Student Portfolios and the College Admissions Problem[*]

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Abstract

We develop the first Bayesian model of decentralized college admissions, with heterogeneous students, costly portfolio applications, and uncertainty about student calibers. Students face a nontrivial portfolio choice, and colleges choose admissions standards that act like market-clearing prices.

We solve for the two college model equilibrium, deriving a "law of demand". The lesser college impacts its rival through student portfolio reallocation. Notably, it might impose higher standards, and weaker students sometimes apply more aggressively. We show that the weaker college counters affirmative action at its rival with a discriminatory admissions policy.

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1 Introduction

The college admissions process has lately been the object of much scrutiny, both from academics and in the popular press. This interest owes in part to the competitive nature of college admissions. Schools set admissions standards to attract the best students, and students in turn respond most judiciously in making their application decisions. This paper examines the joint behavior of students and colleges in equilibrium.

We develop and explore an equilibrium model of the college admissions process, with decentralized matching of a heterogeneous population of students, and two colleges — one better and one worse, respectively, called 1 and 2. The model captures two previously unexplored frictions relevant in the "real-world". First, college applications are costly and so students must solve a nontrivial portfolio choice problem. Second, colleges seek to fill their capacity with the best students possible, but student calibers are only imperfectly observed. This tandem of costly applications and yet noisy evaluations feeds the intriguing conflict at the heart of the student choice problem: shoot for the Ivy League, settle for the local state school, or apply to both. Our paper formalizes stretch and safety schools, and analyzes the critical roles they play. Meanwhile, college enrollments are interdependent, because the students' portfolios depend on the joint college admissions standards, and since students accepted at the better college will not attend the lesser one. This asymmetric interdependence leads to many surprising results.

Central to our paper is a theorem characterizing how student acceptance chances at the colleges vary with student caliber. Building on our Bayesian foundation, we find that as a student's caliber rises, the *ratio* of his admission chances at college 1 to college 2 rises monotonically. This property, combined with student optimization, has strong and testable implications for how portfolio choices across students are related. Next, we consider the game induced among colleges by this optimal student behavior. We show how to analyze equilibrium through the lens of supply and demand: When a college raises its standards, its enrollment falls both because fewer students make the cut — the *standards* effect — and fewer will apply *ex ante* — the *portfolio effect*. Treating admissions standards as prices, these effects reinforce each other. In equilibrium, we uncover a "law of demand", in which a college's enrollment falls in its standard. The portfolio effect increases the elasticity of this demand curve.

Analogous to Bertrand competition with differentiated products, colleges choose admissions standards to fill their desired enrollment, taking rival standards and the student portfolios as given. An equilibrium occurs when both markets clear and students behave optimally. The model frictions yield some novel comparative statics. For instance, the admissions standards at *both* colleges fall if college 2 raises its capacity, while lower application costs at *either* school increase the admissions standards at the better college.

In a major thrust of the paper, we ask whether sorting occurs in equilibrium: Do the better students apply more "aggressively"?[1] Does the better college impose higher admissions standards? The answer to this latter question is no when the lesser college is sufficiently small. For by our "law of demand", college 2 continues to raise its standards as its capacity falls. Failures of student sorting are more subtle: The willingness of students either to (*i*) gamble on a stretch school or (*ii*) insure themselves with a safety school may not be monotone in their types. Conversely, all equilibria entail sorting when the colleges differ sufficiently in quality and the lower ranked school is not too small.

This paper takes very seriously the uncertainty that clouds the student admission process. Students apply to colleges, perhaps confident of their types, and perhaps mistakenly so. Equally well, colleges evaluate students trying to gauge the future stars, and often do not succeed. The best framework for analyzing this world therefore involves two-sided incomplete information. In fact, we formulate such a Bayesian model, and then argue that its predictions are well-approximated by one where students know their types, and colleges observe noisy signals. The sorting failures we claim, and the positive theory of how students and colleges react are therefore quite robust findings.

We conclude the paper with a topical foray into college "affirmative action" for in-state applicants, or other preferred applicant classes. As occurs with third degree price discrimination, we show that colleges act in all cases to equate "shadow values" of different groups. We discover that when the better college initiates an affirmative action policy favoring some group, if the weaker college values students across the groups equally, it should respond by *penalizing* applications from that group. This arises from an "acceptance curse": a favored student enrolling at the weaker college may have been rejected by the better college despite affirmative action — a bad signal of their caliber.

Our analysis highlights the role of frictions in college admissions, at the cost of omitting other important real-world elements like heterogeneous student preferences and financial aid offers. These frictions lead to nontrivial student portfolio decisions. Their inclusion not only enriches the model from a theoretical perspective, but also better matches the real world college admissions process.

[1] Precisely, the best students apply just to college 1; weaker students insure by applying to both colleges; even weaker ones apply just to college 2; and finally, the weakest apply nowhere.

Yet it is natural to ask whether these two frictions, application uncertainty and costs, are important in practice. Surely, application outcomes are far from guaranteed: Admissions rates are well below 50% at the most selective colleges, and below 75% at more than half of 4 year colleges.[2] Not surprisingly, students are routinely advised to construct thoughtful portfolios that include both "safety" and "stretch" schools.

Several pieces of evidence suggest that application costs also matter. Given the uncertainty documented above, if applications costs were negligible, we should see students sending many applications. Yet data from the Higher Education Research Institute's freshman survey reveal that the modal number of college applications has been *one* since 1975 (see our supplementary appendix). The mean number of applications has risen over time, to 4.3 in 2006. This low level is consistent with a portfolio choice model, since the marginal benefit of an extra application falls geometrically. For example, a student applying to identical average four-year colleges with 75% acceptance rate sees the marginal benefit to her 5th application scaled by $4^{-4} = 1/256$. So even if attending college this year is worth \$20000, the marginal benefit of that application is only \$59. The trend in applications also agrees with our model, which predicts more applicants as application fees have fallen due to the web and the Common Application. As another illustration of this point, the New York Times recently reported that applications at many colleges skyrocketed when they offered to waive the application fees (Steinberg, 2010).

Modeling frictions is not only important to match stylized facts, but also overturns a number of conclusions reached in their absence. Hoxby (2009) has recently documented the importance of college sorting. While sorting is trivial absent frictions, it proves elusive with frictions. This adds to the literature on decentralized frictional matching — e.g., Shimer and Smith (2000), Smith (2006), Chade (2006), and Anderson and Smith (2010). The student portfolio problem in the model is a special case of Chade and Smith (2006). But the acceptance chances here are *endogenous*, depending on the equilibrium college admission thresholds. So our paper also contributes to this search literature.[3]

The paper is organized as follows. We first introduce a very tractable model in which students perfectly know their calibers, and later show that a richer model without this assumption yields similar implications. We do the student and college equilibrium analyses, respectively, and explore the sorting character of equilibria. We conclude with an exploration of affirmative action. Proofs are found in the appendix.

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[2]Source: Table 329, Digest of Education Statistics, National Center for Education Statistics.

[3]See Burdett and Judd (1983), Burdett, Shi, and Wright (2001), Albrecht, Gautier, and Vroman (2003) and Kircher and Galenianos (2006)).

2 The Model

A. An Overview. The paper introduces three key features — heterogeneous students, portfolio choices with unit application costs, and noisy evaluations by colleges. We impose little additional structure. For instance, we ignore the important and realistic consideration of heterogeneous student preferences over colleges.

A central feature of our analysis is modeling college portfolio applications. Student choice is trivial if it is costless, and in practice, such costs can be quite high. Indeed, the sole purpose of the Common Application is to lower the cost of multiple applications.[4]

Next, we assume noisy signals of student calibers. This informational friction creates uncertainty for students, and a Bayesian filtering problem for colleges. It captures the difficulty faced by market participants, with students choosing "safety schools" and "stretch schools", and colleges trying to infer the best students from noisy signals. Without noise, sorting would be trivial: Better students would apply and be admitted to better colleges, for their caliber would be correctly inferred and they would be accepted. As we will see, sorting is less easily achieved with both application costs and evaluation noise. Indeed, there is a richer role for student choice in this environment.

We also make two other key modeling choices. First, we assume just two colleges, for the sake of tractability. But as we argue in the conclusion, this is the most parsimonious framework that captures all of our key findings. We also fix the capacity of the two colleges. This is defensible in the short run, and so it is best to interpret our model as focusing on the "short run" analysis of college admissions. We explore the simultaneous game in which students apply to college, and colleges decide whom to admit.

In the interest of tractability, our analysis assumes that the colleges' evaluations of students are conditionally independent. This captures the case where students are apprised of all variables (such as the ACT/SAT or their GPA) common to their applications *before* applying to college. Students are uncertain as to how these idiosyncratic elements such as college-specific essays and interviews will be evaluated, but believe that the resulting signals are conditionally independent across colleges. We revisit this restriction in section 6, and argue that our main results on sorting are robust, and that we have analyzed a representative case. For instance, the opposite assumption of perfectly correlated evaluations would imply that admission to a better college often does not guarantee admission to a lesser one. This is not realistic. We also explore how our

[4]This general application form is used by almost 400 colleges, and simplifies college applications. It eliminates idiosyncratic college requirements, but retains separate college application fees.

results change in the level of correlation.

B. The Model. There are two colleges 1 and 2 with *capacities* κ_1 and κ_2, and a unit mass of students with *calibers* x whose distribution has a density $f(x)$ over $[0,\infty)$. Non-triviality demands that college capacity be insufficient for all students, as $\kappa_1+\kappa_2<1$. To avoid many subscripts, we shall almost always assume that students pay a separate but common application cost $c>0$ for the two colleges. Results are similar with different costs. All students prefer college 1. Everyone receives payoff $v>0$ for attending college 1, $u\in(0,v)$ for college 2, and zero payoff for not attending college. Students maximize expected college payoff less application costs. Colleges earn the average enrolled student caliber *times* the measure of students enrolled.

Students know their caliber, and colleges do not. Colleges 1 and 2 each just observes a noisy conditionally independent signal of each applicant's caliber. In particular, they do not know where else students have applied. Signals σ are drawn from a conditional density function $g(\sigma|x)$ on a subinterval of $\mathbb{R}$, with cdf $G(\sigma|x)$. We assume that $g(\sigma|x)$ is continuous and obeys the *strict* monotone likelihood ratio property (MLRP). So $g(\tau|x)/g(\sigma|x)$ is increasing in the student's type x for all signals $\tau>\sigma$.

Students apply simultaneously to either, both, or neither college, choosing for each caliber x, a college application menu $S(x)$ in $\{\varnothing,\{1\},\{2\},\{1,2\}\}$. Colleges choose the set of acceptable students signals. They intuitively should use admission standards to maximize their objective functions, so that college i admits students above a threshold signal $\underline{\sigma}_i$. Appendix A.1 proves this given the MLRP property — despite an *acceptance curse* that college 2 faces (as it may accept a reject of college 1).

For a fixed admission standard, we want to ensure that very high quality students are almost never rejected, and very poor students are almost always rejected. For this, we assume that for a fixed signal σ, we have $G(\sigma|x)\to 0$ as $x\to\infty$ and $G(\sigma|x)\to 1$ as $x\to 0$. For instance, exponentially distributed signals have this property $G(\sigma|x)=1-e^{-\sigma/x}$. More generally, this obtains for signals drawn from any "location family", in which the conditional cdf of signals σ is given by $G((\sigma-x)/\mu)$, for any smooth cdf G and $\mu>0$ — e.g. normal, logistic, Cauchy, or uniformly distributed signals. The strict MLRP then holds if the density is log-concave, so that $\log G'$ is strictly concave.

C. Equilibrium. We consider a simultaneous move game by colleges and students. This yields the same equilibrium prediction as when students move first, as they are atomless.[5] A Nash equilibrium is a triple $(S^*(\cdot),\underline{\sigma}^*_1,\underline{\sigma}^*_2)$ such that:

[5]See Appendix A.2. Alternatively, colleges could move first, committing to an admission standard.

(a) Given $(\underline{\sigma}_1^*, \underline{\sigma}_2^*)$, $S^*(x)$ is an optimal college application portfolio for each x,

(b) Given $(S^*(\cdot), \underline{\sigma}_j^*)$, college i's payoff is maximized by admissions standard $\underline{\sigma}_i^*$.

In a *sorting equilibrium*, colleges' and students' strategies are *monotone*. This means that the better college is more selective ($\underline{\sigma}_1^* > \underline{\sigma}_2^*$) and higher caliber students are increasingly aggressive in their portfolio choice: The weakest apply nowhere; better students apply to the "easier" college 2; even better ones "gamble" by applying also to college 1; the next tier up shoot an "insurance" application to college 2; finally, the top students confidently just apply to college 1. Monotone strategies ensure the intuitive result that the distribution of student calibers at college 1 first-order stochastically dominates that of college 2 (see Claim 4 in Appendix A.7), so that all top student quantiles are larger at college 1. This is the most compelling notion of student sorting in our environment with noise (Chade, 2006).

Our concern with a sorting equilibrium may be motivated on efficiency grounds. If there are complementarities between student caliber and college quality, so that welfare is maximized by assigning the best students to the best colleges, any decentralized matching system must necessarily satisfy sorting to be (constrained) efficient. Since formalizing this idea would add notation and offer little additional insight, we have abstracted from these normative issues and focused on the positive analysis of the model.

3 Equilibrium Analysis for Students

3.1 The Student Optimization Problem

We begin by solving for the optimal college application set for a given pair of admission chances at the two colleges.[6] Consider the portfolio choice problem for a student facing the admission chances $0 \leq \alpha_1, \alpha_2 \leq 1$. The expected payoff of applying to both colleges is $\alpha_1 v + (1-\alpha_1)\alpha_2 u$. The marginal benefit MB_{ij} of adding college i to a portfolio of

This is arguably not the case, but regardless, it too yields the same equilibria until we study affirmative action (proof omitted). In the interests of a unified treatment throughout the paper, we proceed in the simultaneous move world.

[6] Chade and Smith (2006) provide an algorithm that would be useful in the n-college case. In our two college world, their analysis is trivial, and is not needed here.

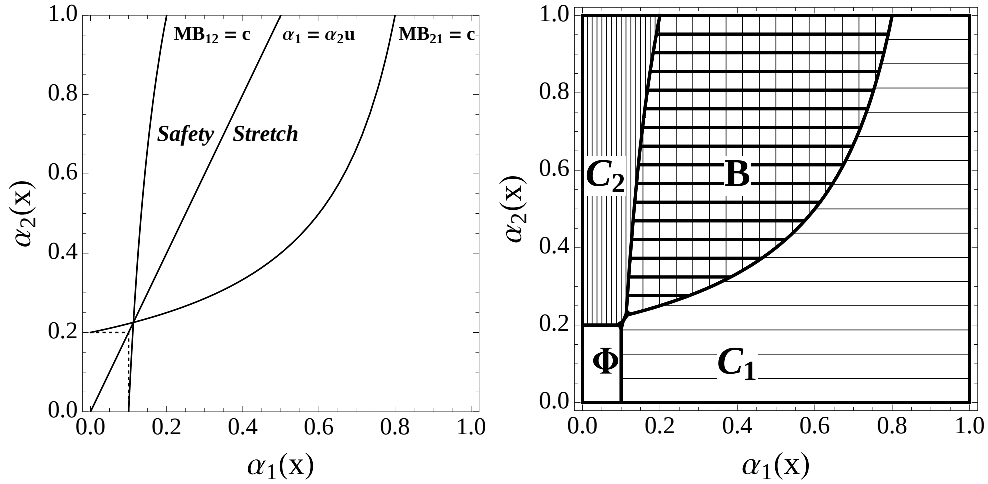

Figure 1: **Optimal Decision Regions.** The left panel depicts (i) a dashed box, inside which applying anywhere is dominated; (ii) the indifference line for solo applications to colleges 1 and 2; and (iii) the marginal benefit curves $MB_{12} = c$ and $MB_{21} = c$ for adding colleges 1 or 2. The right panel shows the optimal application regions. A student in the blank region Φ *does not apply to college.* He *applies to college 2 only* in the vertical shaded region C_2; *to both colleges* in the hashed region B, and *to college 1 only* in the horizontal shaded region C_1.

college j is then:

$$MB_{21} \equiv [\alpha_1 v + (1-\alpha_1)\alpha_2 u] - \alpha_1 v = (1-\alpha_1)\alpha_2 u \tag{1}$$

$$MB_{12} \equiv [\alpha_1 v + (1-\alpha_1)\alpha_2 u] - \alpha_2 u = \alpha_1(v - \alpha_2 u) \tag{2}$$

The optimal application strategy is then given by the following rule:

(a) Apply nowhere if costs are prohibitive: $c > \alpha_1 v$ and $c > \alpha_2 u$.

