Inventories and the business cycle: An equilibrium analysis of (S,s) policies

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1 Introduction

Inventory investment is both procyclical and volatile. Changes in firms' inventory holdings appear to account for almost half of the decline in production during recessions.[1] Moreover, the comovement between inventory investment and final sales raises the variance of production beyond that of sales. Historically, such observations have often prompted researchers to emphasize inventory investment as central to an understanding of aggregate fluctuations.[2] Blinder (1990), for example, concludes that "business cycles are, to a surprisingly large degree, inventory cycles".[3] By contrast, modern business cycle theory has been surprisingly silent on the topic of inventories.[4]

We study a dynamic stochastic general equilibrium model where, given nonconvex factor adjustment costs, producers follow generalized (S, s) inventory policies with regard to intermediate inputs. In particular, we extend the basic equilibrium business cycle model to include fixed costs associated with the acquisition of intermediate inputs for use in final goods production. Given these costs, final goods firms (a) maintain inventories of intermediate inputs, and (b) adjust these inventories only when their stock is sufficiently far from a target level. Our equilibrium analysis implies that this target level varies endogenously with the aggregate state of the economy. Because adjustment costs differ across firms, in addition to productivity and capital, the aggregate state vector includes a distribution of these producers over inventory levels.[5]

Our objective is two-fold. First, we evaluate the ability of our equilibrium general-

[1] Ramey and West (1999) show that, on average, the change in real inventory investment, relative to the change in real gross domestic production, accounts for 49 percent of the decline in output experienced during postwar U.S. recessions.

[2] See Blinder and Maccini (1991).

[3] Blinder (1990) page viii.

[4] When inventories are included in equilibrium models, their role is generally inconsistent with their definition. See, for example, Kydland and Prescott (1982) and Christiano (1988), where inventories are factors of production, or Kahn, McConnell and Perez-Quiros (2001), where they are a source of household utility.

[5] At times, some producers completely exhaust their input stocks. In such instances, the non-negativity constraint on inventories binds, which necessitates a nonlinear solution of the model.

ized (S, s) inventory model to reproduce salient empirical regularities. In particular, we focus on the cyclicality and variability of inventories, and the relative volatility of production and sales, as described below. Second, we examine the model's predictions for the role of inventories in aggregate fluctuations. This provides a formal analysis of the extent to which the existence of inventory adjustment amplifies or prolongs cyclical movements in production.

To assess the usefulness of our model in identifying the role of inventories in the business cycle, we evaluate its ability to reproduce (1) the volatility of inventory investment relative to production, (2) the procyclicality of inventory investment and (3) the greater volatility of production over that of sales. We view these three empirical regularities as essential characteristics of any formal analysis of the cyclical role of inventories. When we calibrate our equilibrium business cycle model of inventories to reproduce the average inventory-to-sales ratio in the postwar U.S. data, we find that it is able to explain half of the measured cyclical variability of inventory investment. Furthermore, inventory investment is procyclical, and production is more volatile than sales, as consistent with the data.

Examining our model's predictions for the aggregate dynamics of output, consumption, investment and employment, we find that the business cycle with inventories is broadly similar to that generated by a comparable model without them. Nonetheless, the inventory model yields somewhat higher variability in employment, and lower variability in consumption and investment. Thus our equilibrium analysis, which necessarily endogenizes final sales, alters our understanding of the role of inventory accumulation for cyclical movements in GDP. In particular, we find that the positive correlation between final sales and net inventory investment does not imply that inventories necessarily amplify aggregate fluctuations in production. The dynamics of final sales are altered by their presence. In the context of our equilibrium business cycle model, the introduction of inventories does not substantially raise the variability of production, because it lowers the variability of final sales.

2 A brief survey of empirical regularities

In this section, we discuss the small set of empirical regularities concerning inventory investment that are most relevant to our analysis.[6] Table 1 summarizes the business cycle behavior of GDP, final sales, changes in private nonfarm inventories and the inventory-to-sales ratio in postwar U.S. data.[7] Note first that the relative variability of inventory investment is large. In particular, though inventory investment's share of gross domestic production averages less than one-half of one percent, its standard deviation is 27 percent that of output. Next, net inventory investment is procyclical; its correlation coefficient with GDP is 0.66. Moreover, as the correlation between inventory investment and final sales is itself positive, 0.42 for the data summarized in table 1, the standard deviation of production substantially exceeds that of sales. It is this second positive correlation that is commonly interpreted as evidence that fluctuations in inventory investment increase the variability of GDP. Thus Ramey and West (1999, page 874) suggest that inventories "seem to amplify, rather than mute movements in production".

Our interest is in examining this thesis using dynamic stochastic general equilibrium analysis. However, inventories have received relatively little emphasis in general equilibrium models of aggregate fluctuations. Given positive real interest rates, the first challenge in any formal analysis of inventory adjustment is to explain why they exist. By far the most common approach is to assume that production is costly to adjust, and that the associated costs are continuous functions of the change in production. This is the production smoothing model which, in its simplest form, assumes that final sales are an exogenous stochastic series, and that adjustments to the level of production incur convex costs. As a result, firms use inventories to smooth

[6] More extensive surveys are available in Fitzgerald (1997), Horstein (1998) and Ramey and West (1999).

[7] The table is based on U.S. quarterly data, 1953:1 - 2002:2, seasonally adjusted and chained in 1996 dollars. GDP and final sales are reported as percentage standard deviations, detrended using a Hodrick-Prescott filter with a weight of 1600. Net investment in private nonfarm inventories x_t, is detrended relative to GDP, y_t. The detrended series is $\frac{x_t - \overline{x}_t}{\overline{y}_t}$, where $\overline{x}_t$ is the HP-trend of the series and $\overline{y}_t$ is the trend for GDP.

production in the face of fluctuations in sales. An apparent limitation of the model is it applies to a narrow subset of inventories, finished manufacturing goods.[8] As illustrated in Table 2, these are only 13 percent of the total. The remaining stocks are commonly rationalized as the result of nonconvex order or delivery costs (see Blinder and Maccini (1991)).[9] Such costs lead firms to adopt (S,s) inventory adjustment policies, ordering only when their stocks are sufficiently far from a target level. It is the prominence of inventories associated with nonconvex adjustment costs that leads to us to the first defining feature of our analysis; we model (S,s) inventory management.

In table 2, we also see that inventories of intermediate inputs are twice the size of finished goods in manufacturing. Furthermore, manufacturing inventories are far more cyclical than retail and wholesale inventories, the other main components of private nonfarm inventories. Humphreys, Maccini and Schuh (2001) undertake a variance decomposition and find that these input inventories are three times as volatile as finished goods within manufacturing. Taken together, these findings motivate the second defining feature of our analysis; we model inventories as stocks of intermediate goods.[10]

3 Model

There are three types of optimizing agents in the economy, households, intermediate goods producers and final goods firms. Households supply labor to both types of producers and purchase consumption from final goods firms. They save using asset

[8] Another difficulty, discussed at length by Blinder and Maccini (1991), is that the basic model, driven by exogenous fluctuations in sales, predicts that production is less variable than sales, and that sales and inventory investment are negatively correlated. Ramey (1991) shows that these inconsistencies with the data may be resolved if there are increasing returns to production, while Eichenbaum (1991) explores productivity shocks and Coen-Pirani (2002) integrates the stockout avoidance motive of Kahn (1987) in a model of industry equilibrium.

[9] An excellent example is Hall and Rust (1999), who show that a generalised $(S,\, s)$ decision rule explains the actual inventory investment behaviour of a U.S. steel wholesaler.

