Precautionary Wealth Accumulation

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Abstract

When does an individual's expected wealth accumulation profile increase as earnings risk increases? This paper answers this question for multi-period models where earnings shocks are independent over time. Sufficient conditions are stated in terms of properties of a decision rule for savings and, alternatively, in terms of properties of preferences.

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1 Introduction

This paper provides theoretical foundations for when individual and aggregate wealth accumulation increase with increases in earnings risk. Stated in terms of a simple picture, the problem addressed in this paper is to find sufficient conditions such that the pattern in Figure 1 holds. Figure 1 describes a situation in which an individual's expected wealth accumulation profile is weakly greater at each age over the life cycle whenever the individual faces one earnings process that is riskier than another earnings process, holding initial wealth and earnings constant.

[Insert Figure 1 Here]

Two important open questions can be answered by finding sufficient conditions for the pattern in Figure 1 to occur. First, there is the question of when theory predicts that households that face more earnings risk have more expected wealth accumulation at any point in the life cycle than households that face less earnings risk. The development of such a theory would be quite helpful as there already exists an empirical literature related to this question which has been surveyed by Browning and Lusardi (1996). Second, there is the question of when theory predicts that aggregate wealth accumulation increases with increases in earnings risk. Much of the interest in the precautionary savings literature is based on the possibility that an important part of aggregate wealth is due to uninsured earnings risk. This question can also be answered with Figure 1. The idea is to treat the expected wealth profile of one individual as the realized average wealth profile of a large number of similarly situated individuals. The law of large numbers justifies this when shocks are independent across individuals and, thus, earnings risk is idiosyncratic. Since aggregate wealth holding is a weighted sum of the average wealth holdings of individuals at different ages, an upward shift of the expected wealth profile of all agents then implies that aggregate wealth holding increases with increases in earnings risk.

This paper provides the first general answer for when the pattern in Figure 1 occurs in the context of a multi-period model with independent earnings shocks. The answer is given at two levels: in terms of properties of the decision rule for savings (i.e. the savings function) and in terms of preferences. A savings function $s_j(x;\theta)$ maps an individual's current state x (e.g. current wealth and earnings), age j and earnings process θ into savings or wealth carried into the next model period. Since economists are familiar with the merits of sufficient conditions stated in terms of preferences, I will try to motivate why sufficient conditions stated in terms of the savings function are interesting. First, they are widely applicable. They apply to theories where savings decisions maximize a utility function and even to theories where savings decisions are made by a rule-of-thumb. They also apply to models where the uncertainty is over wages and there are both consumption-saving and labor-leisure decisions. Second, they are useful in empirical work. For example, one can use properties of estimated decision rules to make statements about properties of expected wealth accumulation profiles. Third, they are useful in the development of preference-based theories of wealth accumulation under earnings or wage uncertainty. The reason is simply that one knows, in advance, what properties of optimal decision rules are sufficient to generate the desired result. Viewed in this way, the focus on decision rules can be seen as a natural first step in an attempt to reverse engineer the pattern in Figure 1.

To understand when the pattern in Figure 1 occurs, it is useful to first have an abstract statement of what Figure 1 asserts. To write such a statement, let x denote an individual's state at a point in time. The state represents cash-on-hand and is related to savings s, earnings z and the interest rate r as follows $x = s(1+r) + z$. In addition, let λ_j and λ_j' denote the future distributions (i.e. probability measures) of the state variable from the point of view of the initial period when the earnings processes are indexed by θ and θ' respectively. Figure 1 asserts that $\theta \geq_\Theta \theta'$ *and* $\lambda_1 = \lambda_1'$ imply $\lambda_j \succeq \lambda_j', \forall j$. In words, whenever one earnings processes θ is riskier than another θ' and the initial distributions are the same then in all future periods the distribution associated with θ is always ordered above the distribution associated with θ'. The actual proposition that is proven is written below, where the equal initial distributions requirement is relaxed to increase the applicability of the theory.

$$\theta \geq_\Theta \theta' \textit{ and } \lambda_1 \succeq \lambda_1' \Longrightarrow \lambda_j \succeq \lambda_j', \forall j$$

What kind of order $\succeq$ on distributions might prove useful for understanding when Figure 1 holds? Clearly, the order must have the property that $\lambda_j \succeq \lambda_j'$ implies that expected wealth holding under λ_j must be at least as great as under λ_j'. In this paper the order $\succeq$ employed is called the increasing-convex order. It is defined so that $\lambda \succeq \lambda'$ if and only if $\int_X f d\lambda \geq \int_X f d\lambda'$ for all functions $f(x)$ which are increasing and convex in x and for which both integrals exist. I will postpone a discussion of why the increasing-convex order is a natural choice until the point where this discussion can be appreciated better. The key point to see now is that $\lambda_j \succeq \lambda_j'$ does imply more expected wealth is held in period j with earnings process θ than with θ'. The reason is that cash-on-hand $f(x) = x$ is an increasing and convex function of the state and that expected wealth holding is a monotone increasing function of expected cash-on-hand since $x = s(1+r) + z$.

The paper presents two main results. The first result says that if the savings function $s_j(x;\theta)$ is increasing in risk θ and is increasing and convex in the state x, then the above proposition holds. This result is tight in that dropping any of these three assumptions allows for the construction of examples where expected wealth accumulation in some period decreases as earnings risk increases. Two fundamental and distinct properties are key to prove this result. One of these is produced when the savings function increases with risk, whereas the other is produced when the savings function is increasing and convex in the state. Intuitively, increases in mean wealth and wealth dispersion caused by increases in earnings risk are reinforced when the savings function is increasing and convex. In contrast, a decreasing or concave savings function counteracts the effect of the savings function increasing with earnings risk.

The second result provides assumptions on preferences so that the optimal savings function has the properties assumed in the first result and, as a consequence, the expected wealth profile increases as earnings risk increases. A common conjecture in the literature is that the expected wealth profile increases with earnings risk when an agent maximizes an additive expected utility function, where the period utility function is increasing and concave in consumption and has a positive third derivative.[1] This paper shows by way of a counter example that this conjecture is wrong and that stronger conditions on preferences are needed. A concave utility function with a positive third derivative is sufficient to guarantee that the savings function increases in the state and increases with earnings risk but is not sufficient to guarantee that the savings function is convex in the state. Conditions on preferences that are sufficient for the expected wealth profile to increase with earnings risk are the conditions highlighted by Carroll and Kimball (1996, 2001) which guarantee that the consumption function is concave or, equivalently, that the savings function is convex in the state.[2] Overall, the results of the paper highlight the unexpected importance of the convexity of the savings function in a theory of precautionary wealth accumulation.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature and puts the contribution of this paper in context. Section 3 describes the framework and then states and proves the main results. Section 4 discusses the results.