(b) Apply just to college 1, if it beats applying just to college 2 ($\alpha_1 v \geq \alpha_2 u$), and nowhere ($\alpha_1 v \geq c$), and to both colleges ($MB_{21} < c$, i.e. adding college 2 is worse).

(c) Apply just to college 2, if it beats applying just to college 1 ($\alpha_2 u \geq \alpha_1 v$), and nowhere ($\alpha_2 u \geq c$), and to both colleges ($MB_{12} < c$, i.e. adding college 1 is worse).

(d) Apply to both colleges if this beats applying just to college 1 ($MB_{21} \geq c$), and just to college 2 ($MB_{12} \geq c$), for then, these solo application options respectively beat applying to nowhere, as $\alpha_1 v > MB_{12} \geq c$ and $\alpha_2 u > MB_{21} \geq c$ by (1)–(2).

This optimization problem admits an illuminating and rigorous graphical analysis. The left panel of Figure 1 depicts three critical curves: $MB_{21} = c$, $MB_{12} = c$, $\alpha_1 v = \alpha_2 u$. All three curves share a crossing point, since $MB_{21} = MB_{12}$, when $\alpha_1 v = \alpha_2 u$.

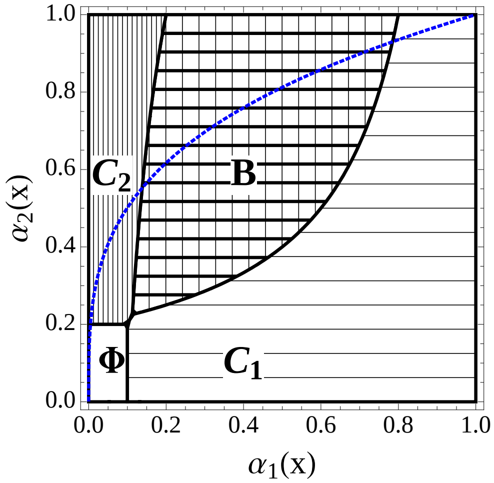

Figure 2: **The Acceptance Function with Exponential Signals.** The figure depicts the acceptance function $\psi(\alpha_1)$ for the case of exponential signals. As their caliber increases, students apply to nowhere (Φ), college 2 only (C_2), both colleges (B) — specifically, first using college 1 as a stretch school, and later college 2 as a safety school — and finally college 1 only (C_1). Student behavior is therefore monotone for the acceptance function depicted.

Cases (a)–(d) partition the unit square into (α_1, α_2) regions corresponding to the portfolio choices (a)–(d), denoted Φ, C_2, B, C_1, shaded in the right panel of Figure 1. This summarizes the portfolio choice of a student with any admissions chances (α_1, α_2).

For an alternative insight into the student optimization, we could apply the marginal improvement algorithm of Chade and Smith (2006). There, a student first decides whether she should apply anywhere. If so, she asks which college is the best singleton. In Figure 1 at the left, college 1 is best right of the line $\alpha_1 v = \alpha_2 u$, and college 2 is best left of it. Next, she asks whether she should apply anywhere else. Intuitively, there are two distinct reasons for applying to both colleges that we can now parse: Either college 1 is a *stretch school* — i.e., a gamble, added second as a lower-chance higher payoff option — or college 2 is a *safety school*, added second for insurance. In Figure 1, these are the parts of region B above and below the line $\alpha_1 v = \alpha_2 u$, respectively.

The choice regions obey some natural comparative statics: The application region C_i to either college increases in its payoff, in light of expressions (1) and (2). Also, as the application cost c rises, the joint application region B shrinks and the empty set Φ grows; the region B shifts right in the college 2 payoff u, and left in the college 1 payoff v. Intuitively, for any student with fixed acceptance rates at the two colleges, a college grows more attractive in its payoff, and the other college grows comparatively less inviting. When we allow endogenous acceptance rates, the picture greatly enriches.

3.2 Admission Chances and Student Calibers

We have solve the optimization for known acceptance chances. But we wish to predict the portfolio decisions of a heterogeneous continuum of students whose acceptance chances are endogenous. To this end, we now derive a mapping from student types to student application portfolios. Fix the thresholds $\underline{\sigma}_1$ and $\underline{\sigma}_2$ set by college 1 and college 2. Student x's acceptance chance at college i is given by $\alpha_i(x) \equiv 1-G(\underline{\sigma}_i|x)$. Since a higher caliber student generates stochastically higher signals, $\alpha_i(x)$ increases in x. In fact, it is a *smoothly monotone onto* function — namely, it is strictly increasing and differentiable, with $0 < \alpha_1(x) < 1$, and the limit behavior $\lim_{x\to 0} \alpha_1(x) = 0$ and $\lim_{x\to\infty} \alpha_1(x) = 1$.

Taking the acceptance chances as given, each student of caliber x faces the portfolio optimization problem of §3.1. She must choose a set $S(x)$ of colleges to apply to, and accept the offer of the best school that admits her. We now translate the sets Φ, C_2, B, C_1 of acceptance chance vectors into corresponding sets of *calibers*, namely, $\Phi, \mathcal{C}_2, \mathcal{B}, \mathcal{C}_1$.

Key to our graphical analysis is a quantile-quantile function relating student admission chances at the colleges: Since $\alpha_i(x)$ strictly rises in the student's type x, a student's admission chance α_2 to college 2 is strictly increasing in his admission chance α_1 to college 1. Inverting the admission chance in the type x, the inverse function $\xi(\alpha, \sigma)$ is the student type accepted with chance α given the admission standard σ, namely $\alpha \equiv 1 - G(\sigma|\xi(\alpha, \sigma))$. This yields an implied differentiable *acceptance function*

$$\alpha_2 = \psi(\alpha_1|\underline{\sigma}_1, \underline{\sigma}_2) = 1 - G(\sigma_2|\xi(\alpha_1, \sigma_1)) \tag{3}$$

We prove in the appendix that the acceptance function rises in college 1's standard $\underline{\sigma}_1$ and falls in college 2's standard $\underline{\sigma}_2$, and tends to 0 and 1 as thresholds near extremes.

By Figure 2, secant lines drawn from the origin or $(1, 1)$ to successive points along the acceptance function decrease in slope. To this end, say that a function $h : [0, 1] \to [0, 1]$ has the *double secant property* if $h(\alpha)$ is weakly increasing on $[0, 1]$ with $h(0) = 0$, $h(1) = 1$, and the two secant slopes $h(\alpha)/\alpha$ and $(1 - h(\alpha))/(1 - \alpha)$ are monotone in α. This description fully captures how our acceptance chances relate to one another.

Theorem 1 (The Acceptance Function) *The acceptance function $\alpha_2 = \psi(\alpha_1)$ has the double secant property. Conversely, for any smooth monotone onto function $\alpha_1(x)$, and any h with the double secant property, there exists a continuous signal density $g(\sigma|x)$ with the strict MLRP, and thresholds $\underline{\sigma}_1, \underline{\sigma}_2$, for which admission chances of student x to colleges 1 and 2 are $\alpha_1(x)$ and $h(\alpha_1(x))$.*

This result gives a complete characterization of how student admission chances at two ranked universities should compare. It says that if a student is so good that he is guaranteed to get into college 1, then he is also a sure bet at college 2; likewise, if he is so bad that college 2 surely rejects him, then college 1 follows suit. More subtly, we arrive at the following testable implication about college acceptance chances:

Corollary 1 *As a student's caliber rises, the ratio of his acceptance chances at college 1 to college 2 rises, as does the ratio of his rejection chances at college 2 to college 1.*

For an example, suppose that caliber signals have the exponential density $g(\sigma|x) = (1/x)e^{-\sigma/x}$. The acceptance function is then given by the geometric function $\psi(\alpha_1) = \alpha_1^{\underline{\sigma}_2/\underline{\sigma}_1}$, as seen in Figure 2. This is increasing and concave when college 2 has a lower admission standard. In turn, the acceptance relation for the location family is easily seen to be $\psi(\alpha_1) = 1 - G((\underline{\sigma}_2 - \underline{\sigma}_1)/\mu + G^{-1}(1-\alpha_1))$.

The acceptance function is closer to the diagonal when signals are noisier, and farther from it with more accurate signals. For an extreme case, as we approach the noiseless case, a student is either acceptable to neither college, both colleges, or just college 2 (assuming that it has a lower admission standard). In other words, the ψ function tends to a function passing through $(0,0)$, $(0,1)$, and $(1,1)$.[7] Specifically, for the earlier location-scale family, $\psi(\alpha_1)$ rises in the signal accuracy $1/\mu$ (see Persico (2000)). Easily, the acceptance function tends to the top of the box $\psi(\alpha_1) = 1 - G(-\infty) = 1$ as $\mu \to 0$, and to the diagonal $\psi(\alpha_1) = 1 - G(0 + G^{-1}(1-\alpha_1)) = \alpha_1$ as $\mu \to \infty$. Near this extreme case, the student behavior is surely monotone, since as the student caliber rises, we proceed in sequence through the regions Φ, C_2, B, and finally C_2.

Since the student's decision problem is unchanged by affine transformations of costs and benefits, *we shall henceforth assume a unit payoff* $v = 1$ *of college 1;* thus, college 2 pays $u \in (0,1)$. Throughout the paper, *we also realistically assume that application costs are not too high relative to the college payoffs* — specifically, $c < u(v-u)/v = u(1-u)$ and $c < u/4$. The first inequality guarantees that the curves $MB_{21} = c$ and $MB_{12} = c$ do not cross twice inside the unit square.[8] The second inequality ensures that the curve $MB_{21} = c$ does not remain above the diagonal.[9] If either inequality fails, the analysis may trivialize because multiple college applications are impossible for some acceptance functions — intuitively, as they are too costly. This case is economically uninteresting.

[7]The limit function is not well-defined: If a student's type is known, just these three points remain.

[8]For if $\alpha_2 = 1$, then $MB_{21} = c$ and $MB_{12} = c$ respectively force $\alpha_1 = 1 - (c/u)$ and $\alpha_1 = c/(v-u)$. Now, $1 - (c/u) > c/(v-u)$ exactly when $c < u(1-u)/v$.

[9]For $MB_{21} = c$ has no roots on the diagonal $\alpha_2 = \alpha_1$ if $c > u/4$.

4 Equilibrium Analysis for Colleges

4.1 A Supply and Demand Approach

Each college i must choose an admission standard $\underline{\sigma}_i$ as a best response to its rival's threshold $\underline{\sigma}_j$ and the student portfolios. With a continuum of students, the resulting *enrollment* $\mathcal{E}_i$ at colleges $i = 1, 2$ is a non-stochastic number:

$$\mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{\mathcal{B}\cup\mathcal{C}_1} \alpha_1(x) f(x)\, dx \tag{4}$$

$$\mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{\mathcal{C}_2} \alpha_2(x) f(x)\, dx + \int_{\mathcal{B}} \alpha_2(x)\,(1 - \alpha_1(x))\, f(x)\, dx, \tag{5}$$

suppressing the dependence of the sets $\mathcal{B}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ on the student application strategy. To understand (4) and (5), observe that caliber x student is admitted to college 1 with chance $\alpha_1(x)$, to college 2 with chance $\alpha_2(x)$, and finally to college 2 but not college 1 with chance $\alpha_2(x)(1 - \alpha_1(x))$. Also, anyone that college 1 admits will enroll automatically, while college 2 only enrolls those who either did not apply or got rejected from college 1.

If we substitute optimal student portfolios into the enrollment equations (4)–(5), then they behave like demand curves where the admissions standards are the prices. Our framework affords analogues to the substitution and income effects in demand theory. The admission rate of any student obviously falls in its anticipated admission standard — the *standards effect*. But there is a compounding *portfolio effect* — that enrollment also falls due to an application portfolio shift. Each college's applicant pool shrinks in its own admissions threshold. We then deduce in the appendix the following "law of demand": *If a college raises its admission standard, then its enrollment falls.* Because of our portfolio effect, a college faces a more elastic demand for slots than purely predicted by the standards effect. A lower admission bar will invite applications from new students.[10]

The law of demand generally applies outside two college setting. For an intuition, suppose that the admissions standard at a college rises. Absent any student portfolio changes, fewer students meet its tougher admission threshold (the standards effect), and its enrollment falls. The portfolio adjustment reinforces this effect. Those who had marginally chosen to add this college to their portfolios now excise it (the portfolio effect).

In consumer demand theory, the "price" of one good affects the demand for the

[10] The portfolio effect may act with a lag — for instance, a college may unexpectedly ease admission standards one year, and see their applicant pool expand the next year when this becomes understood.

other, and in the two good world, they are substitutes. Analogously, we prove in the appendix, that *a college's enrollment rises in its rival's admission standard.* This owes to a portfolio spillover effect. If it grows tougher to gain admission to college i, then those who only applied to its rival continue to do so, some who were applying to both now apply just to j (which helps college j when it is the lesser school), and also some at the margin who applied just to i now also add college j to their portfolios.[11]

Since capacities imply vertical supply curves, we have justified a supply and demand analysis, in which the colleges are selling *differentiated products.* Ignoring for now the possibility that some college might not fill its capacity, equilibrium without excess capacity requires that both markets clear:

$$\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2) \qquad \text{and} \qquad \kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2) \tag{6}$$

Since each enrollment (demand) function is falling in its own threshold, we may invert these equations. This yields for each school i the threshold that "best responds" to its rival's admissions threshold $\underline{\sigma}_j$ so as to fill their capacity κ_i:

$$\underline{\sigma}_1 = \Sigma_1(\underline{\sigma}_2, \kappa_1) \qquad \text{and} \qquad \underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1, \kappa_2) \tag{7}$$

Given the discussion of the enrollment functions, we can treat Σ_i as a "best response function" of college i. It rises in its rival's admission standard and falls in its own capacity. That is, the admissions standards at the two colleges are strategic complements. Figure 3 depicts an equilibrium as a crossing of these increasing best response functions.

By way of contrast, observe that without noise or without application costs, the better college is completely insulated from the actions of its lesser rival — Σ_1 is vertical. The equilibrium analysis is straightforward, and there is necessarily a unique equilibrium. In either case, the applicant pool of college 1 is independent of what college 2 does. For when the application signal is noiseless, just the top students apply to college 1. And when applications are free, all students apply to college 1, and will enroll if accepted.

With application costs and noise, Σ_1 is upward-sloping, as application pools depend on both college thresholds. When college 2 adjusts its admission standard, the student incentives to gamble on college 1 are affected. This feedback is critical in our paper. It

[11] As in consumer theory, complementarity may emerge with three or more goods available. With ranked colleges 1, 2, and 3, college 3 may be harmed by tougher admissions at college 1, if this encourages enough applications at college 2.

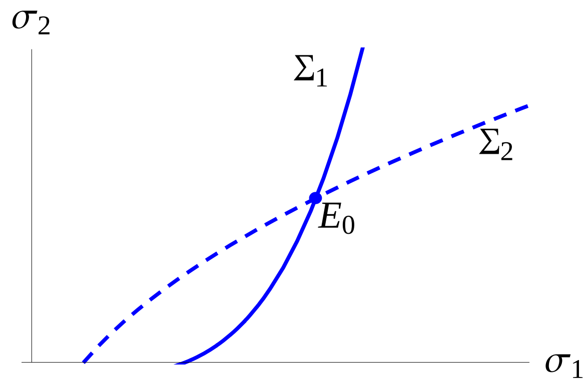

Figure 3: **College Responses and a Stable Equilibrium.** The functions Σ_1 (solid) and Σ_2 (dashed) give pairs of thresholds so that colleges 1 and 2 fill their capacities in equilibrium. In the stylized Gale-Shapley framework, the best response curve Σ_1 of college 1 is vertical.

leads to a richer interaction among the colleges, and perhaps to multiple equilibria.

In Figure 3, the best response function Σ_1 is steeper than Σ_2 at the crossing point. Let us call any such equilibrium *stable.* It is robust in the following sense: Suppose that whenever enrollment falls below capacity, the college eases its admission standards, and vice versa. Then this dynamic adjustment process pushes us back towards the equilibrium. Then at this theoretical level, admission thresholds act as prices in a Walrasian tatonnement. Unstable equilibria should be rare: They require that a college's enrollment responds more to the other school's admission standard than its own.

We show in the appendix that a stable equilibrium exists, and in any such equilibrium, college 1 fills its capacity, while college 2 has excess capacity if college 1 is big enough. Surprisingly, college spaces can go unfilled despite insufficient capacity for the applicant pool. If college 1 is "too big" relative to college 2, then college 2 is left with excess capacity. There is excess demand for college slots, and yet due to the informational frictions, there is also excess supply of slots at college 2, even at "zero price".

