Rates of Convergence in Common Value Auctions*

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Abstract

We derive the limit distributions of equilibrium bids in common value auctions as the number of bidders grows large. We view the common value auction as a statistical experiment in which the winning bids are estimators of the unknown common value, whereby the question of information aggregation is equivalent to whether the winning bids in a sequence of auctions is a *consistent estimator* of the true common value. Our results are directly analogous to the econometric task of characterizing the asymptotic properties of a consistent sequence of estimators.

While the previous literature has demonstrated that information aggregation obtains across all the common auction forms, our analysis uncovers important differences. We find that first-price, second-price, and English auctions aggregate information much faster than uniform-price multi-object auctions. However, this ranking is very sensitive to assumptions about the information structure, with the information aggregation properties of the English and uniform-price auctions being most robust across different assumptions.

In this paper, we derive the limit distributions for equilibrium bids in sequences of common value auctions in which the number of bidders increases. We show some striking differences in the asymptotic behavior of the standard auction forms which have been overlooked in the previous literature. This literature (cf. Wilson (1977), Milgrom (1979), Pesendorfer and Swinkels (1997), Kremer (1999)) has derived conditions under which the common value auction is a decentralized market mechanism which aggregates the private information of the agents, in the sense that the sequence of winning bids and/or market prices converges to the "true" common value.

Remarkably, this literature has demonstrated that information aggregation obtains under quite general distributional assumptions on the information structure and auction format (first-price,

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second-price, English, uniform-price multi-object auctions). Given these results, then, one might conclude that the choice of auction form matters little in large markets (such as financial markets) in which rapid information aggregation is desired.

However, we show that this conclusion is not wholly justified. Our results indicate that the limit distributions of the winning bids are strikingly different across different auction formats. In particular, we show that the different auction forms have very different rates of convergence. First-price, second-price, and English auctions (more generally, auctions where the number of objects is held fixed as the number of bidders grows large) feature the most rapid information aggregation, relative to the "double largeness" (DL) uniform-price auctions considered in Pesendorfer and Swinkels (1997) (hereafter PS), where the number of objects also approaches infinity, but at a slower rate than the number of bidders. While this might favor the first type of auctions from a policy perspective, we also show that this ranking is very sensitive to assumptions about the information structure: specifically, if we do away with a potentially restrictive support assumption, the first- and second-price auctions no longer aggregate information, but the English and DL auctions still do, albeit at a slower rate. In this sense, then, perhaps the information aggregation properties of the English and DL auctions are most "robust" to different information structures, but first- or second-price auctions may be preferred in more restrictive information structures for their faster convergence rates.

Our approach to the asymptotics of the common value auction also differs from the previous literature. In particular, we view the auction as a *statistical experiment* (which will be precisely defined later) in which the winning bids or market prices (which differ for the non-discriminatory auctions which we study) form a sequence of estimators of the common value parameter. Thus, appropriating terminology from asymptotic statistics, the question of information aggregation is identical to that of whether or not the sequence of winning bids or market prices is a *consistent estimator* of the common value; the questions about the rate of convergence, then, require derivations of the limit distributions of these estimators. Our results are directly analogous to the econometric task of characterizing the asymptotic properties of a consistent sequence of estimators.[1] Central to this approach is the idea of a "limit experiment", which is a description of the statistical estimation problem in the limit. A second main set of results in this paper is the derivation of the "limit experiments" for the common value auction under different assumptions. In our case, we find that these limit experiments provide a very interesting and intuitive summary of the nature and type of information revealed by the auction.

For example, let v_0 denote the true common value, and let n and k_n, respectively, denote the number of bidders and number of objects sold in the n-th auction. For the double largeness auctions considered in PS, in which the sequence k_n/n converges to a value τ strictly bounded away from zero and one, the limit experiment is a single draw from the normal distribution with mean v_0:

[1] One advantage of viewing the sequence of auctions as a sequence of statistical experiments is the availability of tools and concepts from asymptotic statistical decision theory (cf. van der Vaart (1999), chs. 7-10, or Ibragimov and Has'minskii (1981), ch. 1) which can be used to analyze the asymptotic behavior of sequences of statistical experiments.

alternatively, the information revealed in the sequence of auctions is approximately equivalent to the information that an agent would have about v_0 upon observing a single draw drawn from a Gaussian distribution centered at v_0.

Different limit experiments obtain, depending on the limit of the k_n/n sequence. However, we note that the Gaussian limit for the PS case generally will not depend at all on the specific assumption about the distribution of bidders' signals, or their priors; however, this should not be too surprising given that the central limit theorem yields Gaussian limiting distributions for a wide variety of statistics irrespective of the specific sampling distribution for the data. Answering these questions expands our insight into the optimality of auctions as information aggregation mechanisms.

1 Background

Bidding environment Throughout, we consider the same pure common value unit-demand model with conditionally independent signals treated in Wilson (1977), Matthews (1984), PS, and Kremer (1999) (setup A in the latter paper).[2] Each of n bidders places value v on a single object, and only desires one object. The common value v is unknown, but the bidders' (correct) priors are that v is drawn from a distribution F_V with support $\mathcal{V}$. While none of the bidders directly observes v, each observes a private signal X. Across bidders i, the signals X_i are independently drawn from $G_{X|v}(\cdot|v)$, the distribution of a signal conditional on a realization of the common value v. Let $g_{X|v}$ and f_V denote the densities of $G_{X|v}$ and F_V, respectively. Let $\mathcal{X}_v$ denote the support of $G(x|v)$, and assume it is a compact subset of $\mathbb{R}$. Let $U(v)$ denote $\max(\mathcal{X}_v)$; at this stage, we allow $U(v)$ to be either finite or infinite. Note that this "conditional independence" model is more restrictive than the general affiliated values model in Milgrom and Weber (1982).

Each bidder i submits a bid b_i after observing her signal X_i. Like Kremer (1999), we consider four standard auction types:

Central rank/"double largeness" auctions: these are the k-object uniform-price highest-rejected bid auctions considered in PS. Specifically, these auctions satisfy the *double largeness* property, defined as $n \to \infty$, $k_n \to \infty$ and $1 - \tau_n \equiv k_n/n \to (1 - \tau) \in [0, 1]$. In the n-th auction, the k_n objects are allocated to the k_n highest bidders at the $k_n + 1$-st highest ("highest rejected") bid.

Fixed rank auctions: these are k-object uniform-price highest-rejected bid auctions in which k stays fixed for all n, thus violating the double-largeness property of PS. If $k = 1$, this reduces to the second-price auction.

First-price auctions: these are single-object discriminatory auctions where the object is allocated to the highest bidder, who pays his bid. This is the case analyzed in Wilson (1977) and

[2]The conditionally independent signals assumption is more restrictive than the general affiliated values model in Milgrom and Weber (1982).

Milgrom (1979) (the latter with a discrete support for the common value).[3]

English auctions: these are single-object ascending auction in which price is raised continuously and the object is allocated to the last bidder to remain in the auction, who pays the price at which his closest competitor dropped out. We make the same "irrevocable dropout" assumption as in Milgrom and Weber (1982). Kremer (1999) has previously studied the convergence properties of equilibrium bids from these auctions.

Throughout this paper, we make the following standard assumption.

Assumption 1 *The family of conditional density functions $g(\cdot|v)$ indexed by v satisfies the strict monotone likelihood ratio property (MLRP), i.e., for any two signals $x' > x$, the ratio $\frac{g(x|v)}{g(x|v')}$ is strictly increasing in v.*

This assumption plays two important roles. First, it is required (cf. Milgrom and Weber (1982)) in order to derive a symmetric equilibrium in strictly increasing strategies for the auction forms we consider. Furthermore, assumption 1 directly implies (cf. Milgrom (1982), pp. 382–383) that the conditional distribution functions $G(\cdot|v)$ is strictly increasing in v in the sense of strict first-order stochastic dominance (FOSD), i.e., for any signal x and $v' > v$, $G(x|v') < G(x|v)$. Equivalently, $\frac{\partial}{\partial v}\left(q_v^{1-\tau}\right) < 0$, where $q_v^{1-\tau} \equiv G^{-1}(1-\tau|v)$ denotes the $1-\tau$-th quantile of $G(\cdot|v)$, $\forall 0 < \tau < 1$. This is an "identification" assumption for central rank auctions, in the (econometric) sense that it implies that the likelihood function of an observation of $q_0^{1-\tau}$ is uniquely maximized at v_0 in the limit, for $0 < \tau < 1$ (this will be discussed in greater detail later).

For the fixed rank, first-price, and English auctions, we will also make the following "finite support" assumption:

Assumption 2 *$U(v) \equiv G^{-1}(1|v)$ is finite for all $v \in \mathcal{V}$, and $U(v)$ is strictly increasing in v. Furthermore, $g(U(v)|v) > 0$, $\forall v$.*

This assumption is essentially an identification condition for these auctions, analogous to assumption 1 for central rank auctions. For the remainder of this paper, assumption (1) will be assumed in all discussion of central rank auctions, whereas both assumption (1) and (2) are assumed in our discussion of fixed rank auctions.

1.1 Previous Literature

We begin by summarizing the main information aggregation (or "consistency" in econometric terms) results for the sequence of winning bids in the previous literature.

Theorem 1 (Summary of findings in previous literature)

[3] Furthermore, see Wang (1991) for an example of a first-price common-value auction with a discrete signal space, where the sequence of winning bids fails to be consistent.

(Pesendorfer and Swinkels (1997)) *Under assumption (1), the winning bids W_n in a sequence of central rank auctions converges in probability to v_0.*

(Wilson (1977), Kremer (1999)) *Under assumptions (1) and (2), the winning bids W_n in sequences of fixed rank, first-price, and English auctions converge in probability to v_0.*

Intuitively, as noted by Kremer (1999), in the limit the winning bid converges to the posterior mean of v conditional on the pivotal bidder's signal. For any fixed rank auction and under every v_0, the pivotal bidder's signal converges almost surely to $U(v_0)$, the upper boundary of the support of $\mathcal{X}_{v_0}$; assumption (2) ensures that $U(v_0)$ is a non-trivial function of v_0, so that the correct value of v_0 can be inferred from an observation of $U(v_0)$. For the central rank auction, however, the pivotal bidder's signal converges almost surely to $G^{-1}(\tau|v_0)$ under v_0, and assumption (1) guarantees that $G^{-1}(\tau|v_0)$ is a nontrivial function of v_0, so that v_0 is identified. As mentioned above, then, assumptions 1 and 2 are identification assumptions in an econometric sense.

Note however, that since the "true" value v_0 is a scalar, if $W_n \xrightarrow{p} v_0$, this implies that W_n converges in distribution to a degenerate limit distribution with unit mass at v_0. This is the usual outcome when considering the limit distribution of consistent estimators; in order to obtain a nondegenerate limit distribution, we must define a new sequence $\{h_n \equiv r_n (W^n - v_0) : n \in \mathbb{N}\}$, where r_n is a power of n not exceeding 1 (usually $\sqrt{n}$). Let $H^n_{v_0}$ denote the distribution of $r_n (W^n - v_0)$, as induced by $P^n_{v_0}$. Note that even though W_n converges to v_0 in probability, depending on the magnitude of r_n the sequence h_n may or may not converge in probability. Let r^*_n denote the highest power of n for which the sequence h_n still converges in probability to a random variable. This sequence r^*_n represents the "rate" at which the sequence W_n converges to v_0.

In this paper we derive the limiting distribution of the normalized winning bids $r^*_n (W^n - v_0)$. These results have potentially important implications for the design of large markets in which rapid information aggregation is desired. Perhaps the most important examples are financial markets, where the principle that the market price should reflect participants' disparate information underlies the proposed efficiency of these markets. The previous literature has shown that consistency, or information aggregation, is a relatively weak requirement of an auction form, since under assumptions 1 and 2 the previous authors have shown that the sequence of winning bids aggregates information for all four auction forms which we analyze.

It may appear, then, that market design is a largely irrelevant issue given these results. However, we show that this is not true. In particular, we show that under the different auction forms the winning bids have *very different* limiting distributions. One important implication of this is that they have different rates of convergence. The fixed rank, first-price, and English auctions are characterized by the most rapid information aggregation. While this might lead them to be more desirable from a policy perspective, we also show that this ranking is very sensitive to assumptions about the information structure: specifically, if we do away with assumption 2, the winning bids in the first- and second-price auctions no longer aggregate information (i.e., are not "consistent estimators" of the true commonvalue v_0), but the English and DL auctions still do, but at a slower rate. We sum up these results (all to be proved below) in the following proposition:

Proposition 1 (Summary of main results)

Under assumption (1), the DL and English auctions aggregate information at the rate of $\sqrt{n}$. The first- and second-price auctions do not aggregate information.

Under assumptions (1) and (2), the first-, second-price, and English auctions aggregate information at the rate of n. The DL auctions aggregate information at the slower rate of $\sqrt{n}$.

The English auction is the most "robust" in the double sense that (1) it aggregates information under less restrictive assumptions on the information structure; and (2) it achieves the maximal rate of convergence under more restrictive information structures.

In the auction literature, Wilson (1985) and Satterthwaite and Williams (1989) have previously examined asymptotic efficiency in private value double auctions, and Pesendorfer and Swinkels (2000) have examined an auction where bidders have both private and common value components to their valuations in order to focus on the tension between efficiency and information aggregation in these environments (see also Feddersen and Pesendorfer (1997) for an examination of these issues in large elections).[4] Furthermore, there has also been a literature on information aggregation in large Cournot markets where firms face uncertainty about a common market variable, such as production costs or demand; see Palfrey (1985), Vives (1988) and Vives (1999).

In the absence of asymmetric information, similar issues have arisen in characterizing the asymptotic properties of market equilibria in large "random" economies in which agents' characteristics (for example, preferences or endowments) are independent and random draws from a given distribution. Hildenbrand (1971) provided conditions under which sequences of excess demand vectors in a sequence of steadily expanding economies populated by agents with random preferences converge in probability to zero. Bhattacharya and Majumdar (1973) extended this analysis for an economy populated by agents with random endowments by using a central limit theorem to show that the limiting distribution of the (normalized) market-price vectors is Gaussian, thus showing also that the "convergence rate" of such a random economy is $\sqrt{n}$. In the same way that the Bhattacharya and Majumdar (1973) paper provides the "asymptotic distribution theory" to complement the convergence results of Hildenbrand (1971), we view our contribution to be deriving the limit distribution of estimators shown to be consistent in Wilson (1977), Milgrom (1979), Matthews (1984), Pesendorfer and Swinkels (1997), and Kremer (1999).

However, as estimators of the common value, the equilibrium bids differ critically from estimators typically encountered in econometric work: equilibrium bids are Bayesian posterior means con-

[4]Generally, allocative efficiency (ensuring that, at least asymptotically, the object is won by the bidder who values it most) is not a concern in pure common value, where bidders are assumed to have the same (but unknown) underlying valuation for the object.

ditional on a quantal event (the event of "being pivotal", in the terminology of Pesendorfer and Swinkels (1997)). Analyzing the convergence properties of posterior means in these four cases requires statistical tools which differ from the usual tools in deriving asymptotic distributions in econometrics. In particular, for the central rank case, we use the theory on convergence of regular experiments (cf. van der Vaart (1999) (chs. 7–10), Ferguson (1996) (ch. 21)). For the fixed rank case, we use extreme value theory (cf. Chernozhukov (2000), Leadbetter, Lindgren, and Rootzen (1983), chaps. 1–2). Chernozhukov (2000) contains influential results on asymptotic approximations for quantile regressions under extreme and near-extreme rank conditions. Indeed, in the fixed rank case, the common value auction is a striking example of an economic framework in which agents must engage in statistical inference under "extremal" events (i.e., the event of being "pivotal", in PS's words).

In section 2 we discuss central rank auctions. In section 3 we consider fixed rank auctions. First-price and English auctions are discussed in, respectively, sections 4 and 5. Illustrative simulations from an example are presented in section 6. We conclude in the final section. Most of the proofs are summarized in the main text, but given complete in the appendix. Furthermore, some technical details are omitted from the proofs, but are completely detailed in a companion paper (Chernozhukov, Hong, and Shum (2000)).

2 Central Rank Auctions (satisfying double largeness)

In this section, we consider the limit distribution of the winning bid W_n in central rank auctions, or those which satisfy PS's double largeness property ($n \to \infty$, $k_n \to \infty$, and $k_n/n \to 1 - \tau \in [0, 1]$).[5]

Following PS, we consider auctions in which the k objects are awarded to the k highest bids, at the price equal to the $k+1$-st highest (i.e., the "highest rejected") bid. As shown in PS, there is a unique symmetric monotonic pure equilibrium bidding strategy in this game, which (without loss of generality) for bidder 1 is:

$$b_1 = s_n(X_1 = x_1) = E\left[v | X_1 = x_1, X_{k:n\backslash 1} = x_1\right] \tag{1}$$

where (1) $s_n(\cdot)$ denotes the equilibrium bidding function, which we explicitly subscript by n to denote that a change in the number of bidders will change equilibrium bidding behavior; (2) $X_{k:n\backslash 1}$ denotes the k-th highest signal received among all the bidders except bidder 1; and (3) the expectation is over the posterior distribution of V, conditional on the joint event $X_1 = x_1, X_{k:n\backslash 1} = x_1$, i.e.,

$$E\left[v | X_1 = x_1, X_{k:n\backslash 1} = x_1\right] = \int_0^1 v \frac{G_{X|v}(x_1|v)^{n-k-1}\left(1 - G_{X|v}(x_1|v)\right)^{n-k-1} g_{X|v}(x_1|v)^2 f_V(v)dv}{\int_0^1 G_{X|w}(x_1|w)^{n-k-1}\left(1 - G_{X|w}(x_1|w)\right)^{n-k-1} g_{X|w}(x_1|w)^2 f_V(w)dw}. \tag{2}$$

[5] Clearly, it is impossible for $1 - \tau < 0$.

Note that given these symmetric bidding strategies, the market price will be the bid submitted by the bidder with the $k+1$-th highest signal. In what follows, we refer to $b_{k:n} = s_n(X_{k:n})$ as the *selling price* or *winning bid*, and denote it by W_n.

In this section, for a given value of τ we consider a simplified version of the winning bid:

$$\tilde{W}_n \equiv E\left[v|X_{k_n:n} = x\right] = \int_0^1 v \frac{G_{X|v}(x|v)^{k_n}\left(1 - G_{X|v}(x|v)\right)^{n-k_n-1} g_{X|v}(x|v) f_V(v)dv}{\int_0^1 G_{X|w}(x|w)^{k_n}\left(1 - G_{X|w}(x|w)\right)^{n-k_n-1} g_{X|w}(x|w) f_V(w)dw}. \tag{3}$$

where $k_n \equiv int(n(1-\tau))+1$ (i.e., the smallest integer exceeding $n(1-\tau)$) to avoid integer problems: the bidder with the k_n+1-th highest signal is the price setter, in equilibrium.

$\tilde{W}_n$ differs from the "actual" winning bid W_n, which is

$$W_n = s_n(X_{k_n:n} = x) = E\left[v|X_{k_n:n} = x, X_{k_n-1:n} = x\right] = \int_0^1 v \frac{G_{X|v}(x|v)^{k_n-1}\left(1 - G_{X|v}(x|v)\right)^{n-k_n-1} g_{X|v}(x|v)^2 f_V(v)dv}{\int_0^1 G_{X|w}(x|w)^{k_n-1}\left(1 - G_{X|w}(x|w)\right)^{n-k_n-1} g_{X|w}(x|w)^2 f_V(w)dw}. \tag{4}$$

While most of our results in this paper are derived for the simplified expression (3) above, for convenience' sake, in each case we make clear how the results would (or would not) change when considering the actual winning bid expression (4). Intuitively, since the stochastic process $(X_{k-m:n} - X_{k:n})$ is $O_p(1/n)$ for all finite m, these differences will not impact the asymptotic results.

Since $\tilde{W}_n$ is a posterior mean (and therefore a Bayesian point estimator), we cannot use the standard tools from classical econometrics (which do not view the common value as a random variable with a prior probability measure) in order to derive the limiting distribution of $\tilde{W}_n$ First (in subsection 2.1), we focus on the case where τ is strictly bounded away from 0 and 1 (i.e., $k_n = O(n)$), since this case requires the most straightforward analysis. Subsequently, we generalize our analysis to cases where the conditional distribution function $G(x|v)$ is possibly non-smooth and features jumps (subsection 4), and where $k_n = o(n)$, so that $1-\tau = 0$ (subsection 5).

