WAGE-TENURE CONTRACTS WITH HETEROGENEOUS FIRMS[1]

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June 2006

[1]This is a preliminary draft. Comments most welcome. (Burdett's email is Kennethb@ssc.upenn.edu). We would like to thank Ben Lester for research assistance.

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BY KEN BURDETT AND MELVYN COLES[1]

1 Introduction

There have been significant developments in the last few years in the study of equilibrium in labor markets where firms post wages and workers, both unemployed and employed, search for better jobs. Perhaps the most used framework in this area was developed by Burdett and Mortensen (1998) They showed that even if both workers and firms are homogeneous that at the unique equilibrium the distribution of wage offers is non-degenerate. It turns out that generalizations to the cases where firms or workers are heterogeneous is relatively straightforward - all yield an equilibrium where the distribution of wage offers is non-degenerate. Three restrictions play a crucial role in Burdett and Mortensen framework: (a) employed workers receive job offers as well as unemployed workers, (b) the offers made by any employee can be completely described by a positive real number - the wage it is willing to pay any employee, and (c) employers do not respond to offers from other firms received by employees. These implies that in equilibrium a worker employed at a particular firm always receives the same wage. These imply the only reason the wage of an employed worker increases is when this worker changes jobs.

Recently Burdett and Coles (2002) (hereafter, B/C) and Stevens (2004) have extended the above analysis by allowing firms to offer enforceable wage/tenure contracts instead of a single wage. The basic idea behind this approach can be explained as follows. Suppose for the moment that workers are indifferent to risk. Further, assume a firm makes an offer that yields any worker who accepts it a particular expected discounted lifetime income. It can be shown that, given workers search while employed, there exists a two-step wage contract (a low wage for workers with low tenure at the firm, and then a high wage later on) that yields a firm a greater expected profit flow than the single wage offer which yields the same expected return. Essentially, firms prefer to back-load wages when workers search on-the-job. Indeed, it was shown by Stevens that if firms can post a two-step wage contracts, there exists an equilibrium where all firms offer the same two-step wage contract. In this case there is no turnover as workers always prefer to stay at the employer who hired them from unemployment but at each firm those with low tenure are paid a low wage and the others are paid greater wage.

In B/C a more general setting was considered. Here, firms can post a general wage/tenure contract. Further, it is assumed that workers are risk-averse. This set up implies there is a tension between the desire of firms to back-load wages and the insurance incentive in that risk averse workers, ceteris paribus, prefer constant wages through time. It was demonstrated that there exists an equilibrium where the contracts offered by firms (contacts that maximize a firm's profit flow) imply that wages strictly increase with tenure. Further, Firms do not offer the same contract. The relationship between the different contracts offered can be described by what we termed a baseline salary scale. Let $w(.)$ denote the wage/tenure function offered by the firm offering the lowest initial wage, and as wages increase define $w(0) = \underline{w}$ and $\lim_{t\to\infty} w(t) = \overline{w}$. In equilibrium it was shown that the optimal wage/tenure offer made by any firm in the market can be written as $w(.|t_0)$ for some $t_0 \geq 0$ such that (a) $w(t_0) = w(0|t_0)$, .and (b) $w(t_0 + t) = w(t|t_0)$ for any $t \geq 0$. Hence, $w(.)$ is termed the baseline salary scale and t_0 a starting time. In equilibrium can be characterized by a baseline salary scale and a distribution of starting times, $(w(.), H(.))$. It was shown that in equilibrium the distribution of starting times is continuous except at $t = 0$ and has support $[0, \infty]$.

As the baseline salary scale is increasing, a firm that offers a greater initial starting time than another makes less profit per employee per unit of time. Firms offering a greater initial starting time, however, keeps its employees longer (on average) and attracts more workers. To see this more clearly, let $V(0|t_0)$ indicates a worker's expected lifetime utility from accepting a job offer with starting time t_0 (and then following an optimal quit strategy). Clearly, $V(t|t_0') > V(t|t_0)$ for any tenure $t \geq 0$, where $t_0' > t_0$. On-the-job search by workers implies a firm offering a greater starting time obtains a larger steady-state labor force. In equilibrium, of course, all firms make the same expected profit flow which equals the expected profit per employee multiplied by the expected number of employees. These results were established assuming workers and firms were homogeneous. This implied our results were not easy to empirically implement. Here, we drop this restriction and assume that some firms are more productive than others.

Assuming there are n different productivity types among employers, yields many new insights. Let $p_1 < p_2 < .... < p_n$ denote the productivities of the different types of firms. It will be shown that the concept of a baseline salary scale also plays an important role in this case in that it will be shown that when there are n productivity types there are n baseline salary scales - one for each productivity type. The type i firm baseline salary scale is the wage/tenure offer made by the type i firm offering the lowest initial wage, $w_i(.)$,where $w_i(0) = \underline{w}_i$ and $\lim_{t\to\infty} w_i(t) = w_i^{\infty}$. In equilibrium the offers made by type i firms can be fully characterized by $(w_i(.), H_i(.))$,where $H_i(.)$ is the distribution starting times for type i firms.

The above implies for any type i firm there exists a $t_0 \geq 0$ such that $w_i(t_0 + t) = w(t|t_0)$ describes the optimal wage/tenure contact made by that firm. It will be shown that not all points on the type i baseline salary scale need be a starting time. In particular, it will be shown the support of the distribution of starting time, $H_i(.)$, is an interval $[0, \widehat{t_i}]$.If for some type i, $\widehat{t_i} < \infty$, then (as the baseline salary scale is increasing, wage) $w_i(\widehat{t_i} + t)$ for $t > 0$ are paid by type i firms but only to employees with large enough tenure - such a wage is not offered as initial offers by type i firms. If $\widehat{t_i} = \infty$, then all points on the baseline salary are initial offers for some type i firm. As will be shown this has important consequences.

Given the offers made by firms can be fully described by $(w_i(.), H_i(.))$, $i = 1, 2, 3, ..n$, the objective becomes to establish the equilibrium relationship between the offers made by the different types of firm. It will be shown that in equilibrium the initial offers made by type i employers can be described by an interval $[\underline{V}_i, \overline{V}_i]$ such that $V \in [\underline{V}_i, \overline{V}_i]$, where V indicates a worker's expected lifetime payoff from accepting a particular job offer. An important result established is that in any market equilibrium it will be shown that $\overline{V}_i = \underline{V}_{i+1}$, $i = 1, 2, n-1$, i.e., more productive firms make more desirable job offers than less productive firms. Although more productive firms make more desirable offers this does not imply the initial wage offered by such firms are greater than less productive firms. Indeed, it will be shown that an employee of a type i firm, will willingly move to a type $i+1$ firm (if the opportunity arrises) although the initial wage offered is lower than that currently faced.

It will be shown that the above and other results lead to a reasonably complete characterization of the equilibrium structure of the labor market. To illustrate essentials, a detailed characterization of equilibrium is provided in the special case where there are only two types of firms. This turns out not to be simple task. The reason for this can be explained as follows. If there are two types of firms in equilibrium a workers expected return to accepting an offer, $V \in [\underline{V}, \overline{V}]$, where $\underline{V} = \underline{V}_1 < \overline{V}_1 = \underline{V}_2 < \overline{V}_2 = \overline{V}$,i.e., the more productive type of firms (type 2) make more desirable offers. It is shown, however, there exists a $\widehat{t} < \infty$ such that $w_1(\widehat{t}_1)$ is the greatest initial wage offer made by type 1 firms, i.e., $w_1(\widehat{t}_1) < w_i^\infty$, whereas with type 2 firms $\widehat{t}_2 = \infty$. This implies some type 1 employees enjoy an expected lifetime utility greater than $\overline{V}_1$.Let V_1^∞ denote the maximum expected return a type 1 employee can enjoy. It follows that $\overline{V}_1 = \underline{V}_2 < V_1^\infty \leq \overline{V}_2$. Suppose a type 1 employee with positive tenure faces expected return $\widehat{V} \in (\underline{V}_2, V_1^\infty)$. This implies that if such an employee obtains an offer from a type 1 firm it may be rejected. This is not only leads to an inefficiency but also much complicates the wage dynamics.

In the nest Section the basic framework is outlined and the optimal search strategy of a worker is described. In Section 3 the optimal wage/tenure contract of a representative type i firm is derived. We then show that in equilibrium the wage tenure/contracts offered by all type i firms can be fully described by a baseline salary scale. As many elements of these two Sections repeats, or are reasonably straightforward extension of the analysis provided in B/C, we only outline essentials. Section 4 provides a description of a market equilibrium. From Section 5 onwards we provide a detailed characterization of possible equilibria when there are two types of firms. Section 6 provides a simulation of the possible equilibria.

2 BASIC FRAMEWORK

Time is continuous and only steady states are considered. A unit mass of both workers and firms participate in a labor market. Workers are homogeneous but there are n types of firms. A type i firm generate revenue p_i per unit of time from each worker it employs, $i = 1, 2, ..n$, where $p_1 < p_2 ... < p_n$. Proportion α_i of all firms are type i. Workers are either unemployed or employed and obtain new job offers at Poisson rate λ, independent of their employment status.

Any job offer is fully described by the wage contract offered by the firm. This specifies the wage the worker receives as a function of tenure at that firm, i.e., an offer is a function $w(.) \geq 0$ defined for all tenures $t \geq 0$. All new hires are offered the same contract and this contract is binding. There is there is no recall should a worker quit or reject a job offer. For simplicity assume firms and workers have a zero rate of time preference. The objective of any firm is to maximize steady state flow profit. Workers are strictly risk averse and are finitely lived, where any worker's life is described by an exponential random variable with parameter $\delta > 0$. Any worker who dies is instantly replaced by a new unemployed worker. The objective of any worker is to maximize total expected lifetime utility. Unemployed workers obtain b per unit of time and that $p_1 > b > 0$. There are no financial markets.

2.1 Workers.

Given the lack of financial markets, a worker who obtains income $w \geq 0$ at any instant of time obtains flow utility $u(w)$ by immediately consuming it. As in B/C we utilize the following restriction on this utility function.

A1: Assume u is strictly increasing, strictly concave, twice differentiable and

$$\lim_{w \to 0^+} u(w) = -\infty.$$

This is restrictive but simplifies the analysis (see B/C for further discussion).

Let $V(t|w(.))$ denote a worker's expected lifetime payoff when employed with tenure t at a firm offering $w(.)$, given an optimal quit strategy is followed in the future An unemployed worker's maximum expected lifetime payoff is indicated by V_u. Let $F(V_0)$ denote the proportion of firms in the market whose offer, if accepted, yields a worker an expected lifetime income no greater than V_0, i.e., F is the distribution of $V(0|w(.))$ in the market. Random search implies $F(V_0)$ denotes the probability a worker contacts a firm offering an expected return no greater than V_0, given a contact is made. Let $\underline{V}$ and $\overline{V}$ denote the infimum and supremum of this distribution function. Using standard arguments imply $V(t|w(.))$can be written as

$$(1) \qquad \delta V(t|w(.)) - \frac{dV(t|w(.))}{dt} = u(w(t)) + \lambda \int_{V(t|.w())}^{\overline{V}} [V_0 - V(t|w(.))] dF(V_0).$$

If $V(t|.) < V_u$ for some tenure t, the worker's optimal strategy is to quit into unemployment. Although this event never occurs in equilibrium, we need to allow for this possibility and so define $T = \inf\{t \geq 0 : V(t|.) < V_u\}$. Hence, T denotes the tenure at which a worker optimally quits into unemployment. If $V(t|.) \geq V_u$ for all t, then define $T = \infty$.

Assume an unemployed worker accepts a job offer if indifferent to doing so, whereas employed workers only quit if the outside job offer is strictly preferred. This implies optimal job search implies the following worker strategies: First, if unemployed, the worker accepts a job offer if it has starting value $V_0 \geq V_u$. Second, if employed with contract $w(.)$ and tenure $t < T$, the worker quits if and only if a job offer is received with starting value $V_0 > V(t|w(.))$. Finally, if employed with contract $w(.)$ and tenure $t = T < \infty$, the worker quits into unemployment.

Given wage contract $w(.)$ and tenure $s < T$, a worker leaves a firm's employment at rate $\delta + \lambda(1 - F(V(s|.)))$. For tenures $t < T$, the survival probability

$$\psi(t|w(.)) = e^{-\int_0^t [\delta + \lambda(1 - F(V(s|w(.)))]ds} \tag{2}$$

describes the probability a newly employed worker does not leave the firm before tenure t. Of course if $T < \infty$, then $\psi = 0$ for all $t \geq T$.

2.2 Firms.

Let $G(V)$ denote the steady-state number of workers who are either unemployed or employed currently enjoying an expected lifetime utility strictly less than V. Suppose a type i firm posts $w(.)$ and let $V_0 = V(0|w(0))$. If $V_0 < V_u$, the firm attracts no workers and so obtains zero profit. If $V_0 \geq V_u$, then $\lambda G(V_0)$ describes this firm's steady-state hiring rate and therefore its steady state flow profit can be written as

$$\Omega_i = [\lambda G(V_0)] \left[\int_0^\infty \psi(t|w(.))[p_i - w(t)]dt \right],$$

Note, the firm's steady state flow profit equals its hiring rate $\lambda G(V_0)$ multiplied by the expected profit per new hire.

First, we determine a firm's wage/tenure contract that maximizes it's expected profit flow conditional on offering a new hire lifetime payoff $V_0 \geq V_u$.Such a contract is termed an optimal contract. Assuming an optimal contract exists for a type i firm, let $w^*_{p_i}(t|V_0)$ denote it and let $\Pi^*_{p_i}(0|V_0)$ denote the firm's maximized payoff per new hire. A type i firm's steady-state flow profits can then be written as

$$\Omega^*_i(V_0) = \lambda G(V_0)\Pi^*_{p_i}(0|V_0).$$

and the firm's optimization problem reduces to choosing a starting payoff V_0 to maximize $\Omega_i^*(V_0)$.

Let F_i denote the offer distribution of firms of type i with support denoted $[\underline{V}_i, \overline{V}_i]$, and note $F = \sum_i \alpha_i F_i$. To simplify the exposition, we impose the following restriction on F.

A2: F is continuously differentiable almost everywhere and has a connected support.

We now define a market equilibrium.

Definition: A *market equilibrium* is:

(ME1) a distribution of starting payoffs F satisfying A2;

(ME2) a set of optimal wage tenure contracts $w_p^*(.|V_0)$ with $p = p_i$, and indexed by $V_0 \geq V_u$;

(ME3) optimal job search by unemployed workers, where optimality implies V_u satisfies

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [x - V_u] dF(x);$$

(ME4) optimal quit behavior by employees, where $V(t|w_{p_i}^*)$ describes the worker's expected lifetime payoff at tenure t given wage contract $w_{p_i}^*(.)$;

(ME5) a distribution of expected lifetime payoffs G consistent with steady state turnover, and

(ME6) a steady state profit condition where for all firm types:

$$\begin{aligned} &\Omega_i^*(V_0) = \overline{\Omega}_i \geq 0 \text{ if } dF_i(V_0) > 0, \\ &\Omega_i^*(V_0) \leq \overline{\Omega}_i, \text{otherwise}, \end{aligned} \tag{3}$$

so that any offered contract (i.e., an optimal contract $w_p^*(.)$ with firm productivity $p = p_i$ and starting value V_0 which lies in the support of F_i), maximizes steady state flow profit.