[10] An equilibrium analysis of retail finished goods inventories is undertaken by Fisher and Hornstein (2000), who use an (S, s) model to explain the higher volatility of production relative to sales.

markets where they trade shares that entitle them to the earnings of both intermediate and final goods producers. All firms in the economy are perfectly competitive. First, identical intermediate goods producers own capital and hire labor for production. They sell their output to, and purchase investment goods from, final goods producers. Next, final goods firms use intermediate inputs and labor to produce output that may be used for consumption or capital accumulation.

We provide an explicit role for inventories by assuming that final goods firms face fixed costs of ordering or accepting deliveries of intermediate inputs. In particular, as the costs are independent of order size, these firms choose to hold stocks of inputs, s, where $s \in \mathbf{S} \subseteq R_{+}$. Further, the costs vary across final goods firms, so some will adjust their inventory holdings, while others will not, at any point in time. As a result, the model yields an endogenous distribution of final goods firms over inventory levels, $\mu : \mathcal{B}(\mathbf{S}) \to [0,1]$, where $\mu(s)$ represents the measure of firms with start-of-period inventories s.

The economy's aggregate state is (z, Ξ), where $\Xi \equiv (K, \mu)$ represents the endogenous state vector. K is the aggregate capital stock held by intermediate goods firms, and z is total factor productivity in the production of intermediate inputs. The distribution of firms over inventory levels evolves according to a mapping Γ_{μ}, $\mu' = \Gamma_{\mu}(z, \Xi)$, and capital similarly evolves according to $K' = \Gamma_K(z, \Xi)$.[11] For convenience, we assume that productivity follows a Markov Chain, $z \in \{z_1, \ldots, z_{N_z}\}$, where

$$\Pr\left(z' = z_j \mid z = z_i\right) \equiv \pi_{ij} \geq 0, \tag{1}$$

and $\sum_{j=1}^{N_z} \pi_{ij} = 1$ for each $i = 1, \ldots, N_z$. Except where necessary for clarity, we suppress the index for current productivity below.

All producers employ labour at the real wage, $\omega(z, \Xi)$, and those involved in the production of final goods purchase intermediate inputs at the relative price $q(z, \Xi)$. Finally, all firms, whether producing intermediate inputs or final goods, value current

[11] Throughout the paper, primes indicate one-period ahead values. We define Γ_{μ} in section 3.2.3, following the description of firms' problems, and Γ_K in section 3.4.

profits by the final output price $p(z,\Xi)$ and discount future earnings by β.[12] For brevity, we suppress the arguments of ω, q and p where possible below.

3.1 Intermediate goods producers

The representative intermediate goods firm produces using capital, k, and labor, l, through a constant returns to scale technology, $zF(k,l)$. Intermediate inputs are sold to final goods firms at the relative price q. The producer may adjust next period's capital stock using final goods as investment. Capital depreciates at the rate $\delta \in (0,1)$. Equation 2 below is the functional equation describing the intermediate goods firm's problem. The value function W is a function of the aggregate state (z,Ξ), which determines the prices (p, q and ω). Ξ evolves over time according to $\mu' = \Gamma_\mu(z,\Xi)$ and $K' = \Gamma_K(z,\Xi)$ where $\Xi \equiv (K,\mu)$, and changes in productivity follow the law of motion described in (1).

$$W(k;z,\Xi) = \max_{k',l}\left(p\Big[qzF(k,l) + (1-\delta)k - k' - \omega l\Big] + \beta\sum_{j=1}^{N_z}\pi_{ij}W\left(k';z_j,\Xi'\right)\right) \quad (2)$$

The following efficiency conditions describe the producer's selection of employment and investment.

$$zD_2F(k,l) = \frac{\omega}{q} \quad (3)$$

$$\beta\sum_{j=1}^{N_z}\pi_{ij}D_1W\left(k';z_j,\Xi'\right) = p \quad (4)$$

Because F is linearly homogenous, the firm's decision rules for employment and production are proportional to its capital stock; $l(k) \equiv L(z,\Xi)k$ where $L(z,\Xi)$ solves (3) as a function of $(z, \omega(z,\Xi), q(z,\Xi))$, and production is given by $x(k;z,\Xi) = zF(1,L(z,\Xi))k$. This means that current profits are linear in k; $\pi(z,\Xi) \equiv qzF(1,L(z,\Xi)) + (1-\delta) - \omega L(z,\Xi)$. It is straightforward to show that this property is inherited by

[12] This is equivalent to requiring that firms discount by $1 + r_{t,t+k} = \frac{p_t}{\beta^k p_{t+k}}$ between dates t and $t+k$, where p represents households' current valuation of output and β their subjective discount factor. This discounting rule is an implication of equilibrium, as discussed in section 3.4.

the value function; $W(k; z, \Xi) = w(z; \xi) k$, where

$$w(z, \Xi) \cdot k = \max_{k'} \; p(z, \Xi) \Big[\pi(z, \Xi) k - k' \Big] + \beta \sum_{j=1}^{N_z} \pi_{ij} w\left(z_j, \Xi'\right) k'.$$

Equation 4 then implies that an interior choice of investment places the following restriction on the equilibrium price of final output.

$$p(z, \Xi) = \beta \sum_{j=1}^{N_z} \pi_{ij} w\left(z_j, \Xi'\right). \tag{5}$$

When (5) is satisfied, the intermediate goods firm is indifferent to any level of k' and will purchase investment as the residual from final goods production after consumption.

3.2 Final goods producers

There are a large number of final goods firms, each facing time-varying costs of arranging deliveries or sales of intermediate inputs. Given differences in delivery costs, some firms adjust their stocks, while others do not, at any date. Thus, firms are distinguished by their inventories of intermediate goods.

At the start of any date, a final goods firm is identified by its inventory holdings, s, and its current delivery cost, $\xi \in [A, B]$. This cost is denominated in hours of labor and drawn from a time-invariant distribution $H(\xi)$ common across firms. Intermediate inputs used in the current period, m, and labor, n, are the sole factors of final goods production, $y = G(m, n)$, where G exhibits decreasing returns to scale. Note that technology is common across these firms; the only source of heterogeneity in production arises from differences in inventories.

The timing of final goods firms' decisions is as follows. At the beginning of each period, any such firm observes the aggregate state (z, Ξ) as well as its current delivery cost ξ. Before production, it undertakes an inventory adjustment decision. In particular, the firm may absorb its fixed cost and adjust its stock of intermediate inputs available for production, $s_1 \geq 0$. Letting x_m denote the chosen size of such an adjustment, the stock of intermediate inputs available for current production becomes $s_1 = s + x_m$. Alternatively, the firm can avoid the cost, set $x_m = 0$, and enter production with its initial stock; $s_1 = s$. Following the inventory adjustment decision, the firm determines current production, selecting $m \in [0, s_1]$ and $n \in R_+$. Intermediate inputs fully depreciate in use, and the remaining stock with which the firm begins next period is denoted s'. Measuring adjustment costs in units of final output using the wage rate, ω, the firm's order choice is summarized below.

order size	total order costs	production-time stock	next-period stock
$x_m \neq 0$	$\omega\xi + qx_m$	$s_1 = s + x_m$	$s' = s_1 - m$
$x_m = 0$	0	$s_1 = s$	$s' = s - m$

We assume that there is a storage cost associated with holding inputs unused throughout the period. This cost is proportional to the level of inventories held; in particular, given end of period inventories s', the firm's total cost of storage is $\sigma s'$ where $\sigma > 0$.

Let $E(s, \xi; z, \Xi)$ represent the expected discounted value of a final goods firm with start-of-date inventory holdings s and fixed order cost ξ. We describe the problem facing such a firm using (6) - (9) below. First, for convenience, we define the beginning of period expected value of the firm, prior to the realization of its fixed cost, but given $(s; z, \Xi)$.

$$V(s; z, \Xi) \equiv \int_A^B E(s, \xi; z, \Xi) H(d\xi) \tag{6}$$

Next, we divide the period into two subperiods, adjustment-time and production-time, and we break the description of the firm's problem into the distinct problems it faces as it enters into each of these subperiods.

