

# A Model of Delegated Project Choice\*

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— PRELIMINARY VERSION —

## Abstract

We present a model in which a principal delegates the choice of project to an agent with different preferences. A project’s characteristics are verifiable once presented to the principal, but the principal does not know how many projects are available to the agent. The principal chooses the set of projects which the agent can implement. Two frameworks are considered: (i) a static setting in which the set of available projects is exogenous to the agent but uncertain, and (ii) a dynamic setting in which the agent searches for projects.

## 1 Introduction

In this paper we present an analysis of a principal-agent problem in which the principal can influence the agent’s behaviour not by outcome-contingent rewards but by specifying what the agent is, and is not, allowed to do. The agent, whose preferences differ from those of the principal, will select from the permitted set the project that best serves her interests. The principal can verify whether or not a selected project is indeed within the permitted set, but cannot observe the number or characteristics of the projects available to the agent. How then should the principal specify the set of projects from which the agent is permitted to choose?

Our analysis was originally motivated by a question in competition policy towards mergers that is explained more fully in section 2. Should mergers be permitted provided that they are expected to enhance (or at least not to diminish) “total welfare”, or should the policy standard be in terms of consumer welfare? Even if the ultimate

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policy objective is total welfare, the answer is not obvious because the merger proposals brought forward by (profit-seeking) firms are selected by them from a wider pool of potential mergers. As we shall see, because of this selection effect, it could be that total welfare is higher on average under a consumer standard than a total welfare standard.

Our goal, however, is not to compare alternative *ad hoc* rules, such as the consumer and total welfare standards in the merger policy example. Rather, it is to characterize optimal permission sets in terms of the fundamentals of our models, including preferences and the probability distributions of project numbers and characteristics. Sometimes optimal permission rules are found to have strikingly simple forms. Two variants of our framework are analyzed.

The first is the model of project choice examined in section 3. This has a static setting in which the agent chooses one project from an exogenous, but uncertain, number of available projects. The principal restricts agent choice by disallowing some projects that are good, in the uncertain hope of inducing the agent instead to choose a project that is better for the principal. The second is the model of project search analyzed in section 4. This is in a dynamic setting where the agent can influence the uncertain arrival rate of projects by exerting costly effort. Here, the principal again may disallow some good projects but, in order to induce search effort, may allow some that are detrimental for his interests. In both models we show how optimal permission sets vary with preferences, probability distributions, project numbers, discount rates, and so on.

Some other papers have examined aspects of optimal delegation when contingent transfers between principal and agent are ruled out (as in our model). Aghion and Tirole (1997) show how, depending on information structure and payoff alignment, it may be optimal for a principal to delegate decision-making power to a better-informed agent. The principal's loss of control over project choice can be outweighed by advantages in terms of encouraging the agent to develop and gather information about projects. In like vein Baker, Gibbons, and Murphy (1999), though they deny formal delegation of authority, examine informal delegation through repeated-game relational contracts. Even an informed principal able to observe project payoffs may refrain from vetoing ones that yield him poor payoffs in order to promote search incentives for the agent.

Other models of delegated choice include Armstrong (1995) and Alonso and Matouschek (2008), both building on the pioneering work by Holmstrom (1984). Those models differ from ours in respect of project specification and the form of asymmetric information. In particular, they characterize each project by a scalar parameter, and the agent has private information about a payoff-relevant state of the world. For example, the agent might be a profit-seeking firm with market power, which has private information about cost, and is to choose a scalar level of price. The welfare-maximizing regulator constrains the set of permissible prices in ignorance of the cost level. Alonso and Matouschek provide conditions in such settings for "interval delegation" to be optimal – i.e., where it is optimal to restrict the agent to choose from a single interval (as with price cap regulation). Like Alonso and Matouschek, our aim is to characterize the optimal permission set for the principal to allow the agent to

choose from, but in the two-dimensional setting where the principal can observe both his own and the agent’s payoff from the project proposed by the agent, but does not know what other projects may be available to the agent.

The merger policy illustration that first motivated our work is but one example of optimal delegation in the absence of contingent payments between principal and agent. For instance, the framework could apply to aspects of decision making within a firm. A CEO of a company may, within limits, delegate project choice to a more junior manager, where the CEO is interested in shareholder value (for example), while the manager enjoys private benefits from certain projects. Alternatively, a single agent whose future tastes may differ from her current tastes may wish to constrain her future choices now, if feasible, which might fit into our proposed principal-agent framework. Having mentioned such applications, our aim in what follows is to offer a general analysis of how best to constrain the choices of an agent whose preferences are aligned imperfectly, if at all, with those of the principal.

## 2 Welfare Standards in Merger Policy

An important debate in antitrust policy concerns the appropriate welfare standard to use when deciding whether to permit a merger (or some other form of conduct). The two leading contenders are a *total welfare* standard, where mergers are evaluated according to whether they increase the unweighted sum of producer and consumer surplus, and a *consumer welfare* standard, where mergers detrimental to consumers are blocked. Many economic commentators feel that antitrust policy should aim to maximize total welfare, whereas in most jurisdictions the focus is more on consumer welfare alone. See Farrell and Katz (2006) for an excellent overview of the issues.

One purpose of this paper is to examine a particular strategic reason, discussed by Lyons (2002) and Fridolfsson (2007), to depart from the regulator’s true welfare standard, which is that a firm may have a *choice* of merger possibilities. A less profitable merger might be better for total welfare, but will not be chosen under a total welfare standard. To illustrate, consider Figure 1, which is similar to those presented in section IV.B in Farrell and Katz (2006).<sup>1</sup>

Here,  $u$  represents the gain in total profit resulting from a merger, while  $v$  measures the resulting net gain (which may be negative) to consumers. Suppose that  $u$  and  $v$  are verifiable once a merger is proposed to the competition authority. If the regulator follows a total welfare standard, he will permit any merger which lies above negatively-sloped line in the figure. Suppose the firm has two mergers to choose from, depicted by  $\blacktriangle$  and  $\blackstar$  on the figure. With a total welfare standard, the firm will choose the merger with the higher  $u$  payoff, i.e., the  $\blacktriangle$  merger. However, the regulator would prefer the alternative  $\blackstar$  since that yields higher total welfare. If the regulator instead imposed a consumer welfare standard, so that only those mergers which lie above the horizontal line  $v = 0$  are permitted, then the firm will be forced to choose the preferred merger. In this case, a regulator wishing to maximize total welfare is better off if he

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<sup>1</sup>The discussion in Farrell and Katz (2006) is a “reduced-form” version of the formal models in Lyons (2002) and Fridolfsson (2007).

imposes a consumer welfare standard. As Farrell and Katz (2006, page 17) put it: “if we want to maximize gains in total surplus (northeasterly movements as shown in figure [1]) and firms always push eastwards, there is something to be said for someone adding a northerly force.”

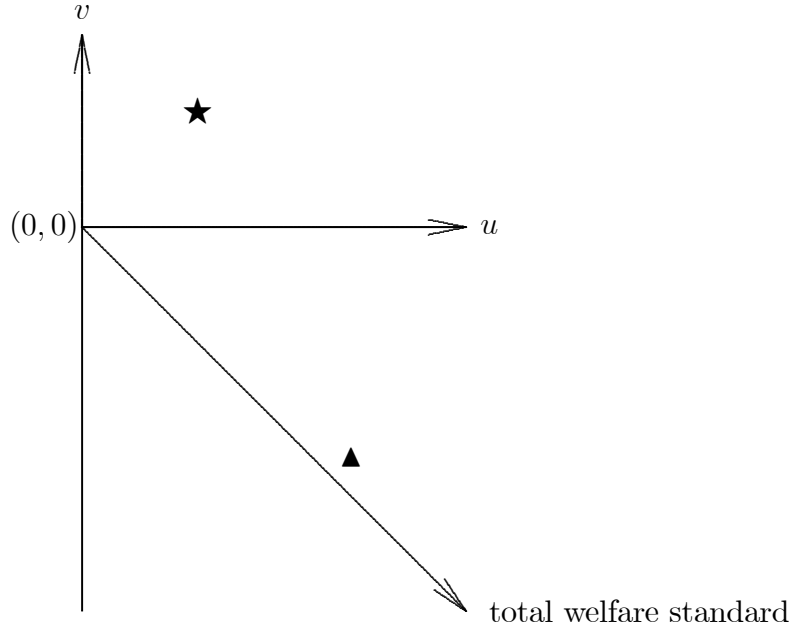


Figure 1: The Impact of Welfare Standard on Chosen Mergers

Nevertheless, there is a potential cost to adopting a consumer welfare standard: if the  $\blacktriangle$  merger turns out to be the *only* possible merger then a consumer welfare standard will not permit this even though the merger will improve total welfare. Thus, the choice of welfare standard will depend on the number of possible mergers and the distribution of profit and consumer surplus gains for a possible merger. For instance, as Farrell and Katz observe, if efficiency gains from a merger take the form of reductions in fixed, not marginal, costs, any merger between competitors can only cause reductions in consumer surplus and so a consumer welfare standard would forbid all mergers (including those which increase total welfare).

Our aim in this paper is to examine in a systematic fashion how the likely number of available projects (mergers in this application), the cost of searching for potential projects, and the distribution of respective payoffs to the principal and agent should determine the shape of the permission set. In our first model in section 3 we show that the principal forbids some projects which are desirable ex post, which could be interpreted as strategically putting less weight on the agent’s payoff than is in the true welfare function. In our second model in section 4, we find cases in which the principal allows all desirable projects together with some projects which are undesirable ex post; in such examples the principal puts more weight on the agent’s payoff than is in the true welfare function.

### 3 Choosing a Project

A *principal* (“he”) delegates the choice of project to an *agent* (“she”). There may be several projects to choose from, although only one can be implemented over the relevant time horizon. We will consider two variants of the delegated choice problem: (i) a static setting in which the agent can choose one project from an exogenous but random number of available projects (as analyzed in this section); and (ii) a dynamic search model in section 4 in which the agent can choose the arrival rate of potential projects.

A *project* is fully described by two parameters,  $u$  and  $v$ . The agent’s payoff if the type- $(u, v)$  project is implemented is  $u$ , while the principal’s payoff is  $v + \alpha u$ . Here,  $\alpha \geq 0$  represents the weight the principal places on the agent’s interests, and  $v$  represents factors specific to the principal’s interests. In the merger context,  $\alpha = 1$  when the antitrust authority wishes to maximize total welfare.

Each project is an independent draw from the same distribution for  $(u, v)$ . Since the agent will never propose a project with a negative payoff, without loss of generality we suppose that only non-negative  $u$  are realized. The marginal density of  $u \geq 0$  is  $f(u)$ . The conditional density of  $v$  given  $u$  is denoted  $g(v, u)$  and the associated conditional distribution function is  $G(v, u)$ . Here,  $v$  can be positive or negative. For technical reasons, suppose that the support of  $(u, v)$  is a rectangle  $[0, u_{\max}] \times [v_{\min}, v_{\max}]$ . In particular, the support of  $v$  given  $u$  does not depend on  $u$ . Suppose also that  $v_{\min} \leq 0 \leq v_{\max}$ , so that  $(0, 0)$  lies in the support of  $(u, v)$ . Finally, suppose that both  $f$  and  $g$  are twice continuously differentiable.