When college 2 has excess capacity, it optimally accepts all applicants. Since college 1 maintains an admissions standard, college behavior is monotone. But this forces $\alpha_2 = 1$ for all students, and so the acceptance function traverses the top side of the unit square in Figure 2. In other words, as student caliber rises, the lowest students apply to college $\{2\}$, higher students to both colleges, and the best students just apply to college $\{1\}$. Let us observe in passing that this is a sorting equilibrium.

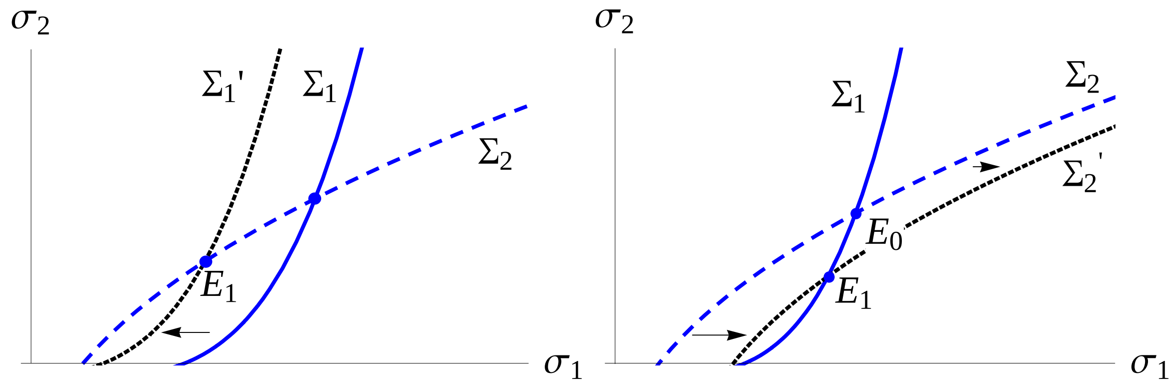

Figure 4: **Equilibrium Comparative Statics.** The figure illustrates how the equilibrium is affected by changing capacities κ_1, κ_2. The best response functions Σ_1 (solid) and Σ_2 (dashed) are drawn. The left panel considers a rise in κ_1, shifting Σ_1 left, thereby lowering both college thresholds. The right panel depicts the analogous rise in κ_2, and shift Σ_2.

4.2 Comparative Statics

We now continue to explore the supply and demand metaphor, and derive some basic comparative statics. The potential multiplicity of equilibria makes a comparative statics exercise difficult.[12] But fortunately, our analysis applies to *all* stable equilibria. Indeed, *greater capacity at either college lowers both college admissions thresholds.* This result speaks to the equilibrium effects at play. Greater capacity at one school, or an exogenous downward shift in the "demand" for slots, reduces the "price" (admission standard) at both schools. The graph in Figure 4 of equations (7) proves this assertion.

For an intuition, consider a sorting equilibrium, where students apply as in Figure 2. Suppose that college 2 raises its capacity κ_2 (as in the right panel of Figure 4). Fixing the admission standard $\underline{\sigma}_1$, this depresses $\underline{\sigma}_2$. The marginal student that was indifferent between applying to college 2 only (C_2) and both colleges (B) now prefers to apply to college 2 only. So fewer apply to college 1. Given this portfolio shift, college 1 drops its admission standards. The left panel depicts the same comparative static for college 1.

Unlike college capacity, college payoffs or application costs affect both best response functions Σ_1 and Σ_2. As a result, the comparative statics are not unambiguous, and counter-intuitive results may emerge. Suppose that the payoff u of college 2 rises, and assume fixed admission standards. Then as we have argued, region B shifts right in Figure 2, thereby increasing the demand for college 2. At the same time, the demand

[12]It is not easy to ensure uniqueness of equilibrium. One case in which this holds is when c is sufficiently small. This follows by continuity from the uniqueness of equilibrium in the costless case.

for college 1 decreases, as some students cease to send stretch applications to that college. These forces lead to new best reply admission standards, namely an upward shift in the Σ_2 locus and a leftward shift in Σ_1. Depending on the strength of these shifts, the new equilibrium admissions thresholds are either both higher, or both lower, or the one at college 1 lower and that at college 2 higher than before the increase in u. An ambiguity likewise arises if the payoff v of college 1 rises (in the original formulation).

Next, assume that the application cost c rises, perhaps due to a rise in the SAT or ACT cost, or the common application fee. Then Σ_1 shifts left — for college 1 must drop its standards to counter the loss of stretch applications, since we have seen that the region B shrinks in Figure 2. On the other hand, Σ_2 shifts ambiguously in the cost c. For with fixed standards, we have seen that college 2 loses safety and solo applications, but no longer loses as many students who send stretch applications to college 1.

Consider instead a rise in just one college's application cost, such as a college requiring a longer essay or imposing a greater fee. We argue that in a sorting equilibrium, *if the application cost at either college slightly falls, then the admission standard at college 1 rises and its student caliber distribution stochastically worsens.* If the application cost at college 2 falls, then more students apply, and it is forced to raise its standards. The marginal benefit (2) of a stretch application to college 1 thus rises. To counter this, college 1 responds with a higher standard, but it still gains more applicants at its lower end, and its caliber distribution stochastically worsens. By contrast, college 2 loses not only its worst students, but also top ones for whom it was insurance; its caliber change is ambiguous. Graphically, a fall in the application cost at college 2 shifts Σ_2 upwards without affecting Σ_1 — as the applicant pool at college 1 depends only indirectly on the college 2 application cost via portfolio effects as the college 2 standard changes.

The argument is more complex when the application cost at college 1 falls, since the applicant pools at both colleges depend on it. The direct impact is an expansion in the applicant pool at college 1 — consisting of those who send stretch applications to it — and a contraction in the applicant pool of college 2, triggering an increase in the admission standard at college 1 and a decrease in that of college 2. The changes in the admissions thresholds lead to further portfolio student reallocation, and the net effect is a priori ambiguous. Graphically, a lower application cost at college 1 simultaneously shifts Σ_1 (leftwards) and Σ_2 (downwards). We show in the appendix that the portfolio effects at both colleges cancel out for those calibers who send stretch applications to college 1; on balance, its admission standard rises, and yet its applicant pool worsens.

The logic underlying this section does not essentially depend on the assumption that there are two colleges. For instance, whenever colleges have overlapping applicant pools, a rise in capacity at either depresses the admission standards at both.

5 Do Colleges and Students Sort in Equilibrium?

Casual empiricism suggests that the best students apply to the best colleges, and those colleges are in turn the most selective. This logic justifies ranking colleges based on their admissions standards. Curiously, these claims are *false* without stronger assumptions. We identify and explore the two possible types of sorting violations by students.

The first violation is seen in the left panel of Figure 5. By Corollary 1, along the acceptance function, higher types enjoy a higher ratio of admissions chances at college 1 to college 2. But this does not imply a higher marginal benefit $\alpha_1(1-\alpha_2 u)$ of applying to college 1. So the acceptance function may multiply slice through the curve (2), where students are indifferent between a stretch application to college 2. This yields a non-monotone sequence of application sets $\Phi, \{2\}, \{1,2\}, \{2\}, \{1,2\}, \{1\}$ as the caliber rises.

The second violation occurs when college admissions standards are sufficiently close. For in the extreme case, when students entertain the same admission chances α at the two colleges, the marginal benefit of adding a safety application, namely $\alpha(1-\alpha u)$, is not monotone in α (and thus not in caliber, either). The right panel of Figure 5 depicts one such case — where the application sets are $\Phi, \{1\}, \{1,2\}, \{1\}$ as caliber rises. This violation can even occur when college 2 sets a higher admissions standard than college 1.

To rule out the first sorting violations, it suffices that college 2 offer a low payoff ($u < 0.5$). We show in the appendix that this ensures that the marginal benefit of additionally applying to college is increasing. The second sort of violation cannot occur when college 1 sets a sufficiently higher admission standard than college 2. This in particular happens when college 1 is sufficiently smaller than college 2. The threshold capacity will in general depend on all the primitives of the problem: the rival capacity, applications cost, payoff differential and signal structure.

Theorem 2 (Non-Sorting and Sorting in Equilibrium)
(*a*) *If college 2 is "too good" (i.e., $u > 0.5$), then there exists a continuous MLRP density $g(\sigma|x)$ that yields a stable equilibrium with non-monotone student behavior.*
(*b*) *If college 2 is small enough relative to college 1, then college 2 sets a higher admissions standard than college 1 in some equilibrium.*

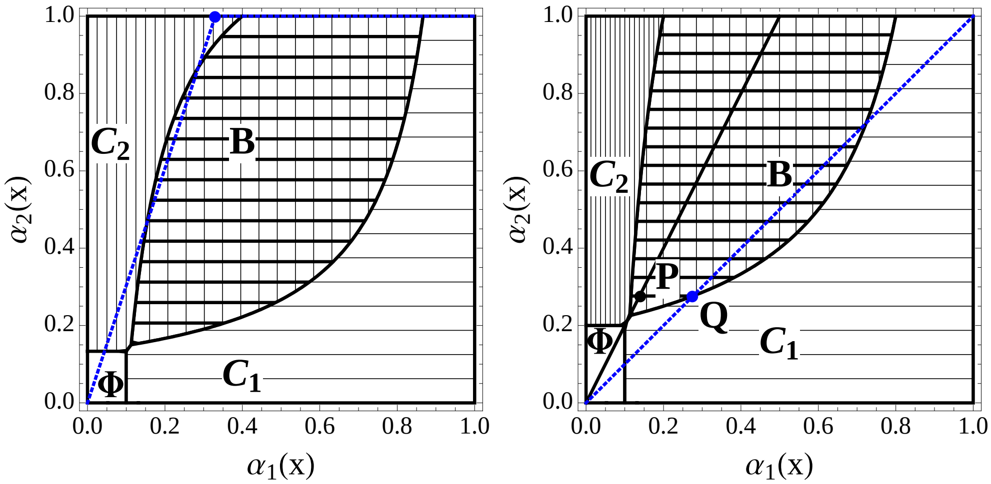

Figure 5: **Non-Monotone Behavior.** In the left panel, the signal structure induces a piecewise linear acceptance function. Student behavior is non-monotone, since there are both low and high caliber students who apply to college 2 only (C_2), while intermediate ones insure by applying to both. In the right panel, equal thresholds at both colleges induce an acceptance function along the diagonal, $\alpha_1 = \alpha_2$. Student behavior is non-monotone, as both low and high caliber students apply to college 1 only (C_1), while middling caliber students apply to both. Such an acceptance function also arises when caliber signals are very noisy.

(c) *If college 1 is small enough relative to college 2, and college 2 is not too good (namely,* $u \leq 0.5$*), then there are only sorting equilibria and neither college has excess capacity.*

The challenge in proving this theorem is to show that all of non-monotone behavior outlined above can happen *in equilibrium*. For part (a), we construct a non-monotone equilibrium by starting with the acceptance function depicted in the left panel of Figure 5, which constrains the relationships between admissions chances across colleges to be some mapping $\alpha_2(x) = h(\alpha_1(x))$. We then construct a particular acceptance chance $\alpha_1(x)$ so that the induced student behavior and acceptance rates given $(\alpha_1, h(\alpha_1))$ equate college capacities and enrollments. Finally, we show that these two mappings satisfy the requirements of Theorem 1 and therefore can be generated by MLRP signals and monotone standards. For part (b), we show that that by perturbing an equilibrium with equal admissions chances by making college 2 smaller, we get an equilibrium with non-monotone standards. Finally, part (c) turns on showing that when κ_1 is relatively small, the crossing of the best-response functions must occur at a point where college 1 sets sufficiently high standards that low caliber students don't apply there.

All told, parts (a) and (b) show that sorting fails in some equilibria, which is surprising given how well behaved the signal structure is. The take-away is that in simultaneous search problems, the marginal benefits both of gambling up and of insuring needn't be monotone in type, even in equilibrium. Positive assortative matching can fail even in an ex-ante sense. This unintuitive result has immediate implications: for example, colleges are often ranked based on the average class rank of matriculating students — which seems quite reasonable — but since smaller schools may be able to maintain higher admissions standards, independent of their quality, this measure is imperfect.

6 General Incomplete Information About Calibers

6.1 Two Equivalent Models

To avoid a wealth of algebraic battles, we have assumed that the colleges observe conditionally independent evaluations of the students' true calibers. In the real world, both the student and colleges are not fully apprised of any student's true caliber. We now consider the spectrum of such possibilities, ranked by who has superior information:

A-1. Students know their calibers, and colleges see noisy conditionally iid signals.

A-2. Colleges know student calibers, and students see noisy iid signals of them.

A-3. Both student and colleges see noisy conditionally iid signals of calibers.

Intuitively, this list transitions from one extreme where the students perfectly know their calibers, to the opposite end of the spectrum, where colleges perfectly know the students. These cases respectively imply the following three distinct economic scenarios that one might imagine, subsuming the model of this paper within a greater picture:

B-1. Students know their calibers; colleges observe conditionally iid noisy signals.

B-2. Students know their calibers; colleges observe perfectly correlated signals.

B-3. Students know their calibers; colleges observe affiliated noisy signals.

Clearly, case A-1, is informationally equivalent to case B-1 — namely, the model in our paper. More subtly, we prove in the appendix that for $i = 2, 3$, *the signal structure in case A-i implies that of case B-i when we re-interpret the student signal in the A cases as the student caliber in the B cases*; consequently, the equilibrium behavioral properties of case A-3 are replicated by case B-3. It is easy to see this for case 2, for with A-2, any student sees a signal equal to $t +$ "noise", while both colleges see the student caliber t. So in case B, if we re-define the student's caliber as x, then both colleges see the same signal $x = t -$ "noise" — and of course, negative noise is just noise.

By this correspondence, we can without loss of generality assume that students know their calibers and colleges vary by the extent to which their signals are correlated.

The winner's curse plays a central role in our equilibrium portfolio analysis. For in case 1, only the colleges can suffer a winner's curse. As we will see, this emerges in the analysis of affirmative action for student subgroups. In case 3, the winner's curse adds a critical dimension of complexity to the student optimization problem. Finally, under interpretation B-2 neither side suffers a winner's case, for students know their types and colleges see the same signal. So this holds under model A-2.

We now extend our paper to cases 2 and 3. Observe critically that *Theorem 1 remains a valid description of how the unconditional acceptance chances at the two colleges relate.*

6.2 Perfectly Correlated College Evaluations

Suppose that the two colleges observe perfectly correlated signals of student calibers. As we mentioned above, this is akin to observing the caliber of each applicant. The key (counterfactual) feature here is that if a student is accepted by the more selective college, then he must be accepted at the less selective one. This immediately implies that $\underline{\sigma}_1 > \underline{\sigma}_2$ in equilibrium, for otherwise nobody would apply to college 2. *So college behavior must be monotone in any equilibrium.* Let us now next to the students.

The analysis of this case differs in a few dimensions from §3.1. Since $\underline{\sigma}_1 > \underline{\sigma}_2$, applying to both colleges now yields payoff $\alpha_1 + (\alpha_2 - \alpha_1)u - 2c$. Unlike before,

$$MB_{21} \equiv (\alpha_2 - \alpha_1)u = c \qquad \text{and} \qquad MB_{12} \equiv \alpha_1(1-u) = c \tag{8}$$

because admission to college 1 guarantees admission to college 2. In this informational world, both optimality equations are linear, and the latter is vertical (see Figure 6).

Let's first explore conditions that guarantee monotone student behavior. Assume that college 1 is sufficiently more selective than college 2. Then the lowest caliber applicants apply to college 2 — namely, those whose admission chance exceeds c/u. Students so good that their admission chance at college 1 is at least $c/(1-u)$ add a stretch application, provided college 2 admits them with chance $c/u(1-u)$ or more. In Figure 6, this occurs when the acceptance function crosses above the intersection point of the curves $MB_{21} = c$ and $MB_{12} = c$.[13] Since the marginal benefit MB_{12} is

[13]Namely, at the mutual intersection of regions C_1, C_2, and B in Figure 6. By inequality (15), this holds under our hypothesis that college 1 is sufficiently choosier than college 2.

