Recursive Equilibrium in Endogenous Growth Models with Incomplete Markets

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Abstract

This paper analyzes the existence of recursive equilibria in a class of convex growth models with incomplete markets. Households have identical CRRA-preferences, production displays constant returns to scale with respect to physical and human capital, and all markets are competitive. There are aggregate productivity shocks that affect the aggregate returns to physical and human capital investment (stock returns and wages), and there are idiosyncratic shocks to human capital (idiosyncratic depreciation shocks) that only affect individual human capital returns. For a given history of aggregate shocks, these idiosyncratic human capital shocks are independently distributed over time and identically distributed across agents. Finally, households have the opportunity to trade assets in zero net supply with payoffs that depend on the aggregate shock, but markets are incomplete in the sense that there are no assets with payoffs depending on idiosyncratic shocks. It is shown that there exist recursive equilibria that are simple in the sense that equilibrium prices (returns) only depend on exogenous shocks. Moreover, the equilibrium allocations of the incomplete-markets economy are identical to the equilibrium allocations of an economy in which households live in autarky facing both aggregate and idiosyncratic risk.

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I. Introduction

Recent work on dynamic general equilibrium models with infinitely-lived agents and uninsurable idiosyncratic risk has provided important insights into the macroeconomic effects of market incompleteness (Ljungqvist and Sargent, 2000). One drawback of this incomplete-markets approach to macroeconomics is that recursive equilibria are in general difficult to compute even for simple economic environments.[1] A second shortcoming is the lack of a general existence proof.[2] This paper presents a tractable macroeconomic model with incomplete markets that avoids some of the shortcomings of the previous literature. More specifically, this paper shows that for the incomplete-markets model developed here, there are always recursive equilibria that are simple in the sense that endogenous equilibrium prices (asset returns) only depend on exogenous shocks. This simplicity of equilibrium means that issues of existence and comparative dynamics can be studied at a level of generality comparable to the complete-markets literature, and that many quantitative applications are computationally straightforward.

The model is an incomplete-markets version of the class of convex growth models analyzed by, among others, Alvarez and Stokey (1998), Caballe and Santos (1993), Jones and Manuelli (1990), and Rebelo (1991).[3] More specifically, households have identical CRRA-preferences, production displays constant returns to scale with respect to reproducible input factors, and all markets are competitive. For the sake of concreteness, this paper considers the case of two

[1]For applied work relying on computational methods, see, for example, Aiyagari (1994), denHaan (1997), Heaton and Lucas (1996), Huggett (1993), and Krusell and Smith (1998).

[2]Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) show the existence of stationary recursive (Markov) equilibria for exchange economies, but they rule out short-sales (borrowing) by assumption and use a state space that includes endogenous variables in addition to the wealth distribution. Becker and Zilcha (1997) and Reffett, Mirman, and Morand (2002) prove the existence of recursive equilibria for neoclassical production economies with no assets except physical capital (no bonds).

[3]Lucas (1988) considers a human capital model with externalities.

input factors, namely physical and human capital. There are aggregate productivity shocks that affect aggregate returns to physical and human capital investment (stock returns and wages), and there are idiosyncratic human capital shocks (depreciations shocks) that only affect individual human capital returns. Conditional on the history of aggregate shocks, these idiosyncratic human capital shocks are independently distributed over time and indentically distributed across households. Finally, the financial market structure is incomplete in the sense that there are no assets with payoffs that depend on idiosyncratic shocks. However, households have the opportunity to trade stocks (accumulate physical capital) and any asset in zero net supply with payoffs that depend on the aggregate shock variable (bonds). In particular, all households can borrow and lend at the common risk-free rate. Moreover, households' ability to trade existing assets is only limited by their ability to repay their debt in the future. In short, the only market imperfection is the lack of explicit insurance markets for idiosyncratic human capital risk.[4]

This paper shows that there exist simple recursive equilibria in which endogenous asset returns (prices) only depend on the exogenous aggregate state. In particular, neither the endogenous wealth distribution nor idiosyncratic shocks affect equilibrium returns. Moreover, the equilibrium allocation of the incomplete-markets economy is also the allocation that obtains when households live in autarky facing both aggregate and idiosyncratic shocks. That is, the equilibrium allocation can be found by solving a one-agent decision problem. Thus, the incomplete-markets model analyzed in this paper is as tractable as its complete-markets counterpart. However, whereas idiosyncratic risk does not affect the equilibrium allocation when markets are complete, it does affect the equilibrium allocation in the incomplete-markets model. Consequently, the two models may lead to very different policy conclusions.

