

Inflation and Interest Rates with Endogenous Market Segmentation

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ABSTRACT

We examine a monetary economy wherein endogenous asset market segmentation permits the extent of household participation in open market operations to smoothly vary with changes in aggregate conditions. While we impose no stickiness at the microeconomic level in either prices or portfolio adjustment, we find that our flexible asset market segmentation can deliver gradual adjustment in the aggregate price level following a monetary shock and thus persistent non-neutralities.

In our model economy, households incur fixed transactions costs when exchanging bonds and money and, as a result, carry money balances in excess of current spending to limit the frequency of such trades. As only a fraction of households choose to actively trade bonds and money at any given time, the market is endogenously segmented. Moreover, because our households have the ability to alter the timing of their trading activities, the extent of market segmentation varies over time in response to real and nominal shocks. We show that this added flexibility can substantially reinforce both sluggishness in aggregate price adjustment, and the persistence of liquidity effects in real and nominal interest rates, relative to models with exogenously segmented asset markets.

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1 Introduction

There is a wealth of empirical research documenting not only the co-movement of real and nominal series at higher frequencies, but what is widely accepted as evidence of persistent responses in real variables following nominal disturbances. We study such co-movements using a monetary model wherein households face fixed costs of transferring wealth between interest-bearing assets and money. As a result of these transactions costs, households infrequently access their interest income and carry money balances in excess of current spending, and participation in asset markets is endogenously segmented. As is well known, market segmentation implies that open market operations can have real effects, because they directly involve only a subset of households. This paper establishes that, when market segmentation is endogenized in a setting where households hold inventories of money, changes in the fraction of households participating in asset markets can add considerable persistence to movements in both nominal and real variables.

Our work builds on an important literature that studies monetary policy in models with exogenously segmented markets.¹ As in the work of Grossman and Weiss (1983), Rotemberg (1984), and Alvarez, Atkeson and Edmond (2003), households in our model economy only periodically access the market for interest-bearing assets (broadly interpreted as markets for relatively high-yield assets) and they carry inventories of money (interpreted to include relatively low yield liquid assets). Nonetheless, our model is closest in spirit to the endogenous segmentation model of Alvarez, Atkeson and Kehoe (2002), in that heterogeneous households actively choose when to adjust their portfolios of bonds and money.² Our model is distinguished relative to theirs by a nontrivial distribution of money that evolves across periods, as most households do not exhaust their money balances with their current consumption expenditures. As in Alvarez, Atkeson and Edmond (2003), our household spending rates (ratios of the value of current consumption to money holdings) are lowest among households that have recently transferred wealth held as bonds into money, and they rise with the time since such a transfer has occurred. In such an environment, a transitory shock to money growth changes the distribution of money holding across households with different spending rates, which can, in turn, generate persistent movements in inflation rates.

Unlike exogenous segmentation environments such as Alvarez, Atkeson and Edmond (2003), the extent of market segmentation varies over time in our model economy, as the fraction of households choosing to participate in the asset markets responds to changes in the economy's state. Moreover, in contrast to the endogenous segmentation model of Chiu (2005), our allowance for idiosyncratic differences across households implies that this fraction remains nontrivial over time, as does the distribution of money. This distinction has important implications for the propagation of nominal disturbances. Following an open market operation, endogenous changes in the timing of households' active participation in asset markets can gradualize aggregate price adjustment relative to that in a model with exogenous market segmentation. Furthermore, such changes can substantially increase the persistence of liquidity effects in both real and nominal interest rates.³

Endogenizing access to the asset market, and thus allowing for movements in the fraction of

¹See Alvarez, Lucas and Weber (2001) and the references therein.

²Alvarez, Atkeson and Kehoe (2002) build on the work of Chatterjee and Corbae (1991) who study an economy where some households choose to pay a fixed cost to trade bonds.

³One exception to this is the exogenous segmentation model of Williamson (2005), where households are permanently divided into groups with and without access to the asset markets. There, the assumption that households with (without) such access prefer to trade among themselves in the goods markets delivers a second type of segmentation that can lead to persistent liquidity effects.

households actively adjusting their nominal balances at any time, implies larger movements in individual households' spending rates following a shock to the money supply. When transactions costs are high, and thus the mean time between active trades is long, households tend to hold relatively large inventories of money and have lower average spending rates. If, in addition, the maximum time between trades is significantly longer than is the mean time, households on average return to the asset markets with substantial remaining balances. We find that, in such circumstances, the persistence in inflation that is implied by the exogenous segmentation model is reduced, as households not currently participating in the asset markets sharply raise their spending rates following an open market operation. Conversely, when the mean time between asset market trades is not as long, so that households have higher average spending rates (or when the mean and maximum times between such trades are similar, so that households on average return to the bond market with little remaining money), endogenous changes in the distribution of households increase persistence in the inflation response beyond that in the exogenous segmentation model and, moreover, lead to persistent changes in interest rates.

As in the many studies in monetary economics that have preceded us, several empirical relationships involving money, interest rates and prices motivate our work. First, short-term real interest rates are negatively correlated with expected inflation. Barr and Campbell provide direct evidence for this using U.K. data involving inflation-indexed bonds. Second, VAR studies consistently have found evidence of liquidity effects; expansionary open market operations appear to reduce short-term nominal interest rates, at least in the short-run. (See, for example, Leeper, Sims, and Zha (1996) and Christiano, Eichenbaum, and Evans (1999).) Finally, the general price level appears to adjust slowly to nominal shocks. This finding is widely supported by the VAR literature as, for example, in the studies of Leeper, Sims, and Zha (1996), Christiano, Eichenbaum, and Evans (1999), and Uhlig (2004). Moreover, King and Watson (1996) show that, at business cycle frequencies, the price level is positively correlated with lagged real output. Additional evidence for the slow adjustment of the price level, discussed in Alvarez, Atkeson and Edmond (2003), is provided by the pattern of short-term movements seen between the ratio of money to consumption and velocity. The correlation between the ratio of money (M2) to consumption (PCE) and the corresponding measure of velocity is -0.89 for HP-filtered monthly data.

The most common theoretical approach to addressing this empirical evidence involves models where nominal prices are sticky at the firm level. While there are several open issues involving the viability of models with sticky prices, we discuss one that directly motivates our work. In their recent paper, Dotsey and King (2005) resolve a long-standing issue for this literature by developing an (S,s) model of nominal price setting that is consistent with empirical estimates of the persistence in inflation. In the process, they discover that their model predicts a rise in short-term nominal interest rates following a persistent shock to money growth rates. The state-of-the-art micro-founded menu cost model cannot address the liquidity effect described by Milton Friedman (1968), *"The initial impact of increasing the quantity of money at a faster rate than it has been increasing is to make interest rates lower for a time than they would otherwise have been."* Whether or not we choose to view this as a critical shortcoming of the paradigm, there is a more substantive disagreement between Friedman's view of the real effects of monetary policy and that underlying sticky price models. Friedman viewed changes in the velocity of money as central in determining the mechanics of movements in output, employment and prices following an expansionary open market operation. Consider his testimony to the House of Commons Select Committee in 1979; *"... the initial effect of a change in monetary growth is an offsetting movement in velocity, followed by*

changes in the growth of spending initially manifested in output and employment, and only later in inflation.' (Friedman, 1980).⁴ This view is firmly rejected by sticky price models where the velocity of money, and indeed real balances themselves, are almost entirely irrelevant to the predictions of the model.

Changes in velocity are at the heart of the short-term nonneutrality exhibited by models with segmented assets markets. Moreover, the endogenous segmentation model of Alvarez, Atkeson and Kehoe (2002) succeeds in generating liquidity effects and in reproducing the negative relation between real interest rates and anticipated inflation. The Alvarez, Atkeson and Edmond (2003) inventory-theoretic model of money with exogenous segmentation separately delivers sluggish adjustment of the price level, and hence persistent inflation responses to nominal shocks. Drawing upon elements of each of these frameworks, we develop an endogenous segmentation model of money that simultaneously succeeds with regard to both sets of regularities. Moreover, as mentioned above, changes in the number of households choosing to exchange bonds and money can substantially reinforce the effects of market segmentation. Following a transitory shock to the money growth rate, such changes prolong responses in inflation and the real interest rate. When shocks to money growth are persistent, they lead to far more sluggish price adjustment and more persistent liquidity effects in both nominal and real interest rates. Alternatively, when we examine real shocks with monetary policy following a Taylor rule, our economy generates persistence in the responses of inflation and interest rates altogether absent under fixed market segmentation. Finally, in versions of our model with endogenous production, we find that persistent technology shocks can lead to non-monotone responses in employment and output.

Finally, our paper offers an independent theoretical contribution in formally establishing how the results of Alvarez, Atkeson and Kehoe (2002) may be extended to a model where there are persistent differences across households. Here, such extension is necessary, because cash-in-advance constraints do not always bind so that households carry inventories of money, thereby transmitting the effects of temporary idiosyncratic differences across periods. By assuming a full set of state-contingent nominal bonds that allow risk-sharing across households, we ensure that these differences across households are persistent, but not permanent. Because households must pay fixed transactions costs to access their bond holdings, the presence of state-contingent bonds in our economy does not lead to full insurance; households that are ex-ante identical diverge over time as idiosyncratic realizations of shocks drive differences in their money and bond holdings. Nonetheless, we prove that, whenever a heterogeneous group of households enters the bond market at the same time, all previous differences among them are eliminated. As a result, our model economy exhibits limited memory. Exploiting this property, we are able to apply the numerical approach to solving generalized (S,s) models developed by King and Thomas (forthcoming) in a setting where the consumption and savings decisions of heterogeneous risk-averse households are directly influenced by nonconvex costs. While this approach has been applied previously in solving models where risk-neutral production units face idiosyncratic fixed costs of adjusting their prices or factors of production (as in Dotsey, King and Wolman (1999) and Thomas (2002)), this is to our knowledge the first application involving heterogeneity among households.

⁴We thank Ed Nelson for bringing this to our attention.

2 Model

We begin with an overview of the model. Thereafter, we proceed to a more formal description of households' problems, followed by the description of a financial intermediary that sells households claims contingent on both aggregate and individual states. Next, we show that there is an equivalent, but more tractable, representation of households' lifetime optimization problems, given their ability to purchase such individual-state-contingent bonds alongside the fact that they are ex-ante identical. Proofs of all lemmas are provided in the appendix.

2.1 Overview

The model economy has three sets of agents: a unit measure of ex-ante identical households, a perfectly competitive financial intermediary, and a monetary authority. Each infinitely-lived household values consumption in every date of life, with period utility $u(c)$, and it discounts future utility with the constant discount factor β , where $\beta \in (0, 1)$. In each period, households receive a common endowment, y . This endowment varies exogenously over time, as does the growth rate of the aggregate money supply, μ . Defining the date t realization of aggregate shocks as $s_t = (y_t, \mu_t)$, we denote the history of aggregate shocks by $s^t = (s_1, \dots, s_t)$, and the initial-period probability density over aggregate histories by $g(s^t)$.

Households have two means of saving. First, they have access to a complete set of state-contingent nominal bonds. These are purchased from a financial intermediary described below, and are maintained in interest-bearing accounts that we will refer to as households' *brokerage accounts*, following the language of Alvarez, Atkeson and Edmond (2003). Next, they also save using money, which they maintain in their *bank accounts* and use to conduct trades in the goods market.⁵ Households have the opportunity to transfer assets between their two accounts at the start of each period; this occurs after the realization of all current shocks, but prior to any trading in the goods market. As such, it is expositionally convenient to refer to each period as consisting of two subperiods that we will term *transfer-time* and *shopping-time*, although nothing in the environment necessitates this approach.

There are three inter-related frictions leading households to maintain money in their bank accounts. First, as in a standard cash-in-advance environment, households cannot consume their own endowments. Each household consists of a worker and a shopper, and the worker must trade the household endowment for money while the shopper is purchasing consumption goods. As a result, the household receives the nominal value of its endowment, $P(s^t)y(s^t)$, only at the end of the period after current goods trade has ceased.⁶ We assume that these end-of-period nominal receipts are deposited across their two accounts, with fraction λ paid into bank accounts and the remainder into brokerage accounts. Second, as all trades in the goods market are conducted with money, each household's consumption purchases are constrained by the bank account balance it holds when shopping-time begins.

Note that, absent other frictions, each household would, in every period, simply shift from its brokerage account into its bank account exactly the money needed to finance current consumption

⁵When allowed to store money in their brokerage accounts, households never do so given positive nominal interest rates paid on bonds. Thus, we simplify the model's exposition here by assuming that money is held only in bank accounts and verify that nominal rates remain positive throughout our results.

⁶While this worker-shopper arrangement may appear stark in an endowment economy, it is less so if one envisions that each household's endowment is one of a unit measure of differentiated inputs that enter a consumption aggregator with identical weights to produce the single good consumed by all.

expenditure not covered by the bank account paycheck from the previous period. There is, however, a third friction that prevents this, leading households to deliberately carry money across periods; this is the assumption that they must pay fixed costs each time they transfer assets between their two accounts. Given these fixed costs, households maintain stocks of money to limit the frequency of their transfers, and they follow generalized (S,s) rules in managing their bank accounts.

Transfer costs are fixed in that they are independent of the size of the transfer; however, they vary over time and across households. Here, we subsume the idiosyncratic features that distinguish households directly in their fixed costs by assuming that each household draws its own current transfer cost, ξ , from a time-invariant distribution $H(\xi)$ at the start of each period. Because this cost draw influences a household's decision of whether to undertake any transfer, and hence its current consumption and money savings, each household is distinguished by its history of such draws, $\xi^t = (\xi_1, \dots, \xi_t)$, with associated density $h(\xi^t) = h(\xi_1) \cdots h(\xi_t)$. As will be seen below, households are able to insure themselves in their brokerage accounts through the purchase of nominal bonds contingent on both aggregate and individual exogenous states.

2.2 Households

At the start of any period, given date-event history (s^t, ξ^t) , a household's brokerage account assets include nominal bonds, $B(s^t, \xi^t)$, purchased in the previous period at price $q(s^t, \xi_t)$, as well as the fraction of its income from the previous period that is deposited there, $(1-\lambda)P(s^{t-1})y(s^{t-1})$. The remainder, the paycheck, $\lambda P(s^{t-1})y(s^{t-1})$, is deposited into the household's bank account and supplements its money savings there from the previous period, $A(s^{t-1}, \xi^{t-1})$. Given this start of period portfolio and its current fixed cost, the household begins the period by determining whether or not to transfer assets across its two accounts. Denoting the household's start-of-period bank balance by $M(s^{t-1}, \xi^{t-1})$, where

$$M(s^{t-1}, \xi^{t-1}) \equiv A(s^{t-1}, \xi^{t-1}) + \lambda P(s^{t-1})y(s^{t-1}), \quad (1)$$

the relevant features of this choice are summarized in the chart below.

	brokerage account withdrawal	shopping-time bank balance
$z(s^t, \xi^t) = 1$	$x(s^t, \xi^t) + P(s^t)\xi_t$	$M(s^{t-1}, \xi^{t-1}) + x(s^t, \xi^t)$
$z(s^t, \xi^t) = 0$	0	$M(s^{t-1}, \xi^{t-1})$

An *active* household is indicated by $z(s^t, \xi^t) = 1$. In this case, the household selects a nonzero nominal transfer $x(s^t, \xi^t)$ from its brokerage account into its bank account and has $M(s^{t-1}, \xi^{t-1}) + x(s^t, \xi^t)$ available in its bank account at the start of the current shopping subperiod. Here, the household's current fixed cost applies, so $P(s^t)\xi_t$ is deducted from its nominal brokerage wealth. Alternatively, the household may choose to undertake no such transfer, setting $z(s^t, \xi^t) = 0$ and remaining *inactive*. In that case, it enters into the shopping subperiod with no change to its start-of-period bank and brokerage account balances.

Each household chooses its state-contingent plan for the timing and size of its account transfers ($z(s^t, \xi^t)$ and $x(s^t, \xi^t)$), and its bond purchases, money savings and consumption ($B(s^t, s_{t+1}, \xi^t, \xi_{t+1})$, $A(s^t, \xi^t)$ and $c(s^t, \xi^t)$), to maximize its expected discounted lifetime utility,

$$\sum_{t=1}^{\infty} \beta^{t-1} \int_{s^t} \int_{\xi^t} u(c(s^t, \xi^t)) h(\xi^t) g(s^t) d\xi^t ds^t, \quad (2)$$

subject to the sequence of constraints in (3) - (6).

$$B(s^t, \xi^t) + (1 - \lambda)P(s^{t-1})y(s^{t-1}) \geq \left[x(s^t, \xi^t) + P(s^t)\xi_t \right] z(s^t, \xi^t) \quad (3)$$

$$+ \int_{s_{t+1}} \int_{\xi_{t+1}} q(s^t, s_{t+1}, \xi_{t+1}) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) ds_{t+1} d\xi_{t+1}$$

$$M(s^{t-1}, \xi^{t-1}) + x(s^t, \xi^t)z(s^t, \xi^t) \geq P(s^t)c(s^t, \xi^t) + A(s^t, \xi^t) \quad (4)$$

$$A(s^t, \xi^t) + \lambda P(s^t)y(s^t) \geq M(s^t, \xi^t) \quad (5)$$

$$A(s^t, \xi^t) \geq 0 \quad (6)$$

Equation 3 is the household's brokerage account budget constraint associated with history (s^t, ξ^t) , and requires that expenditures on new bonds together with any transfer to the bank account and associated fixed cost not exceed current brokerage account wealth. Next, the bank account budget constraint in equation 4 requires that the household's money balances entering the shopping subperiod cover its current consumption expenditure and any money savings for next period.⁷ Money balances for next period, in (5), are these savings together with the bank paycheck received after completion of current goods trade. Equation 6 prevents the household from ending current trade with a negative bank balance; thus, taken together with the restriction in (4), it imposes cash-in-advance on consumption purchases. Finally, in addition to this sequence of constraints, we also impose a standard limit condition on household debt:

$$\lim_{t \rightarrow \infty} \int_{s^t} \int_{\xi^t} q(s^t, \xi^t) B(s^t, \xi^t) ds^t d\xi^t \geq 0. \quad (7)$$

Following the approach of Alvarez, Atkeson and Kehoe (2002), we find it convenient to model risk-sharing by assuming a perfectly competitive financial intermediary that purchases government bonds with payoffs contingent on the aggregate shock and, in turn, sells to households bonds with payoffs contingent on both the aggregate and individual shocks. In particular, given aggregate history s^t , the intermediary purchases government-issued contingent claims $B(s^t, s_{t+1})$ at price $q(s^t, s_{t+1})$, and it sells them across households as claims contingent on individual transfer costs, ξ_{t+1} . Note that, as households' cost draws are not autocorrelated, the price of any such claim is $q(s^t, s_{t+1}, \xi_{t+1})$, independent of the individual history ξ^t .