3 Optimal Wage/Tenure Contracts

Given productivity $p_i = p$, a firm's optimal contracting problem is formally defined as

$$(4) \qquad \max_{w(.)} \int_0^\infty \psi(t|w(.))[p - w(t)]dt$$

subject to (a) $w(.) \geq 0$ and (b) $V(0|w(.)) = V_0$, where ψ is defined by (2).

Before solving this problem it is useful to note that as the arrival rate of offers is independent of a worker's employment status, an unemployed worker accepts a contract which offers $w(t) = b$ for all t. Further, as $b < p_i$ by assumption, a firm can always obtain strictly positive profit by offering this contract. It is now straightforward to show that if an equilibrium exists, (a) all firms make strictly positive profit; $\overline{\Omega}_i > 0$, (b) $\underline{V} \geq V_u$, (c) $\overline{V} < u(p_n)/\delta$ (no firm pays a worker more than the worker's expected value), and (d) $T(w_p^*) = \infty$ for all $p = p_i$ (an employed worker never quits to unemployment). Hence, without loss of generality, we may assume that F not only satisfies A2, but also $V_u \leq \underline{V}$ and $\overline{V} < u(p_n)/\delta$.

For a given F and corresponding $\overline{V}$, define $\overline{w}$ by $u(\overline{w}) = \delta\overline{V}$. In equilibrium, $\overline{w}$, is the highest wage paid in the market. We also define for each type i firm:

$$w_i^\infty = \min[p_i, \overline{w}];$$

$$(5) \qquad \delta V_i^\infty = u(w_i^\infty) + \lambda \int_{\overline{V}_i}^{\overline{V}} [V_0 - V_i^\infty] dF(V_0);$$

$$\Pi_i^\infty = (p_i - w_i^\infty)/\delta,$$

where w_i^∞ is the wage paid by type i firms at arbitrarily long tenures, while V_i^∞ describes the value of being employed at that wage.

Hence, for given F (and therefore given $\overline{V}$ and $\overline{w}$) it is possible to partition the different types of firms into two - those types of firms with $p_i < \overline{w}$ and those with $p_i \geq \overline{w}$. If $p_i \geq \overline{w}$, type i firms offer limiting wage $w_i^\infty = \overline{w}$ at long tenures. As

$\overline{w}$ is the largest wage in the market, a worker employed at such a wage never quit and enjoy expected lifetime payoff $V = u(\overline{w})/\delta \equiv \overline{V}$. Further, $\Pi_i^{\infty} = (p_i - \overline{w})/\delta > 0$ describes the firm's limiting (continuation) profit. In contrast, if $p_i < \overline{w}$, type i firms pay limiting wage $w_i^{\infty} = p_i$; i.e. they pay marginal product to employees with (arbitrarily) long tenures. Note that $p_i < \overline{w}$ implies $V_i^{\infty} < \overline{V}$, and so its employees always have a strictly positive quit rate. Paying marginal product also implies zero (continuation) profit $\Pi_i^{\infty} = 0$ at arbitrarily long tenures.

Claim 1 now establishes these limiting contracts are jointly efficient.

Claim 1

Fix an F satisfying A2. For any type i firm offering starting payoff $V_0 = V_i^{\infty}$, the optimal wage contract is $w(t) = w_i^{\infty}$ for all $t \geq 0$.

Proof:

See Appendix.

Let $V_p^*(t|V_0)$ denote the expected lifetime utility of an employee with tenure t given the optimal contract, i.e. $V_p^*(t|V_0) \equiv V(t|w_p^*(.|V_0))$.We now describe an optimal contract. As firms make strictly positive profit, for each $p = p_i$ we only consider starting payoffs $V_0 < V_i^{\infty}$.

Theorem 1

Fix an F satisfying A2 with $V_u \leq \underline{V}$ and a productivity $p = p_i > b$. Given any starting payoff $V_0 \in [\underline{V}, V_i^{\infty})$, the optimal wage-tenure contract w_p^* and corresponding worker and firm payoffs $\{V_p^*, \Pi_p^*\}$ are solutions to the differential equation system $\{w, V, \Pi\}$:

(6) $$\frac{-u''(w)}{u'(w)^2}\frac{dw}{dt} = \lambda F'(V)\Pi,$$

(7) $$\frac{dV}{dt} = -u'(w)\frac{d\Pi}{dt}$$

(8) $$[\delta + \lambda(1 - F(V))]\Pi - \frac{d\Pi}{dt} = [p - w],$$

subject to the boundary conditions: (a) $\lim_{t\to\infty}\{w(t), V(t), \Pi(t)\} = (w_i^\infty, V_i^\infty, \Pi_i^\infty)$, and (b) the initial condition $V(0) = V_0$

As the proof is almost identical to the one given in B/C, we omit it here.[1] An important difference, however, is the boundary condition (a). The homogeneous firms case considered in B/C implies $\overline{w} < p$ for all firms and so $(w_i^\infty, V_i^\infty, \Pi_i^\infty) \equiv (\overline{w}, \overline{V}, (p - \overline{w})/\delta)$. With heterogeneous firms it can happen that $p_i < \overline{w}$ for some i and so these firms pay limiting wage $p_i < \overline{w}$. In both cases, however, Claim 1 implies the limiting contract is jointly efficient. By promising higher earnings in the future, the firm reduces employee quit rates and so increases joint surplus (noting that a quit is jointly inefficient when the outside offer has value $V_0 < V_i^\infty$).

The limit point $(w_i^\infty, V_i^\infty, \Pi_i^\infty)$ is a stationary point of the differential equation system (6)-(8). It follows that the optimal wage contract corresponds to the saddle path converging to that stationary point. For firms with productivities $p_i > \overline{w}$, $\lim_{V\to\overline{V}} F'(V) > 0$ would imply the optimal wage contract $w_{p_i}^*(.)$ reaches $\overline{w}$ at a finite tenure. In a steady state, however, this would imply a positive mass of workers with lifetime utility $V = \overline{V}$. Claim 2 below establishes that this cannot occur in a market equilibrium. Anticipating that result, equilibrium requires $\lim_{V\to\overline{V}} F'(V) = 0$ and $w_{p_i}^*(.)$ then only converges to $\overline{w}$ asymptotically.

Given an F satisfying A2, the *type i baseline salary scale* is defined as the optimal wage/tenure contract, $w_i(.)$, of the type i firm offering the lowest initial wage. As the baseline salary scale is a increasing function of t which converges (asymptotically) to V_i^∞, the offer made by any type i firm, $w_{p_i}^*(.|V_0)$, is related to the baseline salary scale by a $t_0 \geq 0$ such that $w_{p_i}^*(t|V_0) \equiv w_i(t + t_0)$ for. all $t_0 \geq 0$. Note, this also implies firm i's continuation payoff $\Pi_{p_i}^*(t|V_0) \equiv \Pi_i(t + t_0)$.

The next section now characterizes a market equilibrium where firms not only choose an optimal wage tenure contract given V_0, but also choose V_0 to maximize

[1] A proof is available in ken4.tex which we should post on the web

steady state profit $\Omega_i^*(V_0)$.

4 MARKET EQUILIBRIUM

Claim 2 establishes there cannot be mass points in F and G at $V = \overline{V}$ and therefore a market equilibrium implies $w_i(.)$ only converges to $\overline{w}$ asymptotically (for $p_i > \overline{w}$).

Claim 2

At a market equilibrium $1 - G$ and F do not have a mass point at $\overline{V}$, and so $\lim_{V \to \overline{V}} F'(V) = 0$.

Proof:

See Appendix.

Lemma 1 establishes an essential property implied by the conditions of Theorem 1.

Lemma 1.

For any i, j with $b < p_i < p_j$, and for any $V_0 \in [\underline{V}, V_i^{\infty})$:

(i)

$$w_{p_i}^*(0|V_0) > w_{p_j}^*(0|V_0)$$

(ii)

$$\frac{\partial \Pi_{p_i}^*(0|V_0)}{\partial V_0} < \frac{\partial \Pi_{p_j}^*(0|V_0)}{\partial V_0} < 0.$$

Proof:

See Appendix.

For a given starting payoff V_0, Lemma 1(i) establishes that a lower productivity firm offers a higher starting wage given such an offer yields the same expected return. Lemma 1(ii) establishes that for any starting payoff V_0, the marginal loss in profit per new hire by increasing V_0 is smaller for the higher productivity firm. This reflects that a worker with a higher expected lifetime payoff has a lower quit rate and the marginal return to retaining that worker is greater for the high productivity firm.

Armed with Lemma 1(ii), Proposition 1 now establishes that in equilibrium, higher productivity firms offer higher expected payoff to workers.

Proposition 1

A market equilibrium implies $\overline{V}_i = \underline{V}_{i+1}$; i.e. the support of F can be partitioned into n sets, where type i firms offer $V_0 \in [\overline{V}_{i-1}, \overline{V}_i]$ where $\underline{V} \leq \overline{V}_1 \leq \overline{V}_2 \leq .... \leq \overline{V}_n = \overline{V}$.

Proof:

A firm with productivity $p = p_i$ chooses V_0 to maximize

$$\Omega_i^*(V_0) = \lambda G(V_0)\Pi^*(0, p_i|V_0)$$

Lemma 1(ii) establishes that the marginal return to increasing V_0 is strictly greater for a higher productivity firm. Hence for a given $G(.)$, optimality implies a higher productivity firm chooses a higher V_0. Connectedness of F (assumption A2) then implies the Proposition.

Having identified the basic equilibrium market structure, for ease of exposition the remainder of the paper focusses on the two types case.

5 Two Types

Assume now only two types of firms - type 1 firms and type 2 firms with $p_1 < p_2$. From the above it follows that in any equilibrium the distribution of initial offers made by type 1 firms has support $[\underline{V}_1, \overline{V}_1]$, whereas the support of the distribution of initial offers made by type 2 firms is $[\underline{V}_2, \overline{V}_2]$.Proposition 2 implies $\underline{V}= \underline{V}_1 \leq \overline{V}_1 = \underline{V}_2 \leq \overline{V}_2 = \overline{V}$.

Assume for the moment an equilibrium implies that $p_1 < \overline{w} < p_2$, where $u(\overline{w}) = \delta\overline{V}$ From (5) it follows $\lim_{t\to\infty} w_1(t) = w_1^\infty = p_1$. Hence, offering initial offer w_1^∞ yields zero profit for a type 1 firm and therefore cannot be part of an equilibrium.

This implies not all points on the the type 1 baseline salary scale are initial offers in equilibrium. This implies $\overline{V}_1 = \underline{V}_2 < V_1^\infty$. Hence, although type 1 firms do not make initial offers that yield $V \in (\overline{V}_1, V_1^\infty]$, type 1 employees. with tenure great enough can obtain such a return. It is essentially the same if the equilibrium implies $\overline{w} < p_1$. In this case (5) implies $\lim_{t\to\infty} w_k(t) = \overline{w}$, $k = 1, 2$. This implies $\overline{V}_1 = \underline{V}_2 < V_1^\infty = \overline{V}_2$.Again, although no type 1 firm makes an initial offer that yields return $V \in (\overline{V}_1, V_1^\infty]$, some type 1 employees with large enough tenure obtain this return. In what follows we term $V \in [V_1^\infty, \overline{V}]$, Phase A, $V \in [\overline{V}_1, V_1^\infty)$, Phase B, and $V \in [\underline{V}, \overline{V}_1)$, Phase C. .

In what follows the following objects play an important role. For $i = 1, 2$:
(i) $\widehat{t}_i(V)$ is the inverse function of $V_i(t)$; i.e. $\widehat{t}_i$ describes the salary point t at which an employed worker enjoys lifetime payoff $V = V_i(t)$ along baseline salary scale i. Of course, $\widehat{t}_i$ is a positive and strictly increasing function of V for $V \in [\underline{V}_i, V_i^\infty)$;
(ii) $\widehat{w}_i(V) = w_i(\widehat{t}_i(.))$ is the wage paid on baseline salary scale i when an employee's lifetime payoff is V;
(iii) $\widehat{\Pi}_i(V) = \Pi_i(\widehat{t}_i(.))$ is a type i firm's expected profit when any employee's lifetime payoff is V;
(iv) $\widehat{N}_i(V)$ is the number of employed workers with payoff strictly less than V on baseline salary scale i and note $G(V) \equiv U + \widehat{N}_1(V) + \widehat{N}_2(V)$.

These functions are central to the equilibrium analysis as workers compare lifetime payoff V when deciding whether to quit or not. Claim 3 now characterizes $(\widehat{t}_i, \widehat{w}_i, \widehat{\Pi}_i)$ and Claim 4 characterizes $\widehat{N}_i$.

Claim 3

Fix an F satisfying A2 with $V_u \leq \underline{V}$. The type i baseline salary scale implies $(\widehat{w}_i, \widehat{\Pi}_i)$ are solutions to the differential equation system

$$\frac{d\widehat{w}_i}{dV} = \left[\frac{u'(\widehat{w}_i)}{-u''(\widehat{w}_i)}\right] \frac{\lambda F'(V)\widehat{\Pi}_i}{\left[p_i - \widehat{w}_i - [\delta + \lambda(1 - F(V))]\widehat{\Pi}_i\right]}$$

$$\frac{d\widehat{\Pi}_i}{dV} = -\frac{1}{u'(\widehat{w}_i)}$$

subject to the boundary condition $(\widehat{w}_i, \widehat{\Pi}_i) = (w_i^\infty, \Pi_i^\infty)$ at $V = V_i^\infty$.

Further, $\widehat{t}_i$ is given by

$$\frac{d\widehat{t}_i}{dV} = \frac{1}{u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta + \lambda(1 - F(V))]\widehat{\Pi}_i]}$$

and $\widehat{t}(\underline{V}_i) = 0$.

Proof :

This follows directly from Theorem 1 and the definitions of $(\widehat{t}_i, \widehat{w}_i, \widehat{\Pi}_i)$.

Claim 4.

Optimal job search in a Market Equilibrium implies:

(i) $U = \delta/(\lambda + \delta)$;

(i) During Phase A (where $V \in [V_1^\infty, \overline{V}]$), $\widehat{N}_2$ satisfies

$$\frac{d\widehat{N}_2}{dV} = \left[\delta(\overline{N}_2 - \widehat{N}_2) - \lambda\alpha_2(1 - F_2(V))(1 - \overline{N}_2 + \widehat{N}_2)\right] \frac{d\widehat{t}_2}{dV}$$

where $\overline{N}_2 = \widehat{N}_2(\overline{V})$;

(iii) During Phase B (where $V \in [\overline{V}_1, V_1^\infty)$),

$$\frac{d\widehat{N}_1}{dV} = \left[\alpha_1\lambda U - \delta\widehat{N}_1 - \alpha_2\lambda \int_{\underline{V}}^{V} [1 - F_2(x)] d\widehat{N}_1(x)\right] \frac{d\widehat{t}_1}{dV},$$

$$\frac{d\widehat{N}_1}{dV} / \frac{d\widehat{t}_1}{dV} + \frac{d\widehat{N}_2}{dV} / \frac{d\widehat{t}_2}{dV} = \lambda U[\alpha_1 + \alpha_2 F_2(V)] - [\delta + \lambda\alpha_2[1 - F_2(V)]] \left[\widehat{N}_1 + \widehat{N}_2\right],$$

subject to $\widehat{N}_2(\overline{V}_1) = 0$.

