First and Second Price Mechanisms in Procurement and other Asymmetric Auctions*

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First Draft: February, 2008
This Draft: May, 2008

Abstract

In many settings, with procurement a key example, one seller may be prefered, based on better reliability or quality, or may have lower costs. We study the performance of simple first and second price mechanisms in such setting, and how they relate to the optimal mechanism. Under a variety of conditions on cost distributions, a second price auction with bonuses outperforms, on an outcome by outcome basis, any of a class of first price auctions that includes on one extreme a request for proposals and on the other a standard first price auction.

Keywords: Procurement, Asymmetric Auctions, Request for Proposal, Differentiation, Mechanism Design, First Price Auctions, Second Price Auctions

JEL Codes: D44, D82, L24, M21

1 Introduction

Consider a buyer with potential suppliers 1 and 2. Each supplier has privately known costs c_i, drawn independently from F. The buyer places known value v on procuring from 2, but value $v + \Delta$, $\Delta > 0$, on procuring from 1.[1] The difference Δ could reflect higher quality from 1, more reliable deliv-

*We thank seminar audiences at UT Austin, Washington University in St. Louis, and Nuffield College, Oxford. We are grateful to Paul Klemperer and Ken Hendricks for helpful suggestions, and to Lei Gao for excellent research assistance.

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[1] So, we depart from the setting of Manelli and Vincent (1995) in which low costs suggest low quality.

ery, or a premium that the buyer places on encouraging 1 to remain in the industry.

How should one procure in this setting? A common mechanism is the Request for Proposals (RFP). In the simplest RFP, each supplier submits a proposal as to what will be provided, and at what price. The supplier chooses his favorite proposal at the stated price. In our setting, this reduces to a first price auction, but one in which 2 wins only if his bid is at least Δ below that of 1.[2]

An alternative is to run a standard sealed bid first price auction: despite his preference for 1, the buyer commits to taking the lowest bid. The RFP process allows the buyer to choose his favorite proposal, and so might be more efficient than a standard auction. But, the standard auction makes the market more competitive, since it converts the market from the point of view of the sellers into one with homogeneous products. While simple, this is a potential benefit of auctions that is generally overlooked.[3]

A natural question is whether some intermediate form might be preferred. If the incremental value of buying from 1 is $\Delta = \$100,000$, might it be optimal to commit to act as if Δ were some intermediate number? In practice, auctions very close to this, are also used: firms like Boeing use a first price mechanism, but choose the winning bidder based on price augmented by scores on dimensions such as reliability and technological capability. These scores are private to Boeing, but their total possible impact on the bid is known. So, if Boeing prefers one supplier over the other and the bids are close, they can chose their favorite. But, when bids are sufficiently far apart relative to the announced scoring rule, they are stuck with the low bidder, regardless of which bid is ex-post more attractive.

To explore this, let the *first price handicap auction* (FPHA) with handicap A be such that 1 wins when $b_1 < b_2 + A$, and 2 otherwise. The winning

[2] In some settings, an open RFP is used, with multiple rounds of offers. An example is the 2003 competition between Boeing and Airbus to sell plane to Iberia Airlines. If Δ is common knowledge, this is equivalent to a second price auction with bonus Δ paid to 1, since he needs only match the costs of 2 plus Δ to be picked.

[3] It has been observed that public agencies such as the Tennessee Valley Authority paid more on average for electrical turbines than did private utilities. This is attributed to the power of information in sustaining tacit collusion in a repeated game, since the public utilities ran public auctions while pricing to the private utilities remained secret. But, the result of lower prices to the private utilities is more surprising than commonly understood: The auction format forced the public agencies to act as if the product was homogeneous. In contrast, pricing to the private utilities was done in a setting where the utilities paid attention to their preferences between manufacturers. There was thus a powerful force in the direction of *higher* pricing to the private utilities.

bidder receives his bid. The case $A = 0$ is the standard first price auction. The case $A = \Delta$ is, in our setting, equivalent to a RFP, since the buyer will then always choose the bid which is ex-post most attractive. As might be expected, however, it will turn out that the optimal A is intermediate between 0 and Δ, trading off efficiency and the amount of competition generated.

These auctions have an odd feature. Imagine that costs have support $[0, 1]$. Then, $\beta_1(0) = \beta_2(0) + A$, where β_i is the bid function of i.[4] So, when costs are low, all of the allocative effects of the handicap format are undone in equilibrium. On the other hand, we will also show that when 2 has costs above $1 - A$, 1 wins for sure. So, in equilibrium, the FPHA creates lots of distortion away from the symmetric case when costs are high, but very little when costs are low.

Let us now consider a simple second price mechanism. A bonus B is specified. Each supplier submits a bid, and the low bidder wins. If 2 wins, he receives b_1. If 1 wins, he receives $b_2 + A$. The use of bonuses of this form is actually fairly prevalent: one can write, as part of the rules for a a second price (or open) auction, who is responsible for various costs associated with the work, such as specialized tooling. As such, one can have the extra costs involved with a new supplier competing against an old be born by the buyer, the seller, or some combination.

Our main result is this. For a large class of cost distributions, the second price mechanism dominates the first. For any first price mechanism, with handicap A_{FP}, there is a second price mechanism, with A_{SP}, such that, for each (c_1, c_2), the outcome achieved by the second price mechanism is never worse, and sometimes better, than that of the first price mechanism. The key to this result is that the second price mechanism creates a more even distortion away from a symmetric mechanism, which is typically more efficient than having low distortion at low costs, but high distortion at high costs.[5]

There are indications in our analysis that even in settings where our key condition is not satisfied, the second price mechanism will continue to outperform the first price. This is based on numerical analysis of a (fairly large) class of examples. We hope to understand this better in future work. The proof of the result relies on techniques that we have not seen before and believe will have broader use in understanding asymmetric first price

[4] A bid by 1 below $\beta_2(0) + A$ or by 2 below $\beta_1(0) - A$ can be raised a little and still win for sure, and so is not optimal.

[5] Another advantage of the second price mechanism is that it remains trivial to bid in even with bonuses.

auctions.[6]

Section 2 discusses the literature. Section 3 presents the model. Section 4 derives the optimal mechanism. Section 5 examines second price mechanisms, and Section 6 discusses first price mechanisms. Section 7 looks at the choice between mechanisms. Section 8 discusses the robustness of our central results. Section 9 concludes. The appendix contains less central proofs.

2 Related Literature

Myerson (1981) begins a long discussion of implementation with asymmetric cost distributions. McAffee and McMillan (1988) look at optimal mechanisms with asymmetric cost distributions and argue that one doesn't want to always buy from the low-cost bidder.

In Che (1993), suppliers have different costs and can provide goods of different qualities. Suppliers submit a bid (p, q) which is evaluated via a quasi-linear scoring rule $S(p, q)$. Under the *first score rule* the high scorer executes the submitted bid. Under the *second score rule* the high scorer executes a contract equivalent in score and cost to the next highest bid. This transforms a multidimensional problem into a standard asymmetric cost environment. The optimal scoring rule distorts quality downward, and can be implemented by either the first or second score rules. Branco (1997) adds common value aspects and correlation to costs. Asker and Cantillion (2006) expand the results to multi-dimensional quality.

Shachat and Swarthout (2003) consider a model somewhat similar to ours, with a uniform distributions over costs and over qualities. Their main result (in line with Che (1993)) is that the optimal English auction (theoretically equivalent to a second price mechanism) in which the auctioneer observes qualities and sets a bonus favors bidder 1 by more than 0 but less that Δ. We show that this feature generalizes beyond their example. They provide experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid RFP setting (in a setting where bidders are aware of their own quality but not that of their opponent).

Our results complement Maskin and Riley (2000), who also study "structured asymmetry" in interesting ways. We discuss this in Section 7.1.

[6] Our setting is isomorphic to one with a homogeneous good, but where the distribution over c_2 is a Δ shift upward of the distribution over c_1. We are hopeful that our techniques will allow a better understanding of other asymmtric first price auctions.

3 Model

A buyer faces sellers 1 and 2 with costs c_1 and c_2. Costs are *i.i.d.* from F, where F has density f which is log-concave and continuously differentiable on $[0, 1]$. The reverse cumulative is $\bar{F} = 1 - F$. We assume that $\frac{f}{\bar{F}}$ is strictly increasing. This is very mild.[7]

The buyer's utility from purchasing from i is

$$U_B(p, i) = v_i - p,$$

where p is the transaction price. Let $\Delta = v_1 - v_2$ be the amount by which 1 is preferred to 2. We assume Δ is common knowledge and, without loss of generality, $\Delta \geq 0$.[8]

For simplicity, we focus our analysis on how to allocate the contract between 1 and 2, setting aside when it is better to not buy at all. This is optimal if the buyer's outside option, v_0, is sufficiently negative. In what follows, Δ, rather than the absolute levels of v_1 and v_2, will be central.

4 Optimal Mechanisms

Let

$$\omega(c_i) = c_i + \frac{F(c_i)}{f(c_i)}$$

be the virtual cost of i. Because F is log-concave, $\omega'(\cdot) \geq 1$. Let the *Myerson Difference* be

$$\begin{aligned} \eta(c_1, c_2) &= \Delta - (c_1 - c_2) - \left(\frac{F(c_1)}{f(c_1)} - \frac{F(c_2)}{f(c_2)} \right) \\ &= \Delta - (\omega(c_1) - \omega(c_2)). \end{aligned} \tag{1}$$

Since $\Delta = v_1 - v_2$, η is the value difference between 1 and 2, minus the virtual cost difference between 1 and 2. The next Lemma collects some observations about η.

[7] By log-concavity, $\frac{f}{\bar{F}}$ is weakly increasing (see Lemma 50). If $\frac{f}{\bar{F}} = \gamma \geq 0$ on $[a, b]$, then $\bar{F}(c) = \bar{F}(a)e^{-\gamma c}$ for $c \in [a, b]$. Since $\bar{F}(a) > 0$, $\bar{F}(a)e^{-\gamma c}$ does not go to 0 for any finite c. Hence, what we are ruling out is that the distribution has a segment of an exponential distribution "patched" into it.

[8] Except in drawing the close connection between the *RFP* and various bonus auctions, it is irrelevant whether the sellers know Δ.

Lemma 1

$$\begin{array}{ll} \eta_{c_1}(c_1,c_2) = -\omega'(c_1) \leq -1 & \forall (c_1,c_2) \\ \eta_{c_2}(c_1,c_2) = \omega'(c_2) \geq 1 & \forall (c_1,c_2) \\ \eta(c,c) = \Delta \geq 0 & \forall c \\ \eta(c+\Delta,c) = \frac{F(c)}{f(c)} - \frac{F(c+\Delta)}{f(c+\Delta)} \leq 0 & \forall c \end{array}.$$

Since we assume that the buyer always buys, any mechanism can be represented by $\gamma(\cdot,\cdot)$, where $\gamma(c_1,c_2)$ is the probability that 1 gets the job for given (c_1,c_2). Following Myerson (1981), γ ties down the entire mechanism under the condition that the highest cost types of 1 and 2 receive zero surplus.

Lemma 2 *The buyer's surplus, $BS(\gamma)$ is*

$$BS(\gamma) = v_2 - 1 + \int\int \gamma(c_1,c_2)\,\eta(c_1,c_2)\,f(c_1)\,f(c_2)\,dc_1 dc_2. \tag{2}$$

This is intuitive. Always buying from 2 gives the buyer surplus $v_2 - 1$, since 2 must receive 1 if he is to sell for all c_2. The second term represents the change in buyer surplus from buying from 1 according to γ.

From Lemma 2, it follows directly that

Corollary 3 *In the optimal mechanism,* 1 *wins if*

$$\Delta > \omega(c_1) - \omega(c_2) \tag{3}$$

and 2 *wins otherwise.*[9]

Thus, 1 wins if his virtual cost is no more than Δ above 2's. This follows since by (3), $\gamma(c_1,c_2)\,\eta(c_1,c_2)$ is maximized point-wise. Since ω is increasing, the allocation rule is monotone and hence incentive compatible.

Define the Myerson Line, $\phi_M(\cdot)$, by

$$\Delta = \omega(\phi_M(c_2)) - \omega(c_2),^{10} \tag{4}$$

or, equivalently, by

$$\eta(\phi_M(c_2), c_2) = 0. \tag{5}$$

By Lemma 1, 1 wins for $c_1 < \phi_M(c_2)$ and 2 wins for $c_1 > \phi_M(c_2)$.

Example 4 *Fig. 1 shows $\phi_{M,1}$ (the heavier line) for $f_1(c) = 6c(1-c)$ and $\phi_{M,2}$ (the lighter line) for $f_2(c) = \frac{2(1+c)}{3}$.*

[9] We do not specify what happens in the zero probability event of a tie.If $v_0 < v_1 - \omega(1)$ then it is indeed optimal to always buy.

[10] This is well defined since ω is monotonic.

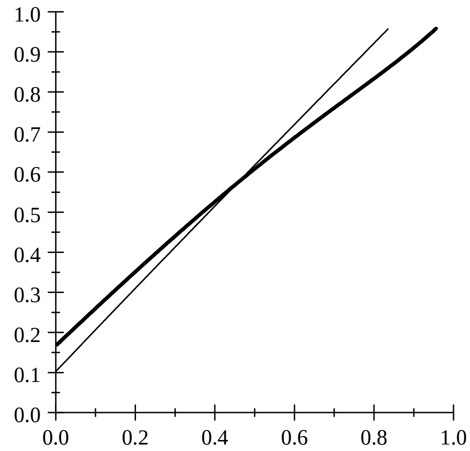

Figure 1: Myerson lines for $f_1(c) = 6c(1-c)$ (thin) and $f_2(c) = \frac{2(1+c)}{3}$ (thick) for $\Delta = .2$

4.1 Properties of the Myerson Line

The geometric properties of the Myerson line are at the core of our basic results. First, we have

Lemma 5 *Fix F. If $\Delta = 0$, then $\phi_M(c_2) = c_2$ for all c_2. If $\Delta > 0$, then for all c_2, $c_2 < \phi_M(c_2) < c_2 + \Delta$ and $\phi_M'(c_2) > 0$.*

So, the Myerson line lies above the diagonal but by less than Δ, and has positive slope. Each of these is trivial from (5) and Lemma 1. So is the following:

Lemma 6 *Fix F. Let $\phi_{M,\Delta}(\cdot)$ be the Myerson Line for given Δ. Then, for all c_2, $\frac{\partial}{\partial \Delta}\phi_{M,\Delta}(c_2) > 0$.*

Thus, higher Δ parameters lead to uniformly more distortion away from the symmetric allocation.

Next, we have that ϕ_M moves more slowly than Δ.

Lemma 7 *$\frac{\partial}{\partial \Delta}\phi_{M,\Delta}(c_2) \leq 1$.*

Key to our central result is to relate ϕ_M' to the concavity or convexity of $\frac{F}{f}$.

Lemma 8 *If* $\frac{F}{f}$ *is convex, then* $\phi'_M \leq 1$*. If* $\frac{F}{f}$ *is concave then* $\phi'_M \geq 1$*.*

The class of log-concave distributions contains many examples of both convex and concave virtual costs:

Example 9 *Both* $f_1(c) = 6c(1-c)$ *and* $f_2(c) = \frac{2(1+c)}{3}$ *are log-concave but* $\frac{F_1}{f_1}$ *is convex while* $\frac{F_2}{f_2}$ *is concave. As seen in Fig. 1,* $\phi'_{M,1} < 1$ *while* $\phi'_{M,2} > 1$*.*

Lemma 10 *If* f *is decreasing then* $\frac{F}{f}$ *is convex.*

This condition is by no means necessary.

Example 11 *Take* $f(c) = ce^{-c}$*. Then,* f *is increasing, but* $\frac{F}{f}$ *is convex.*

For one nice class of distributions, the Myerson line is particularly simple:

Example 12 *Consider the power distributions* $F(c) = c^{\alpha}$*. These are log-concave. Virtual costs are*

$$\omega(c) = c + \frac{F(c)}{f(c)} = c + \frac{c^{\alpha}}{\alpha c^{\alpha-1}} = c\left(1 + \frac{1}{\alpha}\right),$$

and thus linear. So,

$$\begin{aligned} \Delta &\geq \omega(c_1) - \omega(c_2) \\ &\text{iff} \\ c_1 - c_2 &\leq \frac{\Delta}{\left(1+\frac{1}{\alpha}\right)}, \end{aligned}$$

and the Myerson line is parallel to, and $\frac{\Delta}{(1+\frac{1}{\alpha})}$ *above, the diagonal. As* α *increases,* $\frac{\Delta}{(1+\frac{1}{\alpha})}$ *increases, and as* $\alpha \to \infty$*,* F *becomes highly convex and* $\frac{\Delta}{(1+\frac{1}{\alpha})} \to \Delta$*. As* $\alpha \to 0$*,* F *becomes very concave and* $\frac{\Delta}{(1+\frac{1}{\alpha})} \to 0$*. For the uniform distribution (*$\alpha = 1$*),* $\frac{\Delta}{(1+\frac{1}{\alpha})} = \frac{\Delta}{2}$*.*

This idea - that as the cost distribution deteriorates by becoming more convex, the allocation moves towards 1 - is quite general.