The principal delegates the choice of project to the agent. (We assume that it is not possible, or credible, for the principal to give monetary incentives to the agent to choose a desirable project.) Once the agent selects a particular project from the set of possible projects, that project’s characteristics are verifiable.<sup>2</sup> The mechanism we analyze is as follows. Before the agent has any private information, the principal commits to a (measurable) “permitted set” of projects, denoted  $\mathcal{P} \subset [0, u_{\max}] \times [v_{\min}, v_{\max}]$ , and the agent can implement any project she chooses in  $\mathcal{P}$ .

Note that commitment is important here. We will see later in this section that the principal will exclude some desirable projects from the permitted set. Thus, if the agent facing this permitted set reports that she has no project which is permitted, the principal has an incentive to “renegotiate”, and to widen the set of permitted projects. Similarly, in section 4 we find cases in which the principal permits projects which are undesirable ex post, and this again requires commitment power for this to be credible. Note also that we restrict attention to deterministic mechanisms, rather than, say, assigning to each proposed project a (possibly interior) probability  $\pi(u, v)$  of that project being implemented. We do this since the more general analysis seems intractable and also because it is hard to imagine being able to commit to or

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<sup>2</sup>What is needed here is that the principal and agent have *symmetric* information about the eventual project characteristics when the project is proposed. The actual outcome of the project could still be uncertain at this point, in which case  $u$  and  $v$  represent the expected payoffs to the two parties.

implement a stochastic mechanism in practice.<sup>3</sup>

In this first model, the number of projects is random and the probability that the agent has exactly  $n \geq 0$  possible projects is  $q_n$ . (Our analysis applies to the case where there are surely  $N$  projects, so that  $q_N = 1$ , but the analysis is no easier for that case.) The realization of  $(u, v)$  for each project is described by  $f$  and  $g$  as above, and is independent for each project among the  $n$  projects. In addition,  $(u, v)$  is distributed independently of  $n$ .

For each  $u$ , let

$$\mathcal{P}_u = \{v \text{ such that } (u, v) \in \mathcal{P}\}$$

be the set of  $u$ -projects which are permitted, and let

$$p(u) = \int_{v \in \mathcal{P}_u} g(v, u) dv$$

to be the proportion of type- $u$  projects which are permitted. Let

$$x(u) = 1 - \int_u^{u_{\max}} p(z) f(z) dz$$

to be the probability that any given project either has agent payoff  $z$  less than  $u$  or is not permitted. Note that

$$x'(u) = p(u) f(u) . \tag{1}$$

If there are exactly  $n \geq 1$  available projects, the probability that the agent's preferred permitted project has payoff no higher than  $u$  is  $(x(u))^n$ , and so the density of the agent's preferred permitted project is

$$\frac{d}{du} (x(u))^n = np(u) f(u) (x(u))^{n-1} .$$

(One of the  $n$  projects must be permitted and give agent payoff  $u$ , which has probability  $p(u) f(u)$ , while the remaining  $n - 1$  projects must either have agent payoff lower than  $u$  or not be permitted.) Summing over  $n$  implies that the density of the highest- $u$  permitted project is

$$\frac{d}{du} \sum_{n=0}^{\infty} q_n (x(u))^n .$$

If we write  $\phi(x) \equiv \sum_{n=0}^{\infty} q_n x^n$  for the *probability generating function* (or PGF) associated with the random variable  $n$ , it follows that the density of the agent's preferred permitted project is  $\frac{d}{du} \phi(x(u))$ . Useful properties of PGFs which we will sometimes use is that they are well-defined on the interval  $0 \leq x \leq 1$  and convex and increasing over this interval.

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<sup>3</sup>See page 67 in Laffont and Martimort (2002), for instance. Kovác and Mylovanov (2007) find conditions within the Holmstrom (1984) framework where deterministic mechanisms are optimal even when stochastic mechanisms are possible.

The principal's payoff with the permission set  $\mathcal{P}$  is

$$\begin{aligned} & \int_0^{u_{\max}} \{E[v \mid u \text{ and } v \in \mathcal{P}_u] + \alpha u\} \frac{d}{du} \phi(x(u)) \, du \\ &= \int_0^{u_{\max}} \left\{ \int_{v \in \mathcal{P}_u} v g(v, u) dv + \alpha u p(u) \right\} \phi'(x(u)) f(u) \, du . \end{aligned}$$

The principal's problem is to maximize this expression taking into account the relationship between  $p$  and  $x$  in (1) and the endpoint constraint  $x(u_{\max}) = 1$ . The following lemma shows that the optimal permitted set is monotonic in  $v$ .

**Lemma 1** *In the optimal policy there exists a threshold rule  $r(\cdot)$  such that*

$$(u, v) \in \mathcal{P} \text{ if and only if } v \geq r(u) .$$

**Proof.** From (1), the function  $x(\cdot)$  depends on  $\mathcal{P}$  only via the “sufficient statistic”  $p(u)$ , not on the particular  $v$ -projects which are permitted given  $u$ . Therefore, for any candidate function  $p(u)$  the principal might as well permit those particular  $v$ -projects which maximize the term  $\{\cdot\}$  in the above integrand, subject to the constraint that the proportion of type- $u$  projects is  $p(u)$ . But the problem of choosing the set  $\mathcal{P}_u$  in order to

$$\text{maximize } \int_{v \in \mathcal{P}_u} v g(v, u) dv \text{ subject to } \int_{v \in \mathcal{P}_u} g(v, u) dv = p(u)$$

is clearly solved by permitting the projects with the highest  $v$  so that the proportion of permitted projects is  $p(u)$ , i.e., that  $\mathcal{P}_u = \{v \text{ such that } v \geq r(u)\}$  for some  $r(u)$ . Then  $p(u) = 1 - G(r(u), u)$ . ■

This preliminary result shows that there are no “holes” in the permitted set.<sup>4</sup> In particular, the problem simplifies to the choice of threshold function  $r(\cdot)$  rather than the choice of permitted set  $\mathcal{P}$ . (A similar argument is valid in the other two models later in the paper.) Figure 2 depicts a possible threshold function  $r(\cdot)$ , and also shows  $x(u)$  depicted as the shaded area.

It is useful to introduce one further piece of notation, and define

$$V(r, u) \equiv E[v \mid u \text{ and } v \geq r]$$

to be the expected value of  $v$  given that the project has agent payoff  $u$  and that  $v$  is at least  $r$ . Recast in terms of  $r(\cdot)$  rather than  $\mathcal{P}$ , the principal's problem is to choose  $r(\cdot)$  to maximize

$$\int_0^\infty [V(r(u), u) + \alpha u][1 - G(r(u), u)] f(u) \phi'(x(u)) \, du \quad (2)$$

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<sup>4</sup>The information structure we assume gives us this result almost “for free”, in contrast to the more intricate analysis needed to establish when “interval discretion” is optimal in Alonso and Matouschek (2008). (Indeed, Alonso and Matouschek provide examples in which it is optimal to leave holes in the optimal delegation set.)

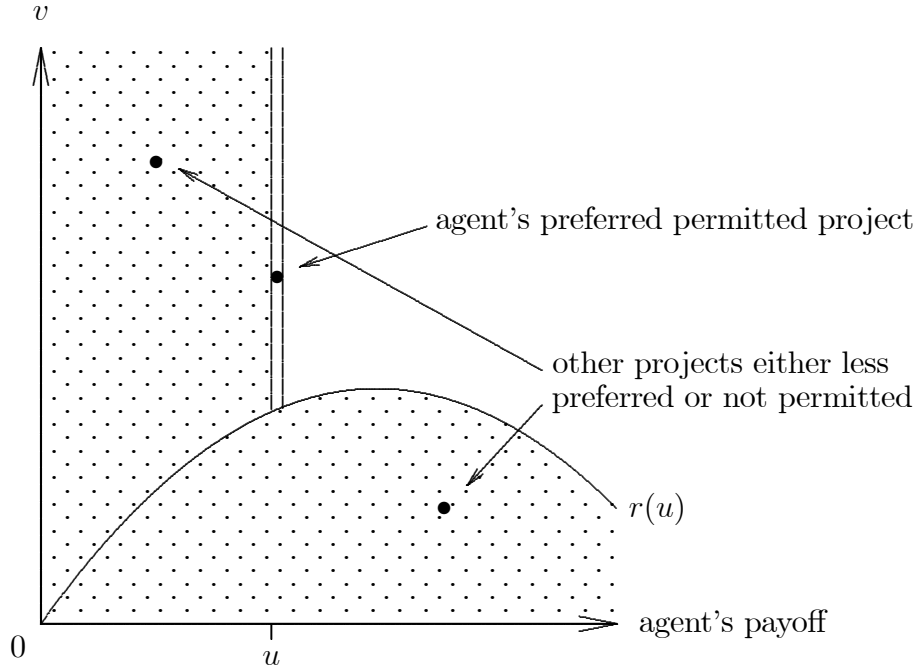


Figure 2: The Agent's Preferred Permitted Project

subject to the “equation of motion”

$$x'(u) = f(u)[1 - G(r(u), u)] \quad (3)$$

and the endpoint condition  $x(u_{\max}) = 1$ .

This classical calculus of variations problem is solved formally in the appendix, but its solution can be understood intuitively with the following argument. Consider some point  $\{u, r(u)\}$  on the frontier of the permitted set. For the set to be optimal it must be that the principal is indifferent between his payoff  $[r(u) + \alpha u]$  at that point and his expected payoff from the agent's next-best alternative, conditional upon the agent's best permitted project being at the point on the frontier.

To calculate the latter expected payoff, note that the density that a project drawn at random has payoffs  $\{u, r(u)\}$  is  $f(u)g(r(u), u)$ . Then the probability that one out of  $n$  projects has payoffs  $\{u, r(u)\}$  and all the others have utility no greater than  $z$  or are not permitted is  $n f(u)g(r(u), u)[x(z)]^{n-1}$ . Taking the  $q_n$ -weighted sum across  $n$ , the probability that one project has payoffs  $\{u, r(u)\}$  and all other permitted projects have utility no greater than  $z$  is therefore  $f(u)g(r(u), u)\phi'(x(z))$ . In particular, the probability that the agent's best permitted project is the point on the frontier is  $f(u)g(r(u), u)\phi'(x(u))$ . Conditional on that, the probability that the next-best alternative for the agent has utility no greater than  $z$  is

$$\frac{f(u)g(r(u), u)\phi'(x(z))}{f(u)g(r(u), u)\phi'(x(u))} = \frac{\phi'(x(z))}{\phi'(x(u))},$$

which has associated density  $\phi''(x(z))x'(z)/\phi'(x(u))$ . Therefore the indifference con-



dition is that

$$r(u) + \alpha u \equiv \frac{1}{\phi'(x(u))} \int_0^u [V(r(z), z) + \alpha z] \phi''(x(z)) x'(z) dz \quad (4)$$

for all  $u$ . In particular, we see that  $r(0) = 0$ . This implies that the principal does not wish to limit the desirable projects available to the agent whose best project has only zero payoff, i.e., there is “no distortion at the bottom”. The reason for this is that when  $u = 0$  there is no strategic benefit to restricting choice. (The strategic effect of raising  $r(u)$  above  $-\alpha u$  is to increase the probability that the agent will choose a smaller  $z$ , and this effect cannot operate when  $u = 0$ .) Differentiating (4) with respect to  $u$  and using (3) implies the Euler equation for the principal’s problem, which is expression (5) below.