[4]More formally, this paper proves the existence of sequential equilibria with a recursive structure in which borrowing (debt) constraints never bind (Hernandez and Santos, 1996, Levine and Zame 1996, and Magill and Quinzii, 1994).

For example, whereas social insurance of idiosyncratic risk has no effect on growth and welfare in the complete-markets economy, it has a substantial effect in the incomplete-markets economy (Krebs, 2001). Moreover, the welfare cost of business cycles are likely to be much larger when markets for idiosyncratic risk are incomplete (Krebs, 2002a).

Two properties of the model are essential in deriving the characterization and existence result. First, in equilibrium the ratio of physical to human capital (capital-to-labor ratio) is identical across households regardless of their current wealth or current idiosyncratic shock realization, which implies the existence of a reduced-form production function that is linear. Second, households choose not to trade the assets in zero-net-supply. This no-trade result extends the work by Constantinides and Duffie (1996) to production economies. In accordance with Constantinides and Duffie (1996), the current paper emphasizes the importance of permanent income shocks in the sense that income follows (approximately) a logarithmic random walk. Thus, neither borrowing and lending nor self-insurance is an optimal response to idiosyncratic income shocks. However, in contrast to Constantinides and Duffie (1996), this paper derives the random walk property of income as an endogenous outcome.[5]

In addition to the work by Constantinides and Duffie (1996), there are further examples of tractable models with incomplete-markets and infinitely-lived agents in the literature. Magill and Quinzii (2000) consider a model with quadratic preferences (certainty-equivalence) and Angeletos and Calvet (2001) and Davis and Willen (2001) assume exponential utility and normally distributed shocks. In contrast, the current paper assumes homothetic preferences, which is the standard assumption in the growth and business cycle literature. In this sense, the model presented here seems better suited for macroeconomic analysis. Finally, there is the work by Woodford (1986) who considers a model with two infinitely-lived agents that is isomorphic to a two-period OLG-model because individual endowments fluctuate

[5]The random walk property implies that equilibrium consumption is always unbounded, which is why the non-existence argument of Krebs (2002b) does not apply.

deterministically. In contrast to this work, the current model allows for a wide range of distributions of idiosyncratic shocks, a feature that is essential when calibrating the model based on household-level consumption and income data (Krebs 2001, 2002a).[6]

II. Model

II.A. Economy

Time is indexed by $t = 0, 1, \ldots$ and individual households by $i = 1, \ldots, I$. A complete description of the exogenous state of the economy in period t is a vector $(s_{1t}, \ldots, s_{It}, S_t)$, where we interpret s_{it} as a household-specific (idiosyncratic) shock and S_t as an economy-wide (aggregate) shock. We assume that s_{it} is an element of a time- and household-independent set $\mathbf{s}$, and that S_t is an element of a time-invariant set $\mathbf{S}$. To avoid mathematical technicalities, the formal proofs also assume that the two sets $\mathbf{s}$ and $\mathbf{S}$ are finite. We denote the vector of idiosyncratic shocks by $s_t = (s_{1t}, \ldots, s_{It})$. A (partial) history of idiosyncratic, respectively aggregate, shocks is denoted by $s^t = (s_0, \ldots, s_t)$, respectively $S^t = (S_0, \ldots, S_t)$. Clearly, the ordered set of all histories defines an event tree with date-events (nodes) (s^t, S^t).

The process of exogenous shocks, $\{s_t, S_t\}$, is a Markov process with stationary transition probabilities denoted by $\pi(s_{t+1}, S_{t+1}|s_t, S_t)$ or $\pi(s', S'|s, S)$. We make two assumptions on these transition probabilities. First, idiosyncratic shocks have no predictive power: $\pi(s_{t+1}, S_{t+1}|s_t, S_t) = \pi(s_{t+1}, S_{t+1}|S_t)$. Second, households are ex-ante identical in the sense that $\pi(\ldots, s_{i,t+1}, \ldots, s_{i',t+1}, \ldots, S_{t+1}|S_t) = \pi(\ldots, s_{i',t+1}, \ldots, s_{i,t+1}, \ldots, S_{t+1}|S_t)$. For simplicity, we also assume $\pi(s', S'|s, S) > 0$ for all $(s, S) \in \mathbf{s} \times \mathbf{S}$. The transition probabilities in conjunction with the initial distribution define in the canonical way the node-probabilities

[6] There is also a literature that rules out trading in any financial asset by assumption (for example, Obstfeld, 1994, and Benabou, 2002). In contrast, in this paper households have the opportunity to trade assets with aggregate payoffs, even though in equilibrium households optimally choose not to take advantage of this opportunity.