(iv) During Phase C (where $V \in [\underline{V}, \overline{V}_1)$), $\widehat{N}_2 = 0$ and $\widehat{N}_1$ satisfies

$$\frac{d\widehat{N}_1}{dV} = \left[U\lambda\alpha_1 F_1(V) - [\delta + \lambda(1 - \alpha_1 F_1(V))]\widehat{N}_1\right] \frac{d\widehat{t}_1}{dV}$$

subject to $\widehat{N}_1(\underline{V}) = 0$;

Proof

See Appendix**.**

In Phase A only type 2 firms offer contracts that yield such a payoff . No type 1 firm offer yields such an expected return - no matter what the tenure. In Phase B (where $V \in (\overline{V}_1, V_1^{\infty})$).only type 2 firms make starting offers that yield such a return. Although all initial offers by type 1 firms yield an expected lifetime payoff $V \leq \overline{V}_1$, type 1 firms increase wages with tenure and this implies some employees of type 1 firms with significantly long tenure that have a such lifetime payoff. This has an important implication that some employees of type 1 firms may receive an offer from a type 2 firms that they reject. This generates a complicated interaction between the two types of firms. Claim 4 (iv) describes these dynamics. Finally, Phase C where $V \in [\underline{V}, \overline{V}_1]$. Here only type 1 firms make offers that yield such an expected payoff to workers. Claim 4 (ii) describes turnover in this phase.

As F has connected support, Proposition 1 and the constant profit condition [ME6] requires

$$\begin{aligned} \lambda G(V)\widehat{\Pi}_1(V) &= \overline{\Omega}_1 > 0 \text{ for all } V \in [0, \overline{V}_1] \\ \lambda G(V)\widehat{\Pi}_2(V) &= \overline{\Omega}_2 > 0 \text{ for all } V \in [\overline{V}_1, \overline{V}] \end{aligned} \tag{9}$$

where $\lambda G(V)$ is the hiring rate of a firm that offers starting payoff V. Note that at $V = \overline{V}$, Claim 2 implies $G = 1$ while Claim 3 implies $\widehat{\Pi}_2 = \Pi_2^{\infty}$. Hence $\overline{\Omega}_2 = \lambda(p_2 - \overline{w})/\delta$ and so the constant profit condition for type 2 firms implies

$$G(V)\widehat{\Pi}_2(V) = (p_2 - \overline{w})/\delta \text{ for all } V \in [\overline{V}_1, \overline{V}]. \tag{10}$$

Proposition 2 now establishes that $\underline{V} = V_u$; i.e., type 1 firms offering the least generous wage contract in the market extract all the surplus from unemployed workers. This leads to the following characterization of a market equilibrium.

Proposition 2

Necessary and sufficient conditions for a market equilibrium are a vector of functions $\{\widehat{t}_i, \widehat{w}_i, \widehat{\Pi}_i, \widehat{N}_i, F\}$ for $i = 1, 2$ satisfying the conditions of Claims 3, 4 and the constant profit condition (9), and

(a) $F(\overline{V}) = G(\overline{V}) = 1$ with $\overline{V} < u(p_2)/\delta$;

(b) $\overline{V}_1 = \underline{V}_2$;

(c) $V_u = \underline{V}$ where

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [x - V_u] dF(x).$$

Further, $\widehat{t}_i, \widehat{N}_i$ must be positive and increasing functions, G must have the properties of a distribution function and F must have properties $A2$.

Proof

See Appendix

5.1 Identifying a Market Equilibrium.

We now solve the conditions of Proposition 2 in two steps. The first step fixes a trial value $\overline{w} \in [b, p_2]$ and corresponding $\overline{V} = u(\overline{w})/\delta$. Ignoring for the moment the equilibrium search condition (c), that $V_u = \underline{V}$ where

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [x - V_u] dF(x),$$

we first solve the other conditions of Proposition 2 recursively, starting at $V = \overline{V}$ with $\widehat{w}_2 = \overline{w}$. Conditional on a trial value $\overline{w}$, backward induction identifies a candidate equilibrium and corresponding distribution function F, which we might denote $F^c(.|\overline{w})$. If this distribution function in addition satisfies

$$\delta \underline{V} = u(b) + \lambda \int_{\underline{V}}^{\overline{V}} [x - \underline{V}] dF^c(x|\overline{w}),$$

then $V_u = \underline{V}$ and we have also identified a Market Equilibrium. Hence the second step searches over $\overline{w} \in [b, p_2]$ for an implied $F^c(.|\overline{w})$ which satisfies this equilibrium criterion.

5.2 A Candidate Equilibrium (given $\overline{w}$)

Suppose $\overline{w} \le p_1$. As $V_1^\infty = \overline{V}$ this case implies Phase A is degenerate. For ease of exposition, however, and noting that the simulations only report cases where a Market Equilibrium implies $\overline{w} > p_1$, we shall focus on the case $\overline{w} \in (p_1, p_2)$ and hence $V_1^\infty < \overline{V}$. Thus type 2 (high productivity) firm pay limiting wage $w_2^\infty = \overline{w}$ while type 1 (low productivity) firms pay limiting wage $w_1^\infty = p_1$. Given $\overline{w} \in (p_1, p_2)$ fixed, the (candidate) equilibrium functions $\{\widehat{w}_i, \widehat{\Pi}_i, \widehat{N}_i, F\}$ are identified using backward induction, starting at $w = \overline{w}$ and $V = \overline{V} = u(\overline{w})/\delta$.

Phase A. $V \in [V_1^\infty, \overline{V}]$.

During this phase, type 1 firms are not active and equilibrium determines $(\widehat{w}_2, \widehat{\Pi}_2, F, G)$ where $G = U + \widehat{N}_1 + \widehat{N}_2$ with $\widehat{N}_1 = \widehat{N}_1(V_1^\infty)$. Theorem 2 in B/C describes a closed form solution for this phase. Consistent with those equations, it can be shown that $(\widehat{w}_2, \widehat{\Pi}_2, F)$ evolve in Phase A according to the differential equations for $\widehat{w}_2, \widehat{\Pi}_2$ given in Claim 3 (with $i = 2$) and

$$\frac{dF}{dV} = \frac{-u''(\widehat{w}_2)}{u'(\widehat{w}_2)^2} \frac{2(p_2 - \widehat{w}_2)\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2\right]}{\lambda \widehat{\Pi}_2^2} \tag{11}$$

subject to initial values $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\overline{w}, (p_2 - \overline{w})/\delta, 1)$ at $V = \overline{V}$. We discuss the derivation of dF/dV below. Note this initial value problem implies $(\widehat{w}_2, \widehat{\Pi}_2, F)$ are fully determined in Phase A given the trial value $\overline{w}$. G is then determined by the constant profit condition written as $G = (p_2 - \overline{w})/(\delta \widehat{\Pi}_2)$.

Phase A ends, and Phase B begins, at $V = V_1^\infty$. The definition of V_1^∞ implies

$$\delta V_1^\infty = u(p_1) + \lambda \int_{V_1^\infty}^{\overline{V}} [x - V_1^\infty] dF(x).$$

Now at salary point t on the type 2 baseline salary scale, an optimal quit strategy implies equation (1) holds, which is rewritten here as

$$\delta V_2 - \frac{dV_2}{dt} = u(w_2) + \lambda \int_{V_2}^{\overline{V}} [x - V_2] dF(x).$$

Using (7) and (8) in Theorem 1 to substitute out dV_2/dt, then setting $V_2(t) = V_1^\infty$ implies

$$u(p_1) = u(w_2) + u'(w_2)[p_2 - w_2 - [\delta + \lambda(1 - F)]\Pi_2]. \tag{12}$$

Hence using backward iteration from $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\overline{w}, (p_2 - \overline{w})/\delta, 1)$, phase A stops when $(\widehat{w}_2, \widehat{\Pi}_2, F)$ satisfies (12) and $V_2(t) = V_1^\infty$ at that point. It can be shown this termination point always exists and is unique and we let $\widehat{w}_2^A$, $\widehat{\Pi}_2^A$, F_A, and G_A denote the corresponding values of $\widehat{w}_2$, $\widehat{\Pi}_2$, F, and G at the termination point. The (candidate) solution is relevant only if it implies $F_A > \alpha_1$ (some type 2 firms make offers $V \in [\overline{V}_1, V_1^\infty]$) and $G_A > U$ (some workers are employed in type 1 firms). In that case we proceed to Phase B. This completes the description of Phase A.

Phase B: $V \in [\overline{V}_1, V_1^\infty]$

Phase B jointly determines $\{\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, \widehat{N}_1, \widehat{N}_2, F\}$. A major complication is a singularity in the differential equations at $V = V_1^\infty$. Appendix B provides a formal analysis of the equilibrium conditions. Here we provide an intuitive interpretation.

Define $\phi = d\widehat{N}_1/dV$ which is the density of type 1 employees with expected lifetime value V. Appendix B establishes that equilibrium in Phase B can be reduced to an autonomous differential equation system $\{\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F\}$ where $\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1$, and $\widehat{\Pi}_2$ evolve according to the differential equations stated in Claim 3,

$$\frac{dF}{dV} = \frac{\frac{2H[u(\widehat{w}_1) - u(\widehat{w}_2)]}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - H\widehat{\Pi}_1]}\phi + \frac{2(p_2 - \widehat{w}_2)(p_2 - \overline{w})}{\delta \widehat{\Pi}_2^3 u'(\widehat{w}_2)}}{\left[\frac{\left[\frac{u'(\widehat{w}_1)^2}{-u''(\widehat{w}_1)}\right]\lambda\widehat{\Pi}_1[u(\widehat{w}_1) - u(\widehat{w}_2)]}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - H\widehat{\Pi}_1]^2} + \frac{\left[\frac{u'(\widehat{w}_1)^2}{-u''(\widehat{w}_1)}\right]\lambda\widehat{\Pi}_1}{\left[p_1 - \widehat{w}_1 - H\widehat{\Pi}_1\right]} - \frac{\left[\frac{u'(\widehat{w}_2)^2}{-u''(\widehat{w}_2)}\right]\lambda\widehat{\Pi}_2}{\left[p_2 - \widehat{w}_2 - H\widehat{\Pi}_2\right]}\right]\phi + \frac{\lambda(p_2 - \overline{w})\left[\frac{u'(\widehat{w}_2)}{-u''(\widehat{w}_2)}\right]}{\delta\widehat{\Pi}_2\left[p_2 - \widehat{w}_2 - H\widehat{\Pi}_2\right]}} \tag{13}$$

where ϕ is given by

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)} \left[\frac{(p_2 - \widehat{w}_2)(p_2 - \overline{w})}{\delta^2 \widehat{\Pi}_2^2} - 1 \right]. \tag{14}$$

and $H = [\delta + \lambda(1 - F)]$. Note that F' has the simple structure $F' = (a + b\phi)/(c + d\phi)$ and so is monotonic in ϕ. Furthermore setting $\phi = 0$ (no type 1 employees) yields F' as described in Phase A.

Appendix B derives (14) by using Claim 4 to substitute $\widehat{N}_2$ out of the constant profit condition $(U + \widehat{N}_1 + \widehat{N}_2)\widehat{\Pi}_2 = (p_2 - \overline{w})/\delta$. Claim 4 also implies the employment density ϕ evolves according to

$$\frac{d\phi}{dV} = \phi \frac{u'(\widehat{w}_1)\frac{d\widehat{w}_1}{dV} - 2[\delta + \lambda(1 - F)]}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F)]\widehat{\Pi}_1]} \tag{15}$$

The denominator in (15) is always strictly positive (as $dV_1/dt > 0$ along the type 1 baseline salary scale). Note, a high density of outside offers (F' large) implies type 1 firms raise wages relatively quickly with tenure ($d\widehat{w}_1/dV$ large) and (15) then implies the density of type 1 employees (ϕ) falls less quickly with V. Indeed, (15) implies the employment density ϕ decreases with V if and only if the density of outside offers $F'(V)$ is not too high.

Note, (14) determines ϕ consistent with the constant profit condition for type 2 firms, while (15) describes how ϕ evolves with V with steady state turnover and optimal wage setting by type 1 firms. Equilibrium requires these objects are consistent. Hence differentiating (14) with respect to V to obtain $d\phi/dV$, setting this equal to $d\phi/dV$ given by (15), and then using Claim 3 to substitute out the $d\widehat{w}_1/dV, d\widehat{w}_2/dV$, and $d\widehat{\Pi}_2/dV$ yields (13) and so determines the equilibrium $F'(V)$.

It is important to notice that (13) is singular at $V = V_1^\infty$ (where $\widehat{w}_1 = p_1, \widehat{\Pi}_1 = 0$) and there is a continuum of solutions to the above differential equation system. Indeed Phase A with $\phi \equiv 0$ describes such a solution but does not describe a Phase B equilibrium (where $\phi > 0$). To characterize the limiting equilibrium in Phase B as $V \to V_1^\infty$, define

$$f^A = \left[\frac{-u''(\widehat{w}_2^A)}{u'(\widehat{w}_2^A)^2}\right] \frac{2(p_2 - \widehat{w}_2^A)\left[p_2 - \widehat{w}_2^A - [\delta + \lambda(1 - F_A)]\widehat{\Pi}_2^A\right]}{\lambda[\widehat{\Pi}_2^A]^2}$$

which is $F'(V)$ according to Phase A at $V = V_1^\infty$ (i.e. put $\phi = 0$ and $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\widehat{w}_2^A, \widehat{\Pi}_2^A, F_A)$ in (13)). Also define the critical density $f^c = f^c(F)$:

$$f^c = \frac{2}{\lambda}\left[\frac{-u''(p_1)}{u'(p_1)^2}\right][\delta + \lambda(1-F)]^2.$$

Note that if $\phi/(p_1 - \widehat{w}_1) \to \infty$ as $V \to V_1^\infty$, then (13) implies $F' \to f^c$.[2]

Appendix B establishes there are three possible types of limiting equilibrium in Phase B where existence of each type depends on the magnitude of the density f^A relative to f^c. When f^A/f^c is small we say there is low poaching intensity, meaning that there are relatively few type 2 firms trying to attract type 1 employees at $V = V_1^\infty$.

Case 1: Low Poaching Intensity ($f^A/f^c < 3/8$).

