Federal Reserve Bank of Minneapolis
Research Department Staff Report 000

September 1999

Volatile exchange rates and the forward premium anomaly: A segmented asset market view

Fernando Alvarez

University of Chicago and NBER

Andrew Atkeson

University of Minnesota, Federal Reserve Bank of Minneapolis and NBER

Patrick Kehoe

University of Pennsylvania, Federal Reserve Bank of Minneapolis and NBER

ABSTRACT

We explore the connection between money and exchange rates in a model with endogenously segmented asset markets. We build on Rotemberg's (1985) model of exogenous segmentation by letting agents choose whether to pay a fixed cost to transfer money between goods and asset markets. As in Rotemberg, segmentation effects magnify the impact of money injections on exchange rates. We show that the endogeneity of the segmentation leads to new implications for the relation between money and exchange rates. Specifically, the model implies that segmentation effects vary systematically with the level of inflation. In the cross section, this feature implies that market segmentation effects are less important in high inflation countries. In the time series, this feature implies that exchange rate risk varies systematically with inflation. Because of this time varying exchange rate risk, our model can potentially generate the forward premium anomaly, namely, the observed tendency of high interest rate currencies to appreciate.

We explore the connection between money and exchange rates in a model with endogenously segmented asset markets. This segmentation arises because agents must pay a fixed cost to transfer money between the goods market and the asset market. In any period, some agents choose to pay the fixed cost and thus, at the margin, freely exchange bonds and money. Other agents choose not to pay the fixed cost and hence do not. The asset market is segmented in the sense that when the government injects money through an open market operation, only the fraction of the agents in the economy who choose to trade are on the other side of the transaction. As a result of this asset market friction, money injections have a disproportionate impact on these traders' portfolios and hence have segmentation effects on exchange rates over and above those present in the standard general equilibrium monetary model of Lucas 1982. We develop a simple general equilibrium model with segmented asset markets and contrast its implications for exchange rates to those of the standard model.

Our model is a two country pure exchange cash-in-advance economy. The key difference between this model and the standard cash-in-advance model is the fixed cost to transfer money between markets. This fixed cost is similar to that in the models of Baumol (1952) and Tobin (1956). In the model agents have idiosyncratic income shocks and each period they choose whether to pay the fixed cost to trade depending on the value of these shocks. If this fixed cost were zero, the model reduces to a standard one in which all agents choose to trade and money growth shocks have no effect on real exchange rates.

Our work builds on that of Rotemberg (1985). He was the first to show that segmentation effects magnify the impact of money injections on exchange rates. Alvarez and Atkeson (1997) extend Rotemberg's model and show that these segmentation effects can also lead to persistent movements in exchange rates. In both of these papers markets are segmented because agents are allowed to move cash between the asset market and the goods market only at exogenously specified dates. We build on this work by making the market segmentation endogenous by letting agents choose whether to pay a fixed cost.

We show that our model can capture the effects that these earlier papers were designed to capture. As in the early models, if markets are highly segmented then real and nominal exchange rates are much more volatile than movements in prices. Moreover, if money growth shocks are persistent, then real and nominal exchange rates are also persistent and more volatile than interest rates. One advantage of our approach is that in the early models markets can be highly segmented only if velocity is extremely low while in our model that need not be the case.

We then develop two new implications of market segmentation that depend critically on this segmentation being endogenous. First, our model has different implications for the volatility of exchange rates when inflation is moderate and when inflation is high. As inflation increases from moderate to high, the returns to trading money and bonds increases, more agents choose to pay the fixed cost to trade, and markets become relatively less segmented. This endogeneity of segmentation leads real exchange rates to be less volatile relative to nominal exchange rates when inflation is high.

Second, we show that our model can generate the forward premium anomaly, namely the tendency for high interest rate currencies to appreciate. This tendency is puzzling since intuitively one might expect that investors would demand higher interest rates on currencies that are expected to fall in value. (See Hodrick 1987 for an introduction to the forward premium anomaly.) The key to generating this anomaly is to have exchange rate risk vary systematically with the level of inflation and interest rates. As Backus, Foresi and Telmer

(1995) and Engel (1996) emphasize, a wide variety of monetary models have constant risk premia and therefore cannot address this anomaly. In contrast, in our model endogenous changes in market segmentation lead to endogenous changes in risk premia. We give conditions under which our model can generate the forward premium anomaly.

In contrast to our model, the standard models can produce none of these features. In them, exchange rates are not very volatile, this volatility does not depend on the level of inflation, and exchange rate risk premia are constant.

Throughout this paper, we abstract from trade in goods in order to focus on the role of asset market segmentation. It is well known that in order to generate volatile real exchange rates it is necessary have frictions both in goods markets and in asset markets. (See, for example, Chari, Kehoe, and McGrattan 1999.) Here we have abstracted from any friction in the goods markets, such as sticky prices, in order to focus on a friction in the asset markets. We view our work as highlighting a potentially important component of a complete model of exchange rates with separate frictions in both the asset market and the goods markets. As such we view our work as complementary to work on goods market frictions such as sticky prices. (See, for example, Obstfeld and Rogoff 1995.)

In related work, Grilli and Roubini (1992) and Schlagenhauf and Wrase (1995) study the effects of money injections on exchange rates in two country variants of the models in Lucas (1990) and Fuerst (1992). These models have substantially different asset market frictions than ours. They can generate volatile real exchange rates but they are unable to generate movements in real exchange rates that are persistent and substantially more volatile than interest rates.

1. The economy

Consider a two country, cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \ldots$. We refer to one country as the home country and the other as the foreign country. In each country, there is a government and a continuum of agents of measure one. Agents in the home country use the home currency, which we refer to as dollars, to purchase a home good. Agents in the foreign country use the foreign currency, which we refer to as euros, to purchase a foreign good.

Trade in this economy at dates $t \geq 1$ takes place in three separate locations: an asset market and the two goods markets. In the asset market, agents trade the two currencies and dollar and euro bonds which promise delivery of the relevant currency in the asset market in the next period, and the two governments introduce their currencies via open market operations. In each goods market, agents use the local currency to buy the local good subject to a cash-in-advance constraint and sell their endowment of the local good for local currency. There is a real fixed cost γ for each transfer of cash between the asset market and any individual agent in either goods market.

There are two sources of uncertainty in this economy: idiosyncratic shocks to agents' endowments and money growth shocks. Each period $t \geq 1$, each agent in the home country has an endowment of the home good y which is i.i.d. with distribution F with density f. Let $Y = \int y f(y) dy$ be the aggregate endowment. At date 1 home agents have initial dollar holdings in the goods markets $y_0 \bar{M}/Y$ where $\bar{M}$ is the initial aggregate stock of dollars and y_0 has distribution F. At date 0, home agents also have $\bar{B}_h$ units of the home government debt and $\bar{B}_h^*$ units of the foreign government debt which are claims on $\bar{B}_h$ dollars and $\bar{B}_h^*$ euros in the asset market at that date. Let $y^t = (y_0, y_1, y_2, \ldots, y_t)$ denote a typical history of individual shocks for a home agent up through period t and $f(y^t) = f(y_0)f(y_1)\dots f(y_t)$ the probability density over such histories. Similarly, each period, each agent in the foreign country has an endowment of the foreign good y^* which is i.i.d. with distribution F with density f. Thus $Y^* = Y$. Likewise, foreign agents start period 1 with $y_0^* \bar{M}^*/Y^*$ euro holdings in the foreign goods market and start period 0 with $\bar{B}_f$ units of the home government debt and $\bar{B}_f^*$ units of the foreign government debt in the asset market in period 0.

We let M_t denote the stock aggregate supply of dollars in period t, and let $\mu_t = M_t/M_{t-1}$ denote the growth rate of the dollar supply. Similarly, let μ_t^* be the growth rate of the supply of euros M_t^*. Let $s_t = (\mu_t, \mu_t^*)$ and let $s^t = (s_1, \dots, s_t)$ denote the history of money growth shocks up through period t and let $g(s^t)$ denote the density of the probability distribution over such histories. Let $g\left(s_t|s^{t-1}\right)$ denote the conditional density of s_t given s^{t-1}.

The timing within each period $t \geq 1$ is as follows. At the beginning of the period money growth s_t is realized, bonds and the two currencies are exchanged in the asset market, and agents pay cash in the asset market to a firm that transfers cash between the goods and asset markets. Agents then use real balances denominated in local currency in the goods market to purchase the local consumption good prior to learning the realization of their endowment and then sell their endowment y_t or y_t^* for local currency. At date 0, agents do not trade in the goods market or consume. At that date agents simply trade their initial holdings of government debt in the asset market for bonds state contingent bonds paying off in the asset market at $t = 1$. They do not know y_0 or y_0^* when they engage in asset trade at date 0.

The home government issues one period dollar bonds contingent on the aggregate state s^t. At date t, given state s^t, the home government pays off outstanding bonds $B\left(s^t\right)$ in dollars and issues claims to dollars in the next asset market of the form $B(s^t, s_{t+1})$ at prices $q(s^t, s_{t+1})$. The home government budget constraint at s^t with $t \geq 1$ is

$$(1) \qquad B(s^t) = M\left(s^t\right) - M(s^{t-1}) + \int_{s_{t+1}} q(s^t, s_{t+1}) B\left(s^t, s_{t+1}\right) ds_{t+1},$$

with $M(s^0) = \bar{M}$ given and, at $t = 0$ the constraint is $\bar{B} = \int_{s_1} q(s^1)B(s^1)ds_1$. Likewise, the foreign government issues euro bonds denoted $B^*(s^t)$ with bond prices denoted $q^*(s^t, s_{t+1})$. The budget constraint for the foreign government is then parallel to the constraint above.

In the asset market at each date and state, home agents trade a set of one period dollar bonds that have payoffs next period contingent on the aggregate event s_{t+1} and their income realization y_t. Instead of letting each agent trade in all possible claims in all currencies contingent on other agents endowments, we suppose that each agent trades only in claims contingent on his own endowment with some intermediary. This latter approach is much less cumbersome than the former and yields the same outcomes. An agent at t with aggregate state s^t and individual shock history y^{t-1} purchases $B\left(s^t, s_{t+1}, y^{t-1}, y_t\right)$ claims to dollars from the intermediary that pay off in the next period's asset market contingent on the aggregate shock s_{t+1} and his endowment shock y_t. We let $q(s^t, s_{t+1}, y_t)$ be the dollar price of such a bond that pays one dollar in the asset market at date $t+1$ contingent on the relevant events. Because individual endowments are independent and identically distributed across individuals, we assume that these bond prices do not depend on the name of the individual. Likewise, foreign agents purchase euro bonds, with $B^*\left(s^t, s_{t+1}, y^{*t-1}, y_t^*\right)$ denoting the quantities of bonds purchased and $q^*(s^t, s_{t+1}, y_t^*)$ their prices in euros.