2.1 Non-extreme case: $0 < \tau < 1$

We start by briefly introducing the idea of a statistical *experiment*, which is a *family* of probability measures, indexed by the members of a parameter space $v \in \mathcal{V}$. Specifically, the experiment $\mathcal{E} = \{\mathcal{X}, \mathcal{A}, P_v : v \in \mathcal{V}\}$, where $(\mathcal{X}, \mathcal{A})$ is the common measurable space of each probability measure P_v, for all $v \in \mathcal{V}$.[6] A *sequence* $\{\mathcal{E}^n : n \in \mathbb{N}\}$ of experiments is therefore a sequence of families of probability distributions, with $\mathcal{E}^n = \{\mathcal{X}^n, \mathcal{A}^n, P_v^n : v \in \mathcal{V}\}$.

[6] Note that while the family of measures $\{P_v : v \in \mathcal{V}\}$ is defined on the same measurable space, they generally do not coincide on this space; indeed, the "support" of P_v, which is the subclass $\mathcal{A}_v \subseteq \mathcal{A}$ on which $P_v > 0$, may differ depending on the value of v.

We consider the sequence of experiments where, in the n-th experiment, only the k_nth largest signal (the pivotal signal) is observed; this corresponds to the conditioning event in the simplified winning bid of equation (3) above. As n grows large, this sequence of experiments becomes one where the sequence of pivotal bidders observes the $(1-\tau)$-th sample quantile, where $0 < \tau < 1$.[7] Define T_n denote the pivotal bidder's signal in the n-th experiment; as n grows large, the density of T_n under the parameter v is $G(T_n|v)^{n(1-\tau)}[1-G(T_n|v)]^{n\tau}$.

We begin by "localizing" the parameter space to $h = \sqrt{n}(v - v_0)$[8]. This implies that our asymptotic approximations are only valid in a local neighbourhood of the true common value v_0, which is standard. Let $P_{n,h}$ denote the probability distribution of T_n, under a common value $v = v_0 + \frac{h}{\sqrt{n}}$. Therefore, $P_{n,0}$ denotes the probability distribution of T_n under v_0.

Our goal is to characterize the limiting analogue of the sequence of posterior distributions $(v|T_n)$ as $n \to \infty$. We show that, under general conditions, for a given point h, the sequence of posterior densities evaluated at h converges to a posterior from a limiting Gaussian experiment: that is, the posterior distribution of h given one draw from a $N(h, V_{v_0})$ distribution, with diffuse (improper) priors on h.

As asides, we show that the rate of convergence is $\sqrt{n}$, and derive the asymptotic precision; furthermore, we shed some light on the manner in which prior information is "swamped" in the limit (as noted by PS), which implies that asymptotically the winning bid is equivalent to a "maximum likelihood" estimator of v_0 and therefore shares the well-known optimality properties of the MLE.

Convergence of posterior distributions in central rank auctions Before stating the formal propositions, we describe the main jist of this argument.

Let $q_0^{1-\tau} \equiv G^{-1}(1-\tau|v_0)$ and $G_v \equiv G_v\left(q_0^{1-\tau}|v_0\right) = -g\left(q_0^{1-\tau}|v_0\right)U_v(v_0)$ for $U(v_0) = G^{-1}(1-\tau|v_0)$ denote the quantile as a function of v_0.

The posterior density evaluated at $\sqrt{n}(v - v_0) = h$ given the observed $T_n \equiv X_{k_n:n}$ is

$$p(h|T_n) = \frac{G(T_n|v_0 + h/\sqrt{n})^{n(1-\tau)}[1 - G(T_n|v_0 + h/\sqrt{n})]^{n\tau}\pi(v_0 + h/\sqrt{n})}{\int G(T_n|v_0 + h/\sqrt{n})^{n(1-\tau)}[1 - G(T_n|v_0 + h/\sqrt{n})]^{n\tau}\pi(v_0 + h/\sqrt{n})\,dh} \quad (5)$$

As $n \to \infty$, $\pi(v_0 + h/\sqrt{n}) \to \pi(v_0)$ and cancels from both numerator and denominator. On the

[7]Extension to the case where a fixed number of quantiles being observed is straightforward.

[8]Therefore, $h = 0$ denotes $v = v_0$. Generally, we want to normalize $h = r_n(v - v_0)$, where r_n is a power of n. When considering a consistent sequence of estimators $\hat{v}_n$, we noted above that such a sequence has a degenerate limit distribution. However, the normalized sequence $h_n = r_n(\hat{v}_n - v_0)$ has a non-degenerate limit distribution.

Intuitively, many estimators take an asymptotic "sample average" form, so that the variance of these estimators disappears as $n \to \infty$: this is the reason for the degenerate limit distribution. To get a non-degenerate limit distribution, we "blow up" the estimator sequence by r_n, a factor of n: the usual $r_n = \sqrt{n}$ arises in cases amenable to an application of a CLT (cf. Amemiya (1985), pg. 90; a central limit theorem gives conditions under which $\sqrt{n}$ times a sample average converges in distribution to a Gaussian random variable).

other hand, corollary 1 below implies that the stochastic likelihood ratio process

$$\frac{G\left(T_n|v_0+h/\sqrt{n}\right)^{n(1-\tau)}\left[1-G\left(T_n|v_0+h/\sqrt{n}\right)\right]^{n\tau}}{G\left(T_n|v_0\right)^{n(1-\tau)}\left[1-G\left(T_n|v_0\right)\right]^{n\tau}} \tag{6}$$

converges to the likelihood ratio process of the Gaussian experiment

$$\frac{dN\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)(X)}{dN\left(0, G_v^{-2}\tau\left(1-\tau\right)\right)(X)} \tag{7}$$

where the notation $dN(\mu, \sigma^2)(X)$ refers to the density of the normal distribution with mean μ and variance σ^2, evaluated at the value of X.

Comparing (5) and (6), we could expect that as n gets large, the scalar random sequence $\{p(h|T_n)\}$ becomes "close" (in a sense to be made more precise) to the posterior density

$$\frac{dN\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)(X)}{\int dN\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)(X)\,dh} \tag{8}$$

which is the posterior density for h given a single draw X from $N\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)$ with diffuse priors on h. This is formalized in Theorem 2 below. In other words, randomness in the sequences T_n reduces, in the limit, to randomness in draws from a $N\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)$ distribution. Given the standard results on Bayesian updating for the Gaussian case (cf. DeGroot (1970), ch. 9), the posterior density in equation (8) is just $dN\left(X, G_v^{-2}\tau\left(1-\tau\right)\right)(h)$.[9]

Subsequently, the posterior mean $E(h|T_n) = \int hp(h|T_n)dh$ should converge in distribution to the mean of the Gaussian posterior density $N\left(X, G_v^{-2}\tau\left(1-\tau\right)\right)$ which is, under any arbitrary h, $X \sim N\left(h, G_v^{-2}\tau\left(1-\tau\right)\right)$. Under the true value $v = v_0 \Leftrightarrow h = 0$, therefore,

$$E(h|T_n) \xrightarrow{d} N\left(0, G_v^{-2}\tau\left(1-\tau\right)\right).$$

This is formalized in Theorem 3 below. Note that the asymptotic variance of the winning bid is therefore identical to that of the maximum likelihood estimator, as derived earlier (cf. equation (11) above).

Next we formalize this argument. We derive of the limit distribution of W_n in several steps, following chapter 10 of van der Vaart (1999). The first step in deriving the limit distribution of the winning bid is to establish that all pairs of probability distributions $(P_{n,h}, P_{n,0})$ in the family $\mathcal{E}^n$ are mutually contiguous, for all $h \in H$.

Lemma 1 *$P_{n,h}$ and $P_{n,0}$ are mutually contiguous, denoted $P_{n,h} \lhd \rhd P_{n,0}$, for all $h \in H$.*

[9] Note that the scalar $p(h|T_n)$ (i.e., the posterior density evaluated at a fixed h) is a random variable in the limit, and distributed the same as the Gaussian density $dN\left(X, G_v^{-2}\tau\left(1-\tau\right)\right)(h)$, which is a scalar random variable with its probability law induced by $X \sim N(h, G_v^{-2}\tau\left(1-\tau\right))$. Similarly, the posterior density itself $f(h|T_n)$ is random, in the sense that for each h the posterior density $f(h|T_n)$ is a random variable. This is not surprising; if the posterior density $f(h|T_n)$ converged in distribution to a fixed function for all T_n, then the posterior mean would be a deterministic scalar.

Proof: Here we outline the proof; it is given in full in the appendix. We proceed by showing that the sequence of likelihood ratios $\left(\frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)}\right)$ converges in distribution to a standard log-normal distribution (equivalently, that the sequence of log-likelihood ratios converges in distribution (under $h=0$) to a normal distribution where the mean is $-\frac{1}{2}$ times the variance). Then $P_{n,h} \lhd \rhd P_{n,0}$ follows by LeCam's 1st lemma (cf. van der Vaart (1999), pp. 88–89), and the same argument can be repeated for every $h \in H$.

An immediate corollary of Lemma 1 is important in what follows:

Corollary 1 *The family $\{P_{n,h}\}$ is local asymptotic normal (LAN) at $h=0 \Leftrightarrow v=v_0$. In other words (cf. Ibragimov and Has'minskii (1981), definition 2.1), the log likelihood ratio $\left(\frac{dP_{n,v_0+r_n^{-1}h}(T_n)}{dP_{n,v_0}(T_n)}\right)$ can be represented as*

$$h\Delta_{n,v_0} - \frac{1}{2}h^2 I_{v_0} + o_{P_{n,v_0}}(1)$$

where $\Delta_{n,v_0} \xrightarrow{d} N(0, I_{v_0})$.

This is immediate from Lemma 1 by taking $r_n = \sqrt{n}$, $I_{v_0} = G_v^2 \frac{1}{\tau(1-\tau)}$, and $\Delta_{n,v_0} = -G_v \frac{1}{\tau(1-\tau)} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(1\left(X_i \geq g\left(q_0^{1-\tau}\right)\right) - \tau\right)$.

For the second step, we use the contiguity and LAN results to show that the random sequence $\{p(h|T_n)\}$ which denotes the posterior density of the localized (around v_0) common value $\sqrt{n}(v-v_0)$ given T_n evaluated at h, has a limiting distribution equal to a posterior distribution of h from a Gaussian experiment with diffuse priors on h in which one obtains a single draw from a $N(h, V_{v_0})$ distribution. Roughly speaking, as n grows large, T_n is informationally equivalent (in terms of estimating the normalized common value $h = \sqrt{n}(v-v_0)$) to a single draw from a Gaussian experiment, and the prior information has a negligible effect.

Let $P(h|T_n)$ denote the posterior distribution of the localized (around v_0) common value, given T_n. Assume the prior distribution $\pi(\cdot)$ on v has a strictly positive and continuous density around v_0. Next we state the main result of this section (which is a version of the Bernstein-Von Mises theorem in van der Vaart (1999), theorem 10.1, for our non-*i.i.d.* setting) as a theorem:

Theorem 2 *The random total variation norm between $P(h|T_n)$ and $N\left(\triangle_{n,v_0}, G_v^{-2}\tau(1-\tau)\right)$ converges in probability to 0 under $P_{n,0}$:*

$$\left\|P(h|T_n) - N\left(\triangle_{n,v_0}, G_v^{-2}\tau(1-\tau)\right)\right\| \xrightarrow{P_{n,v_0}} 0$$

where

$$\triangle_{n,v_0} \equiv G_v^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(1\left(X_i \geq q_0^{1-\tau}\right) - \tau\right).$$

The total variation norm $\|\cdots\|$ is defined as $\|P-Q\| = 2\int (1-dP/dQ)^+ dQ$ where $(x)^+ =$ $\max\{x, 0\}$.

Proof: in Appendix.

Up to this point, we have shown that the (normalized) posterior $P(h|T_n)$ "converges" to a normal distribution with mean Δ_{n,v_0} in the sense that the total variation distance between these two distributions converges under $h = 0 \Leftrightarrow v = v_0$ to zero in probability. One immediate implication of this is that, as n gets large, the prior measure $\pi(v)$ does not matter (as long as it is well-behaved around v_0). This makes very explicit in what sense bidders' prior information is "swamped": namely, that bidders' posterior beliefs converge at rate $\sqrt{n}$ to a distribution which does not depend at all on the prior density $\pi(\cdot)$.

Having derived the limiting behavior of the sequence of posterior distributions $P(h|T_n)$, we proceed to the third step, wherein we derive the limiting behavior of the sequence of winning bids, viewed as the sequence of posterior means corresponding to the sequence $\{P(h|T_n)\}$.

The main result shows that, in the limit, the posterior mean of the central rank auction has the same distribution as the maximum likelihood estimator (cf. equation (11) above):

Theorem 3

$$E(h|T_n) \equiv E\left(\sqrt{n}\,(v - v_0)\,|T_n\right) \xrightarrow{d} N\left(0, G_v^{-1}\tau\,(1-\tau)\right).$$

Proof: Intuitively, in lemma 1 above we showed (cf. equation (14))

$$\frac{L\,(T_n|h)}{L\,(T_n|0)} \xrightarrow{a.s.} \exp\left(-G_v \frac{1}{\tau\,(1-\tau)} g\left(q_0^{1-\tau}|v_0\right)\sqrt{n}\left(T_n - q_0^{1-\tau}\right)\times h - \frac{1}{2}G_v^2 \frac{1}{\tau\,(1-\tau)}h^2\right)$$

so that, in large samples, the posterior density converges to

$$\begin{aligned} f\,(h|T_n) &= \frac{L\,(T_n|h)}{\int L\,(T_n|h)\,dh} = \frac{L\,(T_n|h)\,/L\,(T_n|0)}{\int L\,(T_n|h)\,/L\,(T_n|0)\,dh} \\ &\xrightarrow{a.s.} C\exp\left(-G_v \frac{1}{\tau\,(1-\tau)} g\left(q_0^{1-\tau}|v_0\right)\sqrt{n}\left(T_n - q_0^{1-\tau}\right)\times h - \frac{1}{2}G_v^2 \frac{1}{\tau\,(1-\tau)}h^2\right) \end{aligned}$$

which is approximately the density of a normal distribution with mean

$$G_v^{-1} g\left(q_0^{1-\tau}|v_0\right)\sqrt{n}\left(T_n - q_0^{1-\tau}\right) \equiv \triangle_{n,v_0}$$

and variance $G_v^{-2}\tau\,(1-\tau)$. Therefore under regularity conditions the posterior mean $E\,(h|T_n)$ converges a.s. to

$$G_v^{-1} g\left(q_0^{1-\tau}|v_0\right)\sqrt{n}\left(T_n - q_0^{1-\tau}\right) \equiv \triangle_{n,v_0}$$

which leads to the main result

$$E(h|T_n) \xrightarrow{d} \triangle_{n,v_0} \xrightarrow{d} N\left(0, G_v^{-1}\tau\,(1-\tau)\right).$$

■

It is important to emphasize that, under v_0, each sequence of T_n leads to a different limiting posterior density (and, given the appropriate normalizations, this density is non-degenerate). Theorem 2 essentially shows that, in the central rank case, the *family* of posterior densities $\{f(h|T_n)\}$, indexed by

$T_n \equiv X_{n-k:n}$, is "close"[10] to a family of Gaussian density functions indexed by the location parameters Δ_{n,v_0}. Furthermore, while the limiting posterior density of $\sqrt{n}\,(v - v_0)\,|T_n$ is *random* (due to randomness in T_n), corresponding to each member of this family of limiting posterior distributions, however, is a scalar posterior mean $E\left(\sqrt{n}\,(v - v_0)\,|T_n\right)$, which follows the same probability law as the random location parameter Δ_{n,v_0}, which, by the central limit theorem, is just the Gaussian distribution with zero and variance $G_v^{-1}\tau\,(1-\tau)$. This is what we term the limiting distribution of the winning bid.

Before proceeding, we note that the sequence of (normalized) posterior means has the same limit distribution as a maximum likelihood of v_0 given T_n. To see this, note that the asymptotic normality of T_n, the sequence of sample $(1-\tau)$-th quantiles, implies that any sequence of estimators $\hat{v}_n$ of v based on T_n would be asymptotic normal as well. Specifically, PS (pg. 1258, equation 6) show that the pivotal bidder's bid converges to the "inverse quantile" estimator

$$\hat{v}_n : G\left(T_n|\hat{v}\right) = 1 - \tau \tag{9}$$

which is equivalent to the maximal likelihood estimator, which is the value $\hat{v}$ which maximizes the conditional likelihood of $X_{k_n:n} = T_n|v$. For the central rank auctions, this likelihood is (for $k \approx n\tau$):

$$\mathcal{L}\left(T_n|v\right) = \binom{n}{k} G\left(T_n|v\right)^{n-k}\left(1 - G\left(T_n|v\right)\right)^k \tag{10}$$

with the limiting distribution under $P_{n,0}$ for $0 < \tau < 1$:

$$\sqrt{n}\left(\hat{v}_n - v_0\right) = -G_v^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(1\left(X_i \geq q_0^{1-\tau}\right) - \tau\right) + o_p\left(1\right) \xrightarrow{d} N\left(0, G_v^{-2}\tau\left(1-\tau\right)\right). \tag{11}$$

where $q_0^{1-\tau} \equiv G^{-1}\left(1-\tau|v_0\right)$ and $G_v \equiv G_v\left(q_0^{1-\tau}|v_0\right) = -g\left(q_0^{1-\tau}|v_0\right)U_v\left(v_0\right)$ for $U\left(v_0\right) = G^{-1}\left(1-\tau|v_0\right)$ denoting the quantile as a function of v_0.

The asymptotic equivalence of the "classical" maximum likelihood estimator with the Bayesian posterior mean estimator is another manifestation of PS's result that, in the limit, bidders' prior information is swamped by the equilibrium information obtained by conditioning on the event of being pivotal.

Asymptotic properties of market price and equilibrium bids "close" to the winning bid The major difference between the pivotal bidder and her rivals is that, for example, the highest bidder observes T_n^n, but still believes that the k-th and $k+1$-th highest draws are equal to T_n^n. Roughly speaking, only the pivotal bidder is "correct" in equilibrium. Next we ask: do the equilibrium bids of the non-pivotal bidders aggregate information in any way (and if so, at what rate)?

It turns out that the above results hold for the sequences of both the market price and the winning bid, as well as for the sequences of equilibrium bids within any fixed rank (i.e., within a vanishing

[10] In the sense that the sequence of total variation norms between the two families converges in probability under v_0 to zero.

neighborhood as $n \to \infty$) of the pivotal bidder's rank. In particular, let T_n^0 denote the signal of the pivotal bidder (the winning bidder); then for any finite m (a positive or negative integer), let T_n^m denote the signal of the bidder whose rank is m above(negative m means below) that of the pivotal bidder. Then the sequence $\left(T_n^m - T_n^0\right)$ is $O_{a.s.}\left(\frac{1}{n}\right)$ or $o_{a.s.}\left(\frac{1}{\sqrt{n}}\right)$: in other words, the difference between $T_n^m - T_n^0$ dwindles (stochastically) at a rate faster than $\sqrt{n}$ which, as shown above, is the convergence rate for the sequence T_n^0.

