Dynamic Merger Review*

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Abstract

We analyze the optimal dynamic policy of an antitrust authority towards horizontal mergers when merger proposals are endogenous and occur over time. Approving a currently proposed merger will affect the profitability and welfare effects of potential future mergers, the characteristics of which may not yet be known to the antitrust authority. We show that, in many cases, this apparently difficult problem has a simple resolution: an antitrust authority can maximize discounted consumer surplus by using a completely myopic merger review policy that approves a merger today if and only if it does not lower consumer surplus given the current market structure.

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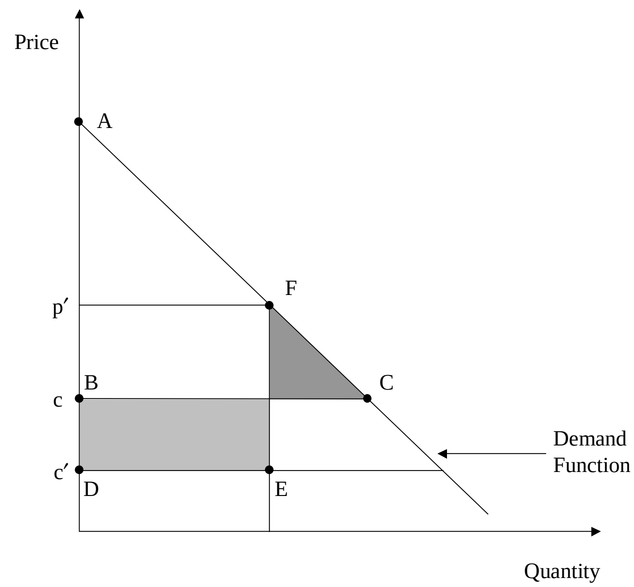

Figure 1: The Williamson tradeoff in merger review: deadweight loss of market power (dark-shaded triangle) *vs.* efficiency gain (light-shaded rectangle).

1 Introduction

The traditional approach to the review of horizontal mergers stresses the tradeoff between market power and efficiencies. Mergers, which cause firms to internalize pricing externalities among former rivals, increase the exercise of market power, and therefore tend to reduce social welfare. On the other hand, since they can create efficiencies, horizontal mergers may instead increase welfare. This tradeoff was first articulated by Williamson [1968] for the case of an antitrust authority who wants to maximize aggregate surplus, using a diagram like Figure 1. In the diagram, a competitive industry merges to become a monopolist that charges the price p', but lowers its marginal cost of production from c to c'. Whether aggregate surplus increases or not depends on whether the dark-grey deadweight loss triangle exceeds the light-grey efficiency gain. A similar, though even more straightforward tradeoff arises when an antitrust authority instead applies a consumer surplus standard to merger approval decisions, as is (roughly) the case in both the U.S. and EU legal regimes. In that case, the marginal cost reduction must be large enough that the price does not increase for the merger to be approved.

More recently, Farrell and Shapiro [1990] (see also McAfee and Williams [1992]) have provided a more complete and formal analysis of this tradeoff for settings with Cournot competition. Farrell and Shapiro provide a necessary and sufficient condition for a merger to increase consumer surplus, as well as a sufficient condition for a merger to increase aggregate surplus.

With few exceptions, however, the literature on merger review has focused on the approval decision for a single merger. Yet, in reality, mergers are usually not one-time events.[1] That

[1] Nilssen and Sorgard [1998], Motta and Vasconcelos [2005], and Matushima [2001] study mergers and an-

is, one proposed merger in an industry may be followed by others. In that case, approval of a merger today based on current conditions, as in the Farrell and Shapiro test, appears inappropriate. Rather, an antitrust authority in general needs to determine the welfare effect of the current proposed merger given the potential for future merger approvals, and given the fact that today's merger approval decision may alter the set of mergers that are later proposed.

In this paper, we show that in many cases this apparently difficult problem has a very simple resolution: an antitrust authority who wants to maximize consumer surplus can accomplish this objective by using a completely myopic merger review policy that approves a merger today if and only if it does not lower consumer surplus given the current market structure.

We begin in Section 2 by establishing some preliminary characterizations of consumer surplus-enhancing mergers and their interactions. Our central results focus on a model of Cournot competition with constant returns to scale. Most importantly, we show in Section 2 that there is a form of complementarity between mergers in that setting. In particular, mergers that enhance consumer surplus continue to be consumer-surplus enhancing if other mergers that enhance consumer surplus take place. Similarly, mergers that reduce consumer surplus continue to be consumer-surplus reducing if other mergers that reduce consumer surplus take place. That is, the sign of a merger's consumer surplus effect is unchanged if another merger whose consumer surplus effect has the same sign takes place. This result, which is of independent interest, sets the stage for our main result, which is contained in Section 3.

In Section 3 we imbed our Cournot competition framework in a dynamic model in which merger opportunities arise, and may be proposed, over time. We show that if the set of possible mergers is disjoint, and if mergers that are not approved in a given period may be approved at a later date, then a completely myopic consumer surplus-based approval policy maximizes discounted consumer surplus for every possible realization of the set of feasible mergers.

In Section 4, we discuss extensions of this result, considering other models of competition (homogeneous and differentiated product price competition), the presence of fixed costs and exit, merger proposal costs, demand shifts, entry, continuing innovation, the use of an aggregate surplus criterion, more limited information possessed by firms about each other's merger possibilities, and breakups.

Section 5 concludes. There we note how our model naturally gives rise to the emergence of endogenous merger waves, and also discuss one important limitation of our results, the assumption that potential mergers are "disjoint."

2 Mergers in the Cournot Model

2.1 Cournot Oligopoly

Consider an industry with n firms producing a homogeneous good and competing in quantities. Let $N \equiv \{1, 2, ..., n\}$ denote the set of firms. Firm i's cost of producing q_i units of output is

titrust review in a dynamic context. In these papers, two mergers between two nonoverlapping pairs of firms can take place sequentially. We discuss these papers further in Section 4. Kamien and Zang [1990], Gowrisankaran [1999], Fauli-Oller [2000], and Pesendorfer [2005] are among a much larger set of articles that study equilibrium merger decisions in dynamic models but without considering merger policy (and sometimes without allowing for efficiencies).

given by $C_i(q_i) = c_i q_i$, where $c_i > 0$ is firm i's marginal cost. Thus, for now, we restrict attention to firms producing under constant returns to scale. The inverse market demand is given by the twice differentiable function $P(Q)$, where $Q \equiv \sum_{i \in N} q_i \geq 0$ is industry output. We make the following (standard) assumption on demand.

Assumption 1 *For any $Q > 0$ such that $P(Q) > 0$:*

(i) $P'(Q) < 0$*;*

(ii) $P'(Q) + QP''(Q) < 0$.

Moreover,

(iii) $\lim_{Q \to \infty} P(Q) = 0$*;*

Part (i) of the assumption says that demand is downward-sloping, part (ii) implies that quantities are strategic substitutes and that each firm's profit maximization problem is strictly concave, part (iii) in conjunction with $c_i > 0$ for all i implies that the equilibrium aggregate output is bounded.

Let $Q_{-i} \equiv \sum_{j \neq i} q_j$ denote the aggregate output of all firms other than i. Firm i's best-response is

$$b(Q_{-i}; c_i) = \arg\max_{q_i \geq 0} [P(Q_{-i} + q_i) - c_i] q_i. \tag{1}$$

As is well known (see e.g., Farrell and Shapiro [1990]), Assumption 1 implies that each firm's best-response function $b(\cdot; c_i)$ satisfies $\partial b(Q_{-i}; c_i)/\partial Q_{-i} \in (-1, 0)$ at all Q_{-i} such that $b(Q_{-i}; c_i) > 0$.

Under Assumption 1, there is a unique Nash equilibrium. Let Q^* and q_i^* denote industry output and firm i's output in equilibrium. From the first-order condition for problem (1), output levels in this equilibrium satisfy

$$q_i^* = -\frac{P(Q^*) - c_i}{P'(Q^*)} \tag{2}$$

if $c_i < P(Q^*)$, and $q_i^* = 0$ otherwise. Assumption 1 also implies that the equilibrium is "stable," so that comparative statics are "well behaved." For example, we will make use of two comparative statics properties: First, a reduction in an active firm's marginal cost increases its equilibrium output and profit, reduces the output of each of its active rivals, and increases aggregate output. Second, following any change in the incentives of a subset of firms, the equilibrium aggregate output increases [decreases] if and only if the equilibrium output of that set of firms increases [decreases].[2]

2.2 The CS-Effect of Mergers

Consider a merger between a subset $M \subseteq N$ of firms. The post-merger marginal cost is denoted $\overline{c}_M$. Aggregate output before the merger is Q^*, and after is $\overline{Q}^*$. We are interested in the effect of the merger on consumer surplus, $CS(\overline{Q}^*) - CS(Q^*)$, where

$$CS(Q) = \int_0^Q [P(s) - P(Q)]\, ds.$$

[2]See Farrell and Shapiro [1990]'s Lemma, p. 111.

Since $CS'(Q) = -QP'(Q) > 0$, a merger raises consumer surplus if and only if it induces an increase in industry output. We will say that a merger is *CS-neutral* if the merger does not affect consumer surplus. Similarly, we will say that a merger is *CS-increasing* [*CS-decreasing*], if consumer surplus following the merger is higher [lower] than before. Finally, a merger is *CS-nondecreasing* [*CS-nonincreasing*] if it is not CS-decreasing [CS-increasing].

We will say that a merger involves active firms if at least one of the merging firms is producing a positive quantity before the merger [and hence has $c_i < P(Q^*)$]. Observe that a merger involving only inactive firms is always CS-nondecreasing and weakly profitable. The following result catalogs some useful properties of CS-neutral mergers involving active firms.

Lemma 1 *If a merger involving active firms is CS-neutral, then*

1. *it causes no changes in the output of any nonmerging firm nor in the total output of the merging firms;*

2. *the merged firm's margin at the pre- and post-merger price $P(Q^*)$ equals the sum of the active merging firms' pre-merger margins:*

$$P(Q^*) - \overline{c}_M = \sum_{i \in M} \max\{0, P(Q^*) - c_i\}; \tag{3}$$

3. *the merged firm's marginal cost is no greater than the marginal cost of the most efficient merger partner: $\overline{c}_M \leq \min_{i \in M}\{c_i\}$, and it is strictly less if the merger involves at least two active firms;*

4. *the merger is profitable (it weakly raises the joint profit of the merging firms), and is strictly profitable if it involves at least two active firms.*

Proof. To see Property 1, observe that under Assumption 1 there is a unique output level for each non-merging firm i that is compatible with any given level of aggregate output Q [since there is a unique Q_{-i} such that $Q_{-i} + b(Q_{-i}; c_i) = Q$]. Since aggregate output is unchanged by a CS-neutral merger, all nonmerging firms' outputs are unchanged. In turn, this implies that the total output of the merging firms must be unchanged as well. For Property 2, a central feature in Farrell and Shapiro [1990]'s analysis, note that the merged firm's first-order condition [using Property 1] is

$$P(Q^*) - \overline{c}_M + \left(\sum_{i \in M} q_i^* \right) P'(Q^*) = 0. \tag{4}$$

Summing up the pre-merger first-order conditions of the active merger partners yields

$$\sum_{i \in M_+} \{P(Q^*) - c_i + q_i^* P'(Q^*)\} = 0 \tag{5}$$

where $M_+ = \{i \in M : q_i^* > 0\}$. Since for all $i \in M \backslash M_+$, we have $P(Q^*) \leq c_i$ and $q_i^* = 0$, it follows that

$$\sum_{i \in M} \max\{0, P(Q^*) - c_i\} + \left(\sum_{i \in M} q_i^* \right) P'(Q^*) = 0. \tag{6}$$

Combining equations (4) and (6), yields (3). Property 3 follows directly from Property 2. Property 4 holds since the merging firms' joint output has not changed (Property 1), but its margin has weakly increased, and has strictly increased if the merger involves at least two active firms (Property 2). ■

The following useful corollary follows from Properties 2 and 4 of Lemma 1 plus the fact that the post-merger aggregate output, $\overline{Q}^*$, and the profit of the merged firm are both decreasing in the merged firm's marginal cost, $\overline{c}_M$:

Corollary 1 *A merger involving active firms is CS-neutral if*

$$\overline{c}_M = \widehat{c}_M(Q^*) \equiv P(Q^*) - \sum_{i \in M} \max\{0, P(Q^*) - c_i\},$$

CS-increasing if $\overline{c}_M < \widehat{c}_M(Q^*)$, and CS-decreasing if $\overline{c}_M > \widehat{c}_M(Q^*)$. Moreover, any CS-nondecreasing merger is profitable for the merging firms, and is strictly profitable if it is CS-increasing or involves at least two active firms.

Thus, an antitrust authority concerned with maximizing consumer surplus and confronted with a single merger involving active firms in set M would strictly prefer to approve the merger if $\overline{c}_M < \widehat{c}_M(Q^*)$, and would be willing to if $\overline{c}_M \leq \widehat{c}_M(Q^*)$. Moreover, any merger among active firms that the antitrust authority would be willing to approve is profitable for the merging parties.

Observe also that the threshold $\widehat{c}_M(Q^*)$ is nondecreasing in Q^* and is strictly decreasing if the merger involves at least two active firms. Thus, the larger is Q^* (and the lower is the pre-merger price), the more likely it is that a merger is CS-nondecreasing. This fact will play a central role in the next subsection when we look at interactions among mergers, where one merger may lead to a change in industry output prior to the proposal of another merger. To see the intuition for this result, consider a proposed merger between symmetric firms, each of whom has a pre-merger marginal cost c and produces $q^* > 0$ units. Since the firms are choosing their outputs optimally before the merger, a lower pre-merger margin $P(Q^*) - c$ (due to a larger pre-merger aggregate output) implies a smaller pre-merger absolute value of $P'(Q^*)q^*$ [see (2)]. The incentives of the merged firm to raise price, however, depend on a comparison of the merger's marginal cost reduction $\Delta c = (c - \overline{c}_M)$ to the market power effect, $P'(Q^*)q^*$, which reflects the internalization of the pricing externality between the merging firms. With a CS-nondecreasing merger, the first effect weakly exceeds the second. A smaller pre-merger price preserves this relation and therefore the CS-nondecreasing effect of the merger.

Figure 2 illustrates the cases of CS-neutral, CS-increasing, and CS-decreasing mergers. The figure considers a merger involving the firms in set M_1, at least two of whom are active. The complementary set of firms is denoted $M_2 \equiv N \backslash M_1$. The axes in the figure measure the joint outputs of the two sets of firms. The curves labeled r_{M_1} and r_{M_2} depict what we call the "group-reaction functions" of each set of firms prior to the merger. Specifically, M_i's pre-merger group-reaction function gives the joint pre-merger Nash-equilibrium output of the firms in M_i, $r_{M_i}(q_{M_j})$, conditional on the firms in M_j, $j \neq i$, jointly producing q_{M_j}. It is routine to verify that these group-reaction functions satisfy $-1 < r'_{M_i}(q_{M_j}) < 0$.

The equilibrium before the merger is point A, the intersection of the two pre-merger group-reaction curves. With a CS-neutral merger, the post-merger best-response curve of the merged

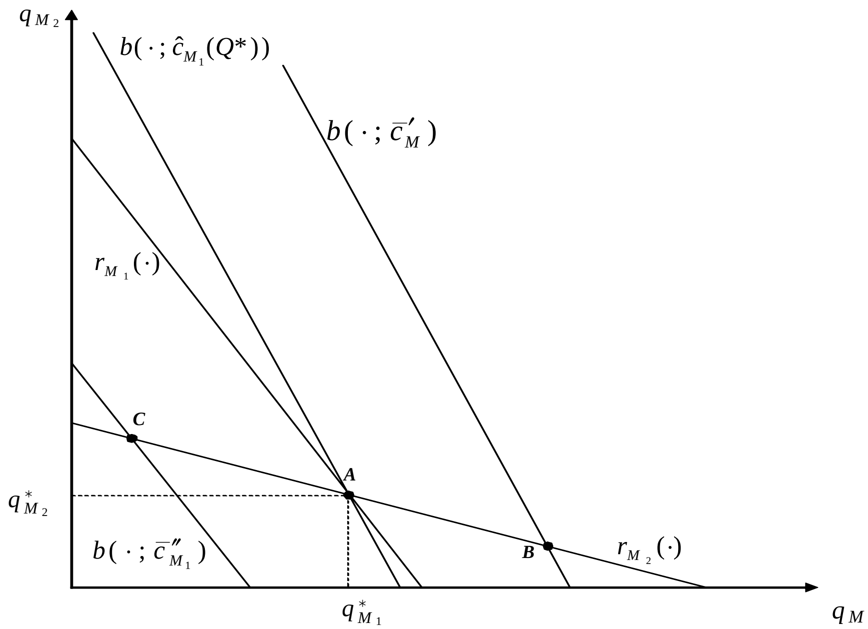

Figure 2: A merger involving the firms in M_1. Depending on the merged firm's marginal cost, the merger is CS-neutral [point A], CS-increasing [point B], or CS-decreasing [point C]. In the figure, $\overline{c}''_{M_1} > \widehat{c}_{M_1}(Q^*) > \overline{c}'_{M_1}$.

firm, $b(\cdot;\widehat{c}_{M_1}(Q^*))$, intersects group M_2's group-reaction curve, $r_{M_2}(\cdot)$, at point A.[3] With a CS-increasing merger, the merged firm's marginal cost is less than $\widehat{c}_{M_1}(Q^*)$, so its best-response curve lies further to the right, shifting the equilibrium to point B, where there is a larger aggregate output. In contrast, with a CS-decreasing merger, the merged firm's marginal cost is greater than $\widehat{c}_{M_1}(Q^*)$, so its best-response curve lies further to the left, shifting the equilibrium to point C, where there is a smaller aggregate output.