**Proposition 1** *The optimal threshold rule satisfies the Euler equation*

$$r'(u) + \alpha = [V(r(u), u) - r(u)][1 - G(r(u), u)] f(u) \frac{\phi''(x(u))}{\phi'(x(u))} \quad (5)$$

with initial condition  $r(0) = 0$ , terminal condition  $x(u_{\max}) = 1$ , and equation of motion (3). A sufficient condition for the second-order condition for the calculus of variations problem to be satisfied is that

$$\zeta(x) \equiv \frac{\phi''(x)}{\phi'(x)} \text{ weakly decreases with } x. \quad (6)$$

**Proof.** See appendix. ■

Expression (5) reveals that  $\zeta$  in (6) is important for the form of the solution. A short list of examples for this term includes:

- If  $n$  is known to be exactly  $N \geq 1$  for sure (so  $q_N = 1$ ), then  $\phi(x) = x^N$  and  $\zeta(x) = (N - 1)/x$ .
- If  $n$  is Poisson with mean  $\mu$  (so  $q_n = e^{-\mu} \frac{\mu^n}{n!}$  for  $n \geq 0$ ) then  $\phi(x) = e^{-\mu(1-x)}$  and  $\zeta(x) \equiv \mu$ .
- If  $n$  is Binomial (the sum of  $N$  Bernoulli variables with success probability  $a$ ) then  $\phi(x) = (1 - a(1 - x))^N$  and  $\zeta(x) = a(N - 1)/(1 - a(1 - x))$ . The “known  $n$ ” case is a special case of the Binomial with  $a = 1$ . The Poisson is a limit case of the Binomial when  $aN = \mu$  and  $a \rightarrow 0$ .
- If  $n$  is Geometric (so  $q_n = (1 - a)a^{n-1}$  for  $n \geq 1$  and some parameter  $a \in (0, 1)$ ) then  $\phi(x) = (1 - a)x/(1 - ax)$  and  $\zeta(x) = 2a/(1 - ax)$ . Here, the expected number of projects is  $1/(1 - a)$ .
- If  $n = 2$  with probability  $\pi$  and  $n = 1$  with probability  $1 - \pi$ , then  $\phi(x) = (1 - \pi)x + \pi x^2$  and  $\zeta(x) = 1/(k + x)$  where  $k = (1 - \pi)/(2\pi)$ .

Notice that in all cases,  $\zeta(x)$  takes the general form  $A/(B + Cx)$ . Assumption (6) is valid for the Binomial distribution—and hence for the “known  $n$ ” and Poisson sub-cases—and for the  $n \in \{1, 2\}$  example, but not for the Geometric distribution.

Define the “naive” threshold rule to be

$$r_{naive}(u) = -\alpha u .$$

This is the threshold rule which permits all desirable projects, i.e., those projects such that  $v + \alpha u \geq 0$ . This rule might be implemented by a principal who does not foresee that the agent will only propose projects with the highest  $u$  whenever the agent has a choice.

The right-hand side of (5) is necessarily non-negative. Therefore, since  $r'(u) + \alpha \geq 0$  and  $r(0) = 0$  it follows that

$$r(u) \geq r_{naive}(u)$$

and so the principal never includes undesirable projects within the permitted set. Moreover, the function  $r(u) - r_{naive}(u)$  is weakly increasing. We state this formally as:

**Corollary 1** *The gap between the optimal threshold rule  $r(u)$  and the naive threshold rule  $r_{naive}(u)$  widens with  $u$ . In particular, in the case where  $\alpha = 0$ , the optimal threshold rule increases with  $u$ .*

In the degenerate case where the agent never has more than one project (i.e.,  $q_0 + q_1 = 1$ ),  $\phi'' = 0$  and so (5) implies that  $r(u) + \alpha u \equiv 0$ , and the principal permits all desirable projects. In this case, the naive rule is optimal. Outside this dull case, though,  $\phi'' > 0$  and (5) implies that

$$r(u) > r_{naive}(u) \text{ if } u > 0 .$$

We state this formally as

**Corollary 2** *Whenever the agent sometimes has a choice of project (i.e., when  $q_0 + q_1 < 1$ ) it is optimal for the principal to exclude strictly desirable projects.*

What is the intuition for why the principal wishes to exclude some desirable projects from the permitted set, whenever the agent sometimes has a choice of project? Suppose the principal initially allows all desirable projects, so that  $r(u) \equiv r_{naive}(u)$ . If the principal increases  $r(\cdot)$  slightly at some  $u > 0$ , the direct cost is approximately zero, since the principal then excludes projects about which he is almost indifferent (since  $r(u) + \alpha u = 0$ ). But there is a strictly beneficial strategic effect: there is some chance that the agent’s highest- $u$  project is excluded by the modified permitted set, in which case there is a chance that she chooses another project which is permitted, say with  $z < u$ . This alternative project is unlikely to be marginal for the principal, and instead the principal will expect to get payoff  $V(r(z), z) + \alpha z$ , which is strictly positive when  $r(z) = -\alpha z$ . This argument indicates that it is beneficial to restrict desirable projects, and not to permit undesirable projects. Moreover, it is

intuitive that the strategic effect is more important for higher  $u$ , since it applies over a wider range  $z < u$ , and this explains why the gap  $r(u) - r_{naive}(u)$  widens with  $u$  (Corollary 1).

A more substantial corollary is the following, which shows that a greater fraction of projects should be permitted when more weight is placed on the agent's interests.

**Proposition 2** *Let  $\alpha_L$  and  $\alpha_H$  be two possible weights placed by the principal on the agent's payoff, where  $\alpha_L < \alpha_H$ . Let  $r_i(\cdot)$  and  $x_i(\cdot)$  solve the Euler equation (5) when  $\alpha = \alpha_i$  for  $i = L, H$ . If assumption (6) holds then  $x_L(0) > x_H(0)$ , i.e., the fraction of permitted projects increases with  $\alpha$ .*

**Proof.** Condition (5) implies that at  $u = 0$  and any other  $u$  such that  $r_L(u) = r_H(u)$

$$\frac{r'_L(u) + \alpha_L}{r'_H(u) + \alpha_H} = \frac{\zeta(x_L(u))}{\zeta(x_H(u))} . \quad (7)$$

If  $x_L(0) \leq x_H(0)$ , then by assumption (6)  $\zeta(x_L(0)) \geq \zeta(x_H(0))$ , and so (7) implies that  $r'_L(0) > r'_H(0)$ . In particular,  $r_L(u) > r_H(u)$  for small  $u > 0$ . If  $x_L(0) \leq x_H(0)$  then  $r_L(\cdot)$  must cross  $r_H(\cdot)$  at some point. (If  $r_L$  were uniformly above  $r_H$  then clearly the fraction of prohibited projects with  $\alpha_L$  would be greater than with  $\alpha_H$ .) Let  $u^*$  be the first point above zero where the curves cross. In particular, we must have  $r'_L(u^*) \leq r'_H(u^*)$ . In addition, we must have  $x_H(u^*) \geq x_L(u^*)$  since  $x_H(0) \geq x_L(0)$  and  $r_H(u) \leq r_L(u)$  for  $u \leq u^*$ . But then (7) implies that

$$1 > \frac{r'_L(u^*) + \alpha_L}{r'_H(u^*) + \alpha_H} = \frac{\zeta(x_L(u^*))}{\zeta(x_H(u^*))} \geq 1 ,$$

a contradiction. We deduce that the curves can never cross, and so our initial assumption that  $x_L(0) \leq x_H(0)$  cannot hold. ■

Thus we see that the more the principal cares about the utility of the agent, the more discretion—in the sense that a greater fraction of projects are permitted—the agent is given. This result is similar to Holmstrom (1984) and Armstrong (1995), where the more likely the agent's preferences were to be close to the principal's, the more discretion the agent was given. It is also simple to show that the agent's expected payoff with the principal's optimal scheme increases in  $\alpha$ .<sup>5</sup>

It is also intuitive that when the agent is likely to have more projects to choose from, the principal will further constrain the permitted set of projects. With more projects available, the agent is likely to have at least one which lies close to the principal's preferred project. There is a close connection between “having more projects” and the function  $\zeta(x)$  being shifted upwards. For instance, in our five examples of distributions for  $n$  above, this is true. (In the Poisson case,  $\zeta$  is just equal to the expected number of projects.) More generally, a natural interpretation of “having more projects” is the following: suppose the number of agent's projects is initially

<sup>5</sup>When  $\alpha = \alpha_H$  the principal prefers  $r_H(u)$  to  $r_L(u)$  and vice versa when  $\alpha = \alpha_L$ . It follows by the principal's revealed preference that expected  $u$  is higher with  $\alpha_H$  than  $\alpha_L$ .

governed by the PGF  $\phi_L(\cdot)$ , and suppose the agent then has access to an additional source of projects which has PGF  $\phi(\cdot)$ , say. A fundamental property of PGFs is that the PGF of the independent sum of two random variables is the product of the underlying PGFs. Therefore, the PGF for the agent with these two sources of projects is just  $\phi_H = \phi\phi_L$ . Under a regularity condition, it turns out that  $\phi_H''/\phi_H'$  is greater than  $\phi_L''/\phi_L'$ :

**Lemma 2** *Suppose that  $1/\phi_L(x)$  is a strictly convex function, and let  $\phi_H(x) = \phi(x)\phi_L(x)$  for any PGF  $\phi(\cdot)$  except the trivial PGF  $\phi \equiv 1$ . Let  $\zeta_i = \phi_i''/\phi_i'$  for  $i = L, H$ . Then*

$$\zeta_H(x) > \zeta_L(x) . \quad (8)$$

**Proof.** Condition (8) is equivalent to the condition that  $\phi_H'(x)/\phi_L'(x)$  strictly increases with  $x$ . But

$$\frac{\phi_H'}{\phi_L'} = \frac{\phi'\phi_L + \phi\phi_L'}{\phi_L'} = \phi + \phi'\frac{\phi_L}{\phi_L'}$$

and so

$$\frac{d}{dx} \frac{\phi_H'}{\phi_L'} = \phi''\frac{\phi_L}{\phi_L'} + \phi' \left[ 2 - \frac{\phi_L\phi_L''}{(\phi_L')^2} \right] \geq \phi' \left[ 2 - \frac{\phi_L\phi_L''}{(\phi_L')^2} \right] .$$

