Competing for Talents

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Abstract: Though individuals prefer to join groups with high quality peers, there are advantages to being high up in the pecking order within a group if higher ranked members of a group have greater access to the group's resources. When two organizations try to attract members from a fixed population of heterogeneous agents, how resources are distributed among the members according to their rank affects how agents choose between the organizations. Competition between the two organizations has implications for both the equilibrium sorting of agents and the way resources are distributed within each organization. To compete more intensely for the more talented agents, both organizations are selective and give no resources to their low ranks. In both organizations, higher ranks are rewarded with more resources, with a greater rate of increase in the organization that has a lower average quality in equilibrium.

1. Introduction

Distributions of talents across organizations exhibit patterns of both mixing and segregation. At the risk of being accused of self-indulgence, let us look at the example of distribution of economists by productivities across departments. Highly productive researchers can be found in many second-tier departments as well as in top-ranked institutions. However, there is an unmistakable hierarchy of departments in terms of average productivity of their faculty members.

A plausible explanation of the coexistence of mixing and segregation is sorting by talents who care both about the quality of the institution they join and about their relative ranking within the institution. In Damiano, Li and Suen (2005), we call these two concerns "peer effect" and "pecking order effect." The peer effect is widely acknowledged in the education literature (e.g., Coleman et al., 1966; Summers and Wolfe, 1977; Lazear, 2001; Sacerdote 2001), and modeled extensively in the literature on locational choice (De Bartolome, 1990; Epple and Romano, 1998). The pecking order effect can be motivated by concerns for self-esteem (Frank, 1985), competition for mates in the same location (Cole, Mailath, and Postlewaite, 1992), or competition for resources among members of an organization (Postlewaite, 1998). These two effects highlight competition and cooperation as two important features of the interaction among talents within an organization.

The equilibrium pattern of mixing and segregation derived in a benchmark two-organization model of Damiano, Li and Suen (2005) is an "overlapping interval" structure, where the very talented are exclusively found in the high quality organization and the least talented are left to the low quality organization, while the intermediate talents are present in both. As a result, the very talented are captives in the high quality organization, and it is the intermediate talents that are the targets for the two organizations. Our comparative statics analysis reveals that the high quality and the low quality organizations have opposite preferences for policies that affect the tradeoff between the peer effect and the pecking order effect.

A logical further step after the analysis of sorting by Damiano, Li and Suen (2005) is to model organizational competition explicitly. To accomplish this, we develop a model in which talents care about their relative ranking within the organization they join because higher ranks receive more resources, and organizations compete for talents by designing how resources are allocated according to rank. We characterize a unique equilibrium of organizational competition. In equilibrium the targets of competition are the top talents; only they receive positive shares of resources from either organization. The organization that in equilibrium attracts a higher average quality of talents has a more egalitarian distribution of resources than the low quality organization, because the low quality organization is disadvantaged by the peer effect and must concentrate its resources on a smaller set of top talents. The equilibrium sorting of talents exhibits mixing of top talents, with a greater share of them going to the high quality organization, while segregation occurs for all types that receive no resources in equilibrium with the better types going to the high quality organization.

In section 2, we formally introduce our model of organizational competition. The model is broadly based on Damiano, Li and Suen (2005). Talents have one-dimensional types distributed uniformly, and a utility function linear in the average type of the organization they join and the resource they receive in the organization. Each organization faces a fixed capacity constraint that allows it to accept half of an exogenously given talent pool, and a fixed total budget of resources that can be allocated among its ranks. We use the notion of sorting equilibrium defined in Damiano, Li and Suen (2005) to describe how talents sort after the organizations have chosen their resource distribution schedules. The issue of multiple equilibria is resolved by arbitrarily labeling one organization as the "dominant" one and selecting the sorting equilibrium with the largest difference in average types in its favor. This quality difference then defines the payoffs of the two organizations in the game in which they simultaneously choose their resource distribution schedules.

The game of organizational competition is strictly competitive. In section 3, we show that the game has a minmax value corresponding to the largest quality difference that the dominant organization can obtain in a Nash equilibrium of the game. We characterize the value and identify a unique resource distribution schedule for the inferior organization to achieve the minmax value. There is a critical rank that receives strictly positive resource, with all ranks below receiving no resources and the resources received by the ranks above increasing linearly in rank. Intuitively, the inferior organization has to pay a peer effect premium in order to compete with the dominant organization, which leads to the jump in the resource distribution schedule at the critical rank. Further, a linear resource allocation schedule is necessary in order to avoid having its high ranks cherry-picked by the rival organization.

In section 4 we characterize a unique Nash equilibrium of the organizational competition. The existence of the equilibrium is established by construction. In equilibrium the dominant organization chooses a resource distribution schedule similar to the minmax schedule of the inferior organization. There is a critical rank below which ranks receive no resources in the dominant organization, because they attract no competition from the inferior organization. Ranks above the critical rank in the dominant organization receive resources that increase linearly in rank, with no discontinuity at the critical rank and a smaller rate of increase than that in inferior organization. Sorting of talents in this equilibrium involves mixing of top talents between the two organizations, and segregation for low types. We also show that the equilibrium is a unique one, by establishing that for any other resource distribution schedule of the dominant organization, the inferior organization can improve upon the minmax schedule.

Section 5 provides some comparative statics results regarding the unique Nash equilibrium of the game of competing for talents. When the organizations have a greater budget for resource distribution, or when the peer effect becomes less important in the talents' utility function, the equilibrium exhibits a smaller disparity between the dominant and the inferior organizations. We then conclude the paper with brief discussions of some of the main assumptions of the model.

2. The Model

Two organizations, A and B, compete for a measure 2 of agents. Agents differ with respect to a one-dimensional continuously distributed characteristic, called "type" and denoted by θ. We assume that the distribution of θ is uniform on the interval $[0, 1]$. Each organization $i = A, B$ has a measure 1 of positions and a fixed resource budget Y to be allocated among its members. An organization determines the distribution of its resource budget Y by choosing a "resource distribution schedule." A resource distribution schedule for organization i is a function $S_i : [0,1] \to \mathbb{R}_+$, which stipulates how Y is allocated among i's members according to their rank. For each $r \in [0,1]$, $S_i(r)$ denotes the amount of resources received by an agent of type θ when a fraction r of the organization's members are of type smaller than θ. We make the assumption that organizations can only adopt "meritocratic" resource distribution schedules in which members of higher ranks receive at least as much resources as lower ranks. We also make the technical assumption that only resource distribution schedules which are almost everywhere continuously differentiable are admissible. Both organizations must fill all their positions and each wants to maximize its own quality, measured by the average type of its members.

Preferences of agents over the two organizations depend on the comparison of the qualities of the two organizations and of the amount of resources they receive when joining. For each $i = A, B$, let m_i be the average type of agents in organization i. Let $r_i(\theta)$ be the quantile rank of an agent of type θ in organization i. If S_i is the resource distribution schedule in organization i, then the utility to an agent θ from joining organization i is given by

$$V_i(\theta) = \alpha S_i(r_i(\theta)) + m_i \tag{2.1}$$

where α is a positive constant that represents the weight on the concern for the pecking order effect relative to the concern for the peer effect. The payoff is zero if an agent does not join either organization.

2.1. Sorting equilibrium

Since each agent's outside option is zero and each organization must fill all positions, a feasible allocation of the agents among the two organizations can be described by a pair of the type distribution functions in the two organizations, as follows.

Definition 2.1. A feasible allocation is a pair of cumulative distribution functions (H_A, H_B) such that $H_A(\theta) + H_B(\theta) = 2\theta$ for all $\theta \in [0,1]$.

Given a pair of resource distribution schedules (S_A, S_B) the agents sort themselves between the two organization. We call this the sorting stage. We adapt the notion of "priority equilibrium" in Damiano, Li and Suen (2005) to the present environment.

Definition 2.2. Given a pair of resource distribution schedules (S_A, S_B), a sorting equilibrium is a feasible allocation (H_A, H_B) such that if H_i is strictly increasing on (θ, θ') and $H_j(\theta) > 0$, then $V_i(\theta) \geq V_j(\theta)$.

