a price war when the aggregate sales report is low induces accurate self-reporting. This strategy profile is an equilibrium when firms are sufficiently patient and market demand is not too stochastic.

## 1 Introduction

In Harrington and Skrzypacz (2007), we consider an oligopoly setting in which firms seek to collude in a price-setting environment when prices are private information. Such a scenario naturally arises when customers are industrial buyers so that prices are, in principle, privately negotiated and thus not publicly observed. Though prices are private information, we assumed firms' quantities were public information. Focusing upon strongly symmetric subgame perfect equilibria (along with some other mild refinements), we showed for a particular class of stochastic demand structures (which included the discrete choice model) that prices in excess of the stage game

*We appreciate the comments of seminar participants at Cornell. The first author acknowledges the support of the National Science Foundation under grant SES-0516943.

Nash equilibrium price are not sustainable for any discount factor (no matter how close it is to one). Loosely speaking, the threat of a price war cannot support collusion. Though this extreme result was delivered under the extreme assumption that market demand is perfectly inelastic, it is robust in the sense that collusive prices are very close to their non-collusive levels when market demand is very inelastic.

The lesson we took from that result was not that firms cannot collude - for some actual price-fixing cases motivated our analysis - but that collusion requires the use of asymmetric punishments. We then showed that a scheme in which a cartel member makes a transfer to the other firms for each unit that it sells will support collusion if firms are sufficiently patient. This type of scheme could be implemented through inter-firm sales which was a practice used in cartels such as in citric acid, lysine, and vitamins.[1]

A key assumption in sustaining collusion with this asymmetric punishment was that firms' realized quantities were public information (and thus transfers could be conditioned on them). This assumption is problematic, however. In a number of well-documented cartels, members *self-reported* their sales, and transfers were made based upon those reports. For example, in the lysine cartel, Kanji Mimoto of Ajinomoto was assigned the task of preparing monthly "scorecards" for the cartel. Each company telephoned or mailed their sales volumes to Mimoto, who then prepared a spreadsheet that was handed out and discussed at the quarterly maintenance meetings. A firm that reported sales above its quota was required to buy output from a firm that reported sales below its quota. In the zinc phosphate cartels of the mid-1990s, firms reported monthly their sales to a trade association, which would aggregate the reports to produce total market sales and report it back to the firms. Firms would then meet and base compensation based on that information.

These self-reporting practices raise the question about the incentives for a firm to accurately reveal its sales to the other members of the cartel. Given the length and effectiveness of these cartels, firms appear to have treated these self-reports as if they were, on the whole, accurate. Nevertheless, there were episodes in which the reports were inaccurate. In the lysine cartel, one cartel member (Cheil) claimed to the European Commission that it reported "misleading" sales information to the other companies, while there is evidence that Ajinomoto "hid" 3,500 tons of lysine from the cartel's auditors; for example, an internal memo read: "Hide 1,000 tons in Thailand internal business."

The objective of this paper is to explain these documented cartel practices. To what extent can firms collude when neither prices nor quantities are public information? Is there a mechanism that will induce cartel members to truthfully report their sales? And if sales are truthfully reported, can punishments be based upon them to enforce collusion? What are the implications for observable behavior of such a mechanism? To address these questions, we return to the model of our previous paper but now assume that both firms' prices and quantities are private information.

[1] These and other ensuing facts on cartels are from Harrington (2006).

Broadly speaking, this informational structure is not new to the repeated game literature. A number of Folk Theorems have been developed for when none of the history is common knowledge. Prior to performing our analysis, a brief review of these Folk Theorems is provided in Section 2 and we discuss how our game and collusive mechanism differ from that literature (and why that literature is inadequate to explaining cartel behavior). The model is described in Section 3. As a benchmark, the static Nash equilibrium is characterized in Section 4. The main result is in Section 5 where we show that, under certain conditions on the demand structure, collusion can be sustained when firms are sufficiently patient. The collusive mechanism is a blend of transfers and a probabilistic price war. A firm is to provide a monetary transfer to the other firms for each unit of sales that it reports and, so as to induce truthful reporting, the probability of a price war is decreasing in total reported sales. Section 6 focuses on the special case when market demand is zero or one unit and derives an optimal collusive equilibrium. Comparative statics are derived such as the probability of a price war is increasing in the number of firms. It is also shown that the our main result is robust to when market demand is sensitive to firms' prices. Section 7 concludes.

## 2 Folk Theorems with Private Monitoring

The literature on repeated games with private monitoring focuses on developing Folk Theorems when none of the history is common knowledge. This literature can be partitioned into work that allows players to communicate through cheap talk messages and work that does not (an example of the latter is Hörner and Olszewski, 2006). As cartels did engage in costless communication - and our mechanism will allow for it - we will limit our attention to reviewing research on private monitoring with communication.[2]

Consider a repeated game in which, in each period, a player receives a private signal of the actions selected by the other players in the previous period and then players simultaneously choose actions. A player's private history comprises her past signals and actions; there is no public history. A player's payoff depends only on her action and signal so, once observing the signal, the payoff contains no information about other players' actions. Now, augment this structure by allowing players to send costless messages. After observing their private signals and prior to choosing an action, players simultaneously send messages that are publicly observed. Based on these messages, monetary payments are allowed between players and, in some cases, "burning money" is permitted (that is, the net transfers are negative).

In developing a Folk Theorem, there are three primary issues that need to be addressed: 1) Are the private signals sufficient to statistically detect cheating by a

[2] For earlier reviews of this literature, see Kandori (2002) and Mailath and Samuelson (2006).

player?;[3] 2) How are players induced to truthfully reveal their private signals?; and 3) If punishments occur in equilibrium with positive probability, how is efficiency achieved? In our brief survey here, we will touch on how these issues are tackled by the literature.

In a setting with three or more players, Kandori and Matshushima (1998) address the first issue by assuming that, for any pair of players, the remaining players' private signals can statistically determine which of those two players cheated. To deal with the second issue, they use the trick of making a player's net payment depend only on the messages of the other players. Since a player's net payment is independent of her own message, she has a (weak) incentive to truthfully reveal her private signal. Finally, efficiency is achieved as the scheme has players make pair-wise transfers so there is no burning money in equilibrium (and no other form of punishment). All this delivers a Folk Theorem. From our perspective of explaining cartel practices, there are three weaknesses to this scheme. First, it requires at least three players.[4] Second, that a firm's payment does not depend on its own reported sales is inconsistent with practices in the cartels mentioned above. Third, the statistical detection assumption is strong and, in the oligopoly context, is unlikely to be satisfied. For example, if firm 1 cheats by pricing low in a price game, firm 3 may be able to statistically detect that someone cheated - because firm 3's quantity is low - but will be unable to determine whether it was firm 1 or 2.[5]

Compte (1998) assumes that a player can statistically detect cheating but does not require statistical determination of who cheated; thus doing away with the strong assumption of Kandori and Matshushima (1998). However, this is at the cost of requiring that players' signals are independent. Again, players make payments based on reports though these transfers do not necessarily balance which means that money is burned so there is an inefficiency. That inefficiency is eliminated, however, by having players exchange informative messages with delay. As information is exchanged and money is burnt in the distant future, the inefficiency disappears as the discount factor goes to one (and the delay goes to infinity). To what extent delay corresponds to actual cartel practice is an open (and quite fascinating) question. Of more serious concern in applying the Folk Theorem of Compte (1998) to the oligopoly setting is the assumption that players' signals are independent. In a price-setting oligopoly game, a firm's signal is its sales and one would expect firms' sales to be positively correlated due to, for example, market demand shocks.

The two previous papers deal with a generic game with finite action spaces. Aoyagi (2002), in contrast, considers the Bertrand price game where firms' demands are

[3] That is, is the probability distribution on a signal sensitive to other players' actions?

[4] They also provide a different scheme for when there are two players which is related to the work of Compte (1998) which we review next.

[5] They correctly note that if firms' products are asymmetrically differentiated then their assumption will be satisifed. But that is of value to explaining actual cartel practices only if the degree of asymmetric differentiation is not trivial. As most price-fixing cartels involve homogeneous commodities - such as chemicals and vitamins - their assumption is not particularly palatable.

affiliated and their private signals are correlated. To get a Folk Theorem, he needs to assume that firm demand is discontinuous in price at the equilibrium price vector. Furthermore, the structure of equilibrium is rather unnatural. The equilibrium partitions the horizon into $T$ components - where component $k$ is made up of periods $k$, $T+k$, $2T+k, \ldots$ - and a firm's behavior is assumed to influence future behavior only within a component. For example, if a firm prices low in period $k$ and this produces low sales for the other firms, the induced punishment is confined to periods $T+k$, $2T+k, \ldots$. There is no evidence this type of mechanism corresponds with actual cartel practices.

For two-player games, Fudenberg and Levine (2007) derive a Folk Theorem when there is "almost public messaging." Though players have private signals, these signals are highly correlated so that, given a pair of message functions (that map private signals into messages), the probability that both players send the same message is arbitrarily close to one. This is a nice result to establish the robustness of the Folk Theorem with public signals, but it is far less useful to our more applied problem of sustaining collusion when prices and quantities are private information and there is no almost public signal.

Finally, Obara (2007) has probably the most compelling set-up. His model is as in Compte (1998) except that he dispenses with the restriction that players' signals are independent; now they can be correlated. He assumes, for each player, there is at least one player that can statistically detect a deviation by that player. Like Compte (1998), his equilibrium requires that informative messages occur with delay. The downside to his equilibrium is its asymmetric structure in that different players perform different roles. For example, in the two-player setting, player 1 (say) provides a payment to player 2 which is based only on player 1's signal; thus, player 2 has a weak incentive to provide an informative message. However, player 1 receives a payment based on both players' signals. To induce him to be truthful, there is a probabilistic punishment based on player 1's message. This ex ante asymmetric treatment of players does not fit well with what we know about cartel practices.

In contrasting our contribution with the body of work just reviewed, it is first important to distinguish our objectives. The focus of previous work is to characterize the set of equilibrium payoffs. In contrast, our focus is to explain observed cartel behavior. While we share the objective of deriving conditions whereby collusion is sustainable, it differs in that this will be of little value to us if the equilibrium that generates collusion is inconsistent with what we know about cartel behavior. Thus, a primary objective for us is to construct an empirically valid equilibrium which sustains collusion.

More formally, the game we analyze is not a member of the games considered in previous work on private monitoring with communication. This is most directly the case because our action space is infinite - firms can choose any non-negative price - while previous work assumes a finite action space (excluding Aoyagi, 2002). Though that difference is more technical than fundamental, it does mean we cannot simply apply one of these Folk Theorems. More substantively, the equilibrium that we construct to support collusion differs in that it can work with any number of firms, firms' private signals are correlated, informative messages do not occur with delay, monetary payments balance out, and there is no ex ante asymmetric treatment of firms. We believe the equilibrium strategy profile conforms reasonably well with what we know about cartel behavior.