2 Literature Review

This section reviews the theoretical literature which focuses on wealth accumulation as earnings risk increases.[3] The early literature is associated with Leland (1968), Sandmo (1970), Mirman (1971), Rothschild and Stiglitz (1971), Dreze and Modigliani (1972) and Diamond and Stiglitz (1974). They focused on a two-period model where future earnings z_2 are random and are drawn from a distribution indexed by the parameter

[1]Section 2 of this paper reviews the literature highlighting the importance of a positive third derivative.

[2]When $c_j(x;\theta) + s_j(x;\theta) = x$, the convexity of the savings function $s_j(x;\theta)$ is equivalent to the concavity of the consumption function $c_j(x;\theta)$.

[3]The theoretical literature on aggregate precautionary savings is not reviewed as this literature has recently been reviewed by Huggett and Ospina (2001).

θ. Agents maximize an expected utility function $E[u_1(c_1) + u_2(c_2)]$ by choosing the amount of a risk-free asset with return r to carry to the next period.[4] As indicated in Table 1, the main result is that the optimal decision rule has the property that savings $s(x;\theta)$ carried to the next period increases in risk θ from any initial state x. A state x denotes cash-on-hand brought into the period. The result follows from the Euler equation below, which ignores corner solutions.

$$u_1'(x_1 - s_2) = E[u_2'(s_2(1+r) + z_2)](1+r)$$

The intuition for the result is that with convex marginal utility (u_2' convex) increases in earnings risk increase expected future marginal utility of consumption for any fixed level of savings s_2 carried to the second period. Thus, as risk increases it is optimal to carry more wealth to the second period. The subsequent literature associated with Miller (1975, 1976), Sibley (1975), Schechtman (1976) and Mendelson and Amihud (1982) generalizes this result to apply to models where preferences are $E[\sum_{j=1}^{J} u_j(c_j)]$, earnings shocks are independent and there are arbitrarily many model periods J. The intuition for the result is the same. Collectively, the results summarized above are the basis for the claim that $u''' > 0$, alternatively u' convex, is sufficient to generate precautionary savings.

[Insert Table 1 Here]

This paper focuses on when the expected wealth profile increases with earnings risk. The two results proved in this paper are summarized in Table 1. The result at the level of decision rules relies on three key properties. One of these properties, $s_j(x;\theta)$ increasing in θ, is precisely the property emphasized in the literature summarized in Table 1. This property is not sufficient to produce the pattern in Figure 1, except in a special case. The special case where it is sufficient is a two-period model. In this case if an individual saves more from any given state as risk increases, then expected wealth holding next period will also increase with risk starting from any given initial state or initial distribution of the state.

I will now develop some intuition for why in the multi-period case additional properties of decision rules are needed. Consider two individuals who start out in period 1 with the same initial cash-on-hand. One individual faces earnings risk whereas the other does not. First, consider the case where the decision rule is increasing in risk but is not increasing in the state variable over some range. In this case, in period 2 the individual with risky earnings has greater wealth holding and greater expected cash-on-hand. However, in period 3 expected wealth levels could be reversed. For

[4] Focusing on additive utility facilitates comparisons with the rest of the literature.

example, this would happen when the entire support of the period 2 distribution of cash-on-hand of the individual with risky earnings lies above the cash-on-hand held by the other individual. Now consider the case where the decision rule is increasing in risk and increasing in the state but is not convex in the state (e.g. it is locally concave at some point).[5] In period 2 the distribution of cash-on-hand associated with risky earnings will be dispersed about the point associated with no risk but with a higher mean. In period 3 the expected wealth levels could be reversed by a Jensen's inequality effect due to the local concavity of the decision rule. This occurs when the Jensen's inequality effect is strong enough to offset the upward shift of the savings function due to increased risk. This is an example of Jensen's inequality being important at a different level than previously considered in the precautionary savings literature.

The remaining literature that relates to wealth accumulation as earnings risk increases can be divided into three groups. The first group, consisting of papers by Caballero (1991), van der Ploeg (1993) and Weil (1993), presents particular parametric examples of decision problems having closed form solutions.[6] While these papers are interesting for many purposes, they do not offer a general methodological approach for understanding when the pattern in Figure 1 holds. The second group of papers calculates the magnitude of precautionary wealth holding in particular parametric models for selected parameter values. The work of Skinner (1988), Caballero (1991), Aiyagari (1994), Hubbard et al (1994) and Huggett (1996) are representative of this group. The third group consists of the paper by Kimball (1990) which examines the comparative statics of the two-period precautionary saving problem. One interesting finding is that, for infinitesimal amounts of earnings risk, one can say how much the consumption function shifts as risk increases. This result can be restated in terms of the shift in the savings function.

3 Results

3.1 Framework

The analysis focuses on a savings function $s_j(x;\theta)$ and earnings distributions $(\pi_{1\theta},...,\pi_{J\theta})$. A savings function $s_j(x;\theta)$ maps a state x, model period j and earnings process θ into a level of savings carried into the next model period. A state x consists of cash-on-hand in a given model period. Cash-on-hand equals savings after interest payments plus

[5] Section 3.4 presents an example where this occurs.

[6] Deaton (1992, Ch. 6) and Weil (1993) discuss the limitations of these examples.

earnings (i.e. $x = s(1+r)+z$).[7] Earnings are assumed to be independent over periods, where the number of model periods J can be finite or infinite.

A savings function and an earnings distribution induce a Markov process on the state variable. The distribution of the state variable is given by a probability measure λ defined on the state space X. The distribution follows a law of motion given by the recursive mapping $T_{j\theta}$. The mapping $T_{j\theta}$ is defined by a transition function $P_{j\theta}(x, B)$, which states the probability that the state next period lies in the set B, given that the period j state is x and the earnings process is θ.

$$\lambda_{j+1}(B) = T_{j\theta}\lambda_j(B) \equiv \int_X P_{j\theta}(x, B)d\lambda_j, \forall B \in \mathcal{X}$$

$$P_{j\theta}(x, B) \equiv \pi_{j\theta}(\{z' : s_j(x;\theta)(1+r) + z' \in B\})$$

To complete the description of the framework it remains to describe the notions of increasing risk $\geq_\Theta$ and increasing distributions $\succeq$. This is done in Definition 1. The notion of increasing risk $\geq_\Theta$ is a multi-period generalization of that in Rothschild and Stiglitz (1970). Thus, one earnings process θ is riskier than another θ' (i.e. $\theta \geq_\Theta \theta'$) provided that in every period earnings distribution $\pi_{j\theta}$ is riskier than $\pi_{j\theta'}$ in the Rothschild-Stiglitz sense. The Rothschild-Stiglitz definition is based on the concave order of probability measures. It states that every risk-averse agent weakly prefers distribution $\pi_{j\theta'}$ to $\pi_{j\theta}$. Clearly, whenever two distributions can be ordered and the means exist the means must be equal.[8] The stochastic order $\succeq$ on distributions of the state variable is called the increasing-convex order.

Definition 1: Let $X = [\underline{x}, \infty)$, $Z = [\underline{z}, \infty)$, $\mathcal{X}$ and $\mathcal{Z}$ be the Borel subsets of X and Z, $\Lambda(X, \mathcal{X})$ and $\Lambda(Z, \mathcal{Z})$ be the set of probability measures on $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ and $\geq_X$ be a partial order on X.