As in the priority equilibrium of Damiano, Li and Suen (2005), the notion of sorting equilibrium above assumes that an agent can join organization i when the agent's type is higher than the lowest type of organization.[1]

Existence of a sorting equilibrium can be established by a fixed point argument. Before doing so, it is convenient to introduce an alternative representation of feasible allocations through allocation functions.[2]

Definition 2.3. Given a feasible allocation (H_A, H_B), the associated allocation function is $t : [0, 1] \to [0, 1]$, defined by

$$t(r) \equiv 1 - H_A\left(\inf\{\theta : H_B(\theta) = r\}\right).$$

In the definition above, $t(r)$ is the fraction of agents in organization A of type higher than rank r's type in organization B. Using the definition of allocation function above, we associate to each feasible allocation an (essentially) unique non-increasing function on the unit intervals.

Conversely, each non-increasing function $t : [0, 1] \to [0, 1]$ identifies an (essentially) unique feasible allocation (H_A, H_B), where H_B is given by

$$H_B(\theta) = \begin{cases} 0 & \text{if } 2\theta \leq 1 - t(0); \\ 1 & \text{if } 2\theta \geq 2 - t(1); \\ \sup\{r : 2\theta \geq r + 1 - t(r)\} & \text{otherwise.} \end{cases}$$

[1] See our earlier paper for a more detailed discussion of priority equilibrium.

[2] While the definition below makes use of the assumption that θ is distributed uniformly on $[0, 1]$, it can be easily extended to the case when the distribution of θ is given by any continuous function F.

The feasibility condition gives $H_A(\theta) = 2\theta - H_B(\theta)$. Thus, there is a one-to-one mapping from feasible allocations to non-increasing function on the unit interval. The convenience of working with allocation functions is made explicit by the following lemma where we show that, for any feasible allocation, the quality difference between the two organizations only depends on the integral of the associated allocation function.

LEMMA 2.4. *Let (H_A, H_B) be a feasible allocation and t the associated allocation function. Then*

$$\int_0^1 \theta \ dH_A(\theta) - \int_0^1 \theta \ dH_B(\theta) = -\frac{1}{2} + \int_0^1 t(r) \ dr.$$

PROOF. Using the definition of $t(r)$ and a change of variable $\theta = \inf\{\theta' : H_B(\theta') = r\}$, we can write

$$-\frac{1}{2} + \int_0^1 t(r) \ \mathrm{d}r = \frac{1}{2} - \int_{\underline{\theta}_B}^{\overline{\theta}_B} H_A(\theta) \ \mathrm{d}H_B(\theta),$$

where $\underline{\theta}_B = \sup\{\theta : H_B(\theta) = 0\}$ and $\overline{\theta}_B = \inf\{\theta : H_B(\theta) = 1\}$. From the feasibility condition that $H_A(\theta) = 2\theta - H_B(\theta)$, the right-hand-side of the above equation is equal to

$$\frac{1}{2} - 2m_B + \int_{\underline{\theta}_B}^{\overline{\theta}_B} \ \mathrm{d}H_B(\theta).$$

The claim follows immediately from the identity $m_A + m_B = 1$. *Q.E.D.*

Since for any allocation, the quality difference between the two organizations is the integral of the allocation function minus 1/2, we will refer to the integral

$$T = \int_0^1 t(r) \ \mathrm{d}r$$

as the difference in quality. The constant $(T - 1/2)/\alpha$ represents the quality premium of A over B, in that any agent would be just indifferent between the two if the agent receives from B a resource greater than what he receives from A by the premium. Denote the premium as a function of quality difference T

$$P(T) = \frac{T - 1/2}{\alpha}.$$

For any difference in quality $T \in [0,1]$ let $\underline{t}^T$ and $\bar{t}^T$ be the allocation functions defined as

$$\underline{t}^T(r) = \begin{cases} 1 & \text{if } S_A(0) + P(T) > S_B(r); \\ 1 - \sup\{\tilde{r} \in [0,1] : S_A(\tilde{r}) + P(T) \leq S_B(r)\} & \text{if otherwise.} \end{cases}$$

and

$$\bar{t}^T(r) = \begin{cases} 0 & \text{if } S_A(1) + P(T) < S_B(r); \\ 1 - \inf\{\tilde{r} \in [0,1] : S_A(\tilde{r}) + P(T) \geq S_B(r)\} & \text{if otherwise.} \end{cases}$$

In words, the agent who has rank r in B must have rank at most $1 - \underline{t}^T(r)$ in A or he would prefer to switch; he must also have rank at least $1 - \bar{t}^T(r)$ in A or otherwise some agent from A would want to switch.

The following proposition identifies necessary and sufficient conditions for an allocation function to constitute a sorting equilibrium.

Proposition 2.5. *A feasible allocation (H_A, H_B) is a sorting equilibrium if and only if the associated allocation function t satisfy $\underline{t}^T(r) \leq t^T(r) \leq \bar{t}^T(r)$ for all $r \in [0,1]$, for $T = \int_0^1 t(r)\ dr$.*

Proof. Follows immediately from the definition of sorting equilibrium 2.2. *Q.E.D.*

Define the following correspondence

$$D(T) = \left[\int_0^1 \underline{t}^T(r)\ \mathrm{d}r, \int_0^1 \bar{t}^T(r)\ \mathrm{d}r\right]. \tag{2.2}$$

The above proposition implies that any sorting equilibrium is a fixed point $T \in [0,1]$ of D. Existence of a sorting equilibrium then follows from an application of Tarsky's fixed point theorem.

Multiple sorting equilibria exist in general. To study the game in which the two organizations compete by choosing resource distribution schedules we must introduce an equilibrium selection in the sorting stage. We assume that organization A is dominant in that the sorting equilibrium with the largest difference in quality T is played in the sorting stage. This "A-dominant equilibrium" is unique. The equilibrium quality difference, T, corresponds to the largest fixed point of the mapping

$$D_A(T) = \int_0^1 \bar{t}^T(r)\ \mathrm{d}r, \tag{2.3}$$

and the equilibrium allocation function is given by $\bar{t}_T$.[3]

Now we can define a "resource distribution game" in which the two organizations simultaneously choose a resource distribution schedule to maximize its own quality. For any (S_A, S_B), the payoff to organization i is defined as the average type of i's members in the A-dominant sorting equilibrium. Since the sum of the payoffs to the two organization is constant, the resource distribution game is strictly competitive, with A trying to maximize the difference in quality, $T_A(S_A, S_B)$, and B trying to minimize it. Therefore, a strategy profile (S_A^*, S_B^*) is a Nash equilibrium of the resource distribution game if and only if[4]

$$S_A^* \in \arg\max_{S_A \in \mathcal{S}} \min_{S_B \in \mathcal{S}} T_A(S_A, S_B), \quad S_B^* \in \arg\min_{S_B \in \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B),$$

and

$$\max_{S_A \in \mathcal{S}} \min_{S_B \in \mathcal{S}} T_A(S_A, S_B) = \min_{S_B \in \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B)$$

where $\mathcal{S}$ is the set of all non-negative, non-decreasing and almost everywhere continuously differentiable functions which respect the resource constraint $\int_0^1 S(r)\ \mathrm{d}r \leq Y$.

3. The Minmax Value

In this section we characterize the minmax value $\min_{S_B \in \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B)$. This corresponds to the maximum quality difference that organization A can hope to achieve in any Nash equilibrium of the resource distribution game. We also characterize the unique resource distribution schedule S_B^* that achieves the minmax value.

Before we proceed with the analysis it is useful to sketch a road map. For any resource distribution schedule S_B and any difference in quality T, we characterize the lowest resource expenditure $C(T; S_B)$ needed for A to attain a sorting equilibrium with quality difference T. Note that we are not requiring T to be the A-dominant equilibrium quality difference at this point. Next, we characterize the maximum value of this minimum resource expenditure

[3] We note that the A-dominant equilibrium is always stable in the sense of Damiano Li and Suen (2005).

[4] Since the resource distribution game is not finite, we cannot assume that maxminimizers and minmaximizers exist. These are shown to exist by construction.

$C(T; S_B)$ that B can impose on A by choosing resource distribution schedule S_B subject to the resource budget constraint Y. This gives us the maximum resource budget C as a function of T. The largest T^* such that $C(T^*)$ is equal to Y is then a lower bound for the minmax value. Finally we show that there is a unique resource distribution schedule S_B^* that achieves $C(T^*)$, and that $C(T; S_B^*)$ is larger than Y for all $T > T^*$. This implies that T^* is also an upper bound on the minmax value.

The A-dominant selection implies that the minmax value is at least $1/2$. This is because, for any resource distribution schedule S_B, when $S_A = S_B$, there is a sorting equilibrium where the difference in quality is zero. Thus we will restrict our analysis below to $T \geq 1/2$.