Lemma 13 *Assume that*

$$\frac{F_1}{F_2}$$

is log-concave.[11] *Then for any* $\Delta > 0$*,* ϕ_{M,F_1} *is everywhere above* ϕ_{M,F_2}*.*

[11] This is a strong form of second order stochastic dominance (see Whitt (1985))

To interpret this, consider log-concave H_1 and H_2. The maximum of the two original costs has distribution $H_1 H_2$, which is more convex than either H_1 or H_2. Trivially, Lemma 13 applies to $F_1 = H_1 H_2$ and F_2 equal to either H_1 or H_2. Thus, $\phi_{M,H_1 H_2}$ is everywhere above ϕ_{M,H_1} or ϕ_{M,H_2}. When the costs deteriorate in this way, the optimal mechanism favors 1 more.[12]

4.2 Two Implementations of the Optimal Allocation

Fix Δ. To implement the optimal allocation through a sealed bid mechanism, have 1 and 2 submit bids b_1 and b_2. The low bidder wins. If 2 wins, he is paid b_1. If 1 wins, he is paid $\phi_M(b_2)$. It is weakly dominant for 2 to bid $b_2 = c_2$, and for 1 to bid $b_1 = \phi_M^{-1}(c_1)$, since he then wins whenever $b_2 > b_1 = \phi_M^{-1}(c_1)$, and so

$$\phi_M(b_2) > \phi_M\left(\phi_M^{-1}(c_1)\right) = c_1.$$

Thus, 1 wins whenever $\phi_M^{-1}(c_1) < c_2$ or $c_1 < \phi_M(c_2)$.[13]

Similarly, let a clock display a descending price τ which starts at $1 - A$ and stops at $\hat{\tau}$ when one player drops out. If 2 wins, he receives $\hat{\tau}$, while if 1 wins, he receives $\phi_M(\hat{\tau})$. It is again weakly dominant for 2 to drop out at c_2 and for 1 to drop out when $c_1 = \phi_M(\tau)$ (recalling again that ϕ_M is increasing). The auction will end at $\hat{\tau}$ with 1 as the winner iff $c_2 = \hat{\tau}$, but $c_1 < \phi_M(\hat{\tau})$. This is optimal by definition of ϕ_M.

One can think of $\phi_M(\tau) - \tau$ as a bonus received by 1 over c_2. By Lemma 8, $\phi_M(\tau) - \tau$ is increasing in c_2 if $\frac{F}{f}$ is convex and decreasing in c_2 if $\frac{F}{f}$ is concave.

Example 14 *From Example 12, in the uniform case the optimal auction can be implemented with constant bonus* $\Delta/2$*. The clock starts at* $1 - \Delta/2$*. When* $\Delta \geq 2$*, this reduces to dealing with* 1 *at a price of* 1.

5 Second Price Mechanisms

For general distributions, the optimal mechanism will specify bonuses that are complicated functions of revealed costs. We rarely see such mechanisms

[12] To think about local variations in F, let $\hat{H}$ be log-concave, and form H by sampling from $\hat{H}$ with probability δ and setting $c = 0$ with probability $1 - \delta$. H is log concave, and FH reflects a process where $1 - \delta$ of the time, one stays with F, but δ of the time, a draw is taken from H and costs are the maximum. The Myerson line moves upwards with δ.

[13] Optimality follows since the surplus of 1 with $c_1 = 1$ is 0, and similarly for 2 with $c_2 = 1 - A$.

in practice. In this section, we focus on simple second price mechanisms that allow the auctioneer to reflect his preferences. In the next section, we look at first price implementations of this type of auction. Such mechanisms are simple insofar as the distortion is not dependent on the actual bids.

One reason why such simplicity may be favored in practice is that complicated rules can be susceptible to interim strategic manipulation by the buyer. Another reason is that it may be either illegal or unpalatable to the bidders to write rules that explicitly favor one bidder as a function of the others costs, but easier to write rules of the form "the buyer will pay transportation costs" implicitly favoring one bidder, but in a coarser manner.

An example of a simple mechanism is a scoring rule, in which the allocation depends on a weighted sum of the submitted bids and other characteristics of the deal.[14] Another example of auctions with simpler bonus rules is various versions of the request for proposal process, which, because they lack commitment power, are based only on the ex-post attractiveness of the competing bids.

5.1 Second Price Auction Formats

In a *second price bonus auction* (SPBA) the auctioneer announces a bonus A, and requests sealed bids from 1 and 2. The low bidder wins.[15] When 1 wins, he receives $\min(b_2 + A, 1)$, while if 2 wins, he receives $\min(b_1, 1)$. Putting a maximum of 1 on payments guarantees that the bidders do not receive an amount above their highest possible cost.

For 2, it is weakly dominant to set $\beta_2(c_2) = c_2$, while for 1, it is weakly dominant to set $\beta_1(c_1) = c_1 - A$. Thus, using SPBA mechanisms, we can implement allocations of the form

$$A \geq c_1 - c_2.$$

In general, this allocation need not be optimal, but the SPBA has the advantage of being very simple to both explain and to bid in. Because $\beta_1(c_1) = c_1 - A$, 2 never wins when $c_2 > 1 - A$. Any $A > 1$ is thus equivalent to $A = 1$: 1 wins for sure, and does so at price 1, and so, without loss of generality, we restrict $A \leq 1$.

In the *clock (open) auction with a bonus* the auctioneer runs an open descending price mechanism where bidders choose when to drop out. The

[14] See Wolfstetter and Lengwiler (2006) and Asker and Cantillion (2006).

[15] Ties are zero probability in equilibrium and the tie breaking rule is inessential (see Jackson and Swinkels (2005)).

last active bidder wins. If 2 wins, he receives the prevailing price. If 1 wins, he receives the prevailing price plus a fixed bonus A.[16] To ensure that compensation does not exceed the highest possible cost we start the clock at a price of $1 - A$. As discussed, it is weakly dominant for 2 to drop out at c_2 and for 1 to drop out at $c_1 - A$. At the initial price of $1 - A$, at least 1 is active.

It is straightforward that for given A these two auctions are equivalent in allocation. These mechanisms will coincide with the Myerson mechanism only in the very special circumstance that $\phi_M(c) - c$ is a constant.

In an *open outcry auction with a handicap,* A is announced. Players successively submit bids until such point as no bidder is willing to improve his bid. The winner is 1 if at the final bids, $b_1 < b_2 + A$, and 2 otherwise. This also yields the same outcome as the $SPBA$.[17] One sees something very much like this in some RFP processes, in which there are multiple rounds in which bidders can improve their offer. If Δ is common knowledge, the outcome is the same as the SPBA with $A = \Delta$.[18]

Given the equivalence of these second price formats, we focus attention on the second price bonus auction.

5.2 Analysis of the Second Price Bonus Auction

Let us turn to the question of determining the optimal bonus $A^*(f, \Delta)$ as a function of the cost distribution f and the preference parameter Δ.

[16] Thus, each format differs from the optimal implementation only in that A does not depend on c_2.

[17] There is a small approximation since bids are improved discretely and the increments may be non-trivial because the bidders are trading off their expected payment against the hassle of prolonging the auction.

[18] It is easy to go wrong in designing a second price auction favoring 1. Consider a *second price handicap auction* in which 1 wins iff

$$b_1 \leq b_2 + A$$

and in which the winner receives the losing bid as compensation. It easy to check that

$$\beta_1(c_1) = c_1 + A, \beta_2(c_2) = c_2 - A,$$

so that 1 wins iff

$$c_1 + A \leq c_2 - A + A = c_2$$

or

$$c_1 - c_2 \leq -A,$$

opposite to what is intended. To get it right, one needs to use a mildly counterintuitive negative handicap.

Example 15 *Recall from Example 12 that for the power distributions* $F(c) = c^{\alpha}$*,*

$$\phi_M(c_2) = c_2 + \frac{\Delta}{\left(1 + \frac{1}{\alpha}\right)}.$$

Thus, the optimal SPBA is equivalent to the optimal mechanism, and has $A = \frac{\Delta}{\left(1+\frac{1}{\alpha}\right)}$*. For the uniform distribution (*$\alpha = 1$*),* $A = \frac{\Delta}{2}$*.*[19]

Since $\frac{\Delta}{\left(1+\frac{1}{\alpha}\right)} < \Delta$, the optimal bonus reflects only a portion of the buyer's value difference. We will show this to be general. We begin by deriving necessary conditions for $A^*(f, \Delta)$.

Lemma 16 *The total surplus generated by the SPBA with bonus A is*

$$TS = v_2 - E(c_2) + \Pr(c_1 - c_2 \leq A)\left(\Delta - E(c_1 - c_2 | c_1 - c_2 \leq A)\right).$$

The buyer's surplus is

$$\begin{aligned} BS &= v_2 - 1 + \Pr(c_1 - c_2 \leq A)\left(\Delta - E\left(\omega(c_1) - \omega(c_2)\,|c_1 - c_2 \leq A\right)\right) \\ &= v_2 - 1 + \int_0^1 \left(\int_0^{c_2+A} \eta(c_1, c_2)\, f(c_1)\, dc_1\right) f(c_2)\, dc_2. \end{aligned} \tag{6}$$

To see TS, note that $v_2 - E(c_2)$ is the surplus from always buying from 2, and that the second term reflect the probability and net benefit of instead buying from 1. BS follows immediately using Lemma (2) since $\gamma(c_1, c_2) = 1$ iff $c_1 < c_2 + A$.

Differentiating (6) and using Lemma 1 yields

Corollary 17

$$\frac{\partial BS}{\partial A} = \int_0^{1-A} \eta(c + A, c)\, f(c + A)\, f(c)\, dc. \tag{7}$$

If $\Delta > 0$*, then this is strictly positive at* $A = 0$*, and strictly negative for* $A > \Delta$*.*

Thus, we can restrict attention to $A \in [0, \Delta]$. Our next (surprisingly difficult) lemma shows that over this range profits are quasi-concave.

Lemma 18 *Buyer surplus is quasi-concave on* $A \in [0, \Delta]$*. Thus* $A^*(\Delta, f)$ *is characterized as the unique solution of*

$$\frac{\partial BS}{\partial A} = \int_0^{1-A} \eta(c + A, c)\, f(c + A)\, f(c)\, dc = 0. \tag{8}$$

[19] See also McAffee and McMillan (1989.)

5.3 Relating the Second Price and Optimal Mechanisms.

For given A, let

$$\phi_{SP}(c_2) = c_2 + A.$$

Much like ϕ_M, ϕ_{SP} divides the set of (c_1, c_2) so that when $c_1 < \phi_{SP}(c_2)$, 1 wins, and when $c_1 > \phi_{SP}(c_2)$, 2 wins.

Recall that ϕ_M is defined by $\eta(\phi_M(c_2), c_2) = 0$. Let A^* and ϕ_{SP^*} characterize the optimal second price auction. If ϕ_{SP^*} and ϕ_M do not cross, then by Lemma 1, $\eta(c + A, c)$ is either everywhere positive or everywhere negative, contradicting Lemma 18

Assume that $\frac{F}{f}$ is concave. Then, by Lemma 8, $\phi'_M > 1 = \phi'_{SP^*}$, and

$$\eta(c + A, c) = \Delta - A + \frac{F(c)}{f(c)} - \frac{F(c + A)}{f(c)}$$

is decreasing in c. So, along ϕ_{SP^*} after it crosses ϕ_M, the firm would strictly prefer 1 while along ϕ_{SP^*} before it crosses ϕ_M, the firm would strictly prefer 2. When $\frac{F}{f}$ is convex, the ranking is reversed.

Summarizing,

Lemma 19 *ϕ_{SP^*} and ϕ_M cross. If $\frac{F}{f}$ is concave, then ϕ_M is steeper than ϕ_{SP^*}, and $\eta(c + A, c)$ is increasing in c and switches from negative to positive where ϕ_M and ϕ_{SP^*} cross. If $\frac{F}{f}$ is convex, then ϕ_M is shallower than ϕ_{SP^*} and $\eta(c + A, c)$ is decreasing in c, and switches from positive to negative where ϕ_M and ϕ_{SP^*} cross.*[20]

To see another application of this lemma, consider a *percentage bonus auction*: The lowest bidder wins. If 2 wins, he receives b_1. If 1 wins, he receives Ab_2 for some constant $A \geq 1$.[21] Dominant bids are $b_2 = c_2$ and $b_1 = \frac{c_1}{A}$, yielding dividing line

$$\phi_P(c_2) = Ac_2$$

that starts at the origin and has slope $A > 1$. Recall that when $\frac{F}{f}$ is convex, $\phi'_M < 1$. So, consider the the intersection of ϕ_P and ϕ_M. An appropriately chosen bonus will result in ϕ_{SP} that has slope 1, and goes through this intersection. The resulting allocation is thus never worse than that given by ϕ_P and sometimes better. We thus have:

[20] If $\frac{F}{f}$ is strictly concave or convex, the crossing point is unique, otherwise ϕ_M and ϕ_{SP^*} may agree on an interval. Recall Example 15.

[21] McAfee and McMillan (1989) discuss such bonuses in government procurement.

Lemma 20 *A necessary condition for a percentage bonus auction to be better than a constant bonus auction is that $\frac{F}{f}$ is concave.*

5.4 Comparative Statics for A^*

Consider the optimal bonus $A^*(\Delta, f)$. We already know that on for $\Delta > 0$, $0 < A^*(\Delta, f) < \Delta$. Our next lemma shows that $A^*(\Delta, f)$ is also relatively insensitive to Δ.

Lemma 21 *If $A^*(\Delta, f) < 1$, then $0 < \frac{dA^*(\Delta,f)}{d\Delta} < 1$.*

So, if Δ is the result of pre-auction effort by 1, then following the optimal mechanism given Δ gives inefficiently low incentives to invest.

How $A^*(\Delta, f)$ depends on f is substantially more complicated. The following has the flavor of Lemma 13.

Theorem 22 *Consider a parametrized family of log-concave distributions $F(\cdot, \alpha)$ with densities $f(\cdot, \alpha)$. Assume that*

$$\frac{\partial}{\partial \alpha}\frac{\partial}{\partial c}\log f(c,\alpha) \geq 0 \tag{9}$$

$$\frac{\partial^2}{\partial c^2}\frac{F(c,\alpha)}{f(c,\alpha)} \leq 0 \tag{10}$$

and

$$\frac{\partial}{\partial \alpha}\frac{\partial}{\partial c}\frac{F(c,\alpha)}{f(c,\alpha)} \leq 0. \tag{11}$$

Then, the optimal bonus $A^*(\alpha)$ is increasing in α.[22]

By (9), as α increases, $\log f(c, \alpha)$ becomes steeper in c and so weight shifts from left to right in an $MLRP$ sense. By (10), virtual costs are concave for each α. By (11) as α goes up, virtual costs get shallower.

Before discussing the proof, we show that the conditions are coherent, and give a further sense of their interpretation.

Lemma 23 *Assume that the distribution $J(c)$ with density $j(c)$ has concave virtual costs. For $\alpha > 0$, J^α satisfies the conditions of Theorem 22, and so A^* is increasing in α.*

[22] Examining the proof, it can be seen that if either 9 and 10 are strict or 11 is strict, and if $A^*(\alpha) < 1$, then $A^*(\alpha)$ is strictly increasing. Similarly, if $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial c}\log f(c,\alpha) \leq 0$ and $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial c}\frac{F(c)}{f(c)} \geq 0$, but $\frac{F}{f}$ remains concave, then $A^*(\alpha)$ decreases in α.

So, if J has concave virtual costs, then if J is convexified ($\alpha > 1$), A^* rises, while if J is concavified ($\alpha < 1$), A^* falls. As α increases, J^α becomes less favorable; when α is an integer, $[J(c)]^\alpha$ represents the maximum of α independent draws from J.

For intuition for Lemma 22, recall from (8) that at $A^*(\alpha)$,

$$\frac{\partial BS}{\partial A} = \int_0^{1-A} \eta(c+A,c,\alpha) f(c,\alpha) f(c+A,\alpha) dc = 0.$$

By (11), increasing α increases $\eta(c+A,c,\alpha)$ point-wise. By (9), as α increases, weight on ϕ_{SP} moves from from low c to high c. But, by (10) and Lemma 19, $\eta(c+A,c,\alpha)$ is negative for small c and positive for large c, and so $\frac{\partial BS}{\partial A}$ is increased.