3.2.1 Production decisions

Beginning with the second subperiod, let $E_1(s_1; z, \Xi)$ represent the value of entering production with inventories s_1. Given this stock of intermediate input available for production, the firm selects its current employment, n, and inventories for next period, s', (hence current input usage, $m = s_1 - s'$) to solve

$$E_1(s_1; z, \Xi) = \max_{s' \geq 0, n} \Big(p \Big[G(s_1 - s', n) - \omega n - \sigma s' \Big] + \beta \sum_{j=1}^{N_z} \pi_{ij} V(s'; z_j, \Xi') \Big), \qquad (7)$$

taking prices (p, ω and q), and the evolution of $\Xi' \equiv (K', \mu')$ according to Γ_K and Γ_μ, as given. Given the production-time stock of intermediate inputs, s_1, and the continuation value of inventories of these inputs, $V(s'; z_j, \Xi')$, equation (7) yields both the firm's employment (in production) decision, which satisfies $D_2 G(s_1 - s', n) = \omega$, and its use of intermediate inputs. Let $N(s_1; z, \Xi)$ describe its employment and $S(s_1; z, \Xi)$ its stock of intermediate input retained for future use. Current production of final goods is $Y(s_1; z, \Xi) = G(s_1 - S(s_1; z, \Xi), N(s_1; z, \Xi))$. Thus, we have decision rules for employment, production, and next-period inventories as functions of the production-time stock s_1.

3.2.2 Inventory adjustment decisions

Given the middle-of-period valuation of the firm, E_1, we now examine the inventory adjustment decision that precedes production. At the start of the period, for a final goods firm with beginning of period inventories s and adjustment cost ξ, equations (8) - (9) describes the (s, ξ) firm's determination of (i) whether to place an order and (ii) the target inventory level with which to begin the production subperiod, conditional on an order. The first term in the braces of (8) represents the net value of stock adjustment, (the gross adjustment value less the value of the payments associated with the fixed delivery cost,) while the second term represents the value of entering production with the beginning of period input stock.

$$E(s, \xi; z, \Xi) = pqs + \max\Big\{ -p\omega\xi + E_A(z, \Xi), -pqs + E_1(s; z, \Xi) \Big\} \qquad (8)$$

$$E_A(z, \Xi) \equiv \max_{s_1 \geq 0} \Big(-pqs_1 + E_1(s_1; z, \Xi) \Big) \qquad (9)$$

Note that the target inventory choice in (9) is independent of both the current inventory level, s, and fixed cost, ξ. Thus, all firms that adjust their inventory holdings choose the same production-time level, and achieve the same gross value of adjustment, $E_A(z, \Xi)$. Let $s^* \equiv s^*(z, \Xi)$ denote this common target, which solves (9) as a function of the aggregate state of the economy. Equation (7) then implies common employment and intermediate input use choices across all adjusting firms, as well as identical inventory holdings among these firms at the beginning of the next period.

Turning to the decision of whether to adjust to the target level of inventories, it is immediate from equation (8) that a firm will place an order if its fixed cost falls at or below $\widetilde{\xi}(s; z, \Xi)$, the cost that equates the net value of inventory adjustment to the value of non-adjustment.

$$-p\omega\widetilde{\xi}(s; z, \Xi) + E_A(z, \Xi) = E_1(s; z, \Xi) - pqs \tag{10}$$

Given the support of the cost distribution, and using (10) above, we define $\overline{\xi}(s; z, \Xi)$ as the type-specific threshold costs separating those firms that place orders from those that do not.

$$\overline{\xi}(s; z, \Xi) = \min\Big\{\max\Big(A, \widetilde{\xi}(s; z, \Xi)\Big), B\Big\} \tag{11}$$

Thus, we arrive at the following decision rules for the production-time holdings of intermediate inputs and associated stock adjustments.

$$s_1(s, \xi; z, \Xi) = \begin{cases} s^*(z, \Xi) & \text{if} \quad \xi \leq \overline{\xi}(s; z, \Xi) \\ s & \text{if} \quad \xi > \overline{\xi}(s; z, \Xi) \end{cases} \tag{12}$$

$$x_m(s, \xi; z, \Xi) = s_1(s, \xi; z, \Xi) - s \tag{13}$$

The common distribution of adjustment costs facing final goods firms, given their threshold adjustment costs, implies that $H\Big(\overline{\xi}(s; z, \Xi)\Big)$ is the probability that a firm of type s will alter its inventory stock before production. Using this result, the start-of-period value of the firm prior to the realization of its fixed delivery cost, (6), may be simplified as

$$V\left(s;z,\Xi\right)=pqs+H\left(\overline{\xi}\left(s;z,\Xi\right)\right)E_A\left(z,\Xi\right)-p\omega\int_A^{\overline{\xi}(s;z,\Xi)}\xi H\left(d\xi\right) \qquad (14)$$
$$+\Big(1-H\left(\overline{\xi}\left(s;z,\Xi\right)\right)\Big)\Big(E_1\left(s;z,\Xi\right)-pqs\Big),$$

where $\int_A^{\overline{\xi}(s;z,\Xi)}\xi H\left(d\xi\right)$ is the expectation of the fixed cost ξ conditional on its payment.

3.2.3 Aggregation

Having described the inventory adjustment and production decisions of final goods firms as functions of their type, s, and cost draw, ξ, we can now aggregate their demand for the production of intermediate goods firms, their demand for labour, and their production of the final good. First, the aggregate demand for intermediate inputs is the sum of the stock adjustments from each start-of-period inventory level s, weighted by the measures of firms undertaking these adjustments.

$$\overline{X}(z,\Xi)=\int_{\mathbf{S}}H\Big(\overline{\xi}(s;z,\Xi)\Big)\Big(s^*(z,\Xi)-s\Big)\mu(ds) \qquad (15)$$

Second, the production of final goods is the population-weighted sum of production across both adjusting and non-adjusting firms.

$$\begin{aligned}\overline{Y}(z,\Xi) \quad &= \quad Y\left(s^*(z,\Xi);z,\Xi\right)\int_{\mathbf{S}}H\Big(\overline{\xi}(s;z,\Xi)\Big)\mu\left(ds\right)+ \qquad (16)\\ &\int_{\mathbf{S}}Y(s;z,\Xi)\Big[1-H\Big(\overline{\xi}(s;z,\Xi)\Big)\Big]\mu\left(ds\right)\end{aligned}$$

Finally, employment demand by final goods firms is the weighted sum of labor employed in production by adjusting and non-adjusting firms, together with the total time costs of adjustment.

$$\overline{N}(z,\Xi)=N\left(s^*(z,\Xi);z,\Xi\right)\int_{\mathbf{S}}H\Big(\overline{\xi}(s;z,\Xi)\Big)\mu\left(dS\right) \qquad (17)$$
$$+\int_{\mathbf{S}}\Big[1-H\Big(\overline{\xi}(s;z,\Xi)\Big)N(s;z,\Xi)\Big]\mu\left(dS\right)+\int_{\mathbf{S}}\left[\int_A^{\overline{\xi}(s;z,\Xi)}\xi H(d\xi)\right]\mu\left(dS\right)$$

We next define Γ_μ, the evolution of the distribution of final goods firms, using (10) - (11). Of each group of firms sharing a common stock $s \neq s^*$at the start of the current period, fraction $1 - H(\overline{\xi}(s; z, \Xi))$ do not adjust their inventories. Thus, $\mu(s)[1 - H\left(\overline{\xi}(s; z, \Xi)\right)]$ firms will begin the next period with $S(s; z, \Xi)$ as defined in section 3.2.1. Those firms that either enter the period with the current target or actively adjust to it for production, $\mu(s^*(z, \Xi)) + \int_{\mathbf{S}} H\left(\overline{\xi}(s; z, \Xi)\right)\mu\,(ds)$ in all, will move to the next period with $S(s^*(z, \Xi); z, \Xi)$.