3.1. The expenditure minimization problem

For given S_B and some $T \geq 1/2$, we want to find the cheapest S_A such that T is a sorting equilibrium for (S_A, S_B). If S_A is such resource distribution schedule we denote with $C(T; S_B)$ the integral of S_A.

First, note that by definition of $\underline{t}^T$, we have $\underline{t}^T(r) = 1$ for all r such that $S_B(r) < P(T)$. Thus, if $S_B(\tilde{r}) < P(T)$ for some $\tilde{r} > T$, then there is no equilibrium with quality difference T regardless of the resource distribution schedule S_A or the total available resource to A. Moreover, even if A gives no resources to all of its ranks (i.e. $S_A(r) = 0$ for all r), there is a sorting equilibrium with quality difference strictly larger than T. In this case, we write that $C(T; S_B) = 0$.

Next, suppose $S_B(T) \geq P(T)$. Then, for any allocation function t, with $\int_0^1 t(r)\,\mathrm{d}r = T$ and $t(r) = 1$ for any r such that $S_B(r) < P(T)$, let S_A^t be the pointwise smallest resource distribution schedule that satisfies

$$S_A^t(1 - t(r)) \geq \max\{S_B(r) - P(T), 0\} \text{ for all } r \in [0, 1].$$

By construction, for S_B and S_A^t, we have that $\bar{t}^T$ is pointwise larger than t while $\underline{t}^T$ is pointwise smaller than t. It follows that T is a fixed point of the mapping (2.2) and hence there exists a sorting equilibrium with quality difference T for (S_A^t, S_B).

The schedule S_A^t is the resource distribution schedule with the lowest expenditure for which t is a sorting equilibrium given S_B. It follows that

$$\begin{aligned} C(T;S_B) &= \min_t \int_0^1 S_A^t(r)\ \mathrm{d}r \\ \text{s.t.} &\int_0^1 t(r)\ \mathrm{d}r = T; \\ &t(r) = 1 \text{ if } S_B(r) - P(T) < 0. \end{aligned} \tag{3.1}$$

By definition we have

$$\int_0^1 S_A^t(r)\ \mathrm{d}r = \int_0^1 \max\left\{S_B(t^{-1}(1-r)) - P(T), 0\right\}\ \mathrm{d}r$$

where we define

$$t^{-1}(1-r) = \sup\{\tilde{r} \in [0,1] : 1 - t(\tilde{r}) \le r\}.$$

After a change of variable $\tilde{r} = t^{-1}(1-r)$ and integration by part we have

$$\int_0^1 S_A^t(r)\ \mathrm{d}r = -\int_{t^{-1}(1)}^{t^{-1}(0)} \Delta(\tilde{r})t'(\tilde{r})\ \mathrm{d}\tilde{r} = \int_{t^{-1}(1)}^{t^{-1}(0)} t(\tilde{r})\Delta'(\tilde{r})\ \mathrm{d}\tilde{r} - \Delta(t^{-1}(1))$$

where for notational convenience we have defined

$$\Delta(\tilde{r}) = \max\{S_B(\tilde{r}) - P(T), 0\}$$

as the effective resource distribution schedule of the inferior organization. We can then rewrite the minimization problem (3.1) as

$$\begin{aligned} &\min_t \int_{t^{-1}(1)}^{t^{-1}(0)} t(r)\Delta'(r)\ \mathrm{d}r - \Delta(t^{-1}(1)) \\ &\text{s.t.} \int_0^1 t(r)\mathrm{d}r = T, \end{aligned} \tag{3.2}$$

where we have dropped the second constraint of (3.1) since it will be satisfied by any solution to (3.2). Note that both the objective function and the constraint are linear in t. This feature is used below to characterize the solution, and it is why we have chosen to deal with the allocation function instead of with type distribution functions directly.

We next proceed, through a series of lemmata, to characterize a solution to problem (3.2). The next two lemmata establish that there is a solution to (3.2) which assumes at most one value strictly between 0 and 1. This result is then used to provide an explicit characterization of the solution and a value for $C(T; S_B)$.

LEMMA 3.1. *For any allocation function t with $\int_0^1 t(r)\ dr = T$, there exists an allocation function $\tilde{t}$ with $\int_0^1 \tilde{t}(r)\ dr = T$, which assumes a countable number of values and satisfies*

$$\int_{\tilde{t}^{-1}(1)}^{\tilde{t}^{-1}(0)} \tilde{t}(r)\Delta'(r)\ dr - \Delta(\tilde{t}^{-1}(1)) \leq \int_{t^{-1}(1)}^{t^{-1}(0)} t(r)\Delta'(r)\ dr - \Delta(t^{-1}(1)).$$

PROOF. Let $\mathcal{I}$ denote the collection of all open intervals $I \subset [0,1]$, such that: i) t is continuous and strictly decreasing on I; ii) Δ' is monotone on I; and iii) there is no open interval $I' \supset I$ that satisfies property i) and ii). Since both t and Δ' have a countable number of discontinuities, the set $\mathcal{I}$ is countable. Moreover, t assumes a countable number of different values on $[0,1]\setminus\mathcal{I}$. If t assumes uncountably many values, then $\mathcal{I}$ is non-empty. For each $I \in \mathcal{I}$, let $r_- = \inf_r I$ and $r_+ = \sup_r I$. Let $r^0 \in (r_-, r_+)$ solve

$$t(r^0) = \int_{r_-}^{r_+} t(r)\ \mathrm{d}r,$$

and let $\hat{r} \in (r_-, r_+)$ solve

$$t(r_-)(\hat{r} - r_-) + t(r_+)(r_+ - \hat{r}) = \int_{r_-}^{r_+} t(r)\ \mathrm{d}r.$$

We construct a new allocation function $\tilde{t}$ such that, for each $I \in \mathcal{I}$, if Δ' is decreasing on I, $\tilde{t}(r) = t(r^0)$ for all $r \in I$. If otherwise Δ' is increasing on I, then $\tilde{t}(r) = t(r_-)$ for all $r \in (r_-, \hat{r}]$ and $\tilde{t}(r) = t(r_+)$ for all $r \in (\hat{r}, r_+)$. On $[0,1]\setminus\mathcal{I}$, $\tilde{t}$ is identical to t.

By construction $\tilde{t}$ is a decreasing function and

$$\int_0^1 \tilde{t}(r)\ \mathrm{d}r = \int_0^1 t(r)\ \mathrm{d}r.$$

Moreover.

$$\int_0^1 S_A^{\tilde{t}}(r)\ \mathrm{d}r - \int_0^1 S_A^{t}(r)\ \mathrm{d}r = \sum_{I\in\mathcal{I}} \int_{r_-}^{r_+} \big(\tilde{t}(r) - t(r)\big)\,\Delta'(r)\ \mathrm{d}r.$$

For each $I \in \mathcal{I}$ such that Δ' is decreasing on $\mathbf{r}$,

$$\begin{aligned}
\int_{r_-}^{r_+} \big(\tilde{t}(r) - t(r)\big)\,\Delta'(r)\ \mathrm{d}r &= \int_{r_-}^{r^0} \big(t(r^0) - t(r)\big)\,\Delta'(r)\ \mathrm{d}r + \int_{r^0}^{r_+} \big(t(r^0) - t(r)\big)\,\Delta'(r)\ \mathrm{d}r \\
&\leq \Delta'(r^0)\int_{r_-}^{r^0} \big(\tilde{t}(r^0) - t(r)\big)\ \mathrm{d}r + \Delta'(r^0)\int_{r^0}^{r_+} \big(\tilde{t}(r^0) - t(r)\big)\ \mathrm{d}r \\
&= 0.
\end{aligned}$$

For each $I \in \mathcal{I}$ such that Δ' is increasing on $\mathbf{r}$,

$$\begin{aligned}\int_{r_-}^{r_+} \big(\tilde{t}(r) - t(r)\big)\, \Delta'(r)\ \mathrm{d}r &= \int_{r_-}^{\hat{r}} (t(r_-) - t(r))\, \Delta'(r)\ \mathrm{d}r + \int_{\hat{r}}^{r_+} (t(r_+) - t(r))\, \Delta'(r)\ \mathrm{d}r \\ &\leq \Delta'(\hat{r}) \int_{r_-}^{\hat{r}} (t(r_+) - t(r))\ \mathrm{d}r + \Delta'(\hat{r}) \int_{\hat{r}}^{r_+} (t(r_-) - t(r))\ \mathrm{d}r \\ &= 0.\end{aligned}$$