It is intuitive that if one reverses (10) and (11), A will decrease in α with convex virtual costs, since then $\eta(c+A,c,\alpha)$ falls in α, and is positive at small c and negative at large c, so the shift in $f(c,\alpha)f(c+A,\alpha)$ also reduces $\frac{\partial BS}{\partial A}$. Unfortunately, these conditions are inconsistent:

Lemma 24 *If* $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial c}\log f(c,\alpha) \geq 0$ *everywhere, and strict somewhere, then* $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial c}\frac{F(c,\alpha)}{f(c,\alpha)} < 0$ *somewhere.*

Hence, when virtual costs are convex there are complex off-setting forces.[23] One path to fuller results is to better understand the magnitude of the changes via $f(c,\alpha)f(c+A,\alpha)$ and $\eta(c+A,c,\alpha)$. Another is to treat $\eta(c+A,c,\alpha)$ less bluntly than assuming it moves in the same direction at every point.

6 First Price Mechanisms

We consider three first price auction settings.

Auction A^I: A *First Price Handicap Auction* (FPHA) with handicap A. Bidders 1 and 2 draw costs independently from F, and submit bids b_1, b_2. Bidder 1 wins iff $b_1 < b_2 + A$. The winner receives their bid. Player 1 is restricted to bid at most 1.[24]

Auction A^{II}: A *First Price Bonus Auction* (FPBA) with bonus A. Bidders 1 and 2 draw costs independently from F and submit bids b_1, b_2. Bidder 1

[23] It is also unhelpful to reverse (9); while the first term in (45) then goes down, the integrand in the second term can increase.

[24] It is irrelevant whether one also restricts b_2.

wins iff $b_1 < b_2$. If 2 wins, he receives b_2. If 1 wins,he receives $b_1 + A$. Player 1 is restricted to bid at most $1 - A$.

Auction A^{III}: A *First Price Shifted Cost Auction* with shift A. Bidders 1 and 2 draw costs independently and submit bids b_1, b_2. Bidder 2 draws his cost from F. Bidder 1 draws his cost from F_A defined on $[-A, 1 - A]$ by

$$\bar{F}_A(c_1) = \bar{F}(c_1 + A). \tag{12}$$

Bidder 1 wins iff $b_1 < b_2$. The winner receives their bid. Player 1 is restricted to bid at most $1 - A$.

In A^I, the bidders and payment rules are symmetric but the allocation rule is not. In A^{II} bidders and the allocation rule are symmetric, but the payment rule is not. In A^{III}, the allocation and payment rules are symmetric, but the bidders have asymmetrically distributed costs.

A commonly used FPHA is the sealed bid *request for proposal* (RFP) in which the buyer requests sealed bids on a project and then chooses his ex-post favorite. This corresponds to a FPHA with A set equal to Δ.

6.1 Equivalence, Existence and Basic Properties

We take our equilibrium concept to be Bayesian equilibrium in which for A^I and A^{III}, $b_i < c_i$ never occurs, and in which for A^{II}, $b_1 < c_1 - A$ and $b_2 < c_2$ never occur.[25] We show first that any equilibrium these settings is in pure, continuous, and strictly increasing strategies, with the minimum cost types submitting "tied" bids.

Lemma 25 *For each A^t, $t \in \{I, II, III\}$, any equilibrium of A^t is in pure continuous and strictly increasing strategies, with the (soon to be ruled out) exception that* 1 *may have an interval of high types over which he bids the maximum possible. If $\beta_1(\cdot), \beta_2(\cdot)$ is an equilibrium of A^I, then $\beta_1(0) = \beta_2(0) + A$. If $\beta_1(\cdot), \beta_2(\cdot)$ is an equilibrium of either A^{II} or A^{III}, then $\beta_1(0) = \beta_2(0)$.*

We sketch the proof and omit a formal version.[26] Consider A^I (the other cases are similar). Every $c_1 < 1$ and $c_2 < 1 - A$ earns strictly positive profits (since, for example, 2 can submit any b_2 in $(c_2, 1 - A)$ and win in the positive probability event that $c_1 > b + A$). Thus, equilibrium bids win

[25] This rules out uninteresting pathologies.

[26] See Jackson and Swinkels (2005) for details of a similar argument.

with positive probability and earn a positive amount when they win. Single crossing then tells us that best responses are weakly increasing in the strong set order, and so unique almost everywhere.[27] Hence strategies can be taken to be pure and weakly increasing.

It cannot be that for some $\hat{b}$, $b_1 = \hat{b}$ and $b_2 = \hat{b} - A$ both occur with positive probability, else an arbitrarily small drop in bid would win strictly more often, and earn essentially the same (positive amount) when it wins, for a profitable deviation. Nor can it be that say $b_1 = \hat{b}$ with positive probability. If it is, then bids b_2 in $(\hat{b} - A, \hat{b} - A + \varepsilon]$, do not make sense, since b_2 just below $\hat{b} - A$ wins discretely more often at essentially the same positive profit when it wins. But then, given that ties are zero probability, $b_1 = \hat{b}$ can be raised a bit at no loss in probability of winning (the exception to this argument is when $b_1 = 1$).

Finally, given that there are no atoms, there cannot be gaps (jumps) in bids: Assume that as c_1 approaches $\hat{c}_1$ from below, $\beta_1(c_1)$ approaches b_L, while as c_1 approaches $\hat{c}_1$ from above, $\beta_1(c_1)$ approaches $b_H > b_L$. Since b_1 is never in (b_L, b_H), b_2 is never in $(b_L - A, b_H - A)$, since such bids earn less than $b_H - A$. But then, bids b_1 near b_L are less profitable than $b_1 = b_H$, contradicting that bids b_1 near b_L are optimal.

Our next theorem says that these three settings are isomorphic.

Theorem 26 *Let pure, continuous, and strictly increasing strategy profiles* $\left(\beta_1^I, \beta_2^I\right)$, $\left(\beta_1^{II}, \beta_2^{II}\right)$ *and* $\left(\beta_1^{III}, \beta_2^{III}\right)$ *(again with the possible exception that* 1 *bids the maximum possible over some range) be related by*

$$\beta_2^I(c_2) = \beta_2^{II}(c_2) = \beta_2^{III}(c_2), \tag{13}$$

$$\beta_1^{II}(c_1) = \beta_1^I(c_1) - A, \tag{14}$$

and

$$\beta_1^{III}(c_1 - A) = \beta_1^{II}(c_1) \tag{15}$$

for $(c_1, c_2) \in [0,1]^2$. *Then, either each strategy profile is an equilibrium of its respective setting or none is.*

So, it makes no difference whether one runs an auction with a handicap A or a bonus A, and each is tightly related to a standard auction in which 1 has a cost advantage A.

To see the idea of the proof of Theorem 26, we begin with a definition is central to much of the rest of the paper.

[27] Since $b_2 \geq c_2$ and $b_1 \leq 1$, b_2 is irrelevant when $c_2 > 1 - A$. We set $\beta_2(c_2) = c_2$ wlog.

Definition 27 *Let functions ϕ^I, ϕ^{II}, and ϕ^{III} be defined implicitly by*

$$\beta_1^I(\phi^I(c_2)) = \beta_2^I(c_2) + A \tag{16}$$
$$\beta_1^{II}(\phi^{II}(c_2)) = \beta_2^{II}(c_2) \tag{17}$$
and
$$\beta_1^{III}(\phi^{III}(c_2)) = \beta_2^{III}(c_2), \tag{18}$$

and let ψ^I, ψ^{II}, and ψ^{III} be their respective inverses.

Since bid functions are assumed continuous and strictly increasing, these are well-defined, continuous and increasing. Each ϕ^t, $t \in \{I, II, III\}$ connects c_2 to the c_1 that "ties" it in the sense that when $c_1 < \phi^t(c_2)$, 1 wins, and when $c_1 > \phi^t(c_2)$ 2 wins. The key to the proof is to show that because (β_1^I, β_2^I), $(\beta_1^{II}, \beta_2^{II})$ and $(\beta_1^{III}, \beta_2^{III})$ are related as in the statement of Theorem 26,

$$\phi^I(c_2) = \phi^{II}(c_2) = \phi^{III}(c_2) + A.$$

From this, we show that each player has an incentive to mimic another type in A^I if and only if he has an incentive to mimic in A^{II} and A^{III}. Hence (β_1^I, β_2^I), $(\beta_1^{II}, \beta_2^{II})$ and $(\beta_1^{III}, \beta_2^{III})$ are either all equilibria or none.

While Theorem 26 draws a tight formal connection across the three settings, the interpretation of the auctions differs. Consider a *value advantage* case where the buyer places incremental value Δ on buying from 1 but costs are symmetric, and a *cost advantage* case where the buyer is indifferent, but 1 draws costs from F_Δ.

In the value advantage case, treating the two sellers the "same" corresponds to A^I or A^{II} with $A = 0$, and so the bidders face a completely symmetric setting. In the cost advantage case, this corresponds to the (fairly unnatural) design in which 1 is penalized by Δ, his whole cost advantage.

In the cost advantage case, the natural auction that treats the two sellers the "same" is A^{III} with $A = \Delta$, so that 1 retains all of his cost advantage (this is effectively the case studied by Maskin and Riley (2000)). In A^I this corresponds to the RFP, while in A^{II}, 1 receives a bonus A equal to his whole value advantage Δ.

In many settings, the boundary between these cases is blurry: lower transportation costs for 1 are a cost advantage, a value advantage, or a blend, depending on how the auction rules specify transportation costs are shared. Because of this, even in settings where there may be constraints against explicitly favoring a given bidder (e.g., legal, or in terms of the perception of fairness among the sellers), there may be considerable latitude to choose A indirectly. Similarly, with an incumbent and new supplier one can specify who pays the transition costs, and when dealing with a foreign and domestic bidder, a government might have the use of differential tax treatments.

Theorem 26 has the following useful corollary.

Corollary 28 *For each of A^I, A^{II} and A^{III} an equilibrium exists, is unique, and is in pure and strictly increasing strategies.*

This is direct from Maskin and Riley (2000) applied to A^{III}.[28] Given Corollary 28, ϕ^t is uniquely determined by F and A regardless of which first price auction format is used. We thus write simply ϕ_{FP} (FP mnemonic for first price), and ψ_{FP} for $(\phi_{FP})^{-1}$, dropping the subscript when possible.

In what follows, we analyze A^I. Since $\beta_1(0) = \beta_2(0) + A$, $\phi(0) = 0$, and since the equilibrium is strictly increasing, $\phi(1-A) = 1$.

6.2 Preliminary Characterization

We now turn to a more detailed examination of the equilibrium bid and allocation functions. We begin with a set of basic results. Let $g = \frac{f}{\bar{F}^2}$.

Theorem 29 *In the FPHA with handicap A, Player 1's surplus at type c is*

$$S_1(c) = \int_c^1 \bar{F}(\psi(s))ds. \tag{19}$$

Player 2's surplus at type c is

$$S_2(c) = \int_c^{1-A} \bar{F}(\phi(s))ds. \tag{20}$$

[28] Their key assumptions are that the type of one player conditionally stochastically dominates the other, and (to ensure that there is no atom in β_1 at $b_1 = 1$) that the supports of values overlap. Two good entry points to the considerable literature on existence for asymmetric first price auctions are Lebrun (1999), who assumes costs have identical support, and Reny and Zamir (2004). Reny and Zamir and Jackson and Swinkels (2005) show existence in increasing pure strategies for the setting of this paper. A (fairly straightforward) separate proof can be constructed that there is no atom at $b_1 = 1$.

Bid functions are

$$\beta_1(c) = c + \frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))} \tag{21}$$

and

$$\beta_2(c) = c + \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))}. \tag{22}$$

If $f \in C^k[0,1]$*, then* $\beta_1 \in C^{k+1}[0,1)$*,* $\beta_2 \in C^{k+1}[0,1-A)$ *and* $\phi \in C^{k+1}[0,1-A)$*. On these intervals*

$$\beta_1'(c) = \frac{1}{\phi'(\psi(c))} S_1(c)\, g(\psi(c)) > 0, \tag{23}$$

$$\beta_2'(c) = \phi'(c)\, S_2(c)\, g(\phi(c)) > 0, \tag{24}$$

and

$$\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{g(c)}{g(\phi(c))} > 0. \tag{25}$$

Let us sketch the proof. Equations (19) and (20) follow from a simple envelope theorem argument, noting that $\bar{F}(\psi(s))$ is the probability that 1 wins with value s and $\bar{F}(\phi(s))$ is the probability that 2 wins with value s. Equation (21) follows from (19), noting that

$$S_1(c) = \bar{F}(\psi(c))(\beta_1(c) - c)$$

and rearranging, and similarly for equation (22).

Because β_1,β_2, ϕ, and ψ are increasing, they are differentiable almost everywhere. Imagine that $\beta_2'(c_2) = 0$ at some $c_2 < 1 - A$. Then near $c_1 = \phi(c_2)$, the probability of winning changes arbitrarily fast in b_1. But, since for $c_1 < 1$, $\beta_1(c_1) > c_1$, a small decrease in bid is optimal. Thus, $\beta_2'(c_2) > 0$ where it exists, and similarly for β_1'.

Differentiating $\beta_1(\phi(c)) = \beta_2(c) + A$, gives that where β_1' and β_2' exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0. \tag{26}$$

Equations (23) and (24) are a matter of calculation. Substituting (23) and (24) into (26) and rearranging yields (25).

Substituting for S_1 and S_2 in (26) gives

$$\phi'(c) = \frac{\int_{\phi(c)}^1 \bar{F}(\psi(s))ds}{\int_c^{1-A} \bar{F}(\phi(s))ds} \frac{g(c)}{g(\phi(c))}. \tag{27}$$

For $1 \le \hat{k} \le k$, assume that $\phi \in C^{\hat{k}}[0, 1-A)$. Then, each building block of the RHS of (27) belongs to $C^{\hat{k}}[0, 1-A)$ and so the RHS as a whole belongs to $C^{\hat{k}}[0, 1-A)$. But then $\phi' \in C^{\hat{k}}[0, 1-A)$, and so $\phi = \int \phi' \in C^{\hat{k}+1}[0, 1-A)$. By induction, $\phi \in C^{k+1}[0, 1-A)$, and the analogous claims follow for β_1 and β_2

6.3 On the Shape of the Bid and Allocation Functions

With these technical properties of β_1, β_2 and ϕ established, let us turn to their "shape". Recall first that by Lemma 25, $\beta_1(0) = \beta_2(0) + A$. Thus, $\phi(0) = 0$. But, by Theorem 29, $\phi'(c) > 0$ on $[0, 1-A)$, and so we have:

Corollary 30 *$\phi(c) > c$ for all $c > 0$.*

Note also that $\beta_1(1) = 1 = \beta_2(1-A) + A$. Thus, $\phi(1-A) = 1$. A useful complement to this is

Lemma 31 *For $c < 1-A$, $\phi(c) < c + A$.*

The proof shows from 1 and 2's first order conditions that if $c < 1-A$, and $\phi(c) - c = A$, then $\phi(c) - c$ is decreasing. Since $\phi(0) - 0 = 0$, this contradicts reaching $\phi(c) - c = A$.

Next, let us show that both bid functions are flat at the top.

Lemma 32 *β_1 is differentiable at 1 with $\beta'(1) = 0$. β_2 is continuously differentiable at $1-A$ with $\beta_2'(1-A) = 0$.*

For β_1, we claim differentiability at 1, but not continuous differentiability.[29] For β_2, we assert continuous differentiability.

The proofs of the two cases are quite different: with $c_1 = 1$, 1 already has probability $\bar{F}(1-A)$ of winning while with $c_2 = 1-A$, 2 has no chance of winning and so S_1 and S_2 have different shapes at the top. Thus, even bidding 1 already earns 1 a great deal of surplus, and this bounds his bid from below sufficiently to guarantee $\beta_1'(1) = 0$. Showing that $\beta_2'(c_2) \to 0$ as $c_2 \to 1-A$ involves an examination of 1's incentives.

An easy but useful implication of this is:

[29] If h is continuously differentiable on $[0, 1)$, its derivative at 1 need not equal $\lim_{x \to 1} h'(x)$ (which need not exist). We establish continuous differentiability of β_1 later in a rather round-about way.

Lemma 33 *As $c \to 1 - A$,*

$$\frac{\beta_1(\phi(c)) - \phi(c)}{1 - \phi(c)} \to 1 \text{ and } \frac{\beta_2(c) - c}{1 - A - c} \to 1.$$

Our final Lemma of this section is central to what follows.