For each (s^t, s_{t+1}) , the intermediary selects its aggregate bond purchases, $B(s^t, s_{t+1})$, and individual bond sales, $B(s^t, s_{t+1}, \xi^t, \xi_{t+1})$, to solve

$$\max_{\xi_{t+1}} \int_{\xi^t} \int_{\xi_{t+1}} q(s^t, s_{t+1}, \xi_{t+1}) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi^t) d\xi^t d\xi_{t+1} - q(s^t, s_{t+1}) B(s^t, s_{t+1}) \quad (8)$$

subject to:

$$B(s^t, s_{t+1}) \geq \int_{\xi_{t+1}} \int_{\xi^t} B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi^t) h(\xi_{t+1}) d\xi^t d\xi_{t+1}. \quad (9)$$

⁷In those periods when a household is active, it has a single unified budget constraint, $B(s^t, \xi^t) + P(s^{t-1})y(s^{t-1}) + A(s^{t-1}, \xi^{t-1}) \geq P(s^t) [\xi_t + c(s^t, \xi^t)] + A(s^t, \xi^t) + \int_{s_{t+1}} \int_{\xi_{t+1}} q(s^t, s_{t+1}, \xi_{t+1}) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) ds_{t+1} d\xi_{t+1}$.

The constraint in (9) requires that, for any (s^t, s_{t+1}) , the intermediary must purchase sufficient aggregate bonds to cover all individual bonds held against it for that aggregate history. Given s_{t+1} occurs, fraction $h(\xi_{t+1})$ of the households with history ξ^t to whom it sells such bonds will realize that state and demand payment. As shown in Lemma 1 below, the financial intermediary's zero profit condition immediately implies that the price of any individual bond associated with (s_{t+1}, ξ_{t+1}) is simply the product of the price of the relevant aggregate bond and the probability of an individual household drawing the transfer cost ξ_{t+1} .

Lemma 1. *The equilibrium price of state-contingent bonds issued by the financial intermediary, $q(s^t, s_{t+1}, \xi_{t+1})$, is given by $q(s^t, s_{t+1}, \xi_{t+1}) = q(s^t, s_{t+1}) h(\xi_{t+1})$.*

By assuming an initial period 0 throughout which households are perfectly identical, we allow them the opportunity to trade in individual-state-contingent bonds at a time when they have the same wealth and face the same probability distribution over all future individual histories. In this initial period, the government has some outstanding debt, \bar{B} , that is evenly distributed across households' brokerage accounts, and it repays this debt entirely by issuing new bonds. Households receive no endowment, draw no transfer costs and do not value consumption in this initial period. Rather, they simply purchase state-contingent bonds for period 1 subject to the common initial period brokerage budget constraint:

$$\bar{B} \geq \int_{s_1} \int_{\xi_1} B(s_1, \xi_1) q(s_1) h(\xi_1) d\xi_1 ds_1.$$

Following the proof of Lemma 1, section B of the appendix shows that the period 0 budget constraint above can be combined with the sequence of constraints in (3) to yield the following lifetime budget constraint common to all households.

$$\bar{B} \geq \sum_{t=1}^{\infty} \int \int q(s^t) h(\xi^t) \left(z(s^t, \xi^t) \left[x(s^t, \xi^t) + P(s^t) \xi_t \right] - P(s^{t-1}) (1 - \lambda) y(s^{t-1}) \right) d\xi^t ds^t, \quad (10)$$

where $q(s^t) \equiv q(s_1) \cdot q(s_1, s_2) \cdots q(s^{t-1}, s_t)$.

Finally, we assume that the monetary authority is subject to the sequence of constraints,

$$B(s^t) - \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}) ds_{t+1} = \bar{M}(s^t) - \bar{M}(s^{t-1}), \quad (11)$$

requiring that its current bonds be covered by a combination of new bond sales and the printing of new money. This sequence of constraints, alongside equilibrium in the money market, immediately implies that households' aggregate expenditures on new bonds in any period is exactly the difference between the aggregate of their current bonds and the change in the aggregate money supply:

$$\bar{M}(s^t) - \bar{M}(s^{t-1}) = B(s^t) - \int \int \int q(s^t, s_{t+1}) h(\xi_{t+1}) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi^t) d\xi^t d\xi_{t+1} ds_{t+1}. \quad (12)$$

2.3 A risk sharing arrangement

Three aspects of the environment described above may be exploited to simplify our solution for competitive equilibrium: (i) fixed transfer costs are independently and identically distributed across

households and time, (ii) households have perpetual access to a complete set of state-contingent claims in their brokerage accounts, and (iii) they are able to purchase these state-contingent bonds during an initial period in which they are perfectly identical. In this section, we show how these assumptions allow us to move to a more convenient representation of households' problems. In particular, exploiting the common lifetime budget constraint in (10) above, we will move from the household problem stated in section 2.2 to construct the equivalent problem of an extended family that manages all households' bonds in a joint brokerage account, and whose period-by-period decisions regarding bond purchases and account transfers implement the state-contingent lifetime plan selected by every household. In doing so, we transform our somewhat intractable initial problem into something to which we can apply the King and Thomas (2005) approach for solving aggregate economies involving heterogeneity arising due to (S,s) policies at the individual level.

Money as the individual state variable: A complete set of state-contingent claims in the brokerage account allows individuals to insure their bond holdings against idiosyncratic risk; these shocks only affect their bank accounts. Alternatively, an individual's money balance fully captures the cumulative effect of his history of idiosyncratic shocks. In Lemma 2, we prove that prior to their current transfer cost draws, all relevant differences across households are fully summarized by their start-of-period money balances as they enter into any period.

Lemma 2. *Given $M(s^{t-1}, \xi^{t-1})$, the decisions $c(s^t, \xi^t)$, $A(s^t, \xi^t)$, $x(s^t, \xi^t)$ and $z(s^t, \xi^t)$ are independent of the history ξ^{t-1} .*

This result is fairly intuitive. Given that each ξ comes from an i.i.d. distribution, a household's draw in any given period does not predict its future draws, and thus directly affects only its asset transfer decision in that one period. While this certainly affects current shopping-time money balances, and hence consumption, its only future effect is in determining the money balances with which the household will enter the subsequent period, given the household's ability to insure itself in its brokerage account by purchasing bonds contingent on both aggregate and individual shocks. In proving this result, we show that the solution to the original household problem from section 2.2, given the lifetime constraint in (10), is identical to the solution of an alternative problem where households pool risk period-by-period by each committing to pay the economywide average of the total transfers and associated fixed costs incurred across all active households in every period, irrespective of the timing and size of their own portfolio adjustments. It is immediate from this that households' bond holdings may be modelled as independent of their individual histories ξ^t . Thus, within every period, the distinguishing features affecting any household's decisions can be summarized entirely by its start-of-period bank balance, $M(s^{t-1}, \xi^{t-1})$, and its current transfer cost, ξ_t .

Households as members of time-since-active groups: Our next lemma establishes that, within any period, all households that undertake an account transfer will select both a common consumption and a common end-of-period bank balance; hence they begin the subsequent period with the same bank (and brokerage) account balances.

Lemma 3. *For any (s^t, ξ^t) in which $z(s^t, \xi^t) = 1$, $c(s^t, \xi^t)$, $A(s^t, \xi^t)$ and $M(s^t, \xi^t)$ are independent of ξ^t .*

To understand this result, recall that household brokerage and bank accounts are joined in periods when they choose to adjust their portfolios, and all are identical when they make their state-contingent plans in date 0. Given this, in selecting their consumption for such periods, households equate their appropriately discounted marginal utility of consumption to the multiplier on the lifetime brokerage budget constraint from (10), which is common to all households. Next, in selecting what portion of their shopping-time bank balances to retain after consumption (hence their next-period balances), households equate the marginal utility of their current consumption to the expected return on a dollar saved for the next period weighted by their expected discounted marginal utility of next-period consumption. Given common inflation expectations and the common current consumption of active households, this implies that active households also share in common the same expected consumption for next period. Thus, all currently active households exit this period and enter the next period with common money holdings.

Note that the results of Lemmas 2 - 3 combine to imply that, within any period, households that undertake balance transfers all enter shopping-time with the same bank balance, make the same shopping-time decisions, and then enter the next period as effectively identical. Moreover, of this group of currently active households, those households that do not undertake an account transfer again in the next period will continue to be indistinguishable from one another as they enter shopping-time, and hence will enter the subsequent period with common bank (and brokerage) account balances, and so forth. In other words, any household that was last active at some particular date t is effectively identical to any other household last active at that same date. This is useful in our numerical approach to solving for competitive equilibrium, since it allows us to move from identifying individual households by their current money holdings to instead identifying each household as a member of a particular time-since-active group, with all members of any one such group sharing in common the same start-of-period money balances.

Given the above results, we may track the distribution of households over time through two vectors, one indicating the measures of households entering the period in each time-since-active group, $[\theta_{j,t}]$, $j = 1, 2, \dots$, and the other storing the balances with which members of each of these current groups exited shopping-time in the previous period, $[A_{j-1,t-1}]$. From the latter, the current start-of-period balances held by members of each group are retrieved as $M_{jt} = A_{j-1,t-1} + \lambda P_{t-1} y_{t-1}$, where P_{t-1} represents the previous period's price level, and y_{t-1} the common endowment of the previous period. Households within any given start-of-period group j that do not pay their fixed costs move together into the current shopping subperiod with their starting balances M_{jt} . Across all start-of-period groups, those households that do pay to undertake a bank transfer will enter the current shopping subperiod in time-since-active group 0 with common shopping-time balances, $M_{0,t}$, which we refer to as the current *target* money balances.

Threshold transfer rules: Finally, we establish that households follow threshold policies in determining whether or not to transfer assets between their brokerage and bank accounts. Specifically, given its start-of-period money balances, each household has some maximum fixed cost that it is willing to pay to undertake an account transfer and adjust its balances to the current target.

Lemma 4. For any (s^t, ξ^{t-1}) , $A = \{\xi_t \mid z(s^t, \xi^t) = 1\}$ is a convex set bounded below by 0.

As our preceding results imply that all members of any given start-of-period group j are effectively identical prior to the draws of their current transfer costs, this last result allows convenient determination of the fractions of each such group undertaking account transfers, and thus the shopping-time

distribution of households. Define the threshold cost ξ_{jt}^T as that fixed cost that leaves any household in time-since-active group j indifferent to an account transfer at date t . Households in the group drawing costs at or below ξ_{jt}^T pay to adjust their portfolios, while other members of the group do not. Thus, within each group j , the fraction of its members shifting assets to reach the current target bank balance is given by $\alpha_{jt} \equiv H(\xi_{jt}^T)$. Each such active household undertakes a transfer $x_{jt} = M_{0,t} - M_{j,t}$, and the total transfer cost paid across all members of the group are $\theta_j \int_0^{H^{-1}(\alpha_j)} \xi h(\xi) d\xi$.

A family problem: Collecting the results above, and assuming that aggregate shocks are Markov, we may re-express the lifetime plans formulated by individual households as the solution to the recursive problem of an extended family that manages the joint brokerage account of all households and acts to maximize the equally-weighted sum of their utilities. In each period, given the starting distribution of households summarized by $\{\theta_j, A_j\}$ and the current price level P , the family selects the fractions of households from each time-since-active group to receive account transfers, α_j , (and hence the distribution of households over time-since-active groups at the start of next period, θ'_j), the shopping-time bank balance of each active household, M_0 , achieved by transfers from the family brokerage account, as well as the consumption and money savings associated with members of each shopping-time group, c_j and A'_{j+1} respectively, to solve the problem in (13) - (19) below. In solving this problem, the family takes as given the current endogenous aggregate state $K = [\{\bar{\theta}_j, \bar{A}_j\}, P_{-1}y_{-1}, \bar{M}_{-1}]$, and it assumes the future endogenous state will be determined by a mapping F that it also takes as given; $K' = F(K, s)$. In equilibrium, K' is consistent with the family's decisions.

$$V(\{\theta_j, A_j\}; K, s) = \max \sum_{j=1}^{\infty} \theta_j [\alpha_j u(c_0) + (1 - \alpha_j) u(c_j)] + \beta \int_{s'} V(\{\theta'_j, A'_j\}; K', s') g(s, s') ds' \quad (13)$$

subject to:

$$\sum_{j=1}^{\infty} \alpha_j \theta_j [M_0 - M_j] + P \sum_{j=1}^{\infty} \theta_j \left[\int_0^{H^{-1}(\alpha_j)} \xi h(\xi) d\xi \right] \leq \bar{M} - \bar{M}_{-1} + (1 - \lambda) P_{-1}y_{-1} \quad (14)$$

$$M_j = [A_j + \lambda P_{-1}y_{-1}], \text{ for } j > 0 \quad (15)$$

$$M_j \geq Pc_j + A'_{j+1}, \text{ for } j \geq 0 \quad (16)$$

$$A'_{j+1} \geq 0, \text{ for } j \geq 0 \quad (17)$$

$$\sum_{j=1}^{\infty} \alpha_j \theta_j \geq \theta'_1 \quad (18)$$

$$\theta_j (1 - \alpha_j) \geq \theta'_{j+1}, \text{ for } j > 0 \quad (19)$$

Recall from equation 12 that money market clearing in each period requires that the aggregate of households' current bonds less their expenditures on new bonds must equal the change in the aggregate money supply. By imposing this equilibrium condition, we may use equation 14 to represent the family's budget constraint requiring that its joint brokerage assets cover all current transfers to active households and associated fixed costs, as well as all bond purchases for the next period. Next, equation 15 identifies the start-of-period money balances associated with each time-since-active group j , and (16)-(17) represent the bank account budget and cash-in-advance

constraints that apply to members of each shopping-time group. Finally, equations 18 - 19 describe the evolution of households across groups over time. In (18), the total active households (shopping in group 0) in the current period is the population-weighted sum of the fractions of households made active from each start-of-period group, and these households move together to begin the next period in time-since-active group 1. In (19), households in any given time-since-active group j that are inactive in the current period will move into the next period as members of group $j + 1$.

3 Solution

Recall that we imposed money-market clearing in formulating the family's problem above. As such, we can retrieve equilibrium allocations as the solution to (13) - (19) by appending to that problem the goods market clearing condition needed to determine the equilibrium price level taken as given by the family:

$$y = c_0 \sum_{j=1}^{\infty} \theta_j \alpha_j + \sum_{j=1}^{\infty} \theta_j (1 - \alpha_j) c_j + \sum_{j=1}^{\infty} \theta_j \left[\int_0^{H^{-1}(\alpha_j)} x h(x) dx \right]. \quad (20)$$

Equation 20 simply states that, within each period, the current aggregate endowment must satisfy total consumption demand across all active and inactive households together with the economywide fixed costs associated with account transfers.

In the results to follow, we abstract from trend growth in endowments, and we assume that money supply is increased at rate μ^* in the economy's steady-state. Thus, the steady-state is associated with inflation at rate μ^* and a stationary distribution of households over real balances described by $[\boldsymbol{\theta}^*, \mathbf{a}^*]$, where $\boldsymbol{\theta}^* = \{\theta_j^*\}$ and $\mathbf{a}^* = \{a_j^*\}$, with $a_j \equiv \frac{A_j}{P_{-1}}$. As any given household travels outward across time-since-active groups, it finds its actual real balances for shopping time, $(a_j^* + y^*) / (1 + \lambda \mu^*)$, falling further and further below target shopping balances; thus, the maximum fixed cost it is willing to pay to undertake an account transfer rises. Given a finite upper support on the distribution of fixed transfer costs, this implies that no household will delay activity beyond some finite maximum number of periods, which we denote by J . Thus, the two vectors describing the distribution of households are each of finite length J . In solving the steady-state of our economy, we isolate J as that group j by which α_j is chosen to be 1.

Having arrived at the time-since-active representation described above, we are now almost in a position to follow King and Thomas (2005) in applying linear methods to solve for our economy's aggregate dynamics local to the deterministic steady-state. Two details remain. First, as the linear solution does not allow for a changing number of time-since-active groups, we must restrict J to be time-invariant. Thus, we assume that, for all t , $\alpha_{J,t} = 1$, and we then verify that $\alpha_{j,t} \in (0, 1)$, for $j = 1, \dots, J - 1$, is selected throughout our simulations. Second, we assume that, in every date t , all households that enter shopping in time-since-active group $J - 1$ completely exhaust their money balances; $a_{J,t} = 0$. Given that any such household will undertake an account transfer with certainty at the start of the next period, this assumption is consistent with optimizing behavior so long as we verify that nominal interest rates are always positive.⁸

⁸ Given positive nominal rates, if $a_{J,t} > 0$ ever were to occur, the family could have improved its welfare by reducing the target balances given to active households at date $t - (J - 1)$ and increasing its bond purchases at that date to finance increased transfers to a subsequent group of active households for whom the non-negativity constraint would eventually bind.

In parameterizing our model, we set the length of a period to one quarter, and we choose the steady-state inflation rate μ^* to imply an average annual inflation at 3 percent. Period utility is iso-elastic, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, with $\sigma = 2$, and we select the subjective discount factor β to imply an average annual real interest rate of 3 percent. The steady-state aggregate endowment is normalized to 1, and the fraction of the endowment paid to household bank accounts (which may be interpreted as household wages) is $\lambda = 0.6$, corresponding to labor's share of output. Holding these parameters fixed, we will consider several alternative assumptions regarding the distribution of the fixed costs that cause market segmentation in our model, as we discuss below.