The lemma then follows. *Q.E.D.*

LEMMA 3.2. *For any allocation function t with $\int_0^1 t(r)\ dr = T$, there exists an allocation function $\tilde{t}$ with $\int_0^1 \tilde{t}(r)\ dr = T$, which assumes at most one value strictly between 0 and 1 and satisfies*

$$\int_{\tilde{t}^{-1}(1)}^{\tilde{t}^{-1}(0)} \tilde{t}(r)\Delta'(r)\ dr - \Delta(\tilde{t}^{-1}(1)) \leq \int_{t^{-1}(1)}^{t^{-1}(0)} t(r)\Delta'(r)\ dr - \Delta(t^{-1}(1)).$$

PROOF. From Lemma 3.1 we can assume that t assumes a countable number of values. Suppose there are two consecutive intervals I^j and I^{j+1}, such that t assumes value t^j on I^j and value t^{j+1} on I^{j+1}, for some $1 > t^j > t^{j+1} > 0$ and t is strictly larger than t^j for $r < \inf_r I^j$ and strictly smaller than t^{j+1} for $r > \sup_r I^{j+1}$. Consider a new allocation function $\tilde{t}_\epsilon$ defined as follows

$$\tilde{t}_\epsilon(r) = \begin{cases} t^j + \epsilon/(r_+^j - r_-^j) & \text{if } r \in I^j; \\ t^{j+1} - \epsilon/(r_+^{j+1} - r_-^{j+1}) & \text{if } r \in I^{j+1}; \\ t(r) & \text{otherwise.} \end{cases}$$

For ϵ small, $\tilde{t}_\epsilon$ is a decreasing function. Moreover, by construction, $\int_0^1 \tilde{t}_\epsilon(r)\ \mathrm{d}r = \int_0^1 t(r)\ \mathrm{d}r$ and

$$\int_0^1 S_A^{\tilde{t}_\epsilon}(r)\ \mathrm{d}r - \int_0^1 S_A^t(r)\ \mathrm{d}r = \frac{\epsilon}{r_+^j - r_-^j} \int_{r_-^j}^{r_+^j} \Delta'(r)\ \mathrm{d}r - \frac{\epsilon}{r_+^{j+1} - r_-^{j+1}} \int_{r_-^{j+1}}^{r_+^{j+1}} \Delta'(r)\ \mathrm{d}r.$$

Since $\int_0^1 S_A^{\tilde{t}_\epsilon}(r)\ \mathrm{d}r - \int_0^1 S_A^t(r)\ \mathrm{d}r$ is linear in ϵ, we can always choose some ϵ for which $\tilde{t}$ assumes one less value than t and does at least as well as t for the objective function of (3.2). *Q.E.D.*

The above results imply that we can restrict the search for a solution to (3.2) to allocation functions that assume at most one value strictly between 0 and 1. First, an allocation function t that has just one positive value is entirely characterized by its only discontinuity point, $\hat{r}$. To see this, note that since all solutions to (3.2) satisfy the constraint $\int t(r)\ \mathrm{d}r = T$, if t is zero for $r > \hat{r}$ and constant for $r < \hat{r}$, then we have $t(r) = T/\hat{r}$ for all $r \leq \hat{r}$ to satisfy the constraint. Also note that $\hat{r} \geq T$ must hold in this case. Second, an allocation function t that has one value strictly between 0 and 1 is entirely characterized by its two discontinuity points. Letting $r^1 = \sup\{r : t(r) = 1\}$ and $r^0 = \sup\{r :\ t(r) > 0\}$, and using the constraint $\int t(r)\ \mathrm{d}r = T$, we have that $t(r) = (T - r^1)/(r^0 - r^1)$ for $r \in (r^1, r^0)$. Note that $r^1 \leq T$ and $r^0 \geq T$. in this case.

Thus, a solution to problem (3.2) exists, and the value $C(T; S_B)$ is given by

$$\min\left\{\min_{r\geq T} \frac{T}{r}\Delta(r),\ \min_{T\geq r^1\geq 0;\ 1\geq r^0\geq T} \Delta(r^1) + \frac{T-r^1}{r^0-r^1}\left(\Delta(r^0)-\Delta(r^1)\right)\right\}. \tag{3.3}$$

Using the above characterization for $C(T; S_B)$ it is possible to obtain a characterization of a solution to problem (3.2) as a function of Δ. To do so, given a function Δ, we let $\hat{\Delta}$ denote the largest convex function which is pointwise smaller than Δ and such that $\hat{\Delta}(0) = 0$. Formally, $\hat{\Delta}$ is obtained as the lower contour of the convex hull of the function Δ and the origin. That is

$$\hat{\Delta}(r) = \min\{y :\ (r, y) \in co(\{(\tilde{r}, \tilde{y}) : 0 \leq \tilde{r} \leq 1; \tilde{y} \geq \Delta(\tilde{x})\} \cup \mathbf{0})\}.$$

The next lemma provides a simple characterization of the discontinuity points of a solution to (3.2) which depends only on the functions Δ and $\hat{\Delta}$. In particular, it states that if $\Delta(T) = \hat{\Delta}(T)$, then there is a solution to (3.2) with only one discontinuity point at exactly T. The optimal allocation function t is constant, and equal 1 for $r \leq T$ and to 0 for $r > T$. When $\Delta(T) > \hat{\Delta}(T)$ instead, there is an optimal allocation with two discontinuity points $r^1 < T$ and $r^0 > T$. The two discontinuity points are determined by the largest $r < T$ and the smallest $r > T$ at which the function Δ coincides with its convex hull. The optimal allocation function t equals 1 up to r^1 and becomes 0 at r^0.

LEMMA 3.3. *Let* $Q = \{r : \Delta(r) = \hat{\Delta}(r)\} \cup \{0, 1\}$. *If* T *belongs to the closure* $\overline{Q}$ *of* Q, *then the function*

$$t(r) = \begin{cases} 1 & \text{if } r \leq T; \\ 0 & \text{otherwise.} \end{cases}$$

solves (3.2). Otherwise, for $r^1 = \sup\{r \in A : \; r < T\}$ *and* $r^0 = \inf\{r \in A : \; r > T\}$, *a solution to (3.2) is given by the function*

$$t(r) = \begin{cases} 1 & \text{if } r < r^0; \\ (T - r^1)/(r^0 - r^1) & \text{if } r^1 \leq r \leq r^0; \\ 0 & \text{if } r > r^0. \end{cases}$$

PROOF. First rewrite equation (3.3) as

$$C(T; S_B) = \min\left\{ \min_{r \geq T} \frac{r - T}{r} 0 + \frac{T}{r}\Delta(r), \min_{T \geq r^1 \geq 0;\; 1 \geq r^0 \geq T} \frac{r^0 - T}{r^0 - r^1}\Delta(r^1) + \frac{T - r^1}{r^0 - r^1}\Delta(r^0) \right\}.$$

At the claimed solution, if $T \in \overline{Q}$, then the value of the objective function is $\Delta(T) = \hat{\Delta}(T)$. If $T \notin \overline{Q}$, then the value of the objective function is given by

$$\frac{r^0 - T}{r^0 - r^1}\Delta(r^1) + \frac{T - r^1}{r^0 - r^1}\Delta(r^0) = \frac{r^0 - T}{r^0 - r^1}\hat{\Delta}(r^1) + \frac{T - r^1}{r^0 - r^1}\hat{\Delta}(r^0) = \hat{\Delta}(T),$$

where the second equality holds because $\hat{\Delta}$ is linear between r^1 and r^0. Moreover, for all $r^1 \leq T$ and $r^0 \geq T$ we have

$$\begin{aligned} \hat{\Delta}(T) &\leq \frac{r^0 - T}{r^0 - r^1}\hat{\Delta}(r^1) + \frac{T - r^1}{r^0 - r^1}\hat{\Delta}(r^0) \\ &\leq \frac{r^0 - T}{r^0 - r^1}\Delta(r^1) + \frac{T - r^1}{r^0 - r^1}\Delta(r^0) \end{aligned}$$

where the first inequality follows from the fact that $\hat{\Delta}$ is convex and the second from $\Delta(r) \geq \hat{\Delta}(r)$ for all r. Finally, for all $r \geq T$

$$\hat{\Delta}(T) \leq \frac{r - T}{r}\hat{\Delta}(0) + \frac{T}{r}\hat{\Delta}(r) \leq \frac{r - T}{r} 0 + \frac{T}{r}\Delta(r)$$

thus t is a solution to (3.2). *Q.E.D.*

The total quality difference to be achieved for A is T. The two cases in the proof of the above lemma depend on whether it can be achieved by targeting a single rank $\hat{r}$ of B

and giving to sufficiently many ranks in A the minimum resource to be competitive with $\hat{r}$, or by targeting two ranks r^1 and r^0 and giving all ranks in A enough resources to compete with r^1 and giving sufficiently many ranks in A additional resources to compete with r^0. Minimizing the expenditure for A is then equivalent to choosing the cheapest ranks in B to attack.