Therefore, provided that  $\phi' > 0$ , a sufficient condition for (8) to hold is that  $2(\phi_L')^2 > \phi_L\phi_L''$ , which is equivalent to the requirement that  $1/\phi_L$  be strictly convex. ■

Note that the condition that  $1/\phi_L(x)$  is convex is satisfied in all our listed examples of PGFs above. Using this notion of “having more projects”, we can obtain the following result.<sup>6</sup>

**Proposition 3** *Suppose the number of projects is initially  $n_L$  with PGF  $\phi_L(x)$  which satisfies (6) and such that  $1/\phi_L$  is convex. Suppose next that the agent has access to a second source of projects with number  $n$  (where  $n$  is positive with some positive probability), and now the total number of projects is the independent sum  $n_H = n_L + n$ . Let  $r_i(\cdot)$  and  $x_i(\cdot)$  solve the Euler equation (5) when the number of projects is  $n_i$  for  $i = L, H$ . Then  $x_H(0) > x_L(0)$ , i.e., the fraction of permitted projects decreases when there are more projects.*

**Proof.** From Lemma 2, the PGF  $\phi_H$  associated with  $n_H$  satisfies (8). The rest of the proof follows the argument (with  $L$  and  $H$  permuted) of Proposition 2. ■

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<sup>6</sup>An example where adding more projects *widens* the optimal set of permitted projects is the following. Suppose initially the agent has no projects at all with probability  $1 - \varepsilon$  and exactly two projects with probability  $\varepsilon$ . Because the state when no projects materialize plays no role in the determination of  $r(\cdot)$ , the optimal threshold rule for this agent is just as if there were two projects for sure. Such a threshold rule will strictly exclude some desirable projects. Consider next the situation in which the agent has exactly one more project than the previous situation (i.e.,  $n = 1$  with probability  $1 - \varepsilon$  and  $n = 3$  with probability  $\varepsilon$ ). Whenever  $\varepsilon$  is small, the state where there is only one project will dominate the choice of  $r(\cdot)$ , and almost all desirable projects will be permitted, thus widening the set of permitted projects. One can check that the PGF for the original distribution for  $n$  does not satisfy the requirement that  $1/\phi_L$  is convex.

Note that we use the term “having more projects” to mean that the agent now has the independent sum of her original projects and a supply of other projects. This is a stronger requirement than the more usual notion of (first-order) stochastic dominance. Indeed, it is possible to find examples (even in the most regular cases) where stochastic dominance leads to a smaller fraction of projects being excluded.

Without making further assumptions, it is hard to make more progress in characterizing the solution to (5). This expression is in general a second-order differential equation in  $x$ ,  $x'$  and  $x''$  (see expression (40) in the appendix). However, there are two cases in which the Euler equation can simplify to a first-order equation. First, when  $v$  is independently distributed from  $u$  the Euler equation simplifies and with further assumptions (notably, when  $\alpha = 0$  or when  $v$  exponentially distributed), the Euler equation becomes a first-order equation in  $x$  and  $x'$ . Second, when  $n$  is a Poisson variable, the dependence on  $x(u)$  in the Euler equation vanishes, and (5) becomes a first-order equation in  $r$  and  $r'$ . These special cases are discussed in more detail in the next two sub-sections.

### 3.1 Independent distributions

Suppose in this section that the distribution of  $v$  is independent of  $u$ . In particular, in the following we write  $g(v)$ ,  $G(v)$  and  $V(r)$  as functions which do not depend on  $u$ . These functions are related as follows. Define

$$I(r) \equiv \int_r^{v_{\max}} (v - r)g(v) dv . \quad (9)$$

Then

$$-I'(r) = 1 - G(r) = \frac{x'(u)}{f(u)} ,$$

$I''(r) = g(r) > 0$ , and

$$V(r) - r = \frac{I(r)}{-I'(r)} . \quad (10)$$

The Euler equation can then be simplified in the following manner.<sup>7</sup>

**Lemma 3** *If the distribution for  $v$  is independent of  $u$ , expression (5) is equivalent to*

$$\frac{d}{du} \left[ \frac{V(r(u)) - r(u)}{f(u)} \frac{d}{du} \phi(x(u)) \right] = \frac{\alpha}{f(u)} \frac{d}{du} \phi(x(u)) . \quad (11)$$

**Proof.** Multiplying both sides of (5) by  $-I'(r(u))\phi'(x(u))$  and using (10) it follows that (5) is equivalent to

$$-I'(r(u))\phi'(x(u))[r'(u) + \alpha] = I(r(u))\phi''(x(u))x'(u) .$$

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<sup>7</sup>Within the calculus of variations literature, this transformation of the Euler equation is known as the Beltrami identity.

Therefore, we have

$$\begin{aligned}
-\alpha I'(r(u))\phi'(x(u)) &= I(r(u))\phi''(x(u))x'(u) + I'(r(u))\phi'(x(u))r'(u) \\
&= \frac{d}{du}[I(r(u))\phi'(x(u))] \\
&= \frac{d}{du} \left[ [V(r(u)) - r(u)] \frac{x'(u)}{f(u)} \phi'(x(u)) \right] \\
&= \frac{d}{du} \left[ \frac{V(r(u)) - r(u)}{f(u)} \frac{d}{du} \phi(x(u)) \right] .
\end{aligned}$$

And

$$\begin{aligned}
-\alpha I'(r(u))\phi'(x(u)) &= \alpha \frac{x'(u)}{f(u)} \phi'(x(u)) \\
&= \frac{\alpha}{f(u)} \frac{d}{du} \phi(x(u)) .
\end{aligned}$$

Thus, (5) implies expression (11), and vice versa. ■

We next consider two cases which allow (11) to be integrated.

**The principal does not care about the agent's payoff:** When  $\alpha/f(u)$  is constant expression (11) can be integrated to a first-order equation in  $x$  and  $x'$ . The quantity  $\alpha/f(u)$  is constant when  $\alpha = 0$  or when  $u$  is uniformly distributed. In the remainder of this section we focus on the former case (although the case with  $u$  uniform is also straightforward to solve). When  $\alpha = 0$  and  $v$  is independent of  $u$ , the principal does not care about the agent's payoff, either directly (since  $\alpha = 0$ ) or indirectly (since the realization of  $u$  gives no information about the realization of  $v$ ).

When  $\alpha = 0$ , expression (11) simplifies to

$$[V(r(u)) - r(u)] \frac{d}{du} \phi(x(u)) = kf(u) \tag{12}$$

for some positive constant  $k$ . Here, the principal obtains the same payoff with all density functions  $f(\cdot)$  for  $u$ . To see this, change variables in (12) from  $u$  to  $F(u)$ . That is to say, write  $\hat{r}(F(u)) \equiv r(u)$  and  $\hat{x}(F(u)) \equiv x(u)$ , so that  $\hat{r}$  represents the threshold rule expressed in terms of the cumulative *fraction* of  $u$ -projects  $F$ . Then (12) becomes

$$[V(\hat{r}(F)) - \hat{r}(F)] \frac{d}{du} \phi(\hat{x}(F)) \equiv k , \tag{13}$$

with initial condition  $\hat{r}(0)$  and terminal condition  $\hat{x}(1) = 1$ . In particular, the optimal threshold rule  $\hat{r}(\cdot)$  does not depend on the distribution for  $u$  at all, as long as  $u$  is continuously distributed.<sup>8</sup>

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<sup>8</sup>Note that this argument requires us to change variables in expression (12), and so  $F(u)$  needs to be differentiable and, in particular, the distribution for  $u$  has no ‘‘atoms’’. If there were atoms, then we would need to consider what project the agent would choose in the event of a ‘‘tie’’, when there would two projects which yielded the same maximal agent payoff  $u$ .

What can we say about the shape of  $\hat{r}(\cdot)$ ? First, from Corollary 1 we know that  $\hat{r}(F)$  increases with  $F$ . More interestingly, from expression (5) we have

$$\begin{aligned}\hat{r}'(F) &= [V(\hat{r}(F)) - \hat{r}(F)][1 - G(\hat{r}(F))] \frac{\phi''(\hat{x}(F))}{\phi'(\hat{x}(F))} \\ &= \left\{ [V(\hat{r}(F)) - \hat{r}(F)] \frac{d}{du} \phi(\hat{x}(F)) \right\} \frac{\phi''(\hat{x}(F))}{[\phi'(\hat{x}(F))]^2} \\ &= k \frac{\phi''(\hat{x}(F))}{[\phi'(\hat{x}(F))]^2},\end{aligned}$$

where the final equality follows from (13). Thus,  $\hat{r}'$  is decreasing in  $F$  if and only if  $\phi''/(\phi')^2$  decreases with  $x$ . (This condition is satisfied in all five examples of PGFs listed above.) In such cases,  $\hat{r}$  is concave in  $F$ . This discussion is summarised in the following:

**Proposition 4** *Suppose that  $\alpha = 0$ , that  $u$  and  $v$  are independently distributed, and that  $u$  has a continuous density function on the support  $[0, u_{\max}]$ . Then the principal's payoff does not depend on the density of  $u$ , and the optimal threshold policy can be expressed as a function  $\hat{r}$  of  $F(u)$ , the cumulative distribution function for  $u$ . Moreover, if  $\phi''/(\phi')^2$  decreases with  $x$  then the threshold rule  $\hat{r}(F)$  is concave in  $F$ .*

We note that when  $\alpha = 0$ , although the principal does not care directly about the agent's payoff, he nevertheless hopes that the agent has a permitted project with high  $u$ . In particular, the principal's expected payoff conditional on  $u$ , namely  $V(r(u))$ , increases with  $u$ . It follows that, even without re-optimizing the permission rule  $r(\cdot)$ , the principal's expected payoff increases if the agent has more projects (in the sense of having an additional source of projects) because projects from the additional source will be chosen only if they have higher  $u$  than what would otherwise have been chosen.

We next discuss a second class of cases which yield a simple solution:

**Exponential distribution for  $v$ :** Suppose now that  $v$  has support  $[v_{\min}, \infty)$  and  $1 - G(v) = e^{-(v-v_{\min})/\lambda}$ . To avoid boundary problems, suppose that  $v_{\min} \leq -\alpha u_{\max}$ . The "memoryless" property of this distribution implies that  $V(r) - r \equiv \lambda$ , and so (11) can be written as

$$\frac{d}{du} \left[ \frac{1}{f(u)} \frac{d}{du} \phi(x(u)) \right] = \frac{\alpha}{\lambda} \left[ \frac{1}{f(u)} \frac{d}{du} \phi(x(u)) \right].$$

Therefore, for some constant  $k_1$  we have

$$\frac{d}{du} \phi(x(u)) = k_1 f(u) e^{\frac{\alpha}{\lambda} u}. \quad (14)$$

This gives a solution for the optimal  $x(\cdot)$  by integrating (either analytically if possible, or numerically if not) the expression  $f(u) e^{\frac{\alpha}{\lambda} u}$ .