If the poaching intensity of type 2 firms for type 1 employees is small (i.e. f_A is small), then a Phase B equilibrium potentially exists where $F'(V) \to f^A$ as $V \to V_1^\infty$. Given this low density of outside offers, type 1 firms raise wages relatively slowly with tenure and steady state turnover implies $\phi \to 0$ as $V \to V_1^\infty$. Appendix B establishes this equilibrium exists only if $f^A/f^c < 3/8$ as this ensures $\phi/(p_1 - \widehat{w}_1) \to 0$ as $V \to V_1^\infty$ (i.e. ϕ falls very quickly with V) and (13) then implies $F'(V) \to f^A$ as required. Hence for $f^A/f^c < 3/8$, a Phase B equilibrium (potentially) exists where $\{\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F\}$ evolve according to the differential equations described above with initial values

$$\lim_{V \to V_1^\infty} (\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F) = (p_1, \widehat{w}_2^A, 0, \widehat{\Pi}_2^A, F_A),$$

and F' is continuous across V_1^∞.

Case 2. Intermediate Poaching Intensity ($f^A/f^c < 1$).

A second type of limiting equilibrium is that $F'(V) \to f^c(F_A)$ as $V \to V_1^\infty$. Relative to case 1, the higher density of outside offers ($f^c > f^A$) implies type 1 firms

[2] this requires using l'Hopital's rule and (32) in the appendix.

raise wages more quickly with V. The difference is that ϕ now converges to zero slowly and instead $\phi/(p_1 - \widehat{w}_1) \to \infty$ as $V \to V_1^\infty$. (13) then implies $F' \to f^c(F_A)$ as required. Note that this equilibrium does not exist if $f^A > f^c(F_A)$ (otherwise (14) and $F'(V_1^\infty) = f^c < f^A$ implies $\phi < 0$ at $V = [V_1^\infty]^-$). Also note there may be multiple equilibria for $f^A < (3/8)f^c(F_A)$ as both cases 1 and 2 apply. The multiplicity arises as a higher density of outside offers imply type 1 firms raise wages more quickly with V which implies a higher density of type 1 employees around V_1^∞. This region then becomes a more profitable place for type 2 firms to try to poach type 1 employees.

Hence for $f^A < f^c(F_A)$, a Phase B equilibrium (potentially) exists where $\{\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F\}$ evolve according to the differential equations described above with initial values

$$\lim_{V \to V_1^\infty} (\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F) = (p_1, \widehat{w}_2^A, 0, \widehat{\Pi}_2^A, F_A)$$

and $\lim_{V \to V_1^\infty} F'(V) = f^c(F_A) > f^A$.

Case 3. High Poaching Intensity ($f^A/f^c > 1$).

Neither Case 1 nor 2 holds when $f^A/f^c > 1$. A high poaching intensity instead implies $\phi \to \overline{\phi} > 0$ as $V \to V_1^\infty$ and equilibrium now implies a mass point $m > 0$ in F at V_1^∞. The mass point $m > 0$ and Claim 3 imply $\widehat{w}_2$ is not continuous at V_1^∞ - it increases by a discrete amount across V_1^∞. Let $\widehat{w}_2^B = \lim_{V \to [V_1^\infty]^-} \widehat{w}_2(V)$,which is the limiting wage $\widehat{w}_2$ according to phase B. Appendix B establishes that

$$u(\widehat{w}_2^B) + u'(\widehat{w}_2^B)[p_2 - \widehat{w}_2^B - [\delta + \lambda(1 - F)]\widehat{\Pi}_2^A] = u(p_1) \tag{16}$$

with $F = F_A - m$. Contrasting with (12) which determines $\widehat{w}_2^A$ at the end of Phase A, it can be shown that $m > 0$ implies $\widehat{w}_2^B < \widehat{w}_2^A$.

To understand (16), note that at any salary point t on a type i baseline salary scale, an optimal quit strategy implies (1) which is rewritten here as

$$\lambda \int_{V_i(t)}^{\overline{V}} [x - V_i(t)] dF(x) = \delta V_i(t) - \frac{dV_i(t)}{dt} - u(w_i(t)). \tag{17}$$

Now Theorem 1 describes the optimal (profit maximizing) baseline salary scale. Using (7) and (8) in Theorem 1 to substitute out dV_i/dt in (17) yields:

$$(18)\quad \lambda \int_{V_i(t)}^{\overline{V}} [x - V_i(t)] dF(x) = \delta V_i(t) - u'(w_i(t))[p_i - w_i - [\delta + \lambda(1-F)]\Pi_i] - u(w_i(t)).$$

The left hand side of this expression is the expected surplus from receiving outside job offers. Note that this surplus, $S(V_i) = \lambda \int_{V_i}^{\overline{V}} [x - V_i] dF(x)$, is continuous across any mass points. Hence at salary point $t = t_2$ on baseline salary scale 2 where $V_2(t) = V_1^{\infty}$, the right hand side of the above expression must be continuous in t, even with a mass point $m > 0$. Note further that $V = V_1^{\infty}$ implies $w_1 = p_1$ and $\Pi_1 = 0$ on baseline salary scale 1. Hence at $V_i(t) = V_1^{\infty}$, the above expression implies

$$u(p_1) = u(w_2) + u'(w_2)[p_2 - w_2 - [\delta + \lambda(1-F)]\Pi_2],$$

which determines both $\widehat{w}_2^A$ (at $t = t_2^+$ where $F = F_A$) and $\widehat{w}_2^B$ (at $t = t_2^-$ with $F = F_A - m$).

Given $m > 0$ and $\widehat{w}_2^B$ given by (16), (14) implies

$$(19)\qquad \overline{\phi} = \frac{\delta}{u(p_1) - u(\widehat{w}_2^B)} \left[\frac{(p_2 - \widehat{w}_2^B)(p_2 - \overline{w})}{\delta^2 [\widehat{\Pi}_2^A]^2} - 1 \right].$$

Note that $m = 0$ implies $\widehat{w}_2^B = \widehat{w}_2^A$ and Theorem 2 in B/C then implies $\overline{\phi} = 0$. Hence a Phase B equilibrium with $\overline{\phi} > 0$ necessarily requires $m > 0$. Given such an m, $\{\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F\}$ evolve according to the differential equations described above with initial values

$$\lim_{V \to V_1^{\infty}} (\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F) = (p_1, \widehat{w}_2^B, 0, \widehat{\Pi}_2^A, F_A - m)$$

and $\lim_{V \to [V_1^{\infty}]^-} F'(V) = f^c(F_A - m)$.

The above describes three possible types of limiting equilibrium in Phase B as $V \to V_1^{\infty}$. It is explicit in case 3 that there is a continuum of possible solutions: there is a free choice of m. There is a continuum of solutions in the first two cases as well - the equilibrium dynamics imply $\phi \to 0$ with limiting solution of the form $\phi = A[V_1^\infty - V]^a$ where A is the constant of integration and is not determined.

This degree of freedom is tied down by the following boundary condition. Noting that Phase B begins with $F_A > \alpha_1$ and $G_A > U$, using backward iteration from $V = V_1^\infty$, Phase B ends when either $F = \alpha_1$ (there are no more type 2 firms and $G \geq U$) or $G = U$ (there are no more type 1 employees and $F \geq \alpha_1$). In either case, Claim 4 implies $\phi = \phi^B$ where

$$\phi^B = \frac{\delta - (\delta + \alpha_2 \lambda) G}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1]}$$

and $G = (p_2 - \overline{w})/(\delta \widehat{\Pi}_2)$.

Assuming a Phase B equilibrium can be found satisfying the initial values described above and the boundary condition $\phi = \phi^B$ at $V = \overline{V}_1$, we let $\widehat{w}_1^C, \widehat{\Pi}_1^C, F_C$,and G_C denote the corresponding values of $\widehat{w}_1, \widehat{\Pi}_1, F$,and G at the termination point and proceed to Phase C.

Phase C. $V \in [\underline{V}, \overline{V}_1]$.

If $G^C = U$ then we are done - there is a mass of firms F^C offering a contract with starting value $V = \underline{V} = \overline{V}_1$, where $F^C - \alpha_1$ type 2 firms offer the type 2 baseline salary scale while all type 1 firms offer the type 1 baseline salary scale.

Suppose instead $F^C = \alpha_1$ and $G^C > U$ which then implies Phase C is non-degenerate. The constant profit condition for type 1 firms implies $G\widehat{\Pi}_1 = G^C \widehat{\Pi}_1^C$. It then follows that $(\widehat{w}_1, \widehat{\Pi}_1, F)$ evolve according to the differential equations given in Claim 3 and

$$\frac{dF}{dV} = \left[\frac{-u''(\widehat{w}_1)}{u'(\widehat{w}_1)^2}\right] \frac{2\delta}{\lambda} \frac{\left[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1\right]}{G^C \widehat{\Pi}_1^C}$$

with initial values $(\widehat{w}_1^C, \widehat{\Pi}_1^C, \alpha_1)$.[3]

[3] Differentiate the constant profit condition $G\widehat{\Pi}_1 = G^C \widehat{\Pi}_1^C$ wrt V to get $\phi = G/(u'(\widehat{w}_1)\widehat{\Pi}_1)$ where

Iterating backwards, Phase C ends when $G = U$ (there are no more type 1 employees). At this termination point, define $V = \underline{V}$ and let $\underline{\widehat{w}_1}, \underline{\widehat{\Pi}_1}, \underline{F}$ denote the corresponding values of $(\widehat{w}_1, \widehat{\Pi}_1, F)$.

5.3 Equilibrium (determining $\overline{w}$).

For a given $\overline{w}$, a solution to phases A-C yields end values $(\underline{\widehat{w}_1}, \underline{\widehat{\Pi}_1}, \underline{F})$. The free choice of $\overline{w}$ is now tied down by the optimal job search condition which requires $V_u = \underline{V}$ where

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [x - V_u] dF(x).$$

Using (1), the same trick as described in Case 3 above establishes that $V_1(0) = \underline{V} = V_u$ occurs if and only if

$$u(\underline{\widehat{w}_1}) + u'(\underline{\widehat{w}_1})[p_1 - \underline{\widehat{w}_1} - [\delta + \lambda(1 - \underline{F})]\underline{\widehat{\Pi}_1}] = u(b). \tag{20}$$

In other words, given end values $(\underline{\widehat{w}_1}, \underline{\widehat{\Pi}_1}, \underline{F})$, we have identified a Market Equilibrium if b satisfies (20). Alternatively, given an arbitrary value of b, identifying a Market Equilibrium requires finding a $\overline{w}$ which implies end values $\underline{\widehat{w}_1}, \underline{\widehat{\Pi}_1}, \underline{F}$ satisfying (20). B/C formally establish the existence of such a $\overline{w}$ for the homogeneous firm case. Given the complexity of Phase B with heterogeneous firms, we do not provide an existence proof. Instead we use numerical examples to illustrate how firm heterogeneity affects equilibrium market behavior.

$\phi \equiv dG/dV$ in this phase. Claim 4 also implies

$$\phi = \frac{\delta - [\delta + \lambda(1 - F)]G}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F)]\widehat{\Pi}_1]}.$$

Setting these two expressions equal and simplifying yields

$$p_1 - \widehat{w}_1 = \frac{\delta \widehat{\Pi}_1}{G}.$$

Now differentiate wrt V. Using Claim 3 for $d\widehat{w}_1/dV$ implies the stated differential equation for F'.

6 Simulations

The object here is to present simulations of equilibria in the model presented above. In Appendix C a brief description of the program constructed to achieve this goal is presented. Here, we report on two experiments in the first we consider changes in the productivity of low productivity firms. In the second experiment we consider how equilibria changes as the proportion of low productivity firms changes.

In the first experiment we set $\lambda/\delta = 20$ so that a worker expects to receive 20 job offers over a working lifetime. Assuming a working career of 40 years, this implies an outside job offer on average every two years. This implies a sensible unemployment rate $U = 4.7\%$. The implied unemployment durations though are very high, being two years. Discounting between job offers is therefore high, $\delta/\lambda = 5\%$. with $\delta = 0.025$ and $\lambda = 0.5$. We assume $b = 43.75$ and $\alpha_1 = 0.50$.A CRRA utility function is used $u(w) = w^{1-\sigma}/(1-\sigma)$ with $\sigma = 1.2$.

In the experiment we vary p_1 holding $p_2 = 100$. The results are presented in Table 1 and Figures 1 and 2. Only when $p_1 = 55$ is a Case 1 equilibrium possible. A Case 2 equilibrium is feasible when $p_1 = 55$, 65, and 75.When $p_1 = 85$ or $p_1 = 95$,only a Case 3 equilibrium is possible

In the second experiment we vary the proportion of low productivity firms. In this case we assume the value of the parameters are $b = 60$, $\lambda = 0.5$, $\delta = 0.025$, $p_1 = 90$, $p_2 = 100$, Again we assume a $CRRA$ utility function with parameter $\sigma = 1.2$. The results are presented in Table 2 and Figures 3 and 4. In each case only a Case 3 equilibrium is feasible.

Appendix A - Proofs.

Proof of Claim 1

Suppose first $p_i \geq \overline{w}$, which implies $w_i^\infty = \overline{w}$ and $V_i^\infty = \overline{V}$. However, $p_i \geq \overline{w}$ implies it is jointly efficient for the firm and worker that the worker never quits

Now consider wage contract $w(t) = \overline{w}$ for all t. The Bellman equation (1) implies $V(t|w(.)) = \overline{V}$ for all t, and so the worker never quits. As this contract also offers full insurance and not quitting is incentive compatible, this contract is therefore jointly efficient. As this contract also extracts maximal employee rents - it offers expected lifetime payoff $V_0 = V_i^{\infty}$ as required - it describes the firm's optimal contract.

Consider now $p_i < \overline{w}$ which implies $w_i^{\infty} = p_i$ and $V_i^{\infty} < \overline{V}$. The wage contract $w(t) = p_i$ for all t; implies the firm makes zero profit and so the worker enjoys the full value of the match. Hence, the worker has efficient quit incentives - the worker will quit only if the value of an employment offer elsewhere exceeds the value of the current match. As side payments conditional on a quit are ruled out by assumption, this contract is therefore jointly efficient. As it also extracts maximal employee rents (given $V_0 = V_i^{\infty}$) it describes the firm's optimal contract. This completes the proof of Claim 1.

Proof of Claim 2

$1 - G(V)$ describes the steady state number of workers employed whose current lifetime utility is at least V. The proof is by contradiction - suppose in equilibrium that $1 - G$ has a mass $\mu > 0$ at $\overline{V}$. Strictly positive profit requires that any firm which in equilibrium offers starting payoff $V_0 = \overline{V}$ must have productivity $p_i > \overline{w}$. Further, Claim 1 implies that any such firm enjoys steady state profit flow $\Omega_i^*(\overline{V}) = \lambda(1-\mu)[(p_i - \overline{w})/\delta]$. Consider instead a firm of that type but which instead offers $w = \overline{w} + \varepsilon$ for some $\varepsilon > 0$ and so obtains flow profit $\Omega = \lambda[(p_i - \overline{w} - \varepsilon)/\delta]$. As $\overline{w} < p_i$ then $\mu > 0$ implies $\Omega > \Omega_i^*(\overline{V})$ for ε small enough which contradicts the definition of a market equilibrium. As $1 - G$ cannot have a mass point at $\overline{V}$, then Claim 1 implies F cannot have a mass point at $\overline{V}$. Finally, no mass point in G at $\overline{V}$ and Theorem 1 requires the stated restriction on F. This completes the proof of Claim 2.