Given the preceding discussion, the evolution of the distribution of final goods firms may be described as follows. Define $S^{-1}(\widetilde{s}; z, \Xi)$ as the production-time inventory level that gives rise to next period inventories $\widetilde{s}$ in the solution to (7). For any stock $\widetilde{s}$ other than that arising from the target level of production-time inventories, $S^{-1}(\widetilde{s}; z, \Xi) \neq s^*\,(z, \Xi)$,

$$\mu'\,(\widetilde{s}) = \left[1 - H\left(\overline{\xi}\,\left(S^{-1}(\widetilde{s}; z, \Xi)\right)\right)\right]\mu\left(S^{-1}(\widetilde{s}; z, \Xi)\right). \tag{18}$$

For the stock arising from the target inventory level, $S^{-1}(\widetilde{s}; z, \Xi) = s^*\,(z, \Xi)$,

$$\mu'(\widetilde{s}) = \left[1 - H\left(S^{-1}(\widetilde{s}; z, \Xi)\right)\right]\mu\left(S^{-1}(\widetilde{s}; z, \Xi)\right) + \int_{\mathbf{S}} H\left(\overline{\xi}(s; z, \Xi)\right)\mu(ds). \tag{19}$$

3.3 Households

The economy is populated by a unit measure of identical households who value consumption and leisure and discount future utility by $\beta \in (0, 1)$. Households have fixed time endowments in each period, normalized to 1, and they receive real wage $\omega\,(z, \Xi)$ for their labor. Their wealth is held as one-period shares in final goods firms, denoted by the measure λ_f, and as shares in the unit measure of identical intermediate goods firms, λ_i.

At each date, households must determine their current consumption, C, hours worked, N, as well as the numbers of new shares in final goods firms, $\lambda_f'(s)$, and intermediate goods firms, λ_i', to purchase at prices $\rho_f(s; z, \Xi)$ and $\rho_i(z, \Xi)$ respectively.[13]

[13] In equilibrium, these prices are $\frac{V(s;z,\Xi)}{p(z,\Xi)}$ and $\frac{W(K;z,\Xi)}{p(z,\Xi)}$.

Their expected lifetime utility maximization problem is described recursively below.

$$R\left(\lambda_i, \lambda_f; z, \Xi\right) = \max_{C,N,\lambda_I',\lambda_F'} \Big(U\left(C, 1-N\right) + \beta E_{z'} R[\left(\lambda_i', \lambda_f'; z', \Xi'\right) \ \mid z]\Big) \tag{20}$$

subject to

$$C + \rho_i(z,\Xi)\lambda_i' + \int_{\mathbf{S}} \rho_f\left(s; z, \Xi\right) \lambda_f'\left(ds\right)$$

$$\leq \omega\left(z, \Xi\right) N + \rho_I(z,\Xi)\lambda_i + \int_{\mathbf{S}} \rho_F\left(s; z, \Xi\right) \lambda_f\left(ds\right) \tag{21}$$

$$\Xi' = \Gamma\left(z, \Xi\right) \tag{22}$$

Let $C\left(\lambda_i, \lambda_f; z, \Xi\right)$ summarize their choice of current consumption, $N\left(\lambda_i, \lambda_f; z, \Xi\right)$ their allocation of time to working, $\Lambda_i\left(k, \lambda_i, \lambda_f; z, \Xi\right)$ their purchases of shares in the representative intermediate goods firm, and $\Lambda_F\left(s, \lambda_i, \lambda_f; z, \Xi\right)$ the quantity of shares they purchase in final goods firms that will begin next period with inventories s.

3.4 Equilibrium

In equilibrium, households will hold a portfolio of all firms, ($\Lambda_i\left(1, \mu; z, \Xi\right) = 1$ and $\Lambda_f\left(s, 1, \mu; z, \Xi\right) = \mu'(s)$), and will supply a level of labor consistent with employment across these firms, at each date. Consequently, the real wage must equal households' marginal rate of substitution between leisure and consumption,

$$\omega\left(z, \Xi\right) = \frac{D_2 U\Big(C\left(1, \mu; z, \Xi\right), 1 - N\left(1, \mu; z, \Xi\right)\Big)}{D_1 U\Big(C\left(1, \mu; z, \Xi\right), 1 - N\left(1, \mu; z, \Xi\right)\Big)}, \tag{23}$$

and all firms must discount future profit flows with state-contingent discount factors that are consistent with households' marginal rate of intertemporal substitution, $\frac{D_1 U\left(C(1,\mu;z,\Xi),1-N(1,\mu;z,\Xi)\right)}{\beta D_1 U\left(C(1,\mu';z',\Xi'),1-N(1,\mu';z',\Xi')\right)}$. Following the approach outlined in Khan and Thomas (2002), we have already imposed the latter restriction in describing firms' problems above. Specifically, we have assumed that all firms value current profit flows at the final output price $p\left(z, \Xi\right)$, which represents the household marginal utility of equilibrium consumption, and that firms discount their future values by the subjective discount factor β.

$$p\left(z, \Xi\right) = D_1 U\Big(C\left(1, \mu; z, \Xi\right), 1 - N\left(1, \mu; z, \Xi\right)\Big) \tag{24}$$

When p and ω are evaluated at the equilibrium values of consumption and total work hours, we are able to recover all equilibrium decision rules by solving firms' problems alone.

Because there is no heterogeneity in intermediate goods production, in equilibrium, $K = k$ at each date. Thus, the evolution of the aggregate capital stock, summarized above by $K' = \Gamma_K(z, \Xi)$, is defined as $\Gamma_K(z, \Xi) \equiv (1-\delta)K + \overline{Y}(z, \Xi) - C(1, \mu; z, \Xi)$, where $\overline{Y}(z, \Xi)$ is given by (16). Next, the aggregate demand for intermediate inputs by final goods firms adjusting their holdings of inventories must equal the production of these inputs, and household labor supplied must equal total employment demand across intermediate and final goods firms;

$$\overline{X}(z, \Xi) = x(K; z, \Xi) \quad \text{and} \quad N(1, \mu; z, \Xi) = L(z, \Xi)K + \overline{N}(z, \Xi).$$

For any particular output price p, the two requirements above directly imply a relative price for intermediate inputs, $q(z, \Xi; p)$, and a wage, $\omega(z, \Xi; p)$, which in turn imply levels of output and consumption. Given these, the equilibrium output price $p(z, \Xi)$ is that which satisfies condition (5), so that the intermediate goods firm is satisfied to invest that what remains of final output after consumption.

$$\overline{Y}(z, \Xi) = C(1, \mu; z, \Xi) + \left[K' - (1-\delta)K\right]$$

Finally, it is convenient to describe equilibrium inventory investment in terms of total use and production of intermediate goods. Define total usage as the total production-time input stock less that remaining at the end of the period, held as inventories for the subsequent period.

$$M(z, \Xi) \equiv \int_{\mathbf{S}} \left[\int_A^B \left(s_1(s, \xi; z, \Xi) - S(s; z, \Xi)\right) H(d\xi)\right] \mu(dS)$$

Aggregate inventory investment is the change in total inventories, weighted by the relative price of intermediate inputs, which in equilibrium is the q-weighted difference between the intermediate goods firm's supply and total usage by final goods firms, $q(z, \Xi)\Big(x(K; z, \Xi) - M(z, \Xi)\Big)$.