This directly implies that the difference in the sequences of equilibrium bids submitted on the basis of the two signal sequences T_n^0 and T_n^m is also $o_p\left(\frac{1}{\sqrt{n}}\right)$. More precisely, for $B_n^m = b\left(T_n^m\right)$:

$$\sqrt{n}\left(B_n^m - B_n^0\right) = \left(E\left(h|T_n^m\right) - E\left(h|T_n^0\right)\right) = o_{a.s.}(1).$$

As a consequence, for any finite integer $m > 0$, the entire $2m+1$ vector of normalized bids $\sqrt{n}\left(B - v_0\right)$, where $B \equiv \left(B_n^{-m}, \ldots, B_n^{-1}, B_n^0, B_n^1, \ldots, B_n^m\right)'$ converges almost surely to a vector of random variables, all with marginal Gaussian distributions with mean zero and variance $G_v^{-1}\tau\left(1-\tau\right)$. In other words, *each element of* $\sqrt{n}\left(B - v_0\right)$ *possesses the* same *marginal limiting distribution.* This leads to the following striking corollary:

Corollary 2 *For any integer m with $|m| < \infty$,*

$$B_n^m \equiv \left(B_n^{-m}, \ldots, B_n^{-1}, B_n^0, B_n^1, \ldots, B_n^m\right)' \overset{a.s.}{\to} v_0;$$

$$\sqrt{n}\left(B_n^m - v_0\right) \overset{d}{\to} N\left(0, G_v^{-1}\tau\left(1-\tau\right)\right).$$

Any fixed rank sequence B_n^m, for $|m| < \infty$, aggregates information in the auction, in the sense of being a strongly consistent estimator for v_0. Even more remarkably, they all aggregate information *equally well*, in terms of having the same marginal limiting distribution (and therefore the same rate of convergence). Thus, having "wrong beliefs" (as in the case of the non-pivotal bidders who bid B_n^m with $|m| > 0$) do not matter, as long as the degree that the beliefs are wrong vanishes in the limit. These are very strong statements about common value auctions, in the central rank case.

Central Rank Auctions: non-smooth $\mathbf{g(x|v)}$ In the previous section (and in the related literature on quantile regression analysis), we have assumed that the density $g(x|v)$ is continuous at least in a small neighborhood at the estimating quantile. In this section, we consider a more general case by allowing for discontinuities in the density. These results serve to demonstrate the robustness of the results in the previous sections.

We consider the central rank case where, for a fixed $0 < \tau < 1$, and $k_n = \tau * n$, $\forall n$. We allow for a discontinuity ("jump") in the density $g(x|v)$ at the $1-\tau$-th quantile.[11] Let $g^-\left(q|v\right)$ denote the left derivative of $G\left(\cdot|v\right)$ at q and $g^+\left(q|v\right)$ denote as the right derivative of $G\left(\cdot|v\right)$ at q; our allowance for a discontinuity at $G(\cdot|v) = 1-\tau$ implies that $g^-\left(q^{1-\tau}|v\right) \neq g^+\left(q^{1-\tau}|v\right)$.

[11] Note that this is the only place where a jump would matter. Jumps at any other quantile will have no effect in the limit since, as the sample size increases, the sample $1-\tau$-th quantile (which is all the pivotal bidder observes in the auction) will be arbitrarily distant from the jump quantile with arbitrarily large $G(x|v_0)$-probability.

Interestingly, unlike the results in the previous sections where the maximum likelihood estimator and the posterior mean converge at the same rate, in this non-smooth case the quantile estimator still converges at $\sqrt{n}$ rate to a "two sided normal" distribution[12] but the maximum likelihood estimator converges at rate n to a functional of a certain Poisson process (cf. Ibragimov and Has'minskii (1981), chap. 5).

The main results are summarized in the following theorem.

Theorem 4 *In the central rank case where $n \to \infty$, $k_n = O(n)$, $k_n/n \to \tau$ where $0 < \tau < 1$:*
(a) *For $X \sim N(0,1)$, $X^+ \equiv \max\{0, X\}$ and $X^- \equiv \min\{0, X\}$,*

$$\sqrt{n}\left(T_n - q^{1-\tau}\right) \xrightarrow{d} Y \equiv \frac{\sqrt{\tau(1-\tau)}}{g^+\left(q^{1-\tau}|v_0\right)} X^+ - \frac{\sqrt{\tau(1-\tau)}}{g^-\left(q^{1-\tau}|v_0\right)} X^-$$

under v_0;
(b) *For a fixed h, the normalized posterior density $f(h|T_n)$ converges almost surely to a random variable which is proportional to $dN(0,1)(X)$, the standard Gaussian density evaluated at the random quantity*

$$X \equiv -\frac{g^+ U_v}{\sqrt{\tau(1-\tau)}}\left(h - U_v^{-1}\sqrt{n}\left(T_n - q^{1-\tau}\right)\right)^- + \frac{g^- U_v}{\sqrt{\tau(1-\tau)}}\left(h - U_v^{-1}\sqrt{n}\left(T_n - q^{1-\tau}\right)\right)^+$$

under v_0;
(c) *Under v_0, the normalized posterior mean*

$$\sqrt{n}\left(E(v|T_n) - v_0\right) \xrightarrow{d} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{1}{g^-} - \frac{1}{g^+}\right) + \frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{X^+}{g^+} - \frac{X^-}{g^-}\right), \tag{12}$$

which is a two-sided Gaussian random variable with median $\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{1}{g^-} - \frac{1}{g^+}\right)$ and mean 0 (i.e., asymptotically unbiased).

Proof: in appendix.

2.2 Intermediate Rank Auctions: Auctions with Double Largeness where $\mathbf{k_n = o(n)}$

Next we consider the case where $k_n \to \infty, n \to \infty$, but $k_n/n \to 0 \Leftrightarrow k_n = o(n)$ (which satisfies the "double largeness" condition of PS). We give the simple result for the case where $g(s|v)$ is strictly bounded away from 0 and ∞ on its support (see Chernozhukov, Hong, and Shum (2000) for a consideration of more general cases).

Theorem 5 *For intermediate rank auctions where $n \to \infty$, $k_n = o(n)$, $\tau_n \equiv k_n/n \to 0$,*

$$E\left[\sqrt{\frac{n^2}{k_n}}\left(v - v_0\right)|T_n^{k_n}\right] \xrightarrow{d} N\left(0, G_v^{-1}\tau_n\left(1-\tau_n\right)\right).$$

Proof: in appendix. Here the rate sequence $r_n \equiv \sqrt{\frac{n^2}{k_n}}$ lies at or above $\sqrt{n}$: note that if $k_n = O(n)$, then $r_n = O(\sqrt{n})$ which is the case of the non-extreme case considered in the previous section.

[12] We thank Bo Honore for insightful comments regarding results in this section.

This case of an asymptotically finite number of objects leads to faster rates of convergence than in the non-extreme cases considered in the previous section. In fact, this is the first appearance of a recurring theme in this paper: auctions in which the bidders condition on (perhaps asymptotically) "extremal" events (in the sense that $\tau_n \to 0$) lead to faster rates of convergence: alternatively, *extremal events are more informative concerning* v_0 *than non-extremal events.* Therefore, auctions in which the event of being pivotal is an extremal event will aggregate information faster. In the following sections, we study several cases of such auctions: fixed-rank (which includes second-price), first-price, and English auctions.

3 Fixed Rank Auctions ($\tau = 1$)

The main difficulty in analyzing the "fixed rank" case (where k_n is finite and fixed for all n as $n \to \infty$) is that the sequence of auctions do not constitute a "locally asymptotic normal" experiments (in the sense of corollary 1 above), so that the arguments we have used so far for the central-rank case are generally not valid. However, extreme value theory can be used to show that the sequence of experiments is asymptotically exponential or gamma, with a faster rate of convergence.

Recall that for fixed-rank auctions, we make an additional assumption (2), which states that $U(v)$ is strictly increasing in v: this is the same assumption made in Wilson (1977). Furthermore, the assumption that $g(U(v)|v) > 0$ implies discrete jumps in both $g(x|v)$ and $G(x|v)$ at $x = U(v)$.[13]

Consider a fixed value for k, and let T_n^k here denote the k-th highest out of n *i.i.d.* draws from $G(\cdot|v)$; in equilibrium, T_n^k is the signal received by the pivotal bidder, in PS's terminology. Furthermore, in equilibrium, the pivotal bidder bases her estimate of v on the conditioning event that the k-th and $k+1$-th highest draws are *both* equal to T_n^k (cf. expression (2)); let $\mathcal{S}_n^k$ denote this conditioning event.

Theorem 6 *Given assumptions (1) and (2), and under* v_0*, the sequence of normalized posterior means*

$$n\left(E(v|\mathcal{S}_n^k) - v_0\right) \xrightarrow{d} -\frac{k}{G_v\left(U\left(v_0\right)|v_0\right)} - \frac{1}{\left(U_v\left(v_0\right)\right)g\left(U\left(v_0\right)|v_0\right)}\mathcal{P}_k$$

where $\mathcal{P}_k$ *is a random variate from the Gamma* $(k,1)$ *distribution.*

Proof: in appendix.

[13] Indeed, the upward jump $G(U(v)|v)^+ - G(U(v)|v)^- = 1 - G(U(v)|v)^- = g(U(v)|v)$, implying that

$$\forall v, \forall x \in \mathcal{X}_v : \lim_{t\to 0} \frac{1 - G(U(v) - xt|v)}{1 - G(U(v) - t|v)} = \frac{1 - G(U(v)|v)^-}{1 - G(U(v)|v)^-} = 1,$$

so that the necessary and sufficient conditions for the convergence of the sequence of sample maxima from $G(\cdot|v)$ to a type III extreme value distribution are satisfied (cf. Leadbetter, Lindgren, and Rootzen (1983), Thm. 1.6.2, with parameter $\alpha = 1$). Technically speaking, this corresponds to the case where the family of tail distributions $1 - G(\cdot|v)$ is *slowly varying* (cf. Feller (1971), section VIII.8). In a companion paper (Chernozhukov, Hong, and Shum (2000)), we also consider the more general (but admittedly technical) case of *regular variation*, where $\alpha \neq 1$: this corresponds to the case where the density $g(x|v)$ vanishes in an arbitrarily small neighborhood of $U(v)$.

An important and immediate implication of this theorem is that the winning bid $E(v|S_n^k)$ converges to the true common value v_0 at rate n, which is much faster than the $\sqrt{n}$ rate for the central rank auctions. For the second-price auction (i.e., $k=1$):

Corollary 3 *Given assumptions (1) and (2), under v_0, the sequence of normalized winning bids in a second-price auction*

$$n\left(E(v|\mathcal{S}_n^1)-v_0\right) \xrightarrow{d} -\frac{1}{\left(U_v\left(v_0\right)\right)g\left(U\left(v_0\right)|v_0\right)}\mathcal{E}$$

for $\mathcal{E}$ being the standard exponential distribution.

The difference in the limiting distributions of the winning bid for the central and fixed rank cases is profound. Intuitively, for the "regular" central rank case where τ is bounded away from 0 and 1, all the equilibrium information about v obtained by the pivotal bidder becomes, in the limit, equivalent to the information contained in a single draw from a Gaussian distribution with unknown mean v; clearly, in that "limit experiment", the draw itself is the most reasonable estimate of v.

The fixed rank case (i.e., k fixed, $\frac{k}{n}\to 0$), on the other hand, represents a "nonregular" estimation problem, in the sense that the family of conditional densities $g(x|v)$ exhibits a jump to zero at $x=U(v)$, for each v. This jump actually speeds up the information aggregation process. In the limit, an observation of a fixed rank T_n^k is informationally equivalent to an observation of the k-th stopping time (i.e., the "time" until the k-th jump or arrival) from a standard (mean 1) Poisson process.

Intuitively, the Poisson aspects of the limiting behavior of T_n^k can be motivated as follows. The likelihood $L(T_n^k|v)$ is equivalent to the likelihood of k "successes" in the *i.i.d.* binomial experiment $\left(n, 1-G(T_n^k|v)\right)$. As n gets large, the success probability $1-G(T_n^k|v)\to 0$ for any finite k, so that the likelihood $L(T_n^k|v)$ is approximated by the Poisson likelihood of the k-th "arrival" occuring at time T_n^k or, equivalently, the likelihood that the k-th stopping time is T_n^k; for the Poisson distribution, the distribution of the k-th stopping time is described by a Gamma$(k,1)$ distribution.

The faster rate of convergence in the fixed rank case may be attributed to the "one-sided" nature of the limit problem: defining $\bar{V}_n(x)\equiv \max\{v : g(x|v)=0\}$, upon an observation T_n^k one can immediately limit one's possible estimates of v to those which exceed $\bar{V}(T_n^k)$. This one-sided aspect of the fixed rank case is completely missing from the central rank case. The important implication is that fixed rank observations are more "informative", or lead to faster information aggregation, than central rank observations. Therefore, from a policy point of view, a fixed rank auction mechanism may be preferred to a central rank auction mechanism (as in PS) for this attractive asymptotic property.[14]

[14] It may appear puzzling that even though $k_n/n\to\infty$ in both the fixed and intermediate rank cases (the latter discussed in section 2.2 above), that the rates of convergence and limiting distribution are so different. However, there is a crucial difference between these two cases: in the intermediate rank case, the boundary condition that $T_n^{k_n}$ cannot exceed $U(v_0+ha_n^{-1})$ for any h is not binding, since $1\left(T_n^{k_n}\le U\left(v_0+ha_n^{-1}\right)\right)\approx 1\left(a_n\left(T_n^{k_n}-U\left(v_0\right)\right)\le U_v h\right)$, and

Asymptotic properties of bids "close" to the winning bid Note that given the large sample approximation in equation (21), the limit distribution (27) has mean zero. In fact, this is only true for the pivotal bidder: specifically, it is not true for the price setter, who (in equilibrium) is the $k+1$-th highest bidder. In fact, , note that the normalized vector of the m highest signals $n\left(T_n^1 - U(v_0)\right), \ldots, n\left(T_n^m - U(v_0)\right)$ $(m < n)$ has a joint Gamma distribution or, equivalently, is jointly distributed as the first m stopping times of a Poisson process. Hence, we can immediately derive the following corollary, which is the analogue of corollary 2 above for fixed-rank auctions:

Corollary 4 *For any integer $m < n$, the normalized vector of bids submitted by the first m highest bidders, $n(B - v_0)$, where the vector of the m highest bids $B_n^m \equiv \left(E\left(h|T_n^1\right), \ldots, E\left(h|T_n^m\right)\right)'$ converges almost surely to the vector*

$$-\frac{k}{G_v\left(U\left(v_0\right)|v_0\right)} - \frac{1}{\left(U_v\left(v_0\right)\right) g\left(U\left(v_0\right)|v_0\right)}\mathcal{P}$$

where $\mathcal{P} = \left(\mathcal{P}_n^1, \ldots, \mathcal{P}_m\right)'$ is the vector of the first m arrival time of the Poisson process with arrival rate $\left(U_v\left(v_0\right)\right) g\left(U\left(v_0\right)|v_0\right)$.

Specifically, under v_0, the marginal limiting distribution of the m-th highest bidder (where $m < \infty$ and is fixed as $n \to \infty$) is

$$-\frac{k}{G_v\left(U\left(v_0\right)|v_0\right)} - \frac{1}{\left(U_v\left(v_0\right)\right) g\left(U\left(v_0\right)|v_0\right)}\mathcal{P}_m.$$

with asymptotic mean $(k-m)\frac{1}{\left(U_v(v_0)\right) g\left(U(v_0)|v_0\right)}$.

Proof: Same as proof for theorem 6, except that, under v_0, $n\left(T_n^m - U(v_0)\right) \xrightarrow{d} -\frac{1}{g(U(v_0))}\mathcal{P}_m$. ■.

As a consequence, each of the largest m bids is a consistent estimator of the common value v_0; furthermore, while they share the same rate of convergence, they have different (marginal) limit distributions. Furthermore, all of these bids are asymptotically biased: in particular, the sequence of normalized market prices $E(h|S_n^{k+1})$ converges at linear rate n to a distribution with mean $-\frac{1}{(U_v(v_0))g(U(v_0)|v_0)}$. Intuitively, one may attribute this "asymptotic bias" to the price setter's equilibrium beliefs, which are "pessimistic" in the sense that he bids as if his signal (which in reality is the $k+1$-th highest signal), were actually the k-th highest signal, leading him to systematically "undervalue" the object. Unlike the central rank auctions discussed above, the market price has a different limit distribution than the winning bid in fixed rank auctions.

Importantly, however, the presence of asymptotic bias does not contradict the fact that the sequence of market prices $E(v|S_n^{k+1})$ is incosistent for v_0; asymptotic bias refers just to the notion that the mean of the limit distirbution of the normalized sequence of market prices is not zero. While Theorem 1 implies that $E(v|S_n^{k+1})$ is consistent (i.e., converges to v_0 in P_{v_0}-probability). The results in this section imply that there is no linear normalization of the $E(v|S_n^{k+1})$ which has (i.e., no norming

$a_n\left(T_n^{k_n} - U(v_0)\right) \to -\infty$. Therefore, asymptotically, the Poisson approximation which we used for the fixed rank case where k_n was fixed for all n is not appropriate. For the latter, in fact, T^{k_n} will exceed $U(v_0 + h/n)$ for any finite $h < 0$ with probability one (i.e., $G(x|v_0)$-almost everywhere).

sequence r_n such that $r_n \left(E(v|S_n^{k+1}) - v_0\right)$ possesses) a nondegenerate limiting distribution centred at the true value. On the other hand, the sequence of normalized winning bids $n\left(E(v|S_n^k) - v_0\right)$ does indeed have a limiting distribution with mean zero under v_0.

Limiting form of the equilibrium bidding strategy In the previous discussion, we have focused on the convergence properties of equilibrium bids submitted by the pivotal bidder or by bidders in a vanishing neighborhood around the pivotal bidder. This next theorem describes the convergence properties for the sequence of bids corresponding to *any* signal x, under v_0. For any signal x, we define the "pseudo-MLE" $\hat{v}(s)$ as:

$$\hat{v}(x) \equiv v : G(x|v) = \tau.$$

where τ is the limit of the sequence $\tau_n \equiv k_n/n$ is some sequence of auctions under consideration. From assumption 1, the function $\hat{v}(x)$ is monotonic increasing in x. Furthermore, let $f_n(\cdot|x)$ denote the sequence of posterior densities for v, conditional on a signal x, and let $f_n(\tilde{v}|x)$ denote the sequence of posterior densities evaluated at $\tilde{v}$.

Corollary 5 *For each x, the sequence of posterior densities $f_n(\cdot|x)$ converges at rate a_n to the point mass at $\hat{v}(x)$. a_n is n for the fixed rank case, $\sqrt{n}$ for the central rank case, and $\sqrt{\frac{n^2}{k_n}}$ for the intermediate rank case. Furthermore, this convergence is uniform in x, i.e.,:*

For every $\epsilon > 0$, there exists an $N_0(\epsilon)$ and a $\delta(\epsilon)$ such that for all $n > N_0(\epsilon)$,

$$sup_{x\in\mathcal{X}(v_0)}\ Prob\left(||f_n(\cdot|x) - \chi(\hat{v}(x))|| < \epsilon\right) > 1 - \delta(\epsilon).$$

This is true under any $v_0 \in \mathcal{V}$.

where $\chi(x)$ denotes a point mass at x.

Proof (sketch): For each v_0, and for each $x \in \mathcal{X}(v_0)$, each sequence $a_n\left(f(\tilde{v}|x)\right) = O_{v_0}(1)$ for $\tilde{v} \neq \hat{v}(x)$, and arbitrarily large at $\tilde{v} = \hat{v}(x)$. ■

We emphasize here that for a fixed x, the density $f_n(\cdot|x)$ has the *same limiting function* (namely, $\chi(\hat{v}(x))$)), for *all* v_0 such that $g(x|v_0) > 0$.

4 First-Price Auctions

Next we discuss limiting results for the winning bid in first-price auctions. It turns out that the results are very similar to the results for the second-rank auction, with $k = 1$. The convergence rate is n, and the limiting experiment is of the exponential form.

Let b_n^{1p} denote a sequence of winning bids in the first-price auction. Under v_0, the limiting distribution of winning bid in the high-bid first price auction is an asymptotically downward biased estimate of the common value, in the sense that the nondegenerate limiting distribution of the normalized winning bids $n(b_n^{1p} - v_0)$ does not have mean zero.

Theorem 7 *Given assumptions (1) and (2), and under any $v_0 \in \mathcal{V}$,*

$$\begin{aligned} n(b^{1p} - v_0) &\xrightarrow{d} -\frac{1}{U'(v_0)\, g(U(v_0)\,|v_0)} - \frac{1}{G_v(U(v_0)\,|v_0)} - \frac{1}{(U_v(v_0))\, g(U(v_0)\,|v_0)}\mathcal{E} \\ &\approx -\frac{1}{(U_v(v_0))\, g(U(v_0)\,|v_0)}\mathcal{E} \end{aligned}$$

where $\mathcal{E}$ is an exponential random variable with mean 1. The limit distribution is downward-biased with mean $-\frac{1}{(U_v(v_0))g(U(v_0)|v_0)}$.