2.3 Interactions between Mergers

We now turn to the interactions between mergers. In this subsection, we consider two potential disjoint mergers, involving firms in sets M_1 and M_2 with $M_1 \cap M_2 = \emptyset$. We will refer to these simply as merger M_1 and merger M_2. The set of firms not involved in either merger is $N^c \equiv N \backslash (M_1 \cup M_2)$.

Our first result establishes a certain complementarity between mergers that change consumer surplus in the same direction:[4]

Proposition 1 *The sign of the CS-effect of two disjoint mergers is complementary:*

(i) *if a merger is CS-nondecreasing (and hence profitable) in isolation, it remains CS-nondecreasing (and hence profitable) if another merger that is CS-nondecreasing in isolation takes place;*

(ii) *if a merger is CS-decreasing in isolation, it remains CS-decreasing if another merger that is CS-nonincreasing in isolation takes place.*

Proof. For part (i), suppose that mergers M_1 and M_2 are both CS-nondecreasing in isolation. Let Q^* denote aggregate output in the absence of either merger and let $\overline{Q}_i^*$ denote aggregate output if only merger M_i takes place. So $\overline{Q}_i^* \geq Q^*$ for $i = 1, 2$. Without loss of generality, consider merger M_1. Suppose, first, that merger M_1 involves only inactive firms once merger M_2 takes place. Then, once merger M_2 takes place, merger M_1 must be CS-nondecreasing and (weakly) profitable .

Suppose, instead, that merger M_1 involves active firms once merger M_2 takes place, which also means [since $P(\overline{Q}_2^*) \leq P(Q^*)$] that it involves active firms when done in isolation. Since it is CS-nondecreasing in isolation, from Corollary 1 we know that $\overline{c}_{M_1} \leq \widehat{c}_{M_1}(Q^*)$. Moreover,

[3] The post-merger best-response curve $b(\cdot;\overline{c}_{M_1})$ must cross the pre-merger group reaction curve $r_{M_1}(\cdot)$ from above since at $q_{M_2} > [<]\overline{q}_{M_2}$ such that $r_{M_1}(q_{M_2}) > 0$, we have

$$\begin{aligned} b(q_{M_2};\overline{c}_{M_1}) &= b(q_{M_2};\widehat{c}_{M_1}(q^*_{M_2} + r_{M_1}(q^*_{M_2}))) \\ &> [<] b(q_{M_2};\widehat{c}_{M_1}(q_{M_2} + r_{M_1}(q_{M_2}))) \\ &= r_{M_1}(q_{M_2}), \end{aligned}$$

where the inequality follows because $\widehat{c}_{M_1}(\cdot)$ is a strictly increasing function and $b(q_{M_2};c)$ is strictly decreasing in c at all q_{M_2} such that $b(q_{M_2};c) > 0$, and the last equality follows because $\widehat{c}_{M_1}(q_{M_2} + r_{M_1}(q_{M_2}))$ is the cost level at which the merged firm's best response to q_{M_2} is exactly $r_{M_1}(q_{M_2})$.

[4] Proposition 1 focuses on properties needed later in this section and for Section 3. It is straightforward to show as well that a CS-increasing merger M_i remains CS-increasing if a merger M_j that is CS-nondecreasing takes place provided the merger M_i remains among active firms once merger M_j takes place, and that a merger among active firms that is CS-nonincreasing remains CS-nonincreasing if a merger that is CS-nonincreasing takes place; see the discussion in the Appendix of Remark 2.

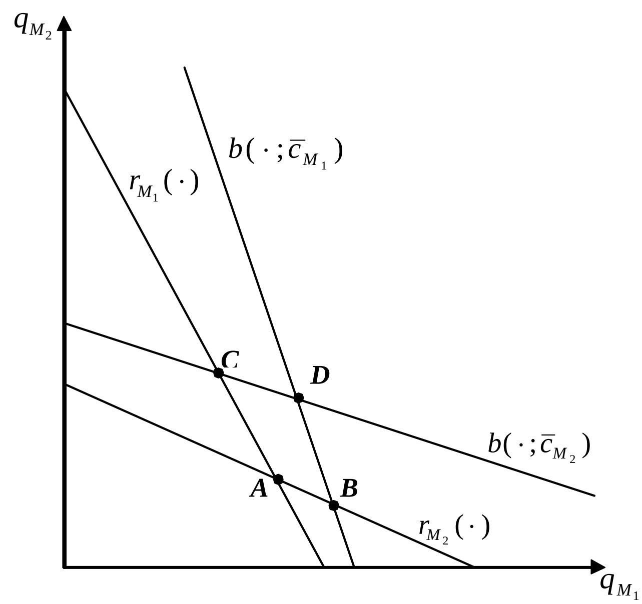

Figure 3: Each merger is CS-increasing in isolation and remains so if the other merger takes place.

because the threshold $\widehat{c}_{M_1}(Q)$ is nondecreasing in Q, we have $\overline{c}_{M_1} \leq \widehat{c}_{M_1}(\overline{Q}_2^*)$. Hence, Corollary 1 implies that merger M_1 is also CS-nondecreasing once merger M_2 has taken place.

The argument for part (ii) follows similar lines (note that a CS-decreasing merger must involve active firms, and must continue to do so after another CS-decreasing merger takes place). ■

Figure 3 illustrates the complementarity between two mergers that are CS-increasing in isolation when no other firms exist ($N^c = \emptyset$). In isolation, merger M_1 moves the equilibrium from point A to point B, while merger M_2 moves the equilibrium from point A to point C. But, conditional on merger M_1 taking place, merger M_2 moves the equilibrium from point B to point D along $b(\cdot;\overline{c}_{M_1})$. Since $\partial b(\cdot;\overline{c}_{M_1})/\partial Q_{-i} \in (-1, 0)$, aggregate output must increase. That is, conditional on merger M_1 taking place, merger M_2 remains CS-increasing. Moreover, we know from Corollary 1 that it also remains profitable. Using the same type of argument, the reverse is also true: conditional on merger M_2 taking place, the merger M_1 remains CS-increasing and profitable.

We now turn to the interaction between mergers that have opposite effects on consumer surplus if implemented in isolation. Specifically, suppose that merger M_1 is CS-nondecreasing (and therefore profitable) in isolation, while merger M_2 is CS-decreasing in isolation. Figure 4 illustrates that merger M_2 can become CS-increasing (and therefore strictly profitable) conditional on merger M_1 occurring. In isolation, merger M_2 moves the equilibrium from point A to point C along $r_{M_1}(\cdot)$, and thus decreases industry output. But after merger M_1 has taken place, merger M_2 moves the equilibrium from point B to point D along $b(\cdot;\overline{c}_{M_1})$, and thus

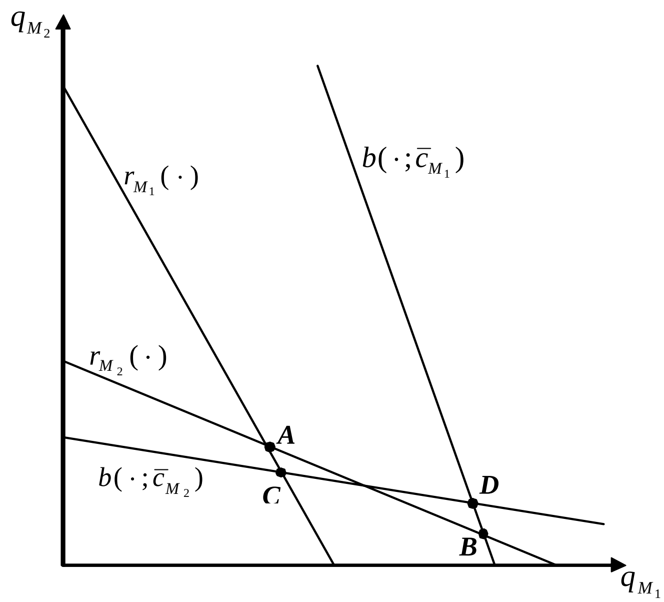

Figure 4: A CS-decreasing merger M_2 that becomes CS-increasing after a CS-increasing merger M_1 takes place.

increases industry output.

When this occurs, we can say the following:

Proposition 2 *Suppose that merger M_1 is CS-nondecreasing in isolation, while merger M_2 is CS-decreasing in isolation but CS-nondecreasing once merger M_1 has taken place. Then:*

(i) *merger M_1 is CS-increasing (and therefore strictly profitable) conditional on merger M_2 taking place;*

(ii) *the joint profit of the firms involved in merger M_1 is strictly larger if both mergers take place than if neither merger takes place.*

Proof. Consider implementing merger M_1 first followed by merger M_2. By hypothesis, consumer surplus weakly increases after each step, so the combined effect on consumer surplus of the two mergers is nonnegative. Suppose we now reverse the order and implement merger M_2 first. Since the combined effect of the two mergers on consumer surplus is nonnegative while the effect of merger M_2 is strictly negative, consumer surplus must strictly increase when merger M_1 is implemented following merger M_2. Hence, part (i) must hold: merger M_1 is CS-increasing (and therefore strictly profitable) conditional on merger M_2 taking place.

To see that part (ii) holds, suppose that merger M_2 is implemented first. Since merger M_2 is CS-decreasing in isolation, it must weakly increase the profit of each firm $i \in M_1$ [the joint output of all firms other than i must decrease, otherwise the fact that $\partial b(\cdot; c_i)/\partial Q_{-i} \in (-1, 0)$ would imply that aggregate output increases]. Since merger M_1 is strictly profitable given merger M_2, the sequence of mergers must strictly increase the joint profit of the firms in M_1. ■

The result is illustrated in Figure 4, where merger M_1 is CS-increasing (and hence strictly profitable) in isolation and remains so conditional on merger M_2 taking place, at which point it moves the equilibrium from point C to point D along $b(\cdot; \overline{c}_{M_2})$.

Remark 1 *Observe that the logic of Proposition 2 can be extended to cases with a merger M_1 that is CS-nondecreasing in isolation and a collection of mergers $M_2, ..., M_K$ that are each CS-decreasing in isolation but form a sequence that is CS-nondecreasing at each step after merger M_1 has taken place. In such cases, merger M_1 is CS-increasing (and therefore strictly profitable) given that mergers $M_2, ..., M_K$ have taken place, and the joint profit of the firms involved in merger M_1 is strictly larger if all of these mergers take place than if none do. We will use this extension of Proposition 2 in Section 3.*

3 CS-Maximizing Merger Review

In this section, we embed the Cournot model of Section 2 in a dynamic model where merger opportunities arise stochastically over time, merger proposals are endogenous, and the antitrust authority decides whether or not to approve proposed mergers. We consider the optimal merger approval policy for an antitrust authority concerned with maximizing discounted consumer surplus. We show that such an antitrust authority can achieve its optimal outcome using a myopic policy that in each period approves a set of mergers that maximizes consumer surplus given the current market structure, ignoring the possibility of any future mergers.

As before, we denote the set of n firms by N. The set of possible mergers are those in set $\{M_1, ..., M_K\}$, where $M_k \subseteq N$ is a set of firms that may merge. We assume that these possible mergers are disjoint; that is, $M_j \cap M_k = \emptyset$ for $j \neq k$. Thus, no firm has the possibility of being part of more than one merger.[5] The assumption of disjointness is reasonable when each firm belongs to at most a single set of "natural" merger partners who can generate significant efficiencies by merging, perhaps because they use similar or complementary technologies. If all other mergers both increase market power and fail to generate efficiencies, no other mergers but these would ever optimally be approved by the antitrust authority.[6] (We discuss the disjointness assumption further in Section 5.)

The merger process lasts for T periods. Merger M_k first becomes feasible at the start of period t with probability $p_{kt} \in [0, 1]$, where $\sum_t p_{kt} \leq 1$. Conditional on merger M_k becoming feasible in period t, the firms in M_k receive and observe a random draw of their post-merger cost

[5] Our results can be extended to allow for a given firm to be part of several different possible mergers provided that at most one of these mergers ever becomes feasible along any path.

[6] Because it takes a strictly positive cost reduction to offset the market power increase from a merger (recall Lemma 1), it is frequently enough to justify the disjointness assumption if other mergers cannot generate *large enough* cost reductions.

$\bar{c}_{M_k}$. This cost is drawn from the set C_{kt} with distribution function $G_{kt}(\cdot)$.[7] This formulation embodies another form of disjointness in merger possibilities: merger M_k receives at most one efficiency realization throughout the merger process.[8] We denote the set of mergers that have become feasible up to and including period t (including their cost realizations) by $\mathfrak{F}_t$.

In each period t, all firms with feasible but not-yet-approved mergers decide whether to propose them or not. Previously proposed but rejected mergers can be proposed again, as can previously unproposed feasible mergers. We denote by $\mathcal{P}_t$ the set of mergers proposed in period t. The antitrust authority then responds by approving some subset of the proposed mergers. We denote by $\mathcal{A}_t$ the set of mergers approved by the end of period t; that is, $\mathcal{A}_t$ is the *market structure* at the end of the period after the merger review process has concluded. Note that we must have $\mathcal{A}_{t-1} \subseteq \mathcal{A}_t \subseteq (\mathcal{A}_{t-1} \cup \mathcal{P}_t) \subseteq \mathfrak{F}_t$; the first inclusion follows because the set of approved mergers weakly grows over time, the second because only proposed mergers can be approved, and the third because only feasible mergers that have not yet been approved can be proposed.[9]

We assume that when a merger M_k becomes feasible in a period t, one of the firms in M_k is designated as the "proposer" of the merger. To keep things simple, we treat bargaining in a reduced-form manner, assuming that the proposer chooses whether to propose the merger to the antitrust authority, and that if he chooses to do so, the firms in M_k split the profit gains or losses from the merger in some fixed proportions (the proportions do not matter).[10]

The antitrust authority observes that a particular merger is feasible and its efficiency (post-merger marginal cost) once it is proposed. For simplicity, we assume that firms observe both their own and their rivals' merger possibilities, including their efficiencies, when they become feasible. (We discuss in Section 4.9 how our results extend if firms possess less information about rivals' mergers, for example observing their feasibility only once they are proposed and their efficiency only once they are approved.)

Payoffs in each period t depend only on the set of mergers $\mathcal{A}_t$ approved by the end of that period, and are determined by a complete information Cournot game, as in Section 2. Each agent i, whether the antitrust authority or a proposer firm, discounts future payoffs (consumer surplus or profit) according to a discount factor $\delta_i \leq 1$.[11]

3.1 Myopic Merger Policies

We are interested in the performance of "myopic" merger review policies, which in each period maximize consumer surplus given the set of proposed mergers and current market structure, ignoring the possibility of future mergers. Toward this end, we start by introducing the following definitions:

[7] Note that our assumptions allow the sequence of feasible mergers and their cost realizations to be deterministic.

[8] We relax this assumption in Section 4.7.

[9] In the model, we do not allow previously approved mergers to be dissolved. However, it follows from our arguments that no (approved) merged firm would want to do so.

[10] The only important feature of this assumption is that it implies that merger M_k is proposed if it raises the joint expected discounted profit of the potential merger partners and will not be proposed if it lowers it.

[11] Implicitly, to justify our profit-splitting assumption, we assume that each firm in a given merger M has the same discount factor.

Definition 1 *A set of approved mergers $\mathcal{A}_t \subseteq (\mathcal{A}_{t-1} \cup \mathcal{P}_t) \subseteq \mathfrak{F}_t$ is* ***myopically CS-maximizing for*** $\mathcal{P}_t$ ***given market structure*** $\mathcal{A}_{t-1}$ *if it maximizes consumer surplus in the current period (period t) given $\mathcal{P}_t$ and $\mathcal{A}_{t-1}$.*

In our model with unchanging demand, maximizing current period consumer surplus is equivalent to maximizing discounted consumer surplus assuming that there will be no subsequent changes in market structure.