# 3 Model

There are $n \geq 2$ firms who, in each period of an infinite horizon setting, simultaneously choose price from a compact set, after which each firm's sales are stochastically determined. (As is typical, it is assumed that a firm supplies to meet demand.) The stochastic realization for period $t$ is composed of total demand, $m^t$, and an allocation of that demand described by a vector of firms' quantities, $\underline{q}^t$. Market demand is integer-valued and lies in the finite set

$$\Gamma \equiv \{\underline{m}, \underline{m}+1, \ldots, \overline{m}-1, \overline{m}\}$$

where $0 \leq \underline{m} < \overline{m}$ and $\overline{m}$ is finite. Let $\rho : \Gamma \rightarrow [0,1]$ be a probability function where $\rho(m)$ is the probability that total demand is $m$. Define

$$\mu \equiv \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\, m$$

as average market sales. Note that market demand does not depend on firms' prices. While firm demand will be sensitive to price, market demand is perfectly inelastic. Firms have a common constant marginal cost of $c$.

An individual firm's demand has support $\{0, 1, \ldots, \overline{q}\}$ where $\overline{q}$ is finite and, of course, $\overline{q} \leq \overline{m}$. (Assuming the lower bound to firm sales is zero is not important.) Let $\Psi\left(\underline{q}; m, \underline{p}\right)$ denote the probability that the quantity vector is $\underline{q}$ given price vector $\underline{p}$ and total sales. To allow us to focus on symmetric equilibria, A1 is assumed.

**A1** $\Psi\left(\underline{q}; m, \underline{p}\right) = \Psi\left(\omega\left(\underline{q}; i, j\right); m, \omega\left(\underline{p}; i, j\right)\right)\ \forall i, j,\ \forall\left(\underline{q}, \underline{p}\right)$, where $\omega\left(\underline{q}; i, j\right)$ is the vector $\underline{q}$ when elements $i$ and $j$ are exchanged.

More convenient for our purposes is to work with $\psi_i\left(q; m, \underline{p}\right)$ which is the probability function on firm $i$'s sales given total demand is $m$ and the price vector. $\sigma_i\left(\cdot; q, \underline{p}\right)$ is a firm's beliefs on total sales given its sales is $q$ and the price vector. By Bayes Rule,

$$\sigma_i\left(m; q, \underline{p}\right) = \frac{\rho(m)\, \psi_i\left(q; m, \underline{p}\right)}{\sum_{m'=\underline{m}}^{\overline{m}} \rho(m')\, \psi_i\left(q; m', \underline{p}\right)}.$$

Two conditions are required of $\sigma_i$ (which implicitly places conditions on $\rho$ and $\psi_i$ ). A2 specifies that a firm always assigns positive probability to demand equalling its maximum value. This is weaker than assuming $\sigma_i$ has full support. A3 assumes that the higher is a firm's quantity, the more weight the firm attaches to total demand being higher.

**A2** $\sigma_i\left(\overline{m}; q, \underline{p}\right) > 0$, $\forall q$, $\forall \underline{p}$.

**A3** If $q' > q''$ then $\sigma_i\left(\cdot; q', \underline{p}\right)$ first-order stochastically dominates (FOSD) $\sigma_i\left(\cdot; q'', \underline{p}\right)$, $\forall q', q''$, $\forall \underline{p}$.

The setting is an infinitely repeated game in which, in each period, firms choose price and then stochastic demand is realized. Let $\delta$ be a common discount factor and assume a firm acts to maximize the expected present value of its profit stream. Each firm's price and realized sales is private information. This structure is augmented by allowing firms to make public messages and conduct monetary transfers.

**Stage 1 (price)** Each firm chooses price.

**Stage 2** With prices being private information, demand is realized and each firm learns its sales for that period.

**Stage 3 (report)** With prices and quantities being private information, each firm submits a publicly observed costless message (where a message is to be interpreted as a sales report).

**Stage 4 (transfer)** With prices and quantities being private information but reports being public information, each firm makes a payment to the other $n-1$ firms.

Rather than work with this 4-stage structure, it will simplify the analysis if we compress the report and transfer stages into one stage. After presenting and proving our main result, we will argue that the proofs work for the 4-stage game as well. The extensive form game that we will then work with is below. For the equilibrium described in Section 5, it will be made clear that there is an implicit sales report associated with a firm's transfer in stage 3.

**Stage 1 (price)** Each firm chooses price.

**Stage 2** With prices being private information, demand is realized and each firm learns its sales for that period.

**Stage 3 (report/transfer)** With prices and quantities being private information, each firm makes a payment to the other $n-1$ firms.

This structure is the same as in Harrington and Skrzypacz (2007) with three exceptions. First, firms' quantities are private information, which is the heart of the problem we're addressing here. Second, market demand is stochastic, which is empirically compelling and is required to make the problem interesting (this is explained in Section 5). Third, each firm sets a single price for all customers instead of a customer-specific price. The latter assumption appears to be more for convenience though a more careful assessment of that claim is needed.

# 4 Static Nash Equilibrium

Before considering the infinitely repeated setting, let's establish the non-collusive benchmark. Firm $i$'s expected profit is

$$\pi_i(p_1,\ldots,p_n) = \sum_{m=\underline{m}}^{\overline{m}} (p_i - c)\, q_i \rho(m)\, \psi_i\left(q; m, \underline{p}\right)$$

Let $p^N(c)$ denote a symmetric Nash equilibrium.

$$p^N(c) \in \arg\max \sum_{m=\underline{m}}^{\overline{m}} (p_i - c)\, q_i \rho(m)\, \psi_i\left(q; m, p^N, \ldots, p_i, \ldots, p^N\right).$$

As specified as assumption A4, we will assume the demand structure allows for a symmetric Nash equilibrium for the static game.

**A4** The static game with cost $c$ has, $\forall c \geq 0$, a symmetric Nash equilibrium price $p^N(c)$ that is increasing and unbounded in $c$.

By way of example, suppose that, with probability $\rho(m)$, there are $m$ consumers and consumers' decisions as to the firm from which to buy are independent. Let $f(p_i, p_{-i})$ denote the probability that a customer buys from firm $i$ given firm $i$ charges $p_i$ and each of the other $n-1$ firms charges a common price of $p_{-i}$. Assume $f(p_i, p_{-i})$ is decreasing in $p_i$, increasing in $p_{-i}$, and (so as to satisfy symmetry), $f(p,p) = 1/n$. Given this binomial structure to demand, it is shown in the Appendix that the first-order condition implies:

$$p^N = c - \left(\frac{1}{n}\right)\left(\frac{1}{\partial f\left(p^N, p^N\right)/\partial p_i}\right). \tag{1}$$

Further assume $f$ is locally linear in a firm's own price at a common price vector. For example, suppose when $p_i$ and $p_{-i}$ are not too different that

$$f(p_i, p_{-i}) = \frac{1}{n} - bp_i + bp_{-i}, \tag{2}$$

where $b > 0$. Using (2) in (1), we have

$$p^N = c + \left(\frac{1}{bn}\right). \tag{3}$$

(3) satisfies A4.

# 5 Limited Folk Theorem

Before moving on to the main result, let us first note that sustaining collusion is trivial when market demand is common knowledge, either because it is not stochastic or is stochastic but is public information. Suppose it is common knowledge that market demand is $M$. If firm $i$ expects the other firms to make truthful sales reports then firm $i$ knows that if its report is incorrect then total reported sales will differ from $M$. Though it will not be known who delivered a misleading sales report, it will be common knowledge among firms that someone did. Thus, if there is a punishment when total reported sales differ from $M$, firms can be induced to report truthfully if they are sufficiently patient. With that property, the pricing mechanism in Harrington and Skrzypacz (2007) can sustain collusion.[6] From hereon, it is assumed market demand is stochastic and unobserved.

Consider a symmetric strategy profile of the following form. There are two phases: collusive and non-collusive. If the industry is in the collusive phase in period $t$, then a firm sets the collusive price in the price stage. In the report/transfer stage, it pays an amount $zq_i^t$ that is to be shared equally among the other $n-1$ firms, where $q_i^t$ is the realized sales of firm $i$ in period $t$ and $z > 0$ is the per unit transfer. More generally, if the transfer made by firm $i$ in period $t$ is $x_i^t$ then define $r_i^t \equiv x_i^t/z$ as the implicit sales report. Given the vector of sales reports, $(r_1^t, \ldots, r_n^t)$, a public randomization device determines whether the industry remains in the collusive phase or shifts to the non-collusive phase. The probability function for shifting to the non-collusive phase is $\phi : \{0, 1, 2, \ldots\} \rightarrow [0, 1]$. If the industry is instead in the non-collusive phase, firms price at the static Nash equilibrium in the price stage and transfer zero in the report/transfer stage. This strategy profile is summarized below.

- In the price stage,
  - price at $\widehat{p}$ if in the collusive phase
  - price at $p^N$ if in the non-collusive phase
- In the report/transfer stage,
  - if in the collusive phase then transfer $q_i^t z$ (and implicitly provide a sales report of $q_i^t$)

[6] This pricing mechanism is described in the proof of Theorem 1, as we deploy it here as well.

– if in the non-collusive phase then transfer zero

- Transition stage (public randomization device). After transfers are made,
  - if in the collusive phase then all firms remain in the collusive phase with probability $1-\phi\left(\sum_{j=1}^{n} r_j^t\right)$ and shift to the non-collusive phase with probability $\phi\left(\sum_{j=1}^{n} r_j^t\right)$, where $r_i^t = x_i^t/z$ and $x_i^t$ is the transfer made by firm $i$.
  - if in the non-collusive phase then all firms remain in the non-collusive phase.

This strategy has a firm's behavior depend only on the public history which is made up of all past sales reports and all past realizations of the public randomization device. We will refer to it as PPE1 or "perfect public equilibrium" one. In the ensuing discussion, when we say a firm "reports" sales of $q'$, we mean that a firm makes a transfer of $q'z$.

Theorem 1 provides a limited Folk Theorem in that collusion is sustainable only for a class of demand structures. More specifically, if firms are sufficiently patient *and* (4) holds then a collusive price $\widehat{p}$ is supportable using PPE1 and, in addition, the probability of transiting to the non-collusive phase is arbitrarily small. In interpreting (4), recall that $\overline{m}$ is the maximal value of market demand and $\mu$ is average market demand. As $p^{N^{-1}}(\cdot)$ is the inverse of the static Nash equilibrium price function, then $p^{N^{-1}}(\widehat{p})$ is the marginal cost for which the static Nash equilibrium price equals $\widehat{p}$. Since $\widehat{p} > p^N(c)$, then $p^{N^{-1}}(\widehat{p}) - c > 0$. (4) will hold if $\rho(\cdot)$ puts sufficient mass on maximal demand so that average demand is close to maximal demand. Given that the lhs is unbounded as $\overline{m} - \mu \to 0$ and the rhs is bounded then (4) would be satisfied. Note that as more and more mass is placed on a particular value (such as $\overline{m}$), demand is approaching the deterministic case. Thus, Theorem 1 shows that if demand is not "too stochastic" - in the restricted sense that average demand is not too much lower than maximal demand - then collusion can be sustained when firms are sufficiently patient.