(i) Let Θ be a nonempty set such that for each $\theta \in \Theta$, $\{\pi_{j\theta} : j = 1, ..., J\}$ is a collection of probability measures in $\Lambda(Z, \mathcal{Z})$. For $\theta, \theta' \in \Theta, \theta$ is *riskier* than θ' (denoted $\theta \geq_\Theta \theta'$) provided that for all j, $\int f(z)d\pi_{j\theta} \leq \int f(z)d\pi_{j\theta'}$ for all concave f for which the integrals exist.

[7] All the results of the paper can be stated using either a one or a two dimensional state variable, $x = s(1+r)+z$ or $x = (s, z)$. A one dimensional state has the advantage of simplicity, whereas the two dimensional state makes statements about expected wealth or savings more transparent.

[8] Shaked and Shantikumar (1994) review the literature on stochastic orders. They discuss stochastic dominance, the concave order and the increasing-convex order among others. An alternative notion of increasing risk would be to say that θ is riskier than θ' provided the present value of earnings at each date associated with θ is riskier in the Rothschild-Stiglitz sense than that associated with θ'. The notion in Definition 1 is employed since existing results in the precautionary savings literature listed in Table 1 of the previous section use this notion.

(ii) For each $\lambda, \lambda' \in \Lambda(X, \mathcal{X})$, $\lambda \succeq \lambda'$ provided $\int f(x)d\lambda \geq \int f(x)d\lambda'$ for all increasing and convex functions f for which the integrals exist.

3.2 General Theory

Theorem 1 underlies all the results in the paper. Theorem 1 rests on the three key assumptions that are listed in assumption A2. When a decision rule is said to be increasing in x or θ this is with respect to the orders $\geq_X$ and $\geq_\Theta$.

A1: $X = [\underline{x}, \infty)$, $Z = [\underline{z}, \infty)$, $\geq_X$ is the usual order on the real line and $\geq_\Theta$ is the order of increasing risk.

A2: $s_j(x;\theta)$ is increasing and convex in $x, \forall(j,\theta)$ and increasing in $\theta, \forall(x,j)$.

Theorem 1: Assume A1-2. For all $\theta, \theta' \in \Theta$ and for all $\lambda_1, \lambda_1' \in \Lambda(X, \mathcal{X})$ the following holds: $\theta \geq_\Theta \theta'$ and $\lambda_1 \succeq \lambda_1' \Longrightarrow \lambda_j \succeq \lambda_j', \forall j$

Proof:

Stokey and Lucas (1989, Thm. 9.13) show that $T_{j\theta}$ maps $\Lambda(X, \mathcal{X})$ into itself. Thus, the sequences $\{\lambda_j\}$ and $\{\lambda_j'\}$ are well defined. To prove the theorem it is sufficient to show that $\theta \geq_\Theta \theta'$ and $\lambda_j \succeq \lambda_j' \Rightarrow T_{j\theta}\lambda_j \succeq T_{j\theta'}\lambda_j'$. This is established in two steps. Step 1 shows that $T_{j\theta}$ preserves order (i.e. $\lambda \succeq \lambda' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta}\lambda'$). Step 2 shows that $T_{j\theta}$ increases in θ (i.e. $\theta \geq_\Theta \theta' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta'}\lambda$, $\forall\lambda$). The result then follows from the transitivity of $\succeq$ after combining these results to get $T_{j\theta}\lambda_j \succeq T_{j\theta}\lambda_j' \succeq T_{j\theta'}\lambda_j'$.

Step 1. Show that $\lambda \succeq \lambda' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta}\lambda'$. The conclusion of this implication is equivalent to the two statements below. The leftmost equivalence follows from the definition of $\succeq$, whereas the rightmost follows from Stokey and Lucas (1989, Thm. 8.3). In these statements $E_{j\theta}[f|x] \equiv \int f(x')P_{j\theta}(x, dx')$ and the class of functions f are those that are increasing and convex in x and for which the integrals exist.

$$T_{j\theta}\lambda \succeq T_{j\theta}\lambda' \Leftrightarrow \int f dT_{j\theta}\lambda \geq \int f dT_{j\theta}\lambda' \Leftrightarrow \int E_{j\theta}[f|x]d\lambda \geq \int E_{j\theta}[f|x]d\lambda'$$

Since $\lambda \succeq \lambda'$, the rightmost inequality above holds if $E_{j\theta}[f|x] = \int f(s_j(x;\theta)(1+r) + z')\pi_{j\theta}(dz')$ is an increasing and convex function of the state x. This holds for two reasons. First, the composition of increasing functions is increasing and the composition of an increasing, convex function f with a convex function $s_j(x;\theta)(1+r)+z'$ preserves convexity (Rockafellar (1970, Thm. 5.1)). Second, integration preserves each of these properties.

Step 2. Show that $\theta \geq_\Theta \theta' \Rightarrow T_{j\theta}\lambda \succeq T_{j\theta'}\lambda, \forall\lambda$. The conclusion of this implication is equivalent to the two expressions below for the reasons given in step 1.

$$T_{j\theta}\lambda \succeq T_{j\theta'}\lambda \Leftrightarrow \int f dT_{j\theta}\lambda \geq \int f dT_{j\theta'}\lambda \Leftrightarrow \int E_{j\theta}[f|x]d\lambda \geq \int E_{j\theta'}[f|x]d\lambda$$

The rightmost inequality above holds if $E_{j\theta}[f|x]$ is an increasing function of θ. $E_{j\theta}[f|x]$ increasing in θ follows from the two inequalities below, given $\theta \geq_\Theta \theta'$. The topmost inequality holds as $f(s_j(x;\theta)(1+r)+z') \geq f(s_j(x;\theta')(1+r)+z')$ for all z' since $s_j(x;\theta)$ is increasing in θ. The bottommost inequality follows from $\theta \geq_\Theta \theta'$, since $f(s_j(x;\theta')(1+r)+z')$ is convex in z' and thus $-f(s_j(x;\theta')(1+r)+z')$ is concave in z'.

$$\begin{aligned} \int f(s_j(x;\theta)(1+r)+z')\pi_{j\theta}(dz') &\geq \int f(s_j(x;\theta')(1+r)+z')\pi_{j\theta}(dz') \\ &\geq \int f(s_j(x;\theta')(1+r)+z')\pi_{j\theta'}(dz') \end{aligned}$$

3.3 Application

3.3.1 Decision Problem

The standard problem in the literature that formalizes the life-cycle, permanent-income hypothesis assumes that an agent maximizes an additively separable expected utility function $E[\sum_{j=1}^{J} \beta^{j-1} u(c_j)]$. An agent receives earnings z_j in period j. Earnings are drawn independently from age-specific distributions $\pi_{j\theta}$ indexed by the parameter θ. Each period j the agent divides cash-on-hand x_j between consumption c_j and savings s_{j+1} so that $c_j + s_{j+1} = x_j$. Savings receive a gross, risk-free return $(1+r) > 0$. The amount of savings must lie above period-specific borrowing limits $\underline{s} \equiv (\underline{s}_1, ..., \underline{s}_{J+1})$ which will be described in detail shortly.