Definition 2 *A* ***myopically CS-maximizing merger policy*** *is a merger approval rule that in each period t approves mergers as a function of the already-approved mergers $\mathcal{A}_{t-1}$, the current set of proposed mergers $\mathcal{P}_t$, and perhaps the period t, resulting in a new market structure $\mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ that is myopically CS-maximizing for $\mathcal{P}_t$ given market structure $\mathcal{A}_{t-1}$.*

While we note later in Remark 2 that our main result holds for *any* myopically CS-maximizing merger policy, for ease of exposition we focus on the performance of the *most lenient* myopically CS-maximizing merger policy. In this policy, the antitrust authority resolves any indifference about mergers in favor of approval, selecting in each period the largest possible set of mergers to approve among those sets that maximize consumer surplus. We call such a set a "largest myopically CS-maximizing set":

Definition 3 *A set of approved mergers $\mathcal{A}_t \subseteq (\mathcal{A}_{t-1} \cup \mathcal{P}_t)$ is a* ***largest myopically CS-maximizing set for*** $\mathcal{P}_t$ ***given market structure*** $\mathcal{A}_{t-1}$ *if it is not contained in any other set that is myopically CS-maximizing for $\mathcal{P}_t$ given $\mathcal{A}_{t-1}$.*

Given the finiteness of the set of proposed mergers $\mathcal{P}_t$, a largest myopically CS-maximizing set must always exist. In fact, there is a unique such "largest" set for any existing market structure $\mathcal{A}_{t-1}$ and set of proposed mergers $\mathcal{P}_t$, which we denote by $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, and this set contains every other myopically CS-maximizing set for $\mathcal{A}_{t-1}$ and $\mathcal{P}_t$. Moreover, this set grows as the set of proposed mergers grows:

Lemma 2 *For each set of proposed mergers $\mathcal{P}_t$ and current market structure $\mathcal{A}_{t-1}$, there is a unique largest myopically CS-maximizing set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ and it contains every other myopically CS-maximizing set for $\mathcal{A}_{t-1}$ and $\mathcal{P}_t$. Moreover, if $\mathcal{P}_t \subset \mathcal{P}'_t$ then $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathcal{P}'_t|\mathcal{A}_{t-1})$.*

Proof. In the Appendix. ■

The *most lenient* myopically CS-maximizing merger policy is therefore the one that approves in each period the largest myopically CS-maximizing set of mergers:

Definition 4 *The* ***most lenient myopically CS-maximizing merger policy*** *is the myopically CS-maximizing merger policy that in each period t implements the largest myopically CS-maximizing set given the proposed mergers $\mathcal{P}_t$ and current market structure $\mathcal{A}_{t-1}$, resulting in new market structure $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$.*

Note that the most lenient myopically CS-maximizing merger policy is independent of the period t, since it depends only on the payoff-relevant variables $\mathcal{P}_t$ and $\mathcal{A}_{t-1}$.

Importantly, the most lenient myopically CS-maximizing merger policy can also be thought of as the result of an antitrust policy that evaluates proposed mergers in an even more myopic way, making decisions on mergers within each period in a step-by-step fashion and approving a merger at each step if and only if it is CS-nondecreasing given the current market structure (including any mergers that have already been approved in that period), and continuing until no further CS-nondecreasing mergers can be identified (including mergers that may have already been examined but rejected earlier in the period). Specifically:

Lemma 3 *Suppose that the antitrust authority considers mergers within period t in a step-by-step fashion, approving mergers that are CS-nondecreasing given the current market structure until no further CS-nondecreasing mergers can be identified. Then if $\mathcal{P}_t$ is the set of proposed mergers and $\mathcal{A}_{t-1}$ is the market structure at the start of the period, the set of approved mergers at the end of period t will be $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$.*

Proof. In the Appendix. ∎

Thus, our results will apply to any antitrust policy that considers mergers one at a time, approving each merger if it is CS-nondecreasing given the current market conditions.

3.2 Optimality of Myopic Merger Policy

Our main result shows that the most lenient myopically CS-maximizing merger policy is a dynamically optimal policy for the antitrust authority. The argument has two parts. First, we show that if all feasible but not-yet-approved mergers are proposed in each period – so that the antitrust authority need not worry about firms' incentives to propose mergers – then the most lenient myopically CS-maximizing merger policy maximizes discounted consumer surplus for every realized sequence of feasible mergers.

Lemma 4 *If all feasible but not-yet-approved mergers are proposed in each period, the most lenient myopically CS-maximizing merger policy, which induces the approval sequence $\mathcal{A}_1 = \overline{\mathcal{A}}^*(\mathfrak{F}_1|\varnothing)$ and $\mathcal{A}_t = \overline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ for all $t > 1$, maximizes discounted consumer surplus for every realization of feasible mergers $\mathfrak{F} = (\mathfrak{F}_1, ..., \mathfrak{F}_T)$.*

Proof. Given the realized sequence of feasible mergers $\mathfrak{F} = (\mathfrak{F}_1, ..., \mathfrak{F}_T)$, consider the problem of maximizing discounted consumer surplus. If we ignore the monotonicity constraint that the set of approved mergers cannot shrink over time, we can choose the approved set of mergers (i.e., the market structure) in each period independently from the mergers approved in every other period. It is evident that in that case the approval sequence $\{\overline{\mathcal{A}}^*(\mathfrak{F}_1|\varnothing), ..., \overline{\mathcal{A}}^*(\mathfrak{F}_T|\varnothing)\}$ is optimal since it maximizes consumer surplus in every period.

Consider now the most lenient myopically CS-maximizing merger policy. We will show that this policy induces the approval sequence $\{\overline{\mathcal{A}}^*(\mathfrak{F}_1|\varnothing), ..., \overline{\mathcal{A}}^*(\mathfrak{F}_T|\varnothing)\}$, from which observation the result follows. To do so we will actually establish a slightly stronger fact, which will also be useful in the proof of Proposition 3: If the antitrust authority follows the most lenient myopically CS-maximizing merger policy in periods $1, ..., t-1$ and if all feasible but not-yet-approved mergers are proposed in period t, the market structure at the end of period t will be $\overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ regardless of the merger proposals that firms have made in periods $1, ..., t-1$.

To see this, consider an arbitrary period t and suppose that $\mathcal{A}_{t-1} \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_{t-1}|\varnothing)$ regardless of the history of previous merger proposals (which is true if $t = 1$). If all feasible but not-yet-approved mergers are proposed in period t, then $\mathcal{A}_t = \overline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$. We will show that

$\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$. Observe that the problem of myopically maximizing consumer surplus given previously approved mergers $\mathcal{A}_{t-1}$ and proposed mergers $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ is a more constrained problem than is the problem of myopically maximizing consumer surplus given no previously approved mergers and proposed mergers $\mathfrak{F}_t$. However, since $\mathcal{A}_{t-1} \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_{t-1}|\varnothing) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ (the first inclusion follows by hypothesis and the second by Lemma 2), the largest solution to this latter, less constrained problem is feasible in the former, more constrained problem. It must therefore also be the largest solution in the more constrained problem. Hence, $\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$, which implies that, if all mergers are proposed in period t, the most lenient myopically CS-maximizing policy induces the set $\mathcal{A}_t = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ in period t. Note also that since $\mathcal{P}_t \subseteq \mathfrak{F}_t$, the monotonicity of the largest myopically CS-maximizing set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ in $\mathcal{P}_t$, established in Lemma 2, implies that $\mathcal{A}_t \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$, regardless of the merger proposals made up through and including period t. Thus, our induction hypothesis holds when we look at period $t+1$. Applying induction, yields the result. ■

Lemma 4 shows that myopic behavior causes no problem for the antitrust agency when it does not need to worry about firms' proposal incentives. At its heart, the result follows from two features: (i) the complementarity of CS-nondecreasing mergers – which implies that the antitrust authority will never later regret approval of a CS-nondecreasing merger due to the appearance of a new CS-nondecreasing merger, and (ii) the fact that since the antitrust authority can always approve a merger at a later date, it will never later regret rejection of a merger that is CS-decreasing given the current market structure.

The second part of the argument concerns firms' incentives to propose mergers. Since the antitrust agency is free to reject mergers it does not like, its only concern is that firms may not propose mergers that it would like to approve. To establish our main result, we will show that when the antitrust authority adopts the most lenient myopically CS-maximizing merger policy the firms' proposal incentives are aligned with the desires of the antitrust authority. More specifically, there is a subgame perfect Nash equilibrium for the firms in which every feasible merger is proposed in every period. Moreover, *all* subgame perfect Nash equilibria result in the same (optimal) sequence of period-by-period consumer surpluses.

Proposition 3 *Suppose the antitrust authority follows the most lenient myopically CS-maximizing merger policy. Then:*

(i) *All feasible mergers being proposed in each period after any history is a subgame perfect Nash equilibrium for the firms. In this equilibrium, the outcome maximizes discounted consumer surplus for any realized sequence of feasible mergers* $\mathfrak{F} = (\mathfrak{F}_1, ..., \mathfrak{F}_T)$.

(ii) *For each sequence* $\mathfrak{F}$, *every subgame perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. (i) The proof of the first claim is by induction. Consider, a period t and suppose that starting in period $t+1$ the joint expected continuation payoff of the firms in each possible feasible merger is independent of firms' prior behavior. (Note that this is true in period T.) We will establish that regardless of the previous history or rivals' proposal strategies in period t, it is optimal in period t for every proposer firm with a feasible but-not-yet approved merger to propose it.[12]

To see this, consider a firm that is the proposer of a feasible but not-yet-approved merger M_k. Note that since continuation payoffs are (by hypothesis) unaffected by period t play, it is optimal to propose the merger if proposing it maximizes the joint expected period t payoff of the firms in M_k. Let $\widehat{\mathcal{P}}$ denote a realization of the set of proposed mergers in period t if merger M_k is proposed (firms in other feasible but not-yet-approved mergers may be using mixed strategies) and let $\widehat{\mathcal{P}}_{-k} \equiv \widehat{\mathcal{P}} \backslash M_k$ denote that realization without merger M_k included.

Suppose, first, that $\widehat{\mathcal{P}}_{-k}$ is such that merger M_k is not approved when proposed. Then the set of approved mergers, and hence the joint period-t expected payoff of the firms in M_k, is unaffected by whether merger M_k is proposed. To see this, observe that we then have $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \subseteq (\widehat{\mathcal{P}}_{-k} \cup \mathcal{A}_{t-1})$, where the first inclusion follows from Lemma 2 and the second from the fact that merger M_k is not approved when proposed. But approving set $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1})$ is therefore feasible and myopically CS-maximizing when mergers $\widehat{\mathcal{P}}_{-k}$ are proposed.[13] Since $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$ is the largest myopically CS–maximizing set for $\widehat{\mathcal{P}}_{-k}$ given $\mathcal{A}_{t-1}$, we must have $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1})$.

Suppose, instead, that $\widehat{\mathcal{P}}_{-k}$ is such that merger M_k is approved when proposed, but that the merged firm M_k is inactive (produces zero output) in period t after its merger is approved. Then, merger M_k is CS-neutral given the other mergers that are approved, which implies that $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash M_k$ is a myopically CS-maximizing set for $\widehat{\mathcal{P}}$ given $\mathcal{A}_{t-1}$. Moreover, in this case we have $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash M_k \subseteq (\widehat{\mathcal{P}}_{-k} \cup \mathcal{A}_{t-1})$. The set $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash M_k$ is therefore both feasible and myopically CS-maximizing for $\widehat{\mathcal{P}}_{-k}$ given $\mathcal{A}_{t-1}$ (recall footnote 13), which implies that we must have $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash M_k = \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$; that is, proposal of merger M_k does not affect the set of other mergers that are approved in period t. As a result, proposal of merger M_k has no effect on the joint period-t profits of the firms in M_k, which are zero in either case.

Finally, suppose that $\widehat{\mathcal{P}}_{-k}$ is such that merger M_k is approved when proposed and that the merged firm M_k is active in period t after its merger is approved. We distinguish between two cases. First, suppose that $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1}) \cup M_k = \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1})$. In this case, proposing merger M_k does not affect the other mergers that will be approved. Since $M_k \in \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1})$, the merger is CS-nondecreasing given the other mergers that will be approved, and is therefore [by Corollary 1] strictly profitable to propose. Second, suppose that $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1}) \cup M_k \subset \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1})$. Part (i) of Lemma 6 in the Appendix implies that there is a sequencing of the mergers in $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$ that is CS-nondecreasing at each step. However, since all of the mergers in this set other than M_k must be CS-decreasing given that the mergers in $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$ have taken place [otherwise they would have been in $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$], merger M_k must be CS-nondecreasing given that the mergers in $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$ have occurred and must be the first merger in this sequence. By Remark 1, the firms in M_k have a strictly greater profit when all of the mergers in $\overline{\mathcal{A}}^*(\widehat{\mathcal{P}}|\mathcal{A}_{t-1}) \backslash \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_{-k}|\mathcal{A}_{t-1})$ are approved than when none are.

[12] The history prior to period t's proposal stage consists of the sequences $\mathfrak{F}^t = (\mathfrak{F}_1, ..., \mathfrak{F}_t)$ of feasible mergers, $\mathcal{P}^{t-1} = (\mathcal{P}_1, ..., \mathcal{P}_{t-1})$ of proposed mergers, and $\mathcal{A}^{t-1} = (\mathcal{A}_1, ..., \mathcal{A}_{t-1})$ of approved mergers. This history, which is observed by all firms, determines a subgame that starts in period t.

[13] Note that if $\mathcal{P}_t \subseteq \mathcal{P}'_t$ and $\mathcal{A}_t$ and $\mathcal{A}'_t$ are myopically CS-maximizing for, respectively, $\mathcal{P}_t$ and $\mathcal{P}'_t$ given $\mathcal{A}_{t-1}$, then the level of consumer surplus under $\mathcal{A}'_t$ must be at least as great as under $\mathcal{A}_t$ (with more mergers proposed, it is a less constrained problem). Hence, if $\mathcal{A}'_t$ is feasible for $\mathcal{P}_t$ given $\mathcal{A}_{t-1}$ – that is, if $\mathcal{A}'_t \subseteq (\mathcal{P}_t \cup \mathcal{A}_{t-1})$ – then $\mathcal{A}'_t$ must also be myopically CS-maximizing for $\mathcal{P}_t$ given $\mathcal{A}_{t-1}$.

Hence, it is strictly more profitable in this case as well to propose merger M_k.

In summary, it is an optimal strategy for every feasible but not-yet-approved merger M_k to be proposed in period t regardless of the previous history and rivals' period-t proposal strategies. The set of approved mergers at the end of period t will therefore be $\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$. By the argument in the proof of Lemma 4, we know that $\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ for any $\mathcal{A}_{t-1}$ that can arise under the most lenient myopically CS-maximizing merger policy. Thus, the market structure (and joint expected payoffs of the firms in each possible merger) at the end of period t is independent of firms' behavior prior to period t. Our induction hypothesis therefore holds when we look at period $t-1$. Applying induction starting in period T implies that in every period proposing every feasible but not-yet-approved merger is optimal.

(ii) To establish the second claim, we first define two sets that form a partition of $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. Let $\overline{\mathcal{A}}_0^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ denote those mergers in $\mathcal{P}_t \cap \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ that result in merged firms that are inactive given the other mergers in $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ and let

$$\overline{\mathcal{A}}_1^*(\mathcal{P}_t|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \backslash \overline{\mathcal{A}}_0^*(\mathcal{P}_t|\mathcal{A}_{t-1})$$

denote the complementary set. Note that approval of inactive mergers has no effect on either consumer surplus or firms' payoffs. This implies that if all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ are proposed – i.e., if $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \subseteq (\mathcal{P}_t \cup \mathcal{A}_{t-1})$ – then the largest myopically CS-maximizing set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ will satisfy $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, so consumer surplus and all firms' payoffs will be the same in period t as if all feasible but not-yet-approved mergers were proposed.[14]

We now show that when the set of feasible but not-yet-approved mergers in period t is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, every merger in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed. The proof is by induction. The induction hypothesis for period t is that in all future periods $\tau > t$, whenever the set of feasible but not-yet-approved mergers is $\mathfrak{F}_\tau \backslash \mathcal{A}_{\tau-1}$, all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_\tau \backslash \mathcal{A}_{\tau-1}|\mathcal{A}_{\tau-1}) \backslash \mathcal{A}_{\tau-1}$ are proposed.