Finally, we provide an explicit example which combines the two sub-cases discussed above.

**Example:** Suppose that  $1 - G(v) = e^{-v/\lambda}$  and  $\alpha = 0$ , in which case (14) implies that

$$\phi(x(u)) = k_1 F(u) + k_2$$

for some constants  $k_1$  and  $k_2$ . Since  $x(u_{\max}) = 1$  and  $\phi(1) = 1$ , we must have  $k_1 + k_2 = 1$ . Since  $r(0) = 0$  we have  $x'(0) = f(0)(1 - G(0)) = f(0)$ , and so  $\phi'(x(0)) = k_1$ . In sum, in this example

$$1 - \phi(x(u)) = \phi'(x(0))[1 - F(u)] . \quad (15)$$

In particular, the probability that any given project is not permitted,  $x(0)$ , satisfies

$$\phi'(x(0)) + \phi(x(0)) = 1 . \quad (16)$$

For instance, consider the example where  $n$  is governed by a Geometric distribution with parameter  $a = \frac{2}{3}$ . (This implies that the agent can choose from three projects on average.) In this case  $\phi(x) = x/(3 - 2x)$ , and (16) implies that  $x(0) = \frac{1}{2}$  and half of the possible projects are forbidden. Expression (15) implies that

$$x(u) = \frac{1 + 3F(u)}{2 + 2F(u)} ,$$

so the optimal threshold rule is

$$r(u) = 2\lambda \log(1 + F(u)) .$$

This is concave in  $F$  as indicated in Proposition 4. Finally, the proportion of permitted type- $u$  projects is

$$p(u) = \frac{1}{(1 + F(u))^2} .$$

### 3.2 Poisson distribution for the number of projects

Return to the situation where the principal might care about the agent's payoff, either because  $\alpha > 0$  or because there is correlation between  $v$  and  $u$ . If  $n$  follows a Poisson distribution with mean  $\mu$ , the Euler equation (5) becomes a first-order differential equation in  $r(u)$ :

$$r'(u) + \alpha = \mu[V(r(u), u) - r(u)][1 - G(r(u), u)]f(u) . \quad (17)$$

The comparative statics of  $r(\cdot)$  with respect to  $\alpha$  and  $\mu$  are stronger than the corresponding results in the general setting reported above in Propositions 2 and 3:

**Proposition 5** *The optimal threshold rule  $r(\cdot)$  is pointwise increasing in  $\mu$  and decreasing in  $\alpha$ .*



**Proof.** The method is similar the proof of Proposition 2. Consider first the impact of increasing  $\mu$ , and let  $\mu_L$  and  $\mu_H > \mu_L$  be two values for  $\mu$ . Let  $r_L(\cdot)$  and  $r_H(\cdot)$  be the corresponding optimal threshold rules. From (17) it follows that at  $u = 0$  and any other  $u$  such that  $r_L(u) = r_H(u)$

$$\frac{r'_L(u) + \alpha}{r'_H(u) + \alpha} = \frac{\mu_L}{\mu_H} < 1,$$

so  $r'_L(u) < r'_H(u)$  at all such  $u$ . So  $r_H$  can never cross  $r_L$  from above. We deduce that  $r_H(u) > r_L(u)$  for all  $u > 0$ . The argument for the impact of  $\alpha$  on  $r(\cdot)$  is similar. ■

Moreover, it is straightforward to show from (17) that

$$\begin{aligned} r(u) &\rightarrow v_{\max} \text{ as } \mu \rightarrow \infty \text{ (if } u > 0), \text{ and} \\ r(u) &\rightarrow r_{naive}(u) = -\alpha u \text{ as } \mu \rightarrow 0. \end{aligned}$$

Thus, with very many projects, the agent is essentially given no discretion, and only those projects with the highest payoff to the principal are permitted. In addition, as it becomes unlikely that the agent has a choice of project, the principal should allow all desirable projects to be implemented.

While solution to equation (17) can easily be solved numerically—and we present some examples of this at the end of this section—it apparently cannot be solved analytically without making further assumptions. One natural simplification is to make the differential equation homogeneous, so that there is no direct dependence on  $u$  in (17). This is done by supposing (i) that  $u$  is uniformly distributed on, say,  $[0, 1]$  and (ii) that  $u$  and  $v$  are independent. In this case, (17) becomes

$$r' = \mu I(r) - \alpha, \tag{18}$$

where  $I(\cdot)$  is given in (9).

Note that if  $\alpha = \mu I(0)$ , then the solution to (18) is simply the flat rule  $r(u) \equiv 0$ . Thus, in the merger context, if the regulator wishes to maximize total welfare (so  $\alpha = 1$ ), then if the expected number of mergers is such that  $\mu I(0) = 1$  the regulator should optimally enforce a pure consumer welfare standard. Outside this knife-edge case, the solution to (18) will be increasing (decreasing) in  $u$  when  $\alpha < (>)\mu I(0)$ . Moreover, by differentiating (18), it follows that  $r(u)$  is concave (convex) when  $\alpha < (>)\mu I(0)$ . Specifically, the solution to (18) when  $\alpha \neq \mu I(0)$  is given implicitly by

$$\int_0^{r(u)} \frac{1}{\mu I(r) - \alpha} dr = u. \tag{19}$$

To illustrate some solutions to (19), suppose that  $(u, v)$  is uniformly distributed on  $[0, 1] \times [-1, 1]$  so that  $I(r) = \frac{1}{4}(1 - r)^2$ . When  $\alpha = 0$ , expression (19) implies that

$$r(u) = \frac{\mu u}{4 + \mu u}. \tag{20}$$

Alternatively, when  $\alpha = 1$  (19) can be integrated (using partial fractions) to give

$$r(u) = \left(1 - \frac{4}{\mu}\right) \frac{e^{u\sqrt{\mu}} - 1}{\left(1 + \frac{2}{\sqrt{\mu}}\right)(e^{u\sqrt{\mu}} - 1) - 2}. \quad (21)$$

As already mentioned, when  $\mu = 4$  it is optimal to have the flat rule that only projects with positive  $v$  are permitted. For  $\mu > 4$ , the solution is increasing and concave, while for  $\mu < 4$  the solution is decreasing and convex. We depict these solutions in Figure 3 for various  $\mu$ . Here, higher curves correspond to higher  $\mu$ , which is consistent with Proposition 5. The straight line depicted for  $\mu = 0$  is just the naive rule which permits any desirable project.

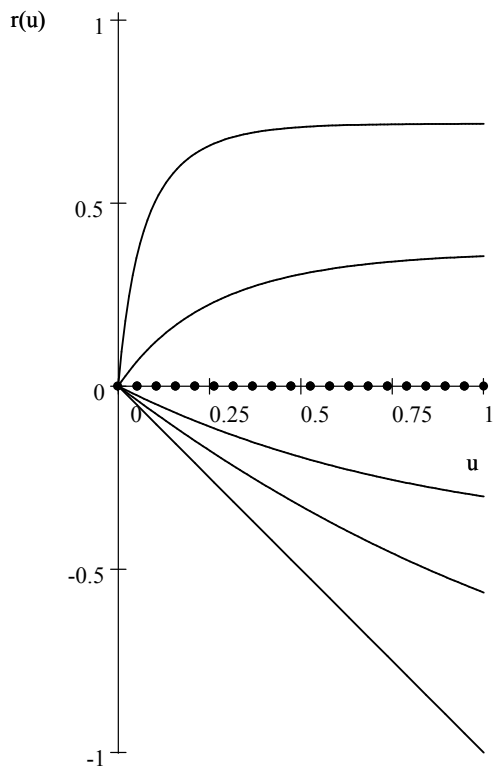


Figure 3:  $r(u)$  for Uniform-Poisson example with  $\alpha = 1$  and  $\mu = 0, 1, 2, 4$  (dotted), 10 and 50

The final issue we discuss here is the impact of correlation between  $u$  and  $v$  on the optimal threshold rule. Intuitively, all else equal we expect that positive correlation between the two payoffs to lead to the agent being given more discretion: the agent can be trusted to pick projects with high  $u$ , and with correlation this tends to lead to high- $v$  projects too. To investigate this issue, suppose that  $\alpha = 0$  and  $v$  is exponentially distributed with support  $v \in [0, \infty)$  and mean  $\lambda(u)$  given  $u$ . If  $\lambda$  is constant, the variables  $u$  and  $v$  are independent, if  $\lambda(u)$  is increasing in  $u$  they are positively correlated, while if  $\lambda(u)$  is decreasing then they are negatively correlated.

The expected value of  $v$  (without any restricted choice) is just

$$E[v] = \int_0^{u_{\max}} \lambda(u) f(u) du . \quad (22)$$

As before, we have  $V(r, u) - r = \lambda(u)$ , and so

$$[V(r, u) - r][1 - G(r, u)] = \lambda(u) e^{-\frac{r}{\lambda(u)}} .$$

In this case equation (17) becomes

$$r'(u) = \mu \lambda(u) f(u) e^{-\frac{r(u)}{\lambda(u)}} . \quad (23)$$

If  $\lambda$  is constant (the independence case), it follows from (23) that

$$r(u) = \lambda \log(1 + \mu F(u)) .$$

If there is correlation, equation (23) does not seem explicitly soluble, but it is easily solved numerically. Figure 4 depicts five solutions. In each of these we have set  $\mu = 2$  and  $u$  is uniform on  $[0, 1]$ . Here,  $\lambda(u)$  takes the form  $1 + k(u - \frac{1}{2})$ , and so from (22) the expectation of  $v$  is the same (equal to 1) for all  $k$ , and changes in the threshold rule are not simply due to changes in the mean of  $v$ . A positive  $k$  represents positive correlation between  $u$  and  $v$ . Here, the thick line depicts the independent case with  $k = 0$ . In the figure, higher curves correspond to less positive (or more negative) correlation. Although the threshold rules are not strictly nested (they cross when  $u$  is close to 1), in broad terms we see that more correlation is indeed associated with more discretion being granted the agent, as is intuitive. In addition, we see that when there is positive correlation, it is possible for the threshold rule to be *convex* in  $u$  (even when  $u$  is uniformly distributed).