We begin to explore our model's dynamics in section 4 through a series of examples involving the response to a money injection that, once observed, is known to be perfectly transitory. There, we abstract from shocks to the endowment to study the effects of a monetary shock in isolation, and to isolate those aspects caused by the endogenous changes in the degree of market segmentation that distinguish our model. We consider each of three examples distinguished only by the distribution of fixed transfer costs, beginning with a baseline case where this distribution is uniform on the interval 0 to B . There, we set the upper support at $B = 0.25$ to imply that the maximum time that any household remains inactive is $J = 6$ quarters. For individual households, the result is a 4.82 quarter average duration between account transfers. In the aggregate, this calibration results in a steady-state velocity of 1.9, which corresponds to the U.S. average over the past decade.⁹ In our second example, we raise the maximum transfer cost to imply an aggregate velocity matching the U.S. postwar average, at 1.5. Retaining the assumption that transfer costs are distributed uniformly, this implies a mean household inactivity duration of 7 quarters and a substantially longer maximum inactivity spell, at 10 quarters. This large difference between a household's average expected period of inactivity versus the maximum such spell will be seen to have important qualitative implications for the model's aggregate dynamics. Thus, in our third example, we will move to consider a more flexible cost distribution under which aggregate velocity again averages 1.5, but mean and maximum durations are close at 9.55 and 10 quarters, respectively.

Following our temporary money growth shock examples, we will move in section 5 to examine the model's aggregate dynamics under more realistic assumptions about monetary policy. First, we will consider the response to a persistent rise in the money growth rate. There, we will assume that money growth follows a mean-zero AR-1 process in logs with persistence 0.57, as consistent with the finding of Chari, Kehoe and McGrattan (2000).¹⁰ Next, in a second set of results, we will consider the response to a persistent shock to the real endowment in an environment where changes in the rate of money growth are dictated by the monetary authority's pursuit of specific stabilization goals. In that case, the common household endowment will follow a persistent lognormal process,

$$\log(y_t) = \rho \log(y_{t-1}) + \varepsilon_t, \varepsilon \sim n(0, \sigma_\varepsilon^2),$$

with $\rho = 0.90$ and $\sigma_\varepsilon = 0.007$, and the monetary authority will follow a Taylor rule in responding to deviations in inflation. In the endowment economy, we assume that the Taylor rule places zero

⁹For comparability, we follow Alvarez, Atkeson and Edmond (2003) in our measures of money and velocity. As in their paper, money is broadly defined as the sum of currency, checkable deposits, and time and savings deposits. They show that the opportunity cost of these assets, relative to short-term Treasury securities, is substantial and, as a whole, not very different from that of M1. Next, velocity is computed as the ratio of nominal personal consumption expenditures to money.

¹⁰The persistence of the monetary measure used to calibrate our model is actually substantially higher, at 0.93 over the sample period 1954:1 to 2003:1. Since our results are not qualitatively changed, we use the Chari, Kehoe and McGrattan M1-based value for comparability.

weight on deviations in output, and is thus:

$$i_t = i^* + 1.5[\pi_t - \pi^*].$$

A version of the model with production, where the Taylor rule does respond to changes in output, is discussed in section 5.3.

4 Examples

4.1 Steady-state

Before examining its responses to shocks, it is useful to begin with a discussion of household portfolio adjustment timing in our model's steady-state. We first consider how each of our three examples relates to the available micro-evidence provided by Vissing-Jørgensen (2002).

Using the Consumer Expenditure Survey, Vissing-Jørgensen computes that the fraction of households that actively bought or sold risky assets (stocks, bonds, mutual funds and other such securities), between one year and the next ranges from 0.29 to 0.53 as a function of financial wealth.¹¹ For a direct comparison with each version of our quarterly model, we compute the steady-state unconditional probability that a household will undertake active trade within one year as $\sum_{j=1}^J (\theta_j - \theta_{j+4})$, with $\theta_{j+4} = 0$ for $j > J - 4$.¹² We find that the fraction of households actively trading in an average year is 0.78 in our baseline example, which is quite high relative to the Vissing-Jørgensen data. This may be explained in part by the fact that the transfer costs in this example are calibrated to match aggregate velocity over only the past decade, when transactions costs were presumably lower than in her 1982-1996 sample period. When we instead calibrate to match aggregate velocity over the postwar period in our second example (with higher transactions costs), the fraction of households trading annually falls to 0.55, slightly above the empirical range. Our most successful example with regard to this evidence is the third, where high transactions costs are drawn from a distribution implying the same postwar aggregate velocity, but longer expected episodes of inactivity. There, the model predicts an average annual fraction of households conducting trades well within the empirical range, at 0.42.

We cannot compare our examples' mean inactivity durations to that implied by the Vissing-Jørgensen data without making some assumption about the shape of the empirical hazard. If one assumes that the probability of an active trade is constant from quarter to quarter in the data, then the range reported above implies a mean duration of household inactivity ranging from 7.5 to 13.8 quarters. Recall that the mean duration of inactivity in our baseline example is only 4.8 quarters, while that in our second example involving high transactions costs is 7 quarters. This again suggests that the frequencies of active trades implied by these two versions of our model are, if anything, high relative to the data. However, our third example with both high maximum and

¹¹The CEX interviews about 4500 households each quarter, and each household is interviewed five times, with financial information gathered in the final interview only. Vissing-Jørgensen (2002) limits her sample to 6770 households that held risky assets both at the time of the fifth interview and one year earlier. She finds that the probabilities of buying or selling risky assets do not significantly change when the sample, spanning 1982 - 1996, is split into subsamples according to interview dates.

¹²For example, in any date t of our model's steady-state, there are θ_1 households entering the period in time-since-active group 1. After one year, at the start of period $t + 5$, $\theta_1 - \theta_5$ of that original group have undertaken at least one trade. Thus, the fraction of them that have traded within a year is $\frac{\theta_1 - \theta_5}{\theta_1}$. The overall fraction trading within one year is the population-weighted sum of these fractions across each starting group, $j = 1, \dots, J$.

mean inactivity spells exhibits an average duration within the range implied by the data, at 9.55 quarters. Thus, we will study this third case as we move to examine our model’s dynamic results in section 5.

We confine our remaining discussion of the model’s steady-state to that arising under our baseline parameters, as the qualitative aspects that we will emphasize hold across all of our examples. Here, with both the aggregate endowment and the money growth rate fixed at their mean values, six groups of households enter into each period, with these groups corresponding to the number of quarters that have elapsed since members’ last account transfer. As any individual household moves through these groups over time, its real money balances available for shopping fall further and further below the target value, 2.936, given both inflation and its expenditures subsequent to its last time active. To correct this widening distance between actual and target real balances, the household becomes increasingly willing to incur a fixed transfer cost. This implies that the threshold cost separating active households from inactive ones rises with households’ time-since-active. Thus, as transfer costs are drawn from a common distribution, the fraction of households exhibiting current activity in Table 1 rises across start-of-period groups.

Table 1: Determination of steady-state shopping-time distribution

time-since-active group	0	1	2	3	4	5	6
start-of-period populations		0.208	0.205	0.196	0.174	0.136	0.082
<i>fraction currently active</i>		<i>0.011</i>	<i>0.045</i>	<i>0.113</i>	<i>0.218</i>	<i>0.397</i>	<i>1.000</i>
shopping-time real balances	2.936	2.510	2.095	1.691	1.301	0.929	n/a
shopping-time populations	0.208	0.205	0.196	0.174	0.136	0.082	0

In figure 1A, we plot the steady-state distribution of households across groups as they enter shopping-time from the final row of table 1. Corresponding to the rising fractions of active households shown above, the dashed curve reflecting the measures of households in each shopping-time group monotonically declines across groups. The solid curve in the figure illustrates the ratios of real consumption expenditure relative to real balances, individual velocities, associated with the members of each shopping-time group. Because households are aware that they must use their current balances to finance consumption not only in the current period but also throughout subsequent periods of inactivity, individual spending rates rise across groups in response to a declining expected duration of future inactivity. Currently active households, those households in group 0, face the longest potential time before their next balance transfer, and thus have the lowest individual velocities. By contrast, households currently shopping in group 5 will receive a transfer with certainty at the start of the next period; thus, individual velocity is 1 for members of this last group.

Two aspects distinguishing our endogenous segmentation model will be relevant in its responses to shocks below. First, on average, a household’s probability of becoming active monotonically rises with the time since its last active date, as seen above. Second, these probabilities change over time as shocks influence the value households place on adjusting their bank balances. To isolate the importance of these two elements below, we will at times contrast the responses in our economy to those in a corresponding economy that has neither. In that otherwise identical *fixed duration* model, the timing of any household’s next account transfer is certain and is not allowed to change with the economy’s state. Consistent with our endogenously segmented economy, where households’ mean

duration of inactivity is 4.8 quarters, households in the corresponding fixed duration model are allowed to undertake transfers exactly once every 5 quarters.

Figure 1B displays the steady-state of the fixed duration model. There, households enter every period evenly distributed across 5 time-since-active groups. Throughout groups 1 through 4, fraction 0 of each group’s members are allowed to undertake account transfers, while fraction 1 of the members of group 5 are automatically made active. Thus, 20 percent of households enter into shopping in each time-since-active group 0 through 4, and this shopping-time distribution remains fixed over time. As in our model with endogenously timed household portfolio adjustments, here too individual velocities monotonically rise with time-since-active and hit 1 in the final shopping group. However, given its lesser maximum duration of inactivity (5 quarters here versus 6 in the endogenous segmentation model), households in the fixed duration economy exhibit somewhat higher spending rates throughout the distribution relative to those in panel A.

4.2 Money injection: a baseline example

In this and the following section, we begin our study of the endogenous segmentation economy’s dynamics using two examples designed to illustrate its underlying mechanics. Here, we examine the effects of an unanticipated one period rise in the money growth rate.

Fixed duration model. For reference, we begin in figure 2 with an examination of the aggregate response in the fixed duration model, where the fractions of active households across groups are fixed and dictated by $\alpha_{FD} = [0 \ 0 \ 0 \ 0 \ 1]$.¹³ As seen in the top panel, the aggregate price-level rises only halfway at the date of the money supply shock, with the remaining price adjustment staggered across several subsequent periods. This inflation episode continues until those households who were active at the shock date have traveled through all time-since-active groups and are once again active, at the start of date 6.

The aggregate price-level adjusts gradually in this exogenously segmented markets economy for precisely the reasons explained by Alvarez, Atkeson and Edmond (2003). Open market operations that inject money into the brokerage accounts must be absorbed by active households.¹⁴ However, as they will be unable to access their brokerage accounts again for 5 periods, these households retain large inventories of money relative to their current consumption spending. As noted above, their spending rate is the lowest among all households in the economy. Consequently, total nominal spending does not rise in proportion to the money supply, and a rise in the share of money held by active households leads to a rise in aggregate real balances. Equivalently, in this endowment model, velocity falls.

Formally, in a fixed duration model, given any fixed number of time-since-active groups J , aggregate velocity may be expressed as the sum of two terms, one associated with the common velocity of currently active households and one associated with the velocities of inactive households across their respective groups:

$$V_t = \frac{1}{J} \frac{M_{0t}}{M_t} v_{0t} + \sum_{j=1}^{J-1} \frac{1}{J} \frac{M_{jt}}{M_t} v_{jt}. \quad (21)$$

¹³Figures in this and the subsequent section reflect the effects of a temporary 0.1 percentage point rise in the money growth rate. Given that our model is solved linearly, we have re-scaled all responses to correspond to a 1 percentage point rise for readability.

¹⁴No household unable to shift assets from its brokerage account into its bank account will accept the additional money, given the rate-of-return dominance implied by positive nominal interest rates.

From this equation, it is clear that the rise in relative money holdings of active households must reduce aggregate velocity, so long as individual velocities do not rise much in response to the shock. As seen in the bottom panel of figure 2, in our fixed duration example, half of the money injection is absorbed by an initial fall in aggregate velocity. As households that were active at the time of the shock travel through time-since-active groups in subsequent periods, their spending rate rises, pulling aggregate velocity back up. During this episode nominal spending rises faster than the money supply, the price level grows above trend, and aggregate real balances return to their long-run level.

Turning to the response in interest rates shown in the middle panel of figure 2, note that the money injection causes a large, but purely transitory, liquidity effect. In economies with segmented markets, real rates are determined by the marginal utilities of consumption among active households in adjacent periods, given that only these households can transform interest-bearing assets into consumption. In the fixed duration model, only those households that are allowed to be active at the date of the shock experience a rise in their lifetime wealth. As a result, their consumption rises, while the consumption of households active in subsequent dates remains unchanged, thus explaining the large but temporary fall in the real interest rate.

Endogenous segmentation model. Figure 3 displays the aggregate response to the same temporary shock in our model. The endogenous segmentation economy exhibits somewhat sharper initial price adjustment, associated with a smaller fall in aggregate velocity, and it has a more protracted response in inflation. Although the average time between a household's account transfers is 4.8 periods in our model's steady-state, its high inflation episode following the purely transitory money shock lasts 8 periods. The initial decline in interest rates is substantially smaller than were those in figure 2, at about one-tenth of the size of the money growth shock. However, in contrast to the immediate correction seen under fixed duration, the real interest rate here remains persistently low for 6 quarters. These differences in amplitude and propagation arise from the two elements distinguishing our model, the nontrivial rising hazard reflecting the fractions of households undertaking bank transfers from each time-since-active group, and the movement in this hazard in response to an aggregate shock. The first of these elements is central to our model's larger initial rise in inflation, while the second is entirely responsible for its substantially different real interest rate response.

Similar to (21) above, aggregate velocity in our model is determined by a weighted sum of the individual velocities of active and inactive households, with weights determined by the measures of households in each time-since-active group and their relative individual money holdings:

$$V_t = \left(\sum_{j=1}^J \alpha_{j,t} \theta_{j,t} \right) \frac{M_{0,t}}{M_t} v_{0,t} + \sum_{j=1}^{J-1} (1 - \alpha_{j,t}) \theta_{j,t} \frac{M_{j,t}}{M_t} v_{j,t} + \frac{P_t}{M_t} \sum_{j=1}^J \theta_{j,t} \int_0^{H^{-1}(\alpha_j)} x h(x) dx. \quad (22)$$

The final term reflects the proportion of the aggregate money stock used in paying transfer costs, and was absent in (21). However, as this term is quantitatively unimportant both on average and following the shock, it cannot explain our economy's lesser decline in aggregate velocity relative to the fixed duration model. The first-order difference lies in the second term, the weighted velocities of inactive households.

In the fixed duration model, every household spends all of its money between any one balance transfer and the next, because this timing is certain. By contrast, the average household in our economy typically has some left-over money in its bank account when it undertakes its next transfer,

because this timing is uncertain. Given their ability to alter this expected left-over money, our inactive households are more flexible in responding to the money growth shock.¹⁵ In response to the rise in anticipated inflation, their spending rates, v_{jt} , rise between 0.3 and 0.5 percent with the money injection, roughly twice as much as in the fixed duration model. As a result, our economy experiences a lesser decline in the second (and largest) term determining aggregate velocity at the date of the shock due to its nontrivial hazard. This is mitigated to some extent by changes in the hazard, as discussed below.

Because the money injection implies an inflationary episode that will reduce inactive households' real balances, it increases the value of actively converting bonds held in the brokerage account into money. Thus, a greater than usual measure of households become active. However, this rise in the number of active households has only limited impact in reducing aggregate velocity, since it implies that in equilibrium each active household receives a lesser share of the total money injection. As a result, the weight $\frac{M_{0,t}}{M_t}$ is smaller in (22) than it is in (21), which in turn implies a lesser initial rise in the consumption of active households in our economy. The smaller rise in each active household's money holdings also implies that their velocity falls by less than in the fixed duration model (0.4 versus 1.6 percent).

While endogenous market segmentation reduces the initial real effect of a monetary shock, it also propagates it through changes in the timing of households' transfer activities, which are summarized in panel A of figure 4. Following a substantial initial rise, the overall measure of active households falls below its steady-state value for a number of periods, despite persistently high activity rates across groups, α_{jt} . This is because large initial rises in these rates shift the household distribution to imply higher than usual money balances for the mean household in subsequent dates, thereby reducing its incentive to transfer funds from the brokerage account.

Those persistent changes in the distribution of households are responsible for the persistent real effects in our economy. In dates following the shock, although money growth has returned to normal, the measure of active households is sufficiently below average that each such household receives an above-average transfer of real balances in equilibrium. Thus, the rise in the consumption of active households in our economy is not purely transitory as it was in the fixed duration model. Rather, as seen in panel B of figure 4, it returns to steady state gradually as the distribution resettles. This explains why the initial decline in the real interest rate is much smaller in our economy relative to the fixed duration model, and why it remains persistently low.

Figure 5 verifies the importance of changes in our economy's endogenous timing of household transfer activities by displaying the aggregate response in an otherwise identical model where such changes are not permitted. In this *time-dependent* activity model, a nontrivially rising hazard governs the timing of household account transfers; in fact, it is precisely that from our economy's steady-state in table 1. Here, however, this hazard is held fixed throughout time. From the comparisons in panels A and C, it is clear that changes in group-specific activity rates serve to reduce aggregate velocity in our model economy, yielding more gradual price adjustment, as was argued above. Next, the time-dependent model's interest rate responses in panel B confirm that our

¹⁵Consumption choices among each inactive group of households, $j = 1, \dots, J - 2$, satisfy:

$$u'(c_{jt}) = \beta E_t \left[\frac{P_t}{P_{t+1}} \left((1 - \alpha_{j+1,t+1}) u'(c_{j+1,t+1}) + \alpha_{j+1,t+1} u'(c_{0,t+1}) \right) \right].$$

As active households have higher consumption than inactive households (and $c_{0,t+1}$ rises with the money injection in our economy), the positive and increased probability of becoming active in the next period compounds the effect of anticipated continued high inflation in discouraging money savings.

economy's persistent liquidity effects in real rates arise entirely from changes in the hazard, rather than its average shape. Absent these changes, the interest rate decline is completely transitory just as in the fixed duration model.

4.3 Money injection: high transfer costs examples

In the preceding example, the fixed transfer costs causing our economy's market segmentation were selected to yield average aggregate velocity at 1.9, and implied a 4.8 quarter average duration of inactivity among households. Here, we examine our model's response to the same temporary money growth shock in an example with high transfer costs implying aggregate velocity matching the U.S. postwar average, at 1.5, and a mean inactivity duration of 7 quarters. In this case, the maximum inactivity spell facing a household is substantially longer, at 10 quarters, and only about 14 percent of households are active in the average period (versus 21 percent above). Thus, households are spread across far more groups and, as a result, carry larger inventories of money on average.