4 Parameter choices

We examine the implications of inventory accumulation for an otherwise standard equilibrium business cycle model using numerical methods. In calibrating our model, we choose the length of a period as one quarter and select functional forms for production and utility as follows. We assume that intermediate goods producers have a Cobb-Douglas production function with capital share α, and that their productivity follows a Markov Chain with three values, $N_z = 3$, that is itself the result of discretizing an estimated log-normal process for technology with persistence ρ and variance of innovations, σ_ε^2. Final goods firms also have Cobb-Douglas technology, with intermediate input share θ_m, $G(m,n) = m^{\theta_m} n^{\theta_n}$. The adjustment costs that provide the basis for inventory holdings in our model are assumed to be distributed uniformly with lower support 0 and upper support B. Finally, we assume that households' period utility is the result of indivisible labor decisions implemented with lotteries (Rogerson (1988), Hansen (1985)), $u(C, 1-N) = \log C + \eta \cdot (1-N)$.

4.1 Benchmark model

If we set $B = 0$, the result is a 2-sector model where firms have no incentive to hold any inventories.[14] With no adjustment costs, final goods firms buy intermediate inputs in every period; hence there are two representative firms, an intermediate goods firm and a final goods firm. We take this model as a benchmark against which to evaluate the effect of introducing inventory accumulation. The parameterization of the benchmark and inventory models is identical, with the already noted exception of the cost distribution associated with adjustments to intermediate inputs.

The parameters which are common to both the benchmark and inventory models, (α, θ_m, θ_n, δ, β, η), are derived, wherever possible, from standard values. The parameter associated with capital's share, α, is chosen to reproduce a long-run annual business capital-to-GDP ratio of 1.094, a value derived from averaging U.S. data

[14] This is essentially the real business cycle model of Hansen (1985) generalized for an intermediate input.

between $1953-2000$. The depreciation rate δ is equal to the average ratio of investment to business capital over the same time period. The distinguishing feature of the baseline model, relative to the Indivisible Labour Economy of Hansen (1985), is the presence of intermediate inputs. The single new parameter implied by the additional factor of production, the share term for intermediate inputs, is set to equal the value implied by the NBER-CES Manufacturing database, 0.5; this lies in the range estimated by Jorgenson, Gollop and Fraumeni (1987) for U.S. manufacturing over the years 1947-79. The remaining production parameter, θ_n, is taken to imply a labour's share of output averaging $\frac{2}{3}$, a value similar to that selected by Hansen (1985) and Prescott (1986). In terms of preferences, the subjective discount factor, β, is selected to yield a real interest rate of 4 percent per year in the steady state of the model, and η is chosen so that average hours worked are $\frac{1}{3}$ of available time. Both values are taken from Prescott (1986).

We determine the stochastic process for productivity using the Crucini residual approach described in King and Rebelo (1999). A continuous shock version of the benchmark model, where $\log z_{t+1} = \rho \log z_t + \varepsilon_{t+1}$ with $\varepsilon_{t+1} \sim N\left(0, \sigma_\varepsilon^2\right)$, is solved using linear methods for an arbitrary pair of initial values $\left(\rho, \sigma_\varepsilon^2\right)$. The linear solution yields a decision rule for output of the form $Y_t = \pi_z(\rho) z_t + \pi_k(\rho) k_t$. Rearranging this solution, data on GDP and capital are then used to infer an implied set of values for the technology shock series z_t. Maintaining the assumption that these technology shock realizations are generated by a first-order autoregressive process, the persistence and variance of these implied values yield new estimates of $\left(\rho, \sigma_\varepsilon^2\right)$. The process is repeated until these estimates converge. The resulting values for the persistence and variance of the technology shock process are not uncommon.

4.2 Inventory model

Table 3 lists the baseline calibration of our inventory model. For all parameters that are also present in the benchmark model, we maintain the same values as there. This approach to calibrating the inventory model is feasible, as the steady states of the two model economies, in terms of the capital-output ratio, hours worked, and the shares of the three factors of production, are close.

The two parameters that distinguish the inventory model from the benchmark are the upper support for adjustment costs (uniformly distributed on $[0, B]$) and the storage cost associated with inventories. We determine the upper support as follows. Using NIPA data, we compute that the quarterly real private nonfarm inventory-to-sales ratio has averaged 0.714 in the U.S. between 1947 : 1 and 1997 : 4, when the data series ends. This value lies just above the Ramey and West (1999) average value across G7 countries of 0.66. Moreover, as noted by these authors, the real series, in contrast to its nominal counterpart, exhibits no trend. Thus, given the storage cost σ, we select the parameter B to reproduce this average inventory-to-sales ratio in our model. The storage cost is difficult to identify in the data; for our baseline calibration of the model, it is set to equal the rate of depreciation on capital. Given $\sigma = \delta$, the upper support of the cost distribution is calibrated at $B = 0.204$. Higher storage costs raise the cost of holding inventories and thus require higher adjustment costs, effectively a larger value for B, to match the measured inventory-sales ratio.

5 Numerical method

(S, s) models of inventory accumulation are rarely examined in equilibrium. As these models are characterized by an aggregate state vector that includes the distribution of the stock of inventory holdings across firms, computation of equilibrium is nontrivial. Our solution algorithm involves repeated application of the contraction mapping implied by (6), (7), (8) and (9) to solve for final goods firms' start-of-period value functions V, given the price functions $p(z, \Xi)$, $\omega(z, \Xi)$ and $q(z, \Xi)$ and the laws of motion implied by Γ and (π_{ij}). This recursive approach is complicated in two ways, as discussed below.

First, the nonconvex factor adjustment here requires that we solve for firms' decision rules using nonlinear methods. This is because firms at times find themselves with a very low stock of intermediate inputs relative to their production-time target, but a sufficiently high adjustment cost draw that they are unwilling to replenish their stock in the current period. At such times, they will exhaust their entire stock in production, deferring adjustment until the beginning of the next period, before further production. Thus, a non-negativity constraint on inventory holdings binds occasionally, and firms decision rules are nonlinear and must be solved as such. This we accomplish using multivariate piecewise polynomial splines, adapting an algorithm outlined in Johnson (1987). In particular, our splines are generated as the tensor product of univariate cubic splines, with one of these corresponding to each argument of the value function.[15] We apply spline approximation to V and E_1, using a multi-dimensional grid on the state vector for these functions.

Second, equilibrium prices are functions of a large state vector, given the presence of the distribution of final goods firms in the endogenous aggregate state vector, $\Xi = (K', \mu')$. For computational feasibility, we assume that agents use a smaller object to proxy for the distribution in forecasting the future state and thereby determining their decisions rules given current prices. In choosing this proxy, we extend the method applied in Khan and Thomas (2002), which itself applied a variation on the method of Krussel and Smith (1998). In particular, we approximate the distribution in the aggregate state vector with a vector of moments, $m = (m_1, ..., m_I)$, drawn from the distribution. In our work involving discrete heterogeneity in production, we find that sectioning the distribution into I equal-sized partitions and using the conditional mean of each partition is very efficient in that it implies small forecasting errors.

The solution algorithm is iterative, applying one set of forecasting rules to generate decision rules that are used in obtaining data upon which to base the next set of forecasting rules. In particular, given I, we assume functional forms that predict next period's endogenous state (K', m'), and the prices p and pq, as functions of the current state, $K' = \widehat{\Gamma}_K\left(z, K, m; \chi_l^K\right)$, $m' = \widehat{\Gamma}_m\left(z, K, m; \chi_l^m\right)$, $p = \widehat{p}\left(z, K, m; \chi_l^p\right)$ and $pq = \widehat{pq}\left(z, K, m; \chi_l^{pq}\right)$, where χ_l^K, χ_l^m, χ_l^p, and χ_l^{pq} are parameter vectors that are determined iteratively, with l indexing these iterations. For the class of utility functions we use, the wage is immediate once p is specified; hence there is no need to

[15] For additional details, see Khan and Thomas (2002).

assume a wage forecasting function.