Consider a merger $M_k \in \overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$. Since $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ is a myopically CS-maximizing set for $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ given $\mathcal{A}_{t-1}$, every merger in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ is CS-nondecreasing given every other merger in that set. Since $\mathcal{A}_{t-1} \subseteq \overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, Lemma 6(i) (in the Appendix) implies that, starting from $\mathcal{A}_{t-1}$, there is an ordering of the mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ that is CS-nondecreasing at each step, which we denote by $(M_1, ..., M_S)$. Suppose that all mergers M_s for $s < k$ are proposed when $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ is the set of feasible and not-yet-approved mergers in period t. (Note that this assumption is valid when $k = 1$.) If $\widehat{\mathcal{P}}_t = \{M_1, ..., M_k\}$, then since the sequence $(M_1, ..., M_k)$ is CS-nondecreasing at each step, we will have $\overline{\mathcal{A}}^*(\{M_1, ..., M_k\}|\mathcal{A}_{t-1}) = \{M_1, ..., M_k\} \cup \mathcal{A}_{t-1}$; that is, all of these mergers, including merger M_k, will be approved.[15] If, instead, $\{M_1, ..., M_k\} \subset \widehat{\mathcal{P}}_t$, then Lemma 2 implies that $(\{M_1, ..., M_k\} \cup \mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$, so merger M_k is still approved. Since merger M_k

[14] The first inclusion follows because $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ is myopically CS-maximizing for $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ given $\mathcal{A}_{t-1}$, and is feasible when the set of proposed mergers is $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1} \subseteq \mathfrak{F}_t \backslash \mathcal{A}_{t-1}$. Therefore, by the logic in footnote 13, $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ is myopically CS-maximizing for $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ given $\mathcal{A}_{t-1}$. The inclusion then follows from Lemma 2, since $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1} \subseteq \mathcal{P}_t$.

[15] In particular, by Lemma 6(ii), every merger in set $\{M_1, ..., M_k\}$ is CS-nondecreasing given every other merger in the set. By Lemma 6(i), if a strict subset of $\{M_1, ..., M_k\}$ were approved, there would be a proposed but unapproved merger that could be approved without lowering consumer surplus, so all mergers in $\{M_1, ..., M_k\}$ will be approved in the most lenient myopically CS-maximizing merger policy.

is certain to be approved if proposed, and results in an active firm, our argument in part (i) implies that proposal of the merger M_k is strictly profitable. Applying induction starting at $k = 1$, we see that if $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ is the set of feasible and not-yet-approved mergers, all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed.

Applying induction starting in period T, we conclude that in every period t if the set of feasible but not-yet-approved mergers in period t is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, then every merger in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed in that period. The result follows. ■

Proposition 3 shows that a myopic merger policy that in each period approves the largest set of mergers that maximizes current consumer surplus (or, equivalently, maximizes discounted consumer surplus ignoring the possibility of any further changes in market structure) is dynamically optimal for the antitrust authority in that it maximizes discounted consumer surplus. Indeed, the proposition establishes an even stronger result: the antitrust authority could not do better even if it knew at the start of the process what the entire sequence of feasible mergers $(\mathfrak{F}_1, ..., \mathfrak{F}_T)$ would be and could implement feasible but unproposed mergers.[16]

In addition, by Lemma 3, the result implies that an even more myopic policy in which the antitrust authority considers mergers individually in a sequential fashion, myopically approving each merger if it is CS-nondecreasing given the market structure at the time of its review, is also dynamically optimal in this very strong sense.

Finally, we make the following observation:

Remark 2 *While for ease of exposition we have restricted attention to the most lenient myopically CS-maximizing merger policy, dynamic optimality holds for* ***any*** *myopically CS-maximizing merger policy. See the Appendix for a discussion.*

4 Extensions

In this section, we discuss a number of extensions of our model. We defer the discussion of one important limitation of our model, the disjointness of mergers, to Section 5.

4.1 Price competition

So far, we have assumed that firms compete in quantities. In this subsection, we discuss the case where firms compete in prices rather than quantities. We show that our basic conclusion continues to hold, albeit in a somewhat weaker form. Specifically, part (i) of Proposition 3 extends to the case of price competition, while part (ii) of that proposition does not. That is, under the most lenient myopically CS-maximizing merger policy there is an equilibrium that maximizes discounted consumer surplus for every realized sequence of feasible mergers, but there may also be other equilibria that do not.

To proceed, suppose that, as before, there are n firms producing a homogeneous good at constant returns to scale. Firm i's marginal cost and price are denoted c_i and p_i. Market demand is given by the nonincreasing function $Q(p)$, where p is the lowest price offered by any

[16] Moreover, the fact that the largest myopically CS-maximizing set monotonically increases over time implies that the antitrust authority also could not do better if it could undo previously approved mergers, which we have assumed is not possible.

firm. Let $\iota(i|N) \in N$ denote the firm with the ith lowest marginal cost when the set of firms is N, i.e., $c_{\iota(1|N)} \leq c_{\iota(2|N)} \leq ... \leq c_{\iota(n|N)}$. (If a subset of firms have the same marginal cost, then the firms in this subset are ordered arbitrarily.) Assuming that $Q(c_{\iota(1|N)}) > 0$ and Assumption 1 holds, and restricting attention to the standard Bertrand pricing equilibrium[17], firm $\iota(i|N)$'s equilibrium price $p_{\iota(i|N)}$ is given by

$$p_{\iota(i|N)} = \begin{cases} c_i & \text{if } 2 \leq i \leq n, \\ \min\{p^m(c_{\iota(1|N)}), c_{\iota(2|N)}\} & \text{if } i = 1, \end{cases} \tag{7}$$

where the nondecreasing function $p^m(c) \equiv \arg\max_p (p-c)Q(p)$ is the monopoly price of a firm with marginal cost c. In equilibrium, all consumers purchase at price $p_{\iota(1|N)}$, and so consumer surplus is given by

$$CS(p_1, p_2, ..., p_n) = \int_{p_{\iota(1|N)}}^{\infty} Q(p)dp. \tag{8}$$

Note that $CS(p_1, p_2, ..., p_n)$ is independent of $p_{\iota(i|N)}$ for $i > 1$, and nonincreasing in $p_{\iota(1|N)}$.

One important difference between the Cournot and Bertrand models is that with Bertrand competition a merger that is CS-neutral in isolation can become CS-decreasing when another merger takes place that is CS-increasing in isolation, as the following example demonstrates:

Example 1 *Suppose there are four firms, $N = \{1,2,3,4\}$, with initial costs $c_1 = 5$, $c_2 = 10$, $c_3 = 15$, $c_4 = 20$, and suppose that there are two possible mergers $M_1 = \{1,3\}$ and $M_2 = \{2,4\}$ with $\overline{c}_{M_1} = 9$ and $\overline{c}_{M_2} = 8$. If the monopoly price for a firm with marginal cost equal to 5 is greater than 10 [i.e., $p^m(5) > 10$], then with no mergers firm 1 will set a price of 10 and make all the sales in the market. The cost-increasing merger M_1 is then CS-neutral in isolation since the post-merger price will still be 10. Merger M_2 is CS-increasing in isolation because it reduces firm 1's price from 10 to 8. However, once merger M_2 occurs, merger M_1 is CS-decreasing since it raises the price from 8 to 9.*

This problem can be traced to the fact that a merger involving the lowest-cost firm $\iota(1|N)$ that increases cost can be CS-neutral in the Bertrand model. We will say that a merger of the firms in set M is *cost increasing* if the post-merger marginal cost of the merged entity, $\overline{c}_M$, is above the marginal cost of the most efficient merger partner, i.e., $\overline{c}_M > \min_{i \in M} c_i$. Intuitively, an antitrust authority can without loss reject any cost-increasing merger, since any such merger both worsens efficiency and the extent of market power.[18] We shall henceforth focus on an antitrust authority that never approves cost-increasing mergers. Formally, this is equivalent to supposing that feasible mergers are never cost increasing, an assumption that will allow us to fit the analysis into the same framework as the Cournot model.

The following result records some properties of mergers that do not increase cost:

Lemma 5 *Consider a merger that does not increase cost among a subset M of firms in a Bertrand market.*

[17] Specifically, the limit of undominated equilibria for games with a discrete pricing grid, as the grid becomes fine.

[18] Formally, given any set of feasible mergers in a period, observe that it is possible to weakly improve consumer surplus starting from any set of approved mergers by instead rejecting all mergers that are cost-increasing. As a result, in any period, given any set of feasible mergers, the largest CS-maximizing set from among those feasible mergers that do not increase cost maximizes consumer surplus in that period.

1. *It is profitable (it weakly increases the joint profit of the firms in M).*

2. *The merger is CS-decreasing only if it involves all of the firms with cost $c_{\iota(1|N)}$, all of the firms with cost $c_{\iota(2|N)}$, and moreover $p^m(\bar{c}_M) > c_{\iota(2|N)}$.*

Proof. In the Appendix. ■

Another important difference from the Cournot model is that a merger that is CS-increasing may not be strictly profitable: for example, a cost-reducing merger of firms $\iota(2|N)$ and $\iota(3|N)$ that results in a cost above $c_{\iota(1|N)}$ lowers the market price but leaves firms $\iota(2|N)$ and $\iota(3|N)$ with zero profit after the merger. For this reason, part (ii) of Proposition 3 will not hold in the Bertrand model [e.g., firms $\iota(2|N)$ and $\iota(3|N)$ in this example can optimally decide not to propose their merger even if it is CS-increasing].

Nevertheless, part (i) of Proposition 3 does hold: all feasible mergers being proposed in each period after any history is a subgame perfect Nash equilibrium for the firms, and the equilibrium outcome maximizes discounted consumer surplus for any realized sequence of feasible mergers $\mathfrak{F}$. To see this, we first consider the interaction between two disjoint mergers, M_1 and M_2. Let N denote the set of firms if neither merger takes place, N_i the set of firms after merger M_i (but not M_j, $j \neq i$) has taken place, and N_{12} the set of firms after both mergers have taken place. The key fact is that in the Bertrand model, versions of Propositions 1 and 2 that are sufficient to establish part (i) of Proposition 3 continue to hold for mergers that do not increase cost:

Proposition 4 *In the Bertrand model:*

(i) *if a merger that does not increase cost is CS-nondecreasing in isolation, it remains CS-nondecreasing if another merger that does not increase cost and is CS-nondecreasing in isolation takes place.*

(ii) *there cannot be two distinct mergers that do not increase cost and are CS-decreasing in isolation.*

Proof. In the Appendix. ■

Proposition 5 *Consider two mergers M_1 and M_2 that are not cost increasing. Suppose that merger M_1 is CS-nondecreasing in isolation, while merger M_2 is CS-decreasing in isolation but CS-nondecreasing once merger M_1 has taken place. Then:*

(i) *Merger M_1 is CS-increasing conditional on merger M_2 taking place;*

(ii) *The joint profit of the firms involved in merger M_1 is weakly larger if both mergers take place than if neither merger takes place.*

Proof. In the Appendix. ■

Proposition 5 differs from Proposition 2 for the Cournot model only in that the profitability conclusions are weak rather than strict. Given Propositions 4 and 5, the basic arguments leading to part (i) of Proposition 3 (now for mergers that do not increase cost) parallel those for the Cournot model.

4.2 Differentiated Products

The Cournot and Bertrand analyses so far assumed a homogeneous product market. Unfortunately, extending our main results to the case of differentiated products, and hence to multiproduct firms, is not straightforward. For example, think of the extreme case in which there are two differentiated products in the market. A merger might leave overall consumer surplus unchanged while raising one price and lowering the other. Since in the extreme case in which the two products are independent in demand there are two independent homogeneous goods markets, we are not able to extend our arguments about the complementarity of CS-nondecreasing mergers. On the other hand, our main results do extend to the case of differentiated products if "strong symmetry" is imposed on both demand and costs (in the sense that all firms that are involved in the same merger have identical marginal costs for all of their products, both pre-merger and post-merger). In that case, price effects for all goods move in the same direction and the complementarity results from our previous analyses carry over, as we now discuss. In our discussion, we will focus on the case of price competition with differentiated products.

Let $Q_j(\mathbf{p}_N)$ denote the demand for product j, where $\mathbf{p}_N$ is the vector of prices, and suppose that the demand system is symmetric across products. Moreover, assume that demand is downward-sloping and strictly log-concave in own price, products are demand substitutes, prices are strategic complements, and the own-price effect dominates the cross-price effects in terms of both the level of demand and its slope.[19]

For simplicity, suppose that, prior to merging, all firms produce a single product so that firm $j \in N$ produces product $j \in N$. After merging, the firms in the set M_k produce all of the products in M_k. We assume that, prior to merging, each firm $j \in M_k$ faces the same marginal cost $c_j = c_{M_k}$ while after the merger all products in M_k are produced at the same marginal cost $\overline{c}_{M_k}$. This assumption ensures that any equilibrium has the property that the price of every product in the set M_k is always the same: $p_i^* = p_{M_k}^*$ for $i \in M_k$ (see Kühn and Rimler [2006]). In particular, this means that we can think of each firm's strategic variable being one-dimensional, so the standard analysis of differentiated goods price competition with single-product firms (see Vives [1999]) extends to our setting with multiproduct firms.

Consider a merger amongst active firms in set M_k, and let $\mathbf{p}_N^*$ denote the vector of pre-merger equilibrium prices. Since prices are strategic complements, the merger is CS-neutral if and only if it leaves all prices unchanged, so the threshold value of post-merger marginal cost that makes this merger CS-neutral is given by

$$\begin{aligned} \widehat{c}_{M_k}(\mathbf{p}_N^*) &\equiv p_{M_k}^* - [p_{M_k}^* - c_{M_k}]\left(\frac{1}{1-\Psi_i}\right), \; i \in M_k, \\ &= c_{M_k} - \left[p_{M_k}^* - c_{M_k}\right]\left(\frac{\Psi_i}{1-\Psi_i}\right), \; i \in M_k, \end{aligned} \tag{9}$$

where

$$\Psi_i \equiv -\frac{\sum_{j \in M_k, j \neq i} \frac{\partial Q_j(\mathbf{p}_N^*)}{\partial p_i}}{\frac{\partial Q_i(\mathbf{p}_N^*)}{\partial p_i}}.$$

[19] The own effect of price change dominates the cross effects in terms of the level of demand if $\sum_{j \in N} (\partial Q_i(\mathbf{p}_N)/\partial p_j) < 0$ and in terms of the slope of demand if $\left|\partial^2 \ln Q_i(\mathbf{p}_N)/\partial p_i^2\right| > \sum_{j \in N, j \neq i} \left(\partial^2 \ln Q_i(\mathbf{p}_N)/\partial p_j \partial p_i\right)$, $i \in N$; see Kühn and Rimler [2006].

The term Ψ_i is familiar from merger analysis: it is the "diversion ratio" from product $i \in M_k$ to other products in M_k, defined as the share of the lost sales of product $i \in M_k$ that are captured by the other products in M_k after an increase in the price of product i. Since (by assumption) $\partial Q_i(\mathbf{p}_N^*)/\partial p_i < 0$ and $\partial Q_j(\mathbf{p}_N)/\partial p_i > 0$ for $j \neq i$, and (from the first-order condition of profit-maximization) $\sum_{j \in M_k} \partial Q_j(\mathbf{p}_N)/\partial p_i < 0$, $i \in M_k$, we have $\Psi_i \in (0,1)$, which implies that $\widehat{c}_{M_k} < c_{M_k}$. That is, for the merger to be CS-neutral, the merger must be cost-reducing, and therefore profitable for the merging parties. Strategic complementarity implies that a decrease in post-merger marginal cost $\overline{c}_{M_k}$ induces all prices to fall. Consequently, a merger amongst active firms in M_k is CS-increasing if and only if $\overline{c}_{M_k} < \widehat{c}_{M_k}$, CS-neutral if and only if $\overline{c}_{M_k} = \widehat{c}_{M_k}$, and CS-decreasing if and only if $\overline{c}_{M_k} > \widehat{c}_{M_k}$.

While every CS-neutral merger is profitable, it is not straightforward to show that every CS-nondecreasing merger is profitable. The complication arises because a reduction in marginal cost $\overline{c}_{M_k}$ has two opposing effects on the profits of the merged firm M_k: holding fixed the prices of all other firms, the *direct* effect of a decrease in $\overline{c}_{M_k}$ is to increase the merged firm's profit; but the *strategic* effect of a decrease in $\overline{c}_{M_k}$ is to reduce the merged firm's profit as all other firms will decrease their prices in response. One therefore needs to impose conditions on demand to ensure that the direct effect outweighs the strategic effect and a decrease in its marginal cost raises that firm's equilibrium profit. It is straightforward to check that this is indeed the case, for example, when demand is linear, $Q_j(\mathbf{p}_N) = \alpha_N - \beta_N p_j + \gamma_N \sum_{i \neq j} p_i$ with $\alpha_N > 0$ and $\beta_N > (n-1)\gamma_N > 0$.