3.2. The budget function

From the proof of Lemma 3.3 it is immediately evident that $C(T; S_B) = \hat{\Delta}(T)$, which depends on S_B since Δ is the maximum of $S_B - P(T)$ and 0. For each $T \geq 1/2$, we next try to characterize the resource distribution schedule S_B that makes the budget of generating an allocation with difference in quality T as large as possible for A. More precisely, for each T we study the maximization problem

$$\begin{aligned} C(T) \equiv \max_{S_B \in \mathcal{S}} & \ C(T; S_B) \\ \text{s.t.} & \int_0^1 S_B(r)\ \mathrm{d}r \leq Y. \end{aligned} \tag{3.4}$$

Our next result establishes that there is a solution S_B to (3.4) with the property that it is 0 up to some critical rank $r(T)$, equal to $P(T)$ at $r(T)$, and has a constant slope between $r(T)$ and 1. Since this solution is entirely characterized by $r(T)$ and the resource constraint that $\int_0^1 S_B(r)\mathrm{d}r = Y$, this characterization result will then be used to solve explicitly for the critical threshold $r(T)$, and to obtain an analytical expression for the "budget function" $C(T)$.

LEMMA 3.4. *For any $S_B \in \mathcal{S}$, there is $\tilde{S}_B \in \mathcal{S}$ such that for some $\tilde{r} \in [0,1]$*

$$\tilde{S}_B(r) = \begin{cases} 0 & \text{if } r < \tilde{r}; \\ P(T) & \text{if } r = \tilde{r}; \\ P(T) + \beta(r - \tilde{r}) & \text{if } r > \tilde{r} \end{cases} \tag{3.5}$$

where β is determined by $\int \tilde{S}_B(r)\ dr = Y$, with the property that $C(T; \tilde{S}_B) \geq C(T; S_B)$.

PROOF. Let $\Delta_{S_B}(r)$ denote $\max\{S_B(r) - P(T), 0\}$. First, since $C(T; S_B)$ only depends on Δ_{S_B}, it cannot be decreased if we replace S_B with some $\tilde{S}_B$ such that $\tilde{S}_B(r) = 0$

whenever $\tilde{S}_B(r) < P(T)$. Second, since by Lemma 3.3, $C(T; S_B) = \hat{\Delta}_{S_B}(T)$ for any resource distribution schedule S_B, we have $C(T; S_B) = C(T; \tilde{S}_B)$ if $\tilde{S}_B$ is such that $\Delta_{\tilde{S}_B} = \hat{\Delta}_{S_B}$. This means that for any S_B, there is a resource distribution schedule $\tilde{S}_B$ which is convex whenever positive and $\Delta_{\tilde{S}_B}(0) = 0$ such that $C(T; \tilde{S}_B) \geq C(T; S_B)$. Finally, for any S_B that is convex whenever positive and $\Delta_{S_B}(0) = 0$, there is a $\tilde{S}_B$ which is linear when positive such that $C(T; \tilde{S}_B) \geq C(T; S_B)$. The lemma then follows from the resource constraint because binding the constraint increases the budget requirement for A.

Q.E.D.

By Lemma 3.4, we can restrict to resource distribution schedules of the form (3.5) when characterizing a solution to (3.4). Thus, the value of any solution to (3.4), $C(T)$, is given by

$$C(T) = \max_{r \in [0,1]} \frac{2(T-r)}{(1-r)^2} \left(Y - P(T)(1-r)\right). \tag{3.6}$$

The maximization problem (3.6) can be solved analytically. To do so, it is convenient to define $i = 1 - r$, $w = 1 - T$ and $z = Y/P(T)$. Then, $r(T)$ solves (3.6), if and only if $r(T) = 1 - \hat{i}$ and $\hat{i}$ is a solution to

$$\max_{w \leq i \leq \min\{1,z\}} \frac{(i-w)(z-i)}{i^2}. \tag{3.7}$$

A solution to (3.7) is easily characterized and given by

$$\hat{i} = \begin{cases} w & \text{if } w \geq z; \\ \min\{2/(1/w + 1/z), 1\} & \text{if } w < z. \end{cases}$$

In the region of interest, $T \geq 1/2$, we have that $w < 1/2$ which implies $1/w + 1/z > 2$ so that $\hat{i}$ is interior whenever $w < z$. The budget function $C(T)$ is then given by

$$C(T) = \begin{cases} 0 & \text{if } w \geq z; \\ 2\left(Y - P(T)(1 - r(T))\right)(T - r(T))/(1 - r(T))^2 & \text{if } w < z \end{cases} \tag{3.8}$$

and

$$r(T) = 1 - 2/(1/w + 1/z). \tag{3.9}$$

The following lemma provides a characterization of the budget function $C(T)$.

Lemma 3.5. *The budget function* $C(T)$ *satisfies* $C(1/2) = Y$, *and* $\lim_{T\to 1} C(T) = \infty$. *Moreover: (i) If* $\alpha Y > 1/2$, *then* $C'(T) > 0$ *for all* $T \geq 1/2$; *(ii) if* $\alpha Y \in [1/16, 1/2]$, *then there exists a* $\hat{T}$ *such that* $C'(T) < 0$ *for* $T \in (1/2, \hat{T})$ *and* $C'(T) > 0$ *for* $T \in (\hat{T}, 1)$; *(iii) if* $\alpha Y < 1/16$, *then there exist* T_- *and* T_+ *such that* $C'(T) < 0$ *for* $T \in (1/2, T_-)$; $C'(T) > 0$ *for* $T \in (T_+, 1)$ *and* $C(T) = 0$ *for* $T \in [T_-, T_+]$.

Proof. The first two properties follow from the characterization of the budget function (3.8) after observing that $r(1/2) = 0$ and $\lim_{T\to 1} r(T) = 1$. Next, when $r(T)$ is interior, by the envelope theorem, the slope of $C(T)$ is given by the partial derivative of (3.8) with respect to T. We have that $C'(T)$ is proportional to

$$\alpha Y(w + z) - 2w^2 z.$$

Substituting for w and z in the last expression and after a few manipulations we have that $C'(T) > 0$ if and only if

$$\alpha Y + (1 - T)\left(3T - \frac{5}{2}\right) > 0.$$

The above holds for all $T \geq 1/2$ if $\alpha Y > 1/2$, thus establishing (i). When $\alpha Y \leq 1/2$, there exists a unique $\hat{T} \in [0, 1]$ such that the above inequality holds for $T > \hat{T}$ while the opposite inequality holds for $T < \hat{T}$. From the characterization of the budget function (3.8) we have that $C(T) = 0$ when $w \geq z$, or equivalently when

$$\left(T - \frac{3}{4}\right)^2 \leq \frac{1}{16} - \alpha Y.$$

The latter inequality, has two real roots T_- and T_+ in $[1/2, 1]$ when $\alpha Y \leq 1/16$, and no real root otherwise. Claims (ii) and (iii) follow from $\hat{T} \in [T_-, T_+]$. *Q.E.D.*

An increase in the target quality difference T has two opposite effects on the budget $C(T)$ required for A. On one hand, to achieve a greater T organization A must be competitive with more ranks in B and this requires a larger budget. On the other hand, a greater T also increases the quality premium that A enjoys over B and this reduces the budget requirement. The first effect dominates when the peer effect is relatively small, which happens when either α or Y is large. This explains why $C(T)$ is monotonically increasing in T

when αY is large. In contrast, the peer effect is strong and $C(T)$ may decrease when αY is small. Indeed, the budget requirement for some intermediate values of T can be zero. Note that for sufficiently large T, the budget function must be increasing. This is because by concentrating its resources on a few top ranks organization B can make it increasingly costly for A to achieve large quality differences.