Proof of Lemma 1.

Part (i). Fix a $p > b$ and a $V_0 \in [\underline{V}, V_p^{\infty})$. Now consider a $V \in [V_0, V_p^{\infty})$ and define $\widehat{t}(V, p|V_0)$ as the tenure point t where $V^*(t, p|V_0) = V$;i.e. $\widehat{t}$ is the tenure point where the employed worker has lifetime value V given the optimal wage contract. $V \in [V_0, V_p^\infty)$ implies $\widehat{t} \geq 0$ and is strictly increasing with V. For such V we can define

$$\widehat{w}(V, p|V_0) = w^*(\widehat{t}(.), p|V_0)$$

which is the wage paid at tenure $t = \widehat{t}(.)$; *i.e.* $\widehat{w}(.)$ is the wage paid at the tenure point where the worker obtains lifetime value V. Also define $\widehat{\Pi}(V, p|V_0) = \Pi^*(\widehat{t}(.), V|V_0)$. As V_0 is fixed throughout this proof, we simplify notation by subsuming reference to V_0 in these functions.

For given p and V_0, the time profiles of w^*, V^* are determined by (6) and (7) in Theorem 1. In particular,

$$\frac{\partial \widehat{w}}{\partial V} = \left[\frac{dw/dt}{dV/dt}\right]_{t=\widehat{t}} = \frac{\lambda F'(V)\widehat{\Pi}(.),}{\left[-\frac{u''(\widehat{w}(.))}{u'(\widehat{w}(.))^2}\right]\left[p - \widehat{w}(.) - [\delta + \lambda(1 - F(V))]\widehat{\Pi}(.)\right]}, \tag{21}$$

describes how wages paid $\widehat{w}$ change as the worker's expected lifetime value V increases with tenure along the optimal contract. Hence for given p, a first order differential equation describes $\widehat{w}$, where Claim 1 implies the boundary condition $\widehat{w}(V_p^\infty, p) = w_p^\infty$. Note (7) in Theorem 1 also implies $\partial\widehat{\Pi}/dV = -1/(u'(\widehat{w}))$.

The proof uses the following single crossing property. Given productivities $p' > p > b$, consider functions $\widehat{w}(., p), \widehat{w}(., p')$ defined above. Suppose a $V \in [V_0, V_p^\infty)$ exists where $\widehat{w}(V, p) = \widehat{w}(V, p')$. We now establish that at any such point (should one exist), then

$$\frac{\partial \widehat{w}}{\partial V}(V, p') > \frac{\partial \widehat{w}}{\partial V}(V, p);$$

i.e. these functions can only cross once.

Hence suppose a $V \in [V_0, V_p^\infty)$ exists where $\widehat{w}(V, p) = \widehat{w}(V, p')$ and fix $w = \widehat{w}$ at that point. Given (w, V), partial differentiation of the right hand side of (21) with respect to p yields

$$\frac{\partial}{\partial p}\left[\frac{\lambda F'(V)\widehat{\Pi}(V, p),}{\left[-\frac{u''(w)}{u'(w)^2}\right]\left[p - w - [\delta + \lambda(1 - F(V))]\widehat{\Pi}(V, p)\right]}\right] > 0$$

if and only if

$$(p-w)\frac{\partial\widehat{\Pi}(V,p)}{\partial p}-\widehat{\Pi}(V,p)>0.$$

Now the definition of $\widehat{\Pi}$ implies:

$$\widehat{\Pi}(V,p)\equiv\Pi(\widehat{t},p)=\int_{\tau=\widehat{t}}^{\infty}\frac{\Psi(\tau|w^*)}{\Psi(\widehat{t}|w^*)}[p-w^*(\tau,p)]d\tau.$$

Optimality of w^* (for given V) and the Envelope Theorem yield

$$\frac{\partial\widehat{\Pi}(V,p)}{\partial p}=\int_{s=\widehat{t}}^{\infty}\frac{\Psi(\tau|w^*)}{\Psi(\widehat{t}|w^*)}d\tau$$

as any variations in the optimal choice of $w^*(.)$ yield only second order profit effects. Hence

$$[p-w]\frac{\partial\widehat{\Pi}(V,p)}{\partial p}-\widehat{\Pi}(V,p)=\int_{\tau=\widehat{t}}^{\infty}\frac{\Psi(\tau|w^*)}{\Psi(\widehat{t}|w^*)}[w^*(\tau,p)-w^*(\widehat{t},p)]d\tau,$$

as $w=\widehat{w}(V,p)\equiv w^*(\widehat{t},p)$. But w^* is strictly increasing in tenure and as $\Psi(.)>0$ for all tenures (the worker never quits into unemployment), we have that

$$[p-w]\frac{\partial\widehat{\Pi}(V,p)}{\partial p}-\widehat{\Pi}(V,p)>0 \text{ for all } V\in[V_0,V_p^{\infty}).$$

The above now yields the desired single crossing property: if a $V\in[V_0,V_p^{\infty})$ exists where $\widehat{w}(V,p)=\widehat{w}(V,p')$, then at that point, (21) implies

$$\frac{\partial\widehat{w}}{\partial V}(V,p')>\frac{\partial\widehat{w}}{\partial V}(V,p). \tag{22}$$

Now use backward induction from $V=V_p^{\infty}$, where Claim 1 implies $\widehat{w}(V_p^{\infty},p)=w_p^{\infty}$. Suppose first that $p<\overline{w}$ which implies $w_p^{\infty}<\overline{w}$ and $V_p^{\infty}<\overline{V}$. As $p'>p$ implies $V_{p'}^{\infty}>V_p^{\infty}$ then, as wages are increasing in tenure, we have $\widehat{w}(V_p^{\infty},p')<w_p^{\infty}$. Hence $\widehat{w}(V,p')<\widehat{w}(V,p)$ at $V=V_p^{\infty}$. (22) now implies these two functions cannot meet at any $V\in[V_0,V_p^{\infty})$, and so $\widehat{w}(V_0,p')<\widehat{w}(V_0,p)$. This establishes part (i) for $p<\overline{w}$.

Suppose now that $p\geq\overline{w}$ which implies $w_p^{\infty}=\overline{w}$ and $V_p^{\infty}=\overline{V}$. Hence for $p'>p$ we have $\widehat{w}(V_p^{\infty},p')=\widehat{w}(V_p^{\infty},p)=\overline{w}$. Noting that Claim 2 implies $F'(\overline{V})=0$, L'Hopital's rule at $V = \overline{V}$ implies

$$(23) \qquad \frac{\partial \widehat{w}}{\partial V} = \frac{\lambda F''(\overline{V})\overline{\Pi},}{\left[-\frac{u''(\overline{w})}{u'(\overline{w})^2}\right]\left[-\frac{\partial \widehat{w}}{\partial V} + \delta/u'(\overline{w})\right]}.$$

Solving implies a quadratic for $\partial\widehat{w}/dV$. There are two solutions but the only relevant one, the one which implies $\partial\widehat{w}/dV > 0$ (and so corresponds to the stable manifold), yields

$$u'(\overline{w})\frac{\partial \widehat{w}}{\partial V}(\overline{V},.) = \frac{1}{2}[\delta + \sqrt{\delta^2 + 4\lambda(p-\overline{w})u'(\overline{w})^4(-F''(\overline{V}))/[-\delta u''(\overline{w})]}]$$

where given $F'(\overline{V}) = 0$, A2 requires $F''(\overline{V}) < 0$. Hence for $p' > p \geq \overline{w}$, we have $\partial\widehat{w}(\overline{V},p')/\partial V > \partial\widehat{w}(\overline{V},p)/\partial V$. (22) now implies $\widehat{w}(V_0,p') < \widehat{w}(V_0,p)$ for $V_0 < V_p^\infty$ which completes the proof of part (i).

Part (ii). Fix $p > b$. Theorem 1 describes the optimal time profiles of V^*, Π^*, and so

$$\frac{\partial\widehat{\Pi}(V,p)}{\partial V} \equiv \left[\frac{d\Pi/dt}{dV/dt}\right]_{\widehat{t}}$$

where profit $\widehat{\Pi}$ changes as the worker's expected lifetime value V increases with tenure along the optimal contract, and $d\Pi/dt, dV/dt$ are defined by (7) and (8) in Theorem 1. Putting $V = V_0$ and using (7) implies

$$\frac{\partial\widehat{\Pi}(V_0,p)}{\partial V} = \left[\frac{d\Pi/dt}{dV/dt}\right]_{\widehat{t}=0} = -1/u'(w^*(0,p|V_0)).$$

Part (i) above now completes the proof.

Proof of Claim 4.

(i) As positive profit implies $V_u \leq \underline{V}$, an unemployed worker accepts the first job offer received, which implies $U = \delta/(\lambda+\delta)$ in a steady state.

(ii) For $V < \overline{V}_1$, steady state turnover on baseline salary scale 1, over arbitrarily short period of time $\Delta > 0$, implies

$$\begin{aligned} U\lambda\Delta\alpha_1 F_1(V) &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + \widehat{N}_1(V)[\delta + \lambda[1-\alpha_1 F_1(V)]]\Delta + o(\Delta). \end{aligned}$$

where the left hand side describes the inflow of workers who earn no more than V,and the right hand side describes outflow - the first term describes those whose tenure rises above $\widehat{t}_1(V)$, the second is the outflow due to exiting the labor market or receiving a better outside offer. Taking the limit $\Delta \to 0$ implies the condition stated.

(iii) for $V \in [V_{p_1}^{\infty}, \overline{V}]$,steady state turnover on baseline salary scale 2, over arbitrarily short period of time $\Delta > 0$, implies

$$\begin{aligned} [\overline{N}_2 - \widehat{N}_2]\delta\Delta &= [\widehat{N}_2(V) - \widehat{N}_2(V - [dV_2(\widehat{t}_2)/dt]\Delta)] \\ &\quad + [U + \overline{N}_1 + \widehat{N}_2(V)]\lambda\Delta\alpha_2[1 - F_2(V)] + o(\Delta) \end{aligned}$$

where the left hand side describes the outflow of workers with payoff greater than V, and the right hand side describes the inflow - those whose tenure rises above $\widehat{t}_2(V)$,and those who receive an outside offer greater than V. Rearranging and letting $\Delta \to 0$ implies the condition stated.

(iv) For $V \in (\overline{V}_1, V_1^{\infty})$, steady state turnover on baseline salary scale 1, over arbitrarily short period of time $\Delta > 0$, implies

$$\begin{aligned} U\lambda\alpha_1\Delta &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + \widehat{N}_1(V)\delta\Delta + \int_{\underline{V}}^{V} d\widehat{N}_1(x)\lambda\Delta[\alpha_2(1 - F_2(x))] + o(\Delta). \end{aligned}$$

where the left hand side describes the inflow of workers onto baseline salary scale one with payoff less than V, and the right hand side describes the outflow due to increasing tenure, exiting the labor market or receiving a better outside offer from a type 2 firm. Rearranging and letting $\Delta \to 0$ implies the first condition stated.

For $V \in (\overline{V}_1, V_1^{\infty})$, steady state turnover implies the total number of employees with payoff less than V satisfies

$$\begin{aligned} U\lambda[\alpha_1 + \alpha_2 F_2(V)]\Delta &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + [\widehat{N}_2(V) - \widehat{N}_2(V - [dV_2(\widehat{t}_2)/dt]\Delta)] \\ &\quad + [\widehat{N}_1(V) + \widehat{N}_2(V)][\delta + \lambda\alpha_2(1 - F_2(V))]\Delta \end{aligned}$$

where the left hand side describes the inflow, the right hand side describes the outflow due to tenure effects (both types), exiting the labor market and receiving an outside offer greater than V. Rearranging and letting $\Delta \to 0$ implies the second condition. This completes the proof of Claim 4.

Proof of Proposition 2

Note, given the equilibrium functions $\{\widehat{w}_i, \widehat{\Pi}_i, \widehat{N}_i, \widehat{t}_i\}$, then $V_i(t)$ is identified as the inverse function of $\widehat{t}_i$. The baseline salary scale $w_i(t)$ corresponds to $\widehat{w}_i(V_i(t))$ and profit $\Pi_i(t)$ is $\widehat{\Pi}_i(V_i(t))$. We now prove these conditions are necessary and sufficient for a Market Equilibrium.

Necessary. Strictly positive profit for type 1 firms implies $V_u \leq \underline{V}$. Suppose for the moment $V_u < \underline{V}$. Now consider the conditions of Theorem 1 for a type 1 firm ($p = p_1$), but with starting payoff $V_0 \in [V_u, \underline{V})$. The optimal contract implies an early tenure phase for some $\tau > 0$, where the firm pays a fixed wage $w_0 \in (0, w_1(0)]$ for tenures $t < \tau$, and wages $w = w_1(t - \tau)$ for tenures $t \geq \tau$. It is straightforward to argue that this contract generates strictly greater profit per hire.[4] As the hiring rate is the same as for a type 1 firm offering $V = \underline{V}$, this implies $\Omega_1^*(V_0) > \Omega_1^*(\underline{V})$ which contradicts the definition of a market equilibrium. Hence $V_u = \underline{V}$ is a necessary condition of equilibrium.

Proposition 1 and Claims 2-4 now imply Proposition 2 describes necessary conditions for a market equilibrium.

Sufficient. By construction of the type i baseline salary scales, any type i firm offering

[4] Using the arguments in the next section (see equation (16)), w_0 is in fact given by:

$$u(w_0) + u'(w_0)[p_1 - w_0 - [\delta + \lambda]\Pi_1(0)] \tag{24}$$

$$= u(w_1) + u'(w_1)[p_1 - w_1 - [\delta + \lambda(1 - F(\underline{V})]\Pi_1(0)] \tag{25}$$

where $w_1 = w_1(0)$ and $F(\underline{V}) \geq 0$ is the mass of type 1 firms offering $V = \underline{V}$. As the conditions of Theorem 1 imply the continuation profit $\Pi(.)$ is strictly decreasing along the optimal contract, then this earlier phase must yield greater profit (conditional on a hire).

$V \in [\underline{V}_i, \overline{V}^i]$ implies the firm is offering an optimal wage contract, and the constant profit condition (9) guarantees that each optimal contract offered generates the same steady state profit $\overline{\Omega}_i > 0$. Further for type 1 firms, offering $V_0 < \underline{V} = V_u$ generates zero profit, while Lemma 1(ii) and the constant profit condition for type 2 firms implies offering $V_0 > \overline{V}_1$ yields strictly lower profit. Similarly for type 2 firms. Lemma 1(ii) and the constant profit condition for type 1 firms implies offering $V_0 < \overline{V}_1$ yields strictly lower profit, while offering $V_0 > \overline{V}$ attracts no more workers than a contract which offers $V = \overline{V}$ and pays a strictly higher wage. Hence a solution to the conditions of Proposition 2 imply conditions (ME1)-(ME6) are satisfied and so describes a market equilibrium.

Appendix B - Equilibrium Phase B dynamics.