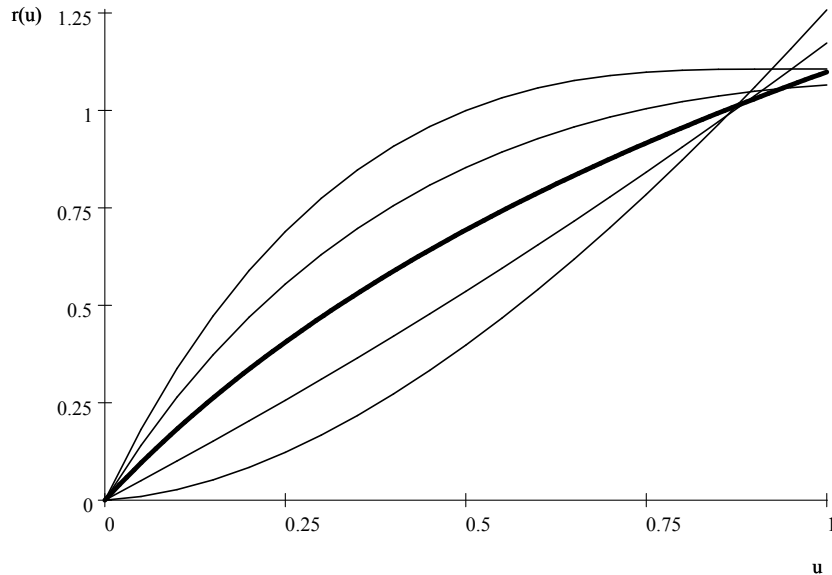


Figure 4: Threshold rules when  $\lambda(u) = 1 + k(u - \frac{1}{2})$ , for  $k = 1.9, 1, 0, -1, -1.9$

## 4 Searching for a Project

The previous model assumed that the number of projects was exogenous to the agent (but uncertain). We now suppose instead that the agent can influence the number of available projects in a dynamic search framework. In particular, the rate of arrival of projects depends on privately costly search effort by the agent. Special cases of our formulation are standard search models in which: (i) the agent can instantaneously obtain a new draw by incurring a cost  $C$ , and (ii) the agent is passive and must wait for a new draw to materialize.

As before, the agent's payoff (excluding search costs) is  $u$ , the principal's payoff is a weighted sum of the agent's payoff (including search costs) and the expected value of a random variable  $v$ , where the relative weight on the agent's payoff by the principal is  $\alpha \leq 1$ . The principal determines a function  $r(\cdot)$  such that any type- $u$  project with  $v \geq r(u)$  is permitted. Also as before,  $x(u)$  is the probability that a project drawn at random either has agent utility below  $u$  or is not in the permitted set.

Provided that search is worthwhile, the agent, who is assumed to be risk neutral, will keep searching until she finds a permitted project which delivers her utility  $u$  above some threshold, denoted  $U$ , to be characterized below. The agent also decides how much effort to put into searching. Specifically, suppose that a project emerges with probability  $h \times dt$  in any small time interval  $dt$ , where the hazard rate  $h$  has associated private flow cost for the agent given by  $c(h)$ , where  $c(\cdot)$  is assumed to be increasing and convex. To avoid corner solutions, suppose that  $c'(0) = 0$ . (However, we do not require that  $c(0) = 0$ .) For use later, define  $C$  to be the minimum average search cost, so that  $C$  is the minimum value of  $c(h)/h$  over  $h \geq 0$ .

The probability that a given project will be implemented is  $1 - x(U)$ , while the agent's expected payoff at the time the project is implemented is  $B(U)/(1 - x(U))$ , where

$$B(U) = \int_U^{u_{\max}} u[1 - G(r(u), u)]f(u) du .$$

Following this strategy, the agent will receive an acceptable project in a time interval  $dt$  with probability  $h(1 - x(U)) \times dt$ . This implies that the probability the first acceptable project will arrive in the time interval  $(t, t + dt)$  is

$$h(1 - x(U))e^{-h(1-x(U))t} \times dt .$$

If the agent discounts at the rate  $\delta$ , her expected utility is therefore

$$\int_0^{\infty} [h(1 - x(U))\frac{B(U)}{(1 - x(U))} - c(h)]e^{-h(1-x(U))t}e^{-\delta t}dt = \frac{hB(U) - c(h)}{h(1 - x(U)) + \delta} . \quad (24)$$

The agent will choose her reservation utility  $U$  and search intensity  $h$  in order to maximize her payoff (24). The first-order condition for choosing  $U$  is

$$U = \frac{hB(U) - c(h)}{h(1 - x(U)) + \delta} . \quad (25)$$

In particular,  $U$ , the reservation utility, is also the agent's discounted payoff from following her optimal strategy (as is usual in search models). Using expression (25), the first-order condition for  $h$  is

$$hc'(h) - c(h) = \delta U . \quad (26)$$

Given that  $c(\cdot)$  is convex, it follows from (26) that higher  $U$  is associated with greater search effort (at least when  $\delta > 0$ ). Expression (26) represents the fundamental relationship between the agent's utility and her search effort. Since a high search effort benefits the principal as well as the agent, the principal has a reason (beyond the weight  $\alpha$  placed on the agent's interests) to increase the agent's payoff.

Note that (25) and (26) imply that

$$\int_U^{u_{\max}} (u - U)(1 - G(r(u), u))f(u) du = c'(h) . \quad (27)$$

Expression (26) shows  $h$  to be an increasing function of  $U$ , whereas (27) gives  $h$  to be a decreasing function of  $U$ . Therefore, the pair of equations (26)–(27) can have at most one solution in  $(U, h)$ . It is clear that  $h$  corresponding to  $U = u_{\max}$  is smaller in (27) than it is in (26). The value of  $h$  corresponding to  $U = 0$  in (26) is the  $h$  which minimizes  $c(h)/h$ , say  $h^*$ . The  $h$  corresponding to  $U = 0$  in (27) is higher than  $h^*$  provided that

$$\int_0^{u_{\max}} u(1 - G(r(u), u))f(u) du \geq c'(h^*) = C , \quad (28)$$

where the equality follows from the fact that  $h^*$  implements the minimum average cost,  $C$ . Condition (28) simply states that the agent has an incentive to engage in search at all when faced with the permission rule  $r(\cdot)$ . Thus, whenever (28) holds, there exists a unique solution to the pair of equations (26)–(27).

Finally, the comparative statics of the agent's decisions are clear-cut. If  $\delta$  is increased, this shifts upwards the increasing relationship between  $h$  and  $U$  in (26) but leaves the relationship in (27) unaltered, and this therefore induces  $U$  to fall and  $h$  to rise. Likewise, if the principal shifts the permission rule  $r(\cdot)$  upwards, this shifts downwards the relationship between  $h$  and  $U$  given in (27), but leaves (26) unaltered, and so induces the agent to lower both  $U$  and  $h$ . This discussion is summarized in the following result:

**Lemma 4** *Suppose that the permission rule  $r(\cdot)$  satisfies inequality (28). Then the agent is willing to search for a project, and her reservation utility  $U$  and search effort  $h$  are the unique solution to the pair of equations (26)–(27) (or, equivalently, to the pair of equations (25)–(26)). All else equal, (i) increasing  $\delta$  induces the agent to (weakly) reduce  $U$  and increase  $h$ , and (ii) shifting the rule  $r(\cdot)$  upwards induces the agent to (weakly) reduce  $U$  and  $h$ .*

Turning to the principal's problem, let  $Z$  denote the discounted expected value of  $v$ :

$$Z = \frac{h \int_U^{u_{\max}} \left( \int_{r(u)}^{v_{\max}} vg(v, u) dv \right) f(u) du}{h(1 - x(U)) + \delta} . \quad (29)$$

The principal aims to choose the permission rule  $r(\cdot)$  and the agent's search strategy  $(U, h)$  in order to maximize  $\alpha U + Z$  subject to the agent's twin incentive constraints (25) and (26). The relationship between  $h$  and  $U$  in (26) cannot be influenced by the principal, although the principal can choose the point on this locus by choosing  $r(\cdot)$  appropriately.

## 4.1 The optimality of linear rules

We attempt to solve the principal's problem in two stages. First, for given reservation utility  $U$ , and hence a given  $h$  satisfying (26), we derive the permission rule  $r(u)$  which maximizes the principal's payoff subject to the single constraint (25). Subsequently, the optimal choice for  $U$  can be analyzed. In this section, we consider the first sub-problem, which turns out to have a surprisingly simple solution.

Therefore, fix  $U$  (and hence  $h$ ). Writing  $\gamma$  for the Lagrange multiplier on the constraint (25), the principal chooses  $r(\cdot)$  to maximize the Lagrangian

$$\mathcal{L} = Z + \gamma \frac{hB - c(h)}{h(1 - x(U)) + \delta} = \frac{h \int_U^{u_{\max}} \left( \int_{r(u)}^{v_{\max}} [v + \gamma u] g(v, u) dv \right) f(u) du - c(h)}{h(1 - x(U)) + \delta} . \quad (30)$$

(The multiplier  $\gamma$  will then be chosen in order to ensure that (25) binds.) For a given  $U$  there are many ways to choose the rule  $r(\cdot)$  which lead to a specified value of  $1 - x(U)$ , the chance that a project is permitted and yields agent utility above the reservation level  $U$ . For a specified value of  $1 - x(U)$ , it is clear that the way to maximize the numerator of the Lagrangian (30) is to set  $r(u) = t - \gamma u$  for some  $t$  which ensures that the target  $1 - x(U)$  is achieved. This simple result tells us immediately that the optimal rule  $r(\cdot)$  will be *linear* in this search setting, regardless of the distribution of  $(u, v)$ .

Altering  $t$  will affect both the numerator and denominator in the Lagrangian. Note that a small increase in  $t$  loses  $(v, u)$  points such that  $v + \gamma u \approx t$ . It follows that

$$\frac{d}{dt} \int_U^{u_{\max}} \left( \int_{t - \gamma u}^{v_{\max}} [v + \gamma u] g(v, u) dv \right) f(u) du = t \frac{d}{dt} (1 - x(U)) .$$

Differentiating (30) with respect to  $t$  therefore implies that the optimal  $t$  satisfies

$$t = Z + \gamma U .$$

Therefore  $\mathcal{L}$  in (30) is maximized, given  $U$  and  $h$ , when

$$r(u) = Z - \gamma(u - U) . \quad (31)$$

We summarize this discussion in the following result:

**Proposition 6** *The optimal strategy for the principal in the search framework is to offer a linear threshold rule (31), where  $U$  is the agent's reservation utility and  $Z$  is the expected discounted value of  $v$ .*

The intuition for the, perhaps surprising, linearity of  $r(u)$  comes from noting that for given  $U$  and for given  $1 - x(U)$ , the principal's problem is simply to maximize the expected value of the weighted sum  $v + \gamma u$ , where the weight  $\gamma$  is the Lagrange multiplier on the constraint that the agent has reservation utility  $U$ .