Let us now turn to the interaction between mergers. Our previous result on the complementarity of those mergers that change consumer surplus in the same direction (Proposition 2) carries over to the present setting if approving a CS-increasing merger M_l raises the threshold $\widehat{c}_{M_k}$ for merger M_k, $k \neq l$ (and approving a CS-decreasing M_l reduces $\widehat{c}_{M_k}$). Since a CS-increasing merger reduces all prices, this means that our complementarity result extends if demand is such that $\widehat{c}_{M_k}(\mathbf{p}_N^*)$ is weakly decreasing in all prices. In the case of linear demand, for example, the diversion ratio Ψ_i is a constant, so $\widehat{c}_{M_k}(\mathbf{p}_N^*)$ depends only on, and is strictly decreasing in, $p_{M_k}^*$. It follows that complementarity holds. More generally, a sufficient condition for $\widehat{c}_{M_k}(\mathbf{p}_N^*)$ to be nonincreasing in the prices of all products with positive sales is that the diversion ratio Ψ_i is nondecreasing in all prices. Provided that this complementarity holds, Proposition 3 extends to this setting.

4.3 Fixed Costs and Exit

So far, we have assumed that all fixed costs are sunk, and that mergers had no effect on these costs. Our Cournot results extend to cases in which fixed costs are present and possibly affected by mergers provided that (i) mergers that are CS-nondecreasing in isolation continue to be profitable in isolation and (ii) mergers do not cause active firms to shut down.[20]

Regarding (i), recall from Corollary 1 that, in the absence of fixed costs, every CS-nondecreasing merger is profitable in isolation. This result continues to hold in the presence of fixed costs, provided that the post-merger fixed cost of a merged firm is not larger than the sum of the pre-merger fixed costs of the merger partners. In particular, our result is unaffected if mergers generate efficiencies in fixed costs as well as marginal costs.

[20] In the Bertrand model, all but one firm will exit if there are positive levels of fixed costs.

If (ii) is violated, Proposition 1 need not hold. For example, suppose both mergers M_1 and M_2 are CS-increasing in isolation and do not induce any firm to exit. However, if both mergers are approved, then some other firm $j \in N \backslash (M_1 \cup M_2)$ might find it optimal to exit. (This outcome is possible since, without exit, the market price after both mergers would be lower than after only one merger.) Taking the endogenous exit of firm j into account, consumer surplus after both mergers might therefore be lower than after merger M_1 only, in which case merger M_2 would be CS-decreasing conditional on merger M_1. Thus, Proposition 1(i) may fail to hold. In a similar vein, Motta and Vasconcelos [2005] allow for exit in a setting with four symmetric firms and two possible disjoint cost-reducing mergers involving two firms each. Each merger is CS-decreasing in isolation because it induces the other two, non-merging firms to exit. But consumer surplus increases if both mergers are approved, implying that each merger becomes CS-increasing once the other merger has taken place. Thus, Proposition 1(ii) fails to hold in their model. As a result, a myopic policy need not be optimal.

While these observations suggest that in general our main results could break down in the presence of fixed costs and endogenous exit, we can allow for exit among a competitive fringe of price-taking firms that do not take part in any mergers. To do so, we construct the competitive fringe's (long-run) supply function, $S^F(p)$, which takes potential exit (and entry) of these firms into account. The residual demand of the large, strategic firms in set N is then given by $R(p) \equiv D(p) - S^F(p)$, where $D(p)$ is market demand. As long as the inverse residual market demand function $P(\cdot) \equiv R^{-1}(\cdot)$ satisfies the conditions of Assumption 1, our analysis and conclusions remain unchanged.

4.4 Merger Proposal Costs

In our analysis, we have assumed that there are no costs of proposing a merger to the antitrust authority. Moreover, we have highlighted a subgame-perfect equilibrium in which all feasible mergers are always proposed, including some that have no chance of being approved. One might be concerned that firms would not propose such mergers if they had even the tiniest cost of making a merger proposal. However, since every CS-nondecreasing merger is *strictly* profitable (provided it results in an active firm), there is also a subgame-perfect equilibrium in which, in every period and after any history, the mergers proposed are those that will be approved in equilibrium (i.e., the mergers that are in the largest myopically CS-maximizing set of mergers, given the current market structure) and result in an active firm. Since all of these mergers are strictly profitable, our conclusion – that, if the antitrust authority adopts the most lenient myopically CS-maximizing merger policy, then every subgame-perfect equilibrium outcome maximizes discounted consumer surplus for every realized $\mathfrak{F}$ – would not change in the presence of merger proposal costs, provided these are sufficiently small.

4.5 Demand Shifts

While our model had a stationary demand function, Corollary 1 suggests that our main results hold provided that demand is weakly declining over time. Specifically, suppose that inverse demand in period t can be written as $P(Q; \theta_t)$, where θ_t is the publicly observable demand state realized at the beginning of period t (before mergers are proposed), which we assume is increasing over time, i.e., $\theta_t \geq \theta_{t-1}$. For any tuple $(Q;\theta_t)$ such that $P(Q;\theta_t) > 0$, we continue to assume that $P_Q < 0$ and $P_Q + QP_{QQ} < 0$ (where subscripts denote partial derivatives); moreover, we now assume that $P_\theta < 0$ and $P_{Q\theta} \geq 0$. For example, these conditions hold if inverse demand takes the form $P(Q;\theta_t) \equiv P(Q)/\theta_t$ and $P(Q)$ satisfies the conditions of Assumption 1.

Let $Q^*(\mathcal{A}_t;\theta_t)$ denote the equilibrium industry output when market structure is $\mathcal{A}_t$ and the demand state is θ_t. Since inverse demand is changing over time, it is more convenient to write $\widehat{c}_M$ (the post-merger marginal cost threshold that makes a merger amongst active firms in set M CS-neutral) as a function of equilibrium price rather than industry quantity:

$$\widehat{c}_M(P^*(\mathcal{A}_t;\theta_t)) \equiv P^*(\mathcal{A}_t;\theta_t) - \sum_{i\in M} \max\{0, P^*(\mathcal{A}_t;\theta_t) - c_i\},$$

where $P^*(\mathcal{A}_t;\theta_t) \equiv P(Q^*(\mathcal{A}_t;\theta_t);\theta_t)$. Our assumptions on demand ensure that, holding fixed market structure $\mathcal{A}_t$, an increase in the demand state θ_t will lead to a decrease in the equilibrium price $P^*(\mathcal{A}_t;\theta_t)$.[21] This, in turn, implies that, holding fixed market structure $\mathcal{A}_t$, the post-merger marginal cost threshold $\widehat{c}_M(P^*(\mathcal{A}_t;\theta_t))$ weakly increases over time (as long as the merger involves active firms). As before, the threshold strictly increases as $P^*(\mathcal{A}_t;\theta_t)$ decreases due to CS-increasing mergers in the rest of the industry. Hence, if the antitrust authority adopts a myopically CS-maxmizing merger policy, then if merger M is CS-nondecreasing in period t, it will remain CS-nondecreasing in every future period $t' > t$. (By contrast, a merger M that is CS-decreasing in period t may now become CS-nondecreasing in some later period $t' > t$ even holding market structure fixed.)

The largest myopically CS-maximizing set of mergers now depends not only on the set of proposed mergers $\mathcal{P}_t$ and current market structure $\mathcal{A}_{t-1}$, but also on the demand state θ_t, and is denoted $\overline{\mathcal{A}}^*(\mathcal{P}_t;\theta_t|\mathcal{A}_{t-1})$. As the discussion above makes clear, $\overline{\mathcal{A}}^*(\mathcal{P}_t;\theta_t|\mathcal{A}_{t-1})$ is increasing in θ_t: if $\theta_t'' > \theta_t'$, then $\overline{\mathcal{A}}^*\left(\mathcal{P}_t;\theta_t'|\mathcal{A}_{t-1}\right) \subseteq \overline{\mathcal{A}}^*\left(\mathcal{P}_t;\theta_t''|\mathcal{A}_{t-1}\right)$. Since $\mathfrak{F}_t \subseteq \mathfrak{F}_{t+1}$ and $\theta_t \leq \theta_{t+1}$, we therefore have $\overline{\mathcal{A}}^*(\mathfrak{F}_t;\theta_t|\varnothing) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_{t+1};\theta_{t+1}|\varnothing)$. Hence, if all feasible but not-yet-approved mergers are always proposed, then the most lenient myopically CS-maximizing merger policy maximizes discounted consumer surplus for every realized sequence of feasible mergers $\mathfrak{F}$ (Lemma 4). Moreover, as before, if the antitrust authority adopts the most lenient myopically CS-maximizing merger policy, the resulting equilibrium outcome is dynamically optimal (Proposition 3).

4.6 Entry

In our analysis above, we assumed that the set of firms is fixed, except for mergers. Would our conclusions change if we allowed for firm entry? Recall that our model implies that the equilibrium price $P(Q^*)$ falls weakly over time. This suggests that if a firm does not find it profitable to enter the market at the beginning of the first period, before any mergers have

[21] Summing up the first-order conditions of profit maximization and applying the implicit function theorem, we have

$$\frac{dP(Q^*;\theta_t)}{d\theta} = \frac{P_\theta(P_Q + Q^*P_{QQ}) - Q^*P_QP_{Q\theta}}{(n+1)P_Q + Q^*P_{QQ}},$$

where Q^* is industry output and n is the number of active firms when the market structure is $\mathcal{A}_t$ and the demand state is θ_t. Under our assumptions on demand, the expression on the right-hand side is strictly negative.

become feasible, then this firm will not find it profitable to enter the market in any later period (provided that its costs have not changed). That is, allowing for free entry of firms (with unchanging costs) does not affect our results.

Moreover, suppose that new firms periodically enter the market later, for example after discovering how to make the product. (In our model, such an entry event is equivalent to a sufficient reduction in the marginal cost of a hitherto inactive firm.) These (potentially stochastic) entry events lower the market price, and leave our main result unchanged for reasons that parallel those in our discussion above of demand shifts.

4.7 Continuing Innovation

In the analysis above, we assumed that when a merger, say M_k, becomes feasible, the firms in M_k receive a (random) draw of their post-merger marginal cost $\overline{c}_{M_k}$ once and for all; if merger M_k is implemented, the marginal cost of the merged entity is $\overline{c}_{M_k}$ forever after. But it seems plausible that, over time, firms involved in a (potential) merger may have more than one idea of how to create synergies amongst them, both pre-merger and post-merger. As we now discuss, it is possible to extend our analysis to allow for continuing innovation.

Consider the following generalization of our previous setup: as before, we assume that if merger M_k becomes feasible at the beginning of period t, then the firms in M_k receive a random draw of their post-merger marginal cost from distribution function G_{kt}. Moreover, we now assume that the post-merger marginal cost $\overline{c}_{M_k}$ follows a (discrete-time) stochastic process from period t onward. The stochastic process governing these additional cost draws (or "innovations") is independent of whether the firms in M_k have already merged or not. Crucially, we assume that the post-merger marginal cost $\overline{c}_{M_k}$ weakly decreases over time.

Our previous results carry over to this generalized setting. The arguments closely parallel those in our discussion above of demand shifts. Since a reduction in merged firm M_k's marginal cost reduces the equilibrium price (and thereby reduces the post-merger marginal cost threshold $\widehat{c}_{M_l}$ of every other merger M_l, $l \neq k$), the largest myopically CS-maximizing set of mergers will weakly increase over time if the antitrust authority adopts the most lenient myopically CS-maximizing merger policy. Hence, if all feasible but not-yet-approved mergers are always proposed, then the most lenient myopically CS-maximizing merger policy maximizes discounted consumer surplus for every realized sequence of feasible mergers $\mathfrak{F}$ (Lemma 4). Moreover, the induction argument in the proof of Proposition 3 continues to apply, showing that the resulting equilibrium outcome is dynamically optimal.[22]

4.8 Aggregate Surplus Standard

In our analysis above, we have assumed that the antitrust authority's objective is to maximize discounted consumer surplus. Indeed, as pointed out in the Introduction, this is close to being the legal standard in the U.S. and the EU. Nevertheless, it is interesting to ask whether

[22] In our discussion, we have assumed that a merged firm's post-merger marginal cost follows an exogenous stochastic process that is weakly decreasing over time. It is straightforward to show that if the merged firm has to make an active decision as to whether or not to implement a cost-reducing innovation, then it is indeed profitable for the merged firm to implement it.

the antitrust authority can maximize aggregate surplus (AS) by adopting the most lenient myopically AS-maximizing merger policy.

In the homogeneous-goods Bertrand model, the answer is, yes. One can prove that our results on the interactions between mergers in the Bertrand model, Propositions 4 and 5, continue to hold if we replace the consumer surplus criterion by the aggregate surplus criterion. Consequently, under the most lenient myopically AS-maximizing merger policy there is an equilibrium such that the resulting outcome maximizes discounted aggregate surplus for every realized sequence of mergers $\mathfrak{F}$.

In the homogeneous-goods Cournot model, however, the complementarity of AS-increasing mergers does not hold in general. To see this, recall that, in the Cournot model, a marginal cost reduction by a highly inefficient firm (one that produces almost no output, and thus has a profit margin approximately equal to zero) necessarily reduces aggregate surplus. In contrast, a cost-reducing merger between the two most efficient firms in a market may increase aggregate surplus. Thus, complementarity can fail when a cost-reducing, AS-increasing merger by other firms in the market transforms these two firms from being the most efficient firms in the market to being the least efficient. In addition, mergers that increase aggregate surplus need not be profitable for the firms involved in them. The papers by Nilssen and Sorgard [1998] and Matsushima [2001], for example, both focus on Cournot settings with linear demand and constant marginal costs in which there are two possible mergers, each between a pair of firms, and show that a myopic policy need not be optimal for an antitrust authority interested in maximizing aggregate surplus.

4.9 Information of Firms

In our analysis, we have assumed that firms observe both their own and their rivals' merger possibilities, including their efficiencies, as soon as these become feasible. We now show that the conclusion of Proposition 3 carries over to the case where firms observe the feasibility of other mergers only when they are proposed, and observe the efficiency gains of other mergers only when those mergers are approved. (We continue to assume that firms observe their own merger possibility when it becomes feasible. We also continue to assume that each firm knows the initial costs of all firms at the start of period 1, so that complete-information Cournot competition in each period is justified.) To establish the result, we use the solution concept of extensive-form trembling-hand perfect Nash equilibrium.[23]

Proposition 6 *Assume firms observe the feasibility of other mergers only when they are proposed and the efficiency gains of other mergers only when they are approved, and that the support of G_{kt} is finite for all k and t. Suppose the antitrust authority follows the most lenient myopically CS-maximizing merger policy. Then:*

(i) *All feasible mergers being proposed in each period after any history is an extensive-form trembling-hand perfect Nash equilibrium for the firms. In this equilibrium, the outcome*

[23] We use this solution concept rather than the weaker notion of sequential equilibrium to establish part (ii) of the result. The trembles ensure that following any history, when the true set of feasible but not-yet-approved mergers is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, every proposer firm assigns a strictly positive probability to this set being the set of feasible but not-yet-approved mergers.

maximizes discounted consumer surplus for any realized sequence of feasible mergers $\mathfrak{F} = (\mathfrak{F}_1, ..., \mathfrak{F}_T)$.

(ii) *For each sequence* $\mathfrak{F}$, *every extensive-form trembling-hand perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. In the Appendix. ■

4.10 Breakups

Our results have implications not only for horizontal mergers but also for horizontal breakups of companies into smaller firms. As we now discuss, the model of Section 2 can be applied to the analysis of such breakups. Importantly, however, that application shows that a myopically CS-maximizing breakup policy is *not* dynamically optimal.

To fit breakups in the framework of Section 2, we think of the breakup of (merged) firm M as being the reverse operation to merger M, with $\overline{c}_M$ being the pre-breakup marginal cost of merged firm M and c_i, $i \in M$, being the post-breakup marginal cost of firm i. Hence, breakup of firm M is CS-neutral if and only if merger M is CS-neutral, and the breakup is CS-increasing [CS-decreasing] if and only if the merger is CS-decreasing [CS-increasing].

Our dynamic optimality result for a myopically CS-maximizing merger policy does not extend to breakups. Most importantly, a myopic breakup policy is, in general, path dependent, and so may fail to be optimal even if if the antitrust authority does not need firms to propose breakups. This path dependence arises because the complementarity result of Proposition 1 does not carry over to breakups. To see this point, consider the case in which there are two firms, M_1 and M_2, each of whose breakup is CS-increasing in isolation. (Note that a breakup "in isolation" means that the other firm is still merged). Our results regarding mergers in Section 2 imply that these breakups may not each be CS-increasing once the other breakup has taken place: Specifically, note that breakups M_1 and M_2 each being CS-increasing in isolation is equivalent to saying that each merger M_i is CS-decreasing once the other merger M_j $(j \neq i)$ has taken place. If so, one possibility is that one of the mergers, say M_1, is CS-decreasing in isolation, while the other merger, M_2, is CS-increasing in isolation but CS-decreasing once merger M_1 has taken place.[24] This means that while the breakup of firm M_1 is CS-increasing once firm M_2 is broken up, the breakup of firm M_2 is CS-*decreasing* once firm M_1 is broken up. As a result, there is path dependence: if the breakup of firm M_2 is considered first, both breakups will be implemented, but if the breakup of firm M_1 is considered first, only firm M_1 will be broken up. So a myopic policy fails in general to be optimal, even if the antitrust authority does not need firms to propose breakups.