**Theorem 1** *For any $\varepsilon > 0$ and $\widehat{p} > p^N$, there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ and*

$$\frac{\mu}{\overline{m}-\mu} > \frac{(n-1)\left[p^{N^{-1}}(\widehat{p}) - c\right]}{\widehat{p} - p^N} \tag{4}$$

*then PPE1 is a perfect public equilibrium and*

$$max\left\{\phi(m) : \underline{m} \leq m \leq \overline{m}\right\} < \varepsilon.$$

**Proof**: In establishing that PPE1 is a perfect public equilibrium, we need to establish that: i) a firm's prescribed price is optimal for every history where a history is made of all firms' past sales reports (or, equivalently, transfers), past realizations of the public randomization device, and a firm's own past prices; and ii) a firm's prescribed report is optimal for every history where a history is made of all firms' past sales reports, past realizations of the public randomization device, and a firm's own past prices including the current period's price. As a firm does not observe other firms' past prices, we need to specify a firm's beliefs over other firms' past prices and we will assume they put unit mass on $\widehat{p}$ when firms were in the collusive phase and on $p^N$ when in the non-collusive phase. As this strategy is clearly optimal when in the non-collusive phase, our attention is focused on when firms are in the collusive phase.

By symmetry, we can restrict our analysis to firm 1. Given a generic price vector $\underline{p}$ and the anticipation that transfers will be made as specified in PPE1, firm 1's payoff at the price stage is

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q; m, \underline{p}) \left\{ \left[ (p_1 - c) q + z \left( \frac{m-q}{n-1} \right) - zq \right] + \phi(m) \delta V^N + (1 - \phi(m)) \delta V \right\}. \tag{5}$$

$V$ is the value when in the collusive phase and $V^N$ is the value when in the non-collusive phase. Note that a firm with sales $q$ will expect to pay $zq$ while receiving an equal share of the payments made by the other $n-1$ firms which total $z(m-q)$. Simplifying this expression, we derive

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q; m, \underline{p}) \left( p_1 - c - \left( \frac{n}{n-1} \right) z \right) q$$
$$+ \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[ \phi(m) \delta V^N + (1 - \phi(m)) \delta V \right] + z \left( \frac{\mu}{n-1} \right)$$

Thus, if the collusive price is $\widehat{p}$ then the incentive compatibility constraint (ICC) at the price stage is

$$
\begin{aligned}
& \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1\left(q; m, \underline{\widehat{p}}\right) \left(\widehat{p} - c - \left(\frac{n}{n-1}\right) z\right) q \\
& + \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\phi(m)\, \delta V^N + (1-\phi(m))\, \delta V\right] + z\left(\frac{\mu}{n-1}\right) \\
\geq\ & \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1\left(q; m, p_1, \widehat{p}, \ldots, \widehat{p}\right) \left(p_1 - c - \left(\frac{n}{n-1}\right) z\right) q \\
& + \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\phi(m)\, \delta V^N + (1-\phi(m))\, \delta V\right] + z\left(\frac{\mu}{n-1}\right), \ \forall p_1
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1\left(q; m, \underline{\widehat{p}}\right) \left(\widehat{p} - c - \left(\frac{n}{n-1}\right) z\right) q \\
\geq\ & \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1\left(q; m, p_1, \widehat{p}, \ldots, \widehat{p}\right) \left(p_1 - c - \left(\frac{n}{n-1}\right) z\right) q, \ \forall p_1
\end{aligned}
\tag{6}
$$

Thus, the ICC is satisfied iff $\widehat{p} = p^N\left(c + \left(\frac{n}{n-1}\right) z\right)$; that is, $\widehat{p}$ is the static Nash equilibrium when unit cost is $c + \left(\frac{n}{n-1}\right) z$. For any $z$, we will then assume $\widehat{p} = p^N\left(c + \left(\frac{n}{n-1}\right) z\right)$. Since $p^N\left(c + \left(\frac{n}{n-1}\right) z\right)$ is increasing and unbounded in $z$ by A4, then for any $p > p^N$ there exists $z > 0$ such that $p = p^N\left(c + \left(\frac{n}{n-1}\right) z\right)$.

Taking account of the fact that transfers average out to zero, expected collusive profit is

$$
\widehat{\pi}(z) \equiv \left[p^N\left(c + \left(\frac{n}{n-1}\right) z\right) - c\right]\left(\frac{\mu}{n}\right)
$$

The collusive value is recursively defined by:

$$
V = \widehat{\pi}(z) + \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\phi(m)\, \delta V^N + (1-\phi(m))\, \delta V\right].
$$

The incremental gain from being in the collusive phase is

$$
\begin{aligned}
V - V^N &= \widehat{\pi}(z) + \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\phi(m)\, \delta V^N + (1-\phi(m))\, \delta V\right] - \widehat{\pi}(0) - \delta V^N \\
&= \widehat{\pi}(z) - \widehat{\pi}(0) - \delta\left(V - V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\, \phi(m) + \delta\left(V - V^N\right)
\end{aligned}
$$

$$V - V^N = \frac{\widehat{\pi}(z) - \widehat{\pi}(0)}{1 - \delta + \delta \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \phi(m)} = \frac{\left[p^N \left(c + \left(\frac{n}{n-1}\right) z\right) - p^N(c)\right](\mu/n)}{1 - \delta + \delta \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \phi(m)}. \quad (7)$$

This expression will be useful later in the proof.

The next step is to consider the optimality of a firm's strategy at the report/transfer stage. If in the collusive phase then firm 1's beliefs put unit mass on other firms' prices equalling $\widehat{p}$. For firm 1's strategy to be sequentially rational (when in the collusive phase), the prescribed report must be optimal given the other firms' current period price was $\widehat{p}$, any arbitrary price for firm 1, and any feasible level of realized sales for firm 1, which is denoted $q_1$.

To begin, recall that $\sigma_1\left(m \left| q_1, \underline{p}\right.\right)$ denotes firm 1's posterior beliefs on total market sales conditional on its quantity and the price vector. Given the other firms are expected to provide a truthful report/transfer, firm 1's expected payoff from reporting $r_1$ (when its true sales is $q_1$) is[7]

$$\begin{aligned} W(r_1; q_1, p_1) &\equiv \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m \left| q_1, \underline{p}\right.\right) \left\{\left[(p_1 - c) q_1 + z\left(\left(\frac{m - q_1}{n-1}\right) - r_1\right)\right]\right. \\ &\quad \left. + \phi(m - q_1 + r_1) \delta V^N + (1 - \phi(m - q_1 + r_1)) \delta V\right\} \\ &= \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m \left| q_1, \underline{p}\right.\right) \left\{\left[(p_1 - c) q_1 + \left(\frac{z}{n-1}\right)(m - q_1 - (n-1) r_1)\right]\right. \\ &\quad \left. + \phi(m - q_1 + r_1) \delta V^N + (1 - \phi(m - q_1 + r_1)) \delta V\right\} \end{aligned}$$

The ICCs are:

$$W(q_1; q_1, p_1) \geq W(r_1; q_{1,}, p_1) \;\; \forall r_1 \in \{0, 1, \ldots\}, \;\; \forall q_1 \in \{0, 1, \ldots, \overline{q}\}, \;\; \forall p_1.$$

In other words, for any price set by firm 1 and any realized sales for firm 1, firm 1 finds it optimal to truthfully report its sales. Our analysis proceeds through several steps. First, we derive a condition ensuring that it is not optimal to over-report sales; that is, $r_1 = q_1$ is preferable to any $r_1 > q_1$. In then deriving a condition ensuring that it is not optimal to under-report - that is, $r_1 = q_1$ is preferable to any $r_1 < q_1$ - we first derive a condition whereby if it is not optimal to under-report by one unit then it is not optimal to under-report by any amount. Then we derive a condition whereby if it is not optimal to under-report by one unit when $q_1 = \overline{q}$ (so a firm's sales are at its maximum level) then it is not optimal to under-report by one unit when $q_1 < \overline{q}$. We are then left with the property,

$$W(\overline{q}; \overline{q}, p_1) \geq W(\overline{q} - 1; \overline{q}, p_1), \;\; \forall p_1,$$

[7] Though perfect public equilibrium only requires that the ICC hold when other firms price at $\widehat{p}$, we will allow for any price vector. Though the added generality is not of any value, it reduces the amount of notation.

then all under-reporting ICCs at the report stage are satisfied. The Folk Theorem is derived by examining this ICC as $\delta \to 1$. As we'll see, this requires imposing certain properties on $\phi$.

Let us start by ensuring that firm 1 prefers to provide a truthful sales report to over-reporting sales. The ICC is

$$\begin{aligned} & \sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|q_1, \underline{p}\right.\right)\left\{\left[\left(p_1-c\right) q_1+\left(\frac{z}{n-1}\right)\left(m-n q_1\right)\right]\right. \\ & \left.+\phi(m) \delta V^N+(1-\phi(m)) \delta V\right\} \\ \geq & \sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|q_1, \underline{p}\right.\right)\left\{\left[\left(p_1-c\right) q_1+\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1) r_1\right)\right]\right. \\ & \left.+\phi\left(m-q_1+r_1\right) \delta V^N+\left(1-\phi\left(m-q_1+r_1\right)\right) \delta V\right\}, \forall r_1>q_1 . \end{aligned} \tag{8}$$

Performing some manipulations,

$$\begin{aligned} & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right) \times \\ & \left[\phi(m) \delta V^N+(1-\phi(m)) \delta V-\phi\left(m-q_1+r_1\right) \delta V^N-\left(1-\phi\left(m-q_1+r_1\right)\right) \delta V\right] \\ \geq & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1) r_1-m+n q_1\right) \end{aligned}$$

$$\delta\left(V-V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi(m)\right] \geq \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right) z\left(q_1-r_1\right)$$

$$\delta\left(V-V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi(m)\right] \geq z\left(q_1-r_1\right) \tag{9}$$

Interpreting (9), the rhs is the change in transfer from reporting $r_1$ instead of $q_1$. As $z>0$ and $r_1>q_1$, it is negative. The lhs is the expected change in the future payoff due to over-reporting. It captures the impact of an over-report on the probability of transitioning to the non-collusive phase. Equilibrium requires that the change in the expected future payoff is at least as great as the reduction in the current payoff from making a higher payment. As the rhs of (9) is negative, a sufficient condition for (9) to hold is then:

$$\delta\left(V-V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi(m)\right] \geq 0 \tag{10}$$

We will return to this condition later in the proof.