During Phase B, Claim 3 establishes that $(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2)$ evolve according to

$$\frac{d\widehat{w}_1}{dV} = \left[\frac{u'(\widehat{w}_1)}{-u''(\widehat{w}_1)}\right] \frac{\lambda F' \widehat{\Pi}_1}{\left[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1\right]} \tag{26}$$

$$\frac{d\widehat{w}_2}{dV} = \left[\frac{u'(\widehat{w}_2)}{-u''(\widehat{w}_2)}\right] \frac{\lambda F' \widehat{\Pi}_2}{\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2\right]} \tag{27}$$

$$\frac{d\widehat{\Pi}_1}{dV} = -\frac{1}{u'(\widehat{w}_1)} \tag{28}$$

$$\frac{d\widehat{\Pi}_2}{dV} = -\frac{1}{u'(\widehat{w}_2)} \tag{29}$$

Claim 4 implies $(\widehat{N}_1, \widehat{N}_2)$ evolve according to

$$\frac{d\widehat{N}_1}{dV} = \left[\alpha_1 \lambda U - \delta \widehat{N}_1 - \alpha_2 \lambda \int_{\underline{V}}^{V} [1 - F_2(x)] d\widehat{N}_1(x)\right] \frac{d\widehat{t}_1}{dV}, \tag{30}$$

$$\frac{d\widehat{N}_1}{dV} / \frac{d\widehat{t}_1}{dV} + \frac{d\widehat{N}_2}{dV} / \frac{d\widehat{t}_2}{dV} = \lambda U[\alpha_1 + \alpha_2 F_2(V)] - [\delta + \lambda \alpha_2 [1 - F_2(V)]] \left[\widehat{N}_1 + \widehat{N}_2\right], \tag{31}$$

where $d\widehat{t}_i/dV$ is given by Claim 3. The constant profit condition requires

$$G\widehat{\Pi}_2 = \overline{\Pi}_2 \tag{constant profit}$$

where $\overline{\Pi}_2 = (p_2 - \overline{w})/\delta$ and $G = U + \widehat{N}_1 + \widehat{N}_2$.

The objective here is to show this system reduces to the one described in the text. We do this in three steps. The first step derives the reduced form dynamics for

$(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F)$ during Phase B. The second step considers the limiting dynamics as $V \rightarrow [V_1^{\infty}]^-$ and the third step considers the limiting dynamics as $V \rightarrow [\overline{V}_1]^+$. We start with two useful algebraic conditions.

Claim B1.

During Phase B:

$$(32) \quad \text{(i)} \frac{d}{dV}\left[u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta + \lambda(1 - F(V))]\widehat{\Pi}_i]\right] = \delta + \lambda(1 - F) - u'(\widehat{w}_i)\frac{d\widehat{w}_i}{dV}$$

$$\text{(ii) } u(\widehat{w}_1)+u'(\widehat{w}_1)[p_1-\widehat{w}_1-[\delta+\lambda(1-F)]\widehat{\Pi}_1] = u(\widehat{w}_2)+u'(\widehat{w}_2)[p_2-\widehat{w}_2-[\delta+\lambda(1-F)]\widehat{\Pi}_2]$$

Proof:

Claim 3 and straightforward algebra establish (i). Claim B1(ii) is a direct corollary of (i). This completes the proof.

Step 1. Characterizing the equilibrium dynamics in Phase B for $V \in (\overline{V}_1, V_1^{\infty})$.

Differentiating the (constant profit) condition $G\widehat{\Pi}_2 = \overline{\Pi}_2$ with respect to V yields

$$G\frac{d\widehat{\Pi}_2}{dV} + [\frac{d\widehat{N}_1}{dV} + \frac{d\widehat{N}_2}{dV}]\widehat{\Pi}_2 = 0,$$

which can be rearranged as

$$\phi = \frac{\overline{\Pi}_2}{u'(\widehat{w}_2)\widehat{\Pi}_2^2} - \frac{d\widehat{N}_2}{dV}$$

where $\phi = d\widehat{N}_1/dV$. Now solve (31) for $d\widehat{N}_2/dV$. Noting that $\widehat{N}_1 + \widehat{N}_2 \equiv G - U$, $U = \delta/(\lambda + \delta)$ and $G = \overline{\Pi}_2/\widehat{\Pi}_2$,(31) implies

$$(33) \qquad \frac{d\widehat{N}_2}{dV} = \frac{\delta - [\delta + \lambda(1 - F)]\,\frac{\overline{\Pi}_2}{\widehat{\Pi}_2} - u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F)]\widehat{\Pi}_1]\phi}{u'(\widehat{w}_2)[p_2 - \widehat{w}_2 - [\delta + \lambda(1 - F)]\widehat{\Pi}_2]}.$$

Using this condition to substitute out $d\widehat{N}_2/dV$ in the previous expression, simplifying using Claim B1(ii) then yields

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)}\left[\frac{(p_2 - \widehat{w}_2)(p_2 - \overline{w})}{\delta^2\widehat{\Pi}_2^2} - 1\right]$$

which is (14) stated in the text.

(30) describes how ϕ changes with V and can be rearranged as

(34)
$$\phi[u'(\widehat{w}_1)[p_1-\widehat{w}_1-[\delta+\lambda(1-F(V))]\widehat{\Pi}_1]] = \left[\alpha_1\lambda U - \delta\widehat{N}_1 - \alpha_2\lambda\int_{\underline{V}}^{V}[1-F_2(x)]d\widehat{N}_1(x)\right],$$

Differentiating with respect to V (use Claim B1(i) or it will take you all day)) then yields (15).

Differentiating (14) with respect to V to obtain $d\phi/dV$, setting this equal to $d\phi/dV$ given by (15) and using claim 3 to substitute out the $d\widehat{w}_1/dV$, $d\widehat{w}_2/dV$, and $d\widehat{\Pi}_2/dV$ terms yields (13). Hence (13) with ϕ given by (14) and (26)-(29) are necessary conditions describing the equilibrium evolution of $\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F$ during Phase B.

Step 2. The Limiting Equilibrium as $V \to [V_1^\infty]^-$.

In the limit as $V \to [V_1^\infty]^-$, suppose $F'(V) \to f^\infty$, $F(V) \to F^\infty \leq F_A$ and that a first order Taylor expansion for ϕ implies

$$\begin{aligned} \phi(V) &\to A(V_1^\infty - V)^a \\ \frac{d\phi}{dV} &\to -aA(V_1^\infty - V)^{a-1} \end{aligned}$$

where $A > 0$ and a, f^∞, F^∞ are to be determined.

Note that (26) is singular at V_1^∞. Let $[d\widehat{w}_1/dV]^\infty$ denote the limit value of $d\widehat{w}_1/dV$ as $V \to V_1^\infty$. Using Claim B1(i) and l'Hopital's rule in (26) imply

$$[\frac{d\widehat{w}_1}{dV}]^\infty = \lim_{V\to[V_1^\infty]^-}\frac{d\widehat{w}_1}{dV} = \lim_{V\to[V_1^\infty]^-}\frac{\lambda F'\left[\frac{u'(\widehat{w}_1)^2}{-u''(\widehat{w}_1)}\right]\frac{d\widehat{\Pi}_1}{dV}}{\delta+\lambda(1-F)-u'(\widehat{w}_1)\frac{d\widehat{w}_1}{dV}} \tag{35}$$

$$= \frac{-\lambda f^\infty\left[\frac{u'(p_1)}{-u''(p_1)}\right]}{\delta+\lambda(1-F^\infty)-u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty} \tag{36}$$

Rearranging as a quadratic for $[d\widehat{w}_1/dV]^\infty$ and noting that $d\widehat{w}_1/dV > 0$ along the baseline salary scale implies

$$[\frac{d\widehat{w}_1}{dV}]^\infty = \frac{[\delta+\lambda(1-F^\infty)]+\sqrt{[\delta+\lambda(1-F^\infty)]^2+4\left[\frac{u'(p_1)^2\lambda f^\infty}{-u''(p_1)}\right]}}{2u'(p_1)}. \tag{37}$$

A first order Taylor expansion about V_1^∞ now implies

$$\widehat{w}_1 \to p_1 + (V - V_1^\infty)[\frac{d\widehat{w}_1}{dV}]^\infty$$

as $V \to V_1^\infty$, while Claim B1(i) implies

$$u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F(V))]\widehat{\Pi}_1] \to (V - V_1^\infty)[\delta + \lambda(1 - F^\infty) - u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty].$$

Now consider (15) written as

$$\frac{1}{\phi}\frac{d\phi}{dV} - \frac{u'(\widehat{w}_1)\frac{d\widehat{w}_1}{dV} - 2[\delta + \lambda(1 - F)]}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F(V))]\widehat{\Pi}_1]} = 0.$$

The above first order Taylor expansions yield:

$$\frac{-aA(V_1^\infty - V)^{a-1}}{A(V_1^\infty - V)^a} - \frac{u'(\widehat{w}_1)\frac{d\widehat{w}_1}{dV} - 2[\delta + \lambda(1 - F)]}{(V - V_1^\infty)[\delta + \lambda(1 - F^\infty) - u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty]} \to 0$$

as $V \to [V_1^\infty]^-$. Hence

$$a = \frac{2[\delta + \lambda(1 - F^\infty)] - u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty}{u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty - [\delta + \lambda(1 - F^\infty)]}$$

and so a depends on f^∞ (via the $[d\widehat{w}_1/dV]^\infty$ term).

Now consider f^∞ as implied by (13). $a > 1$ implies $\phi/(p_1 - \widehat{w}_1) \to 0$ as $V \to V_1^\infty$ and (13) then implies

$$f^\infty = \frac{2(p_2 - \widehat{w}_2)\delta\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1 - F^\infty)]\widehat{\Pi}_2\right]}{\delta\widehat{\Pi}_2^2\lambda\left[\frac{u'(\widehat{w}_2)^2}{-u''(\widehat{w}_2)}\right]}. \tag{38}$$

A little work using Theorem 2 in B/C establishes that if $(\widehat{w}_2, \widehat{\Pi}_2, F) \to (\widehat{w}_2^A, \widehat{\Pi}_2^A, F_A)$ in this limit, then $f^\infty = f^A$ which is the density of offers according to Phase A at $V = V_1^\infty$.

If instead $a < 1$, then $\phi/(p_1 - \widehat{w}_1) \to \infty$ as $V \to V_1^\infty$. (13) and l'Hopital's rule then imply

$$f^\infty = \frac{2[\delta + \lambda(1 - F^\infty)][u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty - [\delta + \lambda(1 - F^\infty)]]}{\lambda\left[\frac{u'(p_1)^2}{-u''(p_1)}\right]} \tag{39}$$

Substituting out $[d\widehat{w}_1/dV]^\infty$ and solving for f^∞ yields

$$f^\infty = \frac{2}{\lambda}\frac{-u''(p_1)}{u'(p_1)^2}[\delta + \lambda(1 - F^\infty)]^2.$$

This critical density is denoted $f^c(F^\infty)$ in the text. Further, $f^\infty = f^c(F^\infty)$ implies $a = 0$.

The knife-edge case $a = 1$ implies $\phi/(p_1 - \widehat{w}_1) \to A \geq 0$ and monotonicity of F' wrt ϕ in (13) implies f^∞ takes an intermediate value between $f^c(F^\infty)$ and f^A. But $a = 1$ also requires

$$\frac{2[\delta + \lambda(1 - F^\infty)] - u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty}{u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty - [\delta + \lambda(1 - F^\infty)]} = 1$$

and substituting out $[d\widehat{w}_1/dV]^\infty$ then implies $f^\infty = (3/8)f^c(F^\infty)$. We are now in a position to characterize the set of possible limiting equilibria as $V \to V_1^\infty$.

(i) First suppose there are no mass points in F. This implies $(\widehat{w}_2, \widehat{\Pi}_2, F) \to (\widehat{w}_2^A, \widehat{\Pi}_2^A, F_A)$ as $V \to [V_1^\infty]^-$. (14) and Theorem 2 in B/C then imply $\phi \to 0$ in this limit. Note also that $A > 0$ requires $f^\infty \geq f^A$ (otherwise (14) implies $\phi < 0$ at $V = [V_1^\infty]^-$).

Now any equilibrium with $a > 1$ implies $f^\infty = f^A$. As a is given by

$$a = \frac{2[\delta + \lambda(1 - F_A)] - u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty}{u'(p_1)[\frac{d\widehat{w}_1}{dV}]^\infty - [\delta + \lambda(1 - F_A)]},$$

then $f^\infty = f^A$ and $a > 1$ requires $f^A < (3/8)f^c(F^A)$. Hence a limiting solution exists when $f^A < (3/8)f^c(F^A)$ where $F(V) \to F^A$, $f^\infty = f^A$ and $\phi/(p_1 - \widehat{w}_1) \to 0$ as $V \to V_1^\infty$.

Any equilibrium with $a < 1$ implies $f^\infty = f^c(F_A)$ which automatically implies $a = 0$. Hence, a limiting solution exists when $f^A < f^c(F^A)$ where $F(V) \to F^A$, $f^\infty = f^c(F^A)$ and $\phi/(p_1 - \widehat{w}_1) \to \infty$ as $V \to V_1^\infty$.

Limiting solutions with $a = 1$ may exist but will not generically describe a Market Equilibrium. To see why, suppose $a = 1$ and so $\phi \to A(V_1^\infty - V)$ as $V \to V_1^\infty$. But $a = 1$ requires $f^\infty = (3/8)f^c(F_A)$ and (13) then uniquely determines A. As there is no degree of freedom, this solution will not generically satisfy the boundary condition $\phi = \phi^B$ at $V = \overline{V}_1$ as discussed in Step 3.

(ii) Now consider equilibria with a mass point $m > 0$ in F at V_1^∞. Claim 3 implies $\widehat{w}_2$ is not continuous across V_1^∞, but $\widehat{\Pi}_2$ is continuous and equals $\widehat{\Pi}_2^A$. Note that (27) implies

$$\frac{d\widehat{w}_2}{dV} = \left[\frac{u'(\widehat{w}_2)}{-u''(\widehat{w}_2)}\right] \frac{\lambda \widehat{\Pi}_2^A}{\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2^A\right]} \frac{dF}{dV}$$

at $V = V_1^\infty$. Integrating across $V = V_1^\infty$, this differential equation implies

$$d\widehat{w}_2 = \left[\frac{u'(\widehat{w}_2)}{-u''(\widehat{w}_2)}\right] \frac{\lambda \widehat{\Pi}_2^A}{\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2^A\right]} dF$$

where $F \in [F_A - m, F_A]$ across the corresponding mass point. Inspection establishes this differential equation has solution:

$$u(\widehat{w}_2) + u'(\widehat{w}_2)\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2^A\right] = \text{constant}.$$

The termination criteria for Phase A implies this constant equals $u(p_1)$ at $F = F^A$. Hence at $F = F^\infty = F^A - m$, $\widehat{w}_2 = \widehat{w}_2^B$ is given by

$$u(\widehat{w}_2^B) + u'(\widehat{w}_2^B)\left[p_2 - \widehat{w}_2^B - [\delta + \lambda(1-F^\infty)]\widehat{\Pi}_2^A\right] = u(p_1)$$

and $m > 0$ implies $\widehat{w}_2^B < \widehat{w}_2^A$. (14) now implies $\phi \to \overline{\phi}(m) > 0$ where

$$\overline{\phi} = \frac{\delta}{u(p_1) - u(\widehat{w}_2^B)} \left[\frac{(p_2 - \widehat{w}_2^B)(p_2 - \overline{w})}{\delta^2[\widehat{\Pi}_2^A]^2} - 1\right].$$

Note all such equilibrium imply $\phi/(p_1 - \widehat{w}_1) \to \infty$ as $V \to V_1^\infty$. (13) then implies $f^\infty = f^c(F^\infty)$, which in turn implies $a = 0$. Hence given $m > 0$, a limiting solution exists where $F(V) \to F^A - m$, $f^\infty = f^c(F^A - m)$ and $\phi \to \overline{\phi}$ as $V \to V_1^\infty$.