Moreover, if breakups do need to be proposed, the incentives of firms to propose them are not aligned with those of an antitrust authority seeking to maximize discounted consumer surplus, as can be seen from the fact that a CS-neutral breakup of an active firm is strictly unprofitable for the firm. (This is simply the flip-side of our result on the profitability of a CS-neutral merger.)

[24] The other possibility is that both mergers are CS-decreasing in isolation.

5 Conclusion

In this paper, we have analyzed the antitrust authority's optimal dynamic merger approval policy in a model with Cournot competition in which merger opportunities arise stochastically over time, firms decide whether or not to propose a feasible merger, and the antitrust authority decides whether or not to approve proposed mergers. We first established that a form of complementarity exists between mergers in this Cournot setting: specifically, the sign of a merger's consumer surplus effect is unchanged if another merger whose consumer surplus effect has the same sign takes place. This result, which is of independent interest, set the stage for our main result, which showed that, in our model, an antitrust authority who wishes to maximize discounted consumer surplus can implement the dynamically optimal solution by adopting a completely myopic policy according to which the antitrust authority approves a merger if and only if it does not lower consumer surplus given the current market structure. In fact, the antitrust authority cannot improve upon the outcome induced by the myopic policy even if it has perfect foresight about potential future mergers.

The argument for this surprising conclusion came in two parts. First, ignoring firms' proposal incentives and assuming that all feasible but not-yet-approved mergers are always proposed, a myopically CS-maximizing merger policy is dynamically optimal in that it maximizes discounted consumer surplus for every realized sequence of feasible mergers. As we have shown, this result is based on a fundamental complementarity in the consumer surplus effect of mergers: if each of two mergers share the same sign of their consumer surplus effect, then the sign of each one's consumer surplus effect does not change if the other merger is implemented. This complementarity result follows because an increase in the toughness of competition (due to the approval of a price-reducing merger) does not affect the "efficiency effect" of a merger but reduces its "market power effect," implying that the merger is more likely to be CS-increasing the more competitive is the industry. It implies that the antitrust authority will never later regret approval of a CS-nondecreasing merger. Moreover, the fact that rejected mergers can always be proposed and approved later means that the antitrust authority will never later regret rejection of a CS-decreasing merger.

Second, if the antitrust authority adopts a myopically CS-maximizing merger policy, then firms' proposal incentives are aligned with the interests of the antitrust authority: every merger that the antitrust authority wishes to approve is indeed profitable for the merger partners even if the approval of that merger induces the approval of other (price-reducing) mergers in the same period or in subsequent periods.

One interesting side implication of our model is that it provides a novel theory of merger waves (for example, see Fauli-Oller [2000]). In contrast to much of the existing literature (e.g., Jovanovic and Rousseau [2002, 2008]), our explanation of merger waves does not rely on aggregate shocks. Specifically, because of the complementarity of CS-nondecreasing mergers in our model, the arrival of a CS-increasing merger opportunity for some firms may have a domino effect by turning other feasible but currently CS-decreasing mergers into CS-nondecreasing mergers, and thereby triggering a merger wave. An interesting aspect of this result is the way in which the antitrust authority's CS-maximizing merger policy affects the emergence of merger waves, since complementarity of mergers does not hold in general absent this antitrust review.

We have also shown that our main conclusion – the dynamic optimality of a myopic merger approval policy – is robust in several dimensions. For instance, the conclusion does not depend on firms' and the antitrust authority's information about potential future mergers nor on whether firms compete in prices or quantities.

Perhaps the most important limitation of our model is that mergers are disjoint. This rules out, for example, the possibility that a firm may have to choose between two merger partners, or that a recently merged firm might consider merging with another still-independent firm. While disjointness of possible mergers would hold when firms have natural merger partners (as we noted earlier), and has been assumed throughout the small existing literature on antitrust review of mergers in dynamic settings (Nilssen and Sorgard [1998], Matsushima [2001], Motta and Vasconcelos [2005]), it is clearly a strong assumption.

Nondisjoint mergers can cause problems for myopic policies. A first problem is that myopic approval of a CS-increasing merger today may preclude the possibility of approving an even better merger tomorrow. A second problem relates to firms' proposal incentives. With disjoint mergers, we saw that firms' proposal incentives were aligned with the desires of the antitrust authority because any CS-nondecreasing merger was profitable. When firms must choose among merger partners, however, they may propose the wrong merger from the antitrust authority's perspective (e.g., the most profitable merger may not be the one that is best for consumer surplus). Another issue is that firms may face a disincentive to propose a merger that today would be CS-increasing because of the effect the merger's approval would have on bargaining with future merger partners.

Nonetheless, because the case of nondisjoint mergers leads to such a striking result – the optimality of myopic merger policy – we feel that it is a natural starting point for understanding the issues involved in optimal merger policy in dynamic environments. At the same time, in our own future research, we hope to learn more about what can be said about optimal merger policy in settings with nondisjoint mergers.

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6 Appendix

6.1 Proofs

We begin by establishing two useful results concerning the interactions among sets of mergers. The first lemma focuses on the relationship between sequences of mergers that are CS-nondecreasing at each step, and sets of mergers for which each merger is CS-nondecreasing given all of the other mergers in the set. We call it the "Incremental Gain Lemma":

Lemma 6 *(Incremental Gain Lemma)*

(i) Suppose that a set of mergers $\mathfrak{M} \equiv \{M_1, ..., M_{J_1}\}$ has the property that every merger $M \in \mathfrak{M}$ is CS-nondecreasing if all of the other mergers in $\mathfrak{M}$ (those in the set $\mathfrak{M}\backslash M$) have taken place. Then for any strict subset $Y \subset \mathfrak{M}$, there exists an $M' \in \mathfrak{M}\backslash Y$ that is CS-nondecreasing if all of the mergers in Y have taken place. As a result, starting from Y, there is a sequencing of the mergers in $\mathfrak{M}\backslash Y$ that is CS-nondecreasing at each step.

(ii) Suppose that a sequence of mergers $M_1, ..., M_{J_1}$ is CS-nondecreasing at each step. Then each merger $M \in \mathfrak{M} \equiv \{M_1, ..., M_{J_1}\}$ is CS-nondecreasing if all of the other mergers in $\mathfrak{M}$ (those in the set $\mathfrak{M}\backslash M$) have taken place.

Proof. (i) Suppose the result is not true, so that every $M' \in \mathfrak{M}\backslash Y$ is CS-decreasing if all of the mergers in Y have taken place. Proposition 1(ii) implies that, taking the mergers in Y as given, for any sequencing of the mergers in the set $\mathfrak{M}\backslash Y$ the merger implemented at each step, including the last step, is CS-decreasing. But this contradicts the hypothesis that the last merger in the sequence is CS-nondecreasing if all of the other mergers in the set $\mathfrak{M}$ have taken place.

Given the existence of $M' \in \mathfrak{M}\backslash Y$ that is CS-nondecreasing if all of the mergers in Y have taken place, we can update the subset Y to $Y \cup \{M'\} \subset \mathfrak{M}$ and apply the same argument again. Continuing iteratively identifies a sequencing of the mergers in $\mathfrak{M}\backslash Y$ that is CS-nondecreasing at each step starting from the subset Y.

(ii) Consider an arbitrary merger M_j in sequence $M_1, ..., M_{J_1}$. We will show that M_j is CS-nondecreasing given that all of the mergers in $\mathfrak{M}\backslash M_j$ have taken place. For $k \geq j$, define the set $\mathfrak{M}^k = \{M_i : i \leq k\}$. Suppose that (a_k) merger M_j is CS-nondecreasing given $\mathfrak{M}^k\backslash M_j$ and that (b_k) merger M_{k+1} is CS-nondecreasing given $\mathfrak{M}^k$. Observe that, by hypothesis, property (a_k) is true for $k = j$, and that property (b_k) holds for all k. We claim that properties (a_k) and (b_k) imply property (a_{k+1}): M_j is CS-nondecreasing given $\mathfrak{M}^{k+1}\backslash M_j$. To see this, observe that if merger M_{k+1} is CS-nondecreasing given $\mathfrak{M}^k\backslash M_j$, property (a_{k+1}) follows from Proposition 1(i), while if merger M_{k+1} is CS-decreasing given $\mathfrak{M}^k\backslash M_j$ then property (a_{k+1}) follows from Proposition 2(i) [and the fact that M_{k+1} is CS-nondecreasing given $\mathfrak{M}^k$]. Applying induction we find that merger M_j is CS-nondecreasing given that all of the mergers in $\mathfrak{M}\backslash M_j$ have taken place [property (a_{J_1})]. ■

Part (ii) of Lemma 6 implies that the set of mergers resulting from a merger policy in which the antitrust authority considers mergers within period t in a step-by-step fashion, approving mergers that are CS-nondecreasing given the current market structure until no further CS-nondecreasing mergers can be identified, possesses the property that every merger in the set is CS-nondecreasing given every other merger in the set. This is also a property possessed by any myopically CS-maximizing set (if any approved merger M were CS-decreasing given the other approved mergers, then consumer surplus could be increased by not approving merger M while continuing to approve the others). The next lemma establishes two features of sets possessing this property.

Lemma 7 *Suppose that two distinct sets of mergers $\mathfrak{M}_1 \equiv \{M_1, ..., M_{J_1}\}$ and $\mathfrak{M}_2 \equiv \{M_1, ..., M_{J_2}\}$ with $\mathfrak{M}_1 \not\subseteq \mathfrak{M}_2$, not necessarily disjoint, each have the property that every merger $M \in \mathfrak{M}_i$ is CS-nondecreasing if all of the other mergers in $\mathfrak{M}_i$ (those in the set $\mathfrak{M}_i\backslash M$) have taken place. Then:*

(i) there is a merger $M_1' \in \mathfrak{M}_1\backslash$ $(\mathfrak{M}_1 \cap \mathfrak{M}_2)$ that is CS-nondecreasing given that all of the mergers in $\mathfrak{M}_2$ have taken place;

(ii) the set of mergers $\mathfrak{M}_1 \cup \mathfrak{M}_2$ results in a level of consumer surplus that is at least as great as that of either set $\mathfrak{M}_1$ or set $\mathfrak{M}_2$.

Proof. (i) Part (i) of the Incremental Gain Lemma [Lemma 6(i)] implies that there exists a merger $M_1' \in \mathfrak{M}_1\backslash(\mathfrak{M}_1 \cap \mathfrak{M}_2)$ that is CS-nondecreasing given that all of the mergers in $\mathfrak{M}_1 \cap \mathfrak{M}_2$ have taken place. It also implies that there is a sequencing of the mergers in $\mathfrak{M}_2\backslash(\mathfrak{M}_1 \cap \mathfrak{M}_2)$, say $M_{21}, ..., M_{2J_2}$, that is CS-nondecreasing at each step, given that the mergers in $\mathfrak{M}_1 \cap \mathfrak{M}_2$ have taken place. Let $\mathfrak{M}^k = \{M_{2i} : i \leq k\}$. Proposition 1(i) implies that if merger M_1' is CS-nondecreasing given that all the mergers in $(\mathfrak{M}_1 \cap \mathfrak{M}_2) \cup \mathfrak{M}^k$ have taken place, then [since by hypothesis merger $M_{2,k+1}$ is also CS-nondecreasing given that all of the mergers in $(\mathfrak{M}_1 \cap \mathfrak{M}_2) \cup \mathfrak{M}^k$ have taken place] M_1' is also CS-nondecreasing given that all of the mergers in $(\mathfrak{M}_1 \cap \mathfrak{M}_2) \cup \mathfrak{M}^{k+1}$ have taken place. Since merger M_1' is CS-nondecreasing if all of the mergers in $(\mathfrak{M}_1 \cap \mathfrak{M}_2) \cup \mathfrak{M}^0 = (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ have taken place, applying induction yields the result (taking $k = J_1$).

(ii) Let M_1' be the merger identified in part (i). We argue first that every merger in set $\mathfrak{M}_2 \cup \{M_1'\}$ is CS-nondecreasing if all of the other mergers in that set have taken place. Part (i) implies that this is true for merger M_1'. Now consider any merger $M_2' \in \mathfrak{M}_2$. By hypothesis, merger M_2' is CS-nondecreasing given that all the mergers in set $\mathfrak{M}_2 \backslash M_2'$ have taken place. If merger M_1' is also CS-nondecreasing if all of the mergers in set $\mathfrak{M}_2 \backslash M_2'$ have taken place, then Proposition 1(i) implies that merger M_2' is CS-nondecreasing if all of the mergers in $(\mathfrak{M}_2 \backslash M_2') \cup \{M_1'\} = (\mathfrak{M}_2 \cup \{M_1'\}) \backslash M_2'$ have taken place. If, instead, merger M_1' is CS-decreasing if all of the mergers in set $\mathfrak{M}_2 \backslash M_2'$ have taken place, then Proposition 2(i) implies that this same property holds. This establishes that every merger in $\mathfrak{M}_2 \cup \{M_1'\}$ is CS-nondecreasing if all of the other mergers in that set have taken place. Moreover, the level of consumer surplus with set $\mathfrak{M}_2 \cup \{M_1'\}$ is at least as large as with set $\mathfrak{M}_2$.

If $\mathfrak{M}_1 \subseteq \mathfrak{M}_2 \cup \{M_1'\}$ then the result is proven. Suppose not. Then note that sets $\mathfrak{M}_1$ and $\mathfrak{M}_2 \cup \{M_1'\}$ satisfy the hypotheses of the Lemma. So we can apply the argument again for these two sets. Continuing iteratively in this fashion we establish the result by adding to $\mathfrak{M}_2$ a sequencing of the mergers in $\mathfrak{M}_1 \backslash (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ that is CS-nondecreasing at each step. This establishes that the level of consumer surplus is at least as high with set $\mathfrak{M}_1 \cup \mathfrak{M}_2$ as with set $\mathfrak{M}_2$. We also need to show that the level of consumer surplus in $\mathfrak{M}_1 \cup \mathfrak{M}_2$ is at least as large as in set $\mathfrak{M}_1$. If $\mathfrak{M}_1 \supseteq \mathfrak{M}_2$, so that $\mathfrak{M}_1 \cup \mathfrak{M}_2 = \mathfrak{M}_1$, this follows immediately. If instead $\mathfrak{M}_1 \not\supseteq \mathfrak{M}_2$, then we can repeat the argument above with the roles of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ reversed to establish the result. ■

We now use these results to prove Lemmas 2 and 3.

Proof of Lemma 2. Given proposed mergers $\mathcal{P}_t$ and market structure $\mathcal{A}_{t-1}$, let $\mathcal{A}'$ be a largest myopically CS-maximizing set and let $\mathcal{A} \neq \mathcal{A}'$ be a myopically CS-maximizing set. We will show that $\mathcal{A} \subset \mathcal{A}'$. Suppose otherwise, so that $\mathcal{A}' \subset (\mathcal{A} \cup \mathcal{A}')$. The sets $\mathcal{A}$ and $\mathcal{A}'$ satisfy the hypothesis of Lemma 7. So, by Lemma 7(ii), $\mathcal{A} \cup \mathcal{A}'$ is myopically CS-maximizing as well, contradicting the assumption that $\mathcal{A}'$ is a largest myopically CS-maximizing set for $\mathcal{P}_t$ given market structure $\mathcal{A}_{t-1}$. Hence, $\mathcal{A}'$ must contain every other myopically CS-maximizing set, which also implies that $\mathcal{A}'$ is the unique largest CS-maximizing set.

For the second claim, suppose $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \nsubseteq \overline{\mathcal{A}}^*(\mathcal{P}_t'|\mathcal{A}_{t-1})$. The sets $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ and $\overline{\mathcal{A}}^*(\mathcal{P}_t'|\mathcal{A}_{t-1})$ satisfy the hypothesis of Lemma 7 and, since $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{P}_t \cup \mathcal{A}_{t-1} \subset \mathcal{P}_t' \cup \mathcal{A}_{t-1}$, all mergers in set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ are feasible when the set of proposed mergers is $\mathcal{P}_t'$. Thus, when the set of proposed mergers is $\mathcal{P}_t'$, approval of the mergers in set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \cup \overline{\mathcal{A}}^*(\mathcal{P}_t'|\mathcal{A}_{t-1})$ is feasible and by Lemma 7(ii) is also myopically CS-maximizing, contradicting $\overline{\mathcal{A}}^*(\mathcal{P}_t'|\mathcal{A}_{t-1})$ being the largest myopically CS-maximizing set for $\mathcal{P}_t'$ given market structure $\mathcal{A}_{t-1}$. ■

Proof of Lemma 3. Suppose that the set of proposed mergers is $\mathcal{P}_t$ and the market structure prior to period t is $\mathcal{A}_{t-1}$. Let $\mathcal{A} \subseteq \mathcal{P}_t$ denote a set of mergers resulting from a merger policy in which the antitrust authority considers mergers within period t in a step-by-step fashion, approving mergers that are CS-nondecreasing given the current market structure until no further CS-nondecreasing mergers can be identified. By part (ii) of Lemma 6 (the Incremental Gain Lemma), every merger in $\mathcal{A}$ must be CS-nondecreasing given every other merger in the set.