Next, consider the case of under-reporting. The ICCs are

$$W(q_1; q_1, p_1) \geq W(r_1; q_1, p_1) \;\; \forall r_1 \in \{0, 1, \ldots, q_1 - 1\}, \; \forall q_1 \in \{0, 1, \ldots, \overline{q}\}, \; \forall p_1.$$

The proof strategy entails providing sufficient conditions whereby the binding ICC is when $r_1 = \overline{q}$ is preferred to $r_1 = \overline{q} - 1$, given $q_1 = \overline{q}$ :

$$W(\overline{q}; \overline{q}, p_1) \geq W(\overline{q} - 1; \overline{q}, p_1).$$

This is achieved through two steps. First, the derivation of sufficient conditions whereby,

$$\begin{aligned} \text{if } W(q_1; q_1, p_1) &\geq W(q_1 - 1; q_1, p_1) \text{ then } W(r_1; q_1, p_1) \geq W(r_1 - 1; q_1, p_1) \\ \forall r_1 &\in \{0, 1, \ldots, q_1 - 1\}, \; \forall q_1, \; \forall p_1. \end{aligned}$$

In other words, if firm 1 prefers to tell the truth than to under-report by one unit, then it prefers to under-report by $k$ units than to under-report by $k + 1$ units. By transitivity, this implies that if it prefers to tell the truth than to under-report by one unit, then it prefers to tell the truth than to under-report by any amount. With this step, we then just need to show that, for any $q_1$, firm 1 prefers to report the truth than to under-report by one unit:

$$W(q_1; q_1, p_1) \geq W(q_1 - 1; q_1, p_1) \;\; \forall q_1, \; \forall p_1.$$

The second step is to derive conditions whereby

$$\text{if } W(\overline{q}; \overline{q}, p_1) \geq W(\overline{q} - 1; \overline{q}, p_1) \text{ then } W(q_1; q_1, p_1) \geq W(q_1 - 1; q_1, p_1) \;\; \forall q_1, \; \forall p_1.$$

Thus, if it is not optimal to under-report by one unit when $q_1 = \overline{q}$, then it is not optimal to under-report by one unit for any $q_1$.

Let's begin with the first step. Given $q_1$, reporting $r_1$ is preferred to $r_1 - 1$, where $r_1 \leq q_1$, iff

$$\begin{aligned} &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1(m | q_1, \underline{p}) \left\{ \left[ (p_1 - c) q_1 + \left( \frac{z}{n-1} \right) (m - q_1 - (n-1) r_1) \right] \right. \\ &\left. + \phi(m - q_1 + r_1) \delta V^N + (1 - \phi(m - q_1 + r_1)) \delta V \right\} \\ \geq &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1(m | q_1, \underline{p}) \left\{ \left[ (p_1 - c) q_1 + \left( \frac{z}{n-1} \right) (m - q_1 - (n-1)(r_1 - 1)) \right] \right. \\ &\left. + \phi(m - q_1 + r_1 - 1) \delta V^N + (1 - \phi(m - q_1 + r_1 - 1)) \delta V \right\} \end{aligned} \tag{11}$$

Performing some manipulations,

$$\begin{aligned} &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1(m | q_1, \underline{p}) \left[ \left( \frac{z}{n-1} \right) (m - q_1 - (n-1) r_1) - \phi(m - q_1 + r_1) \delta \left( V - V^N \right) \right] \\ \geq &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1(m | q_1, \underline{p}) \left[ \left( \frac{z}{n-1} \right) (m - q_1 - (n-1)(r_1 - 1)) - \phi(m - q_1 + r_1 - 1) \delta \left( V - V^N \right) \right] \end{aligned}$$

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)\right]\delta\left(V-V^N\right)$$

$$\geq \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left(\frac{z}{n-1}\right)\left[m-q_1-(n-1)\left(r_1-1\right)-m+q_1+(n-1)\, r_1\right]$$

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)\right]\delta\left(V-V^N\right) \geq z \qquad (12)$$

By under-reporting by one unit, firm 1 reduces its payment by $z$, which is the rhs of (12). The lhs is the expected change in the future payoff from under-reporting.

What we want to show is that if (12) holds for $r_1 = q_1$ then it holds $\forall r_1 < q_1$, and this is true $\forall q_1 \leq \overline{q}$. This is indeed the case if

$$\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)$$

is non-increasing in $r_1$ $\forall r_1 \leq q_1$, $\forall q_1 \leq \overline{q}$; or, equivalently, $\phi(m-1)-\phi(m)$ is non-increasing in $m$ $\forall m \leq \overline{m}$. From hereon, this property is assumed for $\phi$.

Having derived a sufficient condition (on $\phi$) whereby if a firm doesn't want to under-report by one unit then it doesn't want to under-report by any amount, the next step is to derive sufficient conditions such that if it is not optimal to under-report by one unit given $q_1 = q'$, then it is not optimal to under-report by one unit given $q_1 = q'-1$. Using the explicit expressions, $W\left(q'; q', p_1\right) \geq W\left(q'-1; q', p_1\right)$ takes the form:

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q', \underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right] \geq \frac{z}{\delta\left(V-V^N\right)} \qquad (13)$$

What we want to show is that if (13) holds then (14) holds,

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q'-1, \underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right] \geq \frac{z}{\delta\left(V-V^N\right)} \qquad (14)$$

This is the case iff the lhs of (14) is at least as great as the lhs of (13):

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q'-1, \underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right] \geq \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q', \underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right] \qquad (15)$$

Since we've already assumed $\phi(m-1)-\phi(m)$ is non-increasing in $m$, (15) holds if $\sigma\left(\cdot\left|q', \underline{p}\right.\right)$ FOSD $\sigma\left(\cdot\left|q'-1, \underline{p}\right.\right)$, which is true by A3.

To summarize, if $\phi(m-1)-\phi(m)$ is non-increasing in $m$ $\forall m \leq \overline{m}$ and (10) holds then the ICCs for the reporting stage are satisfied iff $W\left(\overline{q}; \overline{q}, p_1\right) \geq W\left(\overline{q}-1; \overline{q}, p_1\right)$ or

$$\delta\left(V-V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q}, \underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right] \geq z \qquad (16)$$

Substituting (7) into (16), we have

$$\delta\left(\frac{\left[\widehat{p}-p^{N}\left(c\right)\right]\left(\mu/n\right)}{1-\delta+\delta\sum_{m=\underline{m}}^{\overline{m}}\rho\left(m\right)\phi\left(m\right)}\right)\sum_{m=\underline{m}}^{\overline{m}}\sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi\left(m-1\right)-\phi\left(m\right)\right]\geq z$$

Re-arranging,

$$\frac{\delta\mu}{\frac{1-\delta}{\sum_{m=\underline{m}}^{\overline{m}}\sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]}+\frac{\delta\sum_{m=\underline{m}}^{\overline{m}}\rho(m)\phi(m)}{\sum_{m=\underline{m}}^{\overline{m}}\sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]}}\geq\frac{nz}{\widehat{p}-p^{N}\left(c\right)} \tag{17}$$

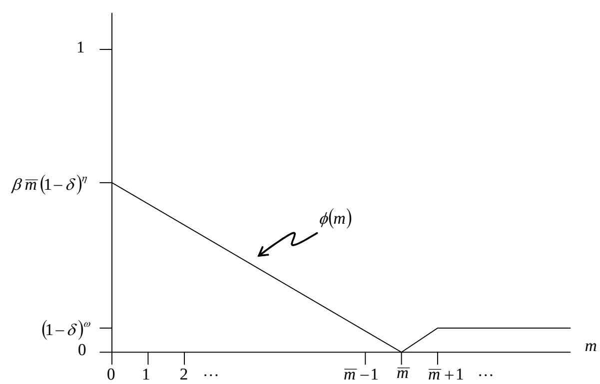
In sum, PPE1 is a perfect public equilibrium if: i) $\widehat{p}=p^{N}\left(c+\left(\frac{n}{n-1}\right)z\right)$; ii) $\phi\left(m-1\right)-\phi\left(m\right)$ is non-increasing in $m$ $\forall m\leq\overline{m}$; iii) (10) holds; and iv) (17) holds. We now want to impose properties on $\phi$ so that (ii)-(iv) are satisfied as $\delta\rightarrow 1$.

Specify:

$$\phi\left(m\right)=\begin{cases}\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta} & \text{if } m\leq\overline{m}\\ \left(1-\delta\right)^{\omega} & \text{if } \overline{m}<m\end{cases} \tag{18}$$

where $\beta>0$, $\eta\in\left(0,1\right)$, and $\omega\in\left(0,\eta\right)$. It is depicted in Figure 1.

Figure 1




Assume $\beta$ is sufficiently close to zero so that $\phi\left(m\right)\in\left[0,1\right),\forall m$. $\phi$ is decreasing in $m$ for $m\leq\overline{m}$, equals zero for $m=\overline{m}$, and is positive and constant for $m>\overline{m}$. Furthermore, since $\lim_{\delta\rightarrow 1}\phi\left(m\right)>0$ for $m\leq\overline{m}$ then the equilibrium probability of a punishment goes to zero. Next note that, when $m\leq\overline{m}$,

$$\phi\left(m-1\right)-\phi\left(m\right)=\beta\left(\overline{m}-\left(m-1\right)\right)\left(1-\delta\right)^{\eta}-\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}=\beta\left(1-\delta\right)^{\eta}.$$

Since then $\phi\left(m-1\right)-\phi\left(m\right)$ is non-increasing (actually, constant) in $m$, condition (ii) it satisfied. Proving (iii) holds is a bit more complicated and is shown in the next lemma.