3.3. The minmax value and the minmax strategy

From the budget function $C(T)$ we can derive a lower bound on the minmax value. In particular, define

$$T^* = \max\{T \in [1/2, 1] : \; C(T) = Y\},$$

and then we have

$$\min_{S_B \in \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B) \leq T^*.$$

This is because $C(T^*; S_B) \leq Y$ for any S_B, and hence there is a resource distribution schedule $S_A \in \mathcal{S}$ such that $T_A(S_A, S_B) \geq T^*$. Let $r^* = r(T^*)$, and denote as S_B^* the resource distribution schedule in (3.5) with $\tilde{r} = r^*$, given by

$$S_B^*(r) = \begin{cases} 0 & \text{if } r < r^*; \\ P(T^*) & \text{if } r = r^*; \\ P(T^*) + 2(r - r^*)(Y - P(T^*)(1 - r^*))/(1 - r^*)^2 & \text{if } r > r^*. \end{cases} \tag{3.10}$$

The next proposition establishes that the minmax value coincides with the lower bound T^* by proving that $C(T; S_B^*) > Y$ for all $T > T^*$.

Proposition 3.6. *The resource distribution schedule S_B^* given by (3.10) is the unique solution to the minmax problem* $\min_{S_B \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B)$.

Proof. The resource distribution schedule S_B^* is the unique solution to the budget maximization problem (3.4) when $T = T^*$. Hence it achieves a lower bound T^* of the minmax problem. If $C(T; S_B^*) > Y$ for all $T > T^*$ then T^* is also an upper bound of the minmax problem. To establish the upper bound, note that from Lemma 3.3 we have that

$$C(T; S_B^*) = \frac{2(T - r^*)}{(1 - r^*)^2} \left(Y - P(T)(1 - r^*)\right). \tag{3.11}$$

Since $C(T; S_B^*)$ is concave in T and $C(T^*; S_B^*) = Y$ by definition of T^* and T^*, $C(1, S_B^*) > Y$ is both necessary and sufficient to prove the claim. When $T^* > 1/2$, using the equation

$$\frac{2(T^* - r^*)}{(1 - r^*)^2} \left(Y - P(T^*)(1 - r^*)\right) = Y,$$

we have that $C(1, S_B^*) > Y$ is equivalent to $\alpha Y > T^* - r^*$. Next, using the same equation $C(T^*; S_B^*) = Y$, and the explicit solution for r^* given by (3.9) and solving for αY we have

$$2\alpha Y = (1 - T^*) + (1 - T^*)\sqrt{2(1 - T^*)}.$$

Solving the same two equations for $(T^* - r^*)$ we have

$$T^* - r^* = \frac{(1 - T^*)\sqrt{2(1 - T^*)}}{2 - \sqrt{2(1 - T^*)}}.$$

It is straightforward to verify that $\alpha Y > T^* - r^*$ for $T^* > 1/2$. Finally, from the characterization of the budget function $C(T)$, we have that $T^* = 1/2$ if and only if $\alpha Y > 1/2$. Since in this case $r^* = 0$, the claim $C(1, S_B^*) > Y$ can be verified by direct substitution into (3.11). *Q.E.D.*

When organization B chooses resource distribution schedule S_B^*, in order to achieve the quality difference T^* organization A must expend all its available resources. However, this may fail to guarantee that a larger quality difference is infeasible for A, because a larger T increases the quality premium and frees some resources for A. The above proposition establishes that given S_B^*, this peer effect is small relative to the additional resource requirement for obtaining a greater quality difference than T^*.

4. The Nash Equilibrium

The analysis of the previous section has identified T^* as the minmax value of the resource distribution game and the resource distribution schedule S_B^*, defined in the statement of Proposition 3.6, as the only candidate Nash equilibrium strategy for B. Thus, whether a Nash equilibrium exists only depends on whether $\max_{S_A \in \mathcal{S}} \min_{S_B \in \mathcal{S}} T_A(S_A, S_B) = T^*$.

Moreover, if the set of Nash equilibria is non-empty, there is a distinct Nash equilibrium for each distinct solution to the maxmin problem. A direct characterization of the set of solutions to the maxmin problem and the maxmin value is difficult. Instead we proceed by first proving that equilibrium strategies must satisfy some additional properties. By using these properties and the fact that the equilibrium strategy of B is given by S_B^*, we identify a candidate equilibrium strategy S_A^* for A. We then establish that a Nash equilibrium exists by directly verifying that (S_A^*, S_B^*) is indeed a Nash equilibrium strategy profile. Finally, we show that (S_A, S_B^*) is not a Nash equilibrium for any resource distribution schedule $S_A \neq S_A^*$, which leaves (S_A^*, S_B^*) as the unique Nash equilibrium of the resource distribution game.

Using the best response properties of any pair of Nash equilibrium strategies (S_A, S_B), the next lemma establishes that the range of S_A and S_B differ by a constant equal to the equilibrium quality premium $P(T)$.

LEMMA 4.1. *Let* (S_A, S_B) *be a Nash equilibrium of the resource distribution game and let* $T \in [1/2, 1)$ *be the equilibrium quality difference. Then, the range of* S_A *is the same as the range of the function* $\max\{S_B - P(T), 0\}$.

PROOF. Suppose that some interval $(\underline{s}, \overline{s})$ is in the range of S_A but not in the range of $\max\{S_B - P(T), 0\}$. Consider the resource distribution schedule $\tilde{S}_A$ defined as

$$\tilde{S}_A(r) = \begin{cases} \underline{s} & \text{if } S_A(r) \in (\underline{s}, \overline{s}); \\ S_A(r) & \text{otherwise.} \end{cases}$$

For any $r, \tilde{r} \in [0,1]$, and any $\tilde{T} \geq T$, we have that $\tilde{S}_A(r) \geq S_B(\tilde{r}) - P(\tilde{T})$, whenever $\tilde{S}_A(r) \geq S_B(\tilde{r}) - P(\tilde{T})$. This implies that $T_A(\tilde{S}_A, S_B) = T$. Since by construction

$$\int_0^1 \tilde{S}_A(r)\ \mathrm{d}r < \int_0^1 S_A(r)\ \mathrm{d}r = Y,$$

there exists some other resource distribution schedule $\hat{S}_A$ such that $T_A(\hat{S}_A, S_B) > T$ and hence S_A is not a best response to S_B. If some interval $(\underline{s}, \overline{s})$ is in the range of $\max\{S_B - P(T), 0\}$ but not in the range of S_A, a similar argument shows that S_B is not a best response to S_A. *Q.E.D.*

The above result, together with the characterization of the unique candidate equilibrium quality difference T^* and the equilibrium resource distribution schedule of organization B, S_B^*, of Proposition 3.6, implies that in any Nash equilibrium, the equilibrium resource distribution schedule S_A^* must be a continuous function with $S_A^*(0) = 0$ and $S_A^*(1) = S_B^*(1) - P(T^*)$.

4.1. Existence

From the characterization result of Proposition 3.6, we know that $\max\{S_B^* - P(T^*), 0\}$ is zero up to r^*. Since in any Nash equilibrium the quality difference must be exactly T^*, even the lowest type agent in organization A in equilibrium will have a higher type than the agent of rank r^* in organization B, regardless of the resource distribution schedule chosen by A. In terms of allocation functions, this means that the equilibrium allocation function t^* will have $t^*(r) = 1$ for all $r \leq r^*$. For $r > r^*$, the equilibrium allocation function will depend on S_A. If A wants to achieve an allocation where its rank r' agents are of type higher than agents of rank r in organization B then it must offer $S_A(r') \geq S_B(r) - P(T^*)$. Since S_B^* is linear for $r \geq r^*$, the minimization problem (3.1) with $T = T^*$ and $S_B = S_B^*$, admits multiple solutions. For example, for each $r^* \leq r^1 \leq T^* \leq r^1$ the allocation

$$t(r) = \begin{cases} 1 & \text{if } r \leq r^1; \\ (T^* - r^1)/(r^0 - r^1) & \text{if } r^1 < T^* < r^0; \\ 0 & \text{if } r \geq r^0; \end{cases} \tag{4.1}$$

solves (3.1). Moreover, for any such allocation function t given by (4.1), $S_A^t(r)$ is equal to $S_B^*(r^1) - P(T^*)$ for $r < 1 - (T^* - r^1)/(r^0 - r^1)$, and equal to $S_B^*(r^0) - P(T^*)$ for $r \geq 1 - (T^* - r^1)/(r^0 - r^1)$. We have

$$\begin{aligned} \int_0^1 S_A^t(r)\,\mathrm{d}r &= \left(1 - \frac{T^* - r^1}{r^0 - r^1}\right)\left(S_B^*(r^1) - P(T^*)\right) + \frac{T^* - r^1}{r^0 - r^1}\left(S_B^*(r^0) - P(T^*)\right) \\ &= \frac{2(Y - P(T^*)(1 - r^*))}{(1 - r^*)^2}\left(\left(1 - \frac{T^* - r^1}{r^0 - r^1}\right)(r^1 - r^*) + \frac{T^* - r^1}{r^0 - r^1}(r^0 - r^*)\right) \\ &= C(T^*; S_B^*) \\ &= Y. \end{aligned}$$

This implies that T^* is a sorting equilibrium for (S_A^t, S_B^*), and S_A^t is a best response to S_B^*, because by the definition of S_B^* we have $T_A(S_A, S_B^*) \leq T^*$ for all resource distribution schedules S_A that satisfy the resource constraint.