Proof: In appendix.

Comparing the results for the first- and second-price auctions, we see that the winning bids for both auction formats have identical limit distributions, up to a location shift:

Corollary 6 *Let b_n^{2p} denote a sequence of winning bids in a $k = 1$ second-price auction. Then, given assumptions (1) and (2), and under any $v_0 \in \mathcal{V}$,*

$$\begin{aligned} n(b^{1p} - v_0) + \frac{1}{U'(v_0)\, g(U(v_0)\,|v_0)} &\xrightarrow{d} n(b^{2p} - v_0) \\ &\xrightarrow{d} \frac{1}{U'(v_0)\, g(U(v_0)\,|v_0)} - \frac{1}{(U_v(v_0))\, g(U(v_0)\,|v_0)}\mathcal{E} \end{aligned}$$

where $\mathcal{E}$ is an exponential random variable with mean 1; the limit distribution has mean zero.

5 English auctions

Intuition suggests that since more information is revealed during an English auction, the winning bid in the English auction may have better convergence properties than both the first price and second price price. This reasoning is partly true. Indeed, the winning bid in the English auction, by making use of information contained in all signals except the second highest signal, resembles closely a full sample Bayesian posterior mean estimator of the common value, the limiting behavior of which is closely related to that of the full sample maximum likelihood estimator.

In fact, by viewing the winning bid as a Bayesian statistical estimator for the common value, it is not difficult to obtain the following results: given assumption (2) the winning bid in the English auction will converge at rate n to the true common value, thus sharing the fast rates of convergence for the fixed rank and first-price auctions. On the other hand, if only assumption (1) holds, then the winning bid in the English auction will converge at rate $\sqrt{n}$ to the true common value, and possess the same asymptotic optimality properties of the full sample maximum likelihood estimator.[15] This implies some attractive "robustness" properties of the English auction, which we will describe in more detail below.

Let $b_n^E \equiv E(v|X_1 = X_2 = x_1, X_3 = x_3, \ldots, X_n = x_n)$ denote the sequence of winning bids in the English auction (as described in Milgrom and Weber (1982)), where X_m denotes the m-th order

[15] Recall that if assumption (2) fails, the winning bids for the fixed rank and first-price auctions are not even consistent, i.e., do not even aggregate information.

statistic from a sample of n *i.i.d.* draws. Our major findings are summarized in the following theorem:

Theorem 8 *(i) Under assumptions (1) and (2),*

$$n\left(b_n^E - v_0\right) \xrightarrow{d} \frac{1}{g\left(U\left(v\right)|v\right)U'\left(v\right)} - \frac{1}{g\left(U\left(v\right)|v\right)}\mathcal{E}$$

where E denotes a standard exponential random variable (with mean 1).

(ii) Under assumption (1),

$$\sqrt{n}\left(b_n^E - v_0\right) \xrightarrow{d} N\left(0, \left[\int \frac{g_v\left(x|v\right)^2}{g\left(x|v\right)}dx\right]^{-1}\right)$$

where N denotes a Gaussian distribution.

Proof: In appendix. For part (i), the proof resembles that for theorem 6. For part (ii), the proof resembles that for theorem 3.

Note that given assumptions (1) and (2), the winning bids in the English auction have the same limiting distribution as the winning bids in second-price auctions (cf. corollary 3 above).

The convergence properties of the English auction are robust to changes in the information structure. If both assumptions (1) and (2) hold, then the English auction shares the consistency and rapid information aggregation of the fixed-rank and first-price auctions. Furthermore, even in cases where assumption (2) fails and the fixed rank and first-price auctions are not consistent, the English auctions is still consistent, and shares the slower $\sqrt{n}$ rate of information aggregation of the central rank auctions.

Furthermore, consider the case where assumption (2) fails, but where the *lower* bound of the support of $\mathcal{X}_v$ is increasing in v, for all v. Such an information structure still satisfies assumption (1), so that the central rank auctions still aggregate information at rate $\sqrt{n}$, while the fixed rank and first-price auctions are inconsistent. For this case, it is straightforward to show that the English is still consistent, with a rapid rate n of convergence![16]

This may appear a somewhat surprising result, but derives from the fact that, asymptotically, the winning bid in an English auction is identical to the full-information maximum likelihood estimator of the common value, and thus possesses the same properties. In particular, note that, for the second case in the above theorem, $\int \frac{f_v(x|v)^2}{f(x|v)}dx$ is the Fisher information for the experiment involving repeated *i.i.d.* draws from $g(\cdot|v)$, and $\left[\int \frac{g_v(x|v)^2}{g(x|v)}dx\right]^{-1}$ is the famous Cramér-Rao lower bound on the variance in estimating the common value v_0 at a $\sqrt{n}$ rate (equivalently, the Fisher information is an upper bound on the precision that is attainable). Therefore:

[16] Additional results when either end or both ends of the boundary of the signal distribution depend on the common value are available from the authors. In those cases the limiting distribution for the winning bid converges to a functional of a Poisson process (cf. Ibragimov and Has'minskii (1981), chap. 5).

Corollary 7 *Under assumption (1), b_n^E is an* asymptotically efficient *sequence of estimators of v: the asymptotic variance of $\sqrt{n}\left(b_n^E - v_0\right)$ does not exceed the asymptotic variance of any other $\sqrt{n}$-consistent estimator of v_0.*

6 An Illustrative Example

In this section we present simulation results to illustrate the main points made above. We consider the following example, which satisfies both assumptions (1) and (2) above:

- Uniform priors: $f(v) = 1$, for $v \in [0, 1]$.
- Uniform conditional density of signals:

$$g(s|v) = \begin{cases} \frac{1}{v} & 0 \leq s \leq v \\ 0 & \text{otherwise} \end{cases}$$

- The "true" common value, v_0, is fixed at 0.75 throughout.

We consider three sequences of auctions: (i) a central rank auction in which τ_n is fixed at 0.5 (i.e., $k_n = 0.5 * n$); (ii) an intermediate rank auction where $k_n = n^{1/3}$; and (iii) a fixed rank case where $k_n = 1$, $\forall n$. In each case, we perform two sets of simulations. First, we provide simulations of the posterior density of v conditional on the pivotal bidder's equilibrium beliefs, i.e., $\left(v|\mathcal{S}_n^{k_n}\right)$, where $T_n^{k_n}$ denotes the k_n-th highest variate out of n *i.i.d.* draws from $G(s|v_0)$. Equilibrium bid functions are simulated to illustrate the convergence of the pivotal bidder's bid to the true value $v_0 = 0.75$. Second, we simulate the distribution of the normalized posterior means $r_n\left(E\left(v|\mathcal{S}_n^{k_n}\right) - v_0\right)$ for different number of bidders n, in order to demonstrate that as n increases, these distributions approach the limit distributions described above. For comparison's sake, we also simulate the density of the pivotal bidder's MLE of v_0 corresponding to each case; these will be discussed in more detail below.

6.1 Limiting distributions

Recall that, asymptotically, an important difference between the MLE and posterior mean for the intermediate and fixed rank cases is that the limiting distribution of the (normalized) MLE has a nonzero mean, which is not true for the posterior mean. These differences are illustrated in figures 1, 2 and 3 corresponding to, respectively, the central rank, intermediate rank, and fixed rank cases. In each case, we have plotted the empirical density (histogram) of 100 simulated posterior means (normalized at the appropriate rates) in the top graphs, the corresponding normalized MLEs in the middle graphs, and their pointwise differences in the bottom graphs. In all cases, the simulations employed auctions with $n = 1000$ bidders.

Note that, in general, the shapes of the densities correspond to the forms derived above. The posterior means and MLEs in the central and intermediate rank cases possess Gaussian limiting distributions, and this is clear from figures 1 and 2. The exponential shape for the fixed rank case is

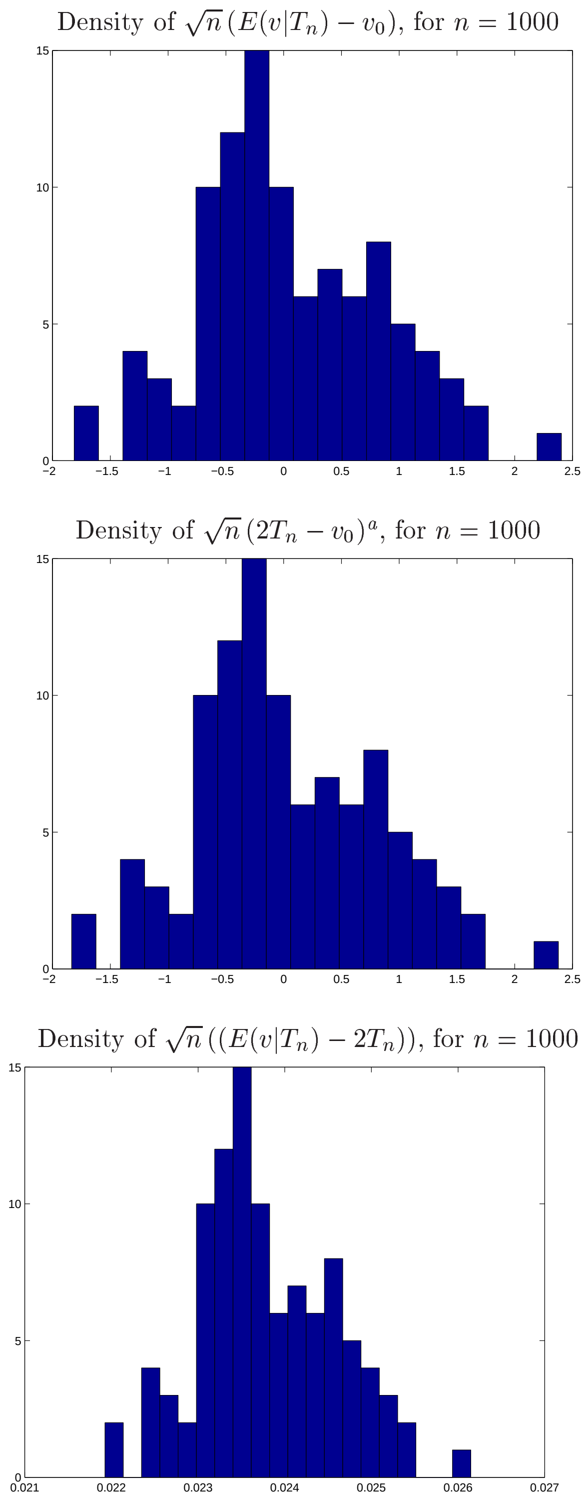
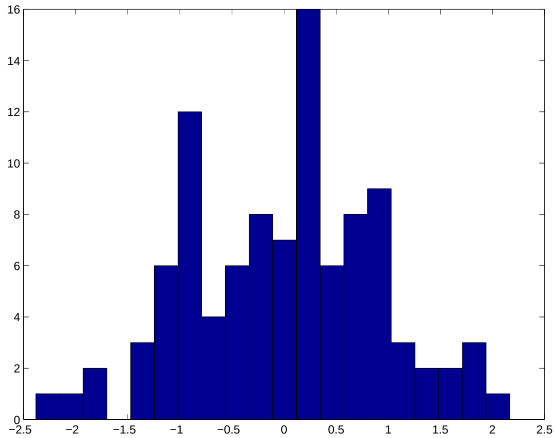


Figure 1: Simulated densities of (normalized) posterior mean for pivotal bidder: central rank case ($\tau = 0.5$)

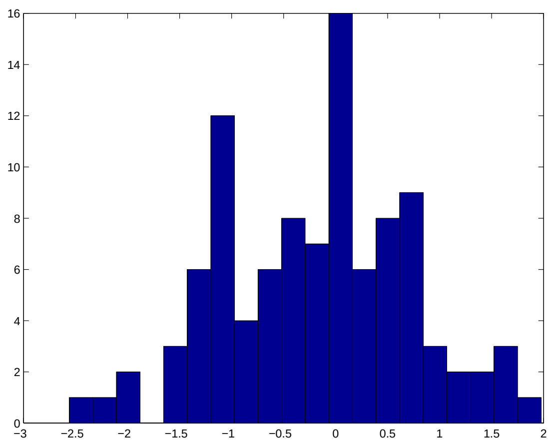
x-axis: $v \in [0, 1]$, True v_0: 0.75, n=1000.

[a] $2x$ is the MLE of v for the central rank case where $\tau_n = 0.5$, derived solely from the information of being pivotal.

Density of $\sqrt{n^3/\left(k\left(n-k\right)\right)}\left(E(v|T_n)-v_0\right)$, for $n=1000$



Density of $\sqrt{n^3/\left(k\left(n-k\right)\right)}\left(10*T_n-v_0\right)^a$, for $n=1000$



Density of $\sqrt{n^3/\left(k\left(n-k\right)\right)}\left(\left(E(v|T_n)-10*T_n\right)\right)$, for $n=1000$

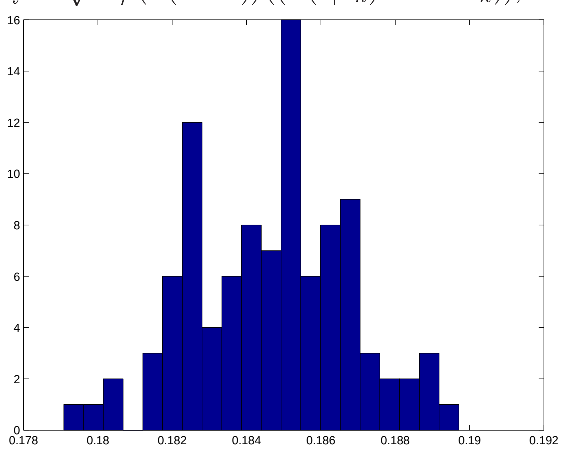


Figure 2: Simulated densities of (normalized) posterior mean for pivotal bidder: intermediate rank case ($k_n = n^{1/3}$)

x-axis: $v \in [0,1]$, True v_0: 0.75, n=1000.

[a] $10x$ is the MLE of v for the intermediate rank case where $k_n = n^{1/3}$, which for $n = 1000$ implies $k = 10$.

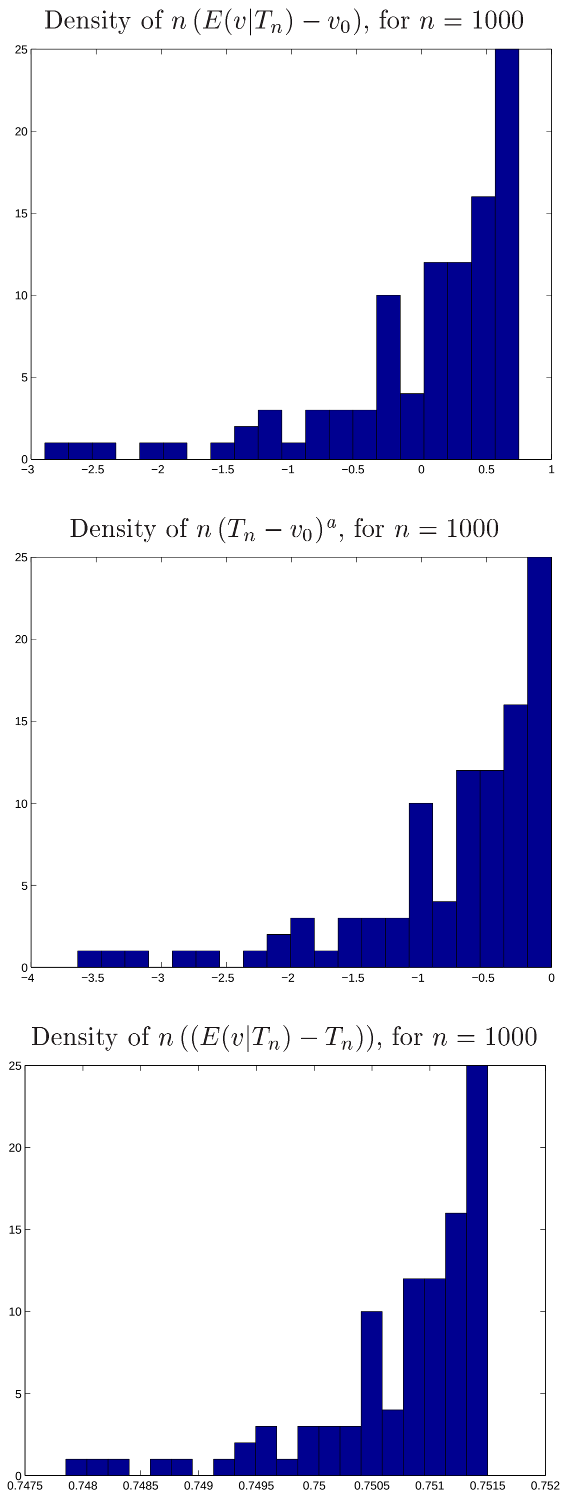


Figure 3: Simulated densities of (normalized) posterior mean for pivotal bidder: fixed rank case ($k_n = 1$)

x-axis: $v \in [0, 1]$, True v_0: 0.75, n=1000.

---

[a] x is the MLE of v in the fixed rank $k = 1$ case, derived solely from the information of being pivotal.

also apparent. Figure 3 also confirms the large asymptotic bias of the MLE in the fixed rank case; in fact, there is a point mass in the differences at 0.75, which is exactly the value predicted by the theory[17].

6.2 Convergence of beliefs and limiting bidding behavior

We plot simulated posterior densities for the pivotal bidder, in all three cases, in figure 4. It is readily apparent that the equilibrium beliefs converge much more slowly in the central rank case, thus illustrating clearly the slower convergence rate ($\sqrt{n}$) of the central rank case relative to the faster (n) convergence rate for the fixed rank case.

Recall that the intermediate rank case is a special case of the central rank case, and beliefs converge in that case faster than $\sqrt{n}$, but slower than the fixed rank case of n. These differences are obvious in the simulated example: the density corresponding to $n = 100$ is more concentrated in the fixed rank case than in the intermediate rank case.[18]

The implications for these different rates of convergence are illustrated by the plots of the simulated equilibrium bid functions, shown in figure 5. In each graph, we plot a horizontal line at 0.75, the true value of the common value, and so the intersection of this horizontal line with the bid functions will identify the signal of the bidder who bids closest to the true value. As n grows large, information aggregation implies this signal should belong to the pivotal bidder (i.e., $b_n^{-1}(v_0) = G^{-1}(\tau_n|v_0)$).

For the central rank case, the pivotal signal is $G^{-1}(0.5|v_0) = 0.375$, and indeed it is clear that as n increases the bidder who obtains a signal 0.375 will indeed bid close to v_0. Furthermore, note that for a large range of signals (from about $x > 0.25$), the bid functions are actually increasing in n: this is evidence of the *loser's curse* (as pointed out by PS) whereby aversion to losing when one "should have" (statistically) won causes bidder to bid more aggressively in response to increased competition. This is the opposite of the winner's curse, and is a feature only in multi-object auctions. Note also that the bid functions for $n = 50$ and $n = 100$ cross very close to 0.375, the pivotal signal; this implies that in that range of n, information aggregation has already occurred to such an extent that only bidders with signals above the pivotal signal try to accommodate the loser's curse, and only those with signals below the pivotal signal accommodate the winner's curse.

The bid functions for the intermediate and fixed rank cases are likewise different, and also show evidence of the faster convergence of beliefs in the fixed rank case in that the bid functions for the $n = 50$ and $n = 100$ cases are "closer together" in that case. However, note that since these auctions are both $\tau = 0$ auctions, in the limit, the loser's curse is not present at all.

Information aggregation PS show that the crucial mechanism of information aggregation is that, in the limit, bidders base their estimate of the common value v solely on the event that their signal is

[17]For this example, the asymptotic bias $1/G_v(x|v_0) = -\frac{v_0^2}{x} = -0.75$, when evaluated at $x = U(v_0) = 0.75$.

[18]However, additional simulations (not reported here) show, as we would expect, that as we slow down the rate of increase in k_n relative to n, the posterior densities gradually resemble those in the fixed rank case.