Figure 6 is the high transfer cost counterpart to figure 3. Here, in contrast to the previous example, the aggregate price level actually rises by more than the money growth shock at its impact, given a rise in aggregate velocity, and the inflationary episode is entirely temporary. Moreover, the persistent decline in the real interest rate of figure 3 has also evaporated. These dramatic changes in the model's response may be traced to two features of the mechanics discussed above that become more pronounced when households face the possibility of very extended absence from their brokerage accounts: the rise in activity rates at the date of the shock and, more importantly, the rise in individual spending rates among inactive households.

With the money injection comes a permanent upward shift in the path of the aggregate price level. This is far more costly for inactive households in this example relative to the previous one, because these households are, at impact, holding much higher inventories of money in preparation for longer horizons of potential inactivity. This leads to a percent rise in activity rates (4.2 percent) similar to that in our previous example with much smaller transactions costs. However, high transactions cost draws keep most households inactive. In an effort to offset the fall in their real consumption spending, these households increase their spending rates.¹⁶ As a result, the percentage rises in v_{jt} over time-since-active groups 1 through 5 are roughly double those in our previous endogenous segmentation example, and these rises are also large in each of the higher-numbered groups new to this example, at around 0.75 percent. Moreover, even with increased activity rates, inactive households make up roughly 75 percent of all households at the date of the shock. Thus, as these households release substantially more money into the goods market than usual, total nominal spending actually increases by more than the money supply. This leads to the sharp impact-date inflation.

There is virtually no real interest rate response at all in this example. The high current price level, and the large change in activity rates (which have a lower steady state value in this example), are sufficient to imply that each active household receives no greater percentage rise in real money holdings than did its counterpart in our previous example. However, active households

¹⁶As noted above, probabilistic timing of future activity is important, because it implies that, on average, most households return to their brokerage accounts with some money remaining. At the impact of the shock, this allows inactive households the flexibility to transfer some of their current consumption loss into dates beyond that when they will next be active. Moreover, on average, the expected money with which a household returns to the brokerage account is higher in this example relative to the previous one, allowing greater adjustment along this margin.

here have significantly lower spending rates on average (given potentially very long absences from the brokerage accounts). As a result, their consumption rises by less than one-third the rise seen in figure 4, remaining very close to that of households active in subsequent dates.

We have referred to our second example as one with an increased average duration of inactivity. However, what distinguishes this example is the substantial difference between the mean duration of inactivity (7 quarters) versus the maximum (10 quarters). This leads to additional precautionary accumulation of money and, on average, households return to the bond market with sizeable left-over money balances. Thus, inactive households at the date of the money injection have substantial flexibility in raising their current spending rates by reducing their expected future left-over balances. This allows the sharp initial rises in individual velocities central in the results above. Alternative examples where the maximum length of inactivity is similarly high, but the mean duration is close to it, more closely resemble the baseline example in the section above. One such example follows.

To obtain a high maximum inactivity duration example where the mean duration is similarly high, we abandon our assumption that transfer costs are distributed uniformly. In this third case, we assume a beta distribution parameterized by $[\alpha = 3, \beta = 1/3]$ and set the maximum transfer cost at $B = 0.50$. This results in an average aggregate velocity again at 1.5, a maximum inactivity duration of 10 quarters and a mean duration of 9.55 quarters. With this change in the cost distribution, our model's steady-state hazard describing average activity rates looks much like that of a fixed duration model, in that activity rates are near zero for all groups below J . As such, one might imagine that its dynamic response would resemble a $J = 10$ version of figure 2. However, because our households are able to change the timing of their transfers, this is not the case. In fact, figure 7 reveals that the response to the temporary money shock is instead quite similar to that in our baseline endogenous segmentation example. Again, adjustment in the aggregate price level is slow, resulting in a persistently high inflation episode and, unlike a fixed duration model, the real interest rate is persistently low. The one new feature here relative to both the fixed duration model and our baseline example is a persistent liquidity effect in nominal interest rates. From this, it is clear that the distribution of the costs responsible for market segmentation can have important effects on aggregate dynamics.

Recall that this third example also improves upon those above in its consistency with the microeconomic evidence regarding the frequency of active trades. Here, the model predicts that, on average, the fraction of households undertaking active trades within one year is 0.42. Unlike either of the preceding examples, this prediction lies well inside the range estimated by Vissing-Jørgensen (2002), 0.29 – 0.53. Thus, we pursue this case of high mean and maximum inactivity duration as we examine our model's dynamic results in the section below.

5 Results

The examples we have considered thus far are useful in illustrating the mechanics of our model, at least qualitatively. However, the analysis of a purely random increase in the money supply is far from what most would view as reflective of inflation and interest rate dynamics in an actual economy. In this section, we present results for our model under more plausible assumptions about monetary policy. First, we examine a persistent shock to the money growth rate. Next, we consider an environment where the monetary authority implements changes in the money supply towards stabilizing inflation in the face of persistent real shocks. As we examine the resulting dynamics in each of these cases, we will draw upon our analyses of the examples above for explanations.

5.1 Persistent money growth shock

We begin by considering the aggregate response to a persistent rise in the money growth rate, now assuming that the money growth rate follows an AR(1) process with autocorrelation 0.57, as in Chari, Kehoe, and McGrattan (2002). To see how endogenous changes in the extent of market segmentation influence this response, we contrast our endogenous segmentation economy to its corresponding fixed duration model where such changes are not permitted.¹⁷

Figure 8 shows log deviations from the initial trend for the money stock and for the price levels of our model and the fixed duration model. The impact of the shock on the money supply is largely finished by period 7, while the price levels are clearly more sluggish in their adjustment, with above-average inflation continuing for 3 or 4 additional periods. What is noteworthy is that the response in prices when segmentation is endogenous is more gradual. Moreover, while the price level in both models overshoots its new trend, this is less pronounced in our model.

In the fixed duration model, large wealth effects for households active in the early periods of the shock lead to sharper increases in prices. By eroding the real balances of inactive households, inflation redistributes consumption to active households. By contrast, in our model, rises in the numbers of active households reduce the increase in their individual money holdings, and thus the extent to which, in equilibrium, consumption must be redistributed. Compared to the transitory shock studied in figure 3, since the rise in the money growth rate is now persistent, some households delay their early return to the brokerage account by a period or two in hopes of lower transfer costs. As a result, the rise in the number of active households is initially smaller, and it persists for several periods, thereby protracting the distributional effects of the shock. Relative to the fixed duration model, a persistently smaller redistribution of consumption from inactive to active households in our model explains its lower rates of inflation in periods after the shock. Moreover, because the early periods with above-average numbers of active households are followed by 6 periods in which this number falls below steady-state, the episode with high real balances per active household is extended (as discussed above in section 4.2). This implies greater persistence in the increased consumption of active households, and a persistent liquidity effect in both real and nominal interest rates relative to the fixed duration model (not shown).

5.2 Real shock under a Taylor rule

Here, we allow for shocks to the real endowment received by households, with the money supply governed by the Taylor rule specified in section 3. Figure 9 shows our economy's aggregate response to a persistent rise in the endowment, alongside the corresponding response in the time-dependent model where the hazard dictating households' probabilities of becoming active remains fixed over time. While we do not display the response in the corresponding fixed duration model, note that our discussion of the time-dependent model below would apply equally well if we were instead describing that model. Given similar fixed hazards describing activity rates, the responses in these two exogenous segmentation economies are quite close.

Taken on its own, in the absence of any response in the money growth rate, the rise in endowments would imply a fall in the inflation rate to increase real balances. Given the Taylor rule, this

¹⁷The response in the corresponding time-dependent model (where the hazard describing activity rates is held fixed at the endogenous model's steady-state) is similar to that of the fixed duration model shown here. To understand why, recall from our discussion of figure 7 that the beta distribution from which transfer costs are drawn in our model leads to a steady-state hazard resembling the hazard of a fixed duration model.

requires a fall in the nominal interest rate. Indeed, given the active policy rule we have assumed, where the nominal interest rate responds by more than inflation, the real interest rate must fall.

In the time-dependent model there is a sharp, unanticipated fall in prices at the initial date of the shock. Interest rates fall, and an increase in the real balances of active households finances subsequent purchases of the increase in output. Households active after the shock do not experience a rise in their consumption and do not require anything beyond the usual transfer of real balances from their brokerage accounts. Thus, inflation returns to its average value in the second period, as do interest rates.

By contrast, in our economy with endogenous market segmentation, there are persistent responses in inflation and interest rates, with a half-life of roughly 3 quarters. The fall in nominal interest rates gives households an incentive to hold more money. The resulting rise in activity rates implies that the increase in the money supply, relative to trend, is spread over more households than usual, thus lowering the rise in each individual withdrawal. Relative to the exogenous segmentation model, this reduces the rise in consumption among households active in date 1 of the shock. Thereafter, with fewer households remaining in other groups at this initial date, subsequent active populations are reduced, thereby raising their individual withdrawals. As a result, consumption among active households rises less sharply upon the shock's impact and is more evenly spread across subsequent active groups. For this reason, there is a lesser initial fall in the real interest rate, but a gradual return to steady state thereafter. As a result, there is persistence in the nominal interest rate that translates, through the Taylor rule, into persistently low inflation.

It is worth re-emphasizing that our economy's persistent response in this figure cannot be attributed to the Taylor rule itself, noting again that there is no such persistence in the exogenous segmentation counterpart. Rather, it arises entirely from households' ability to change the timing of their portfolio adjustments in response to the economy's aggregate state. Viewed alternatively, it is a consequence of endogenous changes in the extent of market segmentation.

This set of results, based on empirically plausible policy rules, reinforces our view that the effort to endogenize market segmentation is a worthwhile one, indicating the promise that such models have for explaining persistent movements in inflation and interest rates in actual monetary economies.

5.3 Real shock under a Taylor rule with production

We briefly consider the joint mechanics of market segmentation, output and employment by endogenizing production. Households now value both consumption and leisure, and their utility function is given by $u(c, n) = \frac{(c - \eta n^\gamma)^{1-\sigma}}{1-\sigma}$, where c is consumption and n denotes hours of work. Even abstracting from capital, this generalization is complicated by the introduction of variable labor supply decisions. This is because we maintain consistency with our existing assumptions with respect to risk-sharing, and this implies that households in different time-since-active groups choose different hours worked. Let $n_{j,t-1}$ describe the hours of work in period $t-1$ by a household that, at that time, had last been active j periods in the past, $j = 0, \dots, J-1$.

We assume a competitive firm that hires labor from households and produces the single consumption good. All households are paid the same wage. The firm is owned equally by all households, there is no trade in shares, and all profits are returned to owners. Assume that the aggregate production function is $Y_t = z_t N_t^\rho$ where N_t is aggregate employment. The logarithm of total factor productivity, z_t , follows an auto-regressive process with persistence parameter, ρ .

The timing of events within a period is as follows. At the beginning of each period, households

observe the current aggregate state, draw their transfer costs, make their portfolio adjustments, and then participate in the labor market. Fraction λ_N of both labor income and λ_Π of profits are paid into the bank account. As before, income earned this period is available at the start of the next. Thus, there remains a cash-in-advance constraint on consumption purchases. Moreover, while we use w_{t-1} to describe the real wage in period $t-1$, and Π_{t-1} to represent real profits, all income is paid in nominal units. Thus, at the beginning of period t , the real payment into the bank account of each household of type j , $j = 1, \dots, J$, is $[\lambda_N(w_{t-1}n_{j-1,t-1}) + \lambda_\Pi\Pi_{t-1}] \frac{P_{t-1}}{P_t}$, while $[(1 - \lambda_N)(w_{t-1}n_{j-1,t-1}) + (1 - \lambda_\Pi)\Pi_{t-1}] \frac{P_{t-1}}{P_t}$ is deposited into the brokerage account.

To be more precise about these incomes, we represent the current type- j household's real wage earnings with which it ended the previous period as $e_{jt} \equiv w_{t-1}n_{j-1,t-1}$ in the problem that follows. The other wealth specific to a household as it enters the period is the real value of the money it (deliberately) saved in its bank account from the period. Let m_{jt} represent, for a household currently of type j , its real money savings at the end of the previous period. These savings imply real balances of $m_{jt} \frac{P_{t-1}}{P_t}$ in the household's bank account at the start of this period.

As before, we retrieve competitive allocations by solving the recursive problem of an extended family. Noting that we have are now using $m_{j,t} = \frac{M_{j,t}}{P_{t-1}}$ to define the support of the distribution of real balances, we briefly describe the constraints involved. Relative to the endowment economy, there are two new sets of constraints. The first set uses the current labor choices to determine the labor income a household has available at the end of the period,

$$w_t n_{jt} \geq e_{j+1,t+1} \text{ for } j = 0, \dots, J-1.$$

The second is a single constraint involving that fraction of income not paid into households' bank accounts. Let χ_{t+1} represent the end of period t value of real labor income and profits, earned within the period, that will be deposited into the family brokerage account at the start of the next period,

$$(1 - \lambda_\Pi)\Pi_t + (1 - \lambda_N)w_t \left[\sum_{j=1}^J \theta_{jt} \alpha_{jt} n_{0t} + \sum_{j=1}^{J-1} \theta_{jt} (1 - \alpha_{jt}) n_{jt} \right] \geq \chi_{t+1}.$$

We also replace the brokerage account constraint in the endowment economy, (14), with the following,

$$\frac{P_{t-1}}{P_t} \chi_t + \frac{P_{t-1}}{P_t} \sum_{j=1}^J \theta_{jt} \alpha_{jt} [m_{jt} + \lambda_N e_{jt} + \lambda_\Pi \Pi_{t-1}] + \frac{P_{t-1}}{P_t} \mu_t \bar{m}_{t-1} \geq m_{0t} \sum_{j=1}^J \theta_{jt} \alpha_{jt} + \sum_{j=1}^J \theta_{jt} \varphi(\alpha_{jt}).$$

where $\varphi(\alpha_{jt}) \equiv \int_0^{H^{-1}(\alpha_j)} \xi h(\xi) d\xi$ as before.

Examining the evolution of bank account balances, equations (15) and (16) are replaced by

$$\begin{aligned} m_{0t} - m_{1,t+1} &\geq c_{0t} \\ \frac{P_{t-1}}{P_t} [m_{jt} + \lambda_N e_{jt} + \lambda_\Pi \Pi_{t-1}] - m_{j+1,t+1} &\geq c_{jt} \text{ for } j = 0, \dots, J-1. \end{aligned}$$

There remains a non-negativity constraint on bank account balances, $m_{j+1,t+1} \geq 0$, which replaces (17). The laws of motion for the distribution of money balances (18) and (19) continue to apply, though we have now defined the problem using lagged real balances $m_{j,t}$.

The one-period lag in the deposit of income expands the family's state vector, which is now $\left(\{\theta_{jt}, m_{jt}, e_{jt}\}_{j=1}^J, \Pi_{t-1}, \chi_t\right)$, and the utility function $u(c, n)$ replaces that in 13. Finally, while the aggregate resource condition in (20) still applies, equilibrium in the labor market requires the introduction of an additional constraint,

$$N = \left(\sum_{j=1}^J \alpha_j \theta_j\right) n_0 + \sum_{j=1}^{J-1} (1 - \alpha_j) \theta_j n_j. \quad (23)$$

We assume that monetary policy follows a Taylor rule with the following specification:

$$i_t = i^* + 1.5[\pi_t - \pi^*] + 0.35[y_t - y^*].$$

In studying the behavior of this production economy under the Taylor rule, we choose the following parameters. First, we set $\nu = 0.6$ and the persistence of total factor production, $\rho = 0.9$. Next, we choose a relatively low wage-elasticity of labor supply, at least when compared to values often used in macroeconomic research, setting $\gamma = 2.5$. Aggregate hours worked in the steady-state are chosen to be approximately $\frac{1}{3}$ of time, which requires $\eta = 2.7$. We maintain all other parameters of the endowment model studied in this section, where transfer costs are drawn from a beta distribution. This, in turn, requires that we choose a steady-state level of total factor productivity of 0.33 so as to match our previous measure of aggregate velocity without changing the cost parameters.

Figure 10 illustrates the economy's response following a persistent shock to total factor productivity. As output increases with the shock to technology, there is a rise in the demand for real balances, which, given the money supply, reduces the price level relative to its trend. The monetary authority responds according to the Taylor rule, lowering the nominal interest rate. As the return on bonds falls, the fraction of households actively trading bonds for money rises.

As we have now seen in several examples, this initial rise in the number of traders has dynamic implications. In periods following the shock, the number of active households falls. This, in turn, implies that their real balances rise above the level typically held by an active household. In response, they increase their current consumption spending, and this drives further increases in output and employment. As a result, we observe a non-monotone response in these series. This arises entirely from households' ability to choose when to trade bonds for money, and to change the timing of these trades in response to changes in income, prices, wages, and interest rates. It does not exist when we suppress such choices.

6 Concluding remarks

In the sections above, we have developed a monetary endowment economy where fixed transactions costs lead households to reallocate their wealth between money and interest-bearing securities infrequently. Given these costs, some fraction of households choose not to withdraw interest income in any given period; thus, on average, households carry inventories of money to help finance their consumption spending over multiple periods. In this segmented asset markets environment, open market operations directly affect only a subset of all households. As a result, changes in money growth rates are followed by gradual adjustment in the aggregate price level. Moreover, because they disrupt the distribution of real wealth across households, such changes can have persistent effects on real interest rates.

Our approach to endogenizing market segmentation has emphasized idiosyncratic risk at the household level, through the assumption that transactions costs vary randomly across households. We have shown through a series of examples that the underlying distribution of these costs has important implications for the resulting distributions of households over money holdings, and thus the real effects of monetary policy. Given this finding, we have restricted the distribution of transactions costs to ensure consistency with both macroeconomic data on aggregate velocity, and microeconomic data on the frequency with which households buy or sell securities. Note that it is not essential that idiosyncratic differences across households are subsumed in their transactions costs. We believe that similar results would be obtained if transactions costs were common across households and idiosyncratic risk instead arose from differences in household endowments.

In our economy, small changes in the number of households actively participating in open market operations lead to changes in the distribution of money holdings across households that differ markedly from those in existing monetary models where the degree of market segmentation is fixed. When shocks to money growth rates are transitory, these changes in distribution add persistence to inflation, and they transform sharp temporary movements in interest rates into more moderate and gradual responses. Under persistent money growth shocks, they imply more gradual price adjustment. When monetary policy is governed by an active Taylor rule, persistent real shocks cause persistent movements in inflation and interest rates only if households are allowed to respond to such shocks with changes in the timing of their portfolio reallocations. Finally, in versions of the model with production, we observe a hump-shaped response in employment and output following a persistent but monotone shock to technology.