The dynamic programming formulation of this decision problem is given below. The state variable equals $x = s(1+r) + z$, where s is savings or beginning-of-period wealth, z is earnings and r is an interest rate. The state variable in period j lies in the state space $X_j \equiv [\underline{x}_j, \infty)$. The value function is set to zero after the last period of life (i.e. $V_{J+1}(x;\theta) = 0$). The functions $c_j(x;\theta)$ and $s_j(x;\theta)$ denote the optimal decision rules for consumption and savings solving this problem.

$$\begin{aligned} &V_j(x;\theta) = \max\ u(x - s') + \beta E_{j\theta}[V_{j+1}(s'(1+r) + z';\theta)] \\ &\text{subject to } s' \in \Gamma_j(x) \equiv \{s' : \underline{s}_{j+1} \leq s' \leq x\} \end{aligned}$$

To specify the relation between the minimum values of cash-on-hand and the implied values of the period-specific borrowing limits, it will be useful to divide earnings into a sure component and a random component that add up to total realized period earnings (see Miller (1975, 1976) or Schechtman (1976)). To do this let $\underline{z}_j \equiv sup\,\{z : \pi_{j\theta}([0,z)) = 0\}$ denote sure earnings in period j for earnings process θ. Then given a vector $\underline{x} \equiv (\underline{x}_1, ..., \underline{x}_J)$ of minimum values of cash-on-hand, define the vector $\underline{s} \equiv (\underline{s}_1, ..., \underline{s}_{J+1})$ so that $\underline{x}_j = \underline{s}_j(1+r) + \underline{z}_j$ and that no borrowing is allowed in the last period of life (i.e.

$\underline{s}_{J+1} = 0$). The vector $\underline{x}$ is then restricted so that any debt can be repaid with sure future earnings (i.e. $\underline{x}_j \geq -\sum_{k>j} \underline{z}_k/(1+r)^{k-j}$ and $\underline{x}_J \geq 0$) and that budget sets are nonempty (i.e. $\underline{s}_{j+1} \leq \underline{x}_j$). An important special case is where $\underline{x}_j$ is chosen to equal the negative of the present value of sure future earnings. This case will be called the case of solvency constraints. In contrast, the case of liquidity constraints occurs when the settings of $\underline{x}_j$ are above these values.

3.3.2 Expected Wealth Profiles

The main results of this section state assumptions on preferences so that expected wealth profiles increase with increases in earnings risk. The strategy is to find assumptions on preferences so that optimal decision rules have the properties highlighted in the general theory section. One complication to carrying this out is that when earnings risk increases then the state space of feasible values of cash-on-hand may also change. The reader will recall that the general theory section assumed that the state space on which future distributions of the state variable are defined does not change with changes in the parameter θ governing earnings distributions. To handle this complication, Theorem 2 first considers the simpler case where the state space does not change as earnings risk changes. Theorem 3 then shows that it is easy to handle the case where riskier earnings processes have tighter limits on cash-on-hand by building on the result proved in Theorem 2.

Theorem 2 considers the case where the state space $X_j = [\underline{x}_j, \infty)$ can change across periods but does not change as earnings risk increases. The assumptions for Theorem 2 are listed as assumptions B1-3 below. The reader will note that assumption B1 puts only weak restrictions on the period utility function. In contrast, assumption B3 is a much stronger restriction. Assumption B3 considers utility functions in the constant relative risk averion (CRRA) and the constant absolute risk aversion (CARA) class:

$$u(c) = \begin{cases} c^{1-\sigma}/(1-\sigma) & \text{for } \sigma > 0 \text{ and } \sigma \neq 1 \\ log(c) & \text{for } \sigma = 1 \end{cases} \tag{1}$$

$$u(c) = -(1/a)e^{-ac}, a > 0 \tag{2}$$

Assumptions B1 and B3 are separated to help clarify exactly where in the proof assumption B3 is used. Lemma 1 (i)-(ii) states that the savings function $s_j(x;\theta)$ exists and increases in the state x. These properties follow from assumptions B1-2 using standard results from the theory of dynamic programming and the theory of monotone comparative statics. Lemma 1(iii) states that $s_j(x;\theta)$ increases in earnings risk θ. The proof follows the general line of argument used by Mendelson and Amihud (1982). The intuition behind this result was presented in section 2. The proof of Lemma 1(iii) makes only partial use of assumption B3. In particular, the proof uses only the fact that u_j' is convex and that consumption is interior.

B1 $u(c)$ *is increasing, continuous and strictly concave and* $\beta, (1+r) > 0$.
B2 $\pi_{j\theta}$ *are probability measures on the Borel sets of* $[0, \bar{z}]$.
B3 Condition (a) or (b) holds:
(a) $u(c)$ *is in the CRRA class*
(b) $u(c)$ *is in the CARA class and* $\beta(1+r) \leq 1$.

Lemma 1: Assume B1-3.
(i) There exists a unique optimal decision rule $s_j(x;\theta)$.
(ii) $s_j(x;\theta)$ *is increasing in* x
(iii) $s_j(x;\theta)$ *is increasing in* θ.
(iv) $s_j(x;\theta)$ *is convex in* x.
(v) $s_j(x;\theta,\underline{x})$ *is increasing in* $\underline{x}$.
Proof: See the Appendix.

Theorem 2: Assume B1-3 and that the minimum value of cash-on-hand is the same for earnings processes differing in earnings risk. Then the expected wealth profile increases with increases in earnings risk, other things equal.

Proof: Follows from Theorem 1 and Lemma 1 (i)-(iv) after allowing the state space X in Theorem 1 to change with age due to the fact that borrowing limits are age-dependent.

The remaining property needed to prove Theorem 2 is that the savings function $s_j(x;\theta)$ is convex in x. The proof of this result makes full use of assumption B3. Assumption B3 says that the period utility function is in the CRRA or CARA class.[9] The proof that the savings function is convex is based on the work of Carroll and Kimball (1996, 2001). Carroll and Kimball (1996) present sufficient conditions so that the consumption function is concave in cash-on-hand. The concavity of the consumption function and the convexity of the savings function are equivalent statements in the

[9]The requirement in assumption B3 condition (b) that $\beta(1+r) \leq 1$ may at first seem odd. The problem with CARA utility functions is that there could be a corner solution at zero consumption for sufficiently high interest rates. This uninteresting case is often ruled out by assuming infinite marginal utility at zero consumption. However, this assumption rules out CARA utility. The reason that it is important to rule out a corner solution is that then the consumption function would not be concave and the savings function would not be convex. The assumption that $\beta(1+r) \leq 1$ is a simple but overly strong way to eliminate the possibility of a corner solution. An alternative assumptions would be to allow for negative consumption.

context of the model considered in their paper and in this section. Carroll and Kimball (2001) generalize their previous result to apply to situations where an agent faces either a liquidity constraint or a solvency constraint rather than just the special case of a solvency constraint considered in Carroll and Kimball (1996).[10] This generalization is important for increasing the applicability of Theorem 2 and 3 below. Without this generalization Theorems 2 and 3 would only hold for the case of solvency constraints where the solvency constraints are the same for earnings processes differing in earnings risk.