Step 3. The boundary condition at $V = \overline{V}_1$.

For $V \in (\overline{V}_1, V_1^\infty)$, (30) implies ϕ evolves as:

$$\phi(V) = \left[\alpha_1 \lambda U - \delta \widehat{N}_1 - \alpha_2 \lambda \int_{\underline{V}}^{V} [1 - F_2(x)]\phi(x)dx\right] \frac{d\widehat{t}_1}{dV}.$$

As $F_2 = 0$ for all $V < \overline{V}_1$, taking the limit $V \to [\overline{V}_1]^+$ implies

$$\phi(V) \to \left[\alpha_1 \lambda U - [\delta + \alpha_2 \lambda] \widehat{N}_1(\overline{V}_1)\right] \frac{d\widehat{t}_1}{dV}.$$

But $\widehat{N}_2(\overline{V}_1) = 0$ and so $G(\overline{V}_1) = U + \widehat{N}_1(\overline{V}_1)$. Noting $U = \delta/(\lambda + \delta)$, we therefore have the boundary condition $\phi \to \phi^B$ as $V \to [\overline{V}_1]^+$ where

$$\phi^B = \frac{\delta - (\delta + \alpha_2 \lambda) G}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F)]\widehat{\Pi}_1]}$$

at $V = \overline{V}_1$. This completes the characterization of the equilibrium conditions for Phase B.

Appendix C - Simulation Program

We used an adaptive step size Runge-Kutta methods to solve for the equilibria described in the body of the paper Subroutines: odeint, rkqs, and rkck from Numerical Recipes in Fortran 90, along with the necessary modules nr, nrutil, nrtype, and nrerror.1 (See http://www.library.cornell.edu/nr/bookf90pdf/chap16f9.pdf)

The program works in the following way

1) The program takes as given the parameters b, p_1, p_2, α_1, δ, λ, and the constant relative risk aversion parameter σ.

2) For a candidate $\overline{w}$, the program solves Phase A according to the appropriate system of differential equations and initial conditions. It uses a bisection routine to identify the values of w2, ■2, F, and V that coincide with the end of Phase A. The bisection stops when these computed values are within 1*10-7 of the truevalues.

3) The program then identifies the case if a case 1 or a case equilibrium is possible.

4) For cases 1 or 2, we assign the appropriate value for dF/dV at $V1\infty$ and solve Phase B according to the appropriate system of differential equations with initial conditions given by the end of Phase A. For case 3, we search across all values of m (where $F = F_A - m$) between 0 and F_A for a candidate Phase B that satisfies the given condition on ■. Note here that each m implies a unique starting value for w_2

and $\blacksquare$. Again, we identify the values of w_1, w_2, $\blacksquare_1$, $\blacksquare_2$, F, G, $\blacksquare$, and V at the end of Phase B using a bisection routine.

5) Finally, we solve Phase C according to the appropriate system of differential equations and initial conditions given by the end of Phase B. It uses a bisection routine to identify the values of w_1, $\blacksquare_1$, F, G, and V that coincide with the end of Phase C.

6) Given this candidate equilibrium, we calculate the difference between the left and right side of equation (20) in the text. Again, we use a bisection routine (here with accuracy 1E-6) to identify the $\overline{w}$ which minimizes the difference between the left and right side of equation (20).

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Table 1

p_1	55.000	65.000	75.000	85.000	95.000	99.000
At $\overline{V}$						
$\overline{V}$	3979.965	3981.757	3983.730	3980.707	3988.030	3991.552
$\overline{w}$	99.499	99.544	99.593	99.518	99.701	99.789
$\overline{\Pi}$	20.035	18.243	16.270	19.293	11.970	8.448
At $\{V_1^\infty\}^+$						
F	0.535	0.576	0.635	0.779	0.899	0.994
G	0.077	0.077	0.077	0.094	0.102	0.219
w_2	15.685	22.073	30.920	45.463	71.230	95.634
At $\{V_1^\infty\}^-$						
m	$0.000^{(1)}$	$0.000^{(2)}$	$0.000^{(2)}$	0.112	0.176	0.064
At $\underline{V}_2$						
w_1	46.581	48.990	50.561	59.304	59.135	41.736
w_2	12.340	14.690	17.520	17.770	31.873	35.304
G	0.028		0.070	0.088	0.078	0.059
At $\underline{V}_1$						
w_1	34.647	28.917	24.409	21.825	18.903	12.833
F	0.387	0.308	0.244	0.086	0.049	0.085
G	0.047	0.046	0.0460	0.047	0.047	0.047

Table 2

α_1	0.100	0.200	0.300	0.400	0.500
At $\overline{V}$					
$\overline{V}$	3993.061	3992.021	3990.482	3988.120	3984.406
$\overline{w}$	99.827	99.801	99.762	99.703	99.610
$\overline{\Pi}$	6.938	7.979	9.518	11.880	15.593
At $\{V_1^\infty\}^+$					
F	0.688	0.715	0.748	0.783	0.822
G	0.069	0.065	0.071	0.081	0.094
w_2	52.392	52.880	53.504	54.310	55.533
At $\{V_1^\infty\}^-$					
m	0.117	0.141	0.161	0.173	0.177
At $\underline{V}_2$					
w_1	42.153	48.776	55.162	61.463	67.324
w_2	14.741	17.642	20.165	22.483	24.605
G	0.049	0.055	0.062	0.072	0.086
At $\underline{V}_1$					
w_1	39.585	39.393	38.929	38.301	36.772
F	0.065	0.082	0.107	0.148	0.190
G	0.470	0.048	0.046	0.048	0.048

α_1	0.600	0.700	0.800	0.900
At $\overline{V}$				
$\overline{V}$	3978.207	3966.893	3943.437	3883.157
$\overline{w}$	99.455	99.172	98.586	97.079
$\overline{\Pi}$	21.792	33.107	56.563	116.843
At $\{V_1^\infty\}^+$				
F	0.864	0.906	0.947	0.982
G	0.113	0.144	0.201	0.334
w_2		60.142	64.907	73.775
At $\{V_1^\infty\}^-$				
m	0.171	0.155	0.125	0.077
At $\underline{V}_2$				
w_1	72.978	78.438	83.269	87.344
w_2	26.922	29.869	34.517	44.468
G	0.106	0.137	0.194	0.326
At $\underline{V}_1$				
w_1	34.501	31.829	27.543	24.571
F	0.232	0.273	0.293	0.312
G	0.048	0.048	0.048	0.048

Figure 1

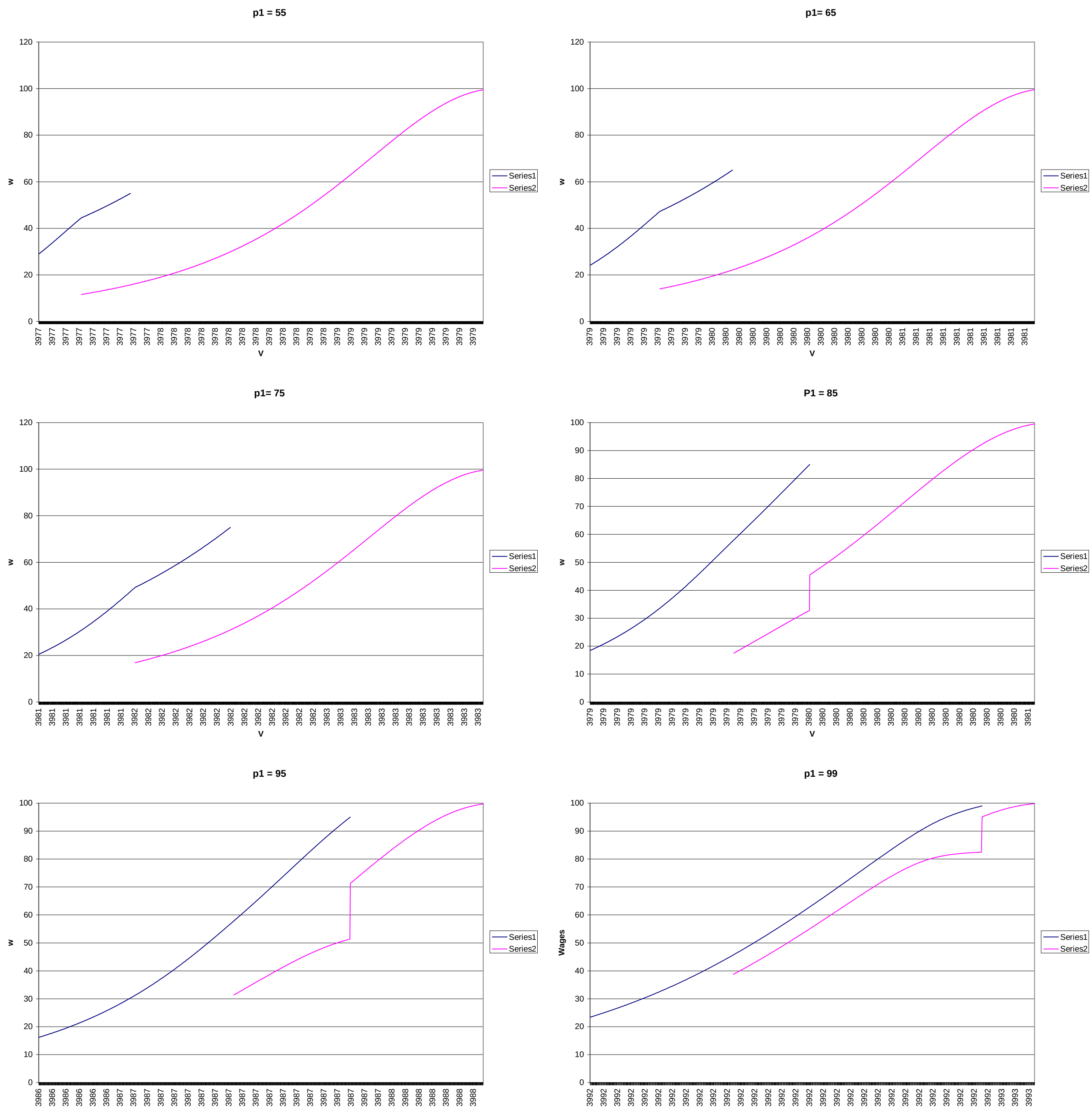
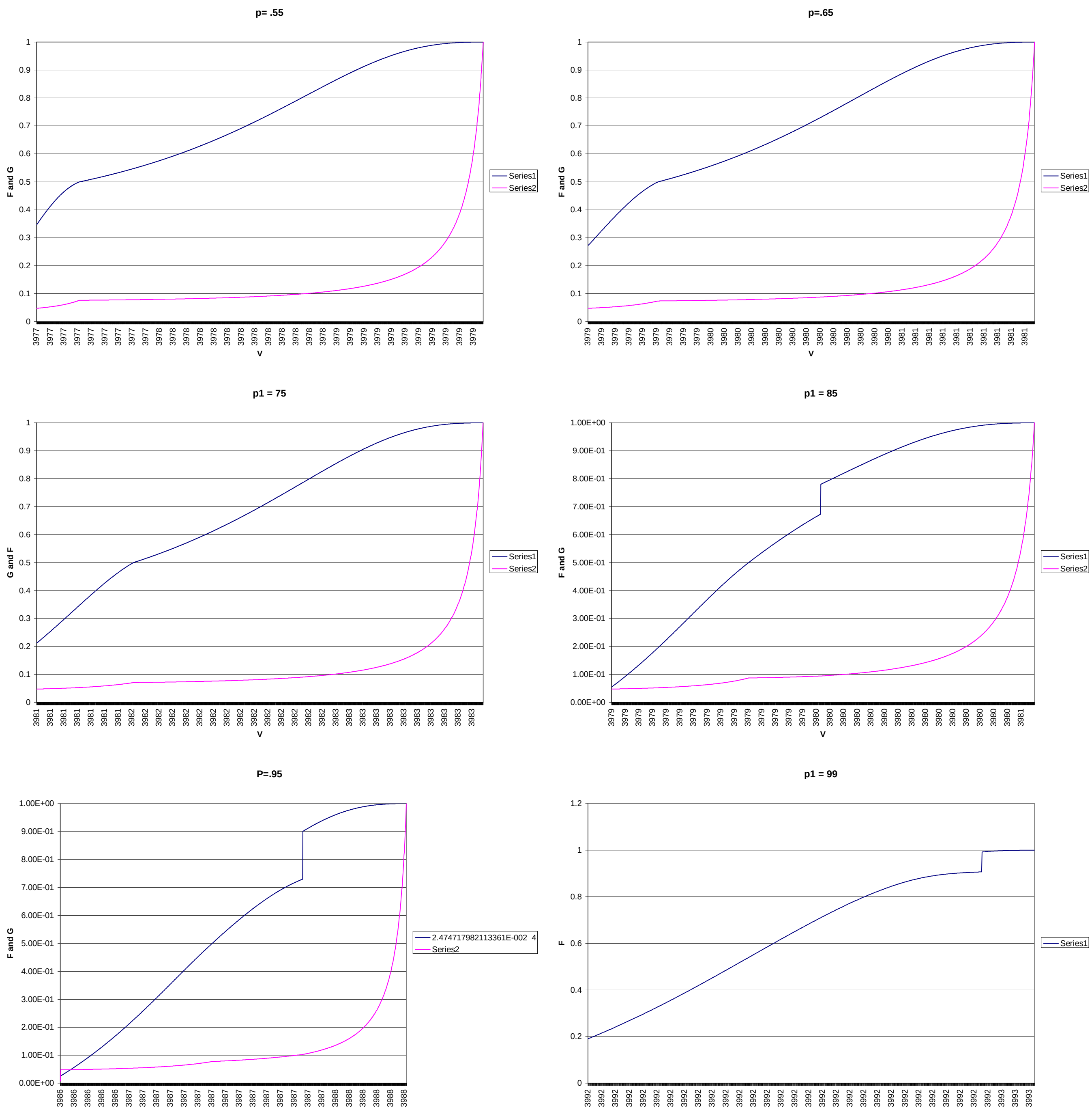