Our findings above suggest that monetary models with endogenous market segmentation may be useful laboratories in which to derive a more complete understanding of the relations between movements in real and nominal aggregate series observed in the data. Nonetheless, the environment we have studied here is sufficiently stylized that it permits only limited comparison with the dynamics of actual economies. To establish the empirical viability of the segmented markets alternative, we must assure ourselves that the appealing aspects of segmentation seen here will survive extensions toward greater realism. In future work, we plan to expand the dynamic elements of the model to include capital accumulation and use the result to evaluate the extent to which richer models of endogenous market segmentation are useful in understanding the joint movement of real and nominal variables. A second pressing concern involves the distribution of fixed transactions costs. Through the examples presented above, we have seen that the aggregate dynamics of our model are largely determined by the average money distribution and that this, in turn, rests on the distribution of the transfer costs. Our examples suggest that the added propagation implied by market segmentation may hinge critically on a cost distribution implying relatively little time-variation in individual trading patterns. While alternative specifications of production and monetary policy may overturn or weaken this particular finding, it is clearly essential that we isolate additional aspects of the data to further discipline the selection of this distribution.

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FIGURE 1A: Endogenous Segmentation Steady-State (mean duration: 4.818)

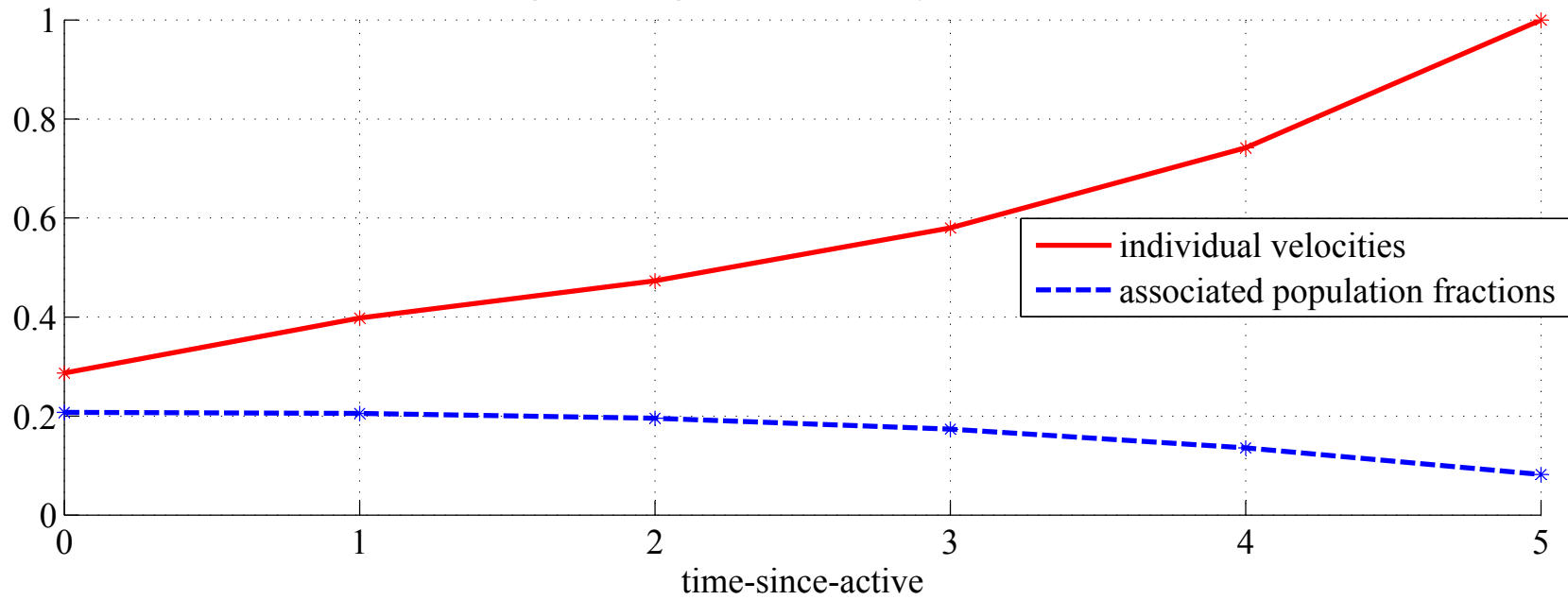


FIGURE 1B: Fixed Duration Steady-State (mean duration: 5)

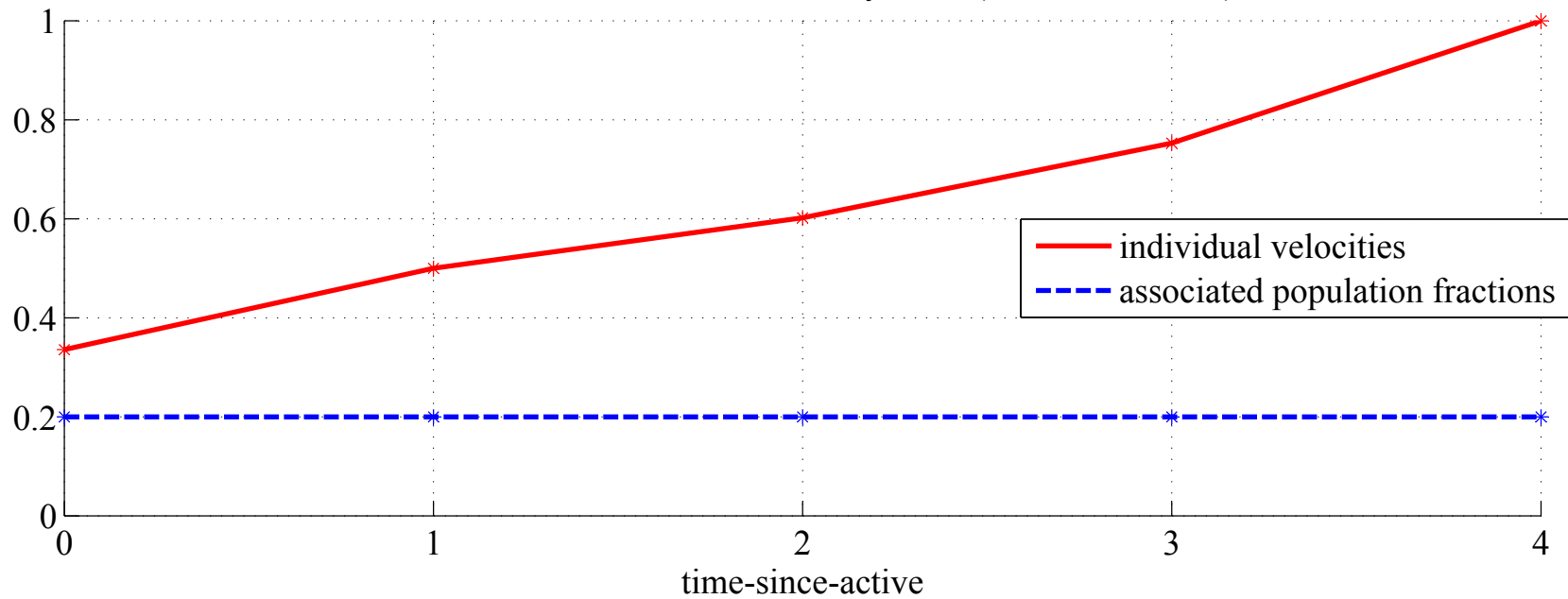


FIGURE 2: A Temporary Money Growth Shock in the 5-Period Fixed Duration Model

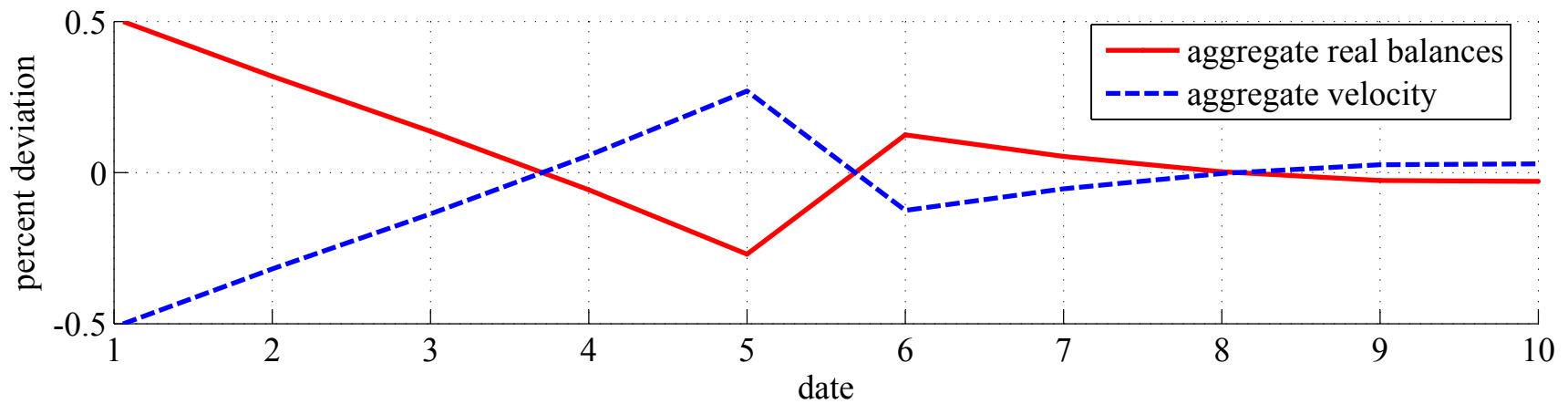
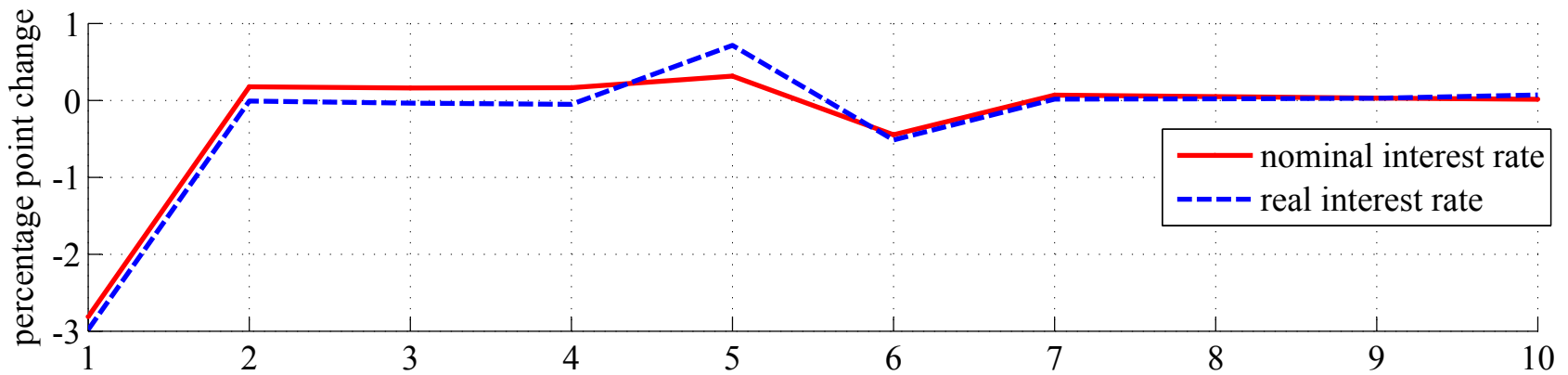
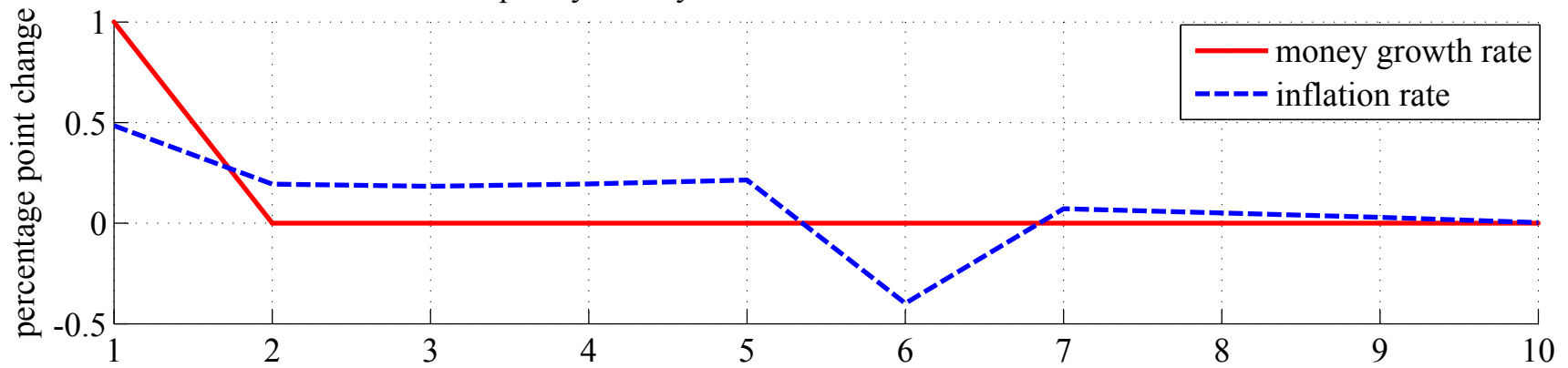