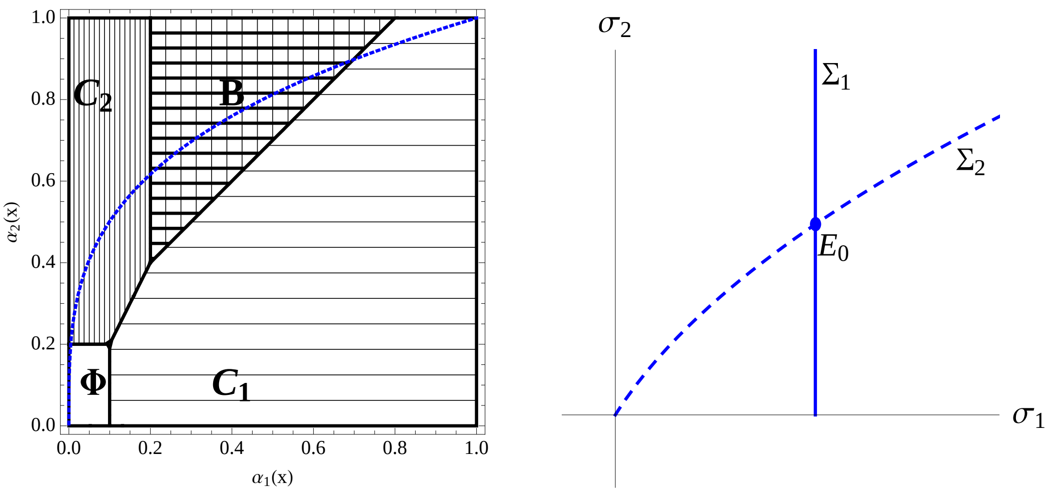

Figure 6: **Student Behavior with Perfectly Correlated Signals.** The shaded regions depict the optimal portfolio choices for students when colleges observe perfectly correlated signals. Unlike Figure 1, the $MB_{12} = c$ curve separating regions B and C_2 is vertical. The Σ_1 curve at the right is vertical, since college 2 no longer imposes an externality on college 1.

independent of the admission chance at college 2, $MB_{12} > c$ for all higher calibers.

But monotone behavior for stronger caliber students requires another assumption. Consider the margin between shooting a safety application or just apply to college 1. The top caliber students will apply to college 1 only, since their admission chance is so high. But the behavior of slightly lesser student calibers is trickier, as the acceptance function can multiply cross the line $MB_{21} = c$.[14] Under slightly stronger assumptions, the acceptance function is concave, and this precludes such perverse multiple-crossing behavior.[15] With such assumptions, student behavior is monotone.

Let's now explore the possibility of non-monotone student behavior. First, absent a concave acceptance function, the sorting failure just alluded to arises. Second, even with a concave acceptance function, college 1 might not be sufficiently choosier than college 2. In this case, weakest applicants apply to college 1, and the sorting failures involving safety schools occur. For stronger students may insure themselves. As seen in Figure 6, a concave acceptance function can consecutively hit regions C_1, B, then C_1.

Having explored the impact of correlation on college-student sorting, we now flesh out its effect on college feedbacks. Since MB_{12} is independent of the admission threshold at college 2, the pool of applicants to college 1 is unaffected by changes in $\underline{\sigma}_2$. Hence, the

[14]For as seen in Figure 6, that line also has a strictly falling secant.

[15]Concavity holds whenever $-G_x(\sigma|x)$ is log-supermodular. This is true when we further restrict to location families (like the Normal) or scale families (such as exponential).

Σ_1 locus is vertical. The better college is insulated from the decisions of its weaker rival, and the setting is not as rich as our maintained conditionally iid case. It is therefore graphically obvious from Figure 6 that the equilibrium is unique.

6.3 Affiliated College Evaluations

We now turn to the general case with two-sided incomplete information about student calibers. We will think of this instead as described earlier in B-3: Each student knows his caliber x, which is drawn from a density $f(x)$. Also, colleges see affiliated signals σ_1, σ_2 about these calibers, and so their joint density $g(\sigma_1, \sigma_2|x)$ is affiliated.[16] We suppress the caliber x argument of the unconditional acceptance chance $\alpha_i(x)$ at college $i = 1, 2$.

Since acceptance / rejection by college 1 is *good / bad news*, it intuitively follows that

$$\alpha_2^A \geq \alpha_2 \geq \alpha_2^R \tag{9}$$

Here, α_2^A and α_2^R are the respective acceptance chances at college 2 given acceptance and rejection at college 1. For instance, with perfectly correlated college evaluations, we have $1 = \alpha_2^A > \alpha_2 > \alpha_2^R$, since college 2 accepts anyone that college 1 does. But in the conditionally iid case, college 2 is unaffected by the decision of college 1, and so $\alpha_2^A = \alpha_2 = \alpha_2^R$. Since these are intuitively opposite ends of an affiliation spectrum, we introduce a new ranking. Call evaluations *more affiliated* if the conditional acceptance chance α_2^A at college 2 is higher for any given unconditional acceptance chance α_2.

Let's first see how affiliation affects student applications. In this more general setting,

$$MB_{21} = (1 - \alpha_1)\alpha_2^R u \qquad \text{and} \qquad MB_{12} = \alpha_1(1 - \alpha_2^A u). \tag{10}$$

This subsumes the marginal analysis for our conditionally iid and perfectly correlated cases: (1)–(2) and (8). Relative to these benchmarks, the acceptance curse (or the "blessing") conferred by college 1's two possible decisions lessens the marginal gain of an extra application to either college — due to inequality (9). More intuitively, double admission is more likely when signals are more affiliated. In our graph, this is reflected by a right shift of the curve $MB_{12} = c$, and a left shift of $MB_{21} = c$. In other words, for any given college admission standards, *students send both fewer stretch and safety applications when college evaluations are more affiliated.*

[16] As is standard, this means that g obeys the monotone likelihood ratio property for every fixed x.

We next explore how affiliated evaluations affects college behavior. Consider the best reply locus Σ_1 of college 1. It is upward-sloping with conditionally iid college evaluations, and vertical with perfectly correlated evaluations. We argue that it is upward-sloping with affiliated evaluations, and grows steeper as evaluations grow more affiliated. In other words, *positing that students know their calibers and colleges see conditionally iid signals of them (i.e. case 1) is not only most tractable, but also delivers robust results about two-way college feedbacks.* Perfectly correlated evaluations therefore ignores the effect of the lesser on the better college, and so is less reflective of the affiliated case.

Let's see how *the best response curve Σ_1 slopes upward with imperfect affiliation.* Let college 1's admission standard $\underline{\sigma}_1$ rise. Then its unconditional acceptance chance α_1 falls for every student. The marginal student pondering a stretch application must then fall in order for college 1 to fill its capacity (6). Optimality $MB_{12} = c$ in (10) next requires that this student's conditional acceptance chance α_2^A fall. This only happens if his unconditional chance α_2 falls too — i.e. the standard $\underline{\sigma}_2$ rises.

Next, *college 1's best response curve Σ_1 slopes up more steeply when college evaluations are more strongly affiliated.* For as affiliation rises, the marginal student sees a greater fall in his admission chance α_2^A. So his unconditional admission chance α_2 falls more too, and college 2's admission standard $\underline{\sigma}_2$ drops more than before (see Figure 6), as claimed. As an aside, since equilibrium is unique with perfectly correlated college evaluations, uniqueness intuitively more often arises when we are closer to this extreme.

Let's revisit Theorem 2, and explore how equilibrium sorting changes in the level of affiliation. By examining (10), we see that *as the affiliation rises, and we transition from conditionally iid to perfectly correlated signals, the region of multiple applications shrinks monotonically.* This simple insight has important implications for sorting behavior. By standard continuity logic, for very low or high affiliation, sorting obtains and fails exactly as in the respective conditional independent or perfectly correlated cases. But we need not simply rely on limit logic. More strongly, the negative result in Theorem 2 (*b*) fails for moderately high affiliation: For then nonmonotone college behavior is impossible since the locus $MB_{21} = c$ lies wholly above the diagonal with sufficiently affiliated signals. So the acceptance function would lie below the diagonal if admission standards were inverted, and no student would ever apply to college 2. Finally, the logic for the positive sorting result of Theorem 2 (*c*) is still valid, that both student and college behavior are monotone if college 2 is not too small and not too good, if the acceptance function is concave — appealing to the logic for perfectly correlated signals.

7 The Spillover Effects of Affirmative Action

We now explore how affirmative action at one college affects the other.[17] Slightly enriching our model, we first assume that a fraction ϕ of the applicant pool belongs to a *target group.* This may be an under-represented minority, but it may also be a majority group. For instance, many states favor their own students at state colleges — Wisconsin public colleges can have at most 25% out-of-state students. Just as well, some colleges strongly value athletes or students from low-income backgrounds. We assume a common caliber distribution, so that there is no other reason for differential treatment of the applicants.

Assume that students honestly report their "target group" status on their applications. Moreover, assume that students from both groups use monotone application strategies. Reflecting the colleges' desire for a more *diverse* student body, let college i earn a bonus $\pi_i \geq 0$ for each enrolled target student. The colleges may set different thresholds for the two groups. Let colleges $1, 2$ offer "discounts" Δ_1, Δ_2 to target applicants. In other words, the respective standards for non-target and target groups are $(\underline{\sigma}_1, \underline{\sigma}_2)$ and $(\underline{\sigma}_1 - \Delta_1, \underline{\sigma}_2 - \Delta_2)$. At each college, the expected payoff of the marginal admits from the two groups should coincide — except of course at a corner solutions (when a college admits all students from a group). This yields two new equilibrium conditions that account for the fact that ex post, colleges behave rationally, and equate their expected values of target and non-target applicants.

$$
\begin{aligned}
E[X + \pi_1 | \sigma = \underline{\sigma}_1 - \Delta_1, \text{target}] &= E[X | \underline{\sigma}_1, \text{non-target}] && (11) \\
E[X + \pi_2 | \sigma = \underline{\sigma}_2 - \Delta_2, \text{target, accepts}] &= E[X | \underline{\sigma}_2, \text{non-target, accepts}] && (12)
\end{aligned}
$$

Here, X is the student caliber. So as with third degree price discrimination, colleges equate the shadow cost of capacity across groups. Along with market clearing (6) at each college, equilibrium requires solving four equations in four unknowns.

For any discounts Δ_1, Δ_2, we let $(\underline{\sigma}_1(\Delta_1, \Delta_2), \underline{\sigma}_2(\Delta_1, \Delta_2))$ be admission standards for non-target students that fill the capacity at both colleges — i.e. solving equations (6). As in Section 4, there could be multiple solutions. Consider a stable one.[18] Then let $V_i(\Delta_1, \Delta_2, \pi_i)$ be the shadow value difference in the LHS and RHS of (11)–(12), *evaluated at capacity-filling standards* $(\underline{\sigma}_1, \underline{\sigma}_2)$. An equilibrium is then a zero $V_1 = V_2 = 0$.

[17] For recent treatment of affirmative action, see Hickman (2010), Groen and White (2004), and Curs and Singell (2002).

[18] A simple modification of the proof of Claim 3 appendix gives existence of such a stable solution.

Naturally, without any group preference ($\pi_1 = \pi_2 = 0$), an equilibrium is $\Delta_1 = \Delta_2 = 0$. Let us now define two new college best response functions. Write $\Delta_i = \Upsilon_i(\Delta_j, \pi_i)$ when $V_i(\Delta_1, \Delta_2, \pi_i) = 0$. An equilibrium is then a crossing point of Υ_1, Υ_2 in (Δ_1, Δ_2)-space.

It is *a priori* not clear what happens to the equilibrium as the group preferences π_1 and π_2 change. This hinges on the sign of the derivatives of V_i with respect to Δ_1 and Δ_2. To see the difficulty, consider for example what happens to V_1 after the discount Δ_1 at college 1 rises. The immediate effect is that V_1 falls, as target students meet a lower standard — fixing the non-target standards. But there are two feedback due to capacity considerations alone. When the discount Δ_i for a target student at college i changes, there is an indirect effect — operating through the capacity equations — on the non-target standard at college j. In the appendix, we show that this subtle feedback are negligible locally around $\Delta_1 = \Delta_2 = 0$ with no affirmative action. From now on, we ignore these two cross feedback effects in computing the total derivatives in Δ_1, Δ_2.

It is critical to pin down the slopes of Υ_1 and Υ_2. From college 1's perspective, shadow value equalization requires that the discounts Δ_1 and Δ_2 rise or fall together. Why? We argue that these discounts have opposite effects on the shadow value difference V_1.

Lower standards for target students at college 1 not only depresses their average caliber via the standards effect, but also encourages worse target applicants to apply — i.e. the portfolio effect reinforces this. To fill capacity, the non-target student standard must rise at college 1; their quality rises due to the portfolio and standards effects. Altogether, the shadow value of non-target students rises relative to target students. Conversely, lower standards for target students at college 2 deter the weakest target "stretch" applicants at college 1, via the portfolio effect. So ignoring the cross effects, the shadow value difference V_1 rises in Δ_2. To summarize, to maintain (11), an increase in Δ_1 must be accompanied by an increase in the discount Δ_2, and thus Υ_1 slopes up.

By contrast, the slope of the Υ_2 is ambiguous. First, when college 1 favors some students more, the portfolio and standards effects reinforce. College 2 loses some stellar target "safety" applicants, but the remaining top tier of target applicants gain admission to college 1 more often, and so are unavailable to college 2. Moreover, the pool of non-target applicants at college 2 improves since their admission standard rises to meet the capacity constraint. In short, the shadow value difference of target students less that of non-target students falls. But this difference may rise or fall when college 2 favors target students more. The standards effect is negative, but the portfolio effect is ambiguous: Its favored applicant pool expands at the lower and upper ends.

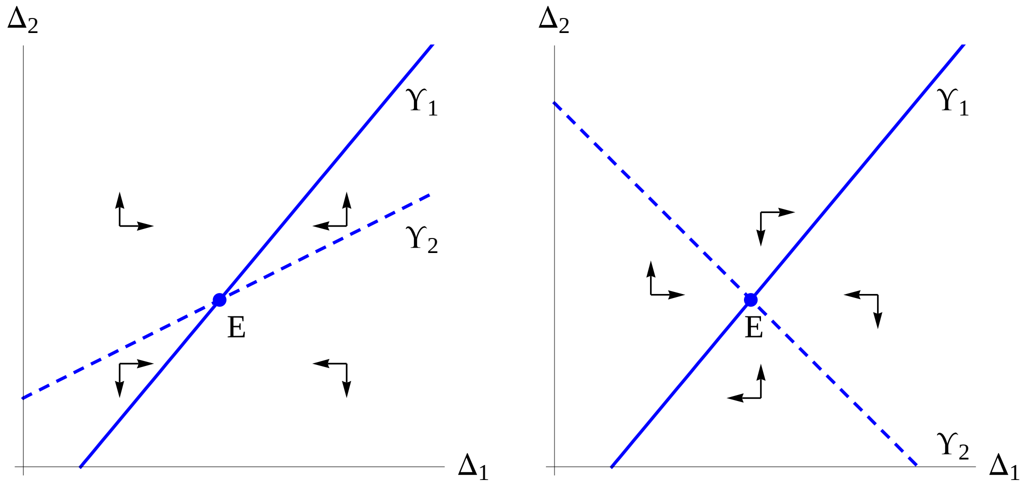

Figure 7: **Shadow Value Stable Equilibrium.** The left panel illustrates a shadow value unstable equilibrium, which happens when the best response Υ_2 slopes upward. A necessary condition for shadow value stability is that $\partial V_2/\partial\Delta_2$ be negative, thus ensuring that Υ_2 slopes downward. The right panel depicts a shadow value stable equilibrium.

We now resolve this indeterminacy and argue that the slope of the Υ_2 is negative at the stable equilibria. Assume that when the shadow value of a target student exceeds that of a non-target student, college i responds by raising the target advantage Δ_i. Call the equilibrium *shadow value stable* if this dynamic adjustment process pushes us back to the equilibrium. We argue that for any such equilibrium, the shadow value difference of targeted over non-targeted students falls when college 2 favors the targeted students more. For suppose not. Then we must have $V_2 > 0$ whenever (Δ_1, Δ_2) lies above the Υ_2 schedule (along which $V_2 = 0$). Thus, the adjustment process would lead to an even higher Δ_2, contrary to shadow value stability. Figure 7 illustrates this logic. So V_2 must fall in Δ_2 at a shadow value stable equilibrium; therefore, Υ_2 slopes down near any such equilibrium, as in the right panel of Figure 7.

We are now ready to state the following result, whose proof is depicted graphically.