This intermediary buys government bonds and trades in the agent-specific contingent claims. Specifically, the intermediary buys government bonds $B(s^t)$, $B^*(s^t)$ and sells agent-specific claims of the form $B\left(s^t, y^{t-1}\right)$, $B^*\left(s^t, y^{*t-1}\right)$ to all the agents to maximize profits

$$\int_{s_{t+1}} \int_{y^t} q(s^t, s_{t+1}, y_t) B\left(s^t, s_{t+1}, y^{t-1}, y_t\right) f(y^{t-1}) dy^t ds_{t+1} +$$

$$\int_{s_{t+1}} e(s^t) \int_{y^{*t}} q(s^t, s_{t+1}, y_t^*) B^*\left(s^t, s_{t+1}, y^{*t-1}, y_t^*\right) f(y^{*t-1}) dy^{*t} ds_{t+1} -$$

$$\int_{s_{t+1}} \left\{ q(s^t, s_{t+1}) B\left(s^t, s_{t+1}\right) + e(s^t) q^*(s^t, s_{t+1}) B^*\left(s^t, s_{t+1}\right) \right\} ds_{t+1}$$

subject to the constraints

$$(2) \quad B(s^{t+1}) + e(s^{t+1}) B^*(s^{t+1}) = \int_{y^t} B\left(s^{t+1}, y^t\right) f(y^t) dy^t + e(s^{t+1}) \int_{y^{*t}} B^*\left(s^{t+1}, y^{*t}\right) f(y^{*t}) dy^{*t},$$

where $e(s^t)$ is the nominal exchange rate between euros and dollars. From the intermediary's profit maximization problem, we characterize the price of agent-specific contingent claims in terms of the prices of government debt and develop the standard parity conditions between changes in the exchange rate and dollar and euro bond prices as follows.

Lemma 1. The price functions $q(s^t, y_{t-1})$, $q^*(s^t, y_{t-1}^*)$ and $q(s^t)$, $q^*(s^t)$ satisfy $q(s^t, y_{t-1}) = q(s^t) f(y_{t-1})$, $q^*(s^t, y_{t-1}^*) = q^*(s^t) f(y_{t-1}^*)$ and $q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t)/e(s^{t+1})$.

Proof. Consider two ways for the intermediary to purchase one dollar for delivery in the asset market at t contingent on s^t at date $t-1$ in state s^{t-1}. The first way is to buy a government bond at s^{t-1} that promises to pay one dollar at s^t at cost $q(s^t)$. The second is to purchase $1/f(y_{t-1})$ dollars from a measure one of consumers, where each consumer delivers only if that consumer experiences idiosyncratic shock y_{t-1}. Since fraction $f(y_{t-1})$ of consumers will experience this shock, this way will deliver a dollar in the asset market at t contingent on s^t at cost $q(s^t, y_{t-1})/f(y_{t-1})$. Arbitrage requires that these two costs be the same. The same argument applies to the euro bonds. Likewise, an arbitrage argument implies the parity condition $q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t)/e(s^{t+1})$.

Notice that with price functions that satisfy Lemma 1, the profits of the intermediary are zero.

Consider now the problem of an agent in the home country. Let $P(s^t)$ denote the price level in dollars the home goods market at date $t \geq 1$. In each period $t \geq 1$, in the goods market agents start the period with dollar real balances $m(s^t, y^{t-1})$. They then choose transfers of real balances between the goods market and the asset market $x(s^t, y^{t-1})$, an indicator variable $z(s^t, y^{t-1})$ equal to zero if these transfers are zero and one if they are not, consumption of the home good $c(s^t, y^{t-1})$, and unspent real balances to carry over from goods shopping $a(s^t, y^{t-1})$ subject to the cash flow constraints

$$(3) \qquad a(s^t, y^{t-1}) = m(s^t, y^{t-1}) + x(s^t, y^{t-1})z(s^t, y^{t-1}) - c(s^t, y^{t-1}),$$

$$(4) \qquad m\left(s^{t+1}, y^t\right) = \frac{P(s^t)}{P(s^{t+1})}[y_t + a(s^t, y^{t-1})].$$

and the cash-in-advance constraints

$$(5) \qquad a(s^t, y^{t-1}) \geq 0,$$

where in (3) at $t = 1$, the term $m(s^t, y^{t-1})$ is given by $y_0 \bar{M}/Y$. In the asset market at $t \geq 1$, home agents begin with cash payments $B\left(s^t, y^{t-1}\right)$ on their bonds and $N(s^{t-1}, y^{t-2})$ held over as cash from the previous asset market. They purchase new bonds, make cash transfers to the goods market and hold $N(s^t, y^{t-1})$ over in cash to the next asset market subject to the sequence of budget constraints

$$(6) \qquad B\left(s^t, y^{t-1}\right) = \int_{s_{t+1}} \int_{y_t} q(s^t, s_{t+1}) B\left(s^t, s_{t+1}, y^{t-1}, y_t\right) f(y_t) ds_{t+1} dy_t +$$

$$N(s^t, y^{t-1}) - N(s^{t-1}, y^{t-2}) + P(s^t)\left[x(s^t, y^{t-1}) + \gamma\right] z(s^t, y^{t-1}),$$

with $N(s^t, y^{t-1}) \geq 0$ and, at $t = 1$, $N(s^{t-1}, y^{t-2}) = 0$. At $t = 0$, home agents have budget constraint

$$\bar{B}_h + e_0 \bar{B}_h^* = \int_{s_1} \int_{y_0} q(s_1) B(s_1, y_0) f(y_0) ds_1 dy_0$$

at $t = 0$. We assume that both consumption and real bondholdings $B(s^t, y^{t-1})/P(s^t)$ are uniformly bounded by some large constants. Notice that $N(s^t, y^{t-1})$ is cash that agents hold over in the asset market while $P(s^t)a(s^t, y^{t-1})$ is cash that agents hold over in the goods market.

The problem of home consumers is to maximize

$$(7) \qquad \sum_{t=1}^{\infty} \beta^t \int \int U(c(s^t, y^{t-1})) g(s^t) f(y^{t-1}) d\mu^t dy^{t-1}$$

subject to the constraints (3)- (6). Consumers in the foreign country solve the analogous problem with $P^*(s^t)$ denoting the price level in the foreign country in euros. We require that $\bar{B}_h + \bar{B}_f = \bar{B}$ and $\bar{B}_h^* + \bar{B}_f^* = \bar{B}^*$.

There is a firm that transfers cash between the goods and the asset market. Since each transfer of cash between the asset market and the home goods market consumes γ units of the home good, the total goods cost of carrying out all transfers between home consumers and the asset market at t is $\gamma \int z(s^t, y^{t-1}) f(y^{t-1}) dy^{t-1}$, and likewise for the foreign consumers. The firm purchases these goods with cash obtained from consumers in that period's asset market. Thus, the resource constraint in the home country is given by

$$(8) \qquad \int \left[c(s^t, y^{t-1}) + \gamma z(s^t, y^{t-1}) \right] f(y^{t-1}) dy^{t-1} = Y$$

for all t, s^t, and we have the analogous constraint in the foreign country. The home country money market clearing condition at $t \geq 0$ is given by

$$(9) \qquad \int \left(m(s^t, y^{t-1}) + \left[x(s^t, y^{t-1}) + \gamma \right] z(s^t, y^{t-1}) \right) f(y^{t-1}) dy^{t-1} +$$

$$\int_{y^{t-1}} N(s^t, y^{t-1}) f(y^{t-1}) dy^{t-1} = M(s^t)/P(s^t)$$

for all s^t. The money market clearing conditions for the foreign country are analogous. We let c denote the sequences of functions $c(s^t, y^{t-1})$ and use similar notation for other variables.

An *equilibrium* is a collection of bond and goods prices q, q^* and P, P^* together with bondholdings B, B^* for both individuals and the intermediary, and an allocation c, x, z, a, m and c^*, x^*, z^*, a^*, m^* such that the bond holdings and the allocation solve the agents' utility maximization problems, the intermediary maximizes profits, the governments' budget constraints hold, and the resource constraints and the money market clearing conditions are satisfied.

2. Characterizing equilibrium

In this section we characterize equilibria in which agents never hold over cash in either the goods or the asset market so that a, a^* and N and N^* are always zero. We do this by characterizing the equilibrium allocation under the supposition that agents never hold over cash and then find the corresponding prices. We then provide sufficient conditions to ensure that this supposition holds. We outline the main argument here and provide proofs in Appendix A.

In equilibrium some agents choose to pay the fixed cost to transfer cash between the goods and asset markets while others do not. We refer to agents that pay the fixed cost as *traders* and refer to agents who do not as *nontraders*. The decision to trade follows a cutoff rule and depends on individual endowments and the aggregate money shock.

We characterize this cutoff rule as follows. Suppose that N and a are identically equal to zero. Under this supposition, an agent's decision to pay the fixed cost to trade at date t affects only his current consumption and bondholdings and does not directly affect the real balances he holds in the goods market at later dates. We show in Appendix A that, since there is a complete set of state contingent bonds, traders have a common consumption level $c_T(s^t)$ after paying the fixed cost that only depends on the aggregate money shock s^t and not on their idiosyncratic endowments. Non-traders simply have consumption equal to the real balances they currently hold in the goods market. To construct the equilibrium cutoff rule for trading, define a function

$$(10) \quad h(x; c_T) = U(c_T) - U(x) - U'(c_T)\left[(c_T + \gamma) - x\right].$$

Intuitively, this function measures the net gain to an agent from switching from being and nontrader with consumption x to a trader with consumption c_T. The first two terms measure the direct utility gain within the current period from the switch while the third term measures the cost in utility terms of the required transfer of real balances from the asset market. Fixing c_T, it is optimal for an agent with real balances x to trade if h is positive and not to trade and consume x if h is negative. Note that h is strictly convex in the argument x, it attains its minimum at $x = c_T$, and is negative at this minimum if $\gamma > 0$. Define the cutoffs $y_L(c_T, \mu), y_H(c_T, \mu)$ as the solutions to

$$(11) \quad h(\frac{y}{\mu}; c_T) = 0,$$

when both of these solutions exist. If (11) is negative for all $y/\mu < c_T$, set $y_L(c_T, \mu) = 0$ while if it is negative for all $y/\mu > c_T$, set $y_H(c_T, \mu) = \infty$. This cutoff rule is illustrated in Figure 1. Note that as the fixed cost γ goes to zero, $y_L(c_T, \mu)/\mu$ and $y_H(c_T, \mu)/\mu$ converge to c_T, so that all agents become traders. Again, the analogous result for the foreign agents is immediate.