For any I, $\widehat{\Gamma}_K$, $\widehat{\Gamma}_m$, $\widehat{p}$, and $\widehat{pq}$, we solve for V on a grid of values for $(s; z, K, m)$. Next, we simulate the economy for T periods, recording the actual distribution of final goods firms, μ_t, at the start of each period, $t = 1, \ldots, T$. To determine equilibrium in each period, we begin by calculating m_t using the actual distribution, μ_t, and then use $\widehat{\Gamma}_K$ and $\widehat{\Gamma}_m$ to specify expectations of K_{t+1} and m_{t+1}. This determines $\beta \sum_{j=1}^{N_z} \pi_{ij} w\left(z_j, K_{t+1}, m_{t+1}\right)$ and $\beta \sum_{j=1}^{N_z} \pi_{ij} V\left(s'; z_j, K_{t+1}, m_{t+1}\right)$ for any s'. Given the second function, the conditional expected continuation value associated with any level of inventories, we are able to determine $s^*\left(z, K, m\right)$ and $\overline{\xi}\left(s; K, m\right)$, hence recovering the decisions of final goods firms and the implied next period distribution, given any values for p and q. Given any p, the equilibrium q is solved to equate the intermediate input producer's supply, $x(K; z, \Xi)$, to the demand generated by final goods firms.[16] The equilibrium output price, $p(z, \Xi; \chi_l^K, \chi_l^m, \chi_l^p, \chi_l^{pq})$, is that which generates production of the final good such that, given $C = \frac{1}{p}$, the residual level of investment, $Y_t - C_t$, implies a level of capital tomorrow, $K_{t+1} = \left(1-\delta\right) K_t + Y_t - C_t$, that satisfies the restriction in (5). Finally, (18) and (19) determine the distribution of final goods firms over inventory levels for next period, μ_{t+1}. With the equilibrium K_{t+1} and μ_{t+1}, we move into the next date in the simulation, again solving for equilibrium, and so forth. Once the simulation is completed, the resulting data, $(p_t, p_t q_t, K_t, m_t)_{t=1}^T$, are used to re-estimate $\left(\chi_l^K, \chi_l^m, \chi_l^p, \chi_l^{pq}\right)$ using OLS.
We repeat this two-step process, first solving for V given $\left(\chi_l^K, \chi_l^m, \chi_l^p, \chi_l^{pq}\right)$, next using our solution for firms' value functions to determine equilibrium decision rules over a simulation, storing the equilibrium results for $(p_t, p_t q_t, K_t, m_t)_{t=1}^T$, and then updating $\left(\chi_{l+1}^K, \chi_{l+1}^m, \chi_{l+1}^p, \chi_{l+1}^{pq}\right)$, until these parameters converge. The number of partition means used to proxy for the distribution μ, I, is increased until agents' forecasting rules are sufficiently accurate.

[16] This demand depends on the target inventory level $s^*\left(z, K, m\right)$, the start-of-period distribution of firms $\mu(s)$, and the adjustment thresholds of each firm type $\overline{\xi}\left(s; K, m\right)$.

5.1 Forecasting functions

Table 2 displays the actual forecasting functions used for the baseline inventory model, that in which the model's inventory-sales ratio matches its measured counterpart when averaged over the simulation. We use a log-linear functional form for each forecasting rule that is conditional on the level of productivity, z_i, $i = 1, \ldots, N_z$.[17] In the results reported here, $I = 1$. This means that, alongside z and K, only the mean of the current distribution of firms over inventory levels, start-of-period aggregate inventory holdings, is used by agents to forecast the relevant features of the future endogenous state. This degree of approximation would be unacceptable if it yielded large errors in forecasts. However, table 2 shows that, for each of the three values of productivity, the forecast rules for prices and both elements of the approximate state vector are extremely accurate. The standard errors across all regressions are small, and the R-squares are high, all above 0.997.[18]

In that they provide a description of the behavior of equilibrium prices and the laws of motion for capital and inventories, the regressions in table 2 also offer some insight into the impact of inventories on the model. In particular, note that there is relatively little impact of inventories, m_1, on the valuation of current output, p, and capital, K. Inventories have somewhat larger influence in determining the price of intermediate inputs and, of course, in forecasting their own future value.

[17] We have tried a variety of alternatives including adding higher-order terms and a covariance term. None of these significantly altered the forecasts used in the model. In future we plan to assess $I = 2$ for robustness. We were unable to complete this experiment in the current draft because it implies 5−dimensional value functions, which given our nonlinear method, implies substantial additional computing cost.

[18] In evaluating the standard errors, it may be useful to note that the means of p, pq, K and m_1 are 3.640, 1.786, 1.239 and 0.4323 respectively.

6 Results

6.1 Steady state

Suppressing stochastic changes in the productivity of intermediate goods producers, the sole source of uncertainty in our model, table 5 presents the steady state behavior of final goods firms. This illustrates the mechanics of our generalized (S,s) inventory adjustment and its consequence for the distribution of production across firms.

In our baseline calibration, where $B = 0.204$, there are 5 levels of inventories identifying firms. This beginning of period distribution is in columns $1 - 5$, while the first column, labelled adjustors, lists those firms from each of these groups that undertake inventory adjustment prior to production.

The inventory level selected by all adjusting firms, referred to above as the target value s^*, is 1.219 in the steady state. Firms that adjusted their inventory holdings last period, those in column 1, begin the current period with 0.831 units of the intermediate input. Given their relatively high stock of inputs they are unwilling to suffer substantial costs of adjustment and, as a result, their probability of adjustment is low, 0.036. The majority of such firms, then, do not undertake inventory adjustment. These firms use 0.327, almost 40 percent, of their available stock of intermediate input in current production.

As inventory holdings decline with the time since their last order, firms are willing to accept larger adjustment costs as they move from group 1 across the distribution to group 5. Thus, their probability of undertaking an order rises as their inventory holdings decline, and the model exhibits a rising adjustment hazard in the sense of Caballero and Engel (1999). Firms optimally pursue generalized (S, s) inventory policies, undertaking factor adjustment stochastically, and the probability of an inventory adjustment rises in the distance between the current stock and the target level associated with adjustment.

The steady state table exhibits evidence of some precautionary behavior among final goods firms, as they face uncertainty about the length of time until they will next undertake adjustment. First, while the representative firm in the benchmark model orders exactly the amount of inputs it will use in current production, 0.31, ordering firms in the baseline inventory economy prepare for the possibility of lengthy delays before the next order, selecting a much higher production-time stock, 1.22. Next, as these firms' inventory holdings decline, the amount of intermediate inputs used in production falls, as does employment and production. The intermediate input-to-labour ratio also falls, as firms substitute labour for the scarcer factor of production. However, the fraction of inventories used in production actually rises until, for firms with very little remaining stock, those in column 4, the entire stock is exhausted. Nonetheless, firms' ability to replenish their stocks prior to production in the next period implies that, even here, the adjustment probability is less than one. However, for the 0.062 firms that begin the period with zero input holdings, all adjust prior to production, adopting the common target. Hence, while the columns labelled $1-5$ reflect the beginning of period distribution of firms over inventory levels, the final column is not relevant in the production-time distribution. The first column, reflecting the behavior of adjusting firms, replaces it in production.