Theorem 3 allows for the possibility that when an agent experiences more earnings risk then the minimum values of cash-on-hand that apply period by period may become tighter in the sense that the minimum level of cash-on-hand increases. This makes intuitive sense in that the present value of sure future labor income either falls or remains the same as earnings risk increases. The insight in Theorem 3 is that, for a given earnings process θ, looser limits shift the expected wealth profile downward. Thus, Theorem 3 follows from Theorem 2 together with this additional insight. The key additional result used to prove this is that the savings function shifts downward as the minimum value of cash-on-hand falls and, hence, as borrowing constraints are loosened. Lemma 1(v) establishes that this result follows from assumptions B1-2 and, in particular, the concavity of the period utility function. In Lemma 1(v) the savings function $s_j(x; \theta, \underline{x})$ explicitly allows for dependence on the limits $\underline{x}$. Previously, this dependence was not highlighted as the limits were fixed as the other variables of interest were varied and as the effect of changing limits was not central to the analysis.

Theorem 3: Assume B1-3 and that riskier earnings processes have tighter limits on cash-on-hand (i.e. $\theta \geq_{\Theta} \theta' \Longrightarrow \underline{x}_j \geq \underline{x}'_j, \forall j$). Then the expected wealth profile increases with increases in earnings risk, other things equal.

Proof: Consider two earnings processes differing in earnings risk. First, Theorem 2 shows that the result holds when the period limits are the same (i.e. $\underline{x}_j = \underline{x}'_j, \forall j$) across earnings processes. Second, Lemma 1 (v) in conjunction with Lemma 1 (ii) implies that when the limits corresponding to the earnings process with less earnings risk are loosened (i.e. reduced) then for any realization of the life-cycle earnings path the realized wealth path is weakly less than before. Therefore, the expected wealth profile for the earnings process with less earnings risk must be even lower than before.

[10] At a technical level, the generalization means that a new method of proof is needed as the value function will no longer be thrice differentiable in the presence of liquidity constraints.

3.4 Counter Example

This section presents a counter example to the conjecture that a positive third derivative is sufficient for the expected wealth profile to increase with increases in earnings risk.[11] The example highlights an important point. In particular, conditions beyond the concavity of the utility function and a positive third derivative are needed in a theory of precautionary wealth accumulation. The reason why conditions beyond a positive third derivative are needed is that the convexity of the savings function is key and that convexity is determined by properties of the utility function that go beyond a positive third derivative.

Example

- $E[\sum_{j=1}^{3} u_j(c_j)]$, $u_j(c) = c^{(1-\sigma_j)}/(1-\sigma_j)$, where $\sigma_1 = \sigma_2 = 0.5$ and $\sigma_3 = 2.0$.
- $s_1 = 0.0$ and $r = 0.0$
- no earnings risk - $(z_1, z_2, z_3) = (1.5, 1.5, 0.0)$
- earnings risk - $(z_1, z_2, z_3) = (1.5, 0.0, 0.0), (1.5, 3.0, 0.0)$ with equal probability.

The example has three model periods. The solution is given in Table 2, which first lists the solution with no earnings risk and then lists the solution with earnings risk.[12] The solution with earnings risk has two rows for periods 2 and 3. This reflects the fact that all variables in period 2 and beyond are contingent on the earnings realizations in period 2.

[Insert Table 2 Here]

The expected wealth profile is given in Table 2. Wealth starts out at the same level at age 1 and is greater at age 2 with earnings risk than without. This pattern is reversed at age 3. In particular, at age 3 expected wealth is 0.9877 with earnings risk, compared with 1.0 without. Figure 2 illustrates why the expected wealth profiles cross by graphing the savings function at age 2. The Jensen's Inequality effect in Figure 2 produced by a concave savings function is sufficiently strong to offset the fact that the mean value of cash-on-hand at age 2 is higher with earnings risk than without.

[11] Huggett and Vidon (2001) analyze counter examples of this type in detail.

[12] The solution is computed by solving the nonlinear equations corresponding to the Euler equations for this problem. The solution with earnings risk is necessarily approximate since only four decimals are provided. Plugging in the consumption values, one can calculate Euler equation residuals (i.e. $u_j'(c_j) - E[u_{j+1}'(c_{j+1})(1+r)]$). The residual for $j = 1$ equals 6.4×10^{-5}. The two second period Euler equations have residuals of 5.3×10^{-5} and 8.9×10^{-6} respectively. Smaller residuals can easily be obtained with more digits.

[Insert Figure 2 Here]

What determines the local concavity or local convexity of the savings function? Carroll and Kimball (1996) explore this issue. To understand one of the key points of their analysis, consider interior solutions to the problem of maximizing $u(x-s')+V(s')$. By differentiating the Euler equation $u'(x-s(x)) = V'(s(x))$, one finds that $s'(x) = u''/(u''+V'')$. By differentiating this last result one finds after some algebra that locally the concavity or convexity of the savings function is determined by comparing the local curvature properties of the functions u and V as indicated below, where the functions u and V are evaluated at $x - s(x)$ and $s(x)$ respectively. Armed with this insight, it is clear why the savings function in Figure 2 is concave. In particular, in period 2 a problem of the general type described above occurs where $u(x - s') = u_2(x - s')$ and $V(s') = u_3(s'(1+r) + e_3)$. Since $u'''u'/(u'')^2 = 1 + 1/\sigma_2$ and $V'''V'/(V'')^2 = 1 + 1/\sigma_3$, the savings function is concave when $\sigma_2 < \sigma_3$.

$$s''(x) \geq (\leq) 0 \Longleftrightarrow u'''u'/(u'')^2 \leq (\geq) V'''V'/(V'')^2$$

4 Discussion

The discussion is in the form of several Remarks.