General analysis of the second stage of the principal's problem, the derivation of the optimal value of  $U$ , turns out to be complex. Instead, in the next two sections we focus on two polar cases: the case of no discounting (which is equivalent to the standard search problem where the agent can instantaneously obtain a new draw of project in return for a cost  $C$ ), and the case of "urgent" projects where  $\delta$  is large.<sup>9</sup>

## 4.2 No discounting (or fixed cost per search)

In this section we focus on the special case where  $\delta = 0$ . When  $\delta = 0$ , then  $\gamma > 0$  in (31) and the linear permission rule is always downward sloping. To see this, note from (29) that the equilibrium  $Z$  satisfies

$$0 = \int_U^{u_{\max}} \left( \int_{Z-\gamma(u-U)}^{v_{\max}} [v - Z]g(v, u)dv \right) f(u) du . \quad (32)$$

If  $\gamma \leq 0$  then the integrand above is always positive, and so the integral cannot equal zero. Thus, we must have  $\gamma > 0$  as claimed. The multiplier  $\gamma > 0$  is then chosen to make constraint (25), or equivalently constraint (27), bind, so that

$$C = \int_U^{u_{\max}} (u - U)(1 - G(Z - \gamma(u - U)), u)f(u) du . \quad (33)$$

(Note that since  $C$  is the minimum value of  $c(h)/h$ , it follows that  $C = c'(h)$ .)

Moreover, when  $\delta = 0$  the agent's choice of search effort cannot be influenced by the principal (so long as the agent is willing to search at all). From expression (24), the agent will choose  $U$  and  $h$  in order to maximize

$$\frac{B(U) - c(h)/h}{1 - x(U)} ,$$

and so  $h$  is chosen to minimize  $c(h)/h$  regardless of  $r(\cdot)$ . Since  $C$  is the minimal value of  $c(h)/h$ , the agent will choose  $U$  in order to maximize  $[B(U) - C]/[1 - x(U)]$ . To make the problem interesting, suppose that  $c(0) > 0$  so that the agent faces a flow cost even to search at a minimal level.<sup>10</sup> This implies that  $C > 0$ . Note that this

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<sup>9</sup>We will see in the all examples which follow that the permission rules have negative slopes (i.e.,  $\gamma > 0$  in (31)). However, in a precursor to this paper, Armstrong and Vickers (2007), we investigated a third special case of the search paradigm, which is where the agent passively waits for projects to arrive, and the arrival rate  $h$  is exogenous. In that setting we showed how the permission rule could sometimes be upward sloping.

<sup>10</sup>If  $c(0) = 0$ , then  $C = 0$  and the problem is trivial. If  $C = 0$ , the agent will make (almost) no search effort, and the principal will wait for an (almost) perfect project to materialize. Thus, there is almost no discretion and  $r(u) \approx v_{\max}$ .

problem is formally identical to the problem where the agent can instantaneously obtain a new draw of project in return for a cost  $C$ .<sup>11</sup>

To obtain some explicit solutions to the principal's problem, suppose that  $(u, v)$  is uniformly distributed on the rectangle  $[0, 1] \times [-1, 1]$ . Expressions (32) and (33) become respectively

$$0 = \frac{1}{12}(1 - U) \left( (1 - Z)^2 - \gamma^2(1 - U)^2 \right) , \text{ or } 1 - Z = \frac{\gamma}{\sqrt{3}}(1 - U) , \quad (34)$$

and

$$C = \frac{1}{12}(1 - U)^2(3(1 - Z) + 2\gamma(1 - U)) .$$

Eliminating  $\gamma$  from this pair of equations shows that

$$Z = 1 - \frac{2C}{k(1 - U)^2} , \quad (35)$$

where

$$k \equiv \frac{1}{2} + \frac{1}{\sqrt{3}} \approx 1.08 .$$

Since the principal's payoff is  $Z + \alpha U$ , the principal will therefore choose  $U$  to maximize

$$\alpha U - \frac{2C}{k(1 - U)^2} .$$

This is a decreasing function of  $U$  whenever

$$\alpha \leq \frac{4c}{k} \approx 3.7C , \quad (36)$$

in which case it is optimal to set  $U = 0$  and so leave the agent with zero rent. The optimal permitted set is determined by  $r(u) = Z - \gamma u$ , where  $Z$  is given by (35) and  $\gamma$  is given by (34), both with  $U$  set equal to zero. It follows that<sup>12</sup>

$$r(u) = 1 - \frac{2C}{k} \left( u\sqrt{3} + 1 \right) . \quad (37)$$

Thus, for different values of  $C$  (37) traces out a family of linear, downward-sloping lines for  $r(u)$  revolving about the point  $(-1/\sqrt{3}, 1)$ . See Figure 5 for the case  $\alpha = 0$  (when condition (36) is always satisfied), where smaller  $C$  correspond to higher  $r$ . When  $C \approx 0$ , we have  $r(u) \approx 1$  as expected. (This is like the "large  $\mu$ " case in section 3.2.)

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<sup>11</sup>If the agent has search effort  $h$ , the expected time until a new project arrives is  $1/h$ . Without discounting, therefore, the (expected) total cost of obtaining a new draw of project is  $c(h)/h$ . Once the agent minimizes this total cost, the cost of getting a new draw is  $C$ .

<sup>12</sup>These solutions are valid only when  $C \leq (1+1/\sqrt{3})/4 \approx 0.4$ . This is to ensure that  $r(u)$  does not hit the lower boundary  $v = -1$ . If  $C > 0.4$ , the optimum will involve  $r(u)$  being a downward-sloping linear function which hits the lower boundary (and  $r(u) \equiv -1$  beyond this point).



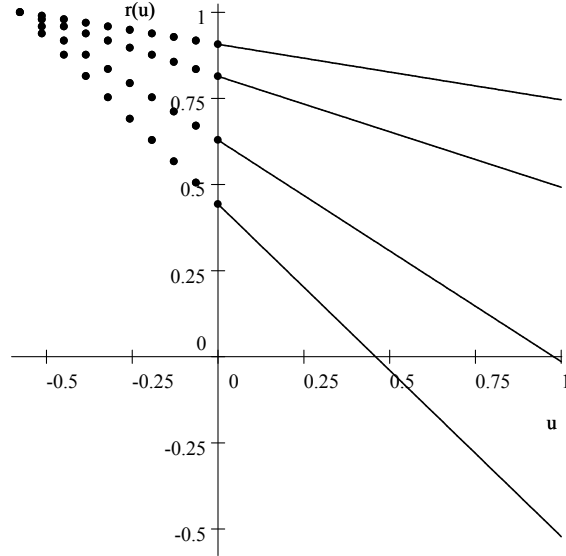


Figure 5:  $r(u)$  for Uniform example with  $\alpha = 0$  and  $C = 0.05, 0.1, 0.2$  and  $0.3$

In the two models we have considered, with  $\alpha = 0$  and a uniform distribution for  $(u, v)$ , we have derived two surprisingly simple families of threshold rules (see (20) and (37) above). In one respect optimal policy is similar in the two models: as projects are easier to come by for the agent (i.e.,  $\mu$  is larger in the first model or  $C$  is smaller in this second model), the permitted set of projects becomes progressively more restricted. In other respects, though, policy is dramatically different in the two settings. In the “choosing a project” model, the rules start at  $r(0) = 0$  and increase, and only desirable projects (i.e.,  $v \geq 0$ ) are permitted. In the search model we have  $r(0) > 0$  and  $r$  is decreasing, and it may be optimal to permit projects with negative payoff for the principal (as when  $C = 0.3$  in Figure 5).

As we explained in section 3, the reason why the principal departs from the efficient rule ( $r(u) \equiv 0$  in this case with  $\alpha = 0$ ) in the “choosing” model is that when some marginally desirable high- $u$  projects are excluded, this may induce the agent to choose a strictly desirable lower- $u$  project instead. This benefit does not exist when  $u = 0$ , which explains why all desirable projects are permitted then. Furthermore, it is clear there can therefore be no incentive to include projects with a negative payoff to the principal. The reason to depart from the efficient cut-off rule is quite different in the “searching” model. Here, when  $\alpha = 0$  the principal wishes to maximize the expected value of  $v$  in the permitted set, subject to the agent being willing to engage in costly search for permitted projects. For a given expected value of  $v$  in the permitted set, the principal is indifferent about whether the threshold rule is upward or downward sloping; however, the agent’s willingness to search is enhanced when higher- $u$  projects are more likely to be permitted, i.e., when the rule is downward sloping. For the same reason, it can be optimal to permit the agent to choose projects with a negative payoff for the principal, if the search cost is large enough.

If (36) does not hold then  $U > 0$  is optimal. Indeed optimal  $U$  and  $\gamma$  satisfy

$$1 - U = \left( \frac{4C}{k\alpha} \right)^{\frac{1}{3}}, \quad \gamma = \frac{\sqrt{3}}{2}\alpha.$$

Note that  $\gamma$  here is independent of  $C$  and less than  $\alpha$ . Therefore, the permission rule is

$$r(u) = 1 - (1 + \sqrt{3}) \left( \frac{C\alpha^2}{2k} \right)^{\frac{1}{3}} + \frac{\sqrt{3}}{2}\alpha(1 - u).$$

In Figure 6 we show the permission sets when  $\alpha = 1$ , for the same search costs as in Figure 5. Since the agent will keep searching until a permitted project with  $u > U$  is found, only that part of the rule with  $u > U$  is relevant, and that part is depicted on the figure. (The principal can choose the linear rule without constraining  $u > U$ , so that the downward-sloping lines can be extended to the left until they reach the vertical axis, but the agent will never choose a permitted project to the left of the vertical lines shown.) In the merger context, if the principal wishes to maximize total welfare, Figure 6 suggests that a good approximation to optimal policy is to permit mergers which increase total welfare by some discrete threshold, where this threshold is higher when merger possibilities are less costly to discover.

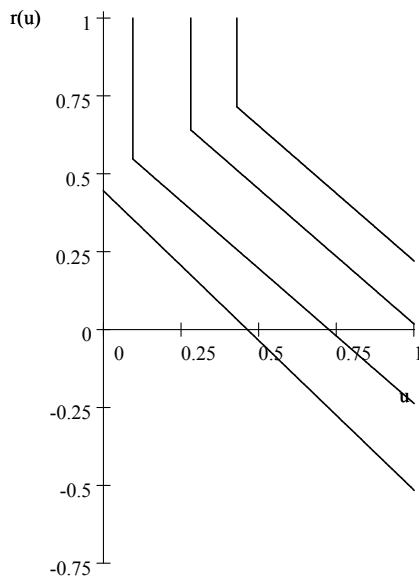


Figure 6:  $r(u)$  for Uniform example with  $\alpha = 1$  and  $c = 0.05, 0.1, 0.2,$  and  $0.3$

### 4.3 Urgent projects

Here we discuss the opposite limit case where  $\delta$  is large. From (24) and (29), it is clear that  $U$  and  $Z$  tend to zero as  $\delta \rightarrow \infty$ . Therefore, Proposition 6 indicates that the rule  $r(\cdot)$  will be a straight line emanating from  $(0, 0)$ . But it is more illuminating

to proceed more directly. The impact of the assumption that  $\delta$  is very large is that both principal and agent aim to maximize their *flow* payoff. If we write

$$B = \int_0^{u_{\max}} u(1 - G(r(u), u))f(u) du$$

then the agent will choose  $h$  to maximize flow payoff  $hB - c(h)$ . If we write

$$\sigma(B) \equiv \max_h : hB - c(h)$$

then  $\sigma$  is a convex increasing function and  $\sigma'(B)$  is the agent's choice of effort given the reward  $B$ .