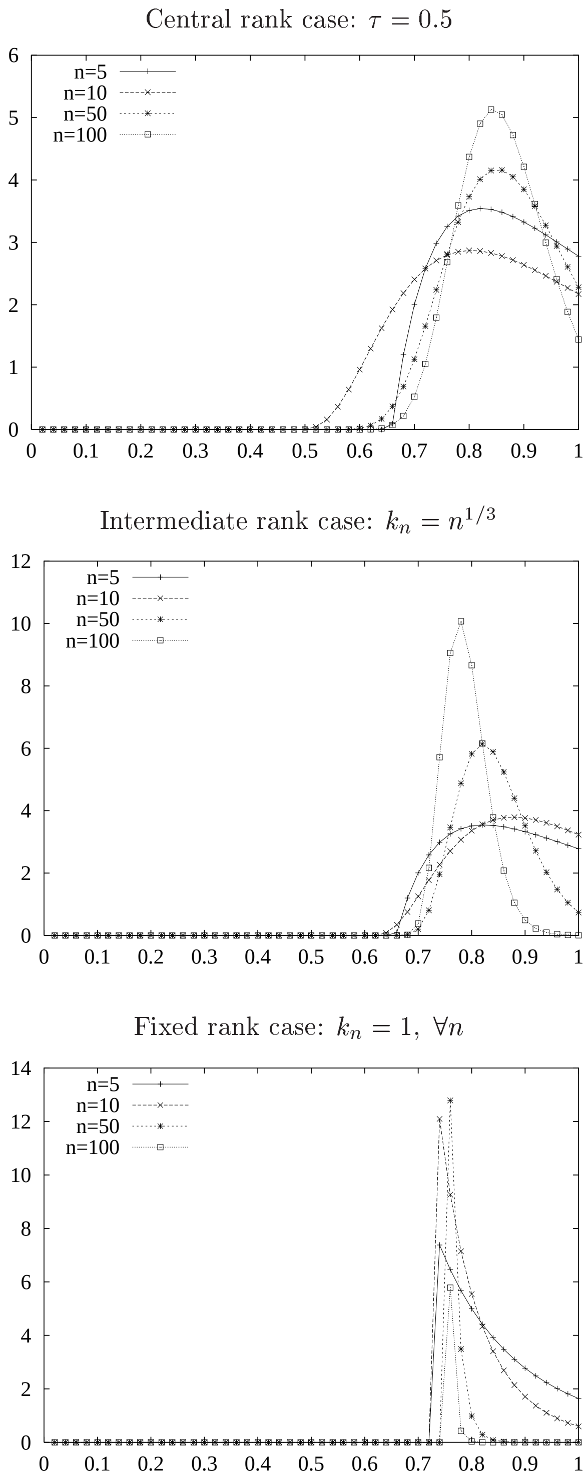


Figure 4: Simulated posterior densities for pivotal bidder
x-axis: $v \in [0, 1]$, y-axis: $f(v|T_n^{k_n})$, True v_0: 0.75.

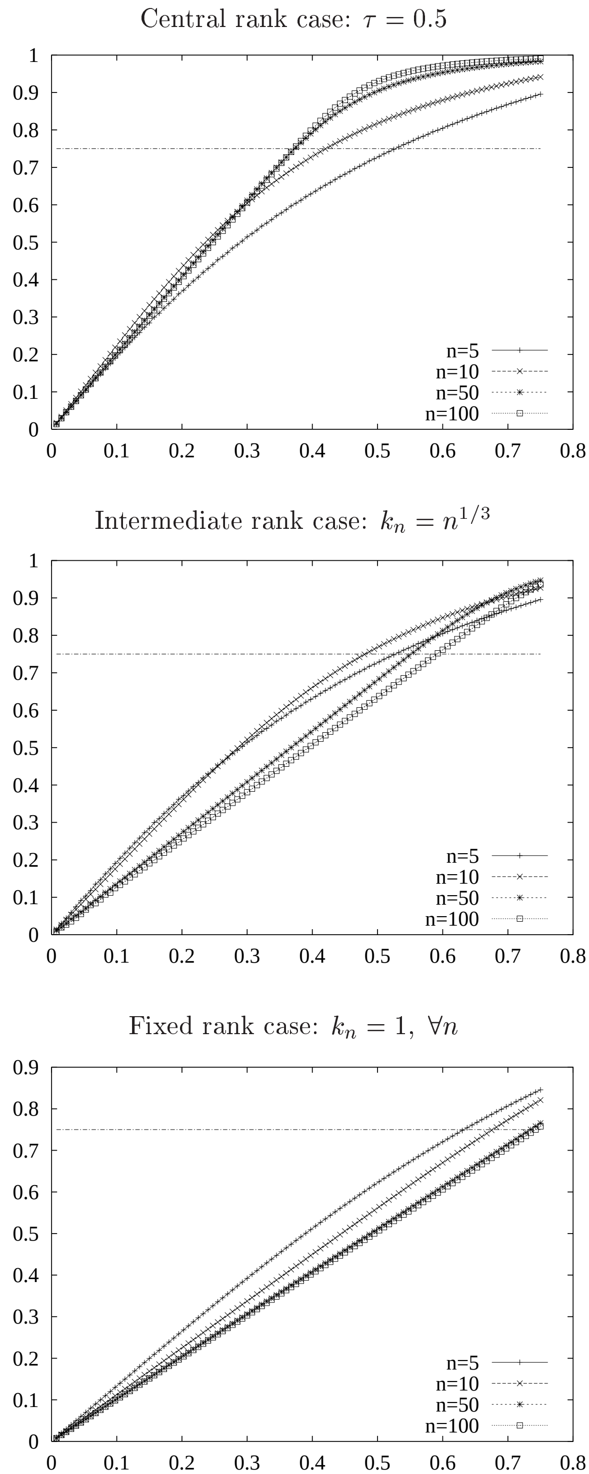


Figure 5: Simulated equilibrium bid functions
x-axis: $x \in [0, 0.75]$, y-axis: $b_n(x) = E(v|T_n^{k_n} = x)$, True v_0=0.75.
Horizontal line at $v_0 = 0.75$.

the pivotal signal; in particular, they ignore their prior beliefs as well as any additional information that their signal conveys about v. These considerations allow us to derive an intuitive discussion of information aggregation, which for simplicity we limit to the central rank case.

Essentially, PS showed that, for a given common value v_0, the sequence of (random) bid functions $b_n(x)$ converges, pointwise in signals x in the support of $G(\cdot|v_0)$, to a non-random "pseudo-MLE" function, defined as

$$\hat{v}(x) = \{v : G(x|v) = 1 - \tau\} \tag{13}$$

where, as before, τ is the limit of the sequence k_n/n. Furthermore, our results (specifically, corollary 5) show that this pointwise convergence occurs at the same rate as the winning bid.

For the simple example considered in this section, the pseudo-MLE function for each of the three cases sequences is:

1. Central rank case ($\tau = 0.5$): $\hat{v}|x = 2x$
2. Intermediate rank case ($k_n = n^{1/3}$): $\hat{v}|x = \frac{n}{n-k_n}x \to x$
3. Fixed rank case ($k = 1$): $\hat{v}|x = x$.

All three plots in figure 5 confirm PS's result: in each case, as n grows the bid functions increasingly resemble the pseudo-MLE functions specified above.

Minimizing the magnitude of the winners' curse: implications for auction design Information aggregation implies a strong version of the "no-regret" feature which Milgrom (1981) (pp. 930-931) first described for the highest rejected bid k-object uniform-price auction with a finite n. There, *ex-post*, each bidder learns the private signal of the price setter (i.e., who submitted the $k+1$-th highest bid). The no-regret feature is that, in the affiliated CV framework, there is no "winner's curse" in that no winner (i.e., a bidder who submitted a bid exceeding the market price) regrets paying the market price for the object, even given knowledge of the price-setter's signal, i.e.,

$$E[V|x_{n-k':n}, T_n^{k-1} = x_{n-k-1:n}] \geq E[V|x_{n-k-1:n}, T_n^{k-1} = x_{n-k-1:n}],\ k' < k+1$$

where the right-hand side of the inequality is the market price. Similarly, no-regret implies that losers do not suffer the loser's curse (i.e., regret not winning):

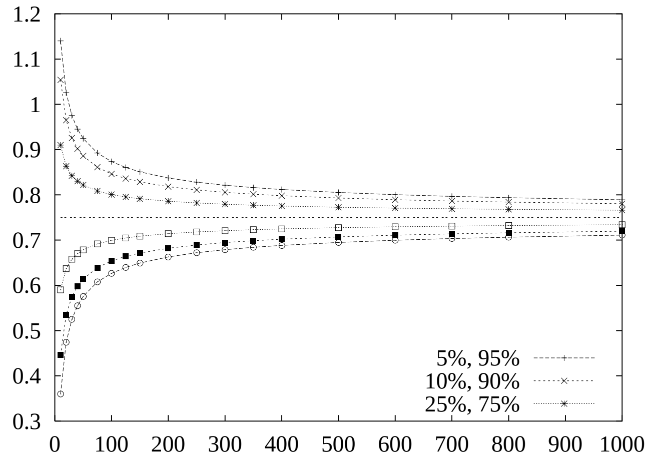
$$E[V|x_{n-k':n}, T_n^{k-1} = x_{n-k-1:n}] \leq E[V|x_{n-k-1:n}, T_n^{k-1} = x_{n-k-1:n}],\ k+1 \leq k' \leq n.$$

An important implicit assumption here is that none of the auction participants, win or lose, ever learn the true value of v_0. If bidders *did* observe v_0 *ex-post*, however, then no-regret will not hold in the sense that for fixed n and any $\epsilon > 0$, the probability (under v_0) that $|v_0 - E[V|x_{n-k-1:n}, T_n^{k-1} = x_{n-k-1:n}]| > \epsilon$ (i.e., the probability of a winners' or losers' curse) is strictly bounded away from zero.

The results of PS show that, even if v_0 were revealed to the auction participants *ex-post*, the no-regret policy still holds in the limit, almost surely:

$$E[V|x_{n-k-1:n}, T_n^{k-1} = x_{n-k-1:n}] \overset{a.s.}{\to} v_0.$$

Central rank case: $\tau = 0.5$[a]



Fixed rank case: $k_n = 1,\ \forall n$[b]

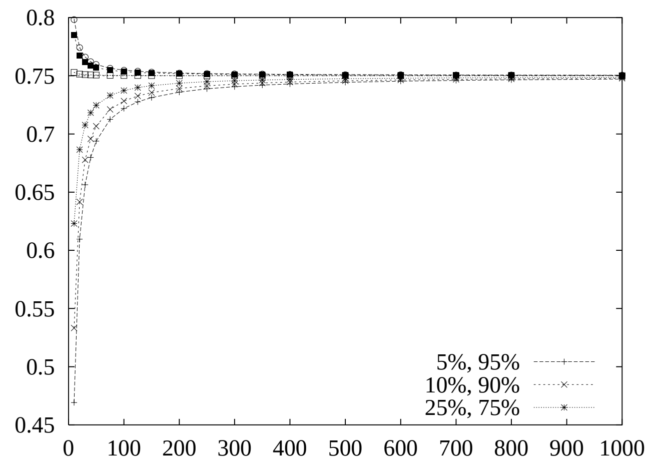


Figure 6: Two-tailed confidence intervals for v_0
x-axis: n, y-axis: two-tailed confidence bands, True v_0=0.75.
Horizontal line at $v_0 = 0.75$.

[a]Confidence bands: $0.75 + \frac{1}{\sqrt{n}}\left(0.75 * \Phi^{-1}(\alpha)\right)$, where $\Phi^{-1}(\alpha)$ is α-th quantile of standard normal variate.
[b]Confidence bands: $0.75 + \frac{1}{n}\left(0.75 + 0.75 * \mathcal{P}_2(\alpha)\right)$, where $\mathcal{P}_2(\alpha)$ is α-th quantile of Gamma $(2, 1)$ variate.

Subsequently, the winners' and losers' curse disappear, but *only in the limit*. Given these results, therefore, one could argue that information aggregation is not valuable only to gauge the large-sample performance of auctions as price-setting mechanisms, but also, from the bidders' point of view, it implies that their *ex-post* losses from participating in the auction are almost surely zero in the limit. Finally, note that the arguments in this subsection were made for a fixed v_0. Clearly they are valid for any arbitrary v, so that the attractive "zero loss" feature of the auction holds regardless of v_0.

Just as derivation of limit distributions for sequences of estimators allows empirical researchers to form approximate confidence intervals for parameter estimates given finite data, the limit distributions derived above can be used to form confidence intervals which quantify very precisely the degree of winners' or losers' curse which exists for any finite n.

In practice, the winner's curse could leave a winning firm in severe financial straights, perhaps leading to bankruptcy. From the perspective of auction design, it may be desired to avoid this outcome, without instituting non-market insurance measures (such as offering firms limited liability) which could be manipulated by the bidders. The results in this paper can be used to approximate the probability of suffering the winner's curse, for any finite N. To the extent that the policy maker desires an auction format which minimizes the probability of the winner's curse for any number of bidders, the results in this paper can be used to approximate these probabilities, and therefore provide a clear ranking of four common auction forms in terms of how likely it is that a market price could arise which exceeds the true underlying value of the object.

In figure 6, we present confidence brands for this example, which illustrate the extent of winner's curse that could be expected to occur, for a whole range of n. For every n, the confidence band comprising the $(5\%, 95\%)$ lines defines a region of realizations of the market price $\bar{P}_n \equiv E[V|x_{n-k-1:n}, T_n^{k-1} = x_{n-k-1:n}]$ for which one would not be able to reject with 90% confidence that the true value $v_0 = 0.75$. Roughly speaking, the prices $\bar{P}_n$ within a confidence interval are those which are most "likely" to arise given $v_0 = 0.75$, where "likely" is measured in terms of how far a given price $\bar{P}_n$ lies away from the "tail" of the conditional distribution $\left(\bar{P}_n|v_0\right)$ for different n.

In the example being considered, the true common value v_0 is 0.75. The graphs in figure 6 show that even with 50 bidders, a market price of 0.80 (roughly 7% above the "true" value) still lies within a 50% confidence interval, for the central rank auction. Considering the large magnitudes in many common value auctions (such as the hundreds of millions of dollars in the spectrum auctions which have recently occurred in many countries) a 7% overbid can have severe financial consequences.

A desire to minimize the magnitude of the winner's curse for any fixed N might lead policymakers to choose auctions of the "fixed rank" type (such as the first-price, second-price, or English auctions). The bottom graph in figure 6 shows that a market price of 0.8 is very nearly a zero-probability event, even with just 50 bidders. This example illustrates one application of the results in this paper which can have potentially important ramifications for market design.

7 Conclusions and extensions

In this paper we extended the previous literature on information aggregation in large common value auctions by deriving the limit distributions for equilibrium bids for four different auction formats. While the previous literature showed that information aggregation obtains quite generally across all of these auction forms, our results indicate quite striking differences in the asymptotic properties of the equilibrium bids across different auction forms, including large differences in rates of information aggregation. These differences in convergence rates can be very important for policymakers who wish to minimize the impact of the winner's curse in designing large auction markets.

These results in this paper correspond only to the pure common value case. As Pesendorfer and Swinkels (2000) and Feddersen and Pesendorfer (1997) have shown, in environments with both private and common value aspects a possible tension arises between allocative efficiency and information aggregation. The framework developed in this paper may also be extended to address allocative efficiency issues, to ask whether, in the limit, the identity of the winning bidder (or pivotal voter) is actually the agent with the pivotal position in the population. We are currently exploring these extensions.

References

AMEMIYA, T. (1985): *Advanced Econometrics*. Harvard University Press.

BALKEMA, A., AND L. DE HAAN (1978): "Limit Distributions for Order Statistics, I," *Theory Probab. Appl.*, pp. 77–92.

BHATTACHARYA, R., AND M. MAJUMDAR (1973): "Random Exchange Economies," *Journal of Economic Theory*, 6, 37—67.

CHERNOZHUKOV, V. (2000): "Conditional Extremes and Near Extremes," mimeo. Stanford University and MIT.

CHERNOZHUKOV, V., H. HONG, AND M. SHUM (2000): "Bayesian Inference in Quantile Experiments," mimeo., MIT.

DEGROOT, M. (1970): *Optimal Statistical Decisions*. McGraw-Hill Book Company.

FEDDERSEN, T., AND W. PESENDORFER (1997): "Voting Behavior and Information Aggregation in Elections with Private Information," *Econometrica*, 65, 1029–1058.

FELLER, W. (1971): *An Introduction to Probability Theory and its Applications*. Wiley, second edition, Volume 2.

FERGUSON, T. (1996): *A Course in Large Sample Theory*. Chapman and Hall.

HILDENBRAND, W. (1971): "Random Preferences and Equilibrium Analysis," *Journal of Economic Theory*, 3, 414—429.

IBRAGIMOV, I., AND R. HAS'MINSKII (1981): *Statistical Estimation: Asymptotic Theory*. Springer Verlag.

KREMER, I. (1999): "Information Aggregation in Common Value Auctions," mimeo., Stanford University.

LEADBETTER, M., G. LINDGREN, AND H. ROOTZEN (1983): *Extremes and Related Properties of Random Sequences and Processes*. Springer Verlag.

LECAM, L. (1986): *Asymptotic Methods in Statistical Decision Theory*. Springer Verlag.

LECAM, L., AND G. YANG (1990): *Asymptotics in Statistics: Some Basic Concepts*. Springer Verlag.

MATTHEWS, S. (1984): "Information Acquisition in Discriminatory Auctions," in *Bayesian Models in Economic Theory*, ed. by M. Boyer, and R. Kihlstrom. North-Holland.

MILGROM, P. (1979): "A Convergence Theorem for Competitive Bidding with Differential Information," *Econometrica*, 47, 679–688.

——— (1981): "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, 49, 921–943.

——— (1982): "Good news and bad news: representation theorems and applications," *The Bell Journal of Economics*, pp. 380–391.

MILGROM, P., AND R. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

PALFREY, T. (1985): "Uncertainty Resolution, Private Information Aggregation and the Counot Competitive Limit," *Review of Economic Studies*, 52, 69—83.

PESENDORFER, W., AND J. SWINKELS (1997): "The Loser's Curse and Information Aggregation in Common Value Auctions," *Econometrica*, 65, 1247–1281.

——— (2000): "Efficiency and Information Aggregation in Auctions," *American Economic Review.*

POLLARD, D. (1984): *Convergence of Stochastic Processes.* Springer Verlag.

SATTERTHWAITE, M., AND S. WILLIAMS (1989): "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market becomes Large," *Review of Economic Studies*, pp. 477–498.

SMIRNOV, N. V. (1952): *Limit Distributions for the Terms of a Variation Series.* American Mathematical Society, Translated from the Russian in *American Mathematical Society Translations*, volume 67.

VAN DER VAART, A. (1999): *Asymptotic Statistics.* Cambridge University Press.

VIVES, X. (1988): "Aggregation of Information in Large Cournot Markets," *Econometrica*, 56, 851–876.

——— (1999): "Information Aggregation, Strategic Behavior, and Efficiency in Cournot Markets," Institut d'Analisi Economica (CSIC, Barcelona).

WANG, R. (1991): "Common Value Auctions with Discrete Private Information," *Journal of Economic Theory*, pp. 429–447.

WILSON, R. (1977): "A Bidding Model of Perfect Competition," *Review of Economic Studies*, 44, 511–518.

——— (1985): "Incentive Efficiency of Double Auctions," *Econometrica*, 53, 1101–1115.