6.2 Business cycles

6.2.1 Inventory investment and final sales

Our first goal was to generalize an equilibrium business cycle model to reproduce the empirical regularities involving inventory investment. This we saw as a necessary first step in developing a model useful for analyzing the role of inventories in the business cycle. Table 6 presents our inventory model's predictions for the volatility and cyclicality of GDP, final sales, inventory investment and the inventory-to-sales ratio. These predictions, derived from model simulations, are contrasted with the corresponding values taken from postwar U.S. data. All series are Hodrick-Prescott filtered.

Table 6A reports percentage standard deviations for each series relative to that of GDP.[19] Contemporaneous correlations with GDP are listed in table 6B. Together, the two panels of table 6 establish that our baseline inventory model is successful in reproducing both the procyclicality of net inventory investment and the higher variance of production when compared to final sales. Further, this simple model with nonconvex factor adjustment costs as the single source of inventory accumulation is able to explain 52 percent of the measured relative variability of net inventory investment. Finally, from table 6B, note that the inventory-to-sales ratio is countercyclical in our model, as in the data. We take these results to imply that the predictions of the model are sufficiently accurate to validate its use in exploring the impact of inventory investment on aggregate fluctuations.

Certainly, there are differences between the model and data. The most pronounced departures in the model are its understated variability of inventory investment and its exaggerated counter-cyclicality of the ratio of inventories to final sales. However, the degree of procyclicality in inventory investment, as well as the excess variability of production over sales, are well reproduced by the model. The latter arises from the positive correlation between inventory investment and final sales, 0.781 in our model.

6.2.2 Aggregate implications of inventory investment

In tables 7A and 7B, we begin to assess the role of inventories in the business cycle using our model. The first row of each table presents results for the benchmark model without inventories, the second row reports the equivalent moment from the inventory model. The most striking aspect of this comparison is the broad similarity in the dynamics of the two model economies. At first look, the introduction of inventories into an equilibrium business cycle model does not appear to alter the model's predictions for the variability or cyclicality of production, consumption, investment or total hours in any substantial way. The differences that do exist are quantitatively minor, and the qualitative features of the equilibrium business cycle model are

[19] The exception is net inventory investment, which is detrended relative to GDP, as described in footnote 7.

unaltered. The familiar features of household consumption smoothing continue to imply an investment series that is substantially more variable than output, allowing a consumption series that is less variable than output. Furthermore, the variability of total hours remains lower than that of production. Likewise, table 7B shows little difference in the contemporaneous correlations with output across the two models. The most apparent divergence appears with respect to capital, which is less procyclical in the inventory economy due to its reduced responsiveness of final sales.

One noteworthy distinction between the benchmark business cycle economy and the baseline inventory economy is the latter's higher standard deviation of GDP. We introduced our paper by discussing the view that inventories exacerbate fluctuations in production. Table 7A appears to provide some equilibrium substantiation for this view. However, the increase in GDP volatility is rather small, only 0.092 percentage points. Given that the level of inventories in our model is calibrated to reproduce their intensity of use in the US economy, we may conclude from this that inventories are of minimal consequence in amplifying fluctuations in production. Furthermore, Table 7A shows that the variability of final sales actually falls in the presence of inventory investment.[20] This is further evident in the relative variability of consumption and investment, both of which are reduced in the inventory model. The variability of total hours worked, by contrast, is raised relative to the economy without inventories.

Tables 8A and 8B provide additional observations that may help in explaining the differences across models, particularly with regard to the hours series. Note that the inventory economy's higher variance in total hours arises entirely from increased variability in hours worked in the intermediate goods sector, $\mathrm{N}_{\mathrm{inter}}$. Moreover, shifts toward more labor-intensive production of intermediate inputs, (evidenced by the countercyclical $\mathrm{K}/\mathrm{N}_{\mathrm{inter}}$ series), are stronger in the inventory model, partly because procyclical inventory investment diverts some resources away from the production of final goods, and hence from investment in capital. Hours worked in the final goods sector, $\mathrm{N}_{\mathrm{final}}$, are actually less variable in the presence of inventories. In both model

[20] Recall that final sales in the benchmark model is equivalent to production, given the absence of inventory investment.

economies, the use of intermediate inputs per worker is procyclical, as technology shocks to the intermediate goods sector make the relative price of intermediate inputs, q, countercyclical. However, this effect is weaker in the inventory economy; consequently $\mathrm{M/N_{final}}$ is less variable and less procyclical there.

Inventories exist in our model because of fixed adjustment costs. These costs imply state-dependent (S, s) adjustment policies for final goods firms maintaining stocks of intermediate intputs. In table 5, we saw that only about one-third of firms actively adjust their inventories in any given period in the steady state.[21] Staggered factor adjustment dampens the average response of final goods firms to changes in relative prices associated with the business cycle. As a result, the response in final goods is dampened relative to the benchmark economy, as reflected in the reduced variability of consumption, investment and final sales, the sum of these two series. One consequence of this dampened response is that efforts to increase production of intermediate inputs in response to a positive productivity shock must rely relatively more on employment, and less on capital. This makes hours worked in the intermediate goods sector rise by more in such times than in the benchmark economy without inventories. This appears to explain the increased variability of hours worked, both in total and in the intermediate goods sector, and the reduced variability of final sales. Moreover, as productivity shocks are persistent, part of the raised level of intermediate inputs delivered to adjusting final goods firms is retained by these firms as inventory investment, which increases in times of high productivity. Because this retained portion does not immediately translate into higher production of final output, fluctuations in final sales are dampened. Thus, inventory accumulation implies a second restraint on the volatility of final sales, beyond that directly implied by the scarcity of inputs among those firms deferring orders.

In concluding this section, we emphasize what we see as a central result of our study. *All else equal*, a positive covariance between final sales and inventory invest-

[21] However, the rate of adjustment is procyclical in the inventory model, and relatively variable. Its percentage standard deviation relative to output is 0.941, and the contemporaneous correlation between the number of firms undertaking adjustment and GDP is 0.961.

ment must increase the variability of production. However, as was clear in table 7 and in the discussion above, final sales are not exogenous; they are affected by the introduction of inventories. Our general equilibrium analysis suggests that nonconvex costs, the impetus for the accumulation of stocks of intermediate inputs, tend to dampen changes in final output. The percentage standard deviation of final sales, 1.35 for the benchmark model, falls to 1.28 when inventories are present in the economy. This reduction in final sales variability largely offsets the effects of introducing inventory investment for the variance of total production.

6.2.3 Inventory-to-sales ratio

The results of the previous section indicate that, when nonconvex costs induce firms to hold inventories, cyclical fluctuations in final goods production are reduced relative to those that would occur if the costs could be eliminated. It then follows that higher levels of these costs, increasing the level of inventories relative to final output in the economy, should further mitigate the business cycle. In this section, we explore this possibility by increasing the upper support of the cost distribution, B, from the baseline value of 0.204 to 0.3. This pushes the average inventory-to-sales ratio up by approximately 15 percent to 0.83. We interpret this change as a rise in the average level of inventory holdings in the economy. Maintaining all other parameters, we contrast the behavior of this *high inventory* economy to the baseline inventory economy where the inventory-to-sales ratio is 0.714, the average quarterly value observed between 1947.1 and 1997.4 in the data.

Table 9A reveals that higher inventory levels are associated with a fall in the variability of consumption, investment and final sales, and also a reduction in the percentage standard deviation of GDP.[22] Moreover the volatility of hours worked in the intermediate goods sector rises, though, with lesser responses in intermediate input usage, the decline in the variability of hours in the final goods sector more than offsets any impact of this increase on the standard deviation of total hours worked.

[22] Although the relative volatility of consumption rises in the high inventory economy, the percent standard deviation in consumption, falls slightly.