FIGURE 3A: Inflation Rate Responses to the Money Growth Shock

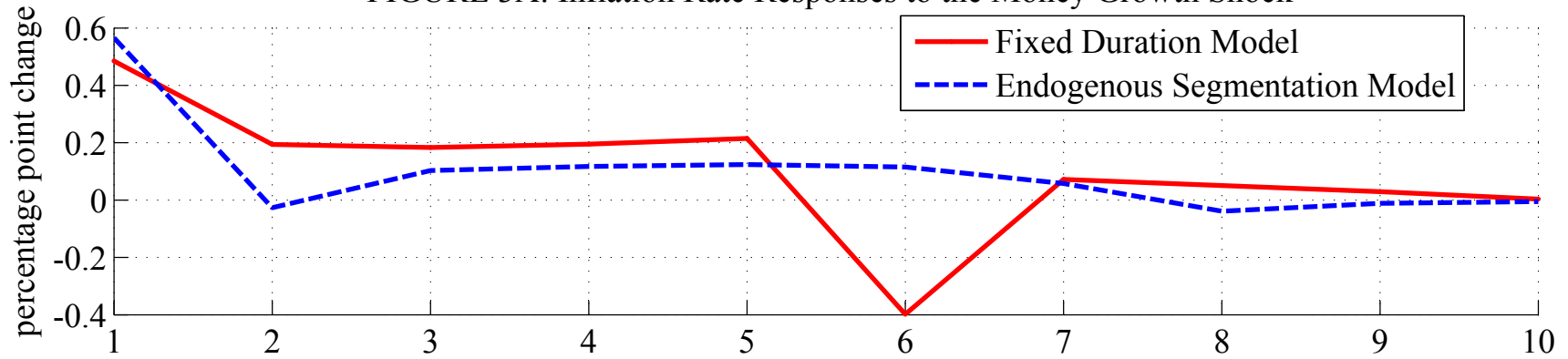


FIGURE 3B: Interest Rate Responses in the Endogenous Segmentation Model

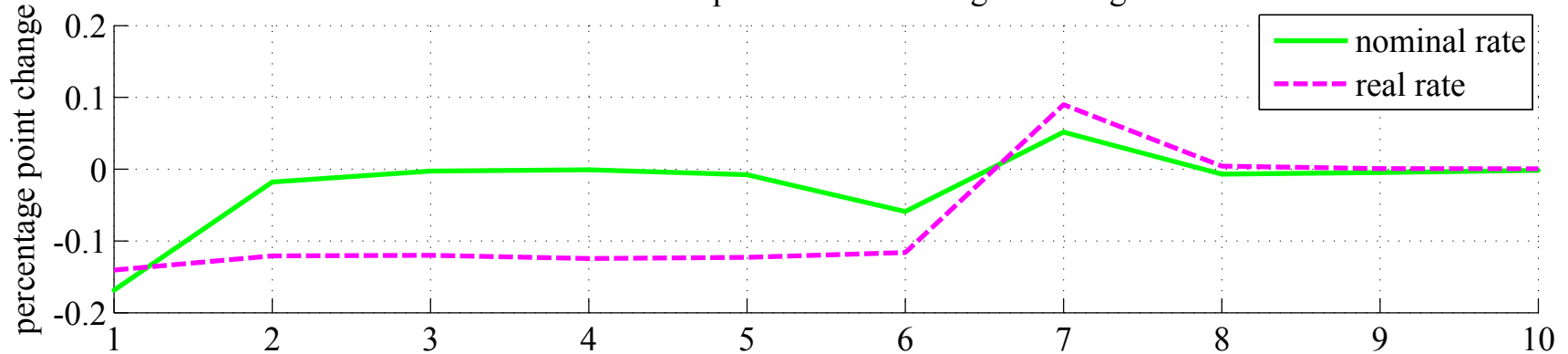


FIGURE 3C: Aggregate Velocity Responses to the Money Growth Shock

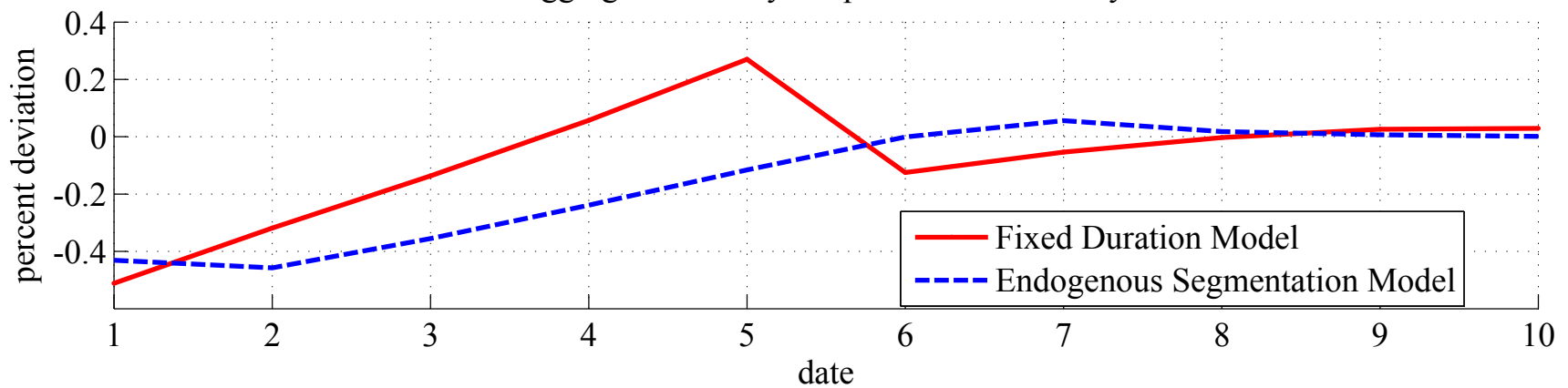


FIGURE 4A: Endogenous Segmentation Response in Total Active Households

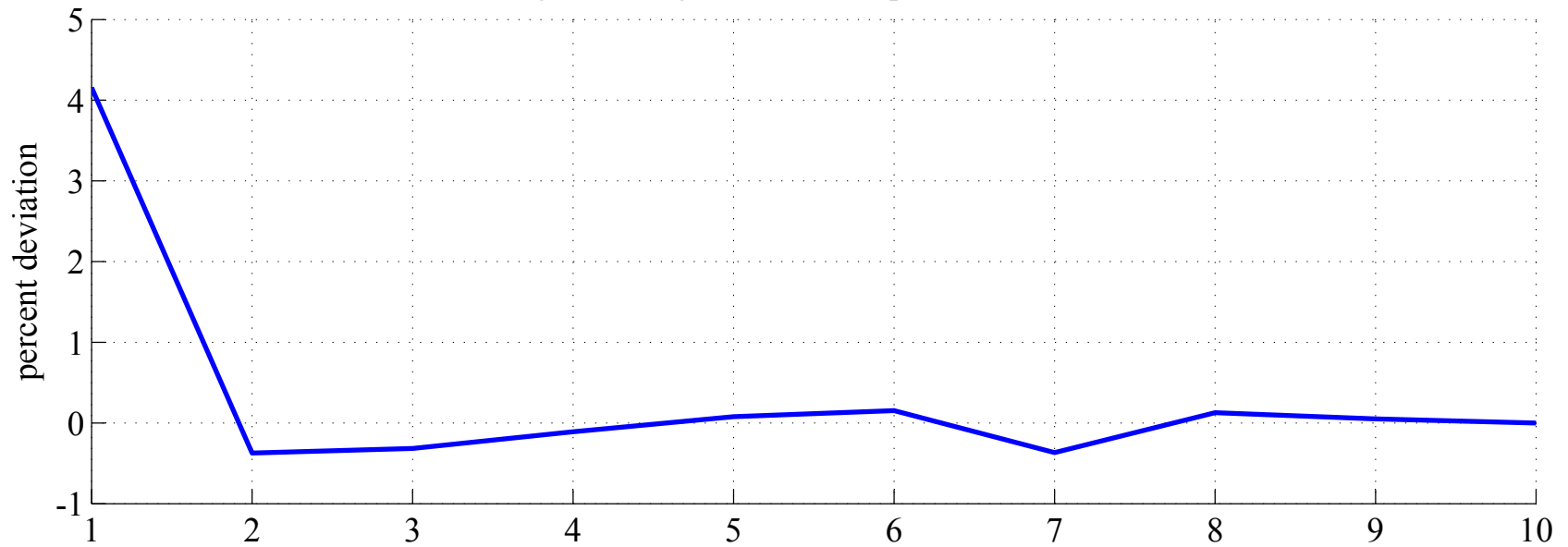


FIGURE 4B: Endogenous Segmentation Response in Consumption of Active Households

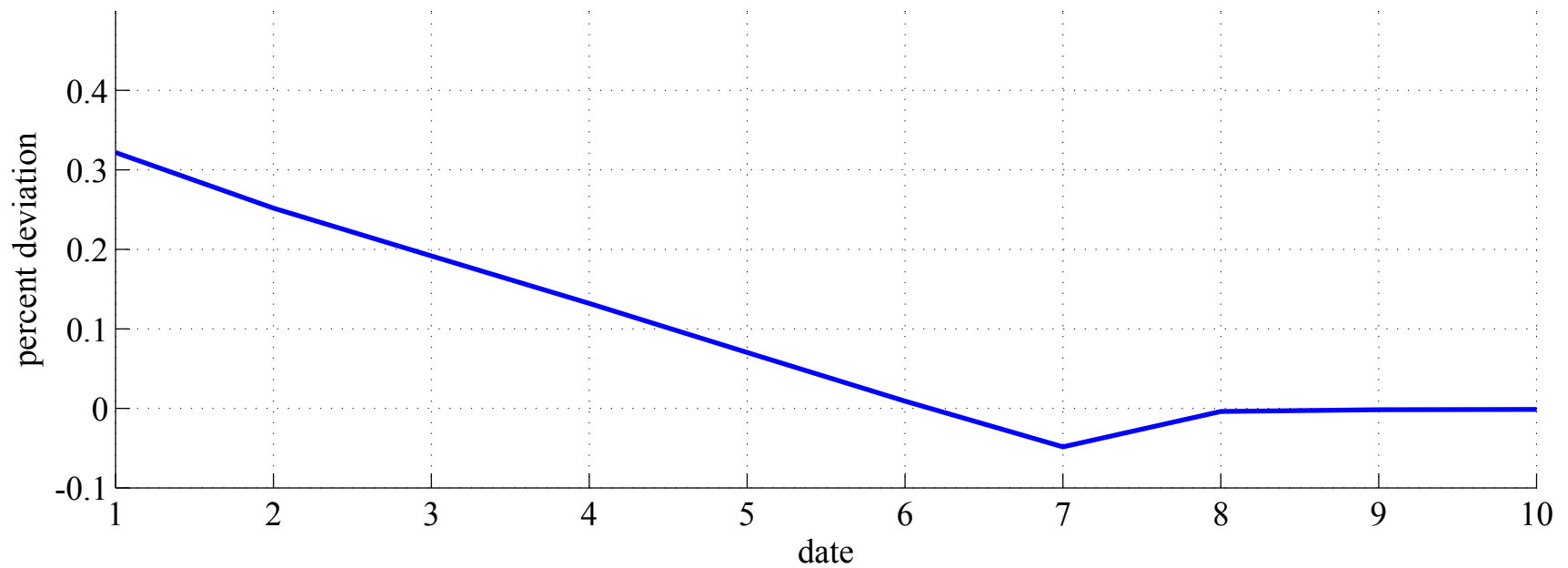


FIGURE 5A: Role of State-Dependent Activity Rates in the Inflation Rate Response

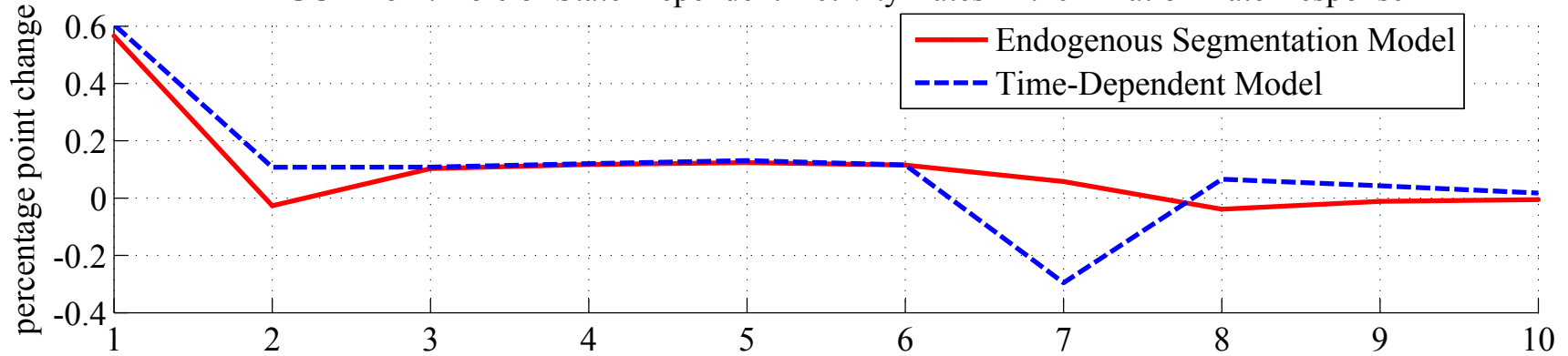


FIGURE 5B: Interest Rate Responses with Changes in Activity Rates Suppressed

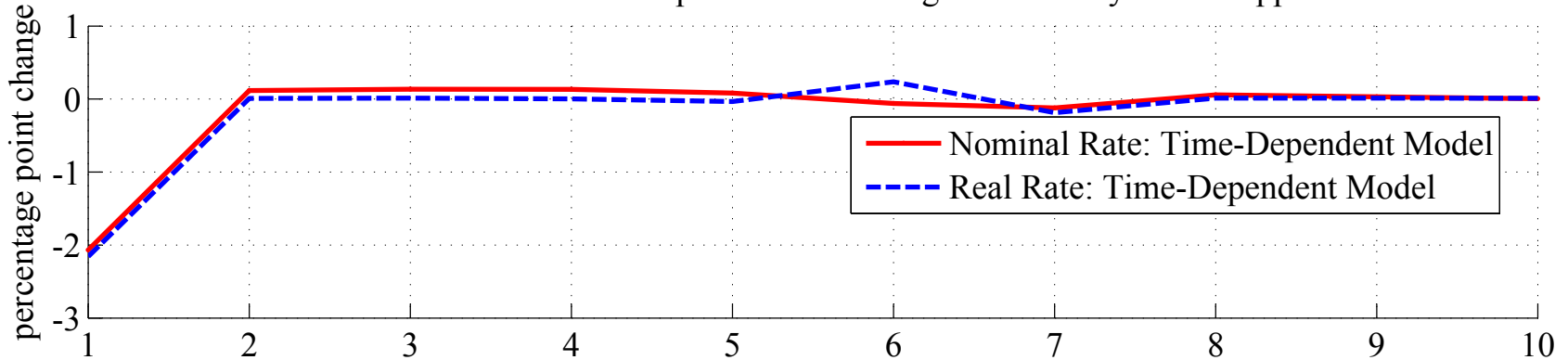


FIGURE 5C: Role of State-Dependent Activity Rates in the Aggregate Velocity Response

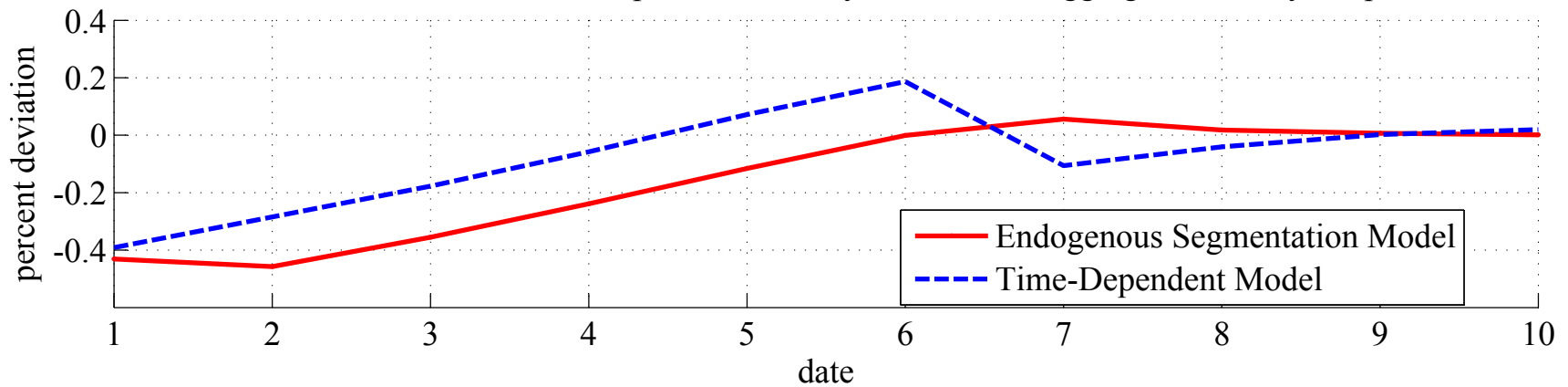


FIGURE 6: Temporary Money Growth Shock in a High Transfer Costs Endogenous Segmentation Model

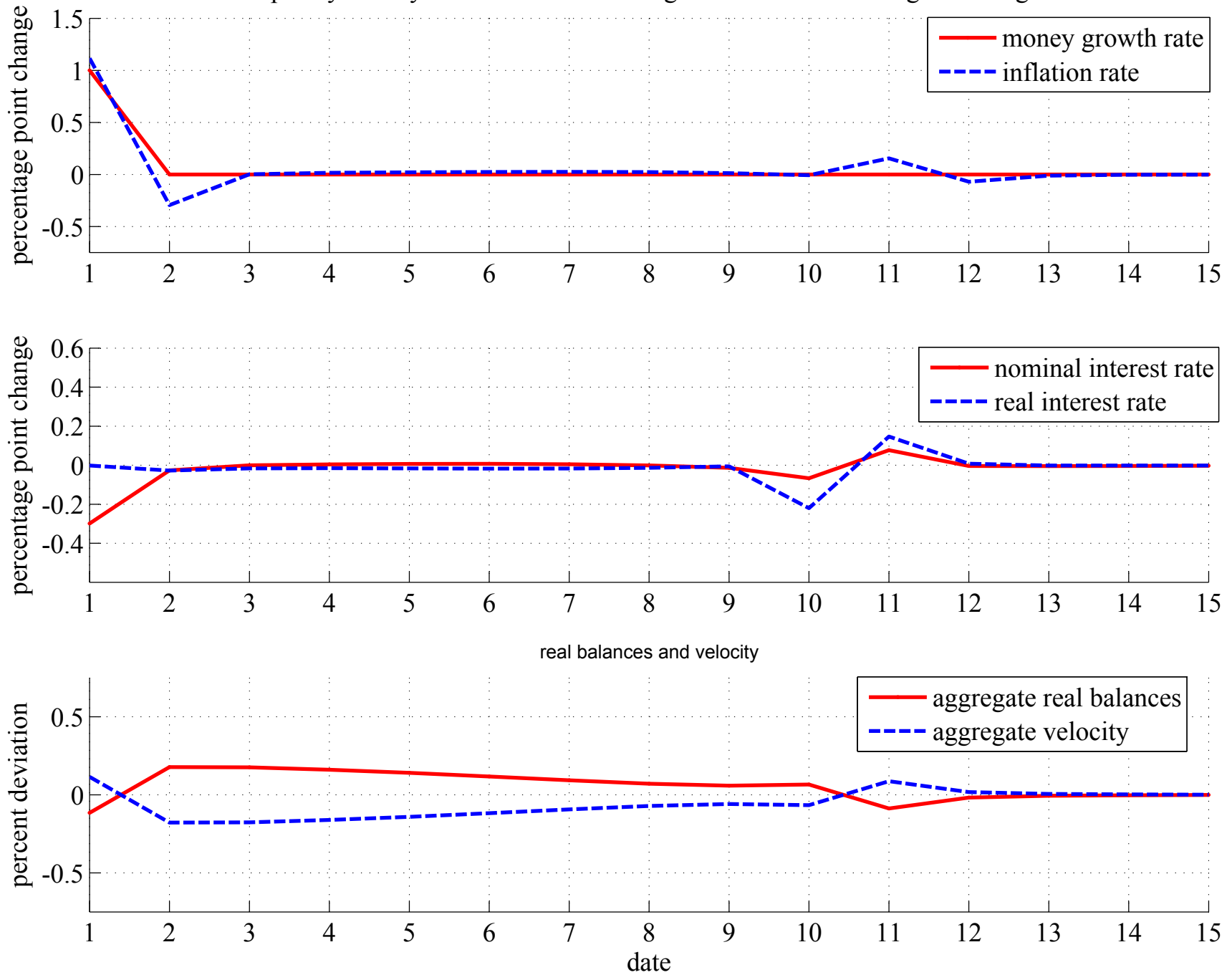


FIGURE 7: Temporary Money Growth Shock in a High Mean Duration Endogenous Segmentation Model