The principal chooses  $r(\cdot)$  to maximize his flow payoff:

$$h \int_U \int_{r(u)}^{v_{\max}} vg(v | u) dv \Big) f(u) du + \alpha \left( h \int_0^{u_{\max}} u(1 - G(r(u), u))f(u) du - c(h) \right),$$

which can be written as

$$\sigma'(B)A + \alpha\sigma(B),$$

where

$$A = \int_0^{u_{\max}} \left( \int_{r(u)}^{v_{\max}} vg(v | u) dv \right) f(u) du.$$

By considering small changes in  $r(\cdot)$  at  $u$ , it follows that

$$r(u) + \left[ A \frac{\sigma''(B)}{\sigma'(B)} + \alpha \right] u \equiv 0, \quad (38)$$

and  $r$  is indeed a straight line starting at the origin. Moreover, it is clear that the line is downward sloping and steeper than the principal's true preferences (which have slope  $-\alpha$ ). Thus, the principal allows some projects which are undesirable ( $v + \alpha u < 0$ ) in order to stimulate search effort by the agent. (This distortion is the opposite to that in the "choosing" model, where the principal forbade some desirable projects.) The only situation in which the principal implements his naive rule, i.e.,  $r(u) = -\alpha u$ , is when  $\sigma'' = 0$ . This implies that the agent's search effort does not respond to incentives, i.e., there is an exogenous hazard rate. If the agent cannot affect the arrival rate of projects, and if the discount rate is very high, the principal should implement the first desirable project which appears.

For example, take the case where  $c(h) = a + \frac{1}{2}bh^2$ . Then  $\sigma(B) = \frac{1}{2b}B^2$  and (38) becomes

$$r(u) = - \left( \frac{A}{B} + \alpha \right) u. \quad (39)$$

This does not depend on the parameters of the cost function. Suppose that  $(u, v)$  is uniformly distributed on the rectangle  $[0, 1] \times [-2, 2]$ . If  $r(u) = -\gamma u$  then it can be calculated that

$$A = \frac{1}{2} - \frac{1}{24}\gamma^2; \quad B = \frac{1}{4} + \frac{1}{12}\gamma$$

and so (39) implies that  $\gamma$  satisfies

$$\gamma = \frac{6 - \frac{1}{2}\gamma^2}{3 + \gamma} + \alpha ,$$

or

$$\gamma = \frac{1}{3}\alpha + \frac{1}{3}\sqrt{12\alpha + \alpha^2 + 45} - 1 .$$

Here,  $\gamma$  increases with  $\alpha$  and ranges from approximately 1.2 to 1.8 as  $\alpha$  ranges from zero to 1.

## 5 Conclusions

Proceeding from the motivating example of welfare standards in merger policy, we have explored the nature of optimal discretion for a principal to give to an agent in two related settings of delegated project choice. The principal’s problem is to design the optimal set of permitted projects without knowing which projects are available to the agent—though being able to verify the characteristics of the project proposed by the agent—and with (contingent) transfers ruled out.

In the first setting the agent has a number (unknown to the principal) of projects to choose from. The optimal permission set excludes some projects that are good for the principal because the loss from excluding marginally good projects is outweighed by the expected gain from thereby inducing the choice of better projects. We showed (i) the principal permitted more types of project when he put more weight on the agent’s welfare, and (ii) the principal permitted fewer types of project when the agent could choose from more projects. Solutions for the optimal delegation set were derived for a number of examples.

In the second setting the agent searches for a project that is both permitted by the principal and meets the agent’s own acceptance threshold. Here the optimal permission set is generally characterised by a linear relationship between the payoffs of principal and agent. We focused on the two polar cases of no discounting and high discounting. In order to encourage search effort, or to cover fixed search costs efficiently, projects with higher agent payoffs are permitted for a wider range of principal payoffs, so that the permission rule is downward sloping. In order to encourage search, the principal might permit some projects which are undesirable *ex post* (this was always true in the high discounting case), in contrast to the bias induced in the “choosing” model.

In sum, our analysis has highlighted three aspects of optimal delegation of project choice. The first, from the model of project choice, is the exclusion of good projects to improve the chances of better projects being chosen. Second, from the model of project search, is the relatively greater tolerance of projects with high agent payoffs to encourage search. However, that model illustrated thirdly that tolerance of such projects is muted by effects on the agent’s own acceptance threshold—a widening of the set of permitted choices by the principal may cause some diminution of the set of projects from which the agent is willing to choose.

## APPENDIX: Derivation of the Euler equation (5)

The following analysis follows the argument in any standard textbook on the calculus of variations. Write

$$s(p, u) \equiv \int_{r(p, u)}^{v_{\max}} vg(v, u) dv ,$$

where  $r(p, u)$  is defined implicitly by

$$G(r(p, u), u) \equiv 1 - p .$$

Thus  $r(p, u)$  is the threshold such that a proportion  $p$  of projects lie above  $r(p, u)$  for given  $u$ , and  $s(p, u)$  is the sum of  $v$  above this threshold. Therefore,  $V(r(p, u), u) = s(p, u)/p$ . Note that  $s$  is increasing and concave in  $p$ , and ranges from 0 to  $E[v \mid u]$  as  $p$  ranges from 0 to 1. Note also that  $s_p(p, u) \equiv r(p, u)$ . Viewing  $p$  as a function of  $u$ , we have  $x'(u) = f(u)p(u)$  as in (1) above.

The principal's aim is to maximize

$$\int_0^{u_{\max}} [V + \alpha u] \frac{d}{du} \phi(x(u)) du = \int_0^{u_{\max}} \left[ f(u)s \left( \frac{x'(u)}{f(u)}, u \right) + \alpha u x'(u) \right] \phi'(x(u)) du ,$$

subject to the endpoint condition  $x(u_{\max}) = 1$ . Write

$$H(u, x, x') \equiv \phi'(x) \left[ f(u)s \left( \frac{x'}{f(u)}, u \right) + \alpha u x' \right]$$

for the Hamiltonian for this problem.

Let  $\eta(u)$  be any smooth function such that  $\eta(1) = 0$ , and consider the function

$$\Psi(t) = \int_0^{u_{\max}} H(u, x + t\eta, x' + t\eta') du .$$

If  $x$  is the optimal path, it follows that  $\Psi(t)$  must be maximized at  $t = 0$ , for any  $\eta$ . By Leibniz's rule

$$\Psi'(0) = \int_0^{u_{\max}} \eta H_x(u, x, x') + \eta' H_{x'}(u, x, x') du .$$

Integrating the second term by parts yields

$$\int_0^{u_{\max}} \eta' H_{x'} du = \eta H_{x'} \Big|_0^{u_{\max}} - \int_0^{u_{\max}} \eta \frac{d}{du} H_{x'} du .$$

The only way that the problem can be stationary at  $x$  is if  $H_{x'} = 0$  when  $u = 0$  (for otherwise we could choose  $\eta(0)$  to be any value which makes the above very large). Since  $H_{x'} = \phi'(x)f(0)s_p(x'/f(0), 0)$  when  $u = 0$ , it follows that at the optimum we must have  $s_p(p(0), u) = r(0) = 0$ , and assume this henceforth.

It follows that

$$\Psi'(0) = \int_0^{u_{\max}} \eta \left\{ H_x - \frac{d}{du} H_{x'} \right\} du .$$

Since this must be zero for all deviations  $\eta(\cdot)$ , it follows that  $\{\cdot\}$  must be zero for all  $u$ , and we obtain the central Euler equation

$$H_x(u, x, x') = \frac{d}{du} H_{x'}(u, x, x') .$$

Written explicitly this becomes

$$\phi''(x) [f s \left( \frac{x'}{f}, u \right) + \alpha u x'] = \phi'(x) \frac{d}{du} \left[ s_p \left( \frac{x'}{f}, u \right) + \alpha u \right] + \phi''(x) x' \left[ s_p \left( \frac{x'}{f}, u \right) + \alpha u \right] . \quad (40)$$

Since  $p(u) = x'(u)/f(u)$  and  $r(u) \equiv s_p(p(u), u)$ , this expression becomes

$$r'(u) + \alpha = \frac{\phi''}{\phi'} f p \left[ \frac{s(p, u)}{p} - r \right] \quad (41)$$

which leads to expression (5) in the text.

What about the second-order condition? When  $H$  is jointly concave in  $x$  and  $x'$ , it is known that the solution to the Euler equation is the global maximum. However, this sufficient condition never applies in our problem, since  $H_{xx} = [f s(x'/f, u) + \alpha u x'] \phi'''(x) > 0$ . Therefore, we look for ‘local’ second-order conditions. Repeated differentiation yields

$$\Psi''(0) = \int_0^{u_{\max}} \eta^2 H_{xx} + 2\eta\eta' H_{xx'} + (\eta')^2 H_{x'x'} du .$$

It is well known that a *necessary* condition for  $\Psi''(0)$  to be negative for all  $\eta$  is that  $H_{x'x'} \leq 0$  along the optimal path. (This is the ‘Legendre condition’.) In our problem,  $H_{x'x'} = \phi'(x) s_{pp}(x'/f, u)/f < 0$  and so this necessary condition is satisfied. But it would be reassuring to have a sufficient condition for  $\Psi''(0) < 0$  as well.

The middle term in the above integral can be integrated by parts to give

$$\int_0^{u_{\max}} 2\eta\eta' H_{xx'} du = \eta^2 H_{xx'} \Big|_0^{u_{\max}} - \int_0^{u_{\max}} \eta^2 \frac{d}{du} H_{xx'} du = - \int_0^{u_{\max}} \eta^2 \frac{d}{du} H_{xx'} du .$$

Here, the second equality follows from the observation that  $H_{xx'} = 0$  when  $u = 0$  (and also that  $\eta(1) = 0$ ). Therefore,

$$\Psi''(0) = \int_0^{u_{\max}} \eta^2 \left[ H_{xx} - \frac{d}{du} H_{xx'} \right] + (\eta')^2 H_{x'x'} du .$$

Since  $H_{x'x'} < 0$ , a sufficient condition for  $\Psi''(0) < 0$  for all  $\eta$  is that the above term  $[\cdot]$  be negative along the optimal path. Writing explicitly shows that

$$H_{xx} - \frac{d}{du} H_{xx'} = \phi''' f p \left( \frac{s}{p} - s_p \right) - \phi'' (r' + \alpha) = f p \left( \frac{s}{p} - s_p \right) \left( \phi''' - \phi'' \frac{\phi'''}{\phi'} \right) ,$$

where the second equality follows from (41). Since  $s$  is concave in  $p$  it follows that  $(\frac{s}{p} - s_p)$  is positive, and so the above expression is negative if and only if  $\zeta(x) \equiv \phi''(x)/\phi'(x)$  weakly decreases with  $x$ . In such cases, we may be sure that the second-order condition  $\Psi''(0) < 0$  is satisfied.

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