Given this form for the cutoff rule for trade, we determine the equilibrium values of traders' consumption and corresponding cutoffs as follows. The constraints (4), (8) and (9) imply that the price level is given by

$$(12) \quad P(s^t) = M(s^t)/Y,$$

money holdings are given by $m(s^t, y^{t-1}) = y_{t-1}/\mu_t$ and the consumption of nontraders is $c(s^t, y^{t-1}) = y_{t-1}/\mu_t$. Substituting the nontrader's consumption into the resource constraint (8) and using the cutoff rules defined in (11) gives

$$(13) \quad (c_T + \gamma)[F(y_L) + 1 - F(y_H)] + \frac{1}{\mu_t} \int_{y_L}^{y_H} y f(y) dy = Y$$

where we have suppressed explicit dependence of c_T, y_H, and y_L on s^t. Clearly, these cutoff points and consumption levels of traders depend only on μ_t, while the consumption level of nontraders depends only on (μ_t, y_{t-1}). Fixing μ_t and using (11) to solve for y_L and y_H as functions of c_T, it is clear that the left hand side of (13) is continuous and strictly monotonic in c_T and is less than Y for $c_T = 0$ and greater than Y as c_T becomes large. Thus, these equations have a unique solution for the equilibrium values of traders' consumption and cutoffs for any money growth rate $\mu \geq 1$. The equilibrium values of the cutoff points and consumption of traders are the unique solution to the equations (11) and (13). These arguments give the following. (For details, see Appendix A.)

Proposition 1: Under the assumption a and N are zero, the equilibrium consumption for home agents is given by

$$c(s^t, y^{t-1}) = \begin{cases} y_{t-1}/\mu_t & if \qquad y_{t-1} \in (y_L(\mu_t), y_H(\mu_t)) \\ c_T(\mu_t) & otherwise \end{cases}$$

where the functions $y_L(\mu), y_H(\mu), c_T(\mu)$ are the solutions to (11) and (13). The consumption for foreign agents is analogous.

In our analysis of asset prices, is useful to use the sequence of budget constraints (6) to substitute out for agent's bond holdings and replace these constraints with a single date 0 constraint on agents transfers of cash between the goods and asset markets. To that end, define date 0 prices $Q(s^t)$ to be the price in terms of dollars in the asset market at date 0 for a dollar delivered in the asset market at date t in state s^t. These prices are given by $Q_0 = 1$, $Q(s^1) = q(s^1)$, and $Q(s^t) = q(s^{t-1}, s_t)Q(s^{t-1})$. As we show in Appendix A, date zero dollar asset prices are determined by the first order condition for home agents who trade

$$(14) \quad \beta^t U'(c_T(\mu_t))g(s^t) = \lambda Q(s^t)P(s^t),$$

where λ is the Lagrange multiplier on home agents' date zero budget constraint. Date zero euro asset price are given by the analogous first order condition for foreign agents. In Appendix A we show that the date zero nominal exchange rate e_0 is given by

$$(15) \quad e_0 = \left(\bar{B} - \bar{B}_h\right)/\bar{B}_h^*.$$

Clearly, this exchange rate exists and is positive as long as $\bar{B}_h < \bar{B}$ and $\bar{B}_h^* > 0$ or $\bar{B}_h > \bar{B}$ and $\bar{B}_h^* < 0$.

We turn now to developing conditions sufficient to guarantee that agents always choose N and a equal to zero. The condition for N to be zero, so that agents in the asset market prefer to save using nominal bonds rather than money is $\int q(s^t, s_{t+1})ds_{t+1} < 1$, which simply ensures that nominal interest rates are positive for all dates t and states s^t. In terms of marginal utilities this condition can be written

$$(16) \quad U'(c_T(\mu_t)) > \beta \int_{\mu_{t+1}} \frac{U'(c_T(\mu_{t+1}))}{\mu_{t+1}} g(s_{t+1}|s^t)ds_{t+1}.$$

The sufficient conditions for a to be zero are more complicated and are presented in Appendix A. These conditions simply requires that the shadow nominal interest rate in the goods market for each individual is positive, regardless of the shock s^t, their current bondholdings and their choice of a in the current period.

3. Exchange rates and interest rates

In the rest of the paper we develop the links between money injections and exchange rates and interest rates. We first develop asset pricing formulas for exchange rates and interest rates. These formulas make clear that a critical link between money injections and these asset prices is how traders' consumption responds to a money injection. We then examine this link and develop formulas for exchange rates and interest rates in terms of the money growth shocks and the underlying parameters.

To develop the asset pricing formulas, recall from (14) that date zero nominal dollar asset prices are given by the marginal utility of a dollar for home traders. Likewise, date zero euro asset prices $Q^*(s^t)$ are given by the analogous marginal utility for the foreign trader. Arbitrage requires that nominal exchange rates satisfy $e(s^t) = e_0 Q^*(s^t)/Q(s^t)$ where e_0 is given by (15). We define the real exchange rate as $x(s^t) = e(s^t)P^*(s^t)/P(s^t)$ which is given by

$$(17) \quad x(s^t) = e_0 \frac{\lambda}{\lambda^*} \frac{U'(c_T^*(\mu_t^*))}{U'(c_T(\mu_t))},$$

Since $P(s^t) = M(\mu^t)/Y$, and likewise for P^*, the nominal exchange rate is thus $e(s^t) = x(s^t)M(\mu^t)/M^*(\mu^{*t})$. At date t in aggregate state s^t, state contingent dollar bond prices are given by

$$(18) \quad q(s^t, s_{t+1}) = \beta \frac{U'(c_T(\mu_{t+1}))}{U'(c_T(\mu_t))} \frac{1}{\mu_{t+1}} g(s_{t+1}|s^t)$$

and likewise for state contingent euro bond prices $q^*(s^t, s_{t+1})$. Define the yield on a one-period nominal bond that pays off one dollar in every state as

$$(19) \quad i(s^t) = -\log\left(\int_{s_{t+1}} q(s^t, s_{t+1}) ds_{t+1}\right)$$

and likewise for the euro yield $i^*(s^t)$. We refer to the yield on an indexed dollar one-period bond that pays off $P(s^t, s_{t+1})/P(s^t)$ dollars in state s_{t+1} as the home real interest rate which is given by $r(s^t) = -\log\left(\int_{s_{t+1}} q(s^t, s_{t+1})\mu(s^{t+1}) ds_{t+1}\right)$ and likewise for the foreign real interest rate $r^*(s^t)$.

From (17)-(19) it is clear that in order to characterize the link between money injections and exchange rates and interest rates we need to determine how trader's consumption in the two countries responds to money injections, namely how $c_T(\mu_t)$ and $c_T^*(\mu_t^*)$ vary with μ_t and μ_t^*.

To see how this response depends on the distribution F, consider a simple example in which y takes on 3 values $y_0 < y_1 < y_2$ with probabilities f_0, f_1, f_2 respectively. We conjecture an equilibrium in which, when money growth is $\bar{\mu}$, agents with the central value of the endowment y_1 choose not to trade and those with low and high endowments y_0 and y_2

choose to trade. Under this conjecture, for money growth shocks μ close to $\bar{\mu}$, the consumption of traders in each country is found from the resource constraint as

$$c_T(\mu) = \frac{y_0 f_0 + y_2 f_2}{f_0 + f_2} + (1 - \frac{1}{\mu})\frac{y_1 f_1}{f_0 + f_2} - \gamma$$

and the values of the cutoffs $y_L\left(c_T\left(\mu\right), \mu\right), y_H\left(c_T\left(\mu\right), \mu\right)$ are found from (11). This conjecture is valid as long as $y_0 < y_L\left(c_T\left(\bar{\mu}\right), \bar{\mu}\right) < y_1 < y_H\left(c_T\left(\bar{\mu}\right), \bar{\mu}\right) < y_2$. A sufficient condition for this conjecture to hold is that

$$(20) \quad y_1/\bar{\mu} = (Y - f_1 y_1/\bar{\mu})/(f_0 + f_2) - \gamma$$

and γ is sufficiently small. Under this condition the consumption of traders when $\mu = \bar{\mu}$ is $y_1/\bar{\mu}$ while that of nontraders is also $y_1/\bar{\mu}$. As γ gets small the cutoffs $y_L\left(c_T\left(\bar{\mu}\right), \bar{\mu}\right)/\bar{\mu}$ and $y_H\left(c_T\left(\bar{\mu}\right), \bar{\mu}\right)/\bar{\mu}$ approach c_T so that the above inequalities hold.

Clearly an increase in the money growth rate μ raises the inflation tax levied on nontraders' real balances. In equilibrium asset prices adjust to redistribute these inflation tax revenues to traders. In this example the number of traders does not vary with the money injection and hence the consumption of each trader increases. Specifically, for μ close to $\bar{\mu}$,

$$(21) \quad \frac{d \log c_T}{d \log \mu} = \frac{(y_1 f_1)/\mu}{c_T(f_0 + f_2)}$$

which is the ratio of the total consumption of nontraders to that of traders.

In general, a money injection increases the total amount consumed by traders and changes the number of agents who choose to become traders. If the number of agents who choose to become traders does not increase very much, the consumption of each trader increases. Of course, if a sufficiently large number of additional agents become traders then the consumption of each trader can actually fall. In Appendix B we give additional examples which elaborate on this point.

4. Volatile exchange rates

Several features of data on real exchange rates for low inflation countries stand out: real exchange rate are much more volatile than inflation or interest rates and they have similar volatility to that of nominal exchange rates. More precisely, real exchange rates are substantially more variable than inflation and interest rate differentials across countries and the variance of changes in real exchange rates is roughly equal to the variance of changes in nominal exchange rates. (See Mussa 1986 for documentation.) The standard model fails to reproduce these observations.