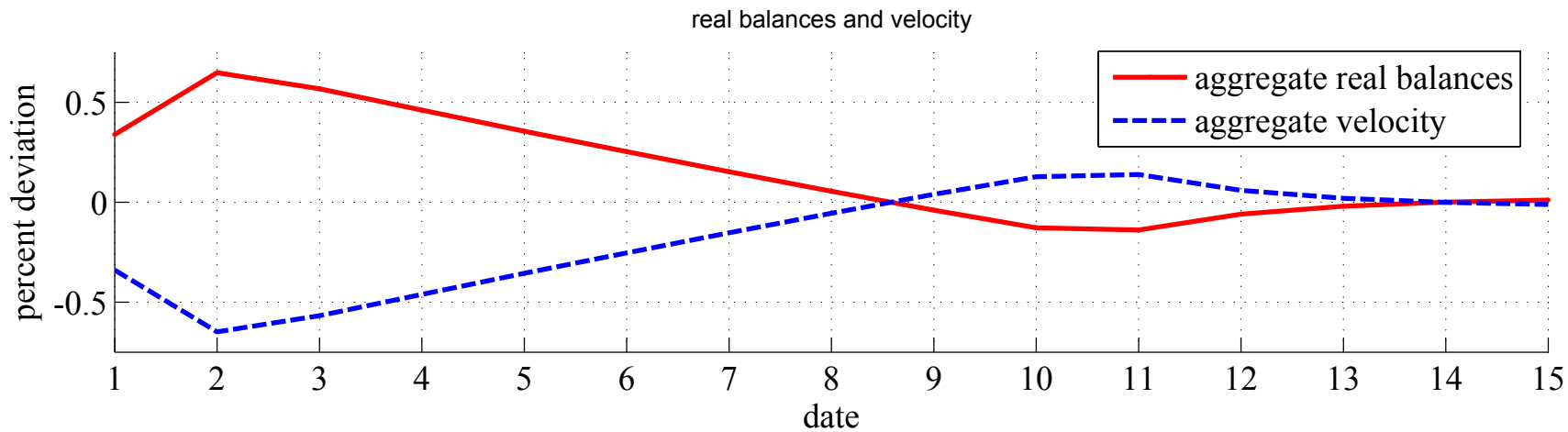
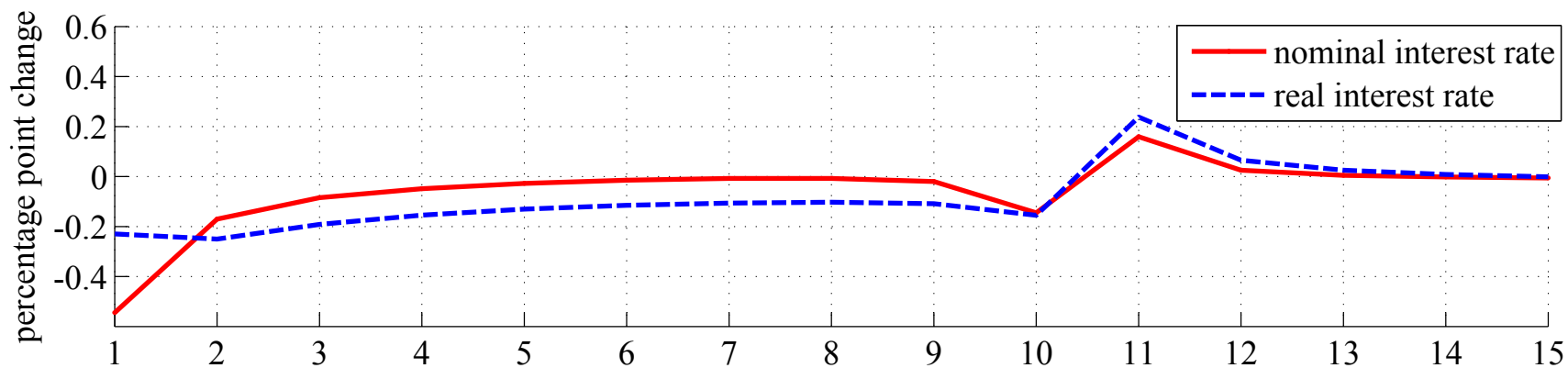
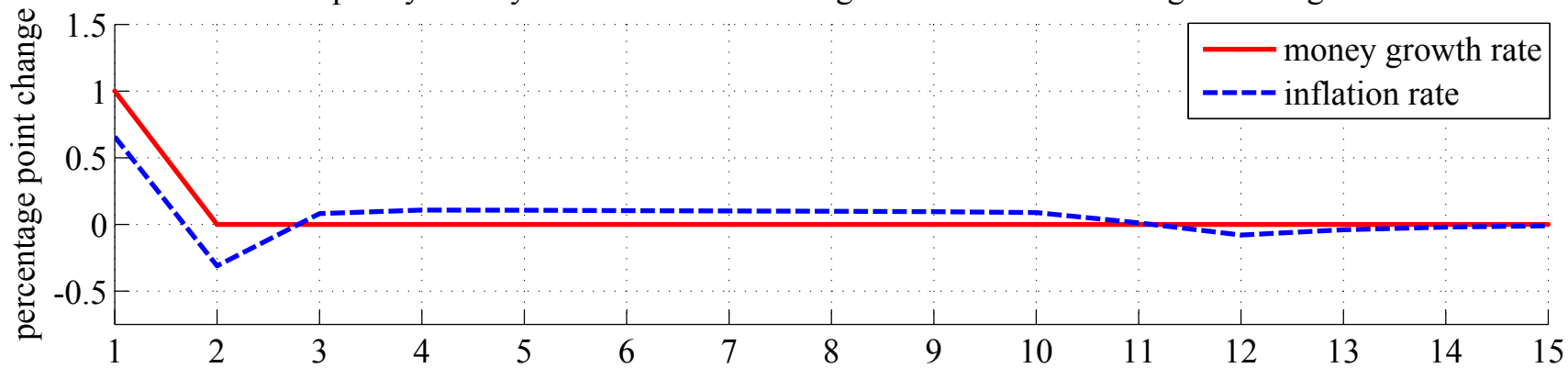


Figure 8: Money and Prices with a Persistent Money Growth Shock

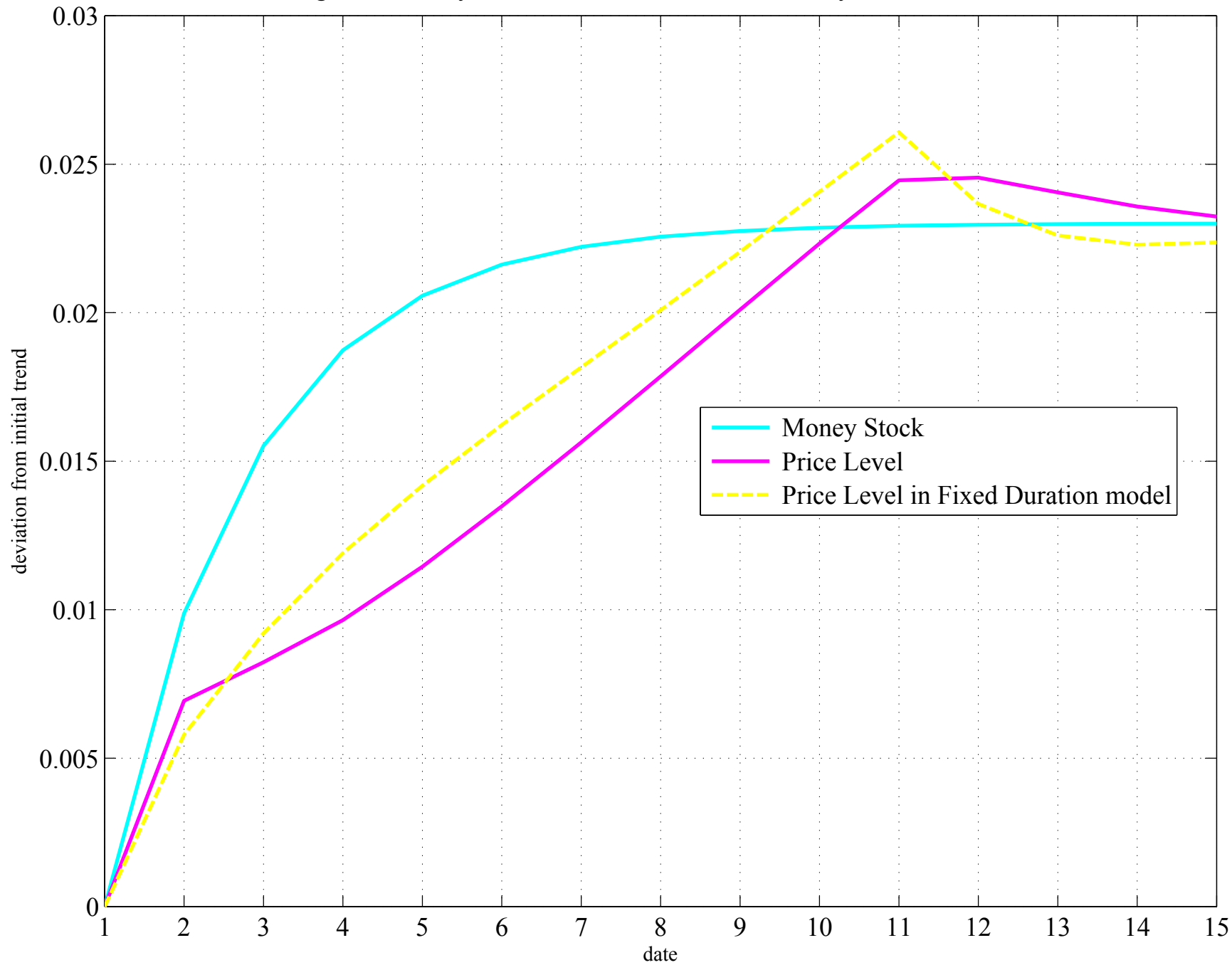


FIGURE 9A: Inflation and Interest Rates with a Persistent Endowment Shock and a Taylor Rule

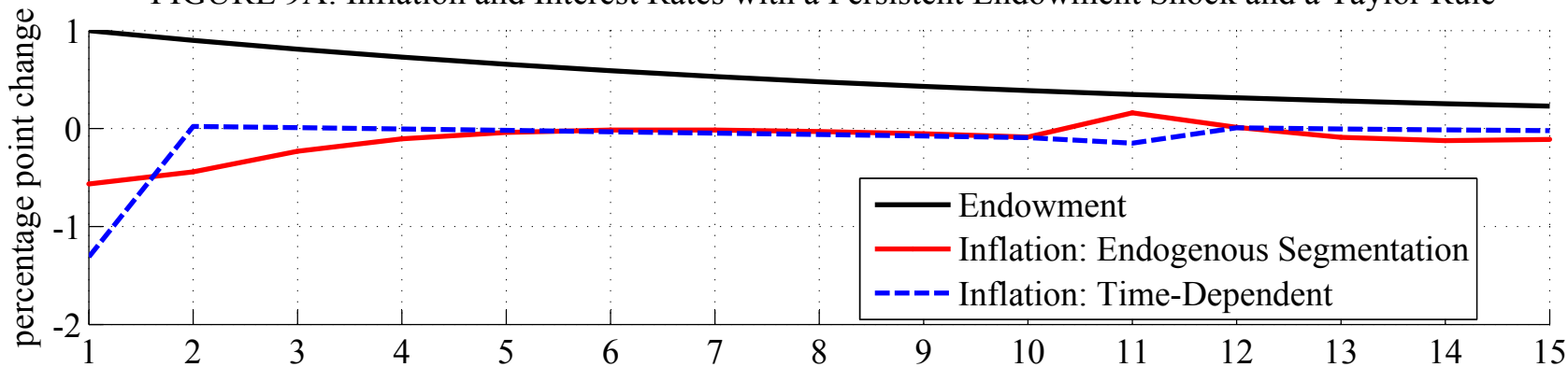


FIGURE 9B: Endogenous Segmentation Model

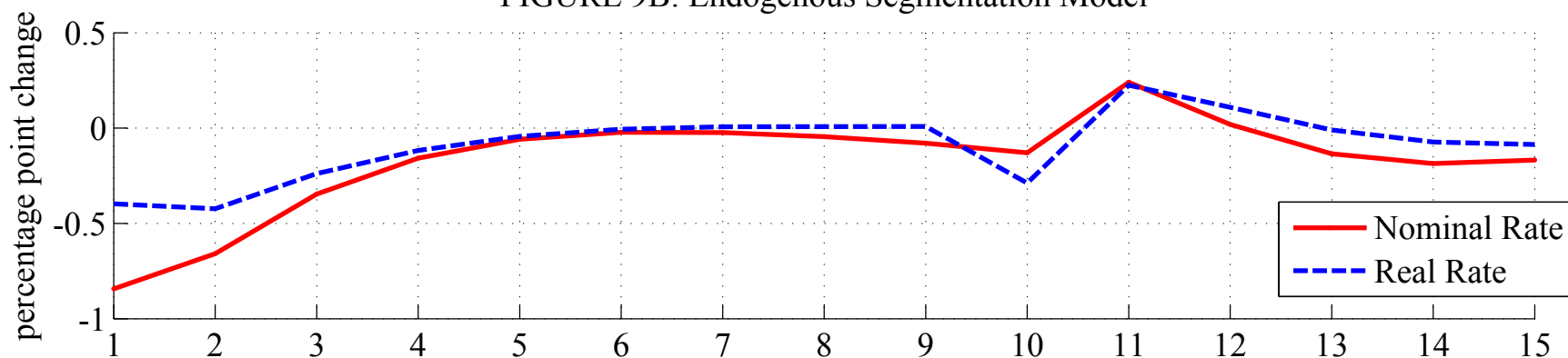


FIGURE 9C: Time-Dependent Model

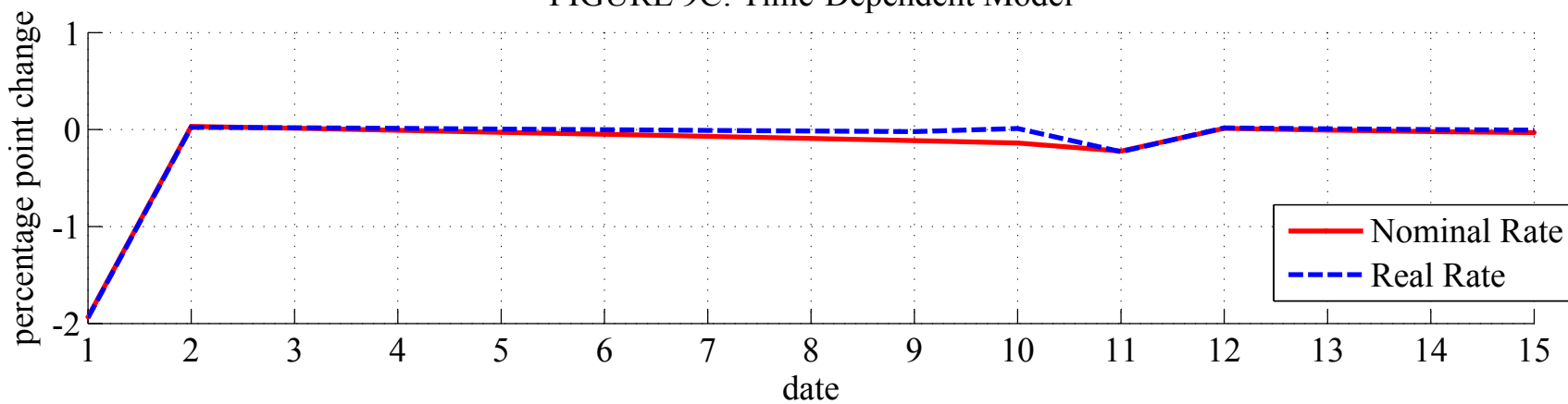
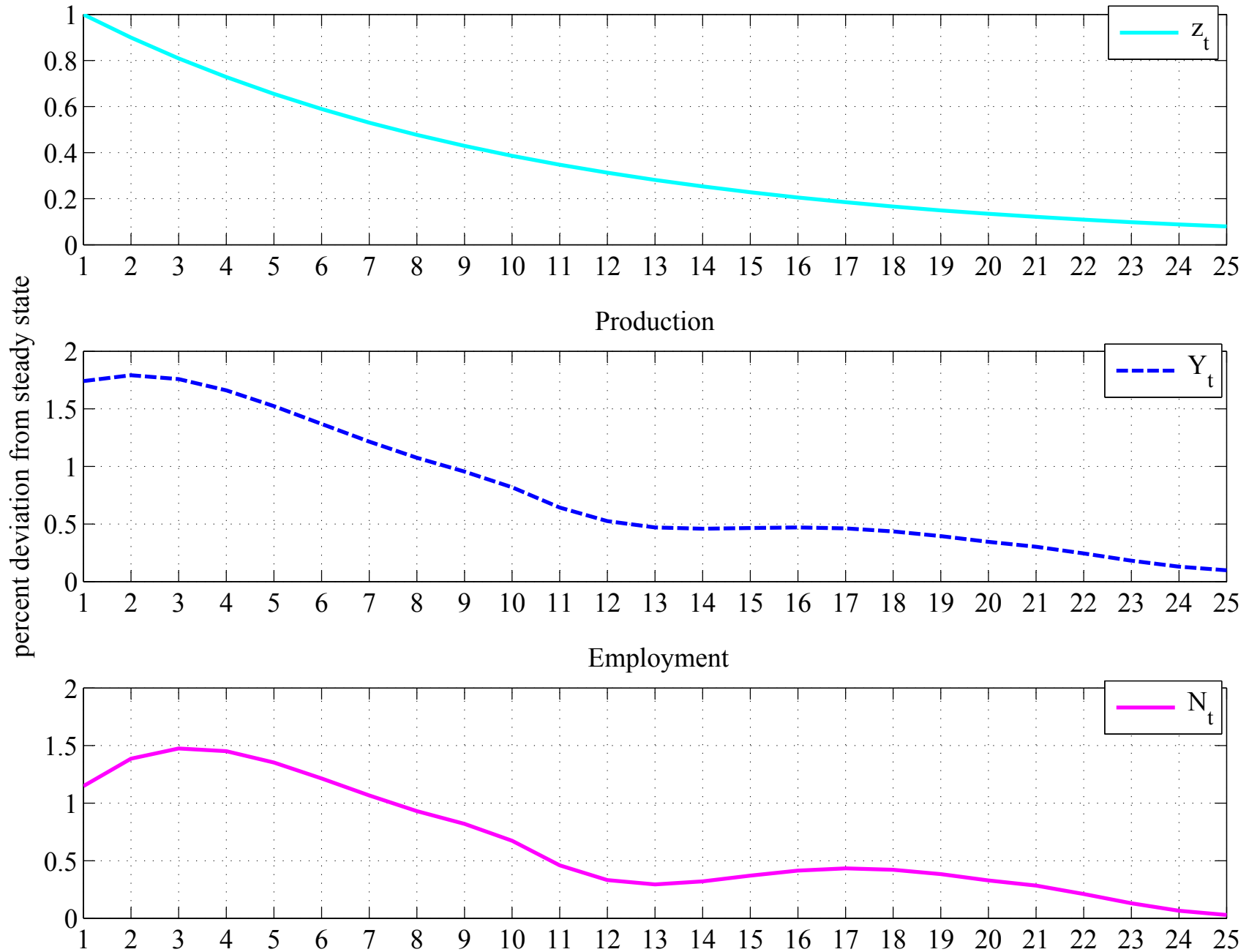


Figure 10: Employment and Production with a Taylor Rule



Appendix

A Risk sharing

As described in the main text, bonds issued by a competitive financial intermediary are contingent on both aggregate and idiosyncratic shocks. The supply of these bonds is created using purchases of government bonds and free entry into intermediation implies a zero-profit condition.

Our first lemma derives the equilibrium price of households' bonds. Given the aggregate history s^t , the intermediary's profit when next period's aggregate shock is s_{t+1} solves the problem in (8) - (9). There, recall that households identified by the history ξ^t purchase $B(s^t, s_{t+1}, \xi^t, \xi_{t+1})$ units of nominal bonds that pay one unit of currency next period if the aggregate shock is given by s_{t+1} and their idiosyncratic shock is ξ_{t+1} , and the measure of such households is $h(\xi^t)$.