**Lemma 2** *Given (18),*

$$lim_{\delta\to 1}\sum_{m=\underline{m}}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi\left(m\right)\right]\delta\left(V-V^N\right)\geq 0,\forall r_1>q_1,\forall q_1,\forall p_1.$$

**Proof.** Using (18) and assuming $r_1 > q_1$, the lhs of (10) is

$$\begin{aligned}
&\sum_{m=\underline{m}}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi\left(m\right)\right]\delta\left(V-V^N\right)\\
=\ &\delta\left(V-V^N\right)\left\{\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\beta\left(\overline{m}-\left(m-q_1+r_1\right)\right)\left(1-\delta\right)^{\eta}-\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}\right]\right.\\
&\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\left(1-\delta\right)^{\omega}-\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
=\ &\delta\left(V-V^N\right)\left\{\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left(q_1-r_1\right)\beta\left(1-\delta\right)^{\eta}\right.\\
&\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\left(1-\delta\right)^{\omega}-\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}\right]\right\}
\end{aligned}\qquad (19)$$

Using (7) and (18), we substitute for $V - V^N$ in (19),

$$\begin{aligned}
&\delta\left(\frac{\left[\widehat{p}-p^N\left(c\right)\right]\left(\mu/n\right)}{1-\delta+\delta\sum_{m=\underline{m}}^{\overline{m}}\rho\left(m\right)\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}}\right)\times\\
&\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left(q_1-r_1\right)\beta\left(1-\delta\right)^{\eta}\\
&\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\left(1-\delta\right)^{\omega}-\beta\left(\overline{m}-m\right)\left(1-\delta\right)^{\eta}\right]\right\}
\end{aligned}$$

$$= \delta \left( \frac{\left[\widehat{p} - p^N(c)\right](\mu/n)}{1-\delta+\delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)\beta(\overline{m}-m)(1-\delta)^{\eta}} \right) \times \tag{20}$$

$$\left\{ \sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)(q_1-r_1)\beta \right.$$

$$\left. + \sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[(1-\delta)^{\omega-\eta} - \beta(\overline{m}-m)\right] \right\}.$$

As $\delta \to 1$, the first term in $\{\cdot\}$ is bounded and negative, while the second term is unbounded and positive since $\omega - \eta < 0$ implies $\lim_{\delta\to1}(1-\delta)^{\omega-\eta} = +\infty$. (It is here where we use A2.) This proves the lemma. ■

Given (18) and Lemma 2, PPE1 is then a perfect public equilibrium if (17) holds as $\delta \to 1$. Evaluate the first term in the denominator on the lhs of (17):

$$\frac{1-\delta}{\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]} \tag{21}$$

$$= \frac{1-\delta}{\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\beta(1-\delta)^{\eta}} = \frac{(1-\delta)^{1-\eta}}{\sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|\overline{q},\underline{p}\right.\right)\beta}$$

Next consider the second term in the denominator on the lhs of (17):

$$\frac{\delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)\phi(m)}{\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]} \tag{22}$$

$$= \frac{\delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)\beta(\overline{m}-m)(1-\delta)^{\eta}}{\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)\beta(1-\delta)^{\eta}}$$

$$= \frac{\delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)(\overline{m}-m)}{\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|\overline{q},\underline{p}\right.\right)}$$

$$= \delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)(\overline{m}-m) = \delta(\overline{m}-\mu)$$

Using (21) and (22), as $\delta \to 1$ the lhs of (17) is

$$\lim_{\delta\to1} \frac{\delta\mu}{\frac{1-\delta}{\sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]} + \frac{\delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)\phi(m)}{\sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|\overline{q},\underline{p}\right.\right)\left[\phi(m-1)-\phi(m)\right]}}$$

$$= \lim_{\delta\to1} \frac{\delta\mu}{\frac{(1-\delta)^{1-\eta}}{\sum_{m=\underline{m}}^{\overline{m}} \sigma(m|\overline{q})\beta} + \delta(\overline{m}-\mu)} = \frac{\mu}{\overline{m}-\mu}$$

Hence, as $\delta \to 1$, (17) is

$$\frac{\mu}{\overline{m}-\mu} \geq \frac{nz}{\widehat{p}-p^N(c)}. \tag{23}$$

Given $\widehat{p} = p^N\left(c + \left(\frac{n}{n-1}\right)z\right)$, we can solve for the per unit transfer, $z$, required to induce the collusive price $\widehat{p}$.

$$\begin{aligned} p^N\left(c+\left(\frac{n}{n-1}\right)z\right) &= \widehat{p} \\ c+\left(\frac{n}{n-1}\right)z &= p^{N^{-1}}(\widehat{p}) \end{aligned}$$

$$z = \left(\frac{n-1}{n}\right)\left[p^{N^{-1}}(\widehat{p}) - c\right]. \tag{24}$$

Note that $p^{N^{-1}}$ exists by A4. Substituting (24) into (23) gives us (4).

In sum, by choosing $\delta$ sufficiently close to one, if (4) holds then PPE1 is a perfect public equilibrium and, furthermore,

$$\max\{\phi(m) : \underline{m} \leq m \leq \overline{m}\} = \beta(\overline{m}-\underline{m})(1-\delta)^{\eta} < \varepsilon.$$

■

If we impose the additional structure on demand used in the Appendix to deliver $p^N(c) = c + (1/bn)$, then (4) takes the form

$$\frac{\mu}{\overline{m}-\mu} > n-1. \tag{25}$$

In that case, the Folk Theorem can be re-stated as:

**Corollary 3** *Suppose* $p^N = c + (1/bn)$, *where* $b > 0$. *Then for any* $\widehat{p} > p^N$ *and* $\varepsilon > 0$, *there exists* $\underline{\delta} < 1$ *such that if* $\delta > \underline{\delta}$ *and* $\frac{\mu}{\overline{m}-\mu} > n-1$ *then PPE1 is a perfect public equilibrium with max* $\{\phi(m) : \underline{m} \leq m \leq \overline{m}\} < \varepsilon$.

Given firms truthfully report their sales and pay transfers based on those reports, the mechanism used to sustain a collusive price is the same as in Harrington and Skrzypacz (2007). For each unit that a firm sells (and reports), it makes a payment of $z$ to the other cartel members. Thus, when a firm sells a unit rather than have the unit sold by another firm, it ends up paying $z$ rather than receiving $\frac{z}{n-1}$, so the net effect is $\left(\frac{n}{n-1}\right)z$. A firm's effective marginal cost is then $c + \left(\frac{n}{n-1}\right)z$. As a firm's price impacts its current profit and its net transfer - but not its future payoff - the equilibrium (collusive) price is simply the static Nash equilibrium price when each firm faces a marginal cost of $c+\left(\frac{n}{n-1}\right)z$. A higher collusive price can then be sustained by setting a higher per unit transfer.

This collusive price is sustainable only if firms accurately report their sales (and then make the appropriate transfers). There is clearly an incentive to under-report sales since doing so reduces the payment made to other firms. The collusive mechanism offsets this temptation by making it more likely that there is a price war (that is, infinite reversion to a static Nash equilibrium) when reported market sales are lower. The increase in the probability of a price war from under-reporting has to be sufficiently great to offset the reduction in payments made by a firm to other members of the cartel. Counter-acting the incentive to under-report could, however, create an incentive to *over-report*; a firm makes a higher payment than is called for in order to make a price war less likely. Here, we use the fact that an over-report results in reported market sales taking on a non-equilibrium value (one exceeding $\overline{m}$) with positive probability. While the probability of a price war is decreasing in total reported sales for *equilibrium* values of total demand, the probability of a price war is higher when total reported sales exceeds maximal demand. This property of the collusive mechanism dissuades firms from over-reporting their sales. Finally, as a positive probability of a price war is needed to induce firms to truthfully report their sales, there is an inefficiency. However, by having this probability go to zero as $\delta \to 1$, this inefficiency disappears and we have a Folk Theorem.

Theorem 1 was proven for the 3-stage structure that compresses the report and transfer stages. Let us now argue that it'll extend to the 4-stage game mentioned in Section 3 which has the reporting and transfer stages separated. With a 4-stage game, the strategy profile (in the collusive phase) has firms make truthful reports in the report stage and then, based upon the reports submitted, has a price war occur with probability $\phi\left(\sum_{j=1}^{n} r_j^t\right)$. If the price war does not occur then firm $i$ makes a transfer of $zr_i^t$.

Let us compare the ICC for the report stage in the 4-stage game with the ICC for the report/transfer stage in the 3-stage game. In the former case, it is presumed that, if collusion continues (that is, a price war does not occur), a firm will make a transfer in stage 4 based on its report. Thus, the only difference between these two ICCs is that with the 3-stage structure the transfer is made for sure and with the 4-stage structure the transfer is made with some probability (which is the probability that a punishment does not occur between the report and transfer stages). In proving Theorem 1, we let the probability of a punishment go to zero which means that these ICCs are, in fact, the same as $\delta \to 1$.

A second difference between the 3-stage and 4-stage games is that the latter has an additional set of ICCs associated with the transfer stage. That is, we need to ensure that a firm finds it optimal to make a transfer equal to its report. Since failure to make a transfer of $zr_i^t$, after having reported $r_i^t$, is observable, this ICC is assured of holding as $\delta \to 1$ as it can be punished for sure. Putting this another way, if, in the 4-stage game, the ICC for the report stage holds then the ICC for the transfer stage also holds; the binding ICC is whether to report truthfully in the report stage. Based upon this argument, assuming the 3-stage game is simply a matter of convenience as Theorem 1 holds as well for the more natural 4-stage game.

# 6 Case of Unit Demand

In this section we consider the special case in which market demand is either zero or one unit. For the class of equilibria considered in the previous section, Section 6.1 characterizes the best equilibrium and performs comparative statics. For example, it is shown that a price war is more likely when there are more firms. In Section 6.2, we allow market demand to be sensitive to firms' prices and prove that the same type of equilibrium prevails. Thus, the collusive mechanism at work is not specific to when market demand is perfectly inelastic.

## 6.1 Characterization of Optimal Equilibrium

The class of perfect public equilibria studied in Section 5 are defined by a linear transfer scheme - for some $z > 0$, a firm that sells $q$ units transfers $zq$ to the other firms - and a probabilistic price war where the probability of reverting to a static Nash equilibrium is more likely when total reported sales is lower. In equilibrium, all firms report truthfully. For this class of equilibria, Theorem 1 provided sufficient conditions whereby collusion is sustainable as the discount factor goes to one. We will now characterize the best equilibrium (in this class) for any discount factor. Given this more ambitious task, it is assumed that market demand is either zero or one.

In each period, market sales is one with probability $\rho$ and zero with probability $1-\rho$. Conditional on market demand being one unit, $f(p_i, p_{-i})$ denotes the probability that firm $i$ sells one unit given its price $p_i$ and all other firms set a common price of $p_{-i}$. Firm $i$'s payoff function at the price stage is

$$\rho\left[f(p_i, p_{-i})(p_i - c - z) + (1 - f(p_i, p_{-i}))\left(\frac{z}{n-1}\right)\right]$$
$$+\delta\left\{(1-\rho)\left[\phi(0)V^N + (1-\phi(0))V\right] + \rho\left[\phi(1)V^N + (1-\phi(1))V\right]\right\}$$

or

$$\rho f(p_i, p_{-i})\left(p_i - c - \left(\frac{n}{n-1}\right)z\right) + \rho\left(\frac{z}{n-1}\right)$$
$$+\delta\left\{(1-\rho)\left[\phi(0)V^N + (1-\phi(0))V\right] + \rho\left[\phi(1)V^N + (1-\phi(1))V\right]\right\}$$

Recall that $\phi(m)$ is the probability of a price war given total reported sales is $m$, $V$ is the collusive value, and $V^N$ is the value when firms are at a static Nash equilibrium forever.