In the segmented markets model the following hold. The more segmented are markets the higher is the variability of real exchange rates relative to inflation rates and the closer is the variability of real and nominal exchange rates. The more persistent is money growth the higher is the variability of real exchange rates relative to interest rates.

To develop these implications we begin by relating money growth and equilibrium asset prices. We make the following assumptions to get analytical results. Let the log of home money growth at date t, $\log \mu_t$, be normally distributed and have constant conditional variance over time. Let $\bar{\mu}$ be defined by $\log \bar{\mu} = E \log \mu_t$ where E is the unconditional expectation. Similarly, let the log of foreign money growth be log-normally distributed with unconditional mean $\bar{\mu}^*$. Let $U(c) = c^{1-\sigma}/(1-\sigma)$ where $\sigma > 0$. Let $\bar{c}_T$ denote the consumption of home traders when home money growth is equal to $\bar{\mu}$ and $\bar{c}_T^*$ be the consumption of foreign traders when foreign money growth is equal to $\bar{\mu}^*$.

To a first order approximation the log of home traders' marginal utility is given by

$$(22) \quad \log U'(c_{Tt}) = \log U'(\bar{c}_T) - \phi \hat{\mu}_t,$$

where $\hat{\mu}_t = \log \mu_t - \log \bar{\mu}$ and

$$(23) \quad \phi = \sigma \frac{d \log c_T}{d \log \mu}$$

evaluated at $\mu = \bar{\mu}$. The parameter ϕ is the elasticity of a home trader's marginal utility with respect to a home money injection. The log of foreign traders' marginal utility is approximated in the analogous fashion with ϕ^* being the elasticity of a foreign trader's marginal utility with respect to a foreign money injection. Clearly these parameters depend on mean money growth rates $\bar{\mu}$ and $\bar{\mu}^*$ respectively. These parameters index the amount of market segmentation. When fixed costs are zero, ϕ and ϕ^* are zero and money injections have no effect on traders' consumption. When fixed costs are positive, ϕ and ϕ^* are nonzero. We will say markets are more segmented the larger are ϕ and ϕ^*.

With these assumptions we analyze the relation between money and exchange rates and interest rates. For the level of the real exchange rate we have

$$(24) \quad \log x_t = \phi \hat{\mu}_t - \phi^* \hat{\mu}_t^* + \bar{x}$$

where $\bar{x} = E \log x_t$. Real yields in the home country are give by

$$(25) \quad r_t = \phi(E_t \hat{\mu}_{t+1} - \hat{\mu}_t) + \bar{r},$$

and likewise in the foreign country. Home and foreign inflation is given by $\pi_{t+1} = \hat{\mu}_{t+1}$ and $\pi_{t+1}^* = \hat{\mu}_{t+1}^*$. One period nominal yields in the home currency are given by

$$(26) \quad i_t = \phi(E_t \hat{\mu}_{t+1} - \hat{\mu}_t) + E_t \hat{\mu}_{t+1} + \bar{\imath}$$

where $\bar{\imath}_t = E i_t$ and likewise for the foreign currency.

The standard model has no fixed costs so traders' consumption does not depend on money growth, the segmentation parameters ϕ and ϕ^* are zero, and from (24), the real exchange rate is constant. In the segmented markets model money shocks have a direct effect on the consumption of traders and thus have an effect on real exchange rates. To illustrate the implications segmented markets more simply, we assume symmetry across countries so that $\phi = \phi^*$.

Consider first the relative volatility of real exchange rates and inflation rates. From (24) we have that

$$\frac{\operatorname{var}\left(\pi_t - \pi_t^*\right)}{\operatorname{var}(\log x_t)} = \frac{1}{\phi^2}.$$

Thus, the more segmented are markets (the higher is ϕ), the higher is the variability of real exchange rates relative to the inflation differential across countries.

Consider next the relative volatility of changes in real and nominal exchange rates.

$$\operatorname{var}(\log e_{t+1} - \log e_t) = \operatorname{var}(\log x_{t+1} - \log x_t + (\log P_{t+1} - \log P_t) - (\log P_{t+1}^* - \log P_t^*))$$

Thus

$$\text{(27)} \qquad \frac{\operatorname{var}(\log e_{t+1} - \log e_t)}{\operatorname{var}(\log x_{t+1} - \log x_t)} = 1 + \frac{1}{2\phi^2}\left[\frac{(1+\phi)\operatorname{var}(\hat{\mu}_t - \hat{\mu}_t^*) + \phi\operatorname{cov}(\hat{\mu}_{t+1} - \hat{\mu}_{t+1}^*,\ \hat{\mu}_t - \hat{\mu}_t^*)}{\operatorname{var}(\hat{\mu}_t - \hat{\mu}_t^*) + \operatorname{cov}(\hat{\mu}_{t+1} - \hat{\mu}_{t+1}^*,\ \hat{\mu}_t - \hat{\mu}_t^*)}\right]$$

Given any process for the money growth differential, as markets become more segmented (ϕ gets higher) this ratio goes to one. For example, if the money growth differential is i.i.d.., (27) reduces to

$$1 + \frac{1}{2}\frac{(1+\phi)}{\phi^2}$$

while if the differential is an AR1 with persistence ρ, (27) reduces to

$$1 + \frac{1}{2}\frac{(1+\phi(1+\rho))}{\phi^2(1+\rho)}.$$

Finally, consider the relative variability of real exchange rates and interest rates. From (25), we have that real interest rates have volatility

$$\operatorname{var}(r_t - r_t^*) = \phi^2 \operatorname{var}\left(E_t(\hat{\mu}_{t+1} - \hat{\mu}_{t+1}^*) - (\hat{\mu}_t - \hat{\mu}_t^*)\right)$$

so that

$$\frac{\operatorname{var}(r_t - r_t^*)}{\operatorname{var}(\log x_t)} = \frac{\operatorname{var}\left(E_t(\hat{\mu}_{t+1} - \hat{\mu}_{t+1}^*) - (\hat{\mu}_t - \hat{\mu}_t^*)\right)}{\operatorname{var}\left(\hat{\mu}_t - \hat{\mu}_t^*\right)}$$

The ratio on the right hand side of this equation is a measure of the persistence of the money growth differentials. Notice that if the money growth differential is i.i.d., this ratio is 1 while if the differential is a random walk, this ratio is zero. More generally, if the money growth differential is an AR1 with persistence ρ, this ratio is $(1-\rho)^2$. Hence, the more persistent are money growth differentials (and inflation differentials) the higher is the volatility of real exchange rates relative to real interest rates. For nominal interest rates

$$\frac{\operatorname{var}(i_t - i_t^*)}{\operatorname{var}(\log x_t)} = \frac{\operatorname{var}((\phi+1)E_t(\hat{\mu}_{t+1} - \hat{\mu}_{t+1}^*) - \phi(\hat{\mu}_t - \hat{\mu}_t^*))}{\operatorname{var}\left(\phi(\hat{\mu}_t - \hat{\mu}_t^*)\right)}$$

Since nominal interest rates are the sum of expected inflation and real interest rates, the volatility of the real exchange rate relative to nominal interest rates is larger the more persistent is the process for the money growth differential and the more segmented are markets. If the money growth differential is i.i.d., then $\operatorname{var}(i_t - i_t^*)/\operatorname{var}(\log x_t)$ is 1. If the money growth differential is a random walk, then $\operatorname{var}(i_t - i_t^*)/\operatorname{var}(\log x_t)$ is $1/\phi^2$. More generally AR1 with persistence ρ,

$$\frac{\operatorname{var}(i_t - i_t^*)}{\operatorname{var}(\log x_t)} = \frac{\left[\phi\,(1-\rho) - \rho\right]^2}{\phi^2}.$$

5. Segmentation in low and high inflation countries

The data on real exchange rates for high inflation countries differs from that in moderate inflation countries in that real exchange rates for these countries are not much more variable than inflation and interest rates and the volatility of real exchange rates is smaller than the volatility of nominal exchange rates. (See Obstfeld 1997 for a discussion of the evidence). In this section we discuss how the degree of market segmentation as measured by the parameter ϕ varies with the average rate of money growth. In particular, we show that if the average rate of inflation is high enough almost all agents choose to pay the fixed cost so that markets are no longer segmented. Thus, as inflation becomes high enough the volatility of real exchanges becomes much smaller than that of nominal exchange rates.

To see this as simply as possible, consider again an example in which y takes on 3 values $y_0 < y_1 < y_2$ with probabilities f_0, f_1, f_2 respectively. Consider the degree of segmentation in a country with low average inflation $\bar{\mu}_A$ and in a country with high average inflation $\bar{\mu}_B$. For the low inflation country assume condition (20) holds and γ is sufficiently small so that

$$y_0 < y_L\left(c_T\left(\bar{\mu}_A\right), \bar{\mu}_A\right) < y_1 < y_H\left(c_T\left(\bar{\mu}_A\right), \bar{\mu}_A\right) < y_2.$$

With a utility function of the form $U(c) = c^{1-\sigma}/(1-\sigma)$,

$$\phi = \sigma \frac{d \log c_T}{d \log \mu} = \frac{\sigma (y_1 f_1)/\bar{\mu}_A}{c_T(f_0 + f_2)}. \tag{28}$$

For the high average inflation country we proceed as follows. Assume

$$1 - \sigma < \frac{Y - \gamma}{Y} \tag{29}$$

It turns out that (29) ensures that agents' utility is sufficiently curved so that a poor agent would choose to pay the fixed cost to trade when everyone else is paying the fixed cost to trade. Let $x_L \in [0, Y - \gamma]$ be the solution to $h(x; Y - \gamma) = 0$ if it exists and let $x_L = 0$ otherwise. Under (29) $x_L > 0$. We then have

Proposition 2. Assume that the support of y is bounded by $\bar{y}$. Under (29) there exists a sufficiently high inflation rate such that all agents are traders and $\phi = 0$.

Proof. We proceed in two steps, we first show that under (29) $x_L > 0$. We then show that if $\bar{\mu}_B > \bar{y}/x_L$, then all agents choose to pay the fixed cost to trade.