The above strategy profile (S_A^t, S_B^*), however, is not a Nash equilibrium. By construction, S_A^t only assumes two values in the interval $[0, S_B^*(1) - P(T^*)]$ hence, by Lemma 4.1, S_B^* is not a best response to S_A^t. However, the space of A's best responses to S_B^* is not limited to resource distribution schedules of the type described above. In fact, as shown by the next lemma, any S_A that exhausts the resource constraint and such that $S_A(1) \leq S_B^*(1) - P(T^*)$ is a best response to S_B^*.

Lemma 4.2. *Let S_A be a resource distribution schedule such that $S_A(1) \leq S_B^*(1) - P(T^*)$ and $\int_0^1 S_A(r)\ dr = Y$. Then S_A is a best response to S_B^*.*

Proof. To prove the claim it is sufficient to show that T^* is a sorting equilibrium given (S_A, S_B^*). By the definition of $\bar{t}^{T^*}$, for a sequence of break points $(r^0, r^1, \ldots, r^k)$ such that S_A^{-1} is defined on each interval (r^j, r^{j+1}), $j = 0, \ldots, k-1$, we can write

$$\int_0^1 \bar{t}^{T^*}\ \mathrm{d}r = r^k - \sum_{j=0}^{k-1} \int_{r^j}^{r^{j+1}} S_A^{-1}(S_B^*(r) - P(T^*))\ \mathrm{d}r,$$

where r^0 is the smallest r that receives strictly positive resources and r^k is the largest rank that receive an amount of resources greater than $S_B^*(1) - P(T^*)$. After a change of variable $\tilde{r} = S_A^{-1}((S_B^*(r) - P(T^*))$ and integration by part we can rewrite the right-hand-side as

$$r^k - \frac{1}{S_B'^*}\left(S_A(1) - \int_{r^0}^1 S_A(r)\ \mathrm{d}r\right).$$

Using the assumption that S_A exhausts the resource constraint noting that

$$S_B^{'*}(1 - r^k) + S_A(1) = S_B^*(1) - P(T^*),$$

we can rewrite the integral of $\bar{t}^{T^*}$ as

$$1 - \frac{1}{S_B'^*}(S_B^*(1) - P(T) - Y).$$

Using the equation $C(T^*; S_B^*) = Y$ we can verify that the above expression is equal to T^*. This establishes that T^* is a fixed point of the mapping (2.2) and hence T^* is a sorting equilibrium for (S_A, S_B^*). *Q.E.D.*

Now we are ready to establish that a Nash equilibrium exists in the resource distribution game. We have already anticipated that the proof of the next result is by construction. In particular, with an argument similar to that in the proof of Lemma 4.2, we can show that S_B^* is a best response to a resource distribution schedule S_A^* which is 0 up to some rank $\hat{r} \in [0, 1]$ and has a constant slope equal to $S_B^*(1)$ above $\hat{r}$, such that $S_A^*(\hat{r}) = 0$ and $S_A^*(1) + P(T^*) = S_B^*(1)$. The proof that (S_A^*, S_B^*) is a Nash equilibrium is then completed by verifying that S_A^* exhausts the resource budget Y.

PROPOSITION 4.3. *Let $\hat{r} = P(T^*)/S_B^*(1)$ and S_A^* be defined by*

$$S_A^*(r) = \begin{cases} 0 & \text{if } r < \hat{r}; \\ (r - \hat{r}) S_B^*(1) & \text{if } r \geq \hat{r}. \end{cases}$$

The strategy profile (S_A^*, S_B^*) is a Nash equilibrium of the resource distribution game.

PROOF. We first verify that $\int_0^1 S_A^*(r)\ \mathrm{d}r = Y$ By definition we have

$$\int_0^1 S_A^*(r)\ \mathrm{d}r - Y = \frac{(S_B^*(1) - P(T^*))^2}{2S_B(1)^*}.$$

Substituting $C(T^*) = Y$,

$$S_B^*(1) = \frac{2Y}{1 - r^*} - P(T^*),$$

and

$$P(T^*) = \frac{2}{1 - r^*} - \frac{1}{1 - T^*}$$

from equation (3.9), we can verify that the integral of $S_A^*(r)$ is equal to Y.

Having established that S_A^* respects the resource constraint we note that Lemma 4.2 and the fact that S_B^* is the minmax strategy imply $T_A(S_A^*, S_B^*) = T^*$. Thus, to prove that S_B^* is a best response to S_A^*, it is sufficient to verify that, given S_A^*, for any $S_B \in \mathcal{S}$ there is a sorting equilibrium with $T \geq T^*$. Given S_A^* and S_B, from the definition of $\bar{t}^{T^*}$ we have

$$\int_0^1 \bar{t}^{T^*}(r)\ \mathrm{d}r = 1 - \int_{r^0}^1 {S_A^*}^{-1}(S_B(r) - P(T^*))\ \mathrm{d}r$$

where r^0 is the lowest rank in B that receives more resources than $P(T^*)$. By definition,

$$S_A^{*\ -1}(S_B(r) - P(T^*)) = \frac{S_B(r)}{S_B^*(1)}.$$

Using the above expression, the resource constraint $\int_0^1 S_B(r)\ \mathrm{d}r \leq Y$ and the definition of $S_B^*(1)$, we have

$$\int_0^1 \bar{t}^{T^*}(r)\ \mathrm{d}r \geq 1 - \frac{Y}{S_B^*(1)} = T^*.$$

Thus, $D_A(T^*) \geq T^*$ and D_A has at least one fixed point greater than T^*. *Q.E.D.*

The intuition behind the above result is the same as Lemma 3.3. By using a linear resource distribution schedule for ranks that receive positive resources, organization A makes every rank equally costly for B to target. No change to S_B^* can improve the equilibrium quality difference for B. Unlike the equilibrium resource distribution schedule S_B^*, there is no discontinuity for S_A^* because A does not need to pay a quality premium to be competitive. This also means that the slope of the positive part of S_A^* is smaller than that of S_B^*.

4.2. Uniqueness

To establish (S_A^*, S_B^*) as the unique Nash equilibrium of the resource distribution game we will argue that S_B^* is not a best response to any other resource distribution schedule S_A which is a best response to S_B^*. In proving this claim, by Lemma 4.1, we need only consider resource distribution schedules S_A's that are continuous and for which $S_A(0) = 0$ and $S_A(1) = S_B^*(1) - P(T^*)$. Unfortunately, we cannot restrict further the set of candidate Nash equilibrium strategies for A, since by Lemma 4.2 all feasible resource distribution schedules that respect these three properties are indeed best responses to S_B^*. It is difficult to characterize B's best responses to an arbitrary strategy S_A. Instead, in the proof of the next proposition we establish that S_B^* is not a best response to $S_A \neq S_A^*$ by showing that an appropriately constructed "small" modification of S_B^* improves B's payoff. The proof considers only S_A's that are strictly increasing when positive, as it is straightforward to show that this is necessary for S_B^* to be a best response.