Assuming the first-order condition is sufficient, the symmetric collusive price, $\widehat{p}(z)$, is defined by

$$\rho\left(\frac{\partial f(\widehat{p}(z), \widehat{p}(z))}{\partial p_i}\right)\left(\widehat{p}(z) - c - \left(\frac{n}{n-1}\right)z\right) + \rho f(\widehat{p}(z), \widehat{p}(z)) = 0$$

or

$$\widehat{p}(z) = c + \left(\frac{n}{n-1}\right) z - \left(\frac{1}{n}\right)\left(\frac{1}{\partial f\left(\widehat{p}(z), \widehat{p}(z)\right)\partial p_i}\right)$$

where $f\left(\widehat{p}(z), \widehat{p}(z)\right) = 1/n$. As the static Nash equilibrium price is defined by

$$p^N(c) = c - \left(\frac{1}{n}\right)\left(\frac{1}{\partial f\left(p^N(c), p^N(c)\right)\partial p_i}\right),$$

then

$$\widehat{p}(z) = p^N\left(c + \left(\frac{n}{n-1}\right) z\right). \tag{26}$$

Given $z$, equilibrium behavior requires that each firm price at $\widehat{p}(z)$.

At the reporting stage, we can, without loss of generality, restrict firms to reporting either zero or one unit. Given $q_i = 1$ (so firm $i$ has sales of one unit), the incentive compatibility constraint (ICC) ensuring that firm $i$ reports sales of one rather than zero is

$$\phi(1) V^N + (1-\phi(1)) V - z \geq \phi(0) V^N + (1-\phi(0)) V$$

or

$$[\phi(0) - \phi(1)]\left(V - V^N\right) \geq z \tag{27}$$

When $q_i = 0$, the ICC ensuring that firm $i$ reports zero rather than one is

$$\begin{aligned}
&\left(\frac{\rho n-\rho}{n-\rho}\right)\left[\phi(1) V^N + (1-\phi(1)) V + \frac{z}{n-1}\right] \\
&+\left(\frac{n-\rho n}{n-\rho}\right)\left[\phi(0) V^N + (1-\phi(0)) V\right] \\
\geq\ &\left(\frac{\rho n-\rho}{n-\rho}\right)\left[\phi(2) V^N + (1-\phi(2)) V + \frac{z}{n-1} - z\right] \\
&+\left(\frac{n-\rho n}{n-\rho}\right)\left[\phi(1) V^N + (1-\phi(1)) V - z\right]
\end{aligned}$$

or

$$z \geq -\left\{\left(\frac{\rho n-\rho}{n-\rho}\right)[\phi(2)-\phi(1)] + \left(\frac{n-\rho n}{n-\rho}\right)[\phi(1)-\phi(0)]\right\}\left(V - V^N\right) \tag{28}$$

$\frac{\rho n-\rho}{n-\rho}$ is the probability that a firm other than firm $i$ has sales of one unit, conditional on $q_i = 0$.

An optimal equilibrium (in this class of equilibria) is characterized by values for $\phi(0), \phi(1), \phi(2)$, and $z$ which maximize the equilibrium payoff subject to (28) and (27). The equilibrium payoff $V$ is defined recursively by

$$V = (\rho/n)\left[\widehat{p}(z) - c\right] + \delta\left\{(1-\rho)\left[\phi(0) V^N + (1-\phi(0)) V\right] + \rho\left[\phi(1) V^N + (1-\phi(1)) V\right]\right\}$$

Substituting

$$\frac{(\rho/n)\left(p^N - c\right)}{1-\delta}$$

for $V^N$ and solving for $V$, we derive

$$V = \left(\frac{\rho}{n}\right)\left\{\frac{(1-\delta)\left[\widehat{p}(z) - c\right] + \delta\left[(1-\rho)\phi(0) + \rho\phi(1)\right]\left(p^N - c\right)}{(1-\delta)\left[1-\delta+\delta(1-\rho)\phi(0) + \delta\rho\phi(1)\right]}\right\} \tag{29}$$

For later purposes, it'll be useful to have the incremental value to colluding:

$$V - V^N = \frac{(\rho/n)\left[\widehat{p}(z) - p^N\right]}{1-\delta+\delta(1-\rho)\phi(0)+\delta\rho\phi(1)}$$

The problem is to choose $\phi(0), \phi(1), \phi(2)$, and $z$ to maximize $V$ in (29) subject to (28) and (27). If the solution has $z = 0$ then no collusion can be sustained by this class of equilibria. Thus, let us presume that there is a solution with $z > 0$ and we'll later derive necessary and sufficient conditions for that to be the case.

To begin, note that $\phi(2)$ does not affect $V$ because, in equilibrium, the sum of reported sales is never 2. $\phi(2)$ only enters (28) and, furthermore, a higher value for $\phi(2)$ raises the lhs of (28) and thus loosens the constraint. Thus, there is no loss of generality in setting $\phi(2) = 1$.

The next step is to presume (28) is satisfied and derive a solution maximizing $V$ subject to (27). After solving that problem, we'll show the solution satisfies (28). Suppose (27) did not bind at an optimum so that:

$$\left[\phi(0) - \phi(1)\right]\left(V - V^N\right) > z.$$

If $z > 0$ then $\phi(0) > 0$. Next note that $V$ is decreasing in $\phi(0)$,

$$\frac{\partial V}{\partial\phi(0)} = -\frac{(\rho/n)(1-\delta)^2\delta(1-\rho)\left[\widehat{p}(z) - p^N\right]}{\left\{(1-\delta)\left[1-\delta+\delta(1-\rho)\phi(0)+\delta\rho\phi(1)\right]\right\}^2} < 0.$$

Thus, if (27) is not binding then the collusive value could be increased by lowering $\phi(0)$. Hence, at an optimum,

$$\left[\phi(0) - \phi(1)\right]\left(V - V^N\right) = z.$$

In considering the optimal value of $\phi(1)$, note that the equilibrium payoff is decreasing in $\phi(1)$ :

$$\frac{\partial V}{\partial\phi(1)} = -\frac{(\rho/n)(1-\delta)^2\delta\rho\left[\widehat{p}(z) - p^N\right]}{\left\{(1-\delta)\left[1-\delta+\delta(1-\rho)\phi(0)+\delta\rho\phi(1)\right]\right\}^2} < 0.$$

Since lowering $\phi(1)$ loosens (27) - and we are presently ignoring (28) - the optimal value is $\phi(1) = 0$.

Thus far we've shown that a solution has $\phi(1) = 0$, (27) is binding, and, without loss of generality, $\phi(2) = 1$. Let us now show that (28) is indeed satisfied. Given that (27) is binding, (28) is satisfied iff

$$\begin{aligned} & [\phi(0) - \phi(1)] \left(V - V^N\right) \\ \geq & -\left\{ \left(\frac{\rho n - \rho}{n - \rho}\right) [\phi(2) - \phi(1)] + \left(\frac{n - \rho n}{n - \rho}\right) [\phi(1) - \phi(0)] \right\} \left(V - V^N\right). \end{aligned} \tag{30}$$

Since, by presumption, $V - V^N > 0$, (30) implies

$$\phi(0) - \phi(1) \geq -\left(\frac{\rho n - \rho}{n - \rho}\right) [\phi(2) - \phi(1)] - \left(\frac{n - \rho n}{n - \rho}\right) [\phi(1) - \phi(0)]$$

Setting $\phi(2) = 1$ and $\phi(1) = 0$, (30) is equivalent to $\phi(0) \geq -1$, which is clearly true. Hence, (28) is satisfied. To summarize, if there is an equilibrium with collusion (that is, a solution with $z > 0$) then: $\phi(1) = 0$, $\phi(0) = z / \left(V - V^N\right)$, and, without loss of generality, $\phi(2) = 1$.

To go any further in our analysis, we'll need to impose the additional structure on $f(p_i, p_{-i})$ specified in the Appendix; this means that

$$\widehat{p}(z) = \frac{1}{bn} + c + \left(\frac{n}{n-1}\right) z.$$

With this specification, (27) takes the form:

$$\frac{(\phi(0) \rho / n) \left(\frac{n}{n-1}\right) z}{1 - \delta + \delta (1 - \rho) \phi} \geq z$$

which is equivalent to

$$\phi(0) \geq \frac{(n-1)(1-\delta)}{\rho - \delta (n-1)(1-\rho)}. \tag{31}$$

The problem then is to choose $\phi(0)$ and $z$ to maximize

$$V = \left(\frac{\rho}{n}\right) \left[ \frac{(1-\delta)\left(\frac{1}{bn} + \left(\frac{n}{n-1}\right) z\right) + \delta (1-\rho) \phi(0) \left(\frac{1}{bn}\right)}{(1-\delta)[1 - \delta + \delta(1-\rho)\phi(0)]} \right] \tag{32}$$

subject to (31). A necessary condition for (31) to be satisfied is that the rhs does not exceed one. This is the case iff

$$\rho > \frac{n-1}{1 + \delta(n-1)}.$$

Examining (32), note that $V$ is increasing in $z$. Since $V$ is increasing in $z$ and decreasing in $\phi(0)$, $z$ should be set as high as possible (as it doesn't enter (27)) and $\phi(0)$ should be set as low as possible subject to (27). Hence, if

$$1 \geq \frac{(n-1)(1-\delta)}{\rho - \delta(n-1)(1-\rho)}$$

then the optimal value for $\phi(0)$ is

$$\phi^* = \frac{(n-1)(1-\delta)}{\rho - \delta(n-1)(1-\rho)}.$$

If

$$1 < \frac{(n-1)(1-\delta)}{\rho - \delta(n-1)(1-\rho)},$$

then collusion cannot be sustained as satisfaction of (31) requires $\phi(0) > 1$. Note that

$$1 \geq \frac{(n-1)(1-\delta)}{\rho - \delta(n-1)(1-\rho)} \Leftrightarrow \delta \geq \frac{n-1-\rho}{(n-1)\rho} \Leftrightarrow \rho \geq \frac{n-1}{1+\delta(n-1)}.$$

Since $\frac{n-1}{1+\delta(n-1)}$ is minimized at $\delta = 1$, a lower bound is $\frac{n-1}{n}$. Hence, a necessary condition for there to be an equilibrium in this class with collusion is:

$$\rho \geq \frac{n-1}{n}.$$

At an optimal equilibrium, if market sales is one then there is no chance of a price war, while if market sales is zero, a price war occurs with probability $\phi^*$. Hence, the probability of a price war starting is

$$(1-\rho)\phi^* = \frac{(1-\rho)(n-1)(1-\delta)}{\rho - \delta(n-1)(1-\rho)}.$$

We can now perform some comparative statics on the likelihood of a price war.

$$\frac{\partial(1-\rho)\phi^*}{\partial\delta} = -\frac{(n-1)n(1-\rho)\left(\rho - \frac{n-1}{n}\right)}{\left[\rho - \delta(n-1)(1-\rho)\right]^2} < 0$$

since $\rho > \frac{n-1}{n}$. Thus, as firms become more patient, the probability of a price war declines. When a firm has sales of one, truthful reporting incurs a current cost of $z$. To induce truthful revelation, the probability of a price war must be higher when total reported sales are zero rather than one. Thus, it is the lower expected future payoff - due to a more likely price war when a firm under-reports its sales - that induces it to report truthfully. When firms are more patient, there doesn't have to be as severe a threat of a price war to bring forth truthful revelation.