Lemma 34 *An expression for $\frac{\phi''(c)}{\phi'(c)}$ is*

$$\frac{\phi''(c)}{\phi'(c)} = \frac{\phi'(c)}{S_1(\phi(c))\, g(c)} \left(\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)} \right) + \frac{g'(c)}{g(c)} - \phi'(c) \frac{g'(\phi(c))}{g(\phi(c))}. \quad (28)$$

If $\phi'(c) \leq 1$, then a sufficient condition for $\frac{\phi''(c)}{\phi'(c)} > 0$ is

$$\frac{1}{S_1(\phi(c))\, g(c)} - 2 > 0. \quad (29)$$

Proof of Lemma 34: Since

$$\phi'(r) = \frac{S_1(\phi(r))\, g(r)}{S_2(r)\, g(\phi(r))} = \frac{\int_{\phi(r)}^{1} \bar{F}(\psi(s))\, ds g(r)}{\int_{r}^{1-A} \bar{F}(\phi(s))\, ds g(\phi(r))},$$

$$\begin{aligned} \frac{\phi''(r)}{\phi'(r)} &= \left(\log \phi'(r)\right)' \\ &= \frac{(S_1(\phi(r))'}{S_1(\phi(r))} - \frac{(S_2(r))'}{S_2(r)} + \frac{g'(r)}{g(r)} - \phi'(r) \frac{g'(\phi(r))}{g(\phi(r))} \\ &= \phi'(r) \frac{-\bar{F}(r)}{S_1(\phi(r))} + \frac{\bar{F}(\phi(r))}{S_2(r)} + \frac{g'(r)}{g(r)} - \phi'(r) \frac{g'(\phi(r))}{g(\phi(r))}. \end{aligned}$$

Make the substitution $\bar{F} = \frac{\frac{f}{F}}{g}$ in the first two terms, and then replace $S_2(r)\, g(\phi(r))$ in the second term by $\frac{S_1(\phi(r))g(r)}{\phi'(r)}$ and collect to obtain (28).

If $\phi'(r) \leq 1$, then $\frac{g'(r)}{g(r)} \geq \phi'(r) \frac{g'(r)}{g(r)}$, and so by (28)

$$\frac{1}{\phi'(r)} \frac{\phi''(r)}{\phi'(r)} \geq \frac{\frac{f(\phi(r))}{\bar{F}(\phi(r))} - \frac{f(r)}{\bar{F}(r)}}{S_1(\phi(r))\, g(r)} + \frac{g'(r)}{g(r)} - \frac{g'(\phi(r))}{g(\phi(r))}.$$

Using $\frac{g'}{g} = \frac{f'}{f} + 2\frac{f}{F}$ and collecting, we have

$$\frac{1}{\phi'(r)} \frac{\phi''(r)}{\phi'(r)} \geq \left(\frac{1}{S_1(\phi(r))\, g(r)} - 2 \right) \left(\frac{f(\phi(r))}{\bar{F}(\phi(r))} - \frac{f(r)}{\bar{F}(r)} \right) + \left(\frac{f'(r)}{f(r)} - \frac{f'(\phi(r))}{f(\phi(r))} \right)$$

If the first bracketed term on the RHS is positive, then the whole RHS is positive by our conditions on f. ■

6.4 Limiting Properties of $\phi'(\cdot)$

It is central to our main result to understand the behavior of $\phi'(c)$ as c goes to $1-A$.

Theorem 35 *In the FPHA,* $\limsup \phi'(c) = \infty$*, and* $\liminf \phi'(c) > 1$*.*

So, (modulo the annoying possibility of a discontinuity of the second type in ϕ' at $1-A$), ϕ' becomes arbitrarily large as $c \to 1-A$. We prove this through a series of lemmas that comprise the balance of this section.[30] This discussion can be skipped without loss of continuity.

First, we show that $\limsup \phi'(c)$ cannot be in $(0,\infty)$.

Lemma 36 $\limsup \phi'(c) \in \{0,\infty\}$*.*

The proof of this is fairly dense, but hinges on using an extension of l'Hôpital's rule to the lim sup operator to derive a contradiction if $\limsup \phi'(c)$ is finite but positive. At its heart, the key is that S_1 and S_2 are very different near $c_1 = 1$ and $c_2 = 1-A$.

Next, we show that $\phi'(c)$ does not tend to 0.

Lemma 37 $\limsup \phi'(c) > 0$*.*

Proof of Lemma 37: If $\limsup \phi'(c) = 0$ then $\phi'(c) \to 0$. So, for any small t, there is a last $c(t)$ at which $\phi'(c) = t$ (this is well defined since ϕ is continuously differentiable and $[0, 1-A]$ is compact). But, by a change of variables,

$$\begin{aligned} S_1(\phi(c(t))) &= \int_{\phi(c(t))}^{1} \bar{F}(\psi(s))\, ds \\ &= \int_{c(t)}^{1-A} \bar{F}(s)\, \phi'(s)\, ds, \\ &< t(1-A-c(t)), \end{aligned}$$

since $\phi'(s) < t$, and $\bar{F} < 1$.

Thus,

$$\begin{aligned} \frac{1}{S_1(\phi(c(t)))\, g(c(t))} &> \frac{t}{t(1-A-c(t))\, g(c(t))} \\ &> \frac{1}{g(1-A)} \frac{1}{t(1-A-c(t))}. \end{aligned}$$

[30] The proof is more straightforward if $\lim \phi'$ is known to exist. See the discussion at the end of this section.

This diverges as $t \to 0$ and $c(t) \to 1-A$. But then by Lemma 34 for small t, $\phi''(c(t)) > 0$, contradicting that $c(t)$ was the last moment at which $\phi' = t$. ■

We thus have

Corollary 38 $\limsup \phi'(c) = \infty$.

To conclude from this that $\phi'(c) \to \infty$, we would need to tie down $\liminf \phi'(c)$. A partial characterization key to our later results is

Lemma 39 *There is $\hat{c} < 1-A$ such that for all $c > \hat{c}$, $\phi'(c) > 1$.*

We use (29) to show that when $\phi' = 1$, but c is close to $1-A$, $\phi'' > 0$ and so there are eventually no more crossings of $\phi' = 1$. Since $\limsup \phi'(c) = \infty$, $\phi'(c) > 1$ after some point.

A corollary that is mostly in the name of tidiness (and sunk costs) is that β_1 is indeed continuously differentiable at 1.

Corollary 40 $\beta_1'(c) \to 0$. *Hence, β_1 is continuously differentiable on* $[0,1]$.

The point is that since $\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))}$ stays above 1 near $1-A$, and since $\beta_2'(c) \to 0$, it must also be that $\beta_1'(\phi(c)) \to 0 = \beta_2'(1)$.

We conjecture that $\phi'(c)$ is sufficiently well behaved that $\phi'(c) \to \infty$. But, as $c \to 1-A$, and $\phi(c) - c \to A$, one can show that

$$\frac{\phi''(c)}{\phi'(c)} = Y - \phi'(c)\, X,$$

where as $c \to 1-A$, and $\phi(c) - c \to A$, both X and Y tend to $-\infty$, and, for any finite $\phi'(c)$, there are points (c_2, δ) arbitrarily near $(1-A, A)$ where this expression takes on either sign at arbitrary magnitude. So, this is far from trivial. See Mares and Swinkels (2008b).

6.5 Allocations in First Price Handicap Auctions

The results developed in the previous two sections imply a surprisingly strong characterization of the allocation in a FPHA. We begin with the case where f is non-decreasing. We discuss more general densities later.

We begin with a piece of structure that will be key to our results.

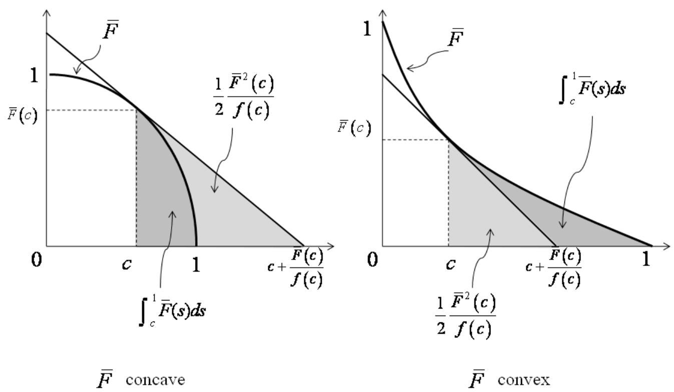

Figure 2: Illustration of $W \leq \frac{1}{2}$ for f increasing (left figure) and $W \geq \frac{1}{2}$ for f decreasing (right figure).

Definition 41 *Given F, define*

$$W(c) = \frac{f(c) \int_c^1 \bar{F}(s)ds}{\bar{F}^2(c)}.$$

If we re-write this as

$$W(c) = \frac{1}{2} \frac{\int_c^1 \bar{F}(s)ds}{\frac{1}{2}\bar{F}(c)\frac{\bar{F}(c)}{f(c)}}. \tag{30}$$

then Fig. 2 illustrates a nice geometric interpretation of W: If $f' \geq 0$, then $\frac{\partial^2 \bar{F}}{\partial c} = -f' \geq 0$, and so $\bar{F}$ is concave, and if $f' \leq 0$, then $\bar{F}$ is convex. The numerator in (30) is the area under $\bar{F}$ to the right of c. But, $\frac{1}{2}\bar{F}(c)\frac{\bar{F}(c)}{f(c)}$ is the area of the triangle to the right of c formed by the tangent pane to $\bar{F}(c)$ at c. If $\bar{F}$ is concave, this triangle is larger than $\int_c^1 \bar{F}(s)ds$, and if $\bar{F}$ is convex, it is smaller. We thus have shown

Lemma 42 *If $f' \leq 0$, then $W(c) \geq \frac{1}{2}$ for all c. If $f' \geq 0$, then $W(c) \leq \frac{1}{2}$ for all c.*

Our main use of W is through the following lemma:

Lemma 43 *Let $r \in \arg\min_{c \in [0,1-A)} \phi'(c)$. Then, for all $c_2 \in [0, 1-A)$,*

$$S_2(c)\, g(\phi(c)) < \frac{1}{\phi'(r)} W(\phi(c)). \tag{31}$$

Proof of Lemma 43: By a change of variables,

$$\begin{aligned} S_2(c) &= \int_c^{1-A} \bar{F}(\phi(s))ds \\ &= \int_{\phi(c)}^{\phi(1-A)} \bar{F}(s)\psi'(s)\,ds \\ &< \frac{1}{\phi'(r)}\int_{\phi(c)}^{1} \bar{F}(s)ds, \end{aligned}$$

where the strict inequality follows since ϕ' is continuous and $\limsup \phi' = \infty$. Multiplying both sides by $g(\phi(c)) = \frac{f(\phi(c))}{\bar{F}^2(\phi(c))}$ gives the result. ■

This in hand, we can turn to our main theorem of this section.

Theorem 44 *Assume that f is weakly increasing. Then, $\phi'(c) > 1$ for all $c \in [0, 1-A)$.*

Proof of Theorem 44: Recall from Theorem 35 that for all c above some $\hat{c} < 1-A$, $\phi'(c) > 1$. By Theorem 29

$$\phi'(0) = \frac{S_1(0)}{S_2(0)}\frac{g(0)}{g(0)} > 1,$$

since $S_1(0) > S_2(0)$. Assume that $\phi' \leq 1$ anywhere on $[0, \hat{c}]$. Since ϕ' is continuously differentiable on $[0, 1-A)$, ϕ' achieves an interior minimum at some $r \in (0, \hat{c}]$, so that $\phi''(r) = 0$. By Theorem 29, $\phi'(r) > 0$. From Lemma 34, it must thus be that

$$\frac{1}{S_1(\phi(r))\,g(r)} - 2 \leq 0,$$

or

$$S_1(\phi(r))\,g(r) > \frac{1}{2}.$$

But, by Theorem 29 and Lemma 43,

$$S_1(\phi(r))\,g(r) = \phi'(r)\,S_2(r)\,g(\phi(r)) < W(\phi(c)), \tag{32}$$

which is a contradiction, since by Lemma 42 $W(\phi(c)) \leq \frac{1}{2}$. ■

6.5.1 Other Routes to $\phi' > 1$.

The condition $f' \geq 0$ is far from necessary. From (28) if $\phi'(r) \leq 1$,

$$\begin{aligned}\frac{\phi''(r)}{\phi'(r)} &= \frac{\phi'(r)}{S_1(\phi(r))\,g(r)}\left(\frac{f(\phi(r))}{\bar{F}(\phi(r))}-\frac{f(r)}{\bar{F}(r)}\right)+\frac{g'(r)}{g(r)}-\phi'(r)\frac{g'(\phi(r))}{g(\phi(r))}\\ &\geq \frac{\phi'(r)}{S_1(r+\delta)\,g(r)}\left(\frac{f(r+\delta)}{\bar{F}(r+\delta)}-\frac{f(r)}{\bar{F}(r)}\right)+\phi'(r)\frac{g'(r)}{g(r)}-\phi'(r)\frac{g'(r+\delta)}{g(r+\delta)}\end{aligned}$$

where $\delta = \phi(r) - r$. Cancelling $\phi'(r)$, and using (32), we have

$$\frac{\phi''(r)}{\phi'(r)} > \frac{\frac{f(r+\delta)}{\bar{F}(r+\delta)}-\frac{f(r)}{\bar{F}(r)}}{W(\phi(r))}+\frac{g'(r)}{g(r)}-\frac{g'(r+\delta)}{g(r+\delta)}. \tag{33}$$

Let $w = W(\phi(r))$. Then, this can be re-written as

$$\begin{aligned}\frac{\phi''(r)}{\phi'(r)} &> \frac{\frac{\partial}{\partial r}\log\frac{\bar{F}(r)}{\bar{F}(r+\delta)}}{w}+\frac{\partial}{\partial r}\log\frac{\frac{f(r)}{\bar{F}^2(r)}}{\frac{f(r+\delta)}{\bar{F}^2(r+\delta)}}\\ &= \frac{\partial}{\partial r}\log\left(\left(\frac{\bar{F}(r)}{\bar{F}(r+\delta)}\right)^{\frac{1}{w}}\frac{\frac{f(r)}{\bar{F}^2(r)}}{\frac{f(r+\delta)}{\bar{F}^2(r+\delta)}}\right)\\ &= \frac{\partial}{\partial r}\log\frac{\bar{F}(r)^{\frac{1}{w}-2}f(r)}{\bar{F}(r+\delta)^{\frac{1}{w}-2}f(r+\delta)}\\ &= \frac{\partial}{\partial r}\log\bar{F}(r)^{\frac{1}{w}-2}f(r)-\frac{\partial}{\partial r}\log\bar{F}(r+\delta)^{\frac{1}{w}-2}f(r+\delta).\end{aligned}$$

If this is positive, then we again have a contradiction. Thus, a sufficient condition for $\phi' > 1$ is that

$$\bar{F}(c)^{\frac{1}{w}-2}f(c)$$

is log-concave at $w = W(\phi(r))$. Indeed, a different way of seeing the proof of Theorem 44 is that when $f' \geq 0$, then for all c,

$$\frac{1}{W(c)} - 2 \geq 0$$

and so $\bar{F}(c)^{\frac{1}{w}-2}f(c)$ inherits the log-concavity of $\bar{F}$ and f. However, $\bar{F}(c)^{\frac{1}{w}-2}f(c)$ is also log-concave for many distributions that do not have this property.

Example 45 *Consider the strictly unimodal distribution* $f(c) = 6c(1-c)$*. Then, it can be verified that* $W(c) = \frac{3c(1+c)}{(1+2c)^2}$ *which is increasing, and so* $W(c) \leq \frac{2}{3}$*. It is therefore enough to show that* $\bar{F}(c)^{\frac{3}{2}-2} f(c) = \bar{F}(c)^{-\frac{1}{2}} f(c)$ *is log-concave, which is satisfied. Effectively,* f *is sufficiently log-concave as to overcome the log-convexity of* $\bar{F}(c)^{-\frac{1}{2}}$*.*

Example 46 *For any* $\alpha \geq 1$*, consider the distribution* $\bar{F}(c) = (1-c)^{\alpha}$*. Then,* $f(c) = \alpha(1-c)^{\alpha-1}$ *and so* $W(c) = w = \frac{\alpha}{1+\alpha}$ *for all* c*. Thus,*

$$\begin{aligned} & \bar{F}(c)^{\frac{1}{w}-2} f(c) \\ = \; & \bar{F}(c)^{\frac{\alpha+1}{\alpha}-2} f(c) \\ = \; & (1-c)^{\alpha\left(\frac{1-\alpha}{\alpha}\right)} \alpha(1-c)^{\alpha-1} \\ = \; & \alpha \end{aligned}$$

which is trivially log-concave.