Remark 1: Relation to Applied Work

Theorems 1-3 can be related to applied work in at least four ways. First, Theorem 1 is stated in terms of things that potentially can be measured. Thus, one could examine if estimated decision rules have the properties assumed in Theorem 1. Such an approach provides a new perspective to the work on precautionary wealth accumulation. The work by Parker (2001) could be viewed as taking one step in this direction. Parker (2001) provides support for the proposition that the consumption function is concave in cash-on-hand. The concavity of the consumption function $c_j(x;\theta)$ is equivalent to the convexity of the savings function $s_j(x;\theta)$ when $c_j(x;\theta) + s_j(x;\theta) = x$. Second, Theorems 1-3 and future extensions to the case of Markov shocks will be key to understanding simulation results of theoretical models as earnings processes, wealth levels and borrowing limits are varied. Third, Theorems 1-3 offer conditions under which the expectation of any increasing and convex function of the state variable increases as initial wealth and earnings risk increase. Thus, under this theory, any evidence supporting the hypothesis that expected wealth holding increases with earnings risk should also be consistent with many other related hypotheses. Fourth, Theorems 1-3 provide perspective on some transformations that are used in empirical work. For example, sometimes the log of wealth is regressed on a list of explanatory variables to determine if increases in a measure of earnings risk leads to increases in wealth accumulation, other things equal.[13] Theorems 1-3 give conditions under which increases in risk leads to increases in the expectation of any increasing and convex transformation of wealth.[14] This result does not cover the log transformation as it is increasing and concave. Furthermore, it is straightforward to create numerical examples satisfying the hypotheses of Theorem 1 but where the expectation of the log of wealth decreases as risk increases.

Remark 2: Theorem 1

The proof of Theorem 1 provides insight into the respective roles of the three key assumptions made on decision rules. To see this recall that Theorem 1 is equivalent to the statement that $\theta \geq_\Theta \theta'$ and $\lambda_j \succeq \lambda'_j$ imply $T_{j\theta}\lambda_j \succeq T_{j\theta'}\lambda'_j$. The proof is based on showing that the following holds: $T_{j\theta}\lambda_j \succeq T_{j\theta}\lambda'_j \succeq T_{j\theta'}\lambda'_j$. The leftmost relationship states that $T_{j\theta}$ preserves order in that if in period j one distribution is larger than another then in period $j+1$ the distributions resulting from applying the map $T_{j\theta}$ will also have this property. The rightmost relationship states that $T_{j\theta}$ is increasing in θ in that any fixed distribution is mapped into a larger distribution as the parameter θ increases.

The role of the three key assumptions is clear. The assumption that the decision rule is increasing and convex in the state is used to establish that order is preserved. Previously, the discussion in section 2 and the example in section 3.4 highlighted how order is not preserved when the decision rule does not satisfy these assumptions. The assumption that the decision rule increases in risk is used to establish that the map $T_{j\theta}$ is increasing in θ. Finally, it should be stressed that these two properties of the map $T_{j\theta}$ are both necessary and sufficient for the proposition contained in Theorem 1 to hold. Huggett (2001) shows that this is true when $\geq_\Theta$ is any reflexive binary relation on parameters and $\succeq$ is any reflexive and transitive binary relation on distributions. Thus, these two properties of the map $T_{j\theta}$ are fundamental in building a theory that produces the pattern in Figure 1 or, more generally, in understanding when comparative dynamics are monotone in models with a recursive structure.

Remark 3: Why Use the Increasing-Convex Order?

One question which has not been fully addressed up to this point is why the paper uses the increasing-convex order $\succeq$? One response is simply to note that the increasing-convex order does lead to a useful characterization for when expected wealth holding increases with earnings risk. A more complete response would discuss why the increasing-convex order seemed to be promising in the first place. One simple bit

[13] See the work surveyed in Browning and Lusardi (1996, Table 5.2).

[14] This is easy to see when the state is given by $x = (s, z)$ rather than $x = s(1 + r) + z$.

of intuition is that the precautionary savings problem involves making comparisons across distributions differing in both mean and dispersion. Intuitively, if the pattern in Figure 1 is to hold, then distributions associated with more risk will have higher mean values of the state and, hence, higher mean values of wealth but will typically also have higher dispersion. The increasing-convex order is designed to compare such distributions, whereas other familiar stochastic orders such as stochastic dominance (sometimes called first order stochastic dominance) are not. Another bit of intuition builds on the discussion in section 2 and the example in section 3.4. There it was argued that when the savings function $s_j(x;\theta)$ was either not increasing or not convex in the state x, then expected wealth holding may decrease in some period as risk increases even when $s_j(x;\theta)$ increases in θ. This also suggests a key role for a stochastic order based on increasing and convex functions.

Remark 4: Open Question

An open question for future work is whether the results of this paper can be extended to situations where earnings follow a Markov process. This is important as the empirical literature on earnings has characterized that the stochastic component of earnings variation has both a purely temporary as well as a persistent component. It has been conjectured that the persistent component is an especially important determinant of precautionary wealth accumulation with the intuition that a given change in the persistent component may be associated with a much larger shift in the distribution of the present value of earnings than the same change in the temporary component. Remark 2 makes the point that the general approach taken in this paper will be fundamental to any analysis of when comparative dynamics are monotone where the underlying problem has a recursive structure. For this reason, an analysis of the Markov shock case will have much in common with the independent shock case considered in this paper.

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A Appendix

The Appendix provides a proof of Lemma 1. The proof of Lemma 1 (iii)-(iv) uses the following standard results. Proofs are easily adapted from similar results in the literature (e.g. Schechtman 1976, Theorem 1.3 and Corollary 1.4). Both results make use of the fact that consumption is interior.

(1) V_j is differentiable in the state and $V_j'(x;\theta) = u'(c_j(x;\theta))$.

(2) A necessary condition for maximization (i.e. the Euler equation) is $u'(c_j(x;\theta)) \geq \beta(1+r)E_{j\theta}[V_{j+1}'(s_j(x;\theta)(1+r)+z';\theta)]$ and $=$ if $s_j(x;\theta) > \underline{s}_{j+1}$.

Proof of Lemma 1 (i):

For $j = J$ the objective is continuous and the constraint set is nonempty and compact for all $x \in X_J$. Thus, there is a solution for each $x \in X_J$. Furthermore, the solution $s_J(x;\theta)$ is unique since the objective is strictly concave and the constraint set $\Gamma_J(x)$ is convex. As $\Gamma_J(x)$ is a continuous correspondence, the Theorem of the Maximum (Stokey and Lucas (1989, Thm. 3.6)) implies that $(V_J(x;\theta), s_J(x;\theta))$ are continuous in x. Backwards recursion and repeating the above argument (using Stokey and Lucas (1989, Lemma 9.5)) gives unique solutions $(V_j(x;\theta), s_j(x;\theta))$ that are continuous in x for $j = 1, 2, ..., J$, provided that the objective is strictly concave. The strict concavity of the objective in the control follows from the strict concavity of u and the concavity of V_j. The concavity of V_j follows from a standard backwards recursion argument.

Proof of Lemma 1 (ii) and (v):

To prove that $s_j(x;\theta,\underline{x})$ is increasing jointly in $(x,\underline{x})$ define $\hat{x} = (x,\underline{x})$ and note that $s_j(x;\theta,\underline{x})$ is the argmax of $f_j(\hat{x},y)$ over the constraint set $Y_j(\hat{x}) \equiv \{y : \underline{s}_{j+1} \leq y \leq x\} \subseteq Y$, where $\underline{s}_{j+1}$ is defined by $\underline{x}_{j+1} = \underline{s}_{j+1}(1+r) + \underline{z}_{j+1}$. Throughout the proof notation for θ is dropped for convenience.

$$f_j(\hat{x},y) \equiv u(x-y) + \beta E_j[V_{j+1}(y(1+r)+z';\underline{x})]$$

Lemma 1 (ii) and (v) then follows from Topkis (1998, Thm. 2.8.2) when (a) $\hat{X}_j, Y$ are lattices, (b) $S_j \equiv \{(\hat{x},y) : \hat{x} \in \hat{X}_j, y \in Y_j(\hat{x})\}$ is sublattice of $\hat{X}_j \times Y$ and (c) $f_j(\hat{x},y)$ is supermodular in $(\hat{x},y)$ on S_j. Lattices, sublattices and supermodularity are defined in Topkis (1998). I now establish that properties (a)-(c) hold.