The probability of a price war is increasing in the number of firms:

$$\frac{\partial(1-\rho)\phi^*}{\partial n} = \frac{(1-\rho)\rho(1-\delta)}{\left[\rho - \delta(n-1)(1-\rho)\right]^2} > 0.$$

As is typical, more firms make it more difficult to sustain collusion because the expected gain in current profit is cheating increases faster with the number of firms than does the severity of the punishment.

Finally, increasing the probability of a high demand state reduces the probability of a price war:

$$\frac{\partial (1-\rho)\phi^*}{\partial \rho} = -\frac{(n-1)(1-\delta)}{[\rho - \delta (n-1)(1-\rho)]^2} < 0.$$

Here there are two effects. First, as a price war can only occur during the low demand state, a price war is less likely when the low demand state is less likely. Second, if the high demand state is more likely then the foregone future payoff from a price war is more severe and this strengthens the attractiveness of truthful reporting. Thus, a price war doesn't have to be as likely when total reported sales is low.

**Theorem 4** *Assume* $p^N(c) = \frac{1}{bn} + c$. *If* $\delta \geq \frac{n-1-\rho}{(n-1)\rho}$ *then an optimal collusive equilibrium exists and it has a probability of a price war equal to* $(1-\rho)\phi^* = \frac{(n-1)(1-\delta)}{\rho-\delta(n-1)(1-\rho)}$, *which is decreasing in* $\delta$ *and increasing in* $n$. *If* $\delta < \frac{n-1-\rho}{(n-1)\rho}$ *then an optimal collusive equilibrium does not exist.*

## 6.2 Elastic Market Demand

Thus far we have assumed that market demand is unaffected by firms' prices. While prices influence how market demand is allocated between firms, it does not affect the level of market demand. Though the equilibrium mechanism described in Section 5 does not appear to rely on market demand being independent of firms' prices, the proof of Theorem 1 clearly takes advantage of that property. In this section, we consider a simple example and show that the insight of Section 5 extends to when market demand is sensitive to firms' prices.

Assume that market demand is either zero or one. $\omega(p_1, p_2)$ denotes the probability that firm 1 sells one unit and, by symmetry, $\omega(p_2, p_1)$ is the probability that firm 2 sells one unit. $1-\omega(p_1,p_2)-\omega(p_2,p_1)$ is then the probability that neither firm sells a unit. $\omega(p_1,p_2)$ is twice differentiable in both firms' prices and let $\omega_i(p_1,p_2)$ denote the derivative of $\omega(p_1,p_2)$ with respect to the $i^{th}$ argument. Assume $\omega_1(p_1,p_2) < 0$, $\omega_2(p_1,p_2) > 0$, and $\omega_1(p_1,p_2) + \omega_2(p_1,p_2) < 0$ so the own price effect dominates the cross price effect.

Let us begin by considering a static symmetric Nash equilibrium. Firm 1's payoff function is

$$\omega(p_1,p_2)(p_1 - c).$$

Assuming the first-order condition is sufficient, a symmetric Nash equilibrium price, $p^N$, is defined by:

$$\omega_1\left(p^N,p^N\right)\left(p^N - c\right) + \omega\left(p^N,p^N\right) = 0 \Leftrightarrow p^N - c = -\frac{\omega\left(p^N,p^N\right)}{\omega_1\left(p^N,p^N\right)} \qquad (33)$$

In exploring an equilibrium with collusion, consider a general transfer scheme that the firms enter into prior to playing the game. Before the pricing game, the firms sign binding contracts which specify payment to a firm of $t(i,j)$ when it reports $i$ units and the other firm reports $j$ units, $i,j \in \{0,1\}$. After prices are set and quantities realized, firms make reports and then, according to $t(\cdot)$, receive payoffs. Assume that

$$t(i,j) + t(i,j) \leq 0, \ \forall (i,j) \in \{0,1\}^2,$$

so that the firms can transfer and destroy value. These un-balanced transfers are to be interpreted as the reduced-form of the continuation value of the repeated game and the contract is relational, allowing for some transfers of future market shares and of price wars, which reduce total payoffs.

Given a transfer scheme, the payoff function faced by firm 1 is:

$$\omega(p_1,p_2)[p_1 - c + t(1,0)] + \omega(p_2,p_1)t(0,1) + [1 - \omega(p_1,p_2) - \omega(p_2,p_1)]t(0,0)$$

The first-order condition defining the equilibrium price, $\widehat{p}$, is:

$$\omega_1(\widehat{p},\widehat{p})[\widehat{p} - c + t(1,0)] + \omega(\widehat{p},\widehat{p}) + \omega_2(\widehat{p},\widehat{p})t(0,1) - [\omega_1(\widehat{p},\widehat{p}) + \omega_2(\widehat{p},\widehat{p})]t(0,0) = 0. \tag{34}$$

We'll assume the first-order condition is sufficient.

With $\widehat{p}$ defined by (34) as a function of $t(1,0)$, $t(0,1)$, and $t(0,0)$, we can turn to characterizing an equilibrium transfer scheme. The cartel problem is to choose $t(1,0)$, $t(0,1)$, $t(0,0)$, and $t(1,1)$ so as to maximize

$$\omega(\widehat{p},\widehat{p})[\widehat{p} - c + t(1,0)] + \omega(\widehat{p},\widehat{p})t(0,1) + [1 - 2\omega(\widehat{p},\widehat{p})]t(0,0)$$

subject to

$$t(1,0) \geq t(0,0) \tag{35}$$

$$\eta t(0,0) + (1-\eta)t(0,1) \geq \eta t(1,0) + (1-\eta)t(1,1) \tag{36}$$

$$0 \geq t(1,0) + t(0,1) \tag{37}$$

$$0 \geq t(0,0) \tag{38}$$

where $\widehat{p}$ is defined by (34). $\eta$ is the probability that the other firm has zero sales given a firm has zero sales. (35) and (36) are incentive compatibility constraints ensuring that a firm wants to provide a truthful report when it has one and zero units, respectively. (37) and (38) are feasibility constraints.

To simplify the analysis, we'll focus our attention on transfer schemes of the following form:

$$t(0,1) = t \geq 0 = t(1,1) \geq -t = t(1,0) = t(0,0).$$

If a firm reports one unit and the other reports zero units then this scheme has the former firm transfer $t$ to the latter firm. There is then no inefficiency when total market sales is one unit. Mapping this into the equilibrium of Section 5, this corresponds to when firms make transfers and the probability of a price war is zero. If both firms report zero units, then each firm pays $t$ so there is money burning. Again mapping into the equilibrium of Section 5, it corresponds to no inter-firm transfers and a positive probability of a price war (the effect of which on each firm's future expected payoff is to reduce it by $t$) when total reported sales is zero. If both report one unit then there is no transfer. Note that every transfer scheme in this class satisfies (35)-(38).

With this limited set of transfer schemes, the problem is to choose $t \geq 0$ to maximize

$$\omega\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\left[\widehat{p}\left(t\right)-c\right]-\left[1-2\omega\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\right]t \tag{39}$$

where $\widehat{p}\left(t\right)$ is defined by:

$$\omega_1\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\left[\widehat{p}\left(t\right)-c\right]+\omega\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)+2\omega_2\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)t=0$$

Note that $t=0$ corresponds to a static Nash equilibrium and thus no collusion. If the solution to this problem involves $t>0$ then equilibrium involves collusion.

Take the derivative of (39) with respect to $t$:

$$\begin{aligned}&\left[\omega_1\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)+\omega_2\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\right]\left[\widehat{p}\left(t\right)-c\right]\widehat{p}'\left(t\right)\\&+\omega\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\widehat{p}'\left(t\right)+2t\left[\omega_1\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)+\omega_2\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\right]\widehat{p}'\left(t\right)\\&-\left[1-2\omega\left(\widehat{p}\left(t\right),\widehat{p}\left(t\right)\right)\right].\end{aligned} \tag{40}$$

Evaluate (40) at $t=0$:

$$\begin{aligned}&\left[\omega_1\left(\widehat{p}\left(0\right),\widehat{p}\left(0\right)\right)+\omega_2\left(\widehat{p}\left(0\right),\widehat{p}\left(0\right)\right)\right]\left[\widehat{p}\left(0\right)-c\right]\widehat{p}'\left(0\right)\\&+\omega\left(\widehat{p}\left(0\right),\widehat{p}\left(0\right)\right)\widehat{p}'\left(0\right)-\left[1-2\omega\left(\widehat{p}\left(0\right),\widehat{p}\left(0\right)\right)\right].\end{aligned} \tag{41}$$

Recall that $\widehat{p}\left(0\right)$ is a static Nash equilibrium price and is defined by (33). Substitute (33) into (41):

$$\begin{aligned}&\left(\omega_1+\omega_2\right)\left(\widehat{p}\left(0\right)-c\right)\widehat{p}'\left(0\right)+\omega\widehat{p}'\left(0\right)-\left(1-2\omega\right)\\=&\ \widehat{p}'\left(0\right)\left\{\left(\omega_1+\omega_2\right)\left(\widehat{p}\left(0\right)-c\right)+\omega\right\}-\left(1-2\omega\right)\\=&\ \widehat{p}'\left(0\right)\left\{-\left(\omega_1+\omega_2\right)\left(\frac{\omega}{\omega_1}\right)+\omega\right\}-\left(1-2\omega\right)\\=&\ -\widehat{p}'\left(0\right)\omega\left(\frac{\omega_2}{\omega_1}\right)-\left(1-2\omega\right)\end{aligned} \tag{42}$$

Since $-\omega\left(\frac{\omega_2}{\omega_1}\right)>0$ then (42) is positive if $\widehat{p}'\left(0\right)>0$ and $\omega\left(\widehat{p}\left(0\right),\widehat{p}\left(0\right)\right)\simeq 1/2$.