PROPOSITION 4.4. *The strategy profile (S_A^*, S_B^*) is the only Nash equilibrium of the resource distribution game.*

PROOF. Let S_A be a resource distribution schedule which is strictly increasing when positive, and which satisfies $S_A(0) = 0$, $S_A(1) = S_B^*(1) - P(T^*)$, and $\int_0^1 S_A(r)\,\mathrm{d}r = Y$. We claim that if $S_A(r) + P(T^*) < rS_B^*(1)$ for some $r \in (0,1)$, then S_B^* is not a best response to S_A. Note that this claim is sufficient for the statement of the proposition because, by construction, S_A^* is the pointwise smallest positive function for which the opposite inequality holds for all r, and because S_A^* exhausts the resource budget, a property that all best responses to S_B^* satisfy.

Given S_A let $\overline{S}_A$ be the pointwise largest linear function with the property that $\overline{S}_A(r) \leq S_A(r) + P(T^*)$ for all r, and let $\tilde{r} = \sup\{r \in [0,1] : \overline{S}_A(r) = S_A(r) + P(T^*)\}$. We distinguish between two cases. In the first case, we have $\overline{S}_A(\tilde{r}) > P(T^*)$. Let $r^0 = S_B^{*-1}(\overline{S}_A(\tilde{r}))$. Note that $r^0 > r^*$. For each $\epsilon > 0$ we construct the following resource distribution schedule S_B^ϵ

$$S_B^\epsilon(r) = \begin{cases} S_B^*(r) & \text{if } r \notin (r^0 - \epsilon, r^0 + \epsilon); \\ S_B^*(r^0) & \text{if } r \in (r^0 - \epsilon, r^0 + \epsilon). \end{cases}$$

For all $\epsilon \leq r^0 - r^*$, S_B^ϵ respects the resource constraint. Note that for each T, the allocation function $\overline{t}^T(r; S_A, S_B^\epsilon)$ is given by

$$\overline{t}^T(r; S_A, S_B^\epsilon) = \begin{cases} \overline{t}^T(r; S_A, S_B^*) & \text{if } r \notin (r^0 - \epsilon, r^0 + \epsilon); \\ 1 - S_A^{-1}(S_B^*(r^0) - P(T)) & \text{if otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} D_A(T;&S_A, S_B^*) - D_A(T; S_A, S_B^\epsilon) = \\ &\int_{r^0-\epsilon}^{r^0+\epsilon} S_A^{-1}(S_B^*(r^0) - P(T))\,\mathrm{d}r - \int_{r^0-\epsilon}^{r^0+\epsilon} S_A^{-1}(S_B^*(r) - P(T))\,\mathrm{d}r. \end{aligned}$$

At $T = T^*$, the first term on the right-hand-side of the above equation equals $2\epsilon\tilde{r}$. To evaluate the second term, note that by definition of $\overline{S}_A$, for all $r \in [0,1]$ we have

$$S_A^{-1}(S_B^*(r) - P(T^*)) \leq \overline{S}_A^{-1}(S_B^*(r)),$$

with strict inequality for all $r > r^0$. Hence we have

$$\begin{aligned} D_A(T; S_A, S_B^*) - D_A(T; S_A, S_B^\epsilon) &> 2\epsilon\tilde{r} - \int_{r^0-\epsilon}^{r^0+\epsilon} \overline{S}_A^{-1}(S_B^*(r))\, \mathrm{d}r \\ &= 2\epsilon\tilde{r} - \int_{r^0-\epsilon}^{r^0+\epsilon} \frac{S_B^*(r)}{K}\, \mathrm{d}r \\ &= 2\epsilon\tilde{r} - \int_{r^0-\epsilon}^{r^0+\epsilon} \frac{1}{K}(S_B^*(r^0) + \beta(r - r^0))\, \mathrm{d}r \\ &= 0, \end{aligned}$$

where the second line follows from $\overline{S}_A$ being linear with some positive slope K, and the third from the fact that S_B^* has constant slope β for $r \geq r^*$. The last line then obtains because $\overline{S}_A(\tilde{r}) = S_B^*(r^0)$.

The following properties of $D(\cdot; S_A, S_B^\epsilon)$ can also be established: i) $D_A(\cdot; S_A, S_B^\epsilon)$ converges uniformly to $D_A(\cdot; S_A, S_B^*)$ as ϵ becomes small; and ii) $D'_A(\cdot; S_A, S_B^\epsilon)$ converges uniformly to $D'_A(\cdot; S_A, S_B^*)$ as ϵ becomes small. Using property ii), the fact that $D'_A(T^*; S_A, S_B^*) > 0$ and the continuity of $D'_A(\cdot; S_A, S_B^*)$, we can establish that for sufficiently small positive γ, we have $D'_A(T; S_A, S_B^\epsilon) \geq 0$ for $T \in (T^*, T^* + \gamma)$ and for all sufficiently small ϵ. Hence $D_A(T; S_A; S_B^\epsilon) > T$ for all $T \in [T^*, T^* + \gamma)$ for all ϵ sufficiently small. Property i) and $D_A(T; S_A, S_B^*) > T$ for all $T \in [T^* + \gamma, 1]$ also imply $D_A(T; S_A, S_B^\epsilon) > T$ for T in the same range. Hence $T_A(S_A, S_B^\epsilon) < T^*$ for ϵ sufficiently small and S_B^* is not a best response to S_A.

In the second case, we have $\overline{S}_A(\tilde{r}) = P(T^*)$. Then, there exist r^0 and r^1, with $r^0 < r^1$, such that $S_A(r^0) = S_A^*(r^0)$, $S_A(r^1) = S_A^*(r^1)$ and $S_A(r) > S_A^*(r)$ for all $r \in (r^0, r^1)$. An argument similar to the one for the first case can be used to show that a resource distribution schedule S_B^ϵ which reduces the amount distributed to ranks just above $S_B^{*-1}(S_A(r^0) + P(T^*))$ and increases the amount of resources to ranks just below $S_B^{*-1}(S_A(r^0) + P(T^*))$ does better than S_B^* against S_A. *Q.E.D.*

The main difficulty in the above result is that, to show that organization B can improve the quality difference T^* in its favor we must check two conditions. First, there is a modification of S_B^* for which T^* is no longer a sorting equilibrium. Second, the modified resource distribution schedule does not generate a sorting equilibrium with a quality difference strictly larger than T^*. This is why it is not enough to identify the target ranks in A to attack; a careful construction of the local modification of S_B^* is necessary.

5. Discussion

Comparative statics analysis for the unique Nash equilibrium in the resource distribution game is straightforward. When the organizations have a greater budget for resource distribution, or when the peer effect becomes less important in the talents' utility function, the equilibrium exhibits a smaller disparity between the dominant and the inferior organizations. The critical ranks in both organizations fall, implying that there is more mixing at the top of the talent pool, as a result of more intense competition between the two organizations. Both resource distribution schedules become flatter, with the inferior organization getting a greater share of the top talents.

In our model of organizational competition for talents, we have assumed that there is a fixed budget of resources for each organization. We view this as a reasonable approximation of competition in the short term before production by the members generates any impact on available resources. Another interpretation is that organizations we model are not-for-profit, so that the objective of the organization is not to maximize the profit in terms of the difference between total output and the resources expended to attract productive members, but is instead to use the fixed resources to attract the best average quality, as in the present paper.

In our model organizations have a fixed capacity of half of the talent pool and must fill all positions. In particular, an organization cannot try to improve its average talent by rejecting low types even though the capacity is not filled. We have made this assumption in order to circumvent the issue of size effect, and focus on implications of sorting of talents. Alternatively, we can justify the assumption of fixed capacity if the peer effect enters the preference of talents in the form of total output (measured by the sum of individual types) as opposed to the average type, and the objective of the organization is to maximize the total output. Since all agents contribute positively to the total output, in this alternative model all positions will be filled.

We have restricted organization strategies to meritocratic resource distribution schedules. This is a natural assumption given how we model the sorting of talents after organizations choose their schedules. Non-meritocratic resource distribution schedules would create incentives for talented agents to "dispose of" their talent. Another assumption we have made about organization strategies is that resource distributions do not depend on type directly. This is a reasonable assumption in the presence of the resource constraint; a resource distribution schedule that depends directly on type might exceed the resources available or leave some resources unused depending on the distribution of types that join the organization.

Our main results of linear resource distribution schedules rely on the assumption of uniform type distribution. The uniformity assumption implies the impact on the quality difference of an exchange of one interval of types for another interval between the two organizations depends only on the difference in the average types of the two intervals. This property allows us to transform the minmax problem in resource distribution functions to a linear programming problem in allocation functions. We leave the question of whether the method we develop in this paper is applicable to more general type distributions to future research.

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