Neither of these examples has an increasing density. While the details seem messy, we conjecture that this approach can yield the more general characterization that beta distributions (of which both are examples) have $\phi' > 1$. When W is increasing, one can in general set $w = \lim_{c \to 1} W(c)$, and then check the log-concavity of $\bar{F}(c)^{\frac{1}{w}-2} f(c)$.[31]

There are two intuitions suggesting that for decreasing f, having W increase is not going to always hold. First, the base of the relevant triangle covers a larger fraction of $[c, 1]$ as c increases. Second, a little manipulation yields

$$W(c) = \frac{E\left(\frac{\bar{F}(s)}{f(s)} | s > c\right)}{\frac{\bar{F}(c)}{f(c)}}.$$

As c grows, the range over which the expectation is taken tightens, and so one might expect the difference between the expectation and the value of the function at its lower boundary to tighten as well.

Example 47 *Calculation shows that* $f(c) = \frac{e^{-c} - e^{-1}}{1 - 2e^{-1}}$ *is log-concave and decreasing with* $f(1) = 0$*. However, graphing* W *shows it to be decreasing, and the graph of the RHS of* (33) *is everywhere negative, so that a proof of* $\phi' > 1$ *would have to rest on something very different that the techniques of this section. Despite this, the numerically solved equilibrium of this example has* $\phi' > 1$*.*[32]

[31] We conjecture that $f(1) = 0$ is necessary for this proof strategy to work.

[32] With, for the case $\Delta = .2$, minimum value approximately 1.0008!

7 Comparing the First Price, Second Price and Optimal Mechanism

The result that $\phi'_{FP} > 1$ has significant practical import. Say that mechanism 1 *ex-post dominates* mechanism 2 if it agrees with the optimal mechanism on strict superset of the set of (c_1, c_2) for which mechanism 2 agrees with the optimal mechanism.[33] So, mechanism 1 always gets it right when mechanism 2 does, and gets it right more often.

Theorem 48 *Assume that $\frac{F}{f}$ is convex. Consider the first price auction with handicap A, for any given A. If $\phi'_{FP} > 1$, then for suitably chosen $\hat{A}$, the first price auction is ex-post dominated by the second price auction with handicap $\hat{A}$.*

So, for any first price auction (including, of course the optimal one), there is a second price mechanism that strictly improves upon it, and does so outcome by outcome.

The proof is illustrated by Fig. 3. When $\frac{F}{f}$ is convex, $\phi'_M\ (c_2) \leq 1$. So, pick any given A_{FP}. By assumption, $\phi'_{FP} > 1$. If ϕ_M and ϕ_{FP} never cross, then trivially one can find $\hat{A}$ such that the associated ϕ_{SP} lies between ϕ_M and ϕ_{FP}. If ϕ_M and ϕ_{FP} cross, then they do so once at some $\hat{c}$. Set $\hat{A} = \phi_{FP}(\hat{c}) - \hat{c}$, and consider ϕ_{SP} for bonus $\hat{A}$. Then, for all $c < \hat{c}$, $\phi_M\,(c) > \phi_{SP}\,(c) > \phi_{FP}(c)$, and for all $c > \hat{c}$, $\phi_M\,(c) < \phi_{SP}\,(c) < \phi_{FP}(c)$. So, for any (c_1, c_2) where the second price auction allocates differently than the first price auction (in the shaded areas of Fig. 3), the second price auction allocates optimally while the first price auction allocates incorrectly.

7.1 On the Relationship to Maskin and Riley

Maskin and Riley (2000) analyze auctions with a seller and two buyers with asymmetrically distributed values. In one specification a "weak" buyer draws his value from F with support $[0, 1]$ and a "strong" buyer draws his value from F_s with support $[s, s + 1]$ given by

$$F_s(x) = F(x + s).$$

When s is large, in the standard first price auction the strong bidder bids 1 in equilibrium and always win. In the second price auction, the strong

[33] Recall that for the various mechanisms the allocation is deterministic on a measure one set of types.

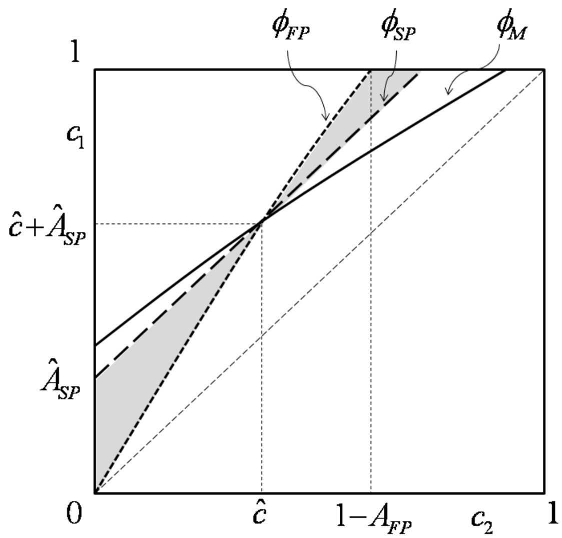

Figure 3: Illustration of Theorem 48.

bidder will still always win but pays the value of his opponent. So, the first price auction raises more revenue than the second price auction. Maskin and Riley extend this revenue ranking to settings where s is smaller, and so there is a range of bids over which the allocation is competitive.[34]

To translate into our setting, begin with A^{III}, in which 1 has a Δ cost advantage but the rules are symmetric. The Maskin and Riley result says that in A^{III}, a symmetric first price mechanism is better than a symmetric second price mechanism. Translated into A^{I}, the setting with symmetrically distributed costs but a Δ value advantage for 1, this says that running a second price mechanism with handicap $A = \Delta$ is worse than running a second price mechanism with handicap $A = \Delta$. So, if one is going to run an RFP process, then it is better to do so in a sealed bid than in an open manner.

This result, that a first price auction can be better than a second, does *not* contradict ours. Maskin and Riley compare the first and second prices auction for a fixed $A = \Delta$. While natural, this turns out to be a pretty bad choice for the auctioneer, especially in the second price case, given that Myerson line lies strictly below $c_2 + \Delta$ (Lemma 5), and so second price

[34] To get this result, they assume that F is convex (f is weakly increasing).

auction universally distorts too far in favor of 1. In the FPHA with $A = \Delta$, $\phi_{FP}(c) - c$ is strictly below Δ except at $c = 1 - \Delta$ (Lemma 31), and so does not always distort too far, providing one intuition for their ranking.[35]

In contrast, what we show is that for *any* first price auction, there is a handicap such that the second price auction does better in the very strong sense of ex-post dominance.[36] So, while Maskin and Riley show that between a focal pair of asymmetric auctions, one prefers the first price mechanism, we show that if one can choose which handicap to offer, one will prefer a second price mechanism.

7.2 Comparing the FPHA with the symmetric FPA

When $\phi' > 1$ holds everywhere, we have a simple characterization of how the FPHA with $A > 0$ compares to the symmetric first price auction:

Theorem 49 *If $\phi' \geq 1$ everywhere, then β_1 and β_2 lie on either side of the symmetric equilibrium strategy β_s of a standard first price auction*

$$\beta_1 \geq \beta_s \geq \beta_2.$$

This is intuitive: 1 bids less aggressively than if he were not favored, and 2 more. Overall, costs go up, but, for appropriately chosen A, the improvement in efficiency more than compensates the buyer. Fig. 4 shows how β_1 and β_2 vary in A for the uniform case. It is an interesting conjecture that β_1 and β_2 should move monotonically further apart as A grows for general f.

8 How General is the Result?

The strong ranking we obtain of a second over a first price mechanism hinges on $\phi'_M > 1$ and $\phi'_{FP} < 1$. Each result depends on significant assumptions. However, we believe the general superiority of second price mechanisms to be quite robust. This is based on three observations, each of which we would like to better understand.

The Miracle of the Myerson Line

[35] We limit bidder 1 to receive at most his highest possible cost. We are working to better understand the degree to which Maskin and Riley's ranking depends on the absense of a reserve price in their framework. In their motivating example of bidders with very different supports over values, it seems critical.

[36] Our requirement that $\bar{F}$ be concave is identical to theirs, but we also require convex virtual costs. Our numerical exercises suggest that neither of these is critical.

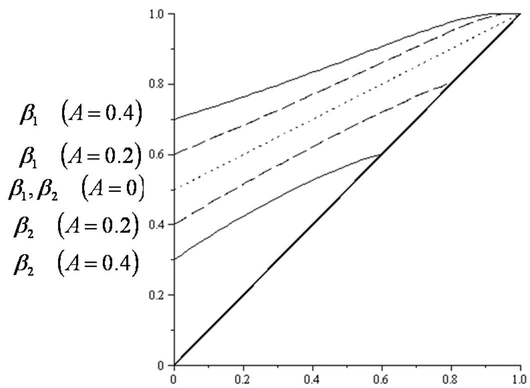

Figure 4: The equilibrium bid functions for $A = 0$ (dotted), $A = .2$ (dashed) and $A = .4$ (solid) for uniformly distributed costs.

The result that $\phi'_M \leq 1$ depends on the convexity of virtual costs $\frac{F}{f}$. Pick Δ, and F with $\frac{F}{f}$ concave. Since $\phi'_M \geq 1$, ϕ_M lies between the lines of slope 1 that are anchored at each endpoint of ϕ_M. For example, the heavy line in Fig. 5 shows ϕ_M for $\Delta = .2$ and

$$F(c) = \frac{c + c^2}{2},$$

while the light lines are the bounding lines of slope 1, with intercepts 0.105 and 0.128. Since the two lines are close together, a SPBA with $A \cong .12$ will be very close to optimal. It gets the allocation right for the vast majority of (c_1, c_2), and where it gets the allocation wrong, $\omega(c_1) - \omega(c_2)$ is close to Δ, and so the damage is small.

To get some sense of the generality of this, for any $\alpha > 0$ and β consider

$$f(x) = 1 + \alpha x + \beta x^2$$

where $\alpha > 0$, and $F(x) = x + \frac{\alpha}{2}x^2 + \frac{\beta}{3}x^3$.[37] Fig. 6, graphs the intercepts of the upper and lower bounding functions as a function of α and β.[38] The two are no more that .04 apart regardless of α and β, and once again, $A \cong .12$ does remarkably well regardless of α and β.

[37] The missing normalization factor cancels in computing $\frac{F}{f}$.

[38] In the region where the "upper" intercept is below the "lower", $\frac{\bar{F}}{f}$ is not concave.

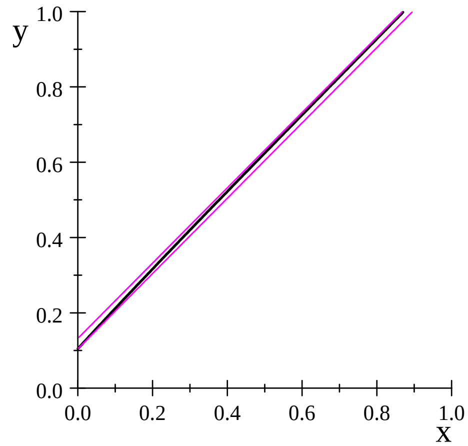

Figure 5: The Myerson Line, and Upper and Lower Lines of Slope 1, for the case $F(x) = \frac{x+x^2}{2}$.

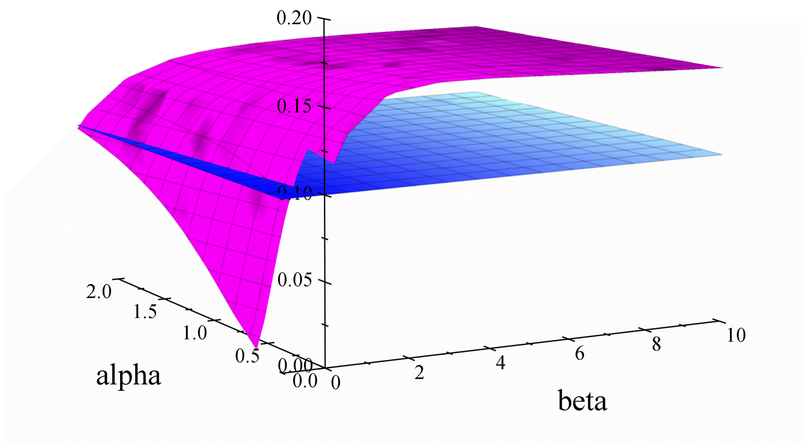

Figure 6: Upper and lower bounds for $\phi(c) - c$ for $f(x) \simeq 1 + \alpha x + \beta x^2$.

In wide numerical experiments with concave $\frac{F}{f}$, we achieved a ϕ_M for which ϕ'_M is not pretty close to 1, only by mixing together polynomial terms of very different degree (so that F itself was near degenerate). There is a strong suggestion here of deeper structure.

When $\frac{F}{f}$ is convex, no "miracle" occurs. It is easy to generate examples where $\left(\frac{F}{f}\right)'$ is large and changes rapidly, and thus ϕ'_M is *much* less than 1. So, it is not the case that second price auctions are in any sense always near optimal. Rather, it seems that when $\frac{F}{f}$ is concave, the SPBA is near optimal, while when $\frac{F}{f}$ is convex, the SPBA is at least significantly better than the FPHA.[39]

The Slope of the Allocation Function

We show that ϕ'_{FP} is everywhere greater than 1 when f is increasing, and in a variety of other examples. The question is what happens when f is decreasing or of general form. Note first that the function W used in the proof of Theorem 43 uses a very crude upper bound on S_2. A companion paper (Mares and Swinkels, 2008b) details a set of building blocks for understanding the structure of ϕ_{FP} for general densities. While extensive and perhaps impressive, it has the small flaw of not having actually led us to a general result.

On the positive side, we have numerically solved ϕ_{FP} for an extensive set of examples, including a representative set of linear and quadratic negatively sloped f and every case that our intuition at one point or another has suggested might lead to trouble. In none of these examples does ϕ'_{FP} fall below 1.[40] It is thus our conjecture that $\phi'_{FP} > 1$ holds quite generally.

Overkill

We use a very strong sense of dominance. In Fig. 5 a sensibly chosen second price mechanism will be close to optimal. On the other hand, the graph of ϕ_{FP} begins at $(0,0)$ and ends at $(1-A, A)$. Since ϕ_M is significantly displaced from the diagonal, there is thus an intuition that ϕ_{FP} will be pretty far away from ϕ_M for much of its range. So, even if there are pairs (c_1, c_2) where ϕ_{FP} outperforms ϕ_{SP}, it is unlikely to outperform it in expectation. We have not formalized this; the approaches that have occurred to us thus

[39] In our experiments for $\frac{F}{f}$ with concave and convex regions, ϕ' was unlikely to be much greater than 1, but could be much less.

[40] Plotted examples and details of the numerical technique can be found at either of our web pages.

far seem doable but tedious, and we hope that future work will lead to something with more insight.

Taking these three observations together, our conjecture is that the superiority of the second price over the first price mechanism holds very generally.

9 Conclusion

In our setting a buyer puts premium Δ on procuring from 1 instead of 2. We derive the optimal mechanism, and provide a number of properties showing how the optimal, first, and second price mechanisms vary with Δ and the underlying distribution F over costs. Our central result shows conditions under which a first price auction with a handicap will always be dominated by an appropriately chosen second price mechanism.

An intuition for the result is that the second price mechanism creates a constant distortion away from a symmetric allocation rule, while in the first price handicap auction, bidder optimization results in no distortion when 2's cost c_2 is low, but a great deal of distortion when c_2 is high. In particular, when f is non-decreasing, we show that the slope of the allocation generated by the first price auction, ϕ'_{FP}, is at least 1. For more general densities, we do not have theoretical results, but have failed to find a case in which $\phi'_{FP} < 1$.

The key to the dominance result is to compare these allocations with the optimal one. When virtual costs, $\frac{F}{f}$, are convex, the optimal mechanism specifies a distortion that decreases in c_2. So, while a well chosen second price mechanism need not be optimal in such a setting, it is better, on a case by case basis, than the first price mechanism, which gets things precisely backward. When $\frac{F}{f}$ is concave our numerical examples suggest that under fairly general conditions, the optimal mechanism will specify a distortion that is similar at different c_2. Thus, the second price mechanism is not only likely to be better than the first price mechanism, but also close to optimal!

Our analysis contains a number of novel points. The connection between the concavity or convexity of $\frac{F}{f}$ and the shape of the optimal allocation is new, and suggests that there may be other interesting properties of how the optimal allocation relates to the structure of F. We also make some headway on the question of how F feeds into the choice of an optimal second price mechanism.

Our derivation of the result that $\phi'_{FP} > 1$ uses techniques that we have not seen before, but that seem likely have more generally applicability. In particular, the degree to which one can generate bounds on the surplus that can be available to each player, and use that to partially characterize equilibrium bid functions seems intriguing.