$\pi(s^t, S^t)$ and the conditional node-probabilities $\pi(s^{t+n}, S^{t+n}|s^t, S^t)$. Notice that our assumptions imply that conditional on the history of aggregate shocks, idiosyncratic shocks are independently distributed over time and identically distributed across agents. That is, conditional on S^{t+1}, the distribution of $s_{i,t+1}$ is independent of s_i^t (or s^t for that matter) and the same for all i. Note also that our formulation allows for the possibility that s_{it} and s_{jt} are correlated.

Assumption 1. The exogenous shock process, $\{s_t, S_t\}$, is a Markov chain with transition probabilities satisfying $\pi(\ldots, s_{i,t+1}, \ldots, s_{i',t+1}, \ldots, S_{t+1}|S_t) = \pi(\ldots, s_{i',t+1}, \ldots, s_{i,t+1}, \ldots, S_{t+1}|S_t)$ and $\pi(s_{t+1}, S_{t+1}|s_t, S_t) = \pi(s_{t+1}, S_{t+1}|S_t)$.

Economic variables at time t are often defined by functions $x_t : (\mathbf{s})^{t+1} \times (\mathbf{S})^{t+1} \to \mathbb{R}^n$, $x_t = x_t(s^t, S^t)$. Any function x_t defines a random variable in the canonical way. For this random variable, we denote the unconditional expectation by $E[x_t] = \sum_{s^t,S^t} \pi(s^t, S^t)x_t(s^t, S^t)$ and the conditional expectation by $E[x_{t+n}|s^t, S^t] = \sum_{s^t,S^t} \pi(s^{t+n}, S^{t+n}|s^t, S^t)x_{t+n}(s^{t+n}, S^{t+n})$.

There is one firm that produces an "all-purpose" good which can be used for consumption, investment in physical capital, and investment in human capital. If the firm employs K_t units of physical capital and H_t units of human capital in period t, then it produces $Y_t = A_t F(K_t, H_t)$ units of the good in period t. Here F is a standard neoclassical production function. More specifically, we assume that F displays constant-returns-to-scale, is twice continuously differentiable, strictly increasing, strictly concave, and satisfies $F(0, H) = F(K, 0) = 0$ as well as $\lim_{K\to 0} F_k(K, H) = \lim_{H\to 0} F_h(K, H) = +\infty$ and $\lim_{K\to\infty} F_k(K, H) = \lim_{H\to\infty} F_h(K, H) = 0$. Total factor productivity is a function $A : \mathbf{S} \to \mathbb{R}_{++}$ that assigns to each aggregate state S_t a (strictly positive) productivity level $A_t = A(S_t)$.

Assumption 2. Output is produced according to $Y_t = A_t F(K_t, H_t)$, where F is a standard neoclassical production function (in particular, it displays constant-returns-to-scale) and $A_t = A(S_t)$.

The firm rents input factors (physical and human capital) in competitive markets. We denote the rental rate of physical capital by $\tilde{r}_{kt}$ and the rental rate of human capital (the wage rate per efficiency unit of labor) by $\tilde{r}_{ht}$. In each period, the firm hires capital and labor up to the point where current profit is maximized. Thus, the firm solves the following static maximization problem:

$$max_{K_t,H_t} \{ A_t F(K_t, H_t) - \tilde{r}_{kt} K_t - \tilde{r}_{ht} H_t \} . \tag{1}$$

Let k_{it} and h_{it} stand for the stock of physical and human capital owned by household i at the beginning of period t, and denote the corresponding investment levels by x_{kit} and x_{hit}. If we denote household $i's$ consumption by c_{it}, then the sequential budget constraint reads:

$$\begin{aligned} c_{it} + x_{kit} + x_{hit} &= \tilde{r}_{kt} k_{it} + \tilde{r}_{ht} h_{it} \\ k_{i,t+1} &= (1 - \delta_{kt}) k_{it} + x_{kit} \quad , \quad k_{it} \geq 0 \\ h_{i,t+1} &= (1 - \delta_{ht} + \eta_{it}) h_{it} + x_{hit} \quad , \quad h_{it} \geq 0 \\ & \quad (k_{i0}, h_{i0}) \ given \ . \end{aligned} \tag{2}$$

In (2) δ_{kt} and δ_{ht} denote the average depreciation rate of human and physical capital, respectively. These average depreciation rates are defined by functions $\delta_k : \mathbf{S} \rightarrow \mathbb{R}_+$ and $\