(a) Let $\hat{X}_j \equiv \{(x,\underline{x}) : x \geq \underline{x}_j^k$ for $k = 1,2$ and $\underline{x} \in \{\underline{x}^1, \underline{x}^2\}\}$. Here $\underline{x}^1$ and $\underline{x}^2$ satisfy $\underline{x}_j^2 \geq \underline{x}_j^1, \forall j$. Order points with the coordinate order (i.e. $(x,\underline{x}) \geq (x',\underline{x}')$ provided $x \geq x'$ and $\underline{x}_j \geq \underline{x}_j', \forall j$). $\hat{X}_j$ is a lattice since the pairwise sup and inf operations are defined and the resulting points lie in $\hat{X}_j$. $Y = R^1$ is a lattice with the usual order.

(b) $(\hat{x}, y), (\hat{x}', y') \in S_j$ implies that $sup((\hat{x}, y), (\hat{x}', y')) \in S_j$ for two reasons. First, $sup(\hat{x}, \hat{x}') \in \hat{X}_j$ as $\hat{X}_j$ is a lattice. Second, $sup(\underline{s}_{j+1}, \underline{s}'_{j+1}) \leq sup(y, y') \leq sup(x, x')$. A parallel argument establishes that the same holds for the inf. Thus, S_j is a sublattice.

(c) $f_j(\hat{x}, y)$ is supermodular in $(\hat{x}, y)$ if each of the two component functions is supermodular in $(\hat{x}, y)$. The first component $u(x - y)$ is supermodular in $(\hat{x}, y)$ as $u(x - y)$ is supermodular in (x, y) by Topkis (1998, Lemma 2.6.2) and the extension to $(\hat{x}, y)$ clearly holds. It remains to show that $E_j[V_{j+1}(y(1 + r) + z'; \underline{x})]$ is supermodular in $(\hat{x}, y)$ for all j. This follows from backwards induction using steps 1-4.

Step 1 observes that $V_{J+1} \equiv 0$. Thus, $V_{J+1}(x; \underline{x})$ is supermodular in $\hat{x} = (x, \underline{x})$.

Step 2 observes that $V_{j+1}(x; \underline{x})$ supermodular in $\hat{x} = (x, \underline{x})$ implies that $V_{j+1}(y(1 + r) + z'; \underline{x})$ is supermodular in $(\hat{x}, y)$, given z'.

Step 3 notes that $V_{j+1}(y(1+r)+z'; \underline{x})$ supermodular in $(\hat{x}, y)$ implies that $E_j[V_{j+1}(y(1+r) + z'; \underline{x})]$ is supermodular in $(\hat{x}, y)$ by Topkis (1998, Corr. 2.6.2).

Step 4 notes that Topkis (1998, Thm. 2.7.6) implies that $V_j(x; \underline{x})$ is supermodular in $\hat{x}$ since $f_j(\hat{x}, y)$ is supermodular in $(\hat{x}, y)$ by steps 1-3 and since assumptions (a)-(b) hold. This result can be loosely paraphrased as saying that the maximum of a supermodular function is supermodular over the remaining unmaximized variables.

Proof of Lemma 1 (iii):

Proceed by induction. $s_j(x; \theta)$ is increasing in θ for $j = J$ as $s_J(x; \theta) = 0$ since no borrowing is allowed in the last period of life. Given that this holds for j show that it holds for $j - 1$. As a first result note that the inequalities below hold, given $\theta \geq_\Theta \theta'$. The leftmost inequality holds when V_j' is convex in x. Schechtman (1976, Thm. 1.10) and Miller (1976, Lemma 1) prove that V_j' is convex in x when u_j' is convex. The rightmost inequality holds when V_j' is increasing in θ. This holds by the induction hypothesis since $V_j'(x; \theta) = u'(x - s_j(x; \theta))$.

$$E_{j\theta}[V_j'(s'(1 + r) + z'; \theta)] \geq E_{j\theta'}[V_j'(s'(1 + r) + z'; \theta)] \geq E_{j\theta'}[V_j'(s'(1 + r) + z'; \theta')]$$

Next suppose by contradiction that $s_{j-1}(x; \theta) < s_{j-1}(x; \theta')$ for some x, given $\theta \geq_\Theta \theta'$. In the equation below the topmost inequality holds by the Euler equation. The bottomost inequality follows from the result above and the fact that V_j' is decreasing in wealth since V_j is concave in wealth. The equality holds by the Euler equation since $s_{j-1}(x; \theta')$ is interior.

$$\begin{aligned} u'(x - s_{j-1}(x; \theta)) &\geq \beta(1 + r)E_{j\theta}[V_j'(s_{j-1}(x; \theta)(1 + r) + z'; \theta)] \\ &\geq \beta(1 + r)E_{j\theta'}[V_j'(s_{j-1}(x; \theta')(1 + r) + z'; \theta')] = u'(x - s_{j-1}(x; \theta')) \end{aligned}$$

A contradiction follows as $s_{j-1}(x;\theta) < s_{j-1}(x;\theta')$ and u strictly concave imply that the marginal utility of consumption must be strictly smaller under $s_{j-1}(x;\theta)$ than under $s_{j-1}(x;\theta')$. This completes the induction.

The proof of Lemma 1(iv) rests on Lemma 2 below, which is a restatement of Carroll and Kimball (2001, Lemma 1-2). The proof of Lemma 2 is essentially identical to that in Carroll and Kimball (2001) with the exception that the proof of Lemma 2(ii) is different and simpler. The proof of Lemma 1(iv) uses the following definition. A differentiable function $F(x)$ has property CC with respect to u provided $F'(x) = u'(f(x))$ for some increasing, concave function $f(x)$. The notation for the earnings process θ is dropped for convenience throughout the proof.

Proof of Lemma 1 (iv):

$V_J(x)$ has property CC with respect to u since $V_J(x) = u(x)$. Backwards induction establishes via Lemma 2 that $V_j(x)$ has property CC with respect to u for all j. This property and the fact that $V_j'(x) = u'(c_j(x))$ together imply that $c_j(x)$ is concave in x and that $s_j(x) = x - c_j(x)$ is convex in x for all j.

Lemma 2: Assume B1-3.

(i) $V_{j+1}(x)$ has property CC with respect to u implies - $_j(s) \equiv \beta E[V_{j+1}(s(1+r)+z)]$ has property CC with respect to u.