If $\mathcal{A} \not\subseteq \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, then Lemma 7(ii) implies that the set $\mathcal{A} \cup \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ is also myopically CS-maximizing but strictly contains set $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, a contradiction to $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ being the largest myopically CS-maximizing set for $\mathcal{P}_t$ given market structure $\mathcal{A}_{t-1}$. Hence, $\mathcal{A} \subseteq \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. If $\mathcal{A} \subset \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, Lemma 7(i) implies that once the mergers in $\mathcal{A}$ have been approved, there is a merger in $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \backslash (\mathcal{A} \cap \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})) = \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \backslash \mathcal{A}$ that is CS-nondecreasing given that the mergers in $\mathcal{A}$ have taken place, contradicting $\mathcal{A}$ being the result of a step-by-step merger policy that approves mergers that are CS-nondecreasing given the current market structure until no further CS-nondecreasing mergers can be identified. ■

Proof of Lemma 5. To see Property 1, note that, in equilibrium, only a uniquely lowest-cost firm (a firm that is the only one to have cost $c_{\iota(1|N)}$) can make a positive profit before the merger. Hence, Property 1 can fail to hold only if the merger involves a firm that is the unique firm with cost $c_{\iota(1|N)}$. Suppose it involves such a firm. In that case, a merger that does not increase cost will not affect the equilibrium prices of the firms not involved in the merger (who price at cost both before and after the merger), so the merger must be weakly profitable.

To see Property 2, let $\overline{N}$ denote the set of firms after the merger. Assume the merger is CS-decreasing, i.e.,

$$p_{\iota(1|\overline{N})} = \min\{p^m(c_{\iota(1|\overline{N})}), c_{\iota(2|\overline{N})}\} > \min\{p^m(c_{\iota(1|N)}), c_{\iota(2|N)}\} = p_{\iota(1|N)}. \tag{10}$$

Since the merger is not cost increasing, $c_{\iota(1|\overline{N})} \leq c_{\iota(1|N)}$, so that $p^m(c_{\iota(1|\overline{N})}) \leq p^m(c_{\iota(1|N)})$. Note, first, that the merger must involve at least one firm with cost $c_{\iota(1|N)}$: if not, then since it is not cost increasing, $c_{\iota(2|\overline{N})} \leq c_{\iota(2|N)}$, which would yield a contradiction.to (10) Next, note that if the merger involves some but not all of the firms with cost $c_{\iota(1|N)}$, then since it is not cost increasing, we have $c_{\iota(2|\overline{N})} = c_{\iota(1|N)} \leq c_{\iota(2|N)}$. This would imply that $\min\{p^m(c_{\iota(1|\overline{N})}), c_{\iota(2|\overline{N})}\} \leq \min\{p^m(c_{\iota(1|N)}), c_{\iota(2|N)}\}$, which contradicts (10), so the merger must involve all of the firms with cost $c_{\iota(1|N)}$. Next, if the merger does not involve all of the firms with cost $c_{\iota(2|N)}$, then since it involves all of the firms with cost $c_{\iota(1|N)}$ and is not cost increasing, it must again be that $c_{\iota(2|\overline{N})} \leq c_{\iota(2|N)}$, which again yields a contradiction. Hence, the merger must involve all of the firms with cost $c_{\iota(1|N)}$ and all of the firms with cost $c_{\iota(2|N)}$. Finally, suppose that $p^m(\overline{c}_M) \leq c_{\iota(2|N)}$, or equivalently, $p^m(c_{\iota(1|\overline{N})}) \leq c_{\iota(2|N)}$. Since the merger is not cost increasing, $p^m(c_{\iota(1|\overline{N})}) \leq p^m(c_{\iota(1|N)})$. But this implies that $p_{\iota(1|\overline{N})} = \min\{p^m(c_{\iota(1|\overline{N})}), c_{\iota(2|\overline{N})}\} \leq \min\{p^m(c_{\iota(1|N)}), c_{\iota(2|N)}\} = p_{\iota(1|N)}$, a contradiction. ■

Proof of Proposition 4. To see part (i), suppose to the contrary that the merger, say M_1, becomes CS-decreasing if the other merger, say M_2, takes place. Since M_1 is not cost increasing, by Lemma 5 the merger must involve (after M_2 has taken place) all of the firms with cost $c_{\iota(1|N_2)}$ and all of the firms with cost $c_{\iota(2|N_2)}$, and moreover $p^m(\overline{c}_{M_1}) > c_{\iota(2|N_2)}$. Because mergers are disjoint and M_2 does not increase cost, this implies that when done in isolation, merger M_1 involves all of the firms with costs of either $c_{\iota(1|N)}$ or $c_{\iota(2|N)}$. If so, then $c_{\iota(2|N)} = c_{\iota(2|N_2)}$. But this implies that $p^m(\overline{c}_{M_1}) > c_{\iota(2|N)}$, so that merger M_1 is CS-decreasing in isolation, a contradiction.

To see part (ii), note that if a merger M_i is CS-decreasing in isolation, by Lemma 5 it must involve (when done in isolation) all of the firms with costs of $c_{\iota(1|N)}$ and all the firms with cost $c_{\iota(2|N)}$. However, since mergers are disjoint, there cannot be two distinct mergers with this property. ■

Proof of Proposition 5.

The proof of part (i) is identical to the proof of Proposition 2, except that in the Bertrand model a CS-increasing merger, while profitable, need not be strictly so.

To see part (ii), consider implementing merger M_2 first, followed by merger M_1. Note that since M_2 is CS-decreasing in isolation it involves all of the firms with costs equal to $c_{\iota(1|N)}$ or $c_{\iota(2|N)}$ by Lemma 5. Since it is not cost increasing, the profit of all firms not involved in this merger, including all of those in M_1, must be zero before and after merger M_2 takes place. Since merger M_1 is profitable after merger M_2 takes place [by part (i)], the sequence of mergers cannot decrease the joint profit of the firms in M_1. ■

Proof of Proposition 6. To consider extensive-form trembling-hand perfect Nash equilibria we perturb the game by introducing a minimum and a maximum probability of a merger proposal at any information set of a proposer of a feasible but not-yet-approved merger. We examine Nash equilibrium behavior in the agent normal form as these minimum and maximum probabilities approach zero and one, respectively.

(i) We will first establish that all feasible mergers being proposed in each period after any history is an extensive-form trembling-hand perfect Nash equilibrium for the firms. Combined with Lemma 3, this yields part (i) of the proposition. To do so, it suffices to examine perturbed games in which at every information set of a proposer of a feasible but not-yet-approved merger the merger must be proposed with a probability of at least $\varepsilon > 0$ and not more than $1 - \varepsilon$, where $\varepsilon \to 0$.

The proof is by induction and follows closely that of Proposition 3. Consider, a period t in which the history prior to period t's proposal stage consists of the sequences $\mathfrak{F}^t = (\mathfrak{F}_1, ..., \mathfrak{F}_t)$ of feasible mergers, $\mathcal{P}^{t-1} = (\mathcal{P}_1, ..., \mathcal{P}_{t-1})$ of proposed mergers, and $\mathcal{A}^{t-1} = (\mathcal{A}_1, ..., \mathcal{A}_{t-1})$ of approved mergers. Note that this history also determines exactly each firm i's observed history, which we denote by I_i^t. Formally, each I_i^t corresponds to an information set for firm i at the proposal stage in period t. The most important difference to the proof of Proposition 3 is that the induction hypothesis is now that for any period $T < T$, starting in period $t + 1$ all feasible but not-yet-approved mergers will be proposed in every period with the maximum possible probability $1 - \varepsilon$.

To show that proposing its merger with the maximum possible probability is optimal in period t for every proposer firm with a feasible but-not-yet approved merger, consider proposer firm i at an information set I_i^t with a feasible but not-yet-approved merger M_k. Recall from the proof of Proposition 3 that, for a given information set I_i^t and a given set of other proposed mergers $\widehat{\mathcal{P}}_{-k}$, either the merger M_k is not approved when proposed (and so proposing the merger has no effect on current profits), or the merger is approved when proposed but the merged firm is inactive (and so, again, proposing the merger has no effect on current profits), or the merger is approved and results in an active firm (in which case there is a strictly positive effect on current profits). In the first case (i.e., the merger is not approved when proposed), there is also no effect on future profits of proposing the merger given the induction hypothesis. The same is true in the second case (when the merger is approved but results in an inactive firm). In the third case (where the merger is approved and results in an active firm), however, there might be an effect on future profits if $t < T$. But this effect is continuous in the size of the tremble ε, and (as is clear from the proof of Proposition 3) is equal to zero if $\varepsilon = 0$. Since there are at most a finite number of such information sets I_i^t and sets $\widehat{\mathcal{P}}_{-k}$, there exists an $\varepsilon_t > 0$ such that for all $\varepsilon \leq \varepsilon_t$, proposing a feasible and not-yet-approved merger that ends up being approved and results in an active firm in period t is strictly profitable. Hence, for $\varepsilon \leq \varepsilon_t$, proposing every feasible and not-yet-approved merger in period t is profitable. The same is clearly true if $t = T$, where there is no effect on future profits.

We conclude that our induction hypothesis – that all feasible and not-yet approved mergers will be proposed in the future with the maximum possible probability – holds when we look at period $t-1$ provided that $\varepsilon \leq \overline{\varepsilon}_t = \min_{\tau \geq t} \varepsilon_\tau$. Applying induction starting in period $T-1$ implies that proposing every feasible but not-yet-approved merger in every period with the maximum possible probability is a Nash equilibrium of the agent normal form of any perturbed game with $\varepsilon \leq \overline{\varepsilon}_1$. Hence, taking $\varepsilon \to 0$, proposal of every feasible but not-yet-approved merger in every period is an extensive-form trembling-hand perfect Nash equilibrium.

(ii) We next show that every extensive-form trembling-hand perfect Nash equilibrium maximizes discounted consumer surplus (and results in the same sequence of period-by-period consumer surpluses) for each sequence of feasible mergers $\mathfrak{F}$. To establish this result, we restrict attention to small perturbations in which the minimum probability of a merger proposal at any information set for a proposer of a feasible but not-yet-approved merger is no more than $\overline{\varepsilon}_1 > 0$ (where $\overline{\varepsilon}_1$ is defined as above) and the maximum probability is no less than $1 - \overline{\varepsilon}_1$. We examine Nash equilibrium behavior in the agent normal form as the minimum and maximum probabilities approach zero and one, respectively.

We now show that if the perturbations are strictly positive but sufficiently small (in the sense defined above), then if the true set of feasible but not-yet-approved mergers in period t is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, every merger in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed with the maximum possible probability in that period. [Recall that the set $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1})$, defined in the proof of Proposition 3(ii), is the set of mergers in $\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1})$ that result in active firms given the market structure $\overline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1})$.] The result follows as we let the minimum and maximum proposal probabilities go to zero and one, respectively.

The proof is by induction. The induction hypothesis for period t is that in all future periods $\tau > t$, whenever the set of feasible but not-yet-approved mergers is $\mathfrak{F}_\tau \backslash \mathcal{A}_{\tau-1}$, then all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_\tau \backslash \mathcal{A}_{\tau-1} | \mathcal{A}_{\tau-1}) \backslash \mathcal{A}_{\tau-1}$ are proposed with the maximum possible probability.

Suppose that $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ is indeed the true set of feasible and not-yet-approved mergers. Let $\mathcal{I}_t(\mathfrak{F}_t \backslash \mathcal{A}_{t-1})$ denote those information sets in period t that are consistent with $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$; that is, these are the information sets that are reached for at least one sequence $(\mathfrak{F}^t, \mathcal{P}^{t-1})$ of feasible mergers and merger proposals that, given the most lenient myopically CS-maximizing merger policy, results in the set of feasible but not-yet-proposed mergers in period t being $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$. Consider any information set $I_i^t \in \mathcal{I}_t(\mathfrak{F}_t \backslash \mathcal{A}_{t-1})$ that belongs to the proposer of a merger $M_k \in ($ $\mathfrak{F}_t \backslash \mathcal{A}_{t-1})$ such that $M_k \in \overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1} | \mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$. From our earlier argument, if all minimum proposal probabilities are less than or equal to $\overline{\varepsilon}_1$ and all maximum proposal probabilities are larger than or equal to $1 - \overline{\varepsilon}_1$, proposing merger M_k never reduces the expected joint discounted profits of the firms in M_k. We now show that proposing merger M_k is in fact strictly profitable in expectation.

Observe, first, that in any Nash equilibrium of the agent normal form of the perturbed game, the information set I_i^t is reached with positive probability along the equilibrium path when the set of feasible but not-yet-proposed mergers in period t is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, so (in belief language) the agent choosing at this information set must assign a strictly positive probability to $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ being the set of feasible but not-yet-approved mergers.[25]

Th erest of the argument follows closely, with some differences, the proof of Proposition 3(ii): Starting from $\mathcal{A}_{t-1}$, there is an ordering of the mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ that is CS-nondecreasing at each step, which we denote by $(M_1, ..., M_S)$. As in the proof of Proposition 3(ii), consider the proposal of merger M_k at I_i^t and suppose that all mergers M_s for $s < k$ are proposed with maximum probability when $\mathfrak{F}_t \backslash \mathrm{A}_{t-1}$ is the true set of feasible and not-yet-approved mergers in period t. (Note that this assumption is valid when $k = 1$.) Now, given the trembles, the proposer of M_k at I_i^t must assign a strictly positive probability to the event that the realized set of proposed mergers is $\widehat{\mathcal{P}}_t = \{M_1, ..., M_k\}$. As in the proof of Proposition 3(ii), in this case all of these mergers will be approved and will result in active firms. Hence, the proposer of M_k at I_i^t must believe that, if proposed, merger M_k will be approved and result in an active firm with strictly positive probability. But from our previous argument, if merger M_k is approved and the merged firm M_k is active, proposal of the merger is strictly profitable when the perturbations are small (in the sense described above). Applying induction starting at $k = 1$, we see that any such merger M_k will be proposed in period t with the maximum possible probability. Thus, if $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ is the true set of feasible and not-yet-approved mergers, all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed with the maximum possible probability.

Applying induction, we conclude that in any perturbed game if the true set of feasible but not-yet-approved mergers in period t is $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, then every merger in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$ will be proposed with the maximum possible probability in that period. Taking the perturbations to zero yields the result. ■

6.2 Sketch of Arguments Underlying Remark 2

In the following, we briefly sketch the arguments leading to our claim in Remark 2. To do so, we need to extend many of the results in the text. To refer to these additional results, we will append a prime to the number of the result it extends. For example, the result that extends Proposition 2 will be denoted Proposition 2'.

6.2.1 CS Effects of and Interactions Between Mergers

Mirroring the statement of Proposition 1, Proposition 1' states: (i) If a merger M_1 is CS-increasing (and hence profitable) in isolation, it remains CS-increasing (and hence profitable) if another merger M_2 that is CS-nondecreasing in isolation takes place and the merged firm M_1 remains active after merger M_2 has taken place. (ii) If a merger M_1 is CS-nonincreasing in isolation and results in an active firm, then the merger remains CS-nonincreasing if another merger M_2 that is CS-nonincreasing in isolation takes place. The proof of part (i) of the proposition uses the fact for M_1 to be CS-increasing after merger M_2 takes place, the merged firm M_1 must also be active if merger M_2 does not take place, and follows a similar argument as in the proof of Proposition 1(i). The proof of part (ii) of the proposition uses the fact

[25] This property – that any firm with a merger in $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ must always assign a strictly positive probability to $\mathfrak{F}_t \backslash \mathcal{A}_{t-1}$ being the set of feasible but not-yet-approved mergers – is a key step of the argument. It would not be true without the perturbations and is the reason why perturbations are needed for ensuring the proposal of all mergers in $\overline{\mathcal{A}}_1^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$.

that if the merged firm M_1 is active in isolation, it must remain active after merger M_2 has taken place since the CS-nondecreasing merger M_2 weakly increases the equilibrium price; moreover, the CS-nondecreasing merger M_2 weakly decreases the threshold value of post-merger marginal cost, $\widehat{c}_{M_1}$, that makes merger M_1 just CS-neutral, and so merger M_1 must remain CS-nondecreasing if M_2 takes place.