To show $x_L > 0$, we need to show that there is a solution to $h(x; Y - \gamma) = 0$ in the interval $(0, Y - \gamma)$. Recall that $h(x; Y - \gamma)$ is minimized at $x = Y - \gamma$ and is negative at this point. Thus it suffices to show that $h(0; Y - \gamma) > 0$. The condition (29) ensures that this inequality holds. Note that $h(0; Y - \gamma) \leq 0$ if (29) is violated.

To see that all agents choose to trade when $\bar{\mu}_B > \bar{y}/x_L$, observe that $c_T = Y - \gamma$, $y_L = x_L \bar{\mu}_B$, $y_H = x_H \bar{\mu}_B$ solve (11) and (13) and that $y_L > \bar{y}$. Thus traders' consumption does not depend on money growth μ and $\phi = 0$.

This proposition implies that as inflation becomes sufficiently high the segmentation effects diminish and real exchange rates becomes much less volatile than nominal exchange rates. It is worth noting that one can construct examples in which the segmentation parameter ϕ declines smoothly with μ.

6. The forward premium anomaly

In the data there is a tendency for high interest rate currencies to appreciate. (See Fama 1984 and Hodrick 1987.) This tendency is puzzling since intuitively one might expect that investors would demand higher interest rates on currencies that are expected to fall in value. In particular, it has been widely documented that, for the currencies of the major industrialized countries over the period of floating exchange rates,

$$(30) \quad Cov\,(i_t - i_t^*, \log e_{t+1} - \log e_t) < 0$$

(see, for example, Backus, Foresi, and Telmer 1998 for a recent discussion). The inequality (30) contradicts the hypothesis of uncovered interest parity, namely that expected returns on bonds denominated in different currencies are equal. Under this hypothesis on average, high interest rate currencies must depreciate while (30) says that in the data the reverse is true.

This puzzle can also be stated in terms of forward exchange rates. To see this note that the forward exchange rate f_t is the price specified in a contract at t in which the buyer has the obligation to transfer at date t+1 f_t dollars and receive 1 euro. The forward premium is forward rate relative to the spot rate f_t/e_t. Arbitrage implies

$$\log f_t - \log e_t = i_t - i_t^*$$

and thus (30) can be restated as

$$Cov\,(\log f_t - \log e_t, \log e_{t+1} - \log e_t) < 0.$$

Thus, there is a tendency for the forward premium and the expected change in exchange rates to move in opposite directions. This observation contradicts the hypothesis that the forward rate is a good predictor of the future exchange rate.

In this section we give sufficient conditions for our model to generate (30). Before doing so, it is useful to develop some intuition about the nature of currency risk and interest rate differentials. In general, the dollar returns R_{t+1} on any asset must satisfy the standard asset pricing condition

$$(31) \quad 1 = E_t m_{t+1} R_{t+1}$$

where the pricing kernel m_{t+1} is given by

$$(32) \quad m_{t+1} = \beta \frac{U'(c_{Tt+1}) P_t}{U'(c_{Tt}) P_{t+1}}.$$

It follows that the risk of this asset and hence its expected return relative to a dollar bond is determined by the covariance of the kernel and the return according to

$$(33) \quad \frac{E_t R_{t+1}}{1+i_t} - 1 = -cov_t(m_{t+1}, R_{t+1}).$$

To investigate the implications of (33) for the expected return on euro bonds, consider converting a dollar at t to $1/e_t$ euros, buying a euro bond paying interest $(1+i_t^*)$ and then converting the resulting euros back to dollars at $t+1$ at exchange rate e_{t+1} to give a dollar return of $R_{t+1} = (1+i_t^*)e_{t+1}/e_t$. Hence the expected return on the euro bond relative to the dollar bond is

$$(34) \quad \frac{1+i_t^*}{1+i_t} \frac{E_t e_{t+1}}{e_t} - 1 = -cov_t(m_{t+1}, (1+i_t^*)\frac{e_{t+1}}{e_t}).$$

The risk on this transaction is determined by the covariance of the marginal utility of a dollar at $t+1$, $U'(c_{Tt+1})/P_{t+1}$, and the exchange rate e_{t+1}. Holding fixed foreign variables, when the marginal utility of a dollar at $t+1$ is high dollars are valuable and thus e_{t+1} is low. Hence the foreign bond is risky since its payoff is low when the marginal utility of a dollar is high and vice-versa. To investigate this risk further use (17) to get

$$\frac{e_{t+1}}{e_t} = \left[\beta \frac{U'\left(c_{Tt+1}^*\right)}{U'\left(c_{Tt}^*\right)} \frac{P_t^*}{P_{t+1}^*}\right] / \left[\beta \frac{U'\left(c_{Tt+1}\right)}{U'\left(c_{Tt}\right)} \frac{P_t}{P_{t+1}}\right].$$

Hence the more variable is the marginal utility of a dollar the higher is the exchange rate risk on the euro bond and hence the higher is its expected return relative to a dollar bond.

In a standard model the conditional variance of the marginal utility of a dollar is constant and hence the risk on the euro bond given by the right-hand side of (34) is constant.

Thus, in the standard model when dollar interest rates are high relative to euro rates the dollar is expected to depreciate contradicting (30). For our model to generate (30) when dollar interest rates are relatively high we need the risk on euro bonds to fall enough so that the dollar actually appreciates.

Our model can generate this pattern as follows. High money growth at home leads both to high interest rates and less segmented markets. With less segmented markets the marginal utility of a dollar is expected to be less variable and hence exchange rate risk on the euro bond is low. If market segmentation falls enough the dollar will appreciate.

The critical feature of our model that allows it to generate (30) is that the degree of market segmentation changes when money growth rates change. To allow for this we need to expand our first order approximation to the marginal utility of traders in (22) to a second order approximation

$$\log U'\left(c_T\left(\mu_t\right)\right) = \log U'\left(c_T\left(\bar{\mu}\right)\right) - \phi\hat{\mu}_t + \frac{1}{2}\eta\hat{\mu}_t^2 \tag{35}$$

where

$$\phi = -\frac{d\log U'(c_T\left(\mu\right))}{d\log\mu}$$

$$\eta = -\frac{d\phi}{d\log\mu}$$

evaluated at $\mu = \bar{\mu}$. The parameter η measures how much the segmentation parameter ϕ changes as money growth changes. We assume $\phi > 0$ and $\eta > 0$ and the log of money growth in both countries follows an AR1 so that $\hat{\mu}_{t+1} = \rho\hat{\mu} + \varepsilon_{t+1}$ where ε_{t+1} is independent across countries and normal with mean zero and variance σ_ε^2.

To compute the covariance between interest rates and exchange rates, we solve for both in terms of the nominal dollar pricing kernel and the analogous nominal euro kernel.

Given our assumption that the stochastic processes for money growth in the two countries are independent and identical then the stochastic processes for the pricing kernels are also independent and identical.

In terms of these pricing kernels, the interest rate differential is given by

$$(36) \quad i_t - i_t^* = \log E_t m_{t+1}^* - \log E_t m_{t+1}$$

while the change in the nominal exchange rate is given by

$$(37) \quad E_t \log e_{t+1} - \log e_t = E_t \log m_{t+1}^* - E_t \log m_{t+1}.$$

Given our approximation (35), it is no longer the case that the pricing kernels are conditionally lognormal, but the standard formulas relating $\log E_t m_{t+1}$ to the conditional moments of $\log m_{t+1}$ can be extended to

$$(38) \quad \log E_t m_{t+1}^* - \log E_t m_{t+1} = \frac{1}{2} \log \left(1 - \eta \sigma_\varepsilon^2\right) - \frac{3}{4} \frac{\eta^2 \sigma_\varepsilon^4}{1 - \eta \sigma_\varepsilon^2} +$$

$$E_t \log m_{t+1}^* - E_t \log m_{t+1} + \frac{1}{2} \frac{1}{1 - \eta \sigma_\varepsilon^2} \left(Var_t \log m_{t+1}^* - Var_t \log m_{t+1}\right)$$

when $1 - \eta \sigma_\varepsilon^2 > 0$[1] . We derive (38) in Appendix C. Using (37) and (38) we have that

$$Cov\left(i_t - i_t^*,\ E_t \log e_{t+1} - \log e_t\right) =$$

$$2Var\left(E_t \log m_{t+1}\right) + \frac{1}{1 - \eta \sigma_\varepsilon^2} Cov\left(E_t \log m_{t+1}, Var_t \log m_{t+1}\right)$$

so that the covariance (30 is negative if

$$2Var\left(E_t \log m_{t+1}\right) < -\frac{1}{1 - \eta \sigma_\varepsilon^2} Cov\left(E_t \log m_{t+1}, Var_t \log m_{t+1}\right).$$

[1]The resulting pricing kernel is a variation on a discrete time version of the one analyzed by Constantinides (1992).

We have

Proposition 3. The following conditions are sufficient for (30) to hold: i) segmentation is large enough in that ϕ satisfies

$$\phi > \frac{\rho}{1-\rho}$$

and ii) segmentation is sensitive enough to money growth in that η satisfies

$$\eta > \frac{1-\rho^2}{\sigma_\varepsilon^2} = \frac{1}{Var\hat{\mu}_t}.$$

Proof. Note

$$E_t \log m_{t+1} = \log\beta - \log\bar{\mu} + \frac{1}{2}\eta\sigma_\varepsilon^2 + [\phi(1-\rho)-\rho]\hat{\mu}_t - \frac{1}{2}\eta\left(1-\rho^2\right)\hat{\mu}_t^2,$$

$$Var_t \log m_{t+1} = (\phi+1)^2\sigma_\varepsilon^2 + \frac{3}{4}\eta^2\sigma_\varepsilon^4 - 2\sigma_\varepsilon^2(\phi+1)\eta\rho\hat{\mu}_t + \sigma_\varepsilon^2\eta^2\rho^2\hat{\mu}_t^2.$$

Computing

$$(39) \quad Var\left(E_t \log m_{t+1}\right) = [\phi(1-\rho)-\rho]^2 Var\hat{\mu}_t + \frac{\eta^2}{4}\left(1-\rho^2\right)^2 Var\hat{\mu}_t^2,$$

and

$$(40) \quad Cov\left(E_t \log m_{t+1}, Var_t \log m_{t+1}\right) =$$

$$-[\phi(1-\rho)-\rho]\, 2\sigma_\varepsilon^2(\phi+1)\eta\rho Var\hat{\mu}_t - \sigma_\varepsilon^2\eta^3\rho^2\frac{1}{2}\left(1-\rho^2\right)Var\hat{\mu}_t^2$$

The first inequality ensures that $Cov\left(E_t \log m_{t+1}, Var_t \log m_{t+1}\right)$ is negative. Thus, comparing separately terms on $Var\hat{\mu}_t$ and $Var\hat{\mu}_t^2$, we have that (30) holds if

$$(41) \quad [\phi(1-\rho)-\rho] < \frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2}(\phi+1)\rho$$

and

$$(42) \quad \left(1-\rho^2\right) < \frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2}\rho^2.$$

Notice that these two equations can be written as

$$\frac{\phi\,(1-\rho)-\rho}{(\phi+1)\rho} < \frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2},$$

$$\frac{1-\rho^2}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2},$$

and for $\phi > 0$ and $\rho \in (0,1)$,

$$\frac{\phi\,(1-\rho)-\rho}{(\phi+1)\rho} < \frac{1-\rho}{\rho} < \frac{1-\rho^2}{\rho^2}$$

thus if (42) holds then (41) holds too.