A Proofs

Proof of Lemma 1 The following calculations shows that the quantile experiments sequence satisfies the LAN ("local asymptotic normality") conditions, in the terminology of LeCam (cf. LeCam and Yang (1990), chapter 5). Under v_0, the log likelihood ratio process between the local parameters h and 0 satisfies:

$$
\begin{aligned}
\log \frac{dP_{n,h}\left(T_n\right)}{dP_{n,0}\left(T_n\right)} =& \log \frac{G\left(T_n|v_0+\frac{h}{\sqrt{n}}\right)^{n(1-\tau)}\left[1-G\left(T_n|v_0+h/\sqrt{n}\right)\right]^{n\tau}}{G\left(T_n|v_0\right)^{n(1-\tau)}\left[1-G\left(T_n|v_0\right)\right]^{n\tau}} \\
=& n\left(1-\tau\right)\left[\log G\left(T_n|v_0+h/\sqrt{n}\right)-\log G\left(T_n|v_0\right)\right] \\
&+ n\tau\left[\log\left[1-G\left(T_n|v_0+h/\sqrt{n}\right)\right]-\log\left[1-G\left(T_n|v_0\right)\right]\right] \\
=& \left(1-\tau\right)\left[\sqrt{n}\frac{\partial \log G\left(T_n|v_0\right)}{\partial v}\times h+\frac{1}{2}\frac{\partial^2 \log G\left(T_n|v_0\right)}{\partial v^2}\times h^2+o_p\left(1\right)\right] \\
&+ \tau\left[\sqrt{n}\frac{\partial \log\left[1-G\left(T_n|v_0\right)\right]}{\partial v}\times h+\frac{1}{2}\frac{\partial^2 \log\left(1-G\left(T_n|v_0\right)\right)}{\partial v^2}\times h^2+o_p\left(1\right)\right]
\end{aligned}
$$

To simplify notation, let $G_v(T_n|v) \equiv \frac{\partial G(T_n|v)}{\partial v}$ and $G_{vv}(T_n|v) \equiv \frac{\partial^2 G(T_n|v)}{\partial v^2}$, then it follows

$$\begin{aligned}\log \frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)} =& (1-\tau)\left[\sqrt{n}\frac{G_v(T_n|v_0)}{G(T_n|v_0)} \times h + \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{G(T_n|v_0)}{G^v(T_n|v_0)}\right) \times h^2 + o_p(1)\right] \\ &+ \tau\left[-\sqrt{n}\frac{G_v(T_n|v_0)}{1-G(T_n|v_0)}h - \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{G(T_n|v_0)}{1-G^v(T_n|v_0)}\right)h^2 + o_p(1)\right] \\ =& \sqrt{n}\left[(1-\tau)\frac{G_v(T_n|v_0)}{G(T_n|v_0)} - \tau\frac{G_v(T_n|v_0)}{1-G(T_n|v_0)}\right]h \\ &+ \frac{1}{2}\left[(1-\tau)\frac{\partial}{\partial v}\left[\frac{G_v(T_n|v_0)}{G(T_n|v_0)}\right] - \tau\frac{\partial}{\partial v}\left[\frac{G_v(T_n|v_0)}{1-G(T_n|v_0)}\right]\right]h^2 + o_p(1)\end{aligned}$$

Denote $q_0^{1-\tau} = G^{-1}(1-\tau|v_0)$,

$$\begin{aligned}\log \frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)} =& \sqrt{n}\left[(1-\tau)\frac{G_v(T_n|v_0)}{G(T_n|v_0)} - \tau\frac{G_v(T_n|v_0)}{1-G(T_n|v_0)}\right]h \\ &+ \frac{1}{2}\left[(1-\tau)\frac{\partial}{\partial v}\left(\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{G\left(q_0^{1-\tau}|v_0\right)}\right) - \tau\frac{\partial}{\partial v}\left(\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{1-G\left(q_0^{1-\tau}|v_0\right)}\right)\right]h^2 + o_p(1)\end{aligned}$$

Next a Taylor expansion of T_n around $q_0^{1-\tau}$ shows that

$$\begin{aligned}\log \frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)} =& \sqrt{n}\left[-(1-\tau)\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{G^2\left(q_0^{1-\tau}|v_0\right)} - \tau\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{\left(1-G\left(q_0^{1-\tau}|v_0\right)\right)^2}\right]g\left(q_0^{1-\tau}|v_0\right)\left(T_n - q_0^{1-\tau}\right) \times h \\ &+ \frac{1}{2}\left[(1-\tau)\frac{\partial}{\partial v}\left(\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{G\left(q_0^{1-\tau}|v_0\right)}\right) - \tau\frac{\partial}{\partial v}\left(\frac{G_v\left(q_0^{1-\tau}|v_0\right)}{1-G\left(q_0^{1-\tau}|v_0\right)}\right)\right] \times h^2 + o_p(1)\end{aligned}$$

Denote $G_v \equiv G_v\left(q_0^{1-\tau}|v_0\right)$ and note that $G\left(q_0^{1-\tau}|v_0\right) = 1-\tau$,

$$\begin{aligned}\log \frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)} =& -G_v\frac{1}{\tau(1-\tau)}g\left(q_0^{1-\tau}|v_0\right)\sqrt{n}\left(T_n - q_0^{1-\tau}\right) \times h \\ &- \frac{1}{2}G_v^2\frac{1}{\tau(1-\tau)}h^2 + o_p(1)\end{aligned} \tag{14}$$

The following linear representation is a standard result (cf. van der Vaart (1999), corollary 21.5)

$$\begin{aligned}\sqrt{n}\left(T_n - q_0^{1-\tau}\right) &= \frac{1}{g\left(q_0^{1-\tau}|v_0\right)}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(1\left(X_i \geq q_0^{1-\tau}\right) - \tau\right) + o_p(1) \\ &\xrightarrow{d} N\left(0, \frac{1}{g^2\left(q_0^{1-\tau}|v_0\right)}\tau(1-\tau).\right)\end{aligned}$$

Intuitively, the sum $\sum_{i=1}^{n} 1\left(X_i \geq q_0^{1-\tau}\right)$ is the number of "successes" in the sequence of binomial experiments (n, τ_n), and since $\tau_n \to \tau$ bounded away from 0 and 1, the normal approximation for the binomial is appropriate.

It follows that under $P_{n,0}$,

$$\log \frac{dP_{n,h}(T_n)}{dP_{n,0}(T_n)} = -G_v \frac{1}{\tau(1-\tau)} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(1\left(X_i \geq g\left(q_0^{1-\tau}\right)\right) - \tau\right) \times h$$
$$- \frac{1}{2} G_v^2 \frac{1}{\tau(1-\tau)} h^2 + o_p(1)$$
$$\xrightarrow{d} N\left(-\frac{1}{2} G_v^2 \frac{1}{\tau(1-\tau)} h^2, G_v^2 \frac{1}{\tau(1-\tau)} h^2\right)$$

Since the mean is $-\frac{1}{2}$ of the variance in the limiting distribution, by LeCam's first lemma (cf. example 6.5 in van der Vaart (1999)) the experiments $P_{n,h}$ and $P_{n,0}$ are mutually contiguous, for all h.

Proof of Theorem 2 The proof follows the arguments van der Vaart (1999), pp. 141–143. Since the proof is relatively long, we break it down into several lemmas.

We begin by introducing some notation. Define

$$f_n(x|h) \equiv G\left(x|v_0 + h/\sqrt{n}\right)^{n(1-\tau)} \left(1 - G\left(x|v_0 + h/\sqrt{n}\right)\right)^{n\tau}$$
$$I_v \equiv G_v^{-2} \tau(1-\tau)$$

Note that $f_n(x|h)$ denotes the density function of $P_{n,h}$ (with respect to Lebesgue measure).

Let $C \subset \mathcal{H}$ denote a neighbourhood of radius M around $h = 0$. For a fixed h, note that the fixed prior $\pi(v)$ for $v \in \mathcal{V}$ is equivalent to $\pi\left(v_0 + h/\sqrt{n}\right) \equiv \pi_n(h)$: for any fixed h, the sequence of priors now varies over n. Let π_n^C denote π_n truncated to the subset C, and let $P_{n,C}$ denote the (truncated) marginal distribution of T_n over $h \in C$, with density

$$\frac{\int_C f_n(x|h')\, \pi(v_0 + h'/\sqrt{n})\, dh'}{\int_C \pi(v_0 + h'/\sqrt{n})\, dh'} dx.$$

Let $P^C(h|T_n)$ denote the posterior measure given the truncated prior π_n^C, with density

$$f_n(T_n|h)\, \pi\left(v_0 + h/\sqrt{n}\right) / \int_{C_n} f_n(T_n|h)\, \pi\left(v_0 + h/\sqrt{n}\right) dh.$$

Finally, let $S_n(T_n, h)$ denote the joint distribution of T_n and h. When restricted to the subset $C \subset \mathcal{H}$, this joint distribution can be equivalently expressed as:

$$S_n^C(dT_n, dh) = P_{n,C}(dT_n) P_n^C(dh|T_n) = \pi_n^C(dh) f_n(T_n|h) dT_n, \tag{15}$$

which highlights the relation between the foregoing definitions. [19]

We start the proof of theorem 2 by establishing the following lemma:

[19] Generally, equation (15) is valid given certain technical conditions on the dominatedness of the joint measure S_n. These conditions are satisfied for our model, but a discussion of the more general case is given in LeCam (1986), pp. 325ff.

Lemma 2 *For any neighborhood $C \subset \mathcal{H}$ of radius M around $h = 0$,*

$$\left|\left|P^C\left(h|T_n\right) - N^C\left(\triangle_{n,v_0}, G_v^{-2}\tau\left(1-\tau\right)\right)\right|\right| \overset{P_{n,C}}{\to} 0$$

Proof:

$$\int \frac{1}{2}\left|\left|P^C\left(h|T_n\right) - N^C\left(\triangle_{n,v_0}, G_v^{-2}\tau\left(1-\tau\right)\right)\right|\right| P_{n,C}\left(dx\right)$$

$$= \int\int \left(1 - \frac{dN^C\left(\triangle_{n,v_0}, G_v^{-2}\tau\left(1-\tau\right)\right)(h)}{f\left(T_n|h\right)\pi\left(v_0 + h/\sqrt{n}\right)/\int f\left(T_n|g\right)\pi\left(v_0+g/\sqrt{n}\right)dg}\right)^+ dP^C\left(dh|T_n\right)P_{n,C}\left(dx\right)$$

$$\leq \int\int\int \left(1 - \frac{f\left(T_n|g\right)\pi\left(v_0+g/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(h)}{f\left(T_n|h\right)\pi\left(v_0+h/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(g)}\right)^+ dN^C\left(\triangle_{n,v_0}, I\right)(g)\, dP^C\left(dh|T_n\right)P_{n,C}\left(dx\right)$$

by $(1-EY)^+ \leq E\left(1-Y\right)^+$

$$\leq \int\int\int \left(1 - \frac{f\left(T_n|g\right)\pi\left(v_0+g/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(h)}{f\left(T_n|h\right)\pi\left(v_0+h/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(g)}\right)^+ \lambda_C\left(dg\right) dP^C\left(dh|T_n\right)P_{n,C}\left(dx\right)$$

by $dN^C\left(\triangle_{n,v_0}, G_v^{-2}\tau\left(1-\tau\right)\right)(h) \leq \lambda^C(dh)$, for some uniform measure $\lambda^C(dh)$ on C

Next, note that the joint measure $\lambda_C\left(dg\right) dP^C\left(dh|T_n\right)P_{n,C}\left(dx\right)$ is equivalent to

$$\lambda_C\left(dg\right) f_n\left(x|h\right)dx\pi_n^C(dh)$$

(by equation (15)), and the latter measure is contiguous to $\lambda_C\left(dg\right)f_n\left(x|0\right)dx\lambda_C\left(dh\right)$. This is due to the assumed continuity of the prior measure π in a neighbourhood of v_0, and to the contiguity of the measures $P_{n,h}$ and $P_{n,0}$, as shown in lemma 1 (note that $f_n\left(x|h\right)dx = dP_{n,h}(x)$). Therefore, if we can show that the integrand above converges in mean under $\lambda_C\left(dg\right)f_n\left(x|0\right)dx\lambda_C\left(dh\right)$, contiguity implies that it also converges in mean to zero under $\lambda_C\left(dg\right)dP^C\left(dh|T_n\right)P_{n,C}\left(dx\right)$, which would prove the lemma.

Note that under $P_{n,0}$, both of the functions $f(x|h)$ and $dN^C\left(\triangle_{n,v_0}, I\right)(h)$ converge to $dN(h,I)(0)$, for each $h \in C$. Furthermore, the assumed continuity of $\pi\left(\cdot\right)$ at v_0 implies that $\pi\left(v_0 + h/\sqrt{n}\right)/\pi\left(v_0+g/\sqrt{n}\right) \to 1$. Taken together, this yields the desired result:

$$\frac{f\left(T_n|g\right)\pi\left(v_0+g/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(h)}{f\left(T_n|h\right)\pi\left(v_0+h/\sqrt{n}\right)dN^C\left(\triangle_{n,v_0}, I\right)(g)} \overset{P_{n,0}}{\longrightarrow} 1$$

■

Next we want to show that for a sequence of neighborhoods $C_n \subset \mathcal{H}$ centered around $h = 0$ with an increasing sequence of radii $M_n \to \infty$, $\left|\left|P^{C_n}\left(h|T_n\right) - P\left(h|T_n\right)\right|\right| \overset{P_{n,0}}{\longrightarrow} 0$. In order to show this, the following lemma and assumption are needed.

Lemma 3 *(Pesendorfer and Swinkels (1997)) The sequence of estimators $\hat{v}_n$ (defined in equation (9) above) is uniformly consistent for v_0: $\forall v \in \mathcal{V}$ and all $\epsilon > 0$, there exists a $N_0(\epsilon)$ and a $\delta(\epsilon) > 0$ such that*

$$\forall n > N_0(\epsilon),\ P_{n,v_0}\left(\left|\left|\hat{v}_n - v_0\right|\right| \leq \epsilon\right) > 1 - \delta(\epsilon).$$

Proof: Pesendorfer and Swinkels (1997), pg. 1260. ■

Lemma 4 *For every sequence $M_n \to \infty$, for every n, for every v outside a neighborhood of radius $M_n/\sqrt{n}$ around v_0, there exists a sequence of tests ϕ_n for testing $H_0: \ v = v_0$ vs. $H_1: \ \|v - v_0\| \geq M_n/\sqrt{n}$ such that*

$$P_{n,v_0}\phi_n \to 0, \qquad\qquad P_{n,v}\left(1 - \phi_n\right) \leq \exp\left(-K(v, v_0)n\right)$$

where $K(v, v_0) > 0$ is some constant which possibly depends on v and v_0. In other words, under v_0 the "tail probabilities" of the test ϕ_n must decay at an exponential rate.

Proof: Pick an ϵ for the uniform consistency of the sequence $\hat{v}_n$ from lemma 3. For a given n, and a given v such that $\|v - v_0\| \geq M_n/\sqrt{n}$, there are two possible cases: (i) $M_n/\sqrt{n} \leq \|v - v_0\| \leq \epsilon$; and (ii) $\|v - v_0\| \geq \epsilon$. We consider each case in turn.

Case i: $M_n/\sqrt{n} \leq \|v - v_0\| \leq \epsilon$. Here we use the same arguments as van der Vaart (1999), pg. 144. Define $\phi_n(\epsilon) = 1\left(|\hat{v}(T_n) - v_0| > \epsilon/2\right)$, which satisfies both $E_{v_0}\phi_n(\epsilon) \overset{n\to\infty}{\longrightarrow} 0$ and

$$\sup_{|v-v_0|\geq\epsilon} E_v\phi_n(\epsilon) \leq \sup_{|v-v_0|\geq\epsilon} P_v\left(|v - v_0| - |\hat{v}(T_n) - v| \leq \epsilon/2\right) \leq \sup_{|v-v_0|\geq\epsilon} P_v\left(|\hat{v}(T_n) - v| > \epsilon/2\right) \to 0$$

For each ϵ, the sequence of tests $\phi_n(\epsilon)$ can be replaced by another sequence $\tilde{\phi}_n(\epsilon)$ which decays exponentially to zero as required by the lemma, i.e., $\forall\ n$ large and $\forall |v - v_0| > \epsilon$, $\exists c > 0$ such that $P_v(1 - \phi_n) \leq e^{-cn}$. van der Vaart (1999) describes one way to do this (pg. 144, top).

Case ii: $\|v - v_0\| \geq \epsilon$. For small $|v - v_0|$: $M_n/\sqrt{n} \leq |v - v_0| \leq \epsilon$ an exponential bound is also needed. van der Vaart (1999) relies on the truncated likelihood influence function. Presumably any linear influence function whose expectation has a strictly positive derivative with respect to the parameter can also serve the same purpose. However, the quantile experiment situation is different, since we do not have available all observations and can not construct the linear influence function directly. Thus we have to rely on tests constructed using T_n only, which is a nonlinear functional of the empirical distribution. Fortunately by a simple argument transforming to the empirical distribution we can easily establish the following large deviation bound for $T_n - q_v^{1-\tau}$ under suitable conditions: for all n large enough,

$$P_v\left(|T_n - q_v^{1-\tau}| > c\right) \leq Ce^{-Cnc^2}. \tag{16}$$

Inequality (16) follows from the following:

$$P_v\left(T_n - q_v^{1-\tau} > \epsilon/\sqrt{n}\right) = P\left(\left(\#ofobs > q_v^{1-\tau} + \epsilon/\sqrt{n}\right) > n \times \tau\right) = P\left(\sum_{i=1}^{n} 1\left(X_i > q + \epsilon/\sqrt{n}\right) > n \cdot \tau\right)$$

$$= P\left(\sum_{i=1}^{n}\left(1\left(X_i > q + \epsilon/\sqrt{n}\right) - \bar{G}\left(q_v^{1-\tau} + \epsilon/\sqrt{n}\right)\right) > n\left(\bar{G}\left(q_v^{1-\tau}\right) - \bar{G}\left(q_v^{1-\tau} + \epsilon/\sqrt{n}\right)\right)\right)$$

$$= P\left(\sum_{i=1}^{n}\left(1\left(X_i > q + \epsilon/\sqrt{n}\right) - \bar{G}\left(q_v^{1-\tau} + \epsilon/\sqrt{n}\right)\right) > ng\left(q^*\right)\epsilon/\sqrt{n}\right) \quad q^* \text{ between } q_v^{1-\tau}, q_v^{1-\tau} + \epsilon/\sqrt{n}.$$

$$\leq P\left(\sum_{i=1}^{n}\left(1\left(X_i > q + \epsilon/\sqrt{n}\right) - \bar{G}\left(q_v^{1-\tau} + \epsilon/\sqrt{n}\right)\right) > c\sqrt{n}\epsilon\right) \quad \text{for some small } c > 0.$$

$$\leq \exp\left(-2\left(\sqrt{n}c\epsilon\right)^2/8n\right) = \exp\left(-\frac{1}{4}c^2\epsilon^2\right).$$

The fourth inequality follows from the consideration that the density $g(\cdot|v)$ is bounded away from 0, and the last line follows from the Hoeffding's inequality (cf. Appendix B in Pollard (1984)).

Then, given assumption 1, $\exists\epsilon > 0$ such that $|q_v^{1-\tau} - q_0^{1-\tau}| > c|v - v_0|$ for some $c > 0$ and $|v - v_0| < \epsilon$. For such v and $|v - v_0| > \frac{2}{c}\sqrt{M_n/n}$, and for $\omega_n \equiv 1\left(|T_n - q_0^{1-\tau}| \geq \sqrt{M_n/n}\right)$, one has that $P_0\omega_n \to 0$ by asymptotic normality of $\sqrt{n}\left(T_n - q_0^{1-\tau}\right)$, and that

$$P_v(1-\omega_n) = P_v\left(|T_n - q_0^{1-\tau}| \leq \sqrt{M_n/n}\right) \leq P_v\left(|T_n - q_v^{1-\tau}| \geq |q_0^{1-\tau} - q_v^{1-\tau}| - \sqrt{M_n/n}\right)$$
$$\leq P_v\left(|T_n - q_v^{1-\tau}| \geq c|v_0 - v| - \sqrt{M_n/n}\right) \leq P_v\left(|T_n - q_v^{1-\tau}| \geq \frac{1}{2}c|v_0 - v|\right) \leq Ce^{-Cn|v-v_0|^2}$$

■

Finally we can state the main result of the second part of the proof:

Lemma 5 *For every sequence of neighborhoods $C_n \subset \mathcal{H}$ centered around $h = 0$ with an increasing sequence of radii $M_n \to \infty$, the sequence*

$$\left|\left|P^{C_n}(h|T_n) - P(h|T_n)\right|\right| \to 0$$

under v_0.