Theorem 3 (Affirmative Action) *Fix $\pi_1 = \pi_2 = 0$. Assume that $\Delta_1 = \Delta_2 = 0$ is a shadow value stable equilibrium with monotone student behavior.*[19] *As the preference for a target group at college 1 rises, it favors those students and college 2 penalizes them. As the preference for target students at college 2 rises, both colleges favor them more.*

Let's highlight here an *asymmetry.* Affirmative action for target students by college 2 leads college 1 to follow suit. But affirmative action for target students at college 1

[19]Such an equilibrium easily exists when $c = 0$, and by continuity for c small enough.

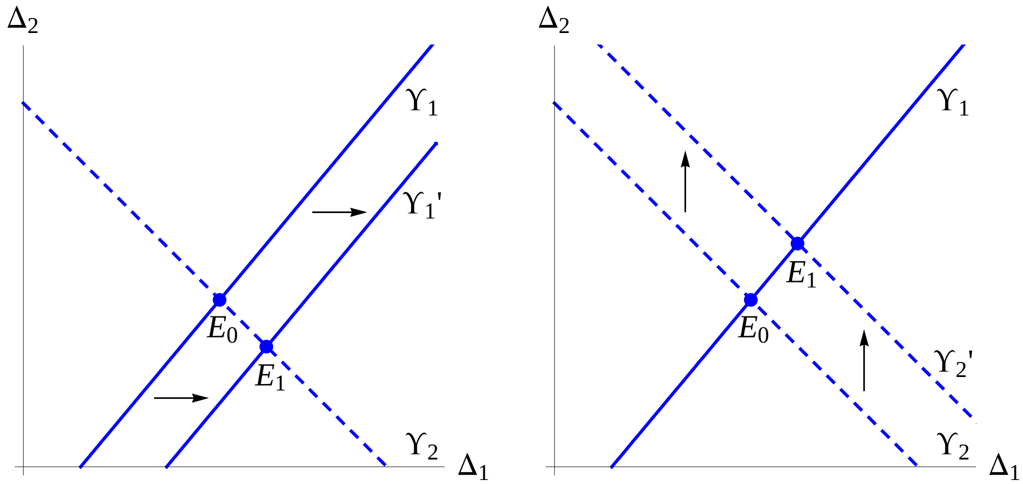

Figure 8: **Affirmative Action Comparative Statics.** The left panel depicts a right shift of the best response discount curve Υ_1 as the minority preference π_1 increases. The equilibrium shifts to E_1, with a higher discount Δ_1 and a lower Δ_2. This is justified for small minority preferences in Theorem 3. The right panel depicts how the best response Υ_2 shifts up in π_2, increasing both equilibrium discounts Δ_1 and Δ_2.

induces college 2 to penalize them. Two effects underpin this asymmetry. First, consider the acceptance curse facing college 2. A student who enrolls at college 2 either just applied there, or was strong enough to have applied to college 1 also — and was rejected. So the event that a student enrolls at college 2 is a worse signal if college 1 has favored them. College 2 must counters this with a penalty, which is further aggravated by a portfolio effect: Indeed, the best favored students that previously applied to college 2 now just apply to college 1; therefore, the pool of favored applicants at college 2 worsens.

8 Concluding Remarks

We have formulated the college admissions problem with frictions. In our equilibrium model, (a) student types are heterogeneous and their types only partially observed; (b) college applications are costly, confronting students with a nontrivial portfolio choice; and (c) two ranked colleges must set admission standards to fill their capacities.

The paper has two distinct sets of contributions, which are robust to an array of informational assumptions. First, we have characterized in a testable fashion how student admission chances co-move as their calibers improve. We have then found how their optimal portfolio choices over stretch and safety schools evolve in their calibers.

Second, we have have discovered that even in this highly monotone matching world, sorting of students and colleges fails absent stronger assumptions. For better students need not always apply more aggressively: If the worse college is either too good or too small, or the application process is noisy enough, one student may gamble on the better college while a more talented one does not. Likewise, college admissions standards needn't reflect their quality — the worse college may optimally impose higher standards if it is small enough. Large public schools might well be punished in college rankings publications that use SAT scores of enrolled students in ranking schools.

Our paper develops several results for this model, largely organized using a supply and demand metaphor. In this Bayesian setting, admissions standards act as prices. College enrollment obeys the law of demand not only because fewer students meet a tougher admission standard, but also since not as many apply when their admission prospects are not as bright. Given this "portfolio effect", students have a more elastic response to changes in the college standards than statistical analysis might suggest.

The portfolio effect induces important and realistic interdependencies missing from all frictionless models of student-college matching. For instance, when its lesser rival raises its capacity or lowers its application cost, the better one should lower its admission standard and will see its student body composition shift.

Further exploiting the supply and demand metaphor, we also enrich the model by allowing differentiated student groups. Here, we find that affirmative action by the better college begets discrimination by the lesser. This reveals another testable implication, that private colleges might wisely discriminate against in-state residents.

Amidst our many general features, tractability has forced us to assume common student preferences and restrict to two colleges. This means that, like many metaphorical models, it is not designed to be taken immediately to the data.[20] On the other hand, the two college world is the most parsimonious one with portfolio effects, stretch and safety schools, and equilibrium admission standards set by competing schools. If assortative matching fails in this setting, surely it fails more generally. All of our other surprising results are surely true in general — from the impact of lesser on better colleges, to the perverse effects of affirmative action by colleges. These features have not so far been analyzed in a unified fashion in the literature. Relaxing the two college restrictions is now the major open problem.

[20] Fu (2010) has taken a first step in the direction of structural estimation.

A Appendix: Omitted Proofs

A.1 Colleges Optimally Employ Admissions Thresholds

Let $\chi_i(\sigma)$ be the expected value of the student's caliber given that he applies to college i, his signal is σ, and he accepts. College i optimally employs a threshold rule if, and only if, $\chi_i(\sigma)$ increases in σ. For college 1 this is immediate, since $g(\sigma|x)$ enjoys the MLRP property. College 2 faces an acceptance curse, and so $\chi_2(\sigma)$ is:

$$\chi_2(\sigma) = \frac{\int_{\mathcal{C}_2} xg(\sigma|x)f(x)dx + \int_{\mathcal{B}} xG(\underline{\sigma}_1|x)g(\sigma|x)f(x)dx}{\int_{\mathcal{C}_2} g(\sigma|x)f(x)dx + \int_{\mathcal{B}} G(\underline{\sigma}_1|x)g(\sigma|x)f(x)dx} \tag{13}$$

where we denote by $\mathcal{C}_2$ the set of calibers applying to 2 only, and $\mathcal{B}$ those applying to both.[21] Write (13) as $\chi_2(\sigma) = \int_{\mathcal{B}\cup\mathcal{C}_2} xh_2(x|\sigma)dx$ using indicator function notation:

$$h_2(x|\sigma) = \frac{(I_{\mathcal{C}_2}(x) + I_{\mathcal{B}}G(\underline{\sigma}_1|x))g(\sigma|x)f(x)}{\int_{\mathcal{B}\cup\mathcal{C}_2}(I_{\mathcal{C}_2}(t) + I_{\mathcal{B}}G(\underline{\sigma}_1|t))g(\sigma|t)f(t)dt}, \tag{14}$$

Then the 'density' $h_2(x|\sigma)$ has the MLRP. Therefore, $\chi_2(\sigma)$ increases in σ. □

A.2 Simultaneous versus Sequential Timing

We claim that *the subgame perfect equilibrium (SPE) outcomes of the two-stage game with students moving first coincide with the Nash equilibria of the one-shot game.*

First, consider the outcome of a SPE of the two-stage game, where students choose application $S = S(\cdot)$ and then colleges choose standards $\underline{\sigma}_1(S)$ and $\underline{\sigma}_2(S)$. Then colleges must be best responding to each other and to S (since S is realized when they choose). Also, students can forecast how colleges would respond to S in an SPE of the two-stage game, and so their applications must be best replies to the standards $\underline{\sigma}_1(S)$ and $\underline{\sigma}_2(S)$. Thus, it is an equilibrium of the one-shot game.

Conversely, since each student has measure zero, he cannot affect the college standards by adjusting his application strategy. Hence, any equilibrium $(S, \underline{\sigma}_1(S), \underline{\sigma}_2(S))$ of the one-shot game is also the outcome an SPE of the two-stage game. □

[21] We assume that students employ pure strategies, which follows from our analysis of the student optimization in §3.1. Measurability of sets $\mathcal{B}$ and $\mathcal{C}_2$ owe to the continuity of our functions $\alpha_i(x)$ in §3.2.

A.3 Acceptance Function Shape: Proof of Theorem 1

To avoid duplication, we assume $\underline{\sigma}_1 > \underline{\sigma}_2$ throughout the proof.

($\Rightarrow$) The Acceptance Function has the Double Secant Property. First, since $G(\underline{\sigma}_1|x)$ is continuously differentiable in x, the acceptance function is continuously differentiable on $(0,1]$. Given $\alpha \equiv 1 - G(\sigma|\xi(\alpha,\sigma))$, partial derivatives have positive slopes $\xi_\alpha, \xi_\sigma > 0$. Differentiating (3),

$$
\begin{aligned}
\frac{\partial\psi}{\partial\alpha_1} &= -G_x(\underline{\sigma}_2|\xi(\alpha_1,\underline{\sigma}_1))\xi_\alpha(\alpha_1,\underline{\sigma}_1) > 0 \\
\frac{\partial\psi}{\partial\underline{\sigma}_1} &= -G_x(\underline{\sigma}_2|\xi(\alpha_1,\underline{\sigma}_1))\xi_\sigma(\alpha_1,\underline{\sigma}_1) > 0 \\
\frac{\partial\psi}{\partial\underline{\sigma}_2} &= -g(\underline{\sigma}_2|\xi(\alpha_1,\underline{\sigma}_1)) < 0
\end{aligned}
\tag{15}
$$

Properties of the cdf G imply $\psi(0,\underline{\sigma}_1,\underline{\sigma}_2) \geq 0$ and $\psi(1,\underline{\sigma}_1,\underline{\sigma}_2) = 1$. The limits of ψ as thresholds approach the supremum and infimum owe to limit properties of G.

Now, $G(\sigma|x)$ and $1-G(\sigma|x)$ are strictly log-supermodular in (σ,x) since the density $g(\sigma|x)$ obeys the strict MLRP. Since $x = \xi(\alpha_1,\underline{\sigma}_1)$ is strictly increasing, $G(s|\xi(\alpha_1,\underline{\sigma}_1))$ and $1-G(s|\xi(\alpha_1,\underline{\sigma}_1))$ are then strictly log-supermodular in (s,α_1). So the secant slopes below strictly fall in α_1, since $\underline{\sigma}_1 > \underline{\sigma}_2$:

$$
\frac{\psi(\alpha_1)}{\alpha_1} = \frac{1-G(\underline{\sigma}_2|\xi(\alpha_1))}{1-G(\underline{\sigma}_1|\xi(\alpha_1))} \quad \text{and} \quad \frac{1-\psi(\alpha_1)}{1-\alpha_1} = \frac{G(\underline{\sigma}_2|\xi(\alpha_1))}{G(\underline{\sigma}_1|\xi(\alpha_1))}
$$

($\Leftarrow$) Deriving a Signal Distribution. Conversely, fix a function h with the double secant property and a smoothly monotone onto function $\alpha_1(x)$. Also, put $\alpha_2(x) = h(\alpha_1(x))$, so that $\alpha_2(x) > \alpha_1(x)$. We must find a continuous signal density $g(\sigma|x)$ with the strict MLRP and thresholds $\sigma_1 > \sigma_2$ that rationalizes the h as the acceptance function consistent with these thresholds and signal distribution.

Step 1: A Discrete Signal Distribution. Consider a discrete distribution with realizations in $\{-1,0,1\}$: $g_1(x) = \alpha_1(x)$, $g_0(x) = \alpha_2(x) - \alpha_1(x)$ and $g_{-1}(x) = 1-\alpha_2(x)$. Indeed, for each caliber x, $g_i \geq 0$ and sum to 1. This obeys the strict MLRP because

$$
\frac{g_0(x)}{g_1(x)} = \frac{\alpha_2(x)-\alpha_1(x)}{\alpha_1(x)} = \frac{h(\alpha_1(x))}{\alpha_1(x)} - 1
$$

is strictly decreasing by the first secant property of h, and

$$
\frac{g_0(x)}{g_{-1}(x)} = \frac{\alpha_2(x)-\alpha_1(x)}{1-\alpha_2(x)} = -1 + \frac{1-\alpha_1(x)}{1-h(\alpha_1(x))}
$$

is strictly increasing in x by the second secant property of h.

Let the college thresholds be $(\underline{\sigma}_1, \underline{\sigma}_2) = (0.5, -0.5)$. Then $G(\underline{\sigma}_1|x) = g_{-1}(x) + g_0(x) = 1 - \alpha_1(x)$ and $G(\underline{\sigma}_2|x) = g_{-1}(x) = 1 - \alpha_2(x)$. Rearranging yields $\alpha_1(x) = 1 - G(\underline{\sigma}_1|x)$ and $\alpha_2(x) = 1 - G(\underline{\sigma}_2|x)$. Inverting $\alpha_1(x)$ and recalling that $\alpha_2 = h(\alpha_1)$, we obtain $\alpha_2 = h(\alpha_1) = 1 - G(\underline{\sigma}_2|\xi(\underline{\sigma}_1, \alpha_1))$, thereby showing that h is the acceptance function consistent with this signal distribution and thresholds.

Step 2: A Continuous Signal Density. To create an atomless signal distribution, we smooth this example using the triangular kernel $k(s) = \max\{1 - s, 0\}$. Fix $\beta > 0$, and define $g(\sigma|x) = \beta \sum_{i=\{-1,0,1\}} g_i(x)k(\beta(\sigma - i))$. For any "bandwidth" $1/\beta \in (0, 1/2)$, acceptance chances remain the same as with the discrete signals, since the masses at $\{-1, 0, 1\}$ is not transferred past the (respective) thresholds $\{0.5, -0.5\}$.

The strict MLRP implies that $g_i(x)$ is strictly log-supermodular in (i, x). Also, the function $k(s)$ is concave in s, and thus log-concave in s too. This implies that $k(\beta(\sigma - i))$ is log-supermodular in (i, σ). This is in §1.5 in Karlin (1968), but for a self-contained treatment, let's assume twice differentiability: Then $k_{\sigma i}k - k_\sigma k_i > 0$ iff $-k''k + k'k' > 0$, which holds iff k is log concave. Thus, $g_i(x)k(\beta(\sigma - i))$ is log-supermodular in (i, x, σ). Finally, partially summing over $i = 1, 2, 3$ yields a log-supermodular function of (x, σ), by Proposition 3.2 in Karlin and Rinott (1980) — it is the MRLP property. □

A.4 Monotone Student Strategies

Claim 1 *Student behavior is monotone in caliber if* (*a*) *college* 2 *has payoff* $u \leq 0.5$, and (*b*) *college* 2 *imposes a low enough admissions standard relative to college* 1.

By part (a), if a student applies to college 1, then any better student does too. By part (b), if a student applies to college 2, then any worse student applies there or nowhere.

The proof proceeds as follows. First, we show that $u \leq 0.5$ implies that if a caliber applies to college 1, any higher caliber applies as well. Second, we produce a sufficient condition that ensures that the admissions threshold at college 2 is sufficiently lower than that of college 1, so that if a caliber applies to college 2, then any lower caliber who applies to college sends an application to college 2, and calibers at the lower tail apply nowhere. From these two results, monotone student behavior ensues.

Proof of Part (a), Step 1. We first show that the acceptance function $\alpha_2 = \psi(\alpha_1)$ crosses $\alpha_2 = 1/u(1 - c/\alpha_1)$ (i.e., $MB_{12} \equiv \alpha_1(1 - \alpha_2 u) = c$) only once when $u \leq 0.5$. Since (i) the acceptance function starts at $\alpha_1 = 0$ and ends at $\alpha_1 = 1$, (ii) $MB_{12} = c$

starts at $\alpha_1 = c$ and ends at $\alpha_1 = c/(1-u)$, and (iii) both functions are continuous, there exists a crossing point. Clearly, they intersect when $\alpha_1(1-\psi(\alpha_1)u) = c$. Now,

$$[(1-\psi(\alpha_1)u)\alpha_1]' = 1-u\psi(\alpha_1)-\alpha_1 u\psi'(\alpha_1) > 1-u\psi(\alpha_1)-u\psi(\alpha_1) = 1-2u\psi(\alpha_1) \geq 1-2u \geq 0,$$

where the first inequality exploits $\psi(\alpha_1)/\alpha_1$ falling in α_1 (Theorem 1), i.e. $\psi'(\alpha_1) < \psi(\alpha_1)/\alpha_1$; the next two inequalities use $\psi(\alpha_1) \leq 1$ and $u \leq 0.5$. Since MB_{12} is rising in α_1 when the acceptance relation hits $\alpha_2 = (1-c/\alpha_1)/u$, the intersection is unique.