Two obvious topics for further research are to get a better understanding of the examples suggesting that $\phi'_{FP} > 1$ holds much more widely than when f is non-decreasing and to get a better understanding of why the optimal mechanism seems so generally to have slope near 1 when $\frac{F}{f}$ is concave. It would also be worth exploring how much broader the result on the superiority of the second price mechanism is if one relaxes the criterion from a case by case to expected buyer surplus basis. A model with more than two bidders seems highly relevant. Other simple auction forms, such as percentage auctions, deserve more consideration. Finally, our techniques suggest that it may be useful to make a further study of the properties of asymmetric first price auctions more generally.

Our results should be interesting to an economic theorist, but also to firms that engage in procurement. While simple, our model seems a good match for many practical settings in which the most common practice is either an open or sealed bid request for proposal. The open RFP corresponds to a second price bonus auction with $A = \Delta$. As we show, this is non-optimal. There is useful insight to firms in saying simply "look, you are better off to commit yourself to act as if your preferences are weaker than they truly are. What you lose in not always getting your favorite guy you will more than make up for in lower costs."

The stronger message is that because they are of a first price nature, sealed bid style RFPs are a very bad way to go once one understands how bidder optimization undoes the desired impact of the handicap structure. By inducing a more even distortion, second price mechanisms are likely to perform better. On a practical level, the fact that their equilibria are so simple to calculate may also have significant advantages. Industry practice is routinely at a variance to the advice we give here. We think it probable that these firms could improve their practices.[41]

Firms should make a distinction between ways in $\Delta \neq 0$ can arise. In some cases, Δ reflects some sort of incompatibility or lock-in rather than an innate preference for one supplier or another. In such cases, designing a better auction mechanism is good, but better still would be to design a setting where Δ is smaller in the first place. In the process of designed the Dreamliner, Boeing made it a priority to have both the physical connection between the aircraft engine and the wing and the software interface between the engine and the cockpit standardized across aircraft engine manufactur-

[41] We accept our expulsion from the Chicago school with quiet dignity.

ers. Moving from one engine manufacturer to another is thus considerably easier, and Δ is lowered. When Δ reflects more fundamental issues, the goal of a good auction design should be to allow the allocation to depend on whose product is better without giving away the store in terms of muting competition.

It is also worthwhile to think about relevant features of many real world settings that are not captured in our model. In our model, Δ is fixed and exogenous. Assume Δ is determined by pre-auction effort. Since $\frac{\partial A}{\partial \Delta} < 1$, the optimal auction in our model provides muted quality incentives at the first stage, while a request for proposals might do better. Exploring this formally and thinking about good mechanisms in such a setting are topics for future research.

10 Appendix

10.1 Results about Log-concavity

Karlin (1968) provides a useful inventory of properties related to log-concavity.

Lemma 50 *If f is a log-concave density then F and $\bar{F}$ are themselves log-concave. If f is a (strictly) log-concave density then $\frac{f}{F}$ is (strictly) decreasing and $\frac{f}{\bar{F}}$ is (strictly) increasing.*

10.2 Proofs for Section 4

Proof of Lemma 2: Consider an incentive compatible mechanism Ξ in which the buyer always buys. As noted in the text, Ξ is characterized by γ, and, adapting Myerson (1981) in the obvious ways to the setting, $BS(\gamma)$ is

$$
\begin{aligned}
& \int\int \left(\gamma(c_1, c_2)\left(v_1 - \omega\left(c_1\right)\right) + \left(1 - \gamma(c_1, c_2)\right)\left(v_2 - \omega\left(c_2\right)\right)\right) f\left(c_1\right) f\left(c_2\right) dc_1 dc_2 \\
= & \underbrace{\int\int \left(v_2 - \omega\left(c_2\right)\right) f\left(c_1\right) f\left(c_2\right) dc_1 dc_2}_{\text{Term 1}} \\
& + \int\int \gamma(c_1, c_2) \underbrace{\left(v_1 - v_2 - \left(\omega\left(c_1\right) - \omega\left(c_2\right)\right)\right)}_{\eta(c_1, c_2)} f\left(c_1\right) f\left(c_2\right) dc_1 dc_2.
\end{aligned}
$$

The lemma follows since (recalling $\omega(c_2) = c_2 + \frac{F}{f}(c_2)$, and integrating out c_1), Term 1 equals

$$\begin{aligned}
\int \left(v_2 - c_2 - \frac{F}{f}(c_2) \right) f(c_2)\, dc_2 &= v_2 - E(c_2) - \int F(c_2)\, dc_2 \\
&= v_2 - E(c_2) - 1 + \int (1 - F(c_2))\, dc_2 \\
&= v_2 - E(c_2) - 1 + E(c_2) \\
&= v_2 - 1. \ \blacksquare
\end{aligned}$$

Proof of Lemma 7: Since $\Delta = \omega(\phi_{M,\Delta}(c_2)) - \omega(c_2)$ for all Δ,

$$1 = \omega'(\phi_M(c_2)) \frac{\partial}{\partial \Delta} \phi_{M,\Delta}(c_2),$$

and hence

$$\frac{\partial}{\partial \Delta} \phi_{M,\Delta}(c_2) = \frac{1}{\omega'(\phi_M(c_2))} < 1,$$

since $\omega'(c) = 1 + \left(\frac{F(c)}{f(c)} \right)' > 1$ by log-concavity. $\blacksquare$

Proof of Lemma 8: From (4) ϕ_M is given by

$$\omega(\phi_M(c_2)) = \omega(c_2) + \Delta$$

and so, since ω is increasing, $\phi_M(c_2) > c_2$. Differentiating,

$$\omega'(\phi_M(c_2))\, \phi'_M(c_2) = \omega'(c_2),$$

and so

$$\phi'_M(c_2) = \frac{\omega'(c_2)}{\omega'(\phi_M(c_2))}.$$

If $\frac{F}{f}$,and thus ω, is convex, then, since $\phi_M(c_2) > c_2$, the bottom is bigger than the top, and so $\phi'_M(c_2) \leq 1$ (and strictly if $\frac{F}{f}$ is strictly convex). Similarly, if $\frac{F}{f}$ is concave, $\phi'_M(c_2) \geq 1$. $\blacksquare$

Proof of Lemma 10: Note that

$$\left(\frac{F}{f} \right)' = \frac{f^2 - f'F}{f^2} = 1 + \left(-\frac{f'}{f} \right) \left(\frac{F}{f} \right).$$

Since $f' \leq 0$, $-\frac{f'}{f}$ and $\frac{F}{f}$ are increasing and positive by log-concavity. $\blacksquare$

Proof of Lemma 13: Since $\frac{F_1}{F_2}$ is log-concave, $\frac{\partial}{\partial c}\ln\frac{F_2(c)}{F_1(c)}$ is increasing. Thus, since $\frac{\partial}{\partial c}\ln F_1(c)\frac{\partial}{\partial c}\ln F_2(c)$ is decreasing,

$$\frac{\partial}{\partial c}\left(\frac{\frac{\partial}{\partial c}\ln\frac{F_2(c)}{F_1(c)}}{\frac{\partial}{\partial c}\ln F_1(c)\frac{\partial}{\partial c}\ln F_2(c)}\right)\geq 0$$

and so

$$\frac{\partial}{\partial c}\left(\frac{1}{\frac{\partial}{\partial c}\ln F_1(c)}-\frac{1}{\frac{\partial}{\partial c}\ln F_2(c)}\right)\geq 0.$$

Equivalently,

$$\frac{\partial}{\partial c}\left(\frac{F_1(c)}{f_1(c)}-\frac{F_2(c)}{f_2(c)}\right)\geq 0$$

and so

$$1+\frac{\partial}{\partial c}\left(\frac{F_1(c)}{f_1(c)}\right)\geq 1+\frac{\partial}{\partial c}\left(\frac{F_2(c)}{f_2(c)}\right). \tag{34}$$

So, let (c_1,c_2) be on ϕ_{M,F_1}. Then, $c_1>c_2$ and

$$c_1+\frac{F_1(c_1)}{f_1(c_1)}=c_2+\frac{F_1(c_2)}{f_1(c_2)}+\Delta$$

or equivalently

$$\int_{c_2}^{c_1}\left(1+\frac{\partial}{\partial c}\left(\frac{F_1(s)}{f_1(s)}\right)\right)ds=\Delta.$$

But then, by (34)

$$\int_{c_2}^{c_1}\left(1+\frac{\partial}{\partial c}\left(\frac{F_2(s)}{f_2(s)}\right)\right)ds<\Delta,$$

and so (c_1,c_2) lies below ϕ_{M,F_2}. ■

10.3 Proofs for Section 5

Two prove Lemma 18, we need two preliminary lemmas.

Lemma 51 *Let*

$$r(c,A)=f(c+A)(\Delta-A)+\bar{F}(c+A)-F(c+A).$$

Then,

$$\frac{\partial BS}{\partial A}=\int_0^{1-A}r(c,A)f(c)dc. \tag{35}$$

Proof of Lemma 51: From (7)

$$\begin{aligned}
\frac{\partial BS}{\partial A} &= \int_0^{1-A} \eta\left(c+A,c\right) f\left(c+A\right) f\left(c\right) dc \\
&= \int_0^{1-A} \left(\left(A-\Delta\right) + \frac{F\left(c\right)}{f\left(c\right)} - \frac{F\left(c+A\right)}{f\left(c+A\right)} \right) f\left(c+A\right) f\left(c\right) dc \\
&= \int_0^{1-A} \left(A-\Delta\right) f\left(c+A\right) f\left(c\right) dc \qquad (36) \\
&\quad + \int_0^{1-A} F\left(c\right) f\left(c+A\right) dc - \int_0^{1-A} F\left(c+A\right) f\left(c\right) dc.
\end{aligned}$$

But,

$$\begin{aligned}
&\int_0^{1-A} r(c,A) f(c) dc \\
&= \int_0^{1-A} \left(\Delta - A\right) f(c) f(c+A) dc \qquad (37) \\
&\quad + \int_0^{1-A} \bar{F}(c+A) f\left(c\right) dc - \int_0^{1-A} F(c+A) f(c) dc.
\end{aligned}$$

Comparing (36) and (37), (35) is equivalent to

$$\int_0^{1-A} F\left(c\right) f\left(c+A\right) dc = \int_0^{1-A} \bar{F}(c+A) f(c) dc.$$

Each is an expression for $\Pr\left(c_1 > c_2 + A\right)$. ■

Lemma 52

$$r(0,A^*) f(0) + \int_0^{1-A^*} r(c,A^*) f'(c) dc > 0. \qquad (38)$$

Proof of Lemma 52: By log-concavity,

$$\frac{r(c,A)}{F(c+A)} = \frac{f(c+A)}{F(c+A)} \left(\Delta - A\right) - 1 + \frac{\bar{F}(c+A)}{F(c+A)},$$

is strictly decreasing in A and c. Since $\int_0^{1-A^*} r(c,A^*) f(c) dc = 0$, $r(\cdot, A^*)$ thus single-crosses 0 from above at a unique $y \in (0, 1-A^*)$. So, $r(0,A^*) f(0) \geq$

0. If $\frac{f'(c)}{f(c)}$ is a constant γ on $[0, y]$, then $0 < f(y) = f(0)\, e^{\gamma y}$, and so $f(0) > 0$. Hence $r(0, A^*)f(0) > 0$.

Since $r(c, A^*)f(c) > 0$ on $(0, y)$ and $r(c, A^*)f(c) < 0$ on $(y, 1 - A^*)$, and since $\frac{d}{dc} \log f(c) = \frac{f'(c)}{f(c)}$ is non-increasing,

$$\begin{aligned} \int_0^{1-A^*} r(c, A^*)f'(c)dc &= \int_0^{1-A^*} r(c, A^*)f(c)\frac{f'(c)}{f(c)}dc \\ &\geq \frac{f'(y)}{f(y)} \int_0^{1-A^*} r(c, A^*)f(c)dc \\ &= 0, \end{aligned}$$

by (40), with strict inequality if $\frac{f'(c)}{f(c)}$ is not constant on $[0, y]$. Thus, both terms of (38) are weakly positive, and whether $\frac{f'(c)}{f(c)}$ is constant on $[0, y]$ or not, at least one term is strictly positive. ■

Proof of Lemma 18: Note that

$$r_c(c, A) = f'(c + A)\,(\Delta - A) - 2f(c + A),$$

while

$$\begin{aligned} r_A(c, A) &= f'(c + A)\,(\Delta - A) - 3f(c + A) \\ &= r_c(c, A) - f(c + A). \end{aligned} \tag{39}$$

Using (51), consider any A^* where

$$\left.\frac{\partial BS}{\partial A}\right|_{A^*} = \left.\int_0^{1-A} r(c, A)f(c)dc\right|_{A=A^*} = 0. \tag{40}$$

Such an A^* exists, since $\left.\frac{\partial BS}{\partial A}\right|_{A=0} \geq 0$ and $\left.\frac{\partial BS}{\partial A}\right|_{A=\Delta} \leq 0$ by Corollary 17. We shall show that

$$\left.\frac{d}{dA} \int_0^{1-A} r(c, A)f(c)dc\right|_{A=A^*} < 0, \tag{41}$$

so that BS is quasi-concave and A^* is the unique optimal bonus.

Note first that by (39)

$$\begin{aligned}
& \frac{d}{dA}\int_0^{1-A} r(c,A)f(c)dc \\
= & \int_0^{1-A} r_A(c,A)f(c)dc - r(1-A,A)f(1-A) \\
= & \int_0^{1-A} \left(r_c(c,A) - f(c+A)\right)f(c)dc - r(1-A,A)f(1-A) \\
= & \int_0^{1-A} r_c(c,A)f(c)dc \qquad (42) \\
& -\int_0^{1-A} f(c+A)f(c)dc - r(1-A,A)f(1-A)
\end{aligned}$$

Integrating the first term by parts yields

$$r(1-A,A)f(1-A) - r(0,A)f(0) - \int_0^{1-A} r(c,A)f'(c)dc.$$

Substituting into (42) and cancelling,

$$\begin{aligned}
& \frac{d}{dA}\int_0^{1-A} r(c,A)f(c)dc \qquad (43) \\
= & \underbrace{-\left(r(0,A)f(0) + \int_0^{1-A} r(c,A)f'(c)dc\right)}_{\text{Term 1}} - \underbrace{\int_0^{1-A} f(c+A)f(c)dc}_{\text{Term 2}}.
\end{aligned}$$

Term 1 is positive at A^* by Lemma 52, while Term 2 is always positive, giving (41). ■

Proof of Lemma 21: Note that

$$\begin{aligned}
\frac{d}{d\Delta}\int_0^{1-A} r(c,A)f(c)dc &= \int_0^{1-A} r_\Delta(c,A)f(c)dc \\
&= \int_0^{1-A} f(c+A)f(c)dc.
\end{aligned}$$

Thus, by the implicit function theorem and (43),

$$\begin{aligned}
\frac{dA^*(\Delta)}{d\Delta} &= \frac{\frac{d}{d\Delta}\int_0^{1-A^*} r(c,A^*)f(c)dc}{-\frac{d}{dA}\int_0^{1-A^*} r(c,A^*)f(c)dc} \\
&= \frac{\int_0^{1-A^*} f(c)f(c+A^*)dc}{\underbrace{r(0,A^*)f(0)+\int_0^{1-A^*} r(c,A^*)f'(c)dc}_{>0 \text{ by Lemma 52}}+\int_0^{1-A^*} f(c)f(c+A^*)dc} \\
&\in (0,1). \ \blacksquare
\end{aligned}$$

Proof of Theorem 22: We suppress α for notational simplicity. Then

$$\int_0^{1-A^*} \eta\left(c+A^*,c\right) f(c)f(c+A^*)dc = 0$$

where we recall that

$$\eta\left(c+A,c\right) = \Delta - A - \frac{F(c+A)}{f(c+A)} + \frac{F(c)}{f(c)}.$$

By Lemma 19, since $\frac{F}{f}$ is concave by (11), $\eta\left(c+A,c\right)$ is increasing in c, and hence $\eta\left(c+A,c\right) f(c)f(c+A)$ single crosses zero from below. Hence,

$$\int_0^{1-A^*} \gamma(c)\eta\left(c+A,c\right) f(c)f(c+A^*)dc \geq 0 \tag{44}$$

for any increasing function $\gamma\left(\cdot\right)$. Note that

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\int_0^{1-A} \eta\left(c+A,c\right) f(c)f(c+A)dc \\
=& \int_0^{1-A} \eta\left(c+A,c\right) \frac{\partial}{\partial\alpha}\left(f(c)f(c+A)\right)dc + \int_0^{1-A}\left(\frac{\partial}{\partial\alpha}\eta\left(c+A,c\right)\right) f(c)f(c+A)dc \\
=& \int_0^{1-A} \eta\left(c+A,c\right)\left(\frac{\partial}{\partial\alpha}\log\left(f(c)f(c+A)\right)\right) f(c)f(c+A)dc \\
&+\int_0^{1-A} \frac{\partial}{\partial\alpha}\left(\frac{F(c)}{f(c)} - \frac{F(c+A)}{f(c+A)}\right) f(c)f(c+A)dc. \qquad (45)
\end{aligned}$$