Proof: Using the sequence of tests from lemma 4 above, we follow the calculation in van der Vaart (1999) to obtain

$$\frac{1}{2}\left|\left|P^{C_n}(h|T_n) - P(h|T_n)\right|\right| \leq \frac{1}{\int_U \pi_n(h)\,dh}\int(1-\phi_n(x))\int_{C_n^c} f(x|h')\,\pi_n(h')\,dh'\frac{\int_U f(x|h)\,\pi_n(h)\,dh}{\int f(x|h)\,\pi_n(h)\,dh}dx$$
$$\leq \frac{1}{\int_U \pi(h)\,dh}\int_{|h|\geq M_n}\int(1-\phi_n(x))\,f(x|h)\,dx\pi_n(h)\,dh$$
$$\leq \sqrt{n}\left(\int_{\epsilon\sqrt{n}\geq|h|\geq M_n} e^{-ch^2}\pi_n(h)\,dh + \int_{\epsilon\sqrt{n}\leq|h|} e^{-c\epsilon^2 n}\pi_n(h)\,dh\right)$$
$$\leq e^{-cM_n^2} + \sqrt{n}e^{-c\epsilon^2 n} \longrightarrow 0$$

■

Note that the exponential convergence rate on the tests is required in order to ensure that it exceeds $\sqrt{n}$, which is the rate at which we want the sequence of posterior distributions to converge.

Since lemma 2 is true for all fixed neighborhoods C around $h = 0$, it has to be true for some sequence of neighborhoods C_n with radius $M_n \to \infty$ slowly enough if necessary. Furthermore, given lemma 5, it is not hard to show that

$$\left|\left|N^{C_n}\left(\triangle_{n,v_0}, G_v^{-2}\tau(1-\tau)\right) - N\left(\triangle_{n,v_0}, G_v^{-2}\tau(1-\tau)\right)\right|\right| \overset{P_{n,0}}{\longrightarrow} 0,$$

which immediately implies

$$\left|\left|P(h|T_n) - N\left(\triangle_{n,v_0}, G_v^{-2}\tau(1-\tau)\right)\right|\right| \overset{P_{n,v_0}}{\longrightarrow} 0,$$

as asserted by Theorem 2. Q.E.D.

Proof of theorem 4 (a) The distribution of $\sqrt{n}\left(T_n - q^{1-\tau}\right)$ follows easily from Theorem 4 of Smirnov (1952), where we take $a_n = \frac{1}{\sqrt{n}}$, $b_n = q^{1-\tau}$, for $g^-(q|v)$ defined as the left derivative of $G(\cdot|v)$ at q and $g^+(q|v)$ defined as the right derivative of $G(\cdot|v)$ at q. For a fixed x, the process

$$\tilde{\mu}_n(x) = \frac{\sqrt{n}\left(G\left(\frac{1}{\sqrt{n}}x + q^{1-\tau}\right) - (1-\tau)\right)}{\sqrt{\tau(1-\tau)}}$$

converges to

$$\mu(x) = 1(x>0)\frac{1}{\sqrt{\tau(1-\tau)}}g^+\left(q^{1-\tau}|v\right)x + 1(x<0)\frac{1}{\sqrt{\tau(1-\tau)}}g^-\left(q^{1-\tau}|v\right)x$$

Therefore according to Theorem 4 of Smirnov (1952)

$$P\left(\sqrt{n}\left(T_n - q^{1-\tau}\right) \le x\right) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{\tau(1-\tau)}}\left(1(x>0)g^+(q|v)+1(x<0)g^-(q|v)\right)} e^{-\frac{y^2}{2}}dy$$

It is not hard to see that for $X \sim N(0,1)$, this has the same distribution as

$$\sqrt{n}\left(T_n - q^{1-\tau}\right) \overset{d}{\longrightarrow} Y \equiv \frac{\sqrt{\tau(1-\tau)}}{g^+\left(q^{1-\tau}|v\right)}X^+ - \frac{\sqrt{\tau(1-\tau)}}{g^-\left(q^{1-\tau}|v\right)}X^-$$

Alternatively, one could write this convergence result as

$$\left[g^+1\left(T_n \ge q^{1-\tau}\right) + g^-1\left(T_n \le q^{1-\tau}\right)\right]\sqrt{n}\left(T_n - q^{1-\tau}\right) \overset{d}{\longrightarrow} \sqrt{\tau(1-\tau)}N(0,1)$$

(b) Given this result we proceed by considering a local expansion of the likelihood ratio process. Define $g_n^h \equiv g^+1\left(T_n > U\left(v + h/\sqrt{n}\right)\right) + g^-1\left(T_n < U\left(v + h/\sqrt{n}\right)\right)$. Then it is straightforward to approximate the likelihood ratio by

$$\begin{aligned}
\log\frac{L(T_n|h)}{L(T_n|0)} &\sim n(1-\tau)\left[\log G\left(T_n|v + h/\sqrt{n}\right) - \log G(T_n|v)\right] \\
&\quad + n\tau\left[\log\left(1 - G\left(T_n|v + h/\sqrt{n}\right)\right) - \log\left(1 - G(T_n|v)\right)\right] \\
&\sim n(1-\tau)\left[\log\left[1 - \tau + g_h^n\left(T_n - q^{1-\tau} - U_v h/\sqrt{n}\right)\right] - \log\left[1 - \tau + g_0^n\left(T_n - q^{1-\tau}\right)\right]\right] \\
&\quad + n\tau\left[\log\left(\tau - g_h^n\left(T_n - q^{1-\tau} - U_v h/\sqrt{n}\right)\right) - \log\left(\tau - g_0^n\left(T_n - q^{1-\tau}\right)\right)\right]
\end{aligned}$$

Taking a standard second order expansion of $\log(1+x) \approx x - \frac{1}{2}x^2 + o\left(x^2\right)$, we can approximate the above by:

$$n(1-\tau)\log\left[1+\frac{g_h\left(T-q-h/U_v\sqrt{n}\right)-g_0(T-q)}{1-\tau+g_0(T-q)}\right]+n\tau\log\left[1+\frac{g_0(T-q)-g_h\left(T-q-hU_v/\sqrt{n}\right)}{\tau-g_0(T-q)}\right]$$

$$\sim n(1-\tau)\left[\frac{g_h\left(T-q-hU_v/\sqrt{n}\right)-g_0(T-q)}{1-\tau+g_0(T-q)}-\frac{1}{2}\left(\frac{g_h\left(T-q-hU_v/\sqrt{n}\right)-g_0(T-q)}{1-\tau+g_0(T-q)}\right)^2\right]$$

$$n\tau\left[\frac{g_0(T-q)-g_h\left(T-q-hU_v/\sqrt{n}\right)}{\tau-g_0(T-q)}-\frac{1}{2}\left(\frac{g_0(T-q)-g_h\left(T-q-hU_v/\sqrt{n}\right)}{\tau-g_0(T-q)}\right)^2\right]$$

$$\sim\frac{\sqrt{n}(1-\tau)}{1-\tau+g_0(T-q)}\left[-g_hU_vh+(g_h-g_0)\sqrt{n}(T_n-q)\right]+\frac{\sqrt{n}\tau}{\tau-g_0(T-q)}\left[g_hU_vh+(g_0-g_h)\sqrt{n}(T_n-q)\right]$$
$$-\frac{1}{2\tau(1-\tau)}n\left(g_h\left(T-q-hU_v/\sqrt{n}\right)-g_0(T-q)\right)^2$$

$$\sim\left(\sqrt{n}-\sqrt{n}\frac{1}{1-\tau}g_0(T-q)\right)\left(-g_hh+(g_h-g_0)\sqrt{n}(T_n-q)\right)$$
$$+\left(\sqrt{n}+\sqrt{n}\frac{1}{\tau}g_0(T-q)\right)\left[g_hU_vh+(g_0-g_h)\sqrt{n}(T_n-q)\right]-\frac{1}{2\tau(1-\tau)}\left[(g_h-g_0)\sqrt{n}(T-q)-g_hU_vh\right]^2$$

This becomes

$$\frac{1}{\tau(1-\tau)}\sqrt{n}g_0(T_n-q)\left(g_hU_vh-(g_h-g_0)\sqrt{n}(T-q)\right)-\frac{1}{2\tau(1-\tau)}\left[g_hU_vh-(g_h-g_0)\sqrt{n}(T-q)\right]^2$$
$$=-\frac{1}{2}\frac{1}{\tau(1-\tau)}\left(g_hU_vh-(g_h-g_0)\sqrt{n}(T-q)-\sqrt{n}g_0(T_n-q)\right)^2+C$$
$$=-\frac{1}{2}\frac{1}{\tau(1-\tau)}\left(g_hU_vh-g_h\sqrt{n}(T-q)\right)^2+C=-\frac{1}{2}\frac{g_h^2U_v^2}{\tau(1-\tau)}\left(h-U_v^{-1}\sqrt{n}(T-q)\right)^2$$

Therefore in large samples the posterior density of $h=\sqrt{n}(v-v_0)$ is proportional to:

$$f(h|T_n)\sim\begin{cases} C\exp\left(-\frac{1}{2}\frac{g^{+2}U_v^2}{\tau(1-\tau)}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^2\right) & \text{if} \quad h<U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\\ C\exp\left(-\frac{1}{2}\frac{g^{-2}U_v^2}{\tau(1-\tau)}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^2\right) & \text{if} \quad h>U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\end{cases}$$
$$=C\exp\left\{-\frac{1}{2}\left[\frac{g^+U_v}{\sqrt{\tau(1-\tau)}}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^-+\frac{g^-U_v}{\sqrt{\tau(1-\tau)}}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^+\right]^2\right\}$$
$$\equiv C\exp\left\{-\frac{1}{2}x^2\right\}$$

which is proportional to a $N(0,1)$ density evaluated at the random variable

$$X\equiv-\frac{g^+U_v}{\sqrt{\tau(1-\tau)}}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^-+\frac{g^-U_v}{\sqrt{\tau(1-\tau)}}\left(h-U_v^{-1}\sqrt{n}\left(T_n-q^{1-\tau}\right)\right)^+, \qquad (17)$$

where the randomness arises from sampling error in T_n. Alternatively, this is equivalent to the two-sided normal density

$$d\left[N\left(U_v^{-1}\sqrt{n}\left(T_n-q^{1\tau}\right),\frac{\tau(1-\tau)}{g^{+2}U_v^2}\right)^-+N\left(U_v^{-1}\sqrt{n}\left(T_n-q^{1\tau}\right),\frac{\tau(1-\tau)}{g^{-2}U_v^2}\right)^+\right](h) \qquad (18)$$

evaluated at h.

(c) Integrating (18) over h, it is clear that the (random) posterior mean is given by

$$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{1}{g^-}-\frac{1}{g^+}\right)+U_v^{-1}\sqrt{n}\left(T_n-q\right)$$

Using the convergence result of the sample quantile from Smirnov (1952) described above, the posterior mean converges in distribution to, for $X\sim N(0,1)$:

$$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{1}{g^-}-\frac{1}{g^+}\right)+\frac{\sqrt{\tau(1-\tau)}}{U_v}\left(\frac{X^+}{g^+}-\frac{X^-}{g^-}\right).$$

■

Proof of theorem 5 For the strictly positive (and bounded from above) density case the convergence rate is based on a well known result (cf. Balkema and de Haan (1978); also see Chernozhukov (2000) for general tail conditions): for $X_1,\ldots,X_n$, *i.i.d.* draws from a uniform $U(0,1)$ distribution, if $n\to\infty$, $k_n\to\infty$, $n-k_n\to\infty$, then for $T_n^{k_n}$ being the $n-k_n$th order statistic:

$$\sqrt{\frac{n^3}{k_n(n-k_n)}}\left(T_n^{k_n}-\frac{n-k_n}{n}\right)=\sqrt{\frac{n}{\tau_n(1-\tau_n)}}\left(T_n^{k_n}-(1-\tau_n)\right)\xrightarrow{d}N(0,1)$$

where $\tau_n\equiv k_n/n$. Notice that the convergence rate is necessarily faster than $\sqrt{n}$ and slower than n since $a_n=\sqrt{\frac{n}{\tau_n(1-\tau_n)}}\sim\sqrt{\frac{n^2}{k_n}}$ and $k_n\to\infty$. To derive the limit distribution for the posterior mean, we approximate the likelihood ratio process as in lemma 1 to obtain (where $q\equiv G^{-1}(1-\tau|v)$ and $h\equiv a_n(v-v_0)$):

$$\log\frac{L\left(T_n^{k_n}|h\right)}{L\left(T_n^{k_n}|0\right)}\sim\log\frac{G\left(T_n^{k_n}|v+ha_n^{-1}\right)^{n(1-\tau_n)}}{G\left(T_n^{k_n}|v\right)^{n(1-\tau_n)}}\frac{1-G\left(T_n^{k_n}|v+ha_n^{-1}\right)^{n\tau_n}}{1-G\left(T_n^{k_n}|v\right)^{n\tau_n}}$$

$$\sim n(1-\tau_n)\left[\frac{\partial\log G\left(T_n^{k_n}|v\right)}{\partial v}ha_n^{-1}+\frac{1}{2}\frac{\partial^2\log G\left(T_n^{k_n}|v\right)}{\partial v^2}h^2a_n^{-2}\right]+n\tau_n\left[\frac{\partial\log\left(\bar{G}\left(T_n^{k_n}|v\right)\right)}{\partial v}ha_n^{-1}+\frac{1}{2}\frac{\partial^2\log\bar{G}\left(T_n^{k_n}|v\right)}{\partial v^2}h^2a_n^{-2}\right]+o(1)$$

$$\sim\sqrt{n\tau(1-\tau_n)}h\left((1-\tau_n)\frac{G_v\left(T_n^{k_n}|v\right)}{G(T_n|v)}-\tau\frac{G_v(T_n|v)}{\bar{G}(T_n|v)}\right)$$
$$+\frac{1}{2}h^2\tau(1-\tau)\left[(1-\tau)\frac{\partial}{\partial v}\left(\frac{G_v(T_n|v)}{G(T_n|v)}\right)-\tau\frac{\partial}{\partial v}\left(\frac{G_v(T_n|v)}{\bar{G}(T_n|v)}\right)\right]+o(1)$$

$$\sim-\sqrt{n\tau(1-\tau)}h\left[(1-\tau)\frac{G_v(q|v)}{G^2(q|v)}+\tau\frac{G_v(q|v)}{\bar{G}(q|v)^2}\right]g(q|v)(T_n-q)$$
$$+\frac{1}{2}h^2\tau(1-\tau)\left[(1-\tau)\frac{\partial}{\partial v}\left(\frac{G_v(q|v)}{G(q|v)}\right)-\tau\frac{\partial}{\partial v}\left(\frac{G_v(q|v)}{1-G(q|v)}\right)\right]+o(1)$$

$$\sim-hG_v(q|v)\,g(q|v)(T_n-q)\sqrt{\frac{n}{\tau(1-\tau)}}-\frac{1}{2}G_v(q|v)^2h^2+o(1)$$

This is evidently a normal density with variance $G_v(q|v)^{-2}$ and mean

$$-G_v(q|v)^{-1}g(q|v)\,a_n(T_n-q).$$

which also shows that the posterior mean converges at rate a_n:

$$a_n(E(v|T_n)-v_0)\sim-G_v(q|v)^{-1}g(q|v)\,a_n(T_n-q)\xrightarrow{d}N\left(0,G_v(1|v)^{-2}\right)$$

Contiguity also holds in this local statistical experiment, because as shown above the likelihood ratio process satisfies the convergence result of:

$$\log \frac{L(T_n|h)}{L(T_n|0)} \xrightarrow{d} N\left(-\frac{1}{2} G_v(1|v)^2 h^2, G_v(1|v)^2 h^2\right)$$

for which Le Cam's first lemma is again applicable. The proof of posterior convergence thus follows the same as in the regular case for τ bounded away from 0 and 1. ■

Proof of theorem 6 Localizing the parameter space to $\mathcal{H} \equiv n(\mathcal{V} - v_0)$, the normalized likelihood ratio process evaluated at $\mathcal{S}_n^k$ can be written as

$$\frac{L(\mathcal{S}_n^k|h)}{L(\mathcal{S}_n^k|h=0)} = \begin{cases} \dfrac{G\left(T_n^k|v_0+h/n\right)^{n-k-1} g^2\left(T_n^k|v_0+h/n\right)\left(1-G\left(T_n^k|v_0+h/n\right)\right)^{k-1}}{G\left(T_n^k|v_0\right)^{n-k-1} g^2\left(T_n^k|v_0\right)\left(1-G\left(T_n^k|v_0\right)\right)^{k-1}} & \text{if } T_n^k \le G^{-1}(1|v_0+h/n) \\ 0 & \text{otherwise} \end{cases}$$

where, under the true common value v_0, the condition $T_n^k \le G^{-1}(1|v_0+h/n)$ is automatically satisfied for $h > 0 \Leftrightarrow v > v_0$.[20]

For $T_n^k \le G^{-1}(1|v_0+h/n)$, by taking logs and applying the mean value theorem, one can rewrite the likelihood ratio as

$$\begin{aligned} 1\left(T_n^k \le G^{-1}(1|v_0+h/n)\right) \exp\Bigg(&(n-k-1)\frac{G_v\left(T_n^k|v^*\right)}{G\left(T_n^k|v^*\right)}\frac{h}{n} + \frac{g_v^2\left(T_n^k|v^*\right)}{g^2\left(T_n^k|v^*\right)}\frac{h}{n} \\ &+ (k-1)\left(\log \bar{G}\left(T_n^k|v_0+h/n\right) - \log \bar{G}\left(T_n^k|v_0\right)\right)\Bigg), \end{aligned} \tag{19}$$

where v^* is between v_0 and $v_0 + h/n$, and $\bar{G}$ denotes $1 - G$, the upper tail probability of the G distribution.

All three terms in expression (19) simplify as $n \to \infty$. Continuity of g in v implies that the second term in (19) above becomes negligible,

$$\frac{g_v^2\left(T_n^k|v^*\right)}{g^2\left(T_n^k|v^*\right)}\frac{h}{n} \to 0.$$

Furthermore, $(n-k-1)/n \to 1$ and $T_n^k \xrightarrow{p} U(v_0)$ for any finite k implies that the first term

$$\frac{G_v\left(T_n^k|v^*\right)}{G\left(T_n^k|v^*\right)} \xrightarrow{p} G_v\left(U(v_0)|v_0\right)$$

where $U(v_0) \equiv G^{-1}(1|v_0)$ denotes the upper support of the distribution of the signal conditional on v_0. Finally, we note that the last term could be written as

$$\begin{aligned} \left(\frac{\bar{G}\left(T_n^k|v_0+h/n\right)}{\bar{G}\left(T_n^k|v_0\right)}\right)^{k-1} &\sim \left(\frac{U(v_0+h/n) - T_n^k}{U(v_0) - T_n^k}\right)^{k-1} \\ &= \left(\frac{U(v_0) + U_v(v_0)h/n - T_n^k}{U(v_0) - T_n^k}\right)^{k-1} \\ &= \left(\frac{U_v(v_0)h - n\left(T_n^k - U(v_0)\right)}{-n\left(T_n^k - U(v_0)\right)}\right)^{k-1} \end{aligned}$$

[20] Since under v_0, $T_n^k \le G^{-1}(1|v_0) < G^{-1}(1|v_0+h/n)$ for $h > 0$ given assumption 1.

Therefore, in large samples, the likelihood ratio process can be approximated by

$$1\left[n\left(T_n^k - U\left(v_0\right)\right) \leq n\left(U\left(v_0 + h/n\right) - U\left(v_0\right)\right)\right] \exp\left(G_v\left(U\left(v_0\right)|v_0\right)h\right) \left(\frac{U_v\left(v_0\right)h - n\left(T_n^k - U\left(v_0\right)\right)}{n\left(U\left(v_0\right) - T_n^k\right)}\right)^{k-1} \tag{20}$$

It is clear from the assumption 1 that

$$\begin{aligned} G_v\left(U\left(v_0\right)|v_0\right) &= \lim_{h\to 0} \frac{G\left(U\left(v_0\right)|v_0 + h\right) - G\left(U\left(v_0\right)|v_0\right)}{h} \\ &= -U_v\left(v_0\right)g\left(U\left(v_0\right)|v_0\right) < 0 \end{aligned} \tag{21}$$

Given these considerations, next we directly derive the limiting behavior of the posterior mean.