Proof of Part (a), Step 2. We now show that Step 1 implies the following single crossing property in terms of x: if caliber x applies to college 1 (i.e., if $1 \in S(x)$, then any caliber $y > x$ also applies to college 1 (i.e., $1 \in S(y)$). Suppose not; i.e., assume that either $S(y) = \varnothing$ or $S(y) = \{2\}$. If $S(y) = \varnothing$, then $S(x) = \varnothing$ as well, as $\alpha_1(x) < \alpha_1(y)$ and $\alpha_2(x) < \alpha_2(y)$, contradicting the hypothesis that $1 \in S(x)$. If $S(y) = \{2\}$, then there are two cases: $S(x) = \{1\}$ or $S(x) = \{1,2\}$. The first cannot occur, for by Theorem 1 $\alpha_2(x)/\alpha_1(x) > \alpha_2(y)/\alpha_1(y)$, and thus $\alpha_2(y)u \geq \alpha_1(y)$ implies $\alpha_2(x)u > \alpha_1(x)$, contradicting $S(x) = \{1\}$. In turn, the second case is ruled out by the monotonicity of MB_{12} derived above, as caliber y has greater incentives than x to add college 1 to its portfolio, and thus $S(y) = \{2\}$ cannot be optimal.

Proof of Part (b), Step 1. We first show that if the acceptance function passes above the point $(\bar{\alpha}_1, \bar{\alpha}_2) = \left(u(1-\sqrt{1-4c/u})/2, (1-\sqrt{1-4c/u})/2\right)$ — point P in the right panel of Figure 5 — then there is a unique crossing of the acceptance function and $\alpha_2 = c/u(1-\alpha_1)$, i.e. $MB_{21} = c$. Now, the acceptance function passes above $(\bar{\alpha}_1, \bar{\alpha}_2)$ if

$$\psi(\bar{\alpha}_1, \underline{\sigma}_1, \underline{\sigma}_2) \geq \bar{\alpha}_2. \tag{16}$$

This condition relates $\underline{\sigma}_1$ and $\underline{\sigma}_2$. Rewrite (16) using Theorem 1 as $\underline{\sigma}_2 \leq \eta(\underline{\sigma}_1) < \underline{\sigma}_1$, requiring a large enough "wedge" between the standards of the two colleges.

To show that (16) implies a unique crossing, consider the secant of $\alpha_2 = c/u(1-\alpha_1)$ (the curve $MB_{21} = c$). It has an increasing secant if and only if $\alpha_1 \geq 1/2$. To see this, differentiate $\alpha_2/\alpha_1 = c/u\alpha_1(1-\alpha_1)$ in α_1. Notice also that $MB_{12} = c$ intersects the diagonal $\alpha_2 = \alpha_1$ at the points $(\alpha_1^\ell, \alpha_2^\ell) = (1/2 - \sqrt{1-c/4u}/2, 1/2u - \sqrt{1-c/4u}/2u)$ and $(\alpha_1^h, \alpha_2^h) = (1/2 + \sqrt{1-c/4u}/2, 1/2u + \sqrt{1-c/4u}/2u) > (1/2, 1/2u)$.

Condition (16) gives $\psi(\alpha_1^\ell, \underline{\sigma}_1, \underline{\sigma}_2) > \alpha_2^\ell$. Since $\underline{\sigma}_2 < \underline{\sigma}_1$, we have $\psi(\alpha_1, \underline{\sigma}_1, \underline{\sigma}_2) \geq \alpha_2$ for all α_1. Thus, the acceptance function crosses $MB_{21} = c$ at or above (α_1^h, α_2^h). And since $\alpha_1^h > 1/2$, the secant of $MB_{21} = c$ must be increasing at any intersection with the acceptance function. Hence, there must be a single crossing point.

PROOF OF PART (b), STEP 2. We now show that this single crossing property in α implies another in x: If caliber x applies to college 2 (i.e., if $2 \in S(x)$), then any caliber $y < x$ that applies somewhere also applies to college 2 (i.e., $2 \in S(y)$ if $S(y) \neq \varnothing$). Suppose not; i.e., assume that $S(y) = \{1\}$. Then there are two cases: $S(x) = \{2\}$ or $S(x) = \{1, 2\}$. The first cannot occur, for by Theorem 1 $\alpha_2(x)/\alpha_1(x) < \alpha_2(y)/\alpha_1(y)$, and thus $\alpha_2(x)u \geq \alpha_1(x)$ implies $\alpha_2(y)u > \alpha_1(y)$, contradicting $S(x) = \{2\}$. The second case is ruled out by the monotonicity of MB_{21} given condition (16), as caliber y has greater incentives than x to apply to college 2, and thus $S(y) = \{1\}$ cannot be optimal. Finally, $S(y) = \varnothing$ if $\alpha_2(y)u < c$ by (16), which happens for low calibers below a threshold. □

A.5 The Law of Demand

Claim 2 (The Falling Demand Curve) *If either college raises its admission standard, then its enrollment falls, and thus its rival's enrollment rises.*

We only prove the result for college 1 since the argument for college 2 is similar.

PROOF STEP 1: THE APPLICANT POOL AT COLLEGE 1 SHRINKS. When $\underline{\sigma}_1$ increases, the acceptance relation shifts up by Claim 1, and thus the above type sets change as well. Fix a caliber $x \in \mathcal{C}_2$ or $x \in \Phi$, so that $1 \notin S(x)$.[22] We will show that x continues to apply either to college 2 only or nowhere, and thus the pool of applicants at college 1 shrinks. If $x \in \mathcal{C}_2$, then $\alpha_2(x)u - c \geq 0$ and $\alpha_2(x)u \geq \alpha_1(x)$, and this continues to hold after the increase in $\underline{\sigma}_1$, since $\alpha(x)$ falls while $\alpha_2(x)$ is constant. And if $x \in \Phi$, then clearly caliber x will continue to apply nowhere when $\underline{\sigma}_1$ increases.

PROOF STEP 2: THE APPLICANT POOL AT COLLEGE 2 EXPANDS. Fix a caliber $x \in \mathcal{C}_2$ or $x \in \mathcal{B}$, so that $2 \in S(x)$. It suffices to show that caliber x continues to apply to college 2 when the admission standard at college 1 increases. If $x \in \mathcal{C}_2$, then $\alpha_2(x)u - c \geq 0$ and $\alpha_2(x)u \geq \alpha_1(x)$; these inequalities continue to hold after $\underline{\sigma}_1$ rises, since $\alpha_1(x)$ falls while $\alpha_2(x)$ remains constant. And if $x \in \mathcal{B}$, then $MB_{21} = (1 - \alpha_1(x))\alpha_2(x)u$ rises in $\underline{\sigma}_1$, encouraging caliber x to apply to college 2. Thus, $x \notin \mathcal{C}_1 \cup \Phi$. Since x was arbitrary, it follows that the applicant pool at college 2, $\mathcal{B} \cup \mathcal{C}_2$, expands when $\underline{\sigma}_1$ increases. □

[22] With a slight abuse of notation, we let Φ denote the set of calibers that apply nowhere. The same symbol was previously used to denote the analogous set in α-space.

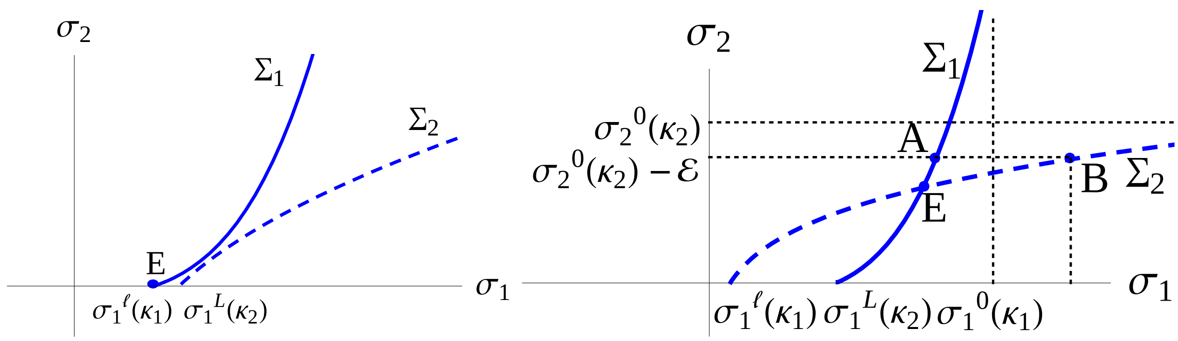

Figure 9: **Equilibrium Existence.** In the left panel, since $\kappa_1 > \bar{\kappa}_1(\kappa_2)$, the best response functions Σ_1 and Σ_2 do not intersect, and equilibrium is at E with $\underline{\sigma}_2 = 0$. The right panel depicts the proof of Claim 3 for the case $\kappa_1 < \bar{\kappa}_1(\kappa_2)$.

A.6 Existence of a Stable Equilibrium

Claim 3 (Existence) *A stable equilibrium exists. College 1 fills its capacity. Also, there exists $\bar{\kappa}_1(\kappa_2, c) < 1 - \kappa_2$ satisfying $\lim_{c \to 0} \bar{\kappa}_1(\kappa_2, c) = 1 - \kappa_2$ such that if $\kappa_1 \leq \bar{\kappa}_1(\kappa_2, c)$, then college 2 also fills its capacity in any equilibrium. If $\kappa_1 > \bar{\kappa}_1(\kappa_2, c)$, then college 2 has excess capacity in some equilibrium.*

Proof: For some insight, we choose the capacity $\bar{\kappa}_1$ given κ_2 so that when college 2 has no standards, both colleges exactly fill their capacity. This borderline capacity is less than $1 - \kappa_2$ since a positive mass of students — perversely, those with the highest calibers — applies just to college 1, and some are rejected. (This happens whenever one's admission chance at college 1 is at least $1 - c/u$, by (1).)

For definiteness, we now denote the infimum signal by $-\infty$, and the supremum signal by ∞. Fix any $\kappa_2 \in (0, 1)$, and let $\underline{\sigma}^l_1(\kappa_2)$ be the unique solution to $\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, -\infty)$, i.e., when college 2 accepts everybody. (Existence and uniqueness of $\underline{\sigma}^l_1(\kappa_2)$ follows from $\mathcal{E}_2(-\infty, -\infty) = 0$, $\mathcal{E}_2(\infty, -\infty) = 1$, and $\mathcal{E}_2(\underline{\sigma}_1, -\infty)$ increasing and continuous in $\underline{\sigma}_1$.)

Define $\bar{\kappa}_1(\kappa_2) = \mathcal{E}_1(\underline{\sigma}^l_1(\kappa_2), -\infty)$. Let $\kappa_1 \geq \bar{\kappa}_1(\kappa_2)$. We claim that there exists an equilibrium in which college 2 accepts everybody, and college 1 sets a threshold $\underline{\sigma}^\ell_1(\kappa_1)$, the unique solution to $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, -\infty)$, which satisfies $\underline{\sigma}^\ell_1(\kappa_1) \leq \underline{\sigma}^l_1(\kappa_2)$. For since college 2 rejects no one, $\underline{\sigma}^\ell_1(\kappa_1)$ fills college 1's capacity exactly. The enrollment at college 2 is then $\mathcal{E}_2(\underline{\sigma}^\ell_1(\kappa_1), -\infty) \leq \kappa_2$ (as $\underline{\sigma}^\ell_1(\kappa_1) \leq \underline{\sigma}^l_1(\kappa_2)$ and $\mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2)$ is increasing in $\underline{\sigma}_1$), so by accepting everybody college 2 fills as much capacity as it can. This equilibrium is trivially stable, as Σ_2 is 'flat' at the crossing point (see Figure 9, left panel). Moreover, if $\kappa_1 > \bar{\kappa}_1(\kappa_2)$, then college 2 has excess capacity in this equilibrium.

Assume now $\kappa_1 < \bar{\kappa}_1(\kappa_2)$. We will show that the continuous functions Σ_1 and Σ_2 must cross at least once (i.e., an equilibrium exists), and that the slope condition is met (i.e., it is stable). First, in this case $\underline{\sigma}^l_1(\kappa_2) < \underline{\sigma}^\ell_1(\kappa_1)$ or, equivalently, $\Sigma_2^{-1}(-\infty, \kappa_2) < \Sigma_1(-\infty, \kappa_1)$. Second, as the standard of college 2 goes to infinity, college 1's threshold converges to $\underline{\sigma}^u_1(\kappa_1) < \infty$, the unique solution to $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \infty)$. This is the largest threshold that college 1 can set given κ_1. Similarly, as the standard of college 1 goes to infinity, college 2's threshold converges to $\underline{\sigma}^u_2(\kappa_2) < \infty$, the unique solution to $\kappa_2 = \mathcal{E}_2(\infty, \underline{\sigma}_2)$, i.e. the largest threshold that college 2 can set given κ_2. Third, for $\epsilon > 0$ small enough, the unique solution to $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}^u_2(\kappa_2) - \epsilon)$ lies below the unique solution to $\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}^u_2(\kappa_2) - \epsilon)$. Equivalently, $\Sigma_2^{-1}(\underline{\sigma}^u_2(\kappa_2) - \epsilon, \kappa_2) > \Sigma_1(\underline{\sigma}^u_2(\kappa_2) - \epsilon, \kappa_1)$.

Since $\Sigma_2^{-1}(-\infty, \kappa_2) < \Sigma_1(-\infty, \kappa_1)$ and $\Sigma_2^{-1}(\underline{\sigma}^u_2(\kappa_2) - \epsilon, \kappa_2) > \Sigma_1(\underline{\sigma}^u_2(\kappa_2) - \epsilon, \kappa_1)$ (graphically, point A is to the left of point B in Figure 9), and Σ_1 and Σ_2 are continuous, by the Intermediate Value Theorem, they must cross at least once with the slope condition being satisfied (see Figure 9, right panel). Thus, a stable equilibrium exists when $\kappa_1 < \bar{\kappa}_1(\kappa_2)$. Moreover, in any equilibrium there is no excess capacity at either college, since $\Sigma_2^{-1}(-\infty, \kappa_2) < \Sigma_1(-\infty, \kappa_1)$.

Hence, a stable equilibrium exists for any $\kappa_2 \in (0, 1)$. Capacities are exactly filled when $\kappa_1 \leq \bar{\kappa}_1(\kappa_2)$, while there can be excess capacity at college 2 whenever $\kappa_1 > \bar{\kappa}_1(\kappa_2)$.

Since $\kappa_2 = \mathcal{E}_1(\underline{\sigma}^l_1(\kappa_2), -\infty)$, $\bar{\kappa}_1(\kappa_2)$ equals $1 - \kappa_2$ plus the mass of students who only applied to, and were rejected by, college 1. This mass vanishes as c vanishes, for then everybody applies to both colleges. So $\bar{\kappa}_1(\kappa_2)$ converges to $1 - \kappa_2$ as c goes to zero. $\square$

A.7 Stochastic Dominance of Types in Sorting Equilibria

Claim 4 (Sorting and the Caliber Distribution) *In any sorting equilibrium, the caliber distribution at college 1 first-order stochastically dominates that at college 2.*

Proof: A monotone student strategy is represented by the partition of the set of types:

$$\Phi = [0, \xi_2), \mathcal{C}_2 = [\xi_2, \xi_B), \mathcal{B} = [\xi_B, \xi_1), \mathcal{C}_1 = [\xi_1, \infty) \tag{17}$$

where $\xi_2 < \xi_B < \xi_1$ are defined by the intersection of the acceptance function with c/u, $\alpha_2 = (1 - c/\alpha_1)/u$ (i.e., $MB_{12} = c$), and $\alpha_2 = c/[u(1 - \alpha_1)]$ (i.e., $MB_{21} = c$), respectively.

Fix $\underline{\sigma}_1$ and $\underline{\sigma}_2$. Let $f_1(x)$ and $f_2(x)$ be the densities of calibers accepted at colleges 1 and 2, respectively. Formally,

$$f_1(x) = \frac{\alpha_1(x)f(x)}{\int_{\xi_B}^{\infty}\alpha_1(t)f(t)dt} I_{[\xi_B,\infty)}(x) \tag{18}$$

$$f_2(x) = \frac{I_{[\xi_2,\xi_B]}(x)\alpha_2(x)f(x) + (1 - I_{[\xi_2,\xi_B]}(x))\alpha_2(x)(1-\alpha_1(x))f(x)}{\int_{\xi_2}^{\xi_B}\alpha_2(s)f(s)ds + \int_{\xi_B}^{\xi_1}\alpha_2(s)(1-\alpha_1(s))f(s)ds} I_{[\xi_2,\xi_1]}(x), \tag{19}$$

where I_A is the indicator function of the set A.