Figure 2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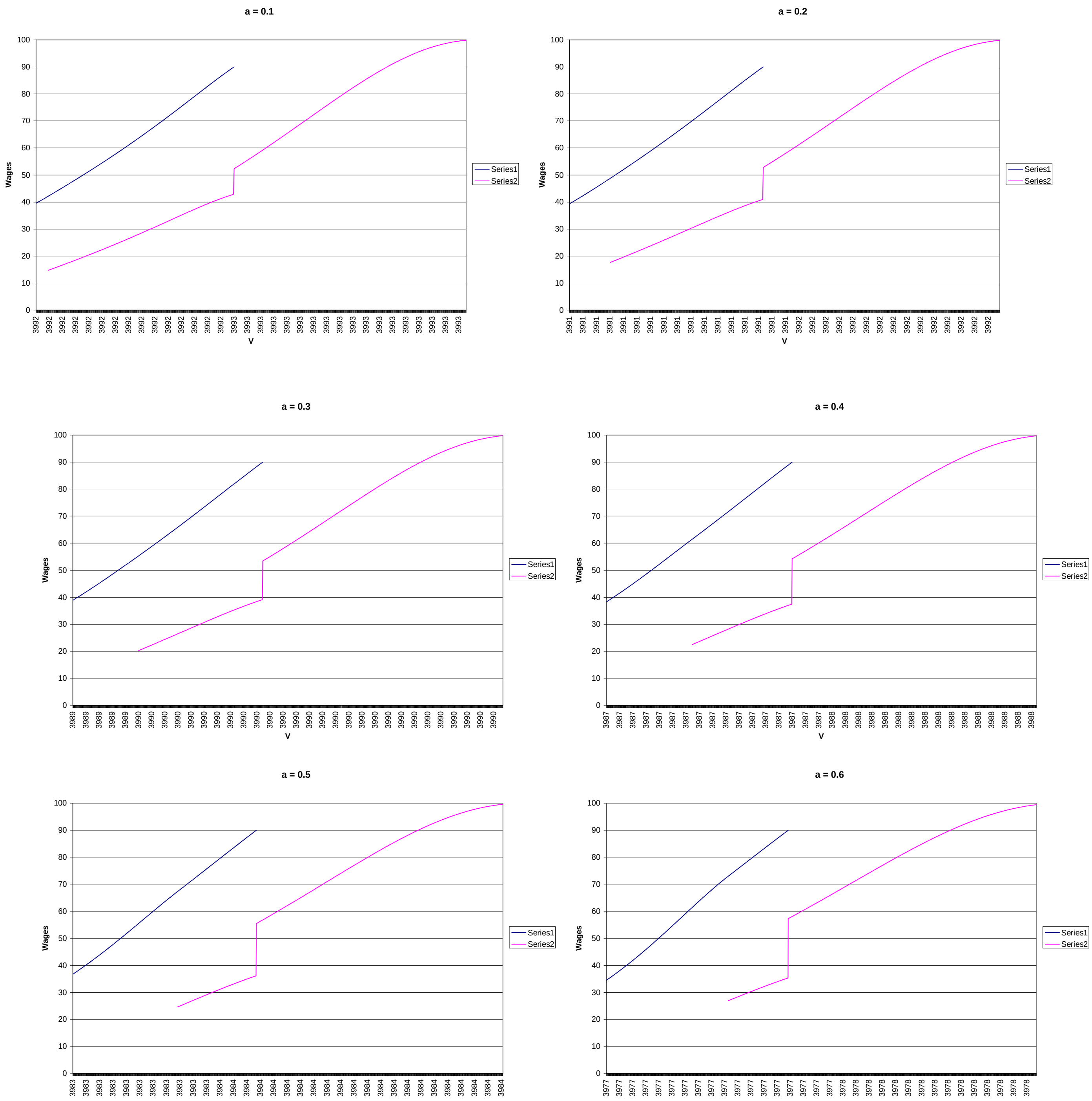
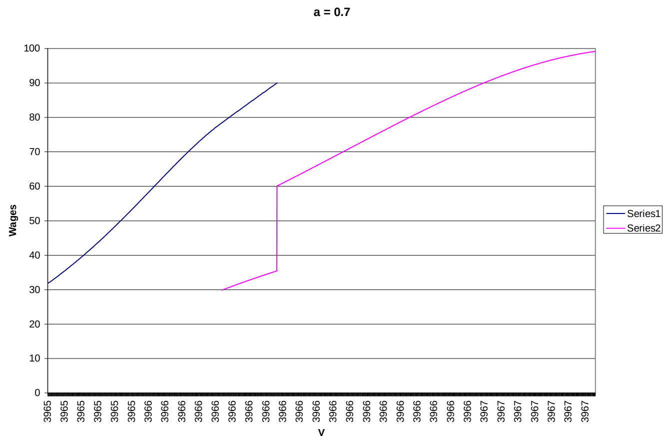
F and G
V
Series1
Series2
p=.65
F and G
V
Series1
Series2
p1 = 75
G and F
V
Series1
Series2
p1 = 85
F and G
V
Series1
Series2
P=.95
F and G
V
2.474717982113361E-002 4
Series2
p1 = 99
F
V
Series1

Figure 3


a = 0.1
Wages
V
Series1
Series2
a = 0.2
Wages
V
Series1
Series2
a = 0.3
Wages
V
Series1
Series2
a = 0.4
Wages
V
Series1
Series2
a = 0.5
Wages
V
Series1
Series2
a = 0.6
Wages
V
Series1
Series2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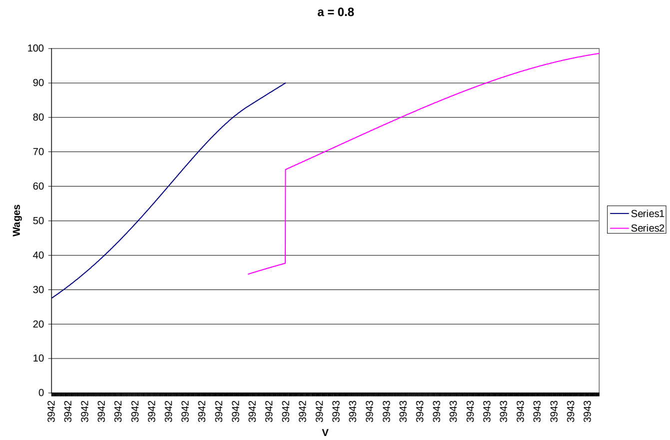
Wages
100
90
80
70
60
50
40
30
20
10
0
v
Series1
Series2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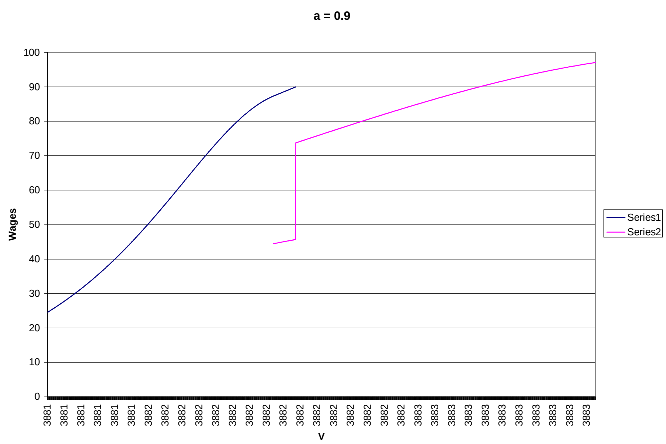
100
90
80
70
60
50
40
30
20
10
0
v
Series1
Series2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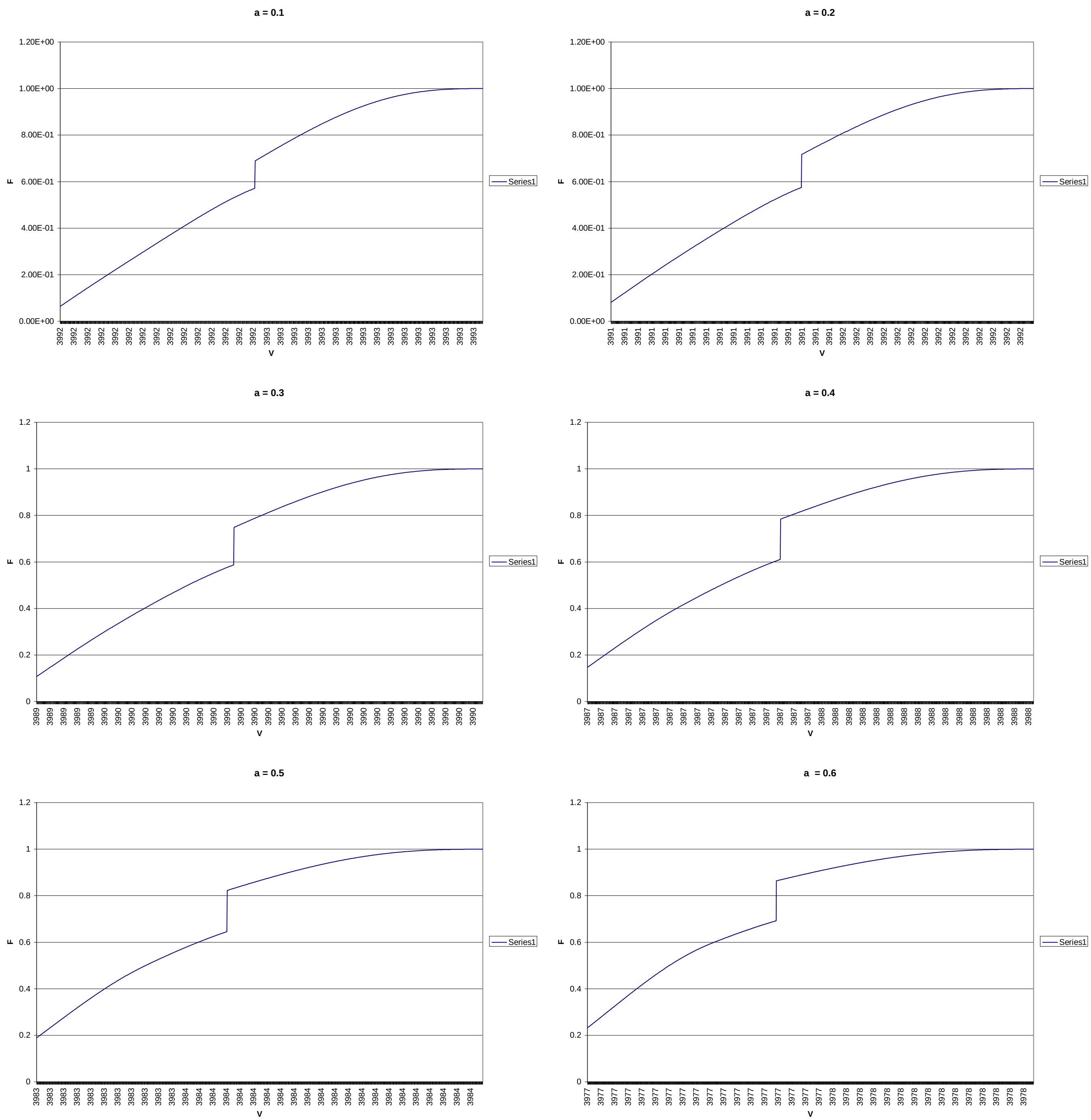
Wages
100
90
80
70
60
50
40
30
20
10
0
v
Series1
Series2

Figure 4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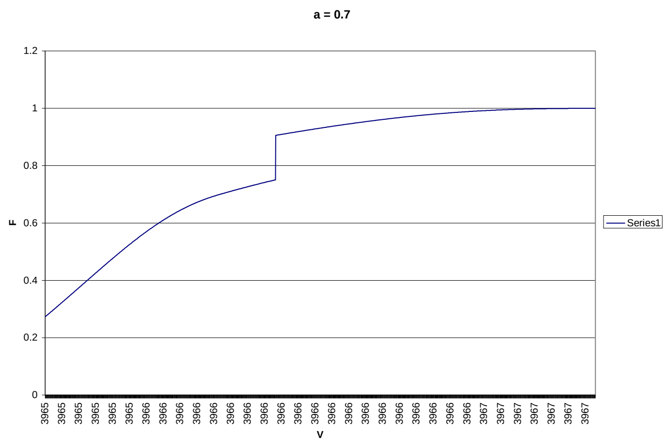
1.20E+00
1.00E+00
8.00E-01
6.00E-01
4.00E-01
2.00E-01
0.00E+00
F
V
3992
3993
Series1
a = 0.2
1.20E+00
1.00E+00
8.00E-01
6.00E-01
4.00E-01
2.00E-01
0.00E+00
F
V
3991
3992
Series1
a = 0.3
1.2
1
0.8
0.6
0.4
0.2
0
F
V
3989
3990
Series1
a = 0.4
1.2
1
0.8
0.6
0.4
0.2
0
F
V
3987
3988
Series1
a = 0.5
1.2
1
0.8
0.6
0.4
0.2
0
F
V
3983
3984
Series1
a = 0.6
1.2
1
0.8
0.6
0.4
0.2
0
F
V
3977
3978
Series1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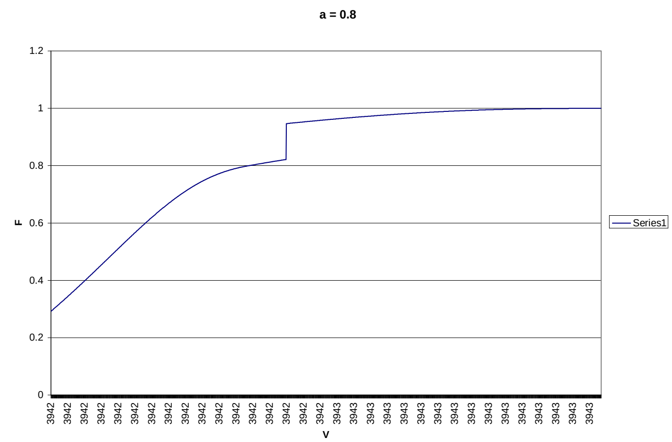
1.2
1
0.8
0.6
0.4
0.2
0
F
Series1
3965 3965 3965 3965 3965 3965 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3966 3967 3967 3967 3967 3967 3967 3967
V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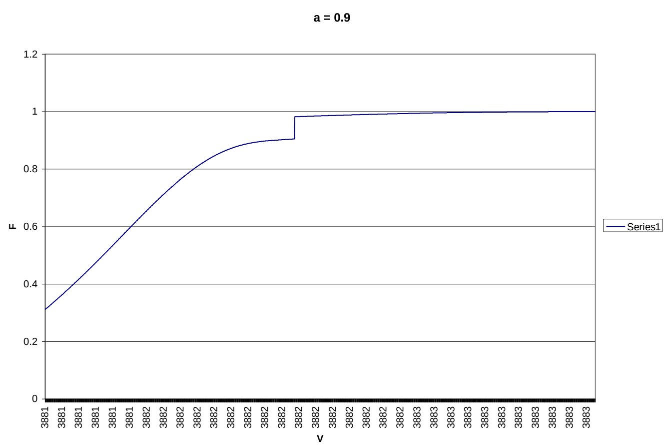
1.2
1
0.8
0.6
0.4
0.2
0
F
Series1
3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3942 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943 3943
V


a = 0.9
1.2
1
0.8
0.6
0.4
0.2
0
F
Series1
3881 3881 3881 3881 3881 3881 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3882 3883 3883 3883 3883 3883 3883 3883 3883 3883 3883 3883
V