To see that $\eta > (1-\rho^2)/\sigma_\varepsilon^2$ implies (42) observe that this inequality implies

$$\frac{1}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)\rho^2} = \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)} + \frac{\eta\sigma_\varepsilon^2}{\rho^2}.$$

Thus

$$\frac{1-\eta\sigma_\varepsilon^2}{\rho^2} < \frac{\eta\sigma_\varepsilon^2}{(1-\rho^2)},$$

which implies (42).

7. Conclusion

We have constructed a simple model with endogenously segmented asset markets and have shown that these frictions are a potentially important part of a complete model of exchange rates. Relative to the existing literature we make several contributions. First, in existing models of market segmentation markets can be highly segmented only if velocity is extremely low while in our model that need not be the case. Second, we show that making market segmentation endogenous leads to new implications for the relation between money and exchange rates. Specifically, the model implies that segmentation effects vary systematically with the level of inflation. In the cross section this feature implies that market segmentation effects are less important in high inflation countries. In the time series this feature implies that exchange rate risk varies systematically with inflation.

As Backus, Foresi, and Telmer (1995) and Engel (1996) have emphasized, standard monetary models have no chance of producing the forward premium anomaly because they generate a constant risk premium as long as the underlying driving processes have constant conditional variances. Backus, Foresi and Telmer argue that empirically it is unlikely that one can generate this anomaly from having nonconstant conditional variances of the driving processes. Instead they argue that what is needed is a model that generates nonconstant risk premia from driving processes that have constant conditional variances. Our model is an attempt to do exactly that.

To keep the analysis tractable we have imposed conditions that ensure no money is every held over in the goods markets. When money is sometimes held over the distribution of these money holdings across agents becomes a state variable and the analysis is very complicated. We think that it would be interesting to explore these segmentation effects in such a model but we leave it for future work.

References

Alvarez, Fernando, and Andrew Atkeson. 1997. Money and exchange rates in the Grossman-Weiss-Rotemberg model. *Journal of Monetary Economics*, vol. 40, 619-640.

Backus David, Silverio Foresi, and Chris Telmer 1995. Interpreting the forward premium anomaly. *Canadian Journal of Economics,* vol. 28 (november), S108-119.

Backus David, Silverio Foresi, and Chris Telmer 1998. Affine models of currency pricing: Accounting for the forward premium anomaly. Mimeo, Stern School of Business, New York University.

Baumol, William J. 1952. The transactions demand for cash: an inventory theoretic approach. *Quarterly Journal of Economics,* 66 (November): 545–56.

Constantinides, George 1992. A theory of the nominal term structure of interest rates. *The Review of Financial Studies*, vol. 5(4), 531-552.

Engel, Charles. 1996. The forward discount anomaly and the risk premium: a survey of recent evidence. *Journal of Empirical Finance*, vol. 3, 123-192.

Fama, Eugene. 1984. Forward and spot exchange rates. *Journal of Monetary Economics,* vol. 14, 319-338.

Fuerst, Timothy S. 1992. Liquidity, loanable funds, and real activity. *Journal of Monetary Economics*, vol. 29 (no. 1, March): 3-24.

Grilli, Vittorio and Nouriel Roubini. 1992. Liquidity and exchange rates. *Journal of International Economics*, vol 32, 339-352.

Hodrick, Robert. 1987. The empircal evidence on the efficiency of forward and futures foreign exchange markets. (Chur: Harwood).

Lucas, Robert E., Jr. 1982. Interest rates and currency prices in a two country world. *Journal of Monetary Economics,* vol. 10, 335-360.

Lucas, Robert E., Jr. 1990. Liquidity and interest rates. *Journal of Economic Theory,* vol 50(2), 237-264.

Mussa, Michael 1986. Nominal exchange rate regimes and the behavior of real exchange rates: evidence and implications", *Carnegie-Rochester conference series on public policy* 25 (autumn), 117-214.

Obstfeld, Maurice. 1997. Scandanivian journal ?????"

Obstfeld, Maurice and Kenneth Rogoff. 1995. Exchange rate dynamics redux. *Journal of Political Economy,* vol 103, 624-660.

Rotemberg, Julio, J. 1985. Money and the terms of trade. *Journal of International Economics*, vol 19, 141-160.

Schlagenhauf, Don E. and Jeffrey Wrase. 1995. Liquidity and real activity in a simple open economy model. *Journal of Monetary Economics,* vol. 35, 431-461.

Tobin, James. 1956. The interest-elasticity of the transactions demand for cash. *Review of Economics and Statistics* 38 (no. 3, August): 241–47.

Appendix A

In this appendix we characterize equilibria in which agents never hold over cash in either the goods or asset market, so that a, a^* and N and N^* are always zero. We do so in several steps. We first characterize the optimal choice of c and x for agents in the home country given prices and arbitrary rules for m, a, and z, and summarize these results in lemma 2. We then characterize the trading rule z for agents in the home country given an arbitrary rule for m, a and the optimal rules for c and x and summarize these results in lemma 3. The analogous results for agents in the foreign country hold immediately. These lemmas complete the proof of proposition 1 in the text. In lemma 4, we characterize the equilibrium bondholdings and then compute the initial exchange rate e_0. In lemma 5, we provide sufficient conditions on the money growth process and endowments process to ensure that a, a^* and N and N^* are always zero.

First use the sequence of budget constraints (6) to substitute out for agent's bond holdings and replace these constraints with a single date 0 constraint on agents transfers of cash between the goods and asset markets. Any bounded allocation and bondholdings that satisfies (6) also satisfies a date 0 budget constraint

$$(43) \quad \sum_{t=0}^{\infty} \int Q(s^t) \int_{y^{t-1}} \Big\{ P(s^t) \left[x(s^t, y^{t-1}) + \gamma \right] z(s^t, y^{t-1}) +$$
$$N(s^t, y^{t-1}) - N(s^{t-1}, y^{t-2}) \Big\} f(y^{t-1}) dy^{t-1} ds^t \leq \bar{B}.$$

Thus, the home consumer's problem can be restated as follows. Choose real money holdings m and a, trading rule z, consumption and transfers c and x and cash in the asset market N, subject to constraints (3-5) and (43). Define date 0 asset prices for the foreign consumer analogously, with $Q^*(s^t)$ to be the price in terms of euros in the asset market at date 0 for a dollar delivered in the asset market at date t in state s^t.

Consider first a home agent's optimal choice of consumption $c(s^t, y^{t-1})$ and transfers of dollar real balances $x(s^t, y^{t-1})$ given prices $Q(s^t), P(s^t)$, arbitrary feasible choices of real money holdings $m\,(s^t, y^{t-1})$ and $a(s^t, y^{t-1})$ and a trading rule $z(s^t, y^{t-1})$. These choices maximize the Lagrangian corresponding to the consumer's problem. Let $\nu(s^t, y^{t-1})$ be the multiplier on (3), and λ be the multiplier on (43). The first order condition corresponding to c is

$$\beta^t U'(c(s^t, y^{t-1})g(s^t)f(y^{t-1}) = \nu(s^t, y^{t-1}).$$

The first order condition corresponding to x is

$$\lambda Q(s^t)P(s^t)z(s^t, y^{t-1})f(y^{t-1}) = \nu(s^t, y^{t-1})z(s^t, y^{t-1}).$$

For those states such that $z(s^t, y^{t-1}) = 1$, these two first order conditions imply

$$\beta^t U'(c(s^t, y^{t-1}))g(s^t) = \lambda Q(s^t)P(s^t). \tag{44}$$

Since all agents are identical at date 0, the multipliers in the Lagrangian are the same for all agents. We summarize this discussion as follows

Lemma 2. All home agents who choose to pay the fixed cost for a given aggregate state s^t have identical consumption $c(s^t, y^{t-1}) = c_T(s^t)$, for some function c_T. Agents who choose not to pay the fixed cost have zero transfers and consumption

$$c(s^t, y^{t-1}) = m(s^t, y^{t-1}) - a(s^t, y^{t-1}). \tag{45}$$

The analogous result for foreign traders is immediate.