We shall show that, if $x_L, x_H \in [0,\infty)$, with $x_H > x_L$, then $f_1(x_H)f_2(x_L) \geq f_2(x_H)f_1(x_L)$; i.e., $f_i(x)$ is log-supermodular in $(-i, x)$, or it satisfies MLRP. The result follows as MLRP implies that the cdfs are ordered by first-order stochastic dominance.

Using (18) and (19), $f_1(x_H)f_2(x_L) \geq f_2(x_H)f_1(x_L)$ is equivalent to

$$\begin{aligned} &\alpha_{1H} I_{[\xi_B,\infty)}(x_H)\left(I_{[\xi_2,\xi_B]}(x_L)\alpha_{2L} + (1 - I_{[\xi_2,\xi_B]}(x_L))\alpha_{2L}(1-\alpha_{1L})\right) I_{[\xi_2,\xi_1]}(x_L) \geq \\ &\alpha_{1L} I_{[\xi_B,\infty)}(x_L)\left(I_{[\xi_2,\xi_B]}(x_H)\alpha_{2H} + (1 - I_{[\xi_2,\xi_B]}(x_H))\alpha_{2H}(1-\alpha_{1H})\right) I_{[\xi_2,\xi_1]}(x_H), \end{aligned} \tag{20}$$

where $\alpha_{ij} = \alpha_i(x_j)$, $i = 1, 2$, $j = L, H$. It is easy to show that the only nontrivial case is when $x_L, x_H \in [\xi_B, \xi_1]$ (in all the other cases, either both sides are zero, or only the right side is). If $x_L, x_H \in [\xi_B, \xi_1]$, then (20) becomes $\alpha_{1H}\alpha_{2L}(1-\alpha_{1L}) \geq \alpha_{1L}\alpha_{2H}(1-\alpha_{1H})$, or

$$\begin{aligned} &(1 - G(\underline{\sigma}_1 \mid x_H))(1 - G(\underline{\sigma}_2 \mid x_L))G(\underline{\sigma}_1 \mid x_L) \geq \\ &(1 - G(\underline{\sigma}_1 \mid x_L))(1 - G(\underline{\sigma}_2 \mid x_H))G(\underline{\sigma}_1 \mid x_H). \end{aligned} \tag{21}$$

Since $g(\sigma \mid x)$ satisfies MLRP, it follows that $G(\sigma \mid x)$ is decreasing in x, and hence $G(\underline{\sigma}_1 \mid x_L) \geq G(\underline{\sigma}_1 \mid x_H)$. Next, $1 - G(\sigma \mid x)$ is log-supermodular in (x, σ), and hence

$$(1 - G(\underline{\sigma}_1 \mid x_H))(1 - G(\underline{\sigma}_2 \mid x_L)) \geq (1 - G(\underline{\sigma}_1 \mid x_L))(1 - G(\underline{\sigma}_2 \mid x_H)),$$

as $\underline{\sigma}_1 > \underline{\sigma}_2$ in a sorting equilibrium. Thus, (21) is satisfied, thereby proving that $f_i(x)$ is log-supermodular in $(-i, x)$, and so F_1 first-order stochastically dominates F_2. □

A.8 Comparative Statics: Changing Application Costs

Claim 5 *If the application costs at college* 2 *rises, then both admission standards fall. In a sorting equilibrium, the distribution of calibers at college 1 stochastically improves.*

Proof: Assume a sorting equilibrium. We modify (17) for different costs: ξ_1 is defined by $MB_{21} = c_2$, ξ_B by $MB_{12} = c_1$, and ξ_2 by $\alpha_2 u = c_2$. If c_2 rises, then ξ_1 drops, ξ_2 rises, and ξ_B is unchanged; thus, the applicant pool at college 2 shrinks, and at college 1 is unchanged. So the Σ_2 curve shifts down, while Σ_1 remains unchanged. The functions now cross at a lower threshold pair, and so both standards $\underline{\sigma}_1, \underline{\sigma}_2$ both fall.

Next consider an increase in c_1. This raises ξ_B, which shrinks the applicant pool at college 1, and increases the enrollment at college 2, at a fixed admission standard. This shifts Σ_1 left and Σ_2 up. While the effect on the standard $\underline{\sigma}_2$ is ambiguous, we now deduce that $\underline{\sigma}_1$ falls. Differentiating (4) and (5) with respect to c_1, and using Cramer's Rule:

$$\frac{\partial \underline{\sigma}_1}{\partial c_1} = \frac{(\partial \mathcal{E}_2/\partial c_1)(\partial \mathcal{E}_1/\partial \underline{\sigma}_2) - (\partial \mathcal{E}_1/\partial c_1)(\partial \mathcal{E}_2/\partial \underline{\sigma}_2)}{(d\mathcal{E}_1/d\underline{\sigma}_1)(d\mathcal{E}_2/d\underline{\sigma}_2) - (d\mathcal{E}_2/d\underline{\sigma}_1)(d\mathcal{E}_1/d\underline{\sigma}_2)} \tag{22}$$

Since the equilibrium is stable, the slope of Σ_1 is steeper that of Σ_2, and thus the denominator is positive. Let $P_i(\xi|y)$ be the portfolio density shift to college i at type ξ given an increment to standard or cost y, and let $S_2(A)$ be the own-standards effect at college 2 in set A. Then parse the enrollment derivatives into the portfolio and standards effects: $d\mathcal{E}_1/dc_1 = P_1(\xi_B|c_1) < 0$, $d\mathcal{E}_2/d\underline{\sigma}_2 = \Sigma_{i=2,B,1} P_2(\xi_i|\underline{\sigma}_2) - S_2(C_2) - S_2(B) < 0$, $d\mathcal{E}_2/dc_1 = P_2(\xi_B|c_1) > 0$, and $d\mathcal{E}_1/d\underline{\sigma}_2 = P_1(\xi_B|\underline{\sigma}_2) > 0$. If c_1 slightly rises, then ξ_B rises by some $\delta > 0$. Thus, college 1 loses mass $f(\xi_B)\alpha_1\delta$ of students, and college 2 gains mass $f(\xi_B)\alpha_1\alpha_2\delta$ of students who would have gone to college 1. Likewise, if $\underline{\sigma}_2$ slightly rises, then ξ_B falls by some δ', and college 1 gains mass $f(\xi_B)\alpha_1\delta'$ and college 2 loses mass $f(\xi_B)\alpha_1\alpha_2\delta'$. Thus, $P_1(\xi_B|\underline{\sigma}_2)P_2(\xi_B|c_1) - P_1(\xi_B|c_1)P_2(\xi_B|\underline{\sigma}_2)$ equals

$$[f(\xi_B)\alpha_1\delta'][f(\xi_B)\alpha_1\alpha_2\delta] - [f(\xi_B)\alpha_1\delta][f(\xi_B)\alpha_1\alpha_2\delta'] = 0$$

Hence, the numerator in (22) reduces to

$$-P_1(\xi_B|c_1)[P_2(\xi_2|\underline{\sigma}_2) + P(\xi_1|\underline{\sigma}_2) - S_2(\mathcal{C}_2) - S_2(\mathcal{B})] < 0$$

In a sorting equilibrium, the applicant pool at college 1 consists of calibers $x \in [\xi_B, \infty)$. From the last part, any cost increase depresses $\underline{\sigma}_1$ in equilibrium. It follows that ξ_B rises in equilibrium — since college 1 has the same capacity as before, if it is to have lower standards, it must also have fewer applicants. Let $(\xi_B^0, \underline{\sigma}_1^0)$ be the old equilibrium pair and $(\xi_B^1, \underline{\sigma}_1^1)$ the new one, with $\xi_B^0 < \xi_B^1$ and $\underline{\sigma}_1^0 > \underline{\sigma}_1^1$. Then the distribution function of enrolled students at college 1 under equilibrium $i = 0, 1$ is:

$$F_1^i(x) = \frac{\int_{\xi_B^i}^x (1 - G(\underline{\sigma}_1^i|t))\, f(t)dt}{\int_{\xi_B^i}^\infty (1 - G(\underline{\sigma}_1^i|t))\, f(t)dt}$$

We must show $F_1^1(x) \leq F_1^0(x)$ for all $x \in [\xi_B^1, \infty)$. For any x, the denominators on both sides equal k_1, so cancel them. Now notice that $0 = F_1^1(\xi_B^1) < F_1^0(\xi_B^1)$ and $\lim_{x\to\infty} F_1^1(x) = \lim_{x\to\infty} F_1^0(x) = 1$. Since both functions are continuous in x, if $\partial F_1^1/\partial x > \partial F_1^0/\partial x$ for all $x \in [\xi_B^1, \infty)$, then $F_1^1(x) < F_1^0(x)$. But this requires $(1 - G(\underline{\sigma}_1^1|x))\, f(x) > (1 - G(\underline{\sigma}_1^0|x))\, f(x)$, which follows from $\underline{\sigma}_1^1 < \underline{\sigma}_1^0$. $\square$

A.9 Sorting and Non-Sorting Equilibria: Proof of Theorem 2

Part (a): College 2 is Too Good. We construct acceptance chances $(\alpha_1(x), h(\alpha_1(x)))$ such that student behavior is non-monotone, college enrollment equals capacity, and $\alpha_1(x)$ and h obey the requirements of Theorem 1. Then Theorem 1 yields existence of a signal distribution with non-monotone equilibrium student behavior. Fix $\kappa_1 + \kappa_2 < 1$.

Step 1: Towards an Acceptance Function. When $u > 0.5$, the secant from the origin to $MB_{12} = c$ falls as α_1 tends to $c/(1-u)$ — as in the left panel of Figure 5. So for some $\underline{z} < c/(1-u)$s, a line from the origin to $(\underline{z}, 1)$ slices the MB_{12} curve twice. This would imply non-monotone student behavior if that line belonged to the acceptance function, such as: $h\colon[0,1]\to[0,1]$ by $h(\alpha) = \alpha/\underline{z}$ and on $[0, \underline{z})$, and $h(\alpha_1) = 1$ for $\alpha_1 \geq \underline{z}$.

Step 2: A piecewise-linear acceptance chance α_1. Choose $\underline{\xi}$ and $\bar{\xi}$ that uniquely solve $\kappa_1 = \int_{\bar{\xi}}^{\infty} f(x)dx$ and $\kappa_2 = \int_{\underline{\xi}}^{\bar{\xi}} f(x)dx$. Set $\alpha_1(x) = 0$ for $x < \underline{\xi}$. This function then jumps up to the rising line segment $\alpha_1(x) = \omega(x)\underline{z} + (1 - \omega(x))c/(1-u)$ for $x \in [\underline{\xi}, \bar{\xi})$, where $\omega(x) \equiv (\bar{\xi} - x)/(\bar{\xi} - \underline{\xi})$. Lastly, α_1 jumps up $\alpha_1(x) = 1$ for $x > \bar{\xi}$.

Step 3: Student Behavior. Observe that $h(0) = 0$ and $h(1) = 1$, and that h is weakly increasing, with both $h(\alpha)/\alpha$ and $[1 - h(\alpha)]/[1 - \alpha]$ weakly decreasing. In this sense, h obeys the double secant property. This suggests that we set $\alpha_2(x) \equiv h(\alpha_1(x))$.

In this case, students $x \in [0, \underline{\xi})$ are accepted with zero chance at either college, and so apply nowhere. Next, because $h(\underline{z}) = 1$, any calibers $x \in [\underline{\xi}, \bar{\xi})$ are accepted with chance one at college 2, and with chance between $\underline{z}$ and $c/(1-u)$ at college 1. Further, any student $\bar{\xi}$ strictly prefers just to apply to college 2 (as in Figure 5). To see this, observe that $MB_{12} = (c/(1-u))(1 - \alpha_2 u) > (c/(1-u))(1-u) = c$ when $\alpha_2 = c/(1-u)$ and $\alpha_1 = 1$. Lastly, calibers $x > \bar{\xi}$ are always accepted at college 1 and only apply there.

Step 4: Smoothing the Construction. By smoothly bending $h(\cdot)$ inside $(0,1)$, an arbitrarily close function h^* also obeys the double secant property. Next, we create a continuous and smooth acceptance chance $\tilde{\alpha}$. Any four small enough numbers, $\underline{\varepsilon}, \underline{\epsilon}, \bar{\varepsilon}, \bar{\epsilon} > 0$ yield a unique Bezier approximation $\tilde{\alpha}$ tangent to α at $\underline{\xi} - \underline{\varepsilon}, \underline{\xi} + \underline{\epsilon}, \bar{\xi} - \bar{\varepsilon}, \bar{\xi} + \bar{\epsilon}$. Then

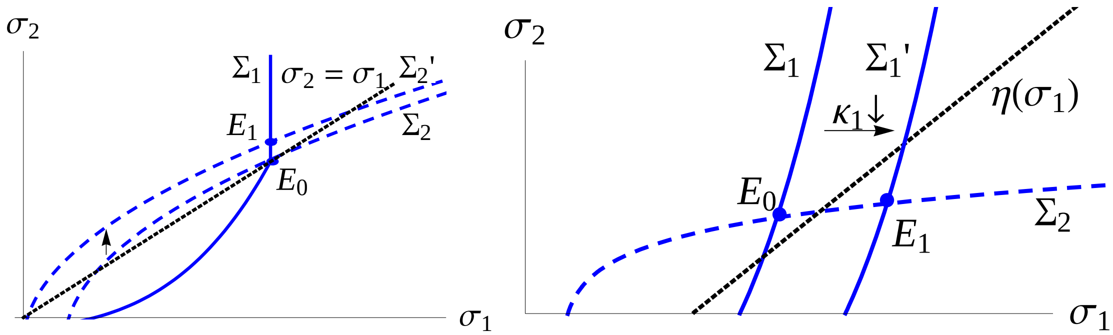

Figure 10: **Existence of Sorting and Non-Sorting Equilibria** In the left panel, we depict the non-sorting equilibrium constructed in the proof of Theorem 2. As κ_2 decreases, Σ_2 shifts up, leading to a non-sorting equilibrium at $\mathcal{E}_1$. The right panel illustrates the proof of Theorem 2. As κ_1 falls, the equilibrium standards at $\mathcal{E}_1$ are guaranteed to satisfy $\underline{\sigma}_2 < \eta(\underline{\sigma}_1)$, thereby obtaining a sorting equilibrium.

$\tilde{\alpha}_1$ falls in $\bar{\epsilon}$, and rises in $\bar{\varepsilon}$, and so does the enrollment at college 1. Also, $\tilde{\alpha}_2 = h^*(\tilde{\alpha}_1)$ falls in $\bar{\epsilon}$, and rises in $\bar{\varepsilon}$, and also falls in $\underline{\epsilon}$, and rises in $\underline{\varepsilon}$. Enrollment at college 2 shares this monotonicity, but at college 1, it is unaffected by $\underline{\epsilon}$ and $\underline{\varepsilon}$.

Fix $\bar{\epsilon} > 0$ small. Choose $\bar{\varepsilon} > 0$ so that college 1 still fills its capacity. WLOG, enrollment at college 2 has fallen. Choose $\underline{\varepsilon} > 0$ large enough so that college 2 exceeds its capacity. Then the former enrollment at college 2 is restored for some $\underline{\epsilon} > 0$.

Theorem 1 now yields a signal density $g(\sigma|x)$ and thresholds $\underline{\sigma}_1 > \underline{\sigma}_2$ such that h^* is the acceptance function. We have thus constructed a non-sorting equilibrium. $\square$

Part (*b*): College 2 is Too Small. The proof is constructive. Consider $(\alpha_1, \alpha_2) = (c, c/u)$ on the line $\alpha_2 = \alpha_1/u$. The acceptance function evaluated at $\alpha_1 = c$ lies below c/u when

$$\psi(c, \underline{\sigma}_1, \underline{\sigma}_2) < c/u. \tag{23}$$

We will restrict attention to pairs $(\underline{\sigma}_1, \underline{\sigma}_2)$ such that (23) holds. In this case, any student who applies to college starts by adding college 1 to his portfolio, and this happens as soon as $\alpha_1(x) \geq c$, or when $x \geq \xi(c, \underline{\sigma}_1)$. Then enrollment at college 1 is given by

$$\mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{\xi(c, \underline{\sigma}_1)}^{\infty} (1 - G(\underline{\sigma}_1|x)) f(x) dx,$$

which is independent of $\underline{\sigma}_2$. Thus, for any capacity $\kappa_1 \in (0, 1)$, a unique threshold $\underline{\sigma}_1(\kappa_1)$ solves $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2)$. (The Σ_1^{-1} function is "vertical" when (23) holds, since the applicant pool at college 1 does not depend on college 2's admissions threshold.)