(ii) - $_j(s) \equiv \beta E[V_{j+1}(s(1+r)+z)]$ has property CC with respect to u implies $V_j(x)$ has property CC with respect to u.

Proof:

(i) Consider the case of CRRA utility. Assume that $\beta = (1+r) = 1$ for expositional simplicity only. With this simplification, - $'_j(s) = E[V'_{j+1}(s+z)] = E[u'(f(s+z))] = E[f(s+z)^{-\sigma}]$ for some increasing, concave f since V_{j+1} has property CC. It remains to show that (- $'_j(s))^{-1/\sigma}$ is increasing and concave. As this is clearly increasing, it remains to establish concavity. The following three inequalities then hold. The first follows for any z from the monotonicity and concavity of f when $s = ps_1 + (1-p)s_2$ with $p \in [0,1]$. The second follows from the first. The third is an application of Minkowski's inequality to the term on the right-hand side of the second inequality (see Hardy et al (1967, Thm. 198)). This requires that $\sigma > 0$ and that both $pf(s_1+z)$ and $(1-p)f(s_2+z)$ are positive random variables.

$$f(s+z) \geq pf(s_1+z) + (1-p)f(s_2+z)$$

$$(E[f(s+z)^{-\sigma}])^{-1/\sigma} \geq (E[(pf(s_1+z) + (1-p)f(s_2+z))^{-\sigma}])^{-1/\sigma}$$

$$(E[(pf(s_1{+}z){+}(1{-}p)f(s_2{+}z))^{-\sigma}])^{-1/\sigma} \geq (E[(pf(s_1{+}z))^{-\sigma}])^{-1/\sigma}{+}(E[((1{-}p)f(s_2{+}z))^{-\sigma}])^{-1/\sigma}$$

Combining these results one obtains the inequality below. This inequality establishes that $(\text{-}\,'_j(s))^{-1/\sigma}$ is concave and, thus, $\text{-}\,_j(s)$ exhibits property CC.

$$(\text{-}\,'_j(s))^{-1/\sigma} \geq p(\text{-}\,'_j(s_1))^{-1/\sigma} + (1-p)(\text{-}\,'_j(s_2))^{-1/\sigma}$$

Now consider the case of CARA utility. Assume that $\beta = (1{+}r) = 1$ for expositional simplicity only. With this simplification, $\text{-}\,'_j(s) = E[V'_{j+1}(s+z)] = E[e^{-af(s+z)}]$ for some increasing, concave f. It is sufficient to show that $-(1/a)log\text{-}\,'_j(s)$ or $-log\text{-}\,'_j(s)$ is increasing and concave. As both are clearly increasing, it remains to establish concavity. The first inequality below holds by the monotonicity and concavity of f when $s = ps_1 + (1{-}p)s_2$ with $p \in [0,1]$ and $a > 0$. The second follows from the first. The third holds by applying the arithmetic-geometric mean inequality to the term on the right-hand side of the second inequality, ommiting the log. More precisely, the arithmetic-geometric mean inequality implies that for positive random variables m, n with means $\bar{m}, \bar{n}$, $E[(m/\bar{m})^p(n/\bar{n})^{1-p}] \leq E[p(m/\bar{m}) + (1{-}p)(n/\bar{n})] = 1$ and, thus, $E[m^p n^{1-p}] \leq \bar{m}^p \bar{n}^{1-p}$.

$$-af(s+z) \leq -paf(s_1+z) - (1-p)af(s_2+z)$$

$$logE[e^{-af(s+z)}] \leq logE[e^{-paf(s_1+z)-(1-p)af(s_2+z)}]$$

$$E[e^{-paf(s_1+z)-(1-p)af(s_2+z)}] \leq E[e^{-af(s_1+z)}]^p E[e^{-af(s_2+z)}]^{(1-p)}$$

Taking the log of both sides of the third inequality above and combining this with the second inequality one obtains the inequality below. This inequality establishes that $log\text{-}\,'_j(s)$ is convex, $-log\text{-}\,'_j(s)$ is concave and, thus, $\text{-}\,_j(a)$ exhibits property CC.

$$logE[e^{-af(s+z)}] \leq plogE[e^{-af(s_1+z)}] + (1-p)logE[e^{-af(s_2+z)}]$$

(ii) Since $V_j(x) = \max_{s' \in \Gamma_j(x)} u(x-s') + \text{-}\,_j(s')$, $V'_j(x) = u'(c_j(x))$ and since $c_j(x)$ is increasing via the concavity of $V_j(x)$, it is sufficient to show that $c_j(x)$ is concave. This occurs if and only if the set $\Lambda_j \equiv \{(c,x) : u'(c) \geq \text{-}\,'_j(x-c); c \geq 0, x-c \geq \underline{s}_{j+1}, x \geq \underline{x}_j\}$ is convex. Since $\text{-}\,'_j(x-c) = u'(f(x-c))$ and u' is a decreasing function, $\Lambda_j = \{(c,x) : c \leq f(x-c); c \geq 0, x-c \geq \underline{s}_{j+1}, x \geq \underline{x}_j\}$. Λ_j convex then follows from the fact that f is increasing and concave using Rockafellar (1970 Thm. 4.6).

Table 1: Precautionary Savings Literature

Authors	Result	Key Assumptions	Model Periods
Leland (1968) Sandmo (1970) Rothschild and Stiglitz (1971) Mirman (1971) Dreze and Modigliani (1972) Diamond and Stiglitz (1974)	optimal decision rule $s_j(x;\theta)$ increases in θ	$u' > 0$, u' convex u strictly concave	$J = 2$
Miller (1975, 1976) Sibley (1975) Schechtman (1976) Mendelson and Amihud (1982)	optimal decision rule $s_j(x;\theta)$ increases in θ	$u' > 0$, u' convex u strictly concave independent earnings shocks	J arbitrary
Huggett (2001)	expected wealth holding profile increases in θ	$s_j(x;\theta)$ increases in x and θ and is convex in x independent earnings shocks	J arbitrary
Huggett (2001)	expected wealth holding profile increases in θ	u is of the CRRA or CARA class independent earnings shocks	J finite

Table 2: Solution

Period j	Earnings z_j	Wealth s_j	Cash-on-Hand $s_j(1+r)+z_j$	Consumption c_j	Expected Wealth $E[s_j]$
1	1.5	0.0	1.5	1.0	0.0
2	1.5	0.5	2.0	1.0	0.5
3	0.0	1.0	1.0	1.0	1.0

Period j	Earnings z_j	Wealth s_j	Cash-on-Hand $s_j(1+r)+z_j$	Consumption c_j	Expected Wealth $E[s_j]$
1	1.5	0.0	1.5	0.5638	0.0
2	0.0	0.9362	0.9362	0.2379	0.9362
	3.0	0.9362	3.9362	2.6592	-
3	0.0	0.6984	0.6984	0.6984	0.9877
	0.0	1.2770	1.2770	1.2770	-