The posterior distribution of V given $\mathcal{S}_n^k$, the pivotal bidder's information, is

$$\frac{G\left(T_n^k|v\right)^{n-k-1} g^2\left(T_n^k|v\right)\bar{G}\left(T_n^k|v\right)^{k-1} q\left(v\right)}{\int G\left(T_n^k|v\right)^{n-k-1} g^2\left(T_n^k|v\right)\bar{G}\left(T_n^k|v\right)^{k-1} q\left(v\right)dv}$$

Once we normalize the posterior distribution around some v, the posterior distribution of $h = n\left(v - v_0\right)$ is

$$f\left(h|T_n^k\right) = \frac{L\left(T_n^k|h\right)}{\int L\left(T_n^k|h\right)dh}$$

$$= \frac{G\left(T_n^k|v_0+h/n\right)^{n-k} g^2\left(T_n^k|v_0+h/n\right)\bar{G}\left(T_n^k|v_0+h/n\right)^{k-1} q\left(v_0+h/n\right)1\left(T_n^k \leq G^{-1}\left(1|v_0+h/n\right)\right)}{\int G\left(T_n^k|v_0+h/n\right)^{n-k} g^2\left(T_n^k|v_0+h/n\right)\bar{G}\left(T_n^k|v_0+h/n\right)^{k-1} q\left(v_0+h/n\right)1\left(T_n^k \leq G^{-1}\left(1|v_0+h/n\right)\right)dh}.$$

Using $q\left(v_0 + h/n\right) \longrightarrow q\left(v_0\right)$ and the preceding expansion (20) of the likelihood ratio:

$$\begin{aligned} &\frac{L\left(T_n^k|h\right)}{L\left(T_n^k|0\right)} \\ &\overset{a.s.}{\longrightarrow} 1\left(n\left(T_n^k - U\left(v_0\right)\right) \leq n\left(U\left(v_0+h/n\right) - U\left(v_0\right)\right)\right)\exp\left(G_v\left(U\left(v_0\right)|v_0\right)h\right)\left(\frac{U'\left(v_0\right)h - n\left(T_n^k - U\left(v_0\right)\right)}{n\left(U\left(v_0\right) - T_n^k\right)}\right)^{k-1} \\ &\overset{a.s.}{\longrightarrow} 1\left(h \geq \left(U_v\left(v_0\right)\right)^{-1} n\left(T_n^k - U\left(v_0\right)\right)\right)\exp\left(G_v\left(U\left(v_0\right)|v_0\right)h\right)\left(\frac{h - nU_v\left(v_0\right)^{-1}\left(T_n^k - U\left(v_0\right)\right)}{nU_v\left(v_0\right)^{-1}\left(U\left(v_0\right) - T_n^k\right)}\right)^{k-1} \end{aligned} \tag{22}$$

where the second statement follows from the consideration that

$$n\left(U\left(v_0 + h/n\right) - U\left(v_0\right)\right) = nU_v\left(v_0\right)\frac{h}{n} + n \times o\left(\frac{h}{n}\right) = U_v\left(v_0\right)h + o\left(1\right). \tag{23}$$

It is not hard to strengthen this result to obtain that for $x = nU_v\left(v_0\right)^{-1}\left(T_n^k - U\left(v_0\right)\right)$ and $\lambda = -G_v\left(U\left(v_0\right)|v_0\right) = U_v\left(v_0\right)g\left(U\left(v_0\right)|v_0\right)$

$$\begin{aligned} \int \frac{L\left(T_n^k|h\right)}{L\left(T_n^k|0\right)}dh &\overset{a.s.}{\longrightarrow} \int 1\left(h \geq x\right)\exp\left(-\lambda h\right)\left(\frac{h-x}{-x}\right)^{k-1} dh \\ &= \frac{1}{\lambda^{k-1}} e^{-\lambda x}\left(-\frac{1}{x}\right)^{k-1}\int_x^\infty e^{-\lambda(h-x)}\left(\lambda(h-x)\right)^{k-1}dh \\ &= \frac{1}{\lambda^k} e^{-\lambda x}\left(-\frac{1}{x}\right)^{k-1}\Gamma\left(k\right) \end{aligned} \tag{24}$$

where the last equality employed a change in variables from h to $\lambda(h-x)$.

Therefore as $n \longrightarrow \infty$, we take the quotient of (22) and (24) to obtain

$$f\left(h|T_n^k\right) \stackrel{a.s.}{\longrightarrow} \frac{1\left(h \geq x\right)e^{-\lambda h}\left(\frac{h-x}{-x}\right)^{k-1}}{\frac{1}{\lambda^k}e^{-\lambda x}\left(-\frac{1}{x}\right)^{k-1}\Gamma\left(k\right)} = \frac{1\left(h \geq x\right)\lambda e^{-\lambda(h-x)}\left(\lambda(h-x)\right)^{k-1}}{\Gamma\left(k\right)} \quad (25)$$

which is a Gamma distribution with parameters (k, λ).

The almost sure limit of the posterior density $f\left(h|T_n^k\right)$ is that of the Gamma distribution with mean $-\frac{1}{G_v(U(v_0)|v_0)} \times k$ and a location shift by $\left(U_v\left(v_0\right)\right)^{-1} n\left(T_n^k - U\left(v_0\right)\right)$. Now note that the posterior mean of h given T_n^k is

$$\int hf\left(h|T_n^k\right)dh = \int h\frac{L\left(T_n^k|h\right)}{\int L\left(T_n^k|h\right)dh}dh = \int h\frac{L\left(T_n^k|h\right)/L\left(T_n^k|0\right)}{\int L\left(T_n^k|h\right)/L\left(T_n^k|0\right)dh}dh$$

Heuristically, if we could exchange pointwise limit with expectation, then

$$\int hf\left(h|T_n^k\right)dh \stackrel{a.s.}{\longrightarrow}$$
$$\int_x^\infty h\frac{\lambda e^{-\lambda(h-x)}\left(\lambda(h-x)\right)^{k-1}}{\Gamma\left(k\right)}dh$$

Therefore the posterior mean will converge a.s. to

$$x + \frac{k}{\lambda} = \left(U_v\left(v_0\right)\right)^{-1} n\left(T_n^k - U\left(v_0\right)\right) - \frac{1}{G_v\left(U\left(v_0\right)|v_0\right)} \times k. \quad (26)$$

Next, employing the well-known result that $n\left(T_n^k - U\left(v_0\right)\right) \stackrel{d}{\longrightarrow} -\frac{1}{g(U(v_0)|v_0)}\mathcal{P}_k$, where $\mathcal{P}_k$ is a Gamma(k,1) distribution, or the sum of k independent unit exponential distributions.[21] it follows that the asymptotic distribution of the posterior mean (26) is

$$-\frac{k}{G_v\left(U\left(v_0\right)|v_0\right)} - \frac{1}{\left(U_v\left(v_0\right)\right)g\left(U\left(v_0\right)|v_0\right)}\mathcal{P}_k \quad (27)$$

where $\mathcal{P}_k$ is Gamma(k,1). ■

Proof of theorem 7 We begin with the differential equation characterizing the equilibrium bidding strategy (cf. Milgrom and Weber (1982), pg. 1107), which can be written as

$$\frac{d}{d\alpha}\left(e^{-\int_\alpha^x \phi(s)ds}b\left(\alpha\right)\right) = e^{-\int_\alpha^x \phi(s)ds}\left(b'\left(\alpha\right) + \phi\left(\alpha\right)b\left(\alpha\right)\right) = \phi\left(\alpha\right)v\left(\alpha,\alpha\right)e^{-\int_\alpha^x \phi(s)ds}$$

for $\phi\left(\alpha\right) \equiv \frac{f_{Y_1}(\alpha|\alpha)}{F_{Y_1}(\alpha|\alpha).}$ Integrating out gives the solution (cf. Milgrom and Weber (1982) (eq. 8) and Wilson (1977)) :

$$b\left(x\right) = b\left(\underline{x}\right)e^{-\int_{\underline{x}}^x \phi(s)ds} + \int_{\underline{x}}^x \phi\left(\alpha\right)v\left(\alpha,\alpha\right)e^{-\int_\alpha^x \phi(s)ds}d\alpha = v\left(x,x\right) - \int_{\underline{x}}^x e^{-\int_\alpha^x \frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)}ds}dv\left(\alpha,\alpha\right)$$

[21] Alternatively, $\mathcal{P}_k$ is the distribution of the kth arrival time of a standard poisson process. This follows from extreme value theory: the Gamma limit distribution of the sequence of normalized k-th order statistic $n(T_n^k - U(v_0))$ is given by Theorem 2.2.2. of Leadbetter, Lindgren, and Rootzen (1983), where $G(x)$ is a type-3 extreme value distribution with parameter $\alpha = 1$ (i.e., unit exponential distribution).

with the boundary condition $b(\underline{x}) = v(\underline{x}, \underline{x})$.

Now for any signal x, define $\hat{v}(x)$ as the "pseudo-MLE" of v, which for this case is

$$\hat{v}(x) \equiv v : G(x|v) = 1.$$

Note the fundamental relationship $\hat{v}(x) \equiv U^{-1}(x) \Leftrightarrow \hat{v}^{-1}(v) = U(v)$. Furthermore, by assumption 1 and for this extreme rank (i.e., $k = 1$) case, $\hat{v}(x)$ is the lower bound of the posterior density of v given x.

For the pure common value model with conditionally independent signals, which has been the focus of this entire paper, we can employ a trick used in in Wilson (1977) (eqs. 1.7,1.9) in order to simplify the expression for $\phi(s)$:

$$\begin{aligned}\phi(s) &= (n-1)\frac{\int_{\hat{v}(s)}^{\infty} f(s|v)^2 F(s|v)^{n-2} g(v)\,dv}{\int_{\hat{v}(s)}^{\infty} F(s|v)^{n-1} f(s|v)\,g(v)\,dv} \\ &= (n-1)\int_{\Sigma V(s)}^{\infty} \frac{f(s|v)}{F(s|v)} g_n(v|s)\,dv = (n-1)\int_{\hat{v}(s)}^{\infty} \frac{f(s|v)}{F(s|v)} dG_n(v|s)\end{aligned}$$

where, following Wilson (1977), $g_n(v|s)$ denotes the posterior density of v, given that $T_n^1 = s$, which is

$$g_n(v|s) = \frac{nF(s|v)^{n-1} f(s|v) g(v)}{\int_{V(s)} nF(s|w)^{n-1} f(s|w) g(w) dw.}$$

First we define the normalized common value $h \equiv n(v - \hat{v}(s))$, and note that

$$\begin{aligned}&\phi(s) - (n-1)\frac{f(s|\hat{v}(s))}{F(s|\hat{v}(s))} = (n-1)\int_{\hat{v}(s)}^{\infty}\left(\frac{f(s|v)}{F(s|v)} - \frac{f(s|\hat{v}(s))}{F(s|\hat{v}(s))}\right) dG_n(v|s) \\ &= \int_0^{\infty}\left(\frac{\partial}{\partial v}\left(\frac{f(s|\hat{v}(s))}{F(s|\hat{v}(s))}\right) h + o(1)\right) dG_n(\hat{v}(s) + h/n|s)\end{aligned} \tag{28}$$

Recalling corollary 5 above, the sequence of posterior densities for any bidder with (not necessarily pivotal) signal s converges at rate n under a given v_0 to a point mass at the pseudo-MLE $\hat{v}(s)$. Furthermore, this convergence is uniform for all $s \in \mathcal{X}(v_0)$. Therefore, the sequence $dG_n(\hat{v}(s) + h/n|s)$ converges, pointwise in h, to:

$$\frac{1}{n} g_n(\hat{v}(s) + h/n|s)\,dh \longrightarrow -\frac{1}{\frac{\partial}{\partial v}F(s|\hat{v}(s))}\exp\left(\frac{\partial}{\partial v}F(s|\hat{v}(s))\,h\right) = \frac{f(s|\hat{v}(s))}{\hat{v}'(s)} e^{-\frac{f(s|\hat{v}(s))}{\hat{v}'(s)}h}$$

Therefore the integral in (28) above convergences to $M(s) + o(1)$, where

$$M(s) = \frac{\partial}{\partial v}\left(\frac{f(s|\hat{v}(s))}{F(s|\hat{v}(s))}\right)\frac{\hat{v}'(s)}{f(s|\hat{v}(s))} = \frac{f_v(s|\hat{v}(s))(\hat{v}'(s))^2}{f(s|\hat{v}(s))} - f(s|\hat{v}(s))$$

so for large n, one can approximate the equilibrium strategy for the pivotal bidder by:

$$b(T_n^1) \sim v(T_n^1, T_n^1) - \int_{\underline{x}}^{T_n} e^{-(n-1)\int_\alpha^{T_n}\frac{f(s|\hat{v}(s))}{F(s|\hat{v}(s))}ds - \int_\alpha^{T_n} M(s)ds + o(1)} dv(\alpha, \alpha).$$

From our previous analysis of uniform price auctions (cf. Theorem 6), we derived that $n\left(v\left(T_n^1,T_n^1\right)-v_0\right)\xrightarrow{d}-\frac{1}{G_v\left(U\left(v_0\right)|v_0\right)}-\frac{1}{\left(U_v\left(v_0\right)\right)g\left(U\left(v_0\right)|v_0\right)}\mathcal{E}$, where $\mathcal{E}$ denotes an exponential random variable with mean 1. Therefore, noting also that $F\left(s|\hat{v}\left(s\right)\right)=1$:

$$\begin{aligned} n\left(b\left(T_n\right)-v_0\right) =& n\left(v\left(T_n,T_n\right)-v_0\right)-n\int_{\underline{\mathrm{x}}}^{T_n} e^{-(n-1)\int_\alpha^{T_n} f(s|\hat{v}(s))ds-\int_\alpha^{T_n} M(s)ds+o(1)}dv\left(\alpha,\alpha\right) \\ &\sim -\tfrac{1}{G_v(U(v_0)|v_0)}-\tfrac{1}{(U_v(v_0))g(U(v_0)|v_0)}\mathcal{E}-n\int_{\underline{\mathrm{x}}}^{T_n} e^{-(n-1)\int_\alpha^{T_n} f(s|\hat{v}(s))ds-\int_\alpha^{T_n} M(s)ds+o(1)}dv\left(\alpha,\alpha\right) \end{aligned}$$

After a change of variables and using integration by parts, the last integral can be approximated as:

$$\begin{aligned} & n\int_{\underline{\mathrm{x}}}^{T_n} e^{-(n-1)\int_\alpha^{T_n} f(s|\hat{v}(s))ds-\int_\alpha^{T_n} M(s)ds}\hat{v}'\left(\alpha\right)d\alpha \\ =& \int_{\underline{\mathrm{x}}}^{T_n}\frac{\hat{v}'\left(\alpha\right)}{f\left(\alpha|\hat{v}\left(\alpha\right)\right)+\frac{1}{n}M\left(\alpha\right)}de^{-(n-1)\int_\alpha^{T_n} f(s|\hat{v}(s))ds-\int_\alpha^{T_n} M(s)ds} \\ =& \frac{v'\left(T_n\right)}{f\left(T_n|\hat{v}\left(T_n\right)\right)+\frac{1}{n}M\left(T_n\right)}-\int_{\underline{\mathrm{x}}}^{T_n} e^{-(n-1)\int_\alpha^{T_n} f(s|\hat{v}(s))ds-\int_\alpha^{T_n} M(s)ds}d\frac{\hat{v}'\left(\alpha\right)}{f\left(\alpha|\hat{v}\left(\alpha\right)\right)+\frac{1}{n}M\left(\alpha\right)}+o(1) \\ =& \frac{v'\left(U\left(v_0\right)\right)}{f\left(U\left(v_0\right)|v_0\right)}+o\left(1\right)=\frac{1}{U'\left(v_0\right)f\left(U\left(v_0\right)|v_0\right)}+o\left(1\right). \end{aligned}$$

The penultimate equality follows from pointwise convergence of the integrand and therefore dominated convergence, and the final equality uses $\hat{v}'(s)=\frac{1}{U'(\hat{v}(s))}$. ■

Proof of theorem 8 Since the signals are iid conditional on the common value, the order statistics are sufficient statistics for the "parameter" common value, their joint density is given by $n!g\left(X_1|v\right)g\left(X_2|v\right)\ldots g\left(X_n|v\right)$. The posterior density conditional on the above "pivotal event" of the winner is given by

$$n!g\left(x_1|v\right)^2\Pi_{i=3}^n g\left(x_i|v\right)1\left(x_1\leq U\left(v\right)\right)$$

(i) Letting $h\equiv\sqrt{n}(v-v_0)$, we obtain the (approximate) rescaled likelihood ratio process

$$1\left(x_1\leq U\left(v\right)+U'\left(v\right)h/n\right)\frac{g\left(x_1|v+h/n\right)^2\Pi_{i=3}^n g\left(x_i|v+h/n\right)}{g\left(x_1|v\right)^2\Pi_{i=3}^n g\left(x_i|v\right)}$$

Conditional on $1\left(x_1\leq U\left(v+h/n\right)\right)$, the (log) likelihood ratio is approximately

$$2\frac{g_v\left(x_1|v\right)}{g\left(x_1|v\right)}\frac{h}{n}+\sum_{i=3}^n\frac{g_v\left(x_i|v\right)}{g\left(x_i|v\right)}\frac{h}{n}$$

which by appropriate use of LLN converges to

$$hE\frac{g_v\left(x|v\right)}{g\left(x|v\right)}=h\int_0^{U(v)}g_v\left(x|v\right)dx=-hg\left(U\left(v\right)|v\right)U'\left(v\right)$$

Hence, following the same arguments as in the proof for theorem 6 above, in large samples the posterior density of h conditional on the event

$$X_1=X_2=x_1,X_3=x_3,\ldots,X_n=x_n$$

is approximated by (up to a constant proportion)

$$1\left(x_1 \le U(v) + U'(v)\,h/n\right)\exp\left(-hg\left(U(v)\,|v\right)U'(v)\right) = 1\left(h \ge \frac{1}{U'(v)}n\left(x_1 - U(v)\right)\right)\exp\left(-hg\left(U(v)\,|v\right)U'(v)\right).$$

The posterior mean $\frac{1}{g(U(v)|v)U'(v)} + \frac{1}{U'(v)}n\left(x_1 - U(v)\right)$. As we discussed before,

$$n\left(x_1 - U(v)\right) \xrightarrow{d} -\frac{1}{g\left(U(v)\,|v\right)}\mathcal{E}.$$

Therefore the normalized winning bid

$$n\left(E\left(v|X_1 = X_2 = x_1, \ldots, X_n = x_n\right) - v_0\right) \xrightarrow{d} \frac{1}{g\left(U(v)\,|v\right)U'(v)} - \frac{1}{g\left(U(v)\,|v\right)}\mathcal{E}$$

This is exactly the same as the winning bid of the second price auction.

(ii) It should be by now evident that the behavior of winning bid in a English auction is nothing different from a Bayesian estimator of the "parameter" common value based on the full sample. In particular, for smooth "regular" conditional distribution of signals without nonregular features such as boundary dependence, the winning bid converges at $\sqrt{n}$ to the true common value, with the limiting variance given by that of the Fisher information. In the "regular" case one expands the localized log likelihood ratio by

$$\log\frac{g\left(x_1|v + h/\sqrt{n}\right)^2\Pi_{i=3}^n g\left(x_i|v + h/\sqrt{n}\right)}{g\left(x_1|v\right)^2\Pi_{i=3}^n g\left(x_i|v\right)}$$

$$= \frac{h}{\sqrt{n}}\sum_{i=1}^n\frac{\partial\log g\left(x_i|v\right)}{\partial v} + \frac{1}{2}h^2\frac{1}{n}\sum_{i=1}^n\frac{\partial^2\log g\left(x_i|v\right)}{\partial v^2} + o_p(1) \sim \frac{h}{\sqrt{n}}\sum_{i=1}^n\frac{\partial\log g\left(x_i|v\right)}{\partial v} - \frac{h^2}{2}\int\frac{g_v\left(x|v\right)^2}{g\left(x|v\right)}dx$$

This is a normal density with variance $\sigma^2 = \left[\int\frac{g_v(x|v)^2}{g(x|v)}dx\right]^{-1}$ and mean

$$\mu = -\left[\int\frac{g_v\left(x|v\right)^2}{g\left(x|v\right)}dx\right]^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\partial\log g\left(x_i|v\right)}{\partial v} \tag{29}$$

By the same arguments as in the proof of theorem 3 above, the normalized winning bid converges almost surely to the posterior mean for a single draw from this normal distribution (given uniform priors). This posterior mean is just equal to (29), which is a random variable with the $N\left(0, \left[\int\frac{g_v(x|v)^2}{g(x|v)}dx\right]^{-1}\right)$ distribution under v_0.