Next consider a home agent's optimal choice of whether to pay the fixed cost to trade given prices $Q(s^t)$, $P(s^t)$ and arbitrary feasible choices of real money holdings in the goods market $m(s^t, y^{t-1})$, $a(s^t, y^{t-1})$. From Lemma 2, we have the form of the optimal consumption and transfer rules corresponding to the choices of $z = 1$ and $z = 0$. Substituting these rules into (7) and (43) gives the problem of choosing $c_T(s^t)$ and $z(s^t, y^{t-1})$ to maximize

$$\begin{aligned}(46)\quad & \sum_{t=1}^{\infty} \beta^t \int \int U(c_T(s^t))z(s^t, y^{t-1})g(s^t)f(y^{t-1})ds^t dy^{t-1}+ \\ & \sum_{t=1}^{\infty} \beta^t \int \int U(m(s^t, y^{t-1}) - a(s^t, y^{t-1}))(1 - z(s^t, y^{t-1}))g(s^t)f(y^{t-1})ds^t dy^{t-1}\end{aligned}$$

subject to the constraint

$$\begin{aligned}(47)\quad & \bar{B}_h + e_0\bar{B}_f \geq \sum_{t=1}^{\infty} \int \int Q(s^t) \left[N(s^t, y^{t-1}) - N(s^{t-1}, y^{t-2})\right] f(y^{t-1})ds^t dy^{t-1}+ \\ & \sum_{t=1}^{\infty} \int \int Q(s^t)P(s^t) \left[c_T(s^t) + \gamma - (m(s^t, y^{t-1}) - a(s^t, y^{t-1}))\right] z(s^t, y^{t-1})f(y^{t-1})ds^t dy^{t-1}.\end{aligned}$$

Let η denote the Lagrange multiplier on (47) and consider the following variational argument. Consider a state (s^t, y^{t-1}). The increment to the Lagrangian of setting $z(s^t, y^{t-1}) = 1$ in this state is

$$\begin{aligned}(48)\quad & \beta^t U\left(c_T(s^t)\right) g(s^t)f(y^{t-1})- \\ & \eta Q(s^t)P(s^t) \left[(c_T(s^t) + \gamma) - (m(s^t, y^{t-1}) - a(s^t, y^{t-1}))\right] f(y^{t-1})\end{aligned}$$

which is simply the direct utility gain $U\left(c_T(s^t)\right)$ minus the cost of the required transfers. The increment to the Lagrangian of setting $z(s^t, y^{t-1}) = 0$ in this state is

$$(49)\quad \beta^t U\left(\left(m(s^t, y^{t-1}) - a(s^t, y^{t-1})\right)\right) g(s^t)f(y^{t-1})$$

which is simply the direct utility gain since there are no transfers. The first order condition with respect to c_T is

$$\beta^t U'(c_T(s^t))g(s^t) = \eta Q(s^t)P(s^t).$$

Subtracting (49) from (48) and using (14), give the cutoff rules defined by (10) and (11). More formally, we have

Lemma 3: Given home traders' consumption $c_T(s^t)$ agents choose $z(s^t, y^{t-1}) = 0$ if $m(s^t, y^{t-1}) - a(s^t, y^{t-1}) \in \left(\frac{y_L\left(c_T(s^t),\mu_t\right)}{\mu_t}, \frac{y_H\left(c_T(s^t),\mu_t\right)}{\mu_t}\right)$ and they choose $z(s^t, y^{t-1}) = 1$ otherwise.

These lemmas complete the proof of proposition 1. To complete our asset pricing formulas given in (14) we need to compute the equilibrium value of the multiplier λ. Given the equilibrium values of consumption computed in proposition 1 and (14) we have that λ solves

$$(50) \quad \sum_{t=1}^{\infty} \int \beta^t U'(c_T(\mu_t)) \int_{y_L(\mu_t)}^{y_H(\mu_t)} \frac{M(s^t)}{Y} \left[c_T(\mu_t) + \gamma - \frac{y}{\mu_t}\right] f(y) dy g(s^t) ds^t = \frac{\bar{B}}{\lambda},$$

and likewise for λ^*.

The bondholdings in this equilibrium have a simple form. For each aggregate state s^t, all agents in the home country purchase an identical portfolio of bonds. That is, $B(s^t, s_{t+1}, y^{t-1}, y_t)$ does not depend on individual histories y^{t-1}. Likewise, all agents in the foreign country purchase a different portfolio of bonds which also does not depend on individual histories y^{*t-1}, To see this note that we can iterate on the sequence of budget constraints (6) for the home agents to get

$$(51) \quad B(s^t, s_{t+1}, y^{t-1}, y_t) = P(s^{t+1}) \left[x(s^{t+1}, y_t) + \gamma\right] z(s^{t+1}, y_t) +$$

$$\sum_{k=t+2}^{\infty} \int_{s^k|s^{t+1}} \frac{Q(s^k)}{Q(s^{t+1})} \int_{y_{k-1}} P(s^k) \left[x(s^k, y_{k-1}) + \gamma\right] z(s^k, y_{k-1}) f(y_{k-1}) dy_{k-1} ds^k.$$

Since the right side of (51) does not depend on y^{t-1}, neither does the left side. When $m(s^t, y^{t-1}) = y_{t-1}/\mu_t$, the money market clearing condition (9) and iteration on the sequence of government budget constraints (1) gives

$$(52)\quad B(s^{t+1}) = \int_{y_t} P(s^{t+1}) \left[x(s^{t+1}, y_t) + \gamma\right] z(s^{t+1}, y_t) f(y_t) dy_t +$$

$$\sum_{k=t+2}^{\infty} \int_{s^k|s^{t+1}} \frac{Q(s^k)}{Q(s^{t+1})} \int_{y_{k-1}} P(s^k) \left[x(s^k, y_{k-1}) + \gamma\right] z(s^k, y_{k-1}) f(y_{k-1}) dy_{k-1} ds^k.$$

Thus, each agent in the home country, regardless of their history, buys an equal share of the new debt issued by the home country government. The analogous argument applies to agents in the foreign country. Using (51) and (52) and the analogs to these equations for the foreign agents, we have that at date $t-1$ in aggregate state s^{t-1}, home and foreign agents buy the following bonds to pay off at t in state s^t, y^{t-1} and s^t, y^{*t-1}.

Lemma 4. For all realizations of s_t and y_{t-1} such that the home agent would be a non-trader at date t, so $y_{t-1} \in (y_L(\mu_t), y_H(\mu_t))$, he buys bonds providing for a payoff that does not depend on his endowment shock, namely

$$(53)\quad B(\mu^t, y^{t-1}) = \int_{s_{t+1}} q(s^{t+1}) B(s^{t+1}) ds_{t+1}.$$

For all realizations of y_{t-1} such that the home agent would be a trader, he buys bonds providing for the same constant payoff as a non-trader plus the contingent payoff required to pay for the appropriate transfer at date t, that is

$$(54)\quad B\left(s^t, y_{t-1}\right) = P(s^t)\left[x(s_t, y_{t-1}) + \gamma\right] + \int_{s_{t+1}} q(s^{t+1}) B(s^{t+1}) ds_{t+1}.$$

Likewise, the foreign agent buys bonds paying off

$$B^*(s^t, y^{*t-1}) = \int_{s_{t+1}} q^*(s^{t+1}) B^*(s^{t+1}) ds_{t+1}$$

in those states s^t, y^{*t-1} such that he would be a non-trader at date t (so $y^*_{t-1} \in (y_L(\mu^*_t), y_H(\mu^*_t))$) and bonds paying off

$$B^*\left(s^t, y^*_{t-1}\right) = P^*(s^t)\left[x^*(s_t, y^*_{t-1}) + \gamma\right] + \int_{s_{t+1}} q^*(s^{t+1})B^*(s^{t+1})ds_{t+1}$$

in those states s^t, y^{*t-1} such that he would be a trader at date t.

In terms of the initial exchange rate $e(s^0)$, observe that when a and N are equal to zero, (47) and iteration on (1) implies that $\bar{B} = \bar{B}_h + e(s^0)\bar{B}^*_h$. Thus, in equilibrium

$$e(s^0) = \left(\bar{B} - \bar{B}_h\right)/\bar{B}^*_h.$$

Clearly, this exchange rate exists and is positive as long as $\bar{B}_h < \bar{B}$ and $\bar{B}^*_h > 0$ or $\bar{B}_h > \bar{B}$ and $\bar{B}^*_h < 0$.

We now turn to the problem of developing conditions sufficient to ensure that agent never want to store cash in the goods market. Assume that agents have CRRA utility of the form $U(c) = c^{1-\sigma}/(1-\sigma)$. Let $Q(s^t)$ and $P(s^t)$ be the prices constructed above when a and N are assumed equal to zero. Consider first the consumption of an agent who deviates from the strategy of never holding cash from one period to the next in the goods market. From Lemmas 2 and 3, we have that, holding fixed a plan $\{a_t(s^t, y^{t-1})\}$ for holding cash in the goods market, this deviant agent's consumption choices are similar to those of an agent who does not hold cash in the goods market. In particular, in those states of nature in which the deviant choose to pay to the fixed cost to trade, from Lemma 2, his consumption satisfies the first order condition

$$\beta^t U'(c^d_T(s^t))g(s^t) = \eta^d Q(s^t)P(s^t)$$

where η^d is the Lagrange multiplier on this agent's date zero budget constraint. Thus, in those states in which the deviant agent pays the fixed cost to trade, he equates his marginal rate of substitution to that of other traders who do not deviate. Given the assumption of CRRA utility, this implies that $c_T^d(s^t) = \theta c_T(\mu_t)$ for all s^t for some fixed factor of proportionality θ. In those states of nature in which the deviant agent does not choose to pay the fixed cost, his consumption is $c^d(s^t, y^{t-1}) = m^d(s^t, y^{t-1}) - a^d(s^t, y^{t-1})$, and his decision whether to pay the fixed cost is determined by the cutoffs $y_L(\theta c_T(\mu_t), \mu_t)$ and $y_H(\theta c_T(\mu_t), \mu_t)$ described in Lemma 3. Using the fact that $m^d(s^t, y^{t-1}) = (y_{t-1} + a^d(s^{t-1}, y^{t-2}))/\mu_t$ and, in the event that the deviant agent pays the fixed cost, $x^d(s^t, y^{t-1}) = \theta c_T(\mu_t) - (m_t^d(s^t, y^{t-1}) - a^d(s^t, y^{t-1}))$, the factor of proportionality θ (and the implied Lagrange multiplier η^d) corresponding to any fixed plan $\{a_t(s^t, y^{t-1})\}$ for holding cash in the goods market must be set so that the deviant agent's date zero budget constraint holds with equality. The relevant budget constraint is written

$$\bar{B}_h + e_0 \bar{B}_f =$$

$$\sum_{t=1}^{\infty} \int \int Q(s^t) P(s^t) \left[\theta c_T(\mu_t) + \gamma - (m(s^t, y^{t-1}) - a(s^t, y^{t-1}))\right] z(s^t, y^{t-1}) f(y^{t-1}) ds^t dy^{t-1}$$

where $z(s^t, y_{t-1}) = 1$ if $(y_{t-1} + a^d(s^{t-1}, y^{t-2}))/\mu_t - a^d(s^t, y^{t-1}) \in [y_L(\theta c_T(\mu_t), \mu_t)/\mu_t, y_H(\theta c_T(\mu_t), \mu_t)/\mu_t]$ and $z(s^t, y_{t-1}) = 0$ otherwise.