As we have argued, nonconvex adjustment costs tend to dampen the response of firms to the exogenous changes in productivity that drive the business cycle, both because of the staggered nature of their factor adjustments and because of their reluctance to deplete or over-accumulate their input stocks in response to shocks. As a result, larger average adjustment costs associated with a higher average inventory-to-sales ratio necessarily imply less severe business cycles.

The increased prevalence of inventories in the model economy certainly raises the variability of net inventory investment. Its standard deviation relative to GDP is now much closer, at 0.222, to the measured value in the data, 0.271. However, the volatility of final sales declines, its relative standard deviation falling closer to its empirical counterpart, 0.824. As a result the positive correlation between final sales and net inventory investment, 0.702, fails to raise the variance of production. GDP volatility actually falls relative to the economy with the lower inventory-to-sales ratio.

7 Concluding remarks

In the preceding pages, we generalized an equilibrium business cycle model to allow for endogenous (S, s) inventories of an intermediate input in final goods production. We showed that our calibrated baseline model of inventories is able to account for the procyclicality of inventory investment, the higher variance of production relative to sales, the countercyclicality of the inventory-to-sales ratio (qualitatively), and approximately one-half of the relative variability of net inventory investment. Using this model to assess the role of inventory investment in the aggregate business cycle, we found that the inventory economy exhibits a business cycle that is broadly similar to that of its benchmark counterpart without inventory investment. However, the adjustment costs that induce inventory holdings also dampen fluctuations in final output, which substantially limits the effects of inventory accumulation for the variability of total production, despite the positive correlation between final sales and inventory investment. Reexamining the model's predictions in the presence of higher adjustment costs, we have seen that an increased presence of inventories in the economy actually reduces aggregate fluctuations.

In future work, we will consider additional sources of fluctuations. This is particularly important, as we know that the source of shocks has proved critical for the implications of the traditional inventory model. The technology shock studied here is ordinarily interpreted as a supply shock, since it raises productivity in the intermediate goods sector. However, it may also be viewed by final goods firms as a demand shock, as it is essentially a rise in the relative price of their output. Thus, in our multi-sector general equilibrium model, the demand or supply origin of the current disturbance appears ambiguous. Nonetheless, when fluctuations arise from demand shocks that do not directly alter the relative price of intermediate inputs, the cyclical role of inventories may differ from that seen here.

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Table 1: GDP, Final sales and inventories

	GDP	Final Sales	Net Inventory Investment	Inventory-to-Sales
percent standard deviation relative to GDP	1.675	0.824	0.271	0.721
correlation with GDP	1.000	0.951	0.658	-0.396
correlation with NII	0.658	0.417	1.000	-0.174

Data are quarterly, 1954.1 – 2001.2. All series are Hodrick-Prescott filtered. GDP and final sales are reported as standard deviations, and net inventory investment is detrended relative to GDP.

Table 2: Sectoral distribution of private non-farm inventories

		percentage of total stock of inventories	STD(inventory investment)	correlation(inventory investment, GDP)
Manufacturing		37	0.14	0.65
	finished goods	13		
	work in process	12		
	materials & supplies	12		
Trade				
	retail	26	0.12	0.32
	wholesale	26	0.09	0.35
Other		11		

Column 3, the percentages of the total stock of inventories, is taken from Ramey and West (1999), page 869, table 4.

Table 3: Baseline calibration

β	η	α	θ_m	θ_n	σ	δ	A	B	ρ_z	σ_ε
0.990	2.185	0.252	0.500	0.293	0.019	0.019	0.000	0.204	0.981	0.014

Table 4: Forecasting rules with one partition

$$\log(y) = \beta_0 + \beta_1 [\log(K)] + \beta_2 [\log(m_1)]$$

z_1 (692 obs)	β_0	β_1	β_2	SE	adj. R^2
pq	0.646	-0.291	-0.101	0.55e-003	0.9984
p	1.353	-0.270	-0.033	0.03e-003	0.9999
K’	0.024	0.886	0.015	1.45e-003	0.9999
m_1’	-0.312	0.161	0.691	1.12e-003	0.9978

z_2 (1692 obs)	β_0	β_1	β_2	SE	adj. R^2
pq	0.591	-0.325	-0.068	0.64e-003	0.9982
p	1.341	-0.285	-0.010	0.04e-003	0.9999
K’	-0.016	0.926	-0.034	1.66e-003	0.9998
m_1’	-0.151	0.037	0.830	1.15e-003	0.9979

z_3 (616 obs)	β_0	β_1	β_2	SE	adj. R^2
pq	0.420	-0.226	-0.144	0.88e-003	0.9979
p	1.269	-0.232	-0.041	0.06e-003	0.9999
K’	0.056	0.846	0.019	1.60e-003	0.9997
m_1’	-0.290	0.246	0.699	1.13e-003	0.9988

Table 5: Distribution of final goods firms in steady-state

	adjustors	1	2	3	4	5
$\mu(s)$: start-of-period distribution		0.279	0.269	0.232	0.158	0.061
s: start-of-period inventories		0.833	0.504	0.238	0.055	0.000
$\alpha(s)$: fraction adjusting		0.036	0.140	0.318	0.611	1.000
s_1: production-time inventories	1.221	0.833	0.504	0.238	0.055	0.000
m: intermediate input	0.389	0.328	0.266	0.183	0.055	0.000
n: labour	0.186	0.165	0.142	0.109	0.047	0.000
y: production	0.365	0.328	0.287	0.223	0.096	0.000
m/n	2.091	1.990	1.871	1.677	1.176	n/a
production share	0.279	0.269	0.232	0.158	0.061	0.000

Table 6: Inventory dynamics for the baseline model

	GDP	Final Sales	Net Inventory Investment	Inventory/Sales
A: percent standard deviations relative to GDP				
data	1.675	0.824	0.271	0.721
baseline inventory	1.441	0.885	0.141	0.921
B: contemporaneous correlations with GDP				
data		0.951	0.658	-0.396
baseline inventory		0.996	0.834	-0.964

Table 7: Baseline inventory model

	GDP	Final Sales	Consumption	Investment	Total Hours	Capital
A: percent standard deviations relative to GDP						
benchmark	1.349	1.000	0.538	6.658	0.501	0.418
baseline inventory	1.441	0.885	0.471	6.323	0.575	0.407
B: contemporaneous correlations with GDP						
benchmark		1.000	0.965	0.961	0.959	0.158
baseline inventory		0.996	0.939	0.968	0.964	0.127

Table 8: Baseline inventory model continued

	N_{inter}	N_{final}	X	M	q	K / N_{inter}	M / N_{final}
A: percent standard deviations relative to GDP							
benchmark	0.501	0.501	1.721	1.721	0.723	0.693	1.258
baseline inventory	0.696	0.441	1.765	1.527	0.667	0.853	1.128
B: contemporaneous correlations with GDP							
benchmark	0.959	0.959	0.999	0.999	-0.993	-0.599	0.984
baseline inventory	0.955	0.962	0.998	0.991	-0.981	-0.719	0.966

Table 9: High inventory model

	GDP	FS	C	I	Hours	N_{inter}	N_{final}	M	NII	K/N_{inter}	M/N_{final}
A: percent standard deviations relative to GDP											
baseline inventory	1.441	0.885	0.471	6.323	0.575	0.696	0.441	1.527	0.141	0.853	1.128
high inventory	1.382	0.831	0.485	5.560	0.560	0.739	0.366	1.478	0.222	0.906	1.141
B: contemporaneous correlations with GDP											
baseline inventory		0.996	0.939	0.968	0.964	0.955	0.962	0.991	0.834	-0.719	0.966
high inventory		0.988	0.939	0.963	0.959	0.936	0.968	0.981	0.804	-0.742	0.960