The analysis above allows us to restrict attention to finding equilibria within the set of thresholds $(\underline{\sigma}_1, \underline{\sigma}_2)$ such that $\underline{\sigma}_1 = \underline{\sigma}_1(\kappa_1)$ and $\underline{\sigma}_2$ satisfies $\psi(c, \underline{\sigma}_1(\kappa_1), \underline{\sigma}_2) < c/u$.

Enrollment at college 2 is given by

$$\mathcal{E}_2(\underline{\sigma}_1(\kappa_1), \underline{\sigma}_2) = \int_B G(\underline{\sigma}_1(\kappa_1)|x)(1 - G(\underline{\sigma}_2|x))f(x)dx,$$

which is continuous, decreasing in $\underline{\sigma}_2$, and increasing in $\underline{\sigma}_1$ (see Claim 2). Thus, $\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1(\kappa_1), \underline{\sigma}_2)$ yields $\underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1(\kappa_1), \kappa_2)$, which is strictly decreasing in κ_2.

Given κ_1, let $\bar{\kappa}_2(\kappa_1) = E_2(\underline{\sigma}_1(\kappa_1), \underline{\sigma}_1(\kappa_1))$ be the level of college 2 capacity so that equilibrium ensues if both colleges set the same threshold.[23] Since Σ_2 strictly falls in κ_2, for any $\kappa_2 < \bar{\kappa}_2(\kappa_1)$, an equilibrium exists with $\underline{\sigma}_2 > \underline{\sigma}_1(\kappa_1)$. Then (a) for any $\kappa_1 \in (0, 1)$ and $\kappa_2 \in (0, \bar{\kappa}_2(\kappa_1)]$, there is a unique equilibrium with $\underline{\sigma}_1 = \underline{\sigma}_1(\kappa_1)$ and $\underline{\sigma}_2 \geq \underline{\sigma}_1(\kappa_1)$, having (b) non-monotone college and student behavior (Figure 10, left).[24] □

Part (c): Conditions for Equilibrium Sorting. We prove that there exists $\underline{\kappa}_1(\kappa_2) > 0$ such that if $\kappa_1 \leq \underline{\kappa}_1(\kappa_2)$ and $u \leq 0.5$, then there are only sorting equilibria and neither college has excess capacity.

Fix $\kappa_2 \in (0, 1)$. We first show that the stable equilibrium with no excess capacity derived in Claim 3 is also sorting when the capacity of college 1 is small enough. More precisely, there is a threshold $\underline{\kappa}_1(\kappa_2)$, smaller than the bound $\bar{\kappa}_1(\kappa_2)$ defined in the proof of Claim 3, such that for all $\kappa_1 \in (0, \underline{\kappa}_1(\kappa_2))$, there is a pair of admissions thresholds $(\underline{\sigma}_1, \underline{\sigma}_2)$ that satisfies $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2)$, $\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2)$, and $\underline{\sigma}_2 < \eta(\underline{\sigma}_1)$ (i.e., a sorting equilibrium), and $\partial\Sigma_1/\partial\underline{\sigma}_2 \partial\Sigma_2/\partial\underline{\sigma}_1 < 1$ (i.e., the equilibrium is stable).

The proof uses three easily-verified properties of the function η: (a) η is strictly increasing; (b) $\underline{\sigma}_2 = \eta(\underline{\sigma}_1) \to \infty$ as $\underline{\sigma}_1 \to \infty$; (c) $\underline{\sigma}_1 = \eta^{-1}(\underline{\sigma}_2) \to -\infty$ as $\underline{\sigma}_2 \to -\infty$.

For any $\kappa_1 \in (0, \bar{\kappa}_1(\kappa_2))$, we know from Claim 3 that there exists a pair $(\underline{\sigma}_1, \underline{\sigma}_2)$ that satisfies $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2)$, with $(\partial\Sigma_1/\partial\underline{\sigma}_2)(\partial\Sigma_2/\partial\underline{\sigma}_1) < 1$.

Claim 6 *The pair $(\underline{\sigma}_1, \underline{\sigma}_2)$ is a sorting equilibrium when κ_1 is sufficiently small.*

Proof: Let $M(\kappa_2) = \{(\underline{\sigma}_1, \underline{\sigma}_2)|\kappa_2 = \mathcal{E}_2(\underline{\sigma}_1, \underline{\sigma}_2) \text{ and } \underline{\sigma}_2 = \eta(\underline{\sigma}_1)\}$. Graphically, this is the set of all pairs at which $\underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1, \kappa_2)$ crosses $\underline{\sigma}_2 = \eta(\underline{\sigma}_1)$.

If $M(\kappa_2) = \varnothing$ we are done, for then $\underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1, \kappa_2) < \eta(\underline{\sigma}_1)$ for all $\underline{\sigma}_1$, including those at which $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2)$. To see this, note that (i) $\underline{\sigma}_1 =$

[23]It is not difficult to show that $\psi(c, \underline{\sigma}_1, \underline{\sigma}_2) < c/u$ is satisfied if $\underline{\sigma}_2 \geq \underline{\sigma}_1(\kappa_1)$.

[24]We are not ruling out the existence of another equilibrium that does not satisfy (23).

$\eta^{-1}(\underline{\sigma}_2) \to -\infty$ as $\underline{\sigma}_2 \to -\infty$, while we proved in Claim 3 that $\underline{\sigma}_1 = \Sigma_2^{-1}(\underline{\sigma}_2, \kappa_2)$ converges to $\underline{\sigma}^l_1(\kappa_2) > -\infty$. Also, (ii) $\underline{\sigma}_2 = \eta(\underline{\sigma}_1) \to \infty$ as $\underline{\sigma}_1 \to \infty$, while we proved in Claim 3 that $\underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1, \kappa_2)$ converges to $\underline{\sigma}^u_2(\kappa_2) < \infty$. Properties (i) and (ii) reveal that if Σ_2 and η do not intersect, then Σ_2 is everywhere below η.

If $M(\kappa_2) \neq \varnothing$, let $(\underline{\sigma}^s_1(\kappa_2), \underline{\sigma}^s_2(\kappa_2)) = \sup M(\kappa_2)$, which is finite by property (b) of $\eta(\underline{\sigma}_1)$ and since $\underline{\sigma}_2 = \Sigma_2(\underline{\sigma}_1, \kappa_2)$ converges to $\underline{\sigma}^u_2(\kappa_2) < \infty$ as $\underline{\sigma}_1$ goes to infinity (see the proof of Claim 3). Now, as κ_1 goes to zero, $\underline{\sigma}_1 = \Sigma_1(\underline{\sigma}_2, \kappa_1)$ goes to infinity for any value of $\underline{\sigma}_2$, for college 1 becomes increasingly more selective to fill its dwindling capacity. Since $\underline{\sigma}_2$ is bounded above by $\underline{\sigma}^u_2(\kappa_2)$, there exists a threshold $\underline{\kappa}_1(\kappa_2) \leq \bar{\kappa}_1(\kappa_2)$ such that, for all $\kappa_1 \in (0, \underline{\kappa}_1(\kappa_2))$, the aforementioned pair $(\underline{\sigma}_1, \underline{\sigma}_2)$ that satisfies $\kappa_1 = \mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2)$ is strictly bigger than $(\underline{\sigma}^s_1(\kappa_2), \underline{\sigma}^s_2(\kappa_2))$, thereby showing that it also satisfies $\underline{\sigma}_2 < \eta(\underline{\sigma}_1)$. Hence, a sorting stable equilibrium exists for any κ_2 and $\kappa_1 \in (0, \underline{\kappa}_1(\kappa_2))$, with both colleges filling their capacities (see Figure 10, right panel).

To finish the proof, notice that, if there are multiple equilibria, both colleges fill their capacity in all of them (graphically, the conditions on capacities ensure that Σ_2 starts above Σ_1 for low values of $\underline{\sigma}_1$ and eventually ends below it). Moreover, adjusting the bound $\underline{\kappa}_1(\kappa_2)$ downward if needed, all equilibria are sorting (graphically, for κ_1 sufficiently small, the set of pairs at which Σ_1 and Σ_2 intersect are all below η). □

A.10 Affiliated Evaluations: Proof that A-3 Implies B-3

Let t be a student's true caliber, unknown to him and colleges. It has density $p(t)$ on $[0,1]$. After seeing the signal realization $X = x$, drawn with type-dependent density $f(x|t)$, the student updates his beliefs to $p(t|x) = f(x|t)p(t)/f(x)$. If a student of caliber t applies to a college, the college observes a signal σ drawn with density $\gamma(\sigma|t)$ and cdf $\Gamma(\sigma|t)$ on $[0,1]$. If a student applies to both colleges, then they observe conditionally iid signals. We assume that $f(x|t)$ and $\gamma(\sigma|t)$ obey the strict MLRP. We now reinterpret model (A-3) as a special case of (B-3).

Define the conditional joint density of signals $g(\sigma_1, \sigma_2|x) = \int_0^1 \gamma(\sigma_1|t)\gamma(\sigma_2|t)p(t|x)dt$. Notice that g integrates to 1, and so is a valid density. Also, as an integral of products of log-supermodular functions, it inherits this property, by Karlin and Rinott (1980). In other words, the signals are affiliated. Next, define the density $f(x) = \int_0^1 f(x|t)p(t)dt$. We now reinterpret the signal x in (A-3) as the student true caliber. For (B-3), it suffices to see that student and college optimizations have the same solutions.

Student Behavior. It suffices to express the chances of two acceptance events for the general model of (A-3) without reference to the type t, and thus as in the affiliated model (B-3). First, the unconditional acceptance chance at college $i = 1, 2$ is

$$\alpha_i(x) = \int_0^1 (1 - \Gamma(\underline{\sigma}_i|t))\, p(t|x)dt = \int_0^1 \int_{\underline{\sigma}_i}^1 \gamma(\sigma_i|t)p(t|x)d\sigma_i dt = \int_{\underline{\sigma}_i}^1 \int_0^1 g(\sigma_i, \sigma_j|x)d\sigma_j d\sigma_i$$

Next, the probability of being rejected at 1 and accepted at 2 is

$$\begin{aligned}
\int_0^1 \Gamma(\underline{\sigma}_1|t)\,(1 - \Gamma(\underline{\sigma}_2|t))\, p(t|x)dt &= \int_0^1 \int_0^{\underline{\sigma}_1} \int_{\underline{\sigma}_2}^1 \gamma(\sigma_2|t)\gamma(\sigma_1|t)p(t|x)\sigma_1 d\sigma_2 dt \\
&= \int_0^{\underline{\sigma}_1} \int_{\underline{\sigma}_2}^1 g(\sigma_1, \sigma_2|x)d\sigma_1 d\sigma_2.
\end{aligned}$$

College Behavior. It likewise suffices to express the enrollment functions without reference to the student type t. For instance, for college 1,

$$\begin{aligned}
\mathcal{E}_1(\underline{\sigma}_1, \underline{\sigma}_2) &= \int_0^1 \left(\int_{\mathcal{C}_1 \cup \mathcal{B}} f(x|t)dx\right)\left(\int_{\underline{\sigma}_1}^1 \gamma(\sigma_1|t)d\sigma_1\right) p(t)dt \\
&= \int_{\mathcal{C}_1 \cup \mathcal{B}} \int_{\underline{\sigma}_1}^1 \left(\int_0^1 \gamma(\sigma_1|t)f(x|t)p(t)dt\right) d\sigma_1 dx \\
&= \int_{\mathcal{C}_1 \cup \mathcal{B}} \left(\int_{\underline{\sigma}_1}^1 \int_0^1 g(\sigma_1, \sigma_2|x)d\sigma_2 d\sigma_1\right) f(x)dx.
\end{aligned}$$

The analysis of college 2 is analogous and thus omitted. □

A.11 Affirmative Action: Proof of Theorem 3

We show that in a neighborhood of $\Delta_1 = \Delta_2 = 0$, changes in Δ_i will have a negligible impact on $\underline{\sigma}_j$, $i, j = 1, 2$ at any stable solution solution to the capacity equations (6). (For this result alone, we also assume that the signal cdf derivative G_x is continuous.) Given any discount pair (Δ_1, Δ_2), the capacity equations with two groups are:

$$\kappa_1 = \phi\mathcal{E}_1^\tau(\underline{\sigma}_1 - \Delta_1, \underline{\sigma}_2 - \Delta_2) + (1 - \phi)\mathcal{E}_1^N(\underline{\sigma}_1, \underline{\sigma}_2) \tag{24}$$

$$\kappa_2 = \phi\mathcal{E}_2^\tau(\underline{\sigma}_1 - \Delta_1, \underline{\sigma}_2 - \Delta_2) + (1 - \phi)\mathcal{E}_2^N(\underline{\sigma}_1, \underline{\sigma}_2), \tag{25}$$

where $\mathcal{E}_i^\tau, \mathcal{E}_i^N$ are the respective fractions of targeted and non-targeted groups enrolled at college i, defined just as in (4) and (5), for the sets of signals (17). Since the signal density

$g = G_\sigma$ and its derivative G_x are both continuous, all derivatives of the enrollment function (using Leibnitz rule) are continuous too.

Differentiating equations (24) and (25) with respect to Δ_1:

$$\begin{aligned}
\frac{J}{\phi}\frac{\partial \underline{\sigma}_1}{\partial \Delta_1} &= \sum_{i=1,2} (-1)^{i+1} \frac{\partial \mathcal{E}_i^\tau}{\partial(\underline{\sigma}_1 - \Delta_1)} \left(\phi \frac{\partial \mathcal{E}_{3-i}^\tau}{\partial(\underline{\sigma}_2 - \Delta_2)} + (1-\phi) \frac{\partial \mathcal{E}_{3-i}^N}{\partial \underline{\sigma}_2} \right) \\
\frac{J}{\phi}\frac{\partial \underline{\sigma}_2}{\partial \Delta_1} &= \sum_{i=1,2} (-1)^{i} \frac{\partial \mathcal{E}_i^\tau}{\partial(\underline{\sigma}_1 - \Delta_1)} \left(\phi \frac{\partial \mathcal{E}_{3-i}^\tau}{\partial(\underline{\sigma}_1 - \Delta_1)} + (1-\phi) \frac{\partial \mathcal{E}_{3-i}^N}{\partial \underline{\sigma}_1} \right)
\end{aligned}$$

where the denominator, from Cramer's Rule, equals

$$\begin{aligned}
J &= \left(\phi \frac{\partial \mathcal{E}_1^\tau}{\partial(\underline{\sigma}_1 - \Delta_1)} + (1-\phi) \frac{\partial \mathcal{E}_1^N}{\partial \underline{\sigma}_1} \right) \left(\phi \frac{\partial \mathcal{E}_2^\tau}{\partial(\underline{\sigma}_2 - \Delta_2)} + (1-\phi) \frac{\partial \mathcal{E}_2^N}{\partial \underline{\sigma}_2} \right) \\
&\quad - \left(\phi \frac{\partial \mathcal{E}_1^\tau}{\partial(\underline{\sigma}_2 - \Delta_2)} + (1-\phi) \frac{\partial \mathcal{E}_1^N}{\partial \underline{\sigma}_2} \right) \left(\phi \frac{\partial \mathcal{E}_2^\tau}{\partial(\underline{\sigma}_1 - \Delta_1)} + (1-\phi) \frac{\partial \mathcal{E}_2^N}{\partial \underline{\sigma}_1} \right)
\end{aligned}$$

is positive in any stable equilibrium — i.e. the two group version of the condition that the slope of Σ_1 exceed the slope of Σ_2 in §4 and §A.8. Now, $\partial \underline{\sigma}_1/\partial \Delta_1 = \phi > 0$ and $\partial \underline{\sigma}_2/\partial \Delta_1 = 0$ when $\Delta_1 = \Delta_2 = 0$, because the derivatives of the function $\mathcal{E}_i^\tau, \mathcal{E}_i^N$ at colleges $i = 1, 2$ coincide. Thus, the feedback effects vanish when $\Delta_1 = \Delta_2 = 0$, and are negligible in a neighborhood of it, by continuity of the enrollment derivatives. The analysis of the derivatives of $\underline{\sigma}_i$, $i = 1, 2$, with respect to Δ_2 is analogous. □

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