Next observe that, since the cutoffs $y_L(\theta c_T(\mu_t), \mu_t)$ and $y_H(\theta c_T(\mu_t), \mu_t)$ are monotonically increasing in θ for all values of μ_t, no deviant agent would choose a plan $\{a_t(s^t, y^{t-1})\}$ for holding cash in the goods market such that the implied factor of proportionality θ was so small such that

$$y_H(\theta c_T(\mu_t), \mu_t) \leq y_L(c_T(\mu_t), \mu_t) \tag{55}$$

for all possible realizations of μ_t. To see this, observe that the consumption of such a deviant agent would lie below the consumption we have constructed for an agent never holds cash in the goods market in every possible state of nature s^t, y^{t-1}, and thus the utility of such a deviant agent would have to be lower than that of an agent who never held cash in the goods market.

Lemma 5. Let $\bar{\theta}$ to be the supremum over the set of θ which satisfy (55). Then, it is optimal for a home agent to never to hold over cash in the asset market if (16) holds for all values of s^t. It is optimal for a home agent to never hold over cash in the goods market if, for all $a \geq 0, \mu_t$ and $\theta \geq \bar{\theta}$

$$U'(\frac{y_H(\theta c_T(\mu_t), \mu_t)}{\mu_t}) > \beta \int_{\mu_{t+1}} \int_{y_L(\theta c_T(\mu_{t+1}),\mu_{t+1})-a}^{y_H(\theta c_T(\mu_{t+1}),\mu_{t+1})-a} U' \left(\frac{y_t + a}{\mu_{t+1}} \right) \frac{f(y_t)}{\mu_{t+1}} g(s_{t+1}|s^t) dy_t ds_{t+1} + \tag{56}$$

$$\beta \int_{\mu_{t+1}} \frac{U'(\theta c_T(\mu_{t+1}))}{\mu_{t+1}} [F(y_L(\theta c_T(\mu_{t+1}), \mu_{t+1}) - a) + 1 - F(y_H(\theta c_T(\mu_{t+1}), \mu_{t+1}) - a)] g(s_{t+1}|s^t) d\mu_{t+1}$$

Likewise, it is optimal for a foreign agent to never hold over cash in the asset or goods markets if the analogs to (16) and (56) hold.

Proof. When (16) holds, nominal interest rates in the asset market are positive, so agents holds over no cash in the asset market. Given any plan $\{a_t(s^t, y^{t-1})\}$ for holding cash in the goods market and associated value of θ, the highest consumption that a deviant agent could have at date t is $y_H(\theta c_T(\mu_t), \mu_t)/\mu_t$ and thus the smallest marginal utility of consumption he could have at that date is $U'(y_H(\theta c_T(\mu_t), \mu_t)/\mu_t)$. The terms on the right-hand side are the expected value of the product of the marginal utility of consumption and the return to holding currency in the goods market $(1/\mu_{t+1})$ at date $t+1$. Thus, condition (56) ensures that such an agent always prefers to consume his real balances at t rather than carry them over into period $t+1$ at rate of return $1/\mu_{t+1}$. Therefore, (56) implies that there is no plan for holding cash in the goods market that gives higher utility than the plan of never holding cash in the goods market.

Appendix B

Let all agents have identical constant endowment y. In equilibrium a fraction α of agents are traders and purchase the money injection from the government and $(1-\alpha)$ are nontraders. Since agents must be indifferent between trading and not trading the following condition must hold

$$(57)\quad U(c_T) - U'(c_T)(c_T + \gamma - y/\mu) = U(y/\mu).$$

There are two solutions for c_T to this equation: one with $c_T + \gamma > y/\mu$ and one with $c_T + \gamma < y/\mu$. When the money injection is positive traders are purchasing real balances from the government so the equilibrium solution satisfies $c_T + \gamma > y/\mu$. The fraction α adjusts to satisfy the resource constraint. Differentiating (57) with respect to μ gives

$$(58)\quad \frac{dc_T}{d\mu} = \frac{[U'(y/\mu) - U'(c_T)]y/\mu^2}{U''(c_T)[c_T + \gamma - y/\mu]} < 0.$$

Again an increase in the money growth rate μ redistributes inflation tax revenues from non-traders to traders. Here, however, the number of traders increases so much that the amount of consumption per trader actually falls.

Next, consider the case in which y has a continuous density. Differentiating (11)-(13) gives

$$(59)\quad \left\{[F(y_L) + 1 - F(y_H)] + \mu f(y_L)[c_T + \gamma - \frac{y_L}{\mu}]\eta_L - \mu f(y_H)[c_T + \gamma - \frac{y_H}{\mu}]\eta_H\right\} \frac{dc_T}{d\mu}$$
$$= [\frac{y_L}{\mu} - c_T - \gamma]f(y_L)\frac{y_L}{\mu} + [c_T + \gamma - \frac{y_H}{\mu}]f(y_H)\frac{y_H}{\mu} + \frac{1}{\mu}\int_{y_L}^{y_H} \frac{y}{\mu} f(y)dy$$

where

$$\eta_i = \frac{U''(c_T)(c_T + \gamma - y_i/\mu)}{U'(c_T) - U'(y_i/\mu)}.$$

From (11) it follows that $y_L/\mu < c_T < y_H/\mu - \gamma$. Thus η_H and η_L are positive and so is the term in brackets on the left hand side of (59). On the right hand side of (59) the first two terms are negative and the last is positive, so without further restrictions its sign is ambiguous. Intuitively, in our first example, the densities $f(y_L) = f(y_H) = 0$, so that (59) reduces to

$$\frac{dc_T}{d\mu} = \frac{1}{\mu} \frac{\int_{y_L}^{y_H} \frac{y}{\mu} f(y) dy}{[F(y_L) + 1 - F(y_H)]} > 0.$$

Intuitively, in our second example $f(y_L) = \infty$ and $f(y_H) = 0$ so that, in the limit (59) reduces to (58).

Consider now a third example, in which y is uniform on $[0, 1]$ and the utility function is $U(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma = 2$. With these preferences, the cutoff rules $y_L(c_T, \mu), y_H(c_T, \mu)$ solving (11) are

(60) $$y_i(c_T, \mu) = \mu(c_T + \gamma/2 \pm c_T[\left(1 + \frac{\gamma}{2c_T}\right)^2 - 1]^{1/2}).$$

In this case, the right hand side of (59) simplifies to $\gamma(y_H - y_L)/2\mu$ which is positive. Thus, under these restrictions $dc_T/d\log\mu$ is positive.

Appendix C

To obtain (38) we use the following result. If x is normally distributed with mean zero and variance σ^2 and satisfies $1 - 2b\sigma^2 > 0$, then

$$E\exp\left(ax + bx^2\right) = \exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1 - 2b\sigma^2}\right)\left(\frac{1}{1 - 2\sigma^2 b}\right)^{1/2}.$$

To see this note that

$$E\exp\left(ax+bx^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(ax+bx^2\right)\exp\left(-\frac{x^2}{2\sigma^2}\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left\{2\sigma^2ax+\left(2\sigma^2b-1\right)x^2\right\}\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left\{-\left(1-2\sigma^2b\right)x^2+2\sigma^2ax-\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)+\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)\right\}\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{1}{2\sigma^2}\left(\left(1-2\sigma^2b\right)^{1/2}x-\frac{\sigma^2a}{(1-2\sigma^2b)^{1/2}}\right)^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{(1-2\sigma^2b)}{2\sigma^2}\left(x-\frac{\sigma^2a}{(1-2\sigma^2b)}\right)^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\left(\frac{1}{1-2\sigma^2b}\right)^{1/2}.$$

Now, to derive (38), note that our approximation to the pricing kernel is

$$\log m_{t+1}=\log\beta-\log\bar{\mu}-(\phi+1)\hat{\mu}_{t+1}+\frac{1}{2}\eta\hat{\mu}_{t+1}^2+\phi\hat{\mu}_t-\frac{1}{2}\eta\hat{\mu}_t^2.$$

Using our assumptions that $\hat{\mu}_{t+1}=\rho\hat{\mu}_t+\varepsilon_{t+1}$ and that ε_{t+1} is normal with mean zero and variance σ_ε^2, this equation can be written

$$\log m_{t+1}=\log\beta-\log\bar{\mu}+\left(\phi(1-\rho)-\rho\right)\hat{\mu}_t-\frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2+\left(\rho\eta\hat{\mu}_t-(\phi+1)\right)\varepsilon_{t+1}+\frac{1}{2}\eta\varepsilon_{t+1}^2$$

which implies that

$$E_t\log m_{t+1}=\log\beta-\log\bar{\mu}+\left(\phi(1-\rho)-\rho\right)\hat{\mu}_t-\frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2+\frac{1}{2}\eta\sigma_\varepsilon^2$$

and, since

$$\log m_{t+1}-E_t\log m_{t+1}=\left(\rho\eta\hat{\mu}_t-(\phi+1)\right)\varepsilon_{t+1}+\frac{1}{2}\eta\varepsilon_{t+1}^2-\frac{1}{2}\eta\sigma_\varepsilon^2,$$

using $E_t\varepsilon_{t+1}^4 = 3\sigma_\varepsilon^4$ and $E_t\varepsilon_{t+1}^3 = 0$, we have

$$\mathrm{Var}_t(\log m_{t+1}) = (\rho\eta\hat{\mu}_t - (\phi+1))^2\,\sigma_\varepsilon^2 + \frac{3}{4}\eta^2\sigma_\varepsilon^4.$$

Similarly,

$$\log E_t m_{t+1} = \log\beta - \log\bar{\mu} + (\phi(1-\rho)-\rho)\,\hat{\mu}_t - \frac{1}{2}\eta(1-\rho^2)\hat{\mu}_t^2 +$$

$$\log E_t \exp\left((\rho\eta\hat{\mu}_t - (\phi+1))\,\varepsilon_{t+1} + \frac{1}{2}\eta\varepsilon_{t+1}^2\right).$$

Using the result presented at the beginning of this appendix gives the last term in this expression as

$$\log E_t \exp\left((\rho\eta\hat{\mu}_t - (\phi+1))\,\varepsilon_{t+1} + \frac{1}{2}\eta\varepsilon_{t+1}^2\right) = \frac{1}{2}\frac{(\rho\eta\hat{\mu}_t - (\phi+1))^2\,\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2} - \frac{1}{2}\log(1-\eta\sigma_\varepsilon^2).$$

Equation (38) then follows directly from these equations.

Figure 1: The Cutoff Rule for Trade