To evaluate $\widehat{p}'\left(0\right)$, first note that firm 1's objective function at the price stage (given the current transfer scheme) is

$$\omega\left(p_{1,}p_2\right)\left(p_1-c\right)-\left[1-2\omega\left(p_1,p_2\right)\right]t$$

$\widehat{p}(t)$ is defined by the first-order condition:

$$\omega_1\left(\widehat{p}(t),\widehat{p}(t)\right)\left[\widehat{p}(t)-c\right]+\omega\left(\widehat{p}(t),\widehat{p}(t)\right)+2\omega_2\left(\widehat{p}(t),\widehat{p}(t)\right)t=0. \qquad (43)$$

Take the total derivative of (43) with respect to $t$:

$$\left(\omega_{11}+\omega_{12}\right)\left[\widehat{p}(t)-c\right]\widehat{p}'(t)+\omega_1\widehat{p}'(t)+\left(\omega_1+\omega_2\right)\widehat{p}'(t)+2\left(\omega_{12}+\omega_{22}\right)\widehat{p}'(t)\,t+2\omega_2=0.$$

Evaluate (43) at $t=0$ and solve for $\widehat{p}'(0)$:

$$\widehat{p}'(0)=\frac{-2\omega_2}{\left(\omega_{11}+\omega_{12}\right)\left[\widehat{p}(0)-c\right]+\omega_1+\left(\omega_1+\omega_2\right)}.$$

By assumption, $\omega_2>0$ and $\omega_1+\omega_2<0$. If, in addition, we assume $\omega_{11}+\omega_{12}\leq 0$ then $\widehat{p}'(0)>0$. Hence, we've shown that if $\omega\left(\widehat{p}(0),\widehat{p}(0)\right)\simeq 1/2$ and

$$\omega_{11}\left(\widehat{p}(0),\widehat{p}(0)\right)+\omega_{12}\left(\widehat{p}(0),\widehat{p}(0)\right)\leq 0.$$

then (42) which implies there is an equilibrium with $t>0$ and, in addition, $\widehat{p}(t)>\widehat{p}(0)$ so that the equilibrium price exceeds a static Nash equilibrium price. We have then proven Theorem 5.

**Theorem 5** *There exists $\varepsilon>0$ such that if $\omega\left(\widehat{p}(0),\widehat{p}(0)\right)\in\left(\frac{1}{2}-\varepsilon,\frac{1}{2}\right)$ then there exists $t^*>0$ such that it is an equilibrium to have $t(0,1)=t^*$, $t(1,0)=t(0,0)=-t^*$, and $t(1,1)=0$. Furthermore, the equilibrium (collusive) price exceeds $p^N$.*

Though the example is special - as market demand is only allowed to be zero or one - it nevertheless shows that the basic logic underlying Theorem 1 is not specific to when market demand is unresponsive to firms' prices. Analogous to (4), collusion occurs in equilibrium when expected demand, $2\omega\left(\widehat{p}(0),\widehat{p}(0)\right)$, is sufficiently close to maximal demand, which is one. Though these are only sufficient conditions (to what extent they are necessary is an open question), their similarity suggests that the equilibrium supporting collusion with elastic market demand is closely related to that which works for inelastic market demand.

Before moving on, it is useful to explain why it is difficult to adapt the proof of Theorem 1 to when market demand depends on firms' prices. When market demand is insensitive to firms' prices, recall from Section 5 that the objective function faced by firm 1 in the price stage is

$$\sum_{m=\underline{m}}^{\overline{m}}\rho(m)\sum_{q=0}^{m}\psi_1\left(q;m,\underline{p}\right)\left\{\left[\left(p_1-c\right)q+z\left(\left(\frac{m-q}{n-1}\right)-q\right)\right]+\phi(m)\,\delta V^N+\left(1-\phi(m)\right)\delta V\right\}.$$

Maximizing this payoff is equivalent to maximizing

$$\sum_{m=\underline{m}}^{\overline{m}}\rho(m)\sum_{q=0}^{m}\psi_1\left(q;m,\underline{p}\right)\left[\left(p_1-c\right)q+z\left(\left(\frac{m-q}{n-1}\right)-q\right)\right].$$

Hence, the equilibrium price is independent of $V$, which is the endogenous equilibrium value from colluding, and depends only on the transfer scheme as parameterized by $z$. However, if market demand is sensitive to firms' prices then the objective at the price stage is

$$\sum_{m=\underline{m}}^{\overline{m}} \rho\left(m;\underline{p}\right) \sum_{q=0}^{m} \psi_1\left(q;m,\underline{p}\right) \left\{ \left[ (p_1 - c)\, q + z \left( \left( \frac{m-q}{n-1} \right) - q \right) \right] + \phi\left(m\right) \delta V^N + \left(1 - \phi\left(m\right)\right) \delta V \right\}.$$

Now the equilibrium price depends on $V$. Of course, $V$ also depends on that equilibrium price which means that the equilibrium price and the equilibrium value must be solved simultaneously. Though there is no reason that we can see that simultaneity makes existence of an equilibrium with collusion more problematic, it surely makes it more challenging to prove.

# 7 Concluding Remarks

There are well-documented episodes in which firms were able to collude using a monitoring mechanism relying upon self-reporting of sales. In this paper, we've derived an equilibrium with self-reporting that does not support collusion. To induce firms to set a collusive price, the mechanism has each firm make a transfer to the other firms for each unit that they report having sold. To induce truthful reporting of sales, the probability of a price war - reversion to a stage game Nash equilibrium - is assumed to be higher, when total reported market sales is lower. Thus, a firm that under-reports its sales realizes a benefit by having to make a lower payment to the other firms, but also incurs a cost by increasing the probability that collusion breaks down. An equilibrium was designed in which the cost in the form of a higher chance of a price war exceeds the benefit of reduced inter-firm payments so that a firm finds it optimal to truthfully report its sales. In the special case when market demand is either zero or one unit, the optimal equilibrium in this class was identified and it was shown that a price war is more likely when there are more firms.

Though we have achieved our goal to derive a collusive equilibrium consistent with what we know about cartel practices, there are other questions to address. To achieve collusion, it was necessary to impose conditions on demand. Is there a Folk Theorem that does not impose conditions on demand? While we've shown that the main result extends to when market demand is sensitive to firms' prices, it was only done for a special case. Can we show more generally that our result extends? In fact, it may actually prove easier to sustain collusion when firms' prices affect market demand for then total market sales is an additional signal of firms' prices. What an equilibrium looks like in that case is unclear, however.

In the broader literature on Folk Thorems with private monitoring, a useful instrument for achieving maximal collusion is by allowing communication delay. This means that messages are only informative every $T$ periods, rather than every period, where $T$ is determined as part of the equilibrium. To what extent can delay allow for a less restrictive Folk Theorem? To what extent could an equilibrium with communication delay explain the frequency of information exchange and meetings that we observe in practice? In fact, some cartels communicated quarterly but meted out punishments annually (see Harrington, 2006). Can we explain the frequency of communication and punishments?

# 8 Appendix: Static Nash Equilibrium

Assume $m$ is binomial distributed and the probability that a customer buys from firm $i$ is $f(p_i, p_{-i})$ when firm $i$ charges $p_i$ and each of the other $n-1$ firms charges a common price of $p_{-i}$. Expected profit of firm $i$ is

$$\pi_i(p_i, p_{-i}) = (p_i - c) \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left\{ \sum_{q=0}^{m} q \left( \frac{m!}{q!(m-q)!} \right) f(p_i, p_{-i})^q \left[1 - f(p_i, p_{-i})\right]^{m-q} \right\}.$$

Take the first derivative and evaluate it when firms charge a common price: $p_i = p = p_{-i}$.

$$\begin{aligned} \frac{\partial \pi_i(p,p)}{\partial p_i} &= \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left\{ \sum_{q=0}^{m} q \left( \frac{m!}{q!(m-q)!} \right) f(p,p)^q \left[1 - f(p,p)\right]^{m-q} \right\} \\ &\quad + (p_i - c) \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left\{ \sum_{q=0}^{m} q \left( \frac{m!}{q!(m-q)!} \right) \left( \frac{\partial f(p,p)}{\partial p_i} \right) \times \right. \\ &\quad \left. \left[ q f(p,p)^{q-1} (1 - f(p,p))^{m-q} - (m-q) f(p,p)^q (1 - f(p,p))^{m-q-1} \right] \right\}. \end{aligned}$$

By symmetry, $f(p,p) = 1/n$. Using this fact and that

$$\sum_{q=0}^{m} q \left( \frac{m!}{q!(m-q)!} \right) f(p,p)^q \left[1 - f(p,p)\right]^{m-q} = \frac{m}{n}$$

then

$$\begin{aligned} \frac{\partial \pi_i(p,p)}{\partial p_i} &= \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left( \frac{m}{n} \right) \\ &\quad + (p_i - c) \left( \frac{\partial f(p,p)}{\partial p_i} \right) \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left\{ \sum_{q=0}^{m} q \left( \frac{m!}{q!(m-q)!} \right) \left( \frac{1}{n} \right)^{q-1} \left( \frac{n-1}{n} \right)^{m-q-1} \times \right. \\ &\quad \left. \left[ q \left( \frac{n-1}{n} \right) - (m-q) \left( \frac{1}{n} \right) \right] \right\}. \end{aligned}$$

Performing some manipulations, we get

$$\frac{\partial \pi_i (p,p)}{\partial p_i} = \left(\frac{\mu}{n}\right) + (p_i - c)\left(\frac{\partial f(p,p)}{\partial p_i}\right)\left(\frac{n^2}{n-1}\right) \times \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\left[\sum_{q=0}^{m}\left(\frac{m!}{q!(m-q)!}\right)\left(\frac{1}{n}\right)^q\left(\frac{n-1}{n}\right)^{m-q} q^2 - \left(\frac{m}{n}\right)^2\right] \tag{44}$$

Given $m$, the variance of this binomial is $m\left(\frac{n-1}{n}\right)\left(\frac{1}{n}\right)$. Thus,

$$m\left(\frac{n-1}{n}\right)\left(\frac{1}{n}\right) = \sum_{q=0}^{m}\left(\frac{m!}{q!(m-q)!}\right)\left(\frac{1}{n}\right)^q\left(\frac{n-1}{n}\right)^{m-q} q^2 - \left(\frac{m}{n}\right)^2.$$

Substituting $m\left(\frac{n-1}{n^2}\right) + \left(\frac{m}{n}\right)^2$ for $\sum_{q=0}^{m}\left(\frac{m!}{q!(m-q)!}\right)\left(\frac{1}{n}\right)^q\left(\frac{n-1}{n}\right)^{m-q} q$ in (44),

$$\frac{\partial \pi_i (p,p)}{\partial p_i} = \left(\frac{\mu}{n}\right) + (p_i - c)\left(\frac{\partial f(p,p)}{\partial p_i}\right)\mu \tag{45}$$

Setting (45) equal to zero and solving for the static Nash equilibrium price, we get

$$p^N = c - \left(\frac{1}{n}\right)\left(\frac{1}{\partial f(p^N, p^N)/\partial p_i}\right).$$

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