The first term is non-negative by (44) since by (9), $\frac{\partial}{\partial\alpha}\log\left(f(c)f(c+A)\right)$ is increasing. The second term is non-negative since by (10),

$$\frac{\partial}{\partial\alpha}\left(\frac{F(c)}{f(c)} - \frac{F(c+A)}{f(c+A)}\right) \geq 0. \ \blacksquare$$

Proof of Lemma 23: Let $F(c,\alpha) = J^{\alpha}(c)$. Since $f(c,\alpha) = (J^{\alpha}(c))' = \alpha j(c) J(c)^{\alpha-1}$,

$$\frac{F(c,\alpha)}{f(c,\alpha)} = \frac{1}{\alpha}\frac{J(c)}{j(c)}$$

is increasing and concave, and since $\log f(c,\alpha) = \log\alpha + \log j(c) + (\alpha-1)\log J(c)$,

$$\frac{\partial^2}{\partial c \partial \alpha}\log f(c,\alpha) = \frac{j(c)}{J(c)} \geq 0.$$

Finally, since J is log-concave,

$$\frac{\partial^2}{\partial c\partial\alpha}\frac{F(c,\alpha)}{f(c,\alpha)} = \frac{\partial^2}{\partial c\partial\alpha}\left(\frac{1}{\alpha}\frac{J(c)}{j(c)}\right) = -\frac{1}{\alpha^2}\frac{\partial}{\partial c}\left(\frac{J(c)}{j(c)}\right) \leq 0. \blacksquare$$

Proof of Lemma 24: Fix α. Since $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial c}\log f(c,\alpha) \geq 0$ for all c and strictly for some c and since $f(\cdot,\alpha)$ is a density, $\frac{\partial}{\partial\alpha}f(\hat{c},\alpha) > 0$ for some $\hat{c}$ and so $\frac{\partial}{\partial\alpha}\frac{f(\hat{c},\alpha)}{F(\hat{c},\alpha)} > 0$ since $\frac{\partial}{\partial\alpha}F(\hat{c},\alpha) \leq 0$ (since $MLRP$ implies $FOSD$). Thus, $\frac{\partial}{\partial\alpha}\frac{F(\hat{c},\alpha)}{f(\hat{c},\alpha)} < 0$. Assume that $\frac{\partial}{\partial c}\frac{\partial}{\partial\alpha}\frac{F(c,\alpha)}{f(c,\alpha)} \geq 0$ everywhere. Then

$$\frac{\partial}{\partial\alpha}\frac{F(\hat{c},\alpha)}{f(\hat{c},\alpha)} = \frac{\partial}{\partial\alpha}\frac{F(0,\alpha)}{f(0,\alpha)} + \int_0^{\hat{c}}\frac{\partial}{\partial c}\frac{\partial}{\partial\alpha}\frac{F(c,\alpha)}{f(c,\alpha)}dc \geq 0,$$

since $\frac{F(0,\alpha)}{f(0,\alpha)} = 0$ for all α,[42] a contradiction. $\blacksquare$

10.4 Proofs for Section 6

Proof of Theorem 26**:** Let ϕ^I, ϕ^{II}, and ϕ^{III} be as in Definition 27. Then, we claim

$$\phi^I(c_2) = \phi^{II}(c_2) \tag{46}$$

and thus

$$\psi^I(c_1) = \psi^{II}(c_1). \tag{47}$$

To see (46), note that

$$\begin{aligned} \beta_1^{II}(\phi^I(c_2)) &= \beta_1^I(\phi^I(c_2)) - A && \text{(by (14))} \\ &= \beta_2^I(c_2) && \text{(by definition of } \phi^I\text{)} \\ &= \beta_2^{II}(c_2) && \text{(by (13))} \\ &= \beta_1^{II}(\phi^{II}(c_2)) && \text{(by definition of } \phi^{II}\text{).} \end{aligned}$$

[42] If $f(0) = 0$, then f is increasing at 0, and so near zero, $\frac{F(c)}{f(c)} < \frac{cf(c)}{f(c)} = c$. Thus $\frac{F(0)}{f(0)}$ is well defined and equal to 0 even if $f(0) = 0$.

Thus, since β_1^{II} is increasing, $\phi^I(c_2) = \phi^{II}(c_2)$.

Similarly, we claim

$$\phi^{II}(c_2) = \phi^{III}(c_2) + A \tag{48}$$

and thus

$$\psi^{III}(c_1 - A) = \psi^{II}(c_1).^{43} \tag{49}$$

To see (48), note that

$$\begin{aligned} \beta_2^{III}(c_2) &= \beta_1^{III}(\phi^{III}(c_2)) && \text{(by definition of } \phi^{III}) \\ &= \beta_1^{II}(\phi^{III}(c_2) + A) && \text{(by (15))}. \end{aligned}$$

Also

$$\begin{aligned} \beta_2^{III}(c_2) &= \beta_2^{II}(c_2) && \text{(by (13))} \\ &= \beta_1^{II}(\phi^{II}(c_2)) && \text{(by definition of } \phi^{II}). \end{aligned}$$

Combining, we have

$$\beta_1^{II}(\phi^{III}(c_2) + A) = \beta_1^{II}(\phi^{II}(c_2))$$

and so, since β_1^{II} is increasing,

$$\phi^{II}(c_2) = \phi^{III}(c_2) + A.$$

For $t \in \{I, II, III\}$, let $S_i^t(\widetilde{c}_i; c_i)$ be i's surplus in A^t when his true type is c_i but he submits the bid prescribed for $\tilde{c}_i$, given $\left(\beta_1^t, \beta_2^t\right)$.[44] Then,

$$\begin{aligned} S_1^{II}(\widetilde{c}_1; c_1) &= \bar{F}(\psi^{II}(\widetilde{c}_1))\left(\beta_1^{II}(\widetilde{c}_1) + A - c_1\right) \\ &= \bar{F}(\psi^{I}(\widetilde{c}_1))\left(\beta_1^{II}(\widetilde{c}_1) + A - c_1\right) && \text{(by (47)} \\ &= \bar{F}(\psi^{I}(\widetilde{c}_1))\left(\beta_1^{I}(\widetilde{c}_1) - c_1\right) && \text{(by (14))} \\ &= S_1^I(\widetilde{c}_1; c_1), \end{aligned}$$

[43] To see that (48) implies (49) note that (48) holds at $\psi^{II}(c_1)$ for all c_1 in the range of ϕ^{II}. Thus,

$$\begin{aligned} \phi^{II}(\psi^{II}(c_1)) &= \phi^{III}(\psi^{II}(c_1)) + A \\ c_1 - A &= \phi^{III}(\psi^{II}(c_1)) \\ \psi^{III}(c_1 - A) &= \psi^{II}(c_1). \end{aligned}$$

[44] By Lemma 25, the range of β_1^t is an interval for $t \in \{I, II, III\}$. Bids below $\beta_1^t(0)$ earn less than $\beta_1^t(0)$ since $\beta_1^t(0)$ already wins with probability 1. Bids above $\beta_1^t(1)$ never win. So, to be an equilibrium, it is necessary and sufficient that player 1 never wants to mimic another cost type, and analogously for player 2.

so that 1 has an incentive to deviate in A^I iff he has an incentive to deviate in A^{II}.

For 2, we similarly have

$$\begin{aligned} S_2^{II}(\tilde{c}_2; c_2) &= \bar{F}(\phi^{II}(\tilde{c}_2)) \left(\beta_2^{II}(\tilde{c}_2) - c_2\right) \\ &= \bar{F}(\phi^{I}(\tilde{c}_2)) \left(\beta_2^{II}(\tilde{c}_2) - c_2\right) \quad \text{(by (46))} \\ &= \bar{F}(\phi^{I}(\tilde{c}_2)) \left(\beta_2^{I}(\tilde{c}_2) - c_2\right) \quad \text{(by (13))} \\ &= S_2^{I}(\tilde{c}_2; c_2). \end{aligned}$$

Thus, $\left(\beta_1^I, \beta_2^I\right)$ is an equilibrium for A^I if and only $\left(\beta_1^{II}, \beta_2^{II}\right)$ is an equilibrium for A^{II}.

Similarly,

$$\begin{aligned} S_1^{III}(\tilde{c}_1; c_1) &= \bar{F}\left(\psi^{III}(\tilde{c}_1)\right) \left(\beta_1^{III}(\tilde{c}_1) - c_1\right) \\ &= \bar{F}\left(\psi^{II}(\tilde{c}_1 + A)\right) \left(\beta_1^{III}(\tilde{c}_1) - c_1\right) \quad \text{(by (49))} \\ &= \bar{F}_A\left(\psi^{II}(\tilde{c}_1 + A)\right) \left(\beta_1^{II}(\tilde{c}_1 + A) - c_1\right) \quad \text{(by (15))} \\ &= \bar{F}_A\left(\psi^{II}(\tilde{c}_1 + A)\right) \left(\beta_1^{II}(\tilde{c}_1 + A) + A - (c_1 + A)\right) \\ &= S_1^{II}(\tilde{c}_1 + A; c_1 + A) \end{aligned}$$

so that $c_1 \in [-A, 1-A]$ has an incentive to deviate in in A^{III} iff $c_1 + A \in [0,1]$ has an incentive to deviate in A^{II}, and

$$\begin{aligned} S_2^{III}(\tilde{c}_2; c_2) &= \bar{F}_A\left(\phi^{III}(\tilde{c}_2)\right) \left(\beta_2^{III}(\tilde{c}_2) - c_2\right) \\ &= \bar{F}_A\left(\phi^{III}(\tilde{c}_2)\right) \left(\beta_2^{II}(\tilde{c}_2) - c_2\right) \quad \text{(by (13))} \\ &= \bar{F}_A\left(\phi^{II}(\tilde{c}_2) - A\right) \left(\beta_2^{II}(\tilde{c}_2) - c_2\right) \quad \text{(by (48))} \\ &= \bar{F}\left(\phi^{II}(\tilde{c}_2)\right) \left(\beta_2^{II}(\tilde{c}_2) - c_2\right) \quad \text{(by definition of } \bar{F}_A) \\ &= S_2^{III}(\tilde{c}_2; c_2) \end{aligned}$$

so that $\left(\beta_1^{III}, \beta_2^{III}\right)$ is an equilibrium for A^{III} if and only if $\left(\beta_1^{II}, \beta_2^{II}\right)$ is an equilibrium for A^{II}. ■

Proof of Theorem 29: We proceed in a sequence of steps.

Step 1: Derivation of (19) **and** (20)**.** If 1 with type c bids as if his type is $\tilde{c}$, his surplus is $\hat{S}_1(\tilde{c}; c) = \bar{F}(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. By the envelope theorem,

$$\frac{\partial}{\partial c} S_1(c) = \frac{\partial}{\partial c} \hat{S}_1(\tilde{c}; c)\bigg|_{\tilde{c}=c} = \bar{F}(\psi(c)).$$

Given $b_1 \leq 1$, $S_1(1) = 0$, yielding (19). Similarly, $\frac{\partial}{\partial c} S_1(c) = \bar{F}(\phi(c))$, and for $c_2 > 1 - A$ no $b_2 > c_2$ ever wins, and so $S_2(1 - A) = 0$, yielding (20). ■

Step 2: Derivation of (21) **and** (22)**.** From (19) we have

$$\bar{F}(\psi(c))(\beta_1(c) - c) = S_1(c) = \int_c^1 \bar{F}(\psi(s))ds,$$

and (21) follows by rearranging, and similary for (22).

Step 3: Positive derivatives of β_1, β_2 **and** ϕ'**.** As increasing functions, $\beta_1(\cdot)$, $\beta_2(\cdot)$, $\phi(\cdot)$, and $\psi(\cdot)$ are differentiable almost everywhere. Note that a bid of $b_1 \in [\beta_2(0) + A, 1)$ by 1 wins with probability $\bar{F}\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)$. Pick $\hat{b}_1$ such that β_2 is differentiable at $\beta_2^{-1}\left(\hat{b}_1 - A\right)$, let $\hat{c}_2 = \beta_2^{-1}\left(\hat{b}_1 - A\right)$, and let $\hat{c}_1 = \beta_1^{-1}\left(\hat{b}_1\right)$. Then, since

$$\bar{F}\left(\beta_2^{-1}(b_1 - A)\right)(b_1 - \hat{c}_1)$$

is maximized at $\hat{b}_1$,

$$-f\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)\frac{1}{\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)}\left(\hat{b}_1 - \hat{c}_1\right) + \bar{F}\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right) = 0.[45] \tag{50}$$

Since $(b_1 - \hat{c}_1) > 0$ (because 1 earns positive surplus with $\hat{c}_1 < 1$, and $\hat{c}_1 < 1$ since $\hat{b}_1 < 1$), and since the other terms in (50) are finite but positive,

$$\frac{1}{\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right)}$$

is finite and positive as well, and so $\beta_2'\left(\beta_2^{-1}\left(\hat{b}_1 - A\right)\right) > 0$. Hence, wherever β_2' exists, $\beta_2' > 0$. Similarly, for $c_2 < 1 - A$, if β_1' exists at $\phi(c_2)$, then $\beta_1' > 0$. As $\beta_1(\phi(c)) = \beta_2(c) + A$, where β_1' and β_2' exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0. \tag{51}$$

[45] At $\hat{b}_1 = \beta_2(0) + A$, this is weakly negative.

Step 4: Derivation of (23) **and** (24)**.** Using (21)

$$\begin{aligned}
&\beta_1'(c) \\
= \quad & 1 + \left(\frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))} \right)' \\
= \quad & 1 + \frac{\left(\bar{F}(\psi(c))\right) \left(\int_c^1 \bar{F}(\psi(s))ds\right)' - \left(\int_c^1 \bar{F}(\psi(s))ds\right) \left(\bar{F}(\psi(c))\right)'}{\left(\bar{F}(\psi(c))\right)^2} \\
= \quad & 1 + \frac{\left(\bar{F}(\psi(c))\right) \left(-\bar{F}(\psi(c))\right) + \left(\int_c^1 \bar{F}(\psi(s))ds\right) f(\psi(c))\psi'(c)}{\left(\bar{F}(\psi(c))\right)^2} \\
= \quad & \int_c^1 \bar{F}(\psi(s))ds \frac{f(\psi(c))}{\bar{F}^2(\psi(c))}\psi'(c) \\
= \quad & S_1(c)\, g(\psi(c)) \frac{1}{\phi'(\psi(c))},
\end{aligned}$$

using that $\phi' > 0$ wherever β_1' and β_2' exist. Similarly,

$$\begin{aligned}
\beta_2'(c) \quad &= \quad \int_c^1 \bar{F}(\phi(s))ds \frac{f(\phi(c))}{\bar{F}^2(\phi(c))}\phi'(c) \\
&= \quad S_2(c)\, g(\phi(c))\, \phi'(c).
\end{aligned}$$

Step 5: Derivation of (25)**.** Substituting (23) and (24) into (51) gives

$$\phi'(c) = \frac{S_2(c)\, g(\phi(c))\, \phi'(c)}{S_1(\phi(c))\, g(\psi(\phi(c)))\, \frac{1}{\phi'(\psi(\phi(c)))}} = \frac{S_2(c)\, g(\phi(c))\, \phi'(c)}{S_1(\phi(c))\, g(c)\, \frac{1}{\phi'(c)}}.$$

Canceling $\phi'(c) > 0$ and rearranging yields (25).

Step 6: Continuous differentiability of ϕ. By (25),

$$\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{g(c)}{g(\phi(c))}$$

almost everywhere. As a bounded, continuous function on a compact interval, ϕ is absolutely continuous (see, e.g., Wade (1995)) and so (see, e.g., Billingsley, Theorem 31.8).

$$\phi(c) = \phi(0) + \int_0^c \phi'(t)\, dt = \int_0^c \frac{S_1(\phi(t))}{S_2(t)} \frac{g(t)}{g(\phi(t))} dt.$$

Since $\phi(\cdot)$ is continuous, $\frac{S_1(\phi(\cdot))}{S_2(\cdot)} \frac{g(\cdot)}{g(\phi(\cdot))} \in C^1[0, 1-A)$ and so $\phi(c) \in C^1[0, 1-A)$ by the fundamental theorem of calculus.