In Proposition 2', the hypothesis is that merger M_1 is CS-increasing [rather than CS-nondecreasing, as in Proposition 2] in isolation, while merger M_2 is CS-nonincreasing [rather than CS-decreasing] in isolation but CS-increasing once merger M_1 has taken place. Under this modified hypothesis, the statements of parts (i) and (ii) remain unchanged. The proof of the proposition proceeds along the same lines as that of Proposition 2, except for some small modifications. For instance, in the second sentence of the proof of part (i) "weakly increases" is replaced by "strictly increases" and "nonnegative" by "strictly positive."

Lemma 6' (the modified Incremental Gain Lemma) differs from Lemma 6 in that "CS-nondecreasing" is replaced everywhere by "CS-increasing." For instance, the set $\mathfrak{M}$ in part (i) has the property that every merger $M \in \mathfrak{M}$ is CS-increasing if all of the other mergers in $\mathfrak{M}$ have taken place. As a result, starting from any strict subset $Y \subset \mathfrak{M}$, there exists a sequencing of the mergers in $\mathfrak{M} \backslash Y$ that is CS-increasing at each step. The proof uses the fact that, under the hypothesis of the lemma, every merger $M \in \mathfrak{M}$ must result in an active firm, for any set $A \subseteq \mathfrak{M}$ of approved mergers.[26] This means that we can apply part (i) of Proposition 1' and use the same arguments as in the proof of Lemma 6 (but with "CS-decreasing" being replaced by "CS-nonincreasing", and so on). As for part (ii) of Lemma 6', we need to add the assumption that every merger in the sequence remains active when all of the other mergers in the sequence have taken place. That is, the statement now reads: Suppose that a sequence of mergers $M_1, ..., M_{J_1}$ is CS-increasing at each step. Then each merger $M \in \mathfrak{M} \equiv \{M_1, ..., M_{J_1}\}$ is CS-increasing if all of the other mergers in $\mathfrak{M}$ (those in the set $\mathfrak{M} \backslash M$) have taken place, provided that each merged firm M remains active after the mergers in $\mathfrak{M} \backslash M$.

To obtain Lemma 7' from Lemma 7, the hypothesis is modified so that the set $\mathfrak{M}_1$ has the property that every merger in the set is CS-increasing if all of the other mergers in that set have taken place, while the set $\mathfrak{M}_2$ continues to have the property that every merger in the set is CS-nondecreasing if all of the other mergers in that set have taken place. Lemma 7' states that if $\mathfrak{M}_1 \nsubseteq \mathfrak{M}_2$ and if each of the mergers in set $\mathfrak{M}_1 \backslash (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ (when done on its own) results in an active firm once all of the mergers in set $\mathfrak{M}_2$ have taken place, then there exists a merger $M_1' \in \mathfrak{M}_1 \backslash (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ that is CS-increasing [rather than CS-nondecreasing, as in Lemma 7(i)] given that all of the mergers in $\mathfrak{M}_2$ have taken place. The proof first identifies a merger $M_1' \in \mathfrak{M}_1 \backslash (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ that is CS-increasing given that the mergers in $(\mathfrak{M}_1 \cap \mathfrak{M}_2)$ have taken place. If M_1' would be not be a merger among active firms once the mergers in $\mathfrak{M}_2$ have taken place, then it must be CS-increasing once the mergers in $\mathfrak{M}_2$ have taken place (since it results in an active firm). If, instead, M_1' would be a merger among active firms once the mergers in $\mathfrak{M}_2$ have taken place, then an induction argument along the same lines as that in the proof of part (i) of Lemma 7 (now using Proposition 1') establishes the result. Note that a sufficient condition for each of the mergers in set $\mathfrak{M}_1 \backslash (\mathfrak{M}_1 \cap \mathfrak{M}_2)$ to result in an active firm once all of the mergers in set $\mathfrak{M}_2$ have taken place is that consumer surplus under set $\mathfrak{M}_1$

[26] By Lemma 6(i), the price after the mergers in set A have taken place can be no higher than the price after all of the mergers in set $\mathfrak{M}$ have taken place.

is at least as large as under set $\mathfrak{M}_2$; that is, letting p_i^* denote the equilibrium price after all of the mergers in set $\mathfrak{M}_i$, $i = 1, 2$, have taken place, if we have $p_2^* \geq p_1^*$. To see this, note that every merged firm $M \in \mathfrak{M}_1$ must have a cost $\overline{c}_M < p_1^*$ since, otherwise, the merger would not be CS-increasing given the other mergers in set $\mathfrak{M}_1$. But then $\overline{c}_M < p_2^*$, which implies that merger M results in an active firm once all of the mergers in set $\mathfrak{M}_2$ have taken place. [A counterpart to Lemma 7(ii) is not necessary for our purposes here.]

6.2.2 Myopically CS-maximizing Sets

In analog to the largest myopically CS-maximizing set, we can define a smallest myopically CS-maximizing set for the set of proposed mergers $\mathcal{P}_t$ given current market structure $\mathcal{A}_{t-1}$, as a myopically CS-maximizing set that does not contain any other myopically CS-maximizing set.

Lemma 2' makes the same uniqueness claim as Lemma 2, but for the smallest myopically CS-maximizing set rather than for $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$: there is a unique smallest myopically CS-maximizing set, denoted $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. (In contrast to Lemma 2, no monotonicity claim is made.) The proof of the uniqueness property is by contradiction. Suppose there are two smallest myopically CS-maximizing sets, say $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}}'$, with $\underline{\mathcal{A}} \neq \underline{\mathcal{A}}'$. Without loss of generality, suppose that $\underline{\mathcal{A}} \nsubseteq \underline{\mathcal{A}}'$. Since each one of the sets must have the property that every merger in the set is CS-increasing given the other mergers in that set and since the two sets must induce the same level of consumer surplus, we can apply Lemma 7' to show that there exists a merger $M' \in \underline{\mathcal{A}}\backslash(\underline{\mathcal{A}} \cap \underline{\mathcal{A}}')$ that is CS-increasing given that all of the mergers in $\underline{\mathcal{A}}'$ have taken place. But then $\underline{\mathcal{A}}'$ cannot be a myopically CS-maximizing set, a contradiction.

The following result shows that the smallest myopically CS-maximizing set is contained in any other myopically CS-maximizing set, and that any myopically CS-maximizing set is contained in the largest myopically CS-maximizing set:

Lemma 8 *For a given proposed set of mergers, $\mathcal{P}_t$, and current market structure $\mathcal{A}_{t-1}$, the following inclusion property holds for the myopically CS-maximizing sets:*

$$\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}).$$

Proof. We have already established the second inclusion property in Lemma 2. We therefore turn to the first inclusion property, $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. Let $\mathcal{A}^0 \subseteq \mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ denote the set of all those mergers in $\mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ that are CS-neutral given the other mergers in $\mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, and let $\mathcal{A}^+ \equiv \mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})\backslash\mathcal{A}^0$ denote the complementary set, which has the property that every merger $M \in \mathcal{A}^+$ is CS-increasing given the other mergers in $\mathcal{A}^+$. We claim that $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{A}^+$. [In fact, $\mathcal{A}^+ = \underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, but we do not need to show this.] To see this, suppose otherwise that $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \nsubseteq \mathcal{A}^+$. Since each set, $\mathcal{A}^+$ and $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, is myopically CS-maximizing and has the property that every merger in the set is CS-increasing given the other mergers in that set, we can apply Lemma 7' to conclude that there exists a merger $M' \in \underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})\backslash\left(\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \cap \mathcal{A}^+\right)$ that is CS-increasing given that all of the mergers in $\mathcal{A}^+$ have taken place. But then $\mathcal{A}^+$ cannot be myopically CS-maximizing, a contradiction. Hence, we have $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{A}^+$, which implies $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. ■

An immediate implication of Lemma 8 is that we can think of set $\mathcal{A}_t^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ as consisting of the mergers in the smallest myopically CS-maximizing set $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$, plus a (potentially empty) set of mergers that are CS-neutral given $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$. Thus, all myopically CS-maximizing merger policies differ from one another only in their treatment of CS-neutral mergers.[27]

6.2.3 Extension of Proposition 3

Establishing the claim of Remark 2 parallels the argument leading to Proposition 3 in the text. We first establish a generalization of Lemma 4, Lemma 4'. Specifically, Lemma 4' states: If all feasible but not-yet-approved mergers are proposed in each period, any myopically CS-maximizing merger policy, which induces the approval sequence $\mathcal{A}_1 = \mathcal{A}_1^*(\mathfrak{F}_1|\varnothing)$ and $\mathcal{A}_t = \mathcal{A}_t^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ for all $t > 1$, maximizes discounted consumer surplus for every realization of feasible mergers $\mathfrak{F} = (\mathfrak{F}_1, ..., \mathfrak{F}_T)$. To establish this result, we use the following lemma, which is also used to prove Proposition 3':

Lemma 9 *Suppose $\mathfrak{F}_{t-1} \subseteq \mathfrak{F}_t$. If the current market structure $\mathcal{A}_{t-1}$ is such that $\mathcal{A}_{t-1} \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_{t-1}|\varnothing)$, then*

$$\underline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing) \subseteq \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \subseteq \mathcal{A}_t^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing).$$

Proof. The second and third inclusion properties in the display follow from Lemma 8, while the equality follows from the same induction argument as in the proof of Lemma 4. To see the first inclusion property, $\underline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing) \subseteq \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, observe that since $\overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ is a solution to the problem of myopically maximizing consumer surplus given that the set of proposed mergers is $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ and mergers $\mathcal{A}_{t-1}$ have previously been approved, the solution set to that problem must be a subset of the solution set of the less constrained problem of myopically maximizing consumer surplus given that the set of proposed mergers is $\mathfrak{F}_t$ and no mergers have previously been approved. ■

Applying Lemma 9 iteratively, we see that if all feasible mergers are proposed in period t, then regardless of firms' previous behavior the market structure in period t will contain the set $\underline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ and be contained within the set $\overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$, which implies that it is a solution to the problem of maximizing consumer surplus in period t given that no mergers have previously been approved. This implies that Lemma 4' holds.

Note also that if in period t all mergers in $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ are proposed when $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ is the set of feasible but not-yet-approved mergers, then regardless of previous behavior by the firms the market structure in period t will again contain the set $\underline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$ and be contained within the set $\overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing)$. To see this, observe that if $\mathcal{P}_t$ is such that $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1} \subseteq \mathcal{P}_t \subseteq \mathfrak{F}_t\backslash\mathcal{A}_{t-1}$, then Lemma 2 implies that $\overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, while the fact that $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ – a set that myopically maximizes consumer surplus when all mergers in $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ are proposed – is feasible when $\mathcal{P}_t$ is proposed, implies that

[27] Note also that CS-neutral mergers are measure zero events in a model with a continuum of possible efficiency realizations.

$\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) = \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$.[28] Applying Lemma 9 then implies that

$$\underline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing) \subseteq \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1}) \subseteq \overline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1}) = \overline{\mathcal{A}}^*(\mathfrak{F}_t|\varnothing).$$

Thus, if in all periods all mergers in $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ are proposed when $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ is the set of feasible but not-yet-approved mergers, then the outcome will yield optimal period-by-period levels of consumer surplus.

These facts imply that dynamic optimality holds for *any* myopically CS-maximizing policy:

Proposition 3' *If the antitrust authority follows a myopically CS-maximizing merger policy, then for each sequence $\mathfrak{F}$, every subgame perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. (Sketch) The proof follows very closely that of part (ii) of Proposition 3. It proceeds by establishing, using an induction argument, that whenever $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ is the set of feasible but not-yet-approved mergers in period t, all mergers in set $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ will necessarily be proposed, which establishes the claim (using the argument above). One important change relative to the case of the most lenient myopically CS-maximizing merger policy is that, in any period t, future market structures may be affected by whether a given merger $M_k \in \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ is proposed today. However, these future market structure effects can involve merger M_k only if the merged firm would be inactive and can involve mergers other than M_k only in situations in which those mergers are CS-neutral given the other mergers being approved. They therefore have no effect on the joint profits of the firms in merger M_k, so we can again focus solely on current period profit effects.

Consider the proposal of a merger $M_k \in \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ in period t under the assumption that future payoffs for the firms involved in that merger are independent of period t behavior. Since $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ is the smallest myopically CS-maximizing set for $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ given $\mathcal{A}_{t-1}$, every merger in $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ is CS-increasing given every other merger in that set and results in an active firm. Part (i) of Lemma 6' implies that there is an ordering of the mergers in $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ that is CS-increasing at each step, which we denote by $(M_1, ..., M_S)$. Suppose that all mergers M_s for $s < k$ are proposed when $\mathfrak{F}_t\backslash\mathcal{A}_{t-1}$ is the set of feasible and not-yet-approved mergers in period t (which is true when $k = 1$.) Consider the case where $\widehat{\mathcal{P}}_t = \{M_1, ..., M_k\}$. We claim that all of the mergers in $\{M_1, ..., M_k\}$ will be approved. To see this, note first that all of the merged firms in $\{M_1, ..., M_k\}$ will be active if all are approved [since the price will be no less than if all of the mergers $\{M_1, ..., M_S\} = \underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})\backslash\mathcal{A}_{t-1}$ are approved]. Hence, by part (ii) of Lemma 6', every merger in $\widehat{\mathcal{P}}_t$ is CS-increasing given the other mergers in that set (and given the previously approved mergers $\mathcal{A}_{t-1}$). If the antitrust authority were to approve only a (possibly empty) subset of $\widehat{\mathcal{P}}_t$, part (i) of Lemma 6' implies that the antitrust authority could strictly increase consumer surplus by approving the other mergers in $\widehat{\mathcal{P}}_t$ as well. This proves the claim that all of the mergers in $\{M_1, ..., M_k\}$ will be approved when $\widehat{\mathcal{P}}_t = \{M_1, ..., M_k\}$.

[28] The argument in footnote 13 implies that the consumer surplus levels with approved mergers $\underline{\mathcal{A}}^*(\mathfrak{F}_t\backslash\mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ and $\underline{\mathcal{A}}^*(\mathcal{P}_t|\mathcal{A}_{t-1})$ must be the same. Since each of these two sets has the property that every merger in the set is CS-increasing given the other mergers in the set, we can apply Lemma 7' to show that the two sets must be the same.

Consider now the case where $\{M_1, ..., M_k\} \subset \widehat{\mathcal{P}}_t$. Using the same argument as above, the set $\{M_1, ..., M_k\}$ has the property that every merger in the set is CS-increasing given the other mergers in that set, and so [by part (i) of Lemma 6'] for any strict subset $Y \subset \{M_1, ..., M_k\}$, there exists a merger $M' \in \{M_1, ..., M_k\} \backslash Y$ that is CS-increasing given Y. We claim that $\{M_1, ..., M_k\} \subseteq \mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$ in any myopically CS-maximizing policy, so that all of the mergers in $\{M_1, ..., M_k\}$ will be approved. To see this, suppose otherwise that $\{M_1, ..., M_k\} \nsubseteq \mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$. Note first that every merger in $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$ is CS-nondecreasing given the other mergers in that set. Since $\widehat{\mathcal{P}}_t \subseteq \mathfrak{F}_t \backslash \mathcal{A}_{t-1}$, the equilibrium price under market structure $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$ must be weakly higher than under market structure $\mathcal{A}_t^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, and the equilibrium price under market structure $\mathcal{A}_t^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ must be the same as under market structure $\underline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ (by virtue of both sets being CS-maximizing given the same set of proposed mergers and given the same market structure). Since all of the merged firms in $\{M_1, ..., M_k\}$ are active when market structure is $\underline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$, this implies that these firms will also be active when the market structure is $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$. Let $Y = \mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1}) \cap \{M_1, ..., M_k\}$. Since all of the merged firms in $\{M_1, ..., M_k\}$ are active when market structure is $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$, Lemma 7' implies that there exists a merger $M' \in \{M_1, ..., M_k\} \backslash Y$ that is CS-increasing given all of the mergers in $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$. But this means that the antitrust authority can strictly increase consumer surplus by approving merger M' in addition to all of the mergers in $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1}) \backslash \mathcal{A}_{t-1}$, a contradiction to $\mathcal{A}_t^*(\widehat{\mathcal{P}}_t|\mathcal{A}_{t-1})$ being myopically CS-maximizing. This proves the claim that all of the mergers in $\{M_1, ..., M_k\}$ will be approved when $\{M_1, ..., M_k\} \subset \widehat{\mathcal{P}}_t$. Hence, using the same arguments as in the proof of Proposition 3, proposal of merger M_k is strictly profitable. Applying induction starting at $k = 1$, it follows that all mergers in $\underline{\mathcal{A}}^*(\mathfrak{F}_t \backslash \mathcal{A}_{t-1}|\mathcal{A}_{t-1})$ will be proposed. ■