Lemma 1 *The equilibrium price of state-contingent bonds issued by the financial intermediary, $q(s^t, s_{t+1}, \xi_{t+1})$, is given by $q(s^t, s_{t+1}, \xi_{t+1}) = q(s^t, s_{t+1}) h(\xi_{t+1})$.*

Proof. *Substitution of the right-hand side of (9) for $B(s^t, s_{t+1})$ in (8) gives*

$$\begin{aligned}
 & \int_{\xi^t} \int_{\xi_{t+1}} q(s^t, s_{t+1}, \xi_{t+1}) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi^t) d\xi_{t+1} d\xi^t \\
 & - q(s^t, s_{t+1}) \int_{\xi^t} \int_{\xi_{t+1}} B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi^t) h(\xi_{t+1}) d\xi_{t+1} d\xi^t \\
 = & \int_{\xi^t} \int_{\xi_{t+1}} \left(q(s^t, s_{t+1}, \xi_{t+1}) - q(s^t, s_{t+1}) h(\xi_{t+1}) \right) h(\xi^t) B(s^t, s_{t+1}, \xi^t, \xi_{t+1}) d\xi_{t+1} d\xi^t \\
 = & 0.
 \end{aligned}$$

The last equality follows from the zero profit condition. ■

B Characterizing household behavior under risk-sharing

This section provides the main results we use in our numerical approach to solving the model. Lemma 2 proves that, given risk-sharing in the brokerage account, beginning of period money balances capture all relevant differences across households at the start of any period. Additional results in lemma 3 establish that whenever households access their brokerage account, there is uniformity of actions in that they choose the same consumption and bank account balance regardless of their past history of idiosyncratic shocks. The importance of this is that any differences across households will be limited by the number of periods since active transactions between the bank and brokerage account. Such differences disappear whenever households with different bank account balances access their brokerage account at the same time. Finally, lemma 4 proves that households' follow a threshold rule in determining the timing of their account transfers; specifically, it shows that households choose to become active whenever their current fixed transfer cost falls below some threshold value that is common to all households with the same beginning-of-period bank balance.

We begin by deriving a lifetime budget constraint associated with the brokerage account. Using lemma 1, the brokerage budget constraint for any household in the initial period is:

$$\bar{B} \geq \int_{s_1} \int_{\xi_1} q(s_1) h(\xi_1) B(s_1, \xi_1) d\xi_1 ds_1. \tag{24}$$

In period 1, the brokerage account constraint is:

$$B(s_1, \xi_1) \geq \int_{s_2} \int_{\xi_2} q(s_1, s_2) h(\xi_2) B(s_1, s_2, \xi_1, \xi_2) d\xi_2 ds_2 \\ - (1 - \lambda) P(s_0) y(s_0) + \left(x(s_1, \xi_1) + P(s_1) \xi_1 \right) z(s_1, \xi_1)$$

which, substituted into (24), gives:

$$\bar{B} \geq \int_{s_1} \int_{\xi_1} q(s_1) h(\xi_1) \left[\int_{s_2} \int_{\xi_2} q(s_1, s_2) h(\xi_2) B(s_1, s_2, \xi_1, \xi_2) d\xi_2 ds_2 - (1 - \lambda) P(s_0) y(s_0) \right. \\ \left. + \left(x(s_1, \xi_1) + P(s_1) \xi_1 \right) z(s_1, \xi_1) \right] d\xi_1 ds_1 \\ = \int_{s^2} \int_{\xi^2} q(s^2) h(\xi^2) B(s_1, s_2, \xi_1, \xi_2) d\xi^2 ds^2 - \int_{s_1} \int_{\xi_1} q(s_1) h(\xi_1) \left[(1 - \lambda) P(s_0) y(s_0) \right. \\ \left. - \left(x(s_1, \xi_1) + P(s_1) \xi_1 \right) z(s_1, \xi_1) \right] d\xi_1 ds_1,$$

where we use the notation $q(s^t) = q(s_1) \cdot q(s_1, s_2) \cdots q(s^{t-1}, s_t)$ and $h(\xi^t) = h(\xi_1) \cdots h(\xi_t)$. Repeated substitutions for $B(s^j, \xi^j)$ for $j = 2, \dots, t$ using (3) leads to the following equation.

$$\bar{B} \geq \int_{s^t} \int_{\xi^t} q(s^t) h(\xi^t) B(s^t, \xi^t) d\xi^t ds^t - \sum_{j=1}^{t-1} \int_{s^j} \int_{\xi^j} q(s^j) h(\xi^j) \left[(1 - \lambda) P(s^{j-1}) y(s^{j-1}) \right. \\ \left. - \left(x(s^j, \xi^j) + P(s^j) \xi_j \right) z(s^j, \xi^j) \right] d\xi^j ds^j.$$

Taking the limit of the above equation, given the No-Ponzi condition,

$$\lim_{t \rightarrow \infty} \int_{s^t} \int_{\xi^t} q(s^t) h(\xi^t) B(s^t, \xi^t) ds^t d\xi^t \geq 0,$$

we arrive at the following lifetime budget constraint associated with the brokerage account.

$$\bar{B} \geq \sum_{t=1}^{\infty} \int \int q(s^t) h(\xi^t) \left(z(s^t, \xi^t) \left[x(s^t, \xi^t) + P(s^t) \xi_t \right] - P(s^{t-1}) (1 - \lambda) y(s^{t-1}) \right) d\xi^t ds^t \quad (25)$$

Equation 25, which is an immediate implication of the availability of a complete set of state-contingent bonds for each household, implies that individual histories are irrelevant beyond current money balances in the bank account. This intuitive property of the model is straightforward, if notationally cumbersome, to prove and is important in our approach to characterizing competitive equilibrium.

Lemma 2 *Given $M(s^{t-1}, \xi^{t-1})$, the decisions $c(s^t, \xi^t)$, $A(s^t, \xi^t)$, $x(s^t, \xi^t)$ and $z(s^t, \xi^t)$ are independent of the history ξ^{t-1} .*

Proof. Equation 25 is equivalent to the following sequence of period by period constraints,

$$B(s^t) = \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}) ds_{t+1} - (1 - \lambda) P(s^{t-1}) y(s^{t-1}) \quad (26) \\ + \int_{\xi^t} h(\xi^t) \left[x(s^t, \xi^t) + P(s^t) \xi_t \right] z(s^t, \xi^t) d\xi^t,$$

and the limit condition

$$\lim_{t \rightarrow \infty} \int_{s^t} q(s^t) B(s^t) ds^t = 0. \quad (27)$$

Let $\xi_t^{t+j} = \{\xi_t, \dots, \xi_{t+j}\}$ be a partial history of individual household shocks, fix (s^t, ξ^t) and define pointwise the continuation value to the household's problem given $(B(s^t), M(s^{t-1}, \xi^{t-1}))$ as

$$V(B_0, M_0; s^t, \xi^t) = \max \sum_{j=0}^{\infty} \beta^j \int_{s_t^{t+j}} \int_{\xi_t^{t+j}} u(c[s^{t+j}, \xi_t^{t+j}]) g(s^{t+j} | s^t) h(\xi_t^{t+j}) ds_t^{t+j} d\xi_t^{t+j}$$

subject to:

$$B(s^{t+j}) = \int_{s_{t+j+1}} q(s^{t+j}, s_{t+j+1}) B(s^{t+j}, s_{t+j+1}) ds_{t+j+1} - (1-\lambda) P(s^{t+j-1}) y(s^{t+j-1}) \\ + \int_{\xi_t^{t+j}} h(\xi_t^{t+j}) \left[x(s^{t+j}, \xi_t^{t+j}) + P(s^{t+j}) \xi_{t+j} \right] z(s^{t+j}, \xi_t^{t+j}) d\xi_t^{t+j}, \text{ for } j = 0, \dots,$$

$$P(s^{t+j}) c(s^{t+j}, \xi_t^{t+j}) \leq M(s^{t+j-1}, \xi_{t-1}^{t+j-1}) + x(s^{t+j}, \xi_t^{t+j}) z(s^{t+j}, \xi_t^{t+j}) - A(s^{t+j}, \xi_t^{t+j}), \text{ for } j = 0, \dots,$$

$$M(s^{t+j}, \xi_t^{t+j}) \leq A(s^{t+j}, \xi_t^{t+j}) + \lambda P(s^{t+j}) y(s^{t+j}) \text{ and } A(s^{t+j}, \xi_t^{t+j}) \geq 0, \text{ for } j = 0, \dots,$$

$$\text{given } B(s^t) = B_0, M(s^{t-1}, \xi_{t-1}^{t-1}) = M_0 \text{ and } \lim_{j \rightarrow \infty} \int_{s_t^{t+j}} q(s^{t+j}) B(s^{t+j}) ds_t^{t+j} = 0.$$

Clearly the optimal choices of $c(s^{t+k}, \xi_t^{t+k})$, $A(s^{t+k}, \xi_t^{t+k})$, $x(s^{t+k}, \xi_t^{t+k})$ and $z(s^{t+k}, \xi_t^{t+k})$, for any $k = 0, 1, \dots$, are independent of ξ^{t-1} given M_0 . Let $c(s^{t+k}, \xi_t^{t+k}; M_0)$, $A(s^{t+k}, \xi_t^{t+k}; M_0)$, $x(s^{t+k}, \xi_t^{t+k}; M_0)$ and $z(s^{t+k}, \xi_t^{t+k}; M_0)$ describe these choices. Next, any solution to the household's problem must also solve the following problem.

$$\max \sum_{j=1}^{t-1} \beta^{j-1} \int_{s^j} \int_{\xi^j} u(c(s^j, \xi^j)) g(s^j) h(\xi^j) ds^j d\xi^j \\ + \beta^t \int_{s^t} \int_{\xi^t} V(B(s^t), M(s^{t-1}, \xi^{t-1}); s^t, \xi^t) g(s^t) h(\xi^t) ds^t d\xi^t$$

subject to

$$B(s^j) = \int_{s_{j+1}} q(s^j, s_{j+1}) B(s^j, s_{j+1}) ds_{j+1} - (1-\lambda) P(s^{j-1}) y(s^{j-1}) \\ + \int_{\xi^j} h(\xi^j) \left[x(s^j, \xi^j) + P(s^j) \xi_j \right] z(s^j, \xi^j) d\xi^j,$$

$$P(s^j) c(s^j, \xi^j) \leq M(s^{j-1}, \xi^{j-1}) + x(s^j, \xi^j) z(s^j, \xi^j) - A(s^j, \xi^j),$$

$$M(s^j, \xi^j) \leq A(s^j, \xi^j) + \lambda P(s^j) y(s^j) \text{ and } A(s^j, \xi^j) \geq 0, \text{ for } j = 1, \dots, t-1.$$

In this problem, the optimal choices of $c(s^{t+k}, \xi_t^{t+k})$, $A(s^{t+k}, \xi_t^{t+k})$, $x(s^{t+k}, \xi_t^{t+k})$ and $z(s^{t+k}, \xi_t^{t+k})$, for any $k = 0, 1, \dots$, are given by the functions $c(s^{t+k}, \xi_t^{t+k}; M(s^{t-1}, \xi^{t-1}))$,

$A(s^{t+k}, \xi_t^{t+k}; M(s^{t-1}, \xi^{t-1}))$, $x(s^{t+k}, \xi_t^{t+k}; M(s^{t-1}, \xi^{t-1}))$ and $z(s^{t+k}, \xi_t^{t+k}; M(s^{t-1}, \xi^{t-1}))$, which attained the maximum value $V(B(s^t), M(s^{t-1}, \xi^{t-1}); s^t, \xi^t)$. ■

We proceed to further characterize household behavior by studying the state-contingent plans chosen by households in period 0 when they are all identical. Let Λ denote the multiplier associated with equation 25 and $\nu_0(s^t, \xi^t)$ be the LaGrange multiplier for (4) and $\nu_1(s^t, \xi^t)$ the multiplier for (5). For clarity, the Lagrangian is shown below.

$$\begin{aligned} \mathcal{L} = & \sum_{t=1}^{\infty} \beta^{t-1} \int_{s^t} \int_{\xi^t} \left(u(c(s^t, \xi^t)) g(s^t) h(\xi^t) \right. \\ & + \nu_0(s^t, \xi^t) g(s^t) h(\xi^t) \left[M(s^{t-1}, \xi^{t-1}) + x(s^t, \xi^t) z(s^t, \xi^t) - A(s^t, \xi^t) - P(s^t) c(s^t, \xi^t) \right] \\ & + \nu_1(s^t, \xi^t) g(s^t) h(\xi^t) \left[A(s^t, \xi^t) + \lambda P(s^t) y(s^t) - M(s^t, \xi^t) \right] \Big) d\xi^t ds^t \\ & + \Lambda \left(\bar{B} + \sum_{t=1}^{\infty} \int_{s^t} q(s^t) \left[(1 - \lambda) P(s^{t-1}) y(s^{t-1}) ds^t \right. \right. \\ & \left. \left. - \sum_{t=1}^{\infty} \int_{s^t} \int_{\xi^t} q(s^t) h(\xi^t) \left(x(s^t, \xi^t) + P(s^t) \xi_t \right) z(s^t, \xi^t) d\xi^t ds^t \right) \right) \end{aligned}$$

Given any choice of $z(s^t, \xi^t)$, the household's choices of $c(s^t, \xi^t)$, $A(s^t, \xi^t)$, $M(s^t, \xi^t)$ and $x(s^t, \xi^t)$, satisfy the following conditions.

$$Du(c(s^t, \xi^t)) - P(s^t) \nu_0(s^t, \xi^t) = 0 \quad (28)$$

$$-\nu_0(s^t, \xi^t) + \nu_1(s^t, \xi^t) \leq 0, \quad (29)$$

$$= 0 \text{ if } A(s^t, \xi^t) > 0,$$

$$-\nu_1(s^t, \xi^t) + \beta \int_{s_{t+1}} \int_{\xi_{t+1}} \nu_0(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi_{t+1}) g(s_{t+1} | s^t) ds_{t+1} d\xi_{t+1} \leq 0 \quad (30)$$

$$= 0 \text{ if } M(s^t, \xi^t) > \lambda P(s^t) y(s^t)$$

$$\beta^{t-1} \nu_0(s^t, \xi^t) g(s^t) h(\xi^t) - \Lambda q(s^t) h(\xi^t) = 0 \text{ if } z(s^t, \xi^t) = 1. \quad (31)$$

The following lemma shows that households choose the same consumption and subsequent money balances for their bank account whenever they pay transfer costs to access their brokerage accounts. Recall that this result is important for the approach we take to characterizing household behavior in that it implies that heterogeneity across households is limited to periods when they are inactive.

Lemma 3 *For any (s^t, ξ^t) in which $z(s^t, \xi^t) = 1$, $c(s^t, \xi^t)$, $A(s^t, \xi^t)$ and $M(s^t, \xi^t)$ are independent of ξ^t .*

Proof. Given $z(s^t, \xi^t) = 1$, (28) and (31) imply

$$Du(c(s^t, \xi^t)) = \frac{\Lambda q(s^t) P(s^t)}{\beta^{t-1} g(s^t)}. \quad (32)$$

As Λ is the same for all households, this proves that $c(s^t, \xi^t)$ is independent of ξ^t . Next, it is sufficient to examine the case where $A(s^t, \xi^t) > 0$ and therefore $M(s^t, \xi^t) > \lambda P(s^t) y(s^t)$. In this case, (29) and (30) together give

$$\nu_0(s^t, \xi^t) = \beta \int_{s_{t+1}} \int_{\xi_{t+1}} \nu_0(s^t, s_{t+1}, \xi^t, \xi_{t+1}) h(\xi_{t+1}) g(s_{t+1} | s^t) ds_{t+1} d\xi_{t+1},$$

which, using (28) and (32) leads to the expression,

$$Du(c(s^t, \xi^t)) = \beta \int_{s_{t+1}} \int_{\xi_{t+1}} Du(c(s^{t+1}, \xi^{t+1})) \left[\frac{P(s^{t+1})}{P(s^t)} \right]^{-1} h(\xi_{t+1}) g(s_{t+1} | s^t) ds_{t+1} d\xi_{t+1}.$$

Since this expression must hold for all ξ^t , it follows that $c(s^{t+1}, \xi^{t+1})$ is not a function of ξ^t which, given (4) and (5), implies that $A(s^t, \xi^t)$ and thus $M(s^t, \xi^t)$ must be independent of ξ^t . ■

Our final result shows that households follow threshold cost policies with respect to the costs of accessing their brokerage accounts

Lemma 4 *For any (s^t, ξ^{t-1}) , $\mathcal{A} = \{\xi_t | z(s^t, \xi^t) = 1\}$ is a convex set bounded below by 0.*

Proof. Assume not, let $\tilde{\mathcal{A}} = \{\xi_t | z(s^t, \xi^t) = 1\}$ and $m_{\tilde{\mathcal{A}}} = \int_{\tilde{\mathcal{A}}} h(\xi) d\xi$. Define $\bar{\xi}$ implicitly using the equation $\int_0^{\bar{\xi}} h(\xi) d\xi = m_{\tilde{\mathcal{A}}}$. By the Axiom of Choice there exists a 1 - 1 function $\mathcal{S} : \tilde{\mathcal{A}} \rightarrow [0, \bar{\xi}]$. Given $B(s^t)$ and $M(s^{t-1}, \xi^{t-1})$, for each $\xi_0 \in \tilde{\mathcal{A}}$ and $\xi = \mathcal{S}(\xi_0)$, construct an alternate continuation plan that assigns the original continuation plan associated with ξ_0 to ξ . In other words, $c^a(s^t, \xi^{t-1}, \xi) = c(s^t, \xi^{t-1}, \xi_0)$, $A^a(s^t, \xi^{t-1}, \xi) = A(s^t, \xi^{t-1}, \xi_0)$ and so on. This alternative plan satisfies (3) - (6) and is thus feasible. Moreover, by construction, it offers the same expected lifetime utility as the original plan. Any solution to the household's problem, solving (2) subject to (3) - (6), must satisfy (25) with equality. Since the original plan solved the household's problem by assumption, the alternative plan satisfies (25) as a strict inequality since

$$\begin{aligned} & \int_{\xi^{t-1}} q(s^t) h(\xi^{t-1}) \int_0^{\bar{\xi}} (x^a(s^t, \xi^t) + P(s^t) \xi_t) d\xi d\xi^{t-1} \\ & < \int_{\xi^t} q(s^t) h(\xi^t) (x(s^t, \xi^t) + P(s^t) \xi_t) z(s^t, \xi^t) d\xi^t ds^t. \end{aligned}$$

This contradicts optimality of the original plan. ■