Step 7: $\phi \in C^{k+1}[0, 1-A)$**.** We show the result on $[0, a]$, $a < 1-A$. Since a is arbitrary, the result follows. Pick $a < 1-A$. From Step 6, $\phi(c)$ is C^1 on $[0, 1-A)$, with

$$\phi'(c) = \frac{\int_{\phi(c)}^{1} \bar{F}(\psi(s))ds}{\int_{c}^{1-A} \bar{F}(\phi(c))} \frac{\frac{f(c)}{\bar{F}^2(c)}}{\frac{f(\phi(c))}{\bar{F}^2(\phi(c))}}. \tag{52}$$

For $1 \leq \hat{k} \leq k$, assume that $\phi \in C^{\hat{k}}[0, a]$. Note that,

$$\left(\int_{\phi(c)}^{1} \bar{F}(\psi(s))ds \right)' = -\phi'(c) \bar{F}(c).$$

Since $\phi' \in C^{\hat{k}-1}[0, a]$ and $\bar{F} \in C^k[0, 1]$, $-\phi'(c) \bar{F}(c) \in C^{\hat{k}-1}[0, a]$, and so $\int_{\phi(c)}^{1} \bar{F}(\psi(s))ds \in C^{\hat{k}}[0, a]$. Similarly, $\left(\int_{c}^{1-A} \bar{F}(\phi(c)) \right)' = -\bar{F}(\phi(c))$, and so $\left(\int_{c}^{1-A} \bar{F}(\phi(c)) \right) \in C^{\hat{k}+1}[0, a]$. Since ϕ, f and $\bar{F}$ each belong to $C^k[0, a]$, and since all components of the RHS of (52) are everywhere positive on $[0, a]$, the RHS of (52) belongs to $C^{\hat{k}}[0, a]$ (see Shilov (1997). Thus, $\phi' \in C^{\hat{k}}[0, a]$, and so $\phi \in C^{\hat{k}+1}[0, a]$. By induction, $\phi \in C^{k+1}[0, a]$.

Step 8: $\beta_1 \in C^{k+1}[0, 1)$ **and** $\beta_2 \in C^{k+1}[0, 1-A)$**:** This follows immediately using the argument and conclusion of Step 7 applied to (23) and (24). ■

Proof of Lemma 31: Since $\phi(0) - 0 = 0$, and since ϕ is continuous, if there is $c < 1-A$ where $\phi(c) - c \geq A$, then there is $c^* < 1-A$ such that $\phi(c^*) - c^* = A$ and such that $\phi(c) - c$ is weakly increasing at c^*.

Player 1's first order condition at $\phi(c^*)$ can be expressed as

$$\frac{\beta_1'(\phi(c^*))}{\beta_1(\phi(c^*)) - \phi(c^*)} = \frac{f(c^*)}{\bar{F}(c^*)} \psi'(\phi(c^*)) = \frac{f(c^*)}{\bar{F}(c^*)} \frac{1}{\phi'(c^*)},$$

or

$$\frac{\bar{F}(c^*)}{f(c^*)} \frac{\phi'(c^*) \beta_1'(\phi(c^*))}{\beta_1(\phi(c^*)) - \phi(c^*)} = 1. \tag{53}$$

Since $\beta_1(\phi(c)) = \beta_2(c) + A$ for all c,

$$\phi'(c) \beta_1'(\phi(c)) = \beta_2'(c). \tag{54}$$

Since $\beta_1(\phi(c^*)) = \beta_2(c^*) + A$ and $\phi(c^*) - c^* = A$,

$$\beta_1(\phi(c^*)) = \beta_2(c^*) + \phi(c^*) - c^*$$

and so

$$\beta_1(\phi(c^*)) - \phi(c^*) = \beta_2(c^*) - c^*. \tag{55}$$

Substituting (54) and (55) into (53) gives

$$\frac{\bar{F}(c^*)}{f(c^*)} \frac{\beta_2'(c^*)}{\beta_2(c^*) - c^*} = 1.$$

But, 2's first order condition at c^* is

$$\frac{\beta_2'(c^*)}{\beta_2(c^*) - c^*} = \frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}\phi'(c^*).$$

Substituting,

$$\frac{\bar{F}(c^*)}{f(c^*)} \frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}\phi'(c^*) = 1,$$

and so

$$\phi'(c^*) = \frac{\frac{f(c^*)}{\bar{F}(c^*)}}{\frac{f(\phi(c^*))}{\bar{F}(\phi(c^*))}}.$$

But, $\frac{f}{\bar{F}}$ is strictly increasing and $\phi(c^*) > c^*$. Thus, $\phi'(c^*) < 1$, contradicting that $\phi(c) - c$ is weakly increasing at c^*. ■

Proof of Lemma 32**:** Since β_1 is increasing,

$$\liminf \frac{\beta_1(1) - \beta_1(c)}{1 - c} \geq 0.$$

Assume

$$\limsup \frac{\beta_1(1) - \beta_1(c)}{1 - c} = \alpha > 0. \tag{56}$$

For any c, $\beta_1(c)$ earns $\bar{F}(\psi(c))(\beta_1(c) - c)$,while a bid of 1 earns $\bar{F}(1 - A)(1 - c)$. Since $\beta_1(c)$ is a best response,

$$\bar{F}(\psi(c))(\beta_1(c) - c) \geq \bar{F}(1 - A)(1 - c),$$

and so for $c < 1$,

$$\frac{\beta_1(c) - c}{1 - c} \geq \frac{\bar{F}(1 - A)}{\bar{F}(\psi(c))}.$$

But, as $c \to 1$, $\psi(c) \to 1 - A$, and so

$$\liminf \frac{\beta_1(c) - c}{1 - c} \geq 1. \tag{57}$$

By (56), along a subsequence c_t, $c_t \to 1 - A$,

$$\frac{\beta_1(1) - \beta_1(c_t)}{1 - c_t} \geq \frac{\alpha}{2}.$$

But, since

$$\frac{\beta_1(1) - \beta_1(c_t)}{1 - c_t} + \frac{\beta_1(c_t) - c_t}{1 - c_t} = 1,$$

this means that $\frac{\beta_1(c_t) - c_t}{1 - c_t} < 1 - \frac{\alpha}{2}$ for all t, contradicting (57). So,

$$\liminf \frac{\beta_1(1) - \beta_1(c)}{1 - c} = \limsup \frac{\beta_1(1) - \beta_1(c)}{1 - c} = 0,$$

and β_1 is differentiable at 1, with $\beta_1'(1) = 0$.

Player 1's profit from mimicking $\tilde{c}$ is $\bar{F}(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. Taking the first order condition at $\tilde{c} = c$,

$$\bar{F}(\psi(c))\beta_1'(c) = f(\psi(c))\psi'(c)(\beta_1(c) - c).$$

But,

$$\psi'(c) = \frac{\beta_1'(c)}{\beta_2'(\psi(c))},$$

and so

$$\bar{F}(\psi(c))\beta_1'(c) = f(\psi(c))\frac{\beta_1'(c)}{\beta_2'(\psi(c))}(\beta_1(c) - c).$$

Cancelling $\beta_1'(c) > 0$, and rearranging,

$$\begin{aligned} \beta_2'(\psi(c)) &= \frac{f(\psi(c))}{\bar{F}(\psi(c))}(\beta_1(c) - c) \\ &< \frac{f(1 - A)}{\bar{F}(1 - A)}(1 - c). \end{aligned}$$

Thus, as $c \to 1$, $\beta_2'(\psi(c)) \to 0$. But then, $\beta_2'(1 - A)$ exists and equals 0. ■

Proof of Lemma 33: From Lemma 32, we have $\beta_1'(1) = 0$, and hence

$$\frac{1 - \beta_1(\phi(c))}{1 - \phi(c)} \to 0$$

as $c \to 1 - A$. But then, since for all c

$$\frac{1-\beta_1(\phi(c))}{1-\phi(c)} + \frac{\beta_1(\phi(c))-\phi(c)}{1-\phi(c)} = 1,$$

$$\frac{\beta_1(\phi(c))-\phi(c)}{1-\phi(c)} \to 1.$$

The proof for β_2 is identical. ■

In what follows we shall need

Lemma 53 $\limsup_{s\to 1}\left(\frac{\bar{F}(s)}{f(s)}\right)' \in [-1, 0]$.

Proof of Lemma 53: Since f is log-concave, f is unimodal, and hence monotone on $[c, 1]$ for some c. Assume first that f is non-decreasing on $[c, 1]$. Then, $\frac{f'(s)}{f(s)} \geq 0$ for all $s \in [c, 1]$, and so since $\frac{f'}{f}$ is non-decreasing by log-concavity, $\lim_{s\to 1}\left(\frac{f'(s)}{f(s)}\right)$ is well defined, finite, and positive. Hence, since $\frac{\bar{F}(s)}{f(s)} \to 0$,

$$\begin{aligned}
\lim_{s\to 1}\left(\frac{\bar{F}(s)}{f(s)}\right)' &= \lim_{s\to 1}\left(\frac{-f^2(s)-\bar{F}(s)f'(s)}{f^2(s)}\right) \\
&= -1 - \lim_{s\to 1}\left(\frac{\bar{F}(s)f'(s)}{f^2(s)}\right) \\
&= -1 - \lim_{s\to 1}\left(\frac{\bar{F}(s)}{f(s)}\right)\lim_{s\to 1}\left(\frac{f'(s)}{f(s)}\right) \\
&= -1.
\end{aligned}$$

Assume f is decreasing on $[c, 1]$ (this includes the case where $f(1) = 0$). Then, for $s \in (c, 1]$,

$$\begin{aligned}
0 &\geq \left(\frac{\bar{F}(s)}{f(s)}\right)' = \lim_{x\uparrow s}\left(\frac{\frac{\bar{F}(s)}{f(s)} - \frac{\bar{F}(x)}{f(x)}}{s-x}\right) \\
&\geq \lim_{x\uparrow s}\frac{1}{f(x)}\left(\frac{\bar{F}(s)-\bar{F}(x)}{s-x}\right) \\
&= \lim_{x\uparrow s}\frac{1}{f(x)}\left(\frac{-\int_x^s f(t)dt}{s-x}\right) \\
&\geq \lim_{x\uparrow s}\frac{1}{f(x)}\left(-\frac{f(x)\int_x^s dt}{s-x}\right) = -1. \ \blacksquare
\end{aligned}$$

Proof of Lemma 36: Assume that $\limsup \phi'(c) = \alpha \in (0, \infty)$. Now,

$$\begin{aligned}
\phi'(c) &= \frac{S_1(\phi(c))\, g(c)}{S_2(c)\, g(\phi(c))} \\
&= \frac{(\beta_1(\phi(c)) - \phi(c))\, \frac{f(c)}{\bar{F}(c)}}{(\beta_2(c) - c)\, \frac{f(\phi(c))}{\bar{F}(\phi(c))}} \\
&= \frac{(\beta_1(\phi(c)) - \phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}}{(\beta_2(c) - c)\, \frac{\bar{F}(c)}{f(c)}} \\
&= \frac{\frac{(\beta_1(\phi(c)) - \phi(c))}{1-\phi(c)}\,(1-\phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}}{\frac{\beta_2(c)-c}{1-A-c}\,(1-A-c)\, \frac{\bar{F}(c)}{f(c)}}.
\end{aligned}$$

But then, by Lemma 33

$$\begin{aligned}
\limsup \phi'(c) &= \lim \frac{\frac{(\beta_1(\phi(c)) - \phi(c))}{1-\phi(c)}}{\frac{\beta_2(c)-c}{1-A-c}} \limsup \frac{(1-\phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}}{(1-A-c)\, \frac{\bar{F}(c)}{f(c)}} \\
&= \limsup \frac{(1-\phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}}{(1-A-c)\, \frac{\bar{F}(c)}{f(c)}} \\
&= \frac{f(1-A)}{\bar{F}(1-A)} \limsup \frac{\left((1-\phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}\right)}{((1-A-c))}.
\end{aligned}$$ [46]

Since the top and bottom of

$$\frac{(1-\phi(c))\, \frac{\bar{F}(\phi(c))}{f(\phi(c))}}{(1-A-c)\, \frac{\bar{F}(c)}{f(c)}}$$

go to 0 as $c \to 1-A$, a generalization of l'Hôpital's rule (Lee (1977)) gives

$$\begin{aligned}
&\limsup \phi'(c) \\
\leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup \frac{\frac{\partial}{\partial c}\left((1-\phi(c))\frac{\bar{F}(\phi(c))}{f(\phi(c))}\right)}{\frac{\partial}{\partial c}((1-A-c))} \\
=\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup \frac{-\phi'(c)\frac{\bar{F}(\phi(c))}{f(\phi(c))} + (1-\phi(c))\left(\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\phi'(c)}{-1} \\
=\ & \frac{f(1-A)}{\bar{F}(1-A)} \left(\limsup \phi'(c)\left(\frac{\bar{F}(\phi(c))}{f(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\right)\right) \\
\leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \left(\limsup \phi'(c)\right) \limsup \left(\frac{\bar{F}(\phi(c))}{f(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\right)
\end{aligned}$$

Since $\limsup \phi'(c) \in (0,\infty)$ by assumption, cancel to obtain

$$\begin{aligned}
1 \quad \leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup \left(\frac{\bar{F}(\phi(c))}{f(\phi(c))} + (1-\phi(c))\left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\right) \\
\leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \left(\underbrace{\limsup\left(\frac{\bar{F}(\phi(c))}{f(\phi(c))}\right)}_{=0} + \limsup\left((1-\phi(c))\left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\right)\right) \\
=\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup \left((1-\phi(c))\left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)}\right).
\end{aligned}$$

But, by Lemma 53 $\limsup \left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)} \in [0,1]$. The term $(1-\phi(c))$ is bounded as well. Thus,

$$\begin{aligned}
1 \quad \leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup (1-\phi(c)) \limsup \left(-\frac{\bar{F}(s)}{f(s)}\right)'_{s=\phi(c)} \\
\leq\ & \frac{f(1-A)}{\bar{F}(1-A)} \limsup (1-\phi(c)) \\
=\ & 0,
\end{aligned}$$

a contradiction. ■

Proof of Lemma 39: Recall from Lemma 34 that

$$\frac{\phi''(c)}{\phi'(c)} = \phi'(c)\frac{\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}}{S_1(\phi(c))\,g(c)} + \frac{g'(c)}{g(c)} - \phi'(c)\frac{g'(\phi(c))}{g(\phi(c))}.$$

If $\phi'(c) = 1$, this reduces to

$$\begin{aligned}
\frac{\phi''(c)}{\phi'(c)} &= \frac{\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}}{S_1(\phi(c))\, g(c)} + \frac{g'(c)}{g(c)} - \frac{g'(\phi(c))}{g(\phi(c))} \\
&> \left(\frac{1}{S_1(\phi(c))\, g(c)} - 2\right)\left(\frac{f(\phi(c))}{\bar{F}(\phi(c))} - \frac{f(c)}{\bar{F}(c)}\right) \\
&= {}_s\left(\frac{1}{S_1(\phi(c))\, g(c)} - 2\right).
\end{aligned}$$

But,

$$\begin{aligned}
S_1(\phi(c))\, g(c) &= \int_{\phi(c)}^{1} \bar{F}(\psi(s))\, ds g(c) \\
&< \int_{\phi(c)}^{1} 1 ds g(c) \\
&= (1 - \phi(c))\, g(c)
\end{aligned}$$

which goes to 0. So, there is c^* such that for $c > c^*$, any point where $\phi' = 1$ has $\phi'' > 0$, and so there is at most one (upward) crossing of $\phi' = 1$ after that. Let $\hat{c}$ be that upward crossing if it exists, and $\hat{c} = c^*$ otherwise. ■

Proof of Theorem 49: Since $\phi'(c) \geq 1$,

$$\begin{aligned}
\frac{\partial}{\partial c} \ln \frac{\bar{F}(\phi(c))}{\bar{F}(c)} &= \phi'(c) \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} - \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=c} \\
&\leq \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} - \frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=c} \\
&\leq 0
\end{aligned}$$

where the first inequality holds since $\phi'(c) \geq 1$, and $\frac{\partial}{\partial s} \ln \bar{F}(s)\big|_{s=\phi(c)} < 0$ since $\bar{F}$ is decreasing, and the second inequality holds since $\bar{F}$ is log-concave and decreasing and $\phi(c) \geq c$. Thus, $\frac{\bar{F}(\phi(c))}{\bar{F}(c)}$is decreasing in c.

By Cauchy's theorem for all $c \in [0, 1 - A)$, there is $\xi_c \in [c, 1 - A)$ such that

$$\frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\int_c^1 \bar{F}(s)ds} = \frac{\bar{F}(\phi(\xi_c))}{\bar{F}(\xi_c)} \leq \frac{\bar{F}(\phi(c))}{\bar{F}(c)},$$

or, rearranging,

$$\frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} \geq \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))}.$$

Adding c to each side yields

$$\beta_2(c) = c + \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))} \leq c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} = \beta_s(c),$$

where $\beta_s(c)$ is the symmetric equilibrium bid function. Similarly,

$$\beta_1(c) = c + \frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))} \geq c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} = \beta_s(c). \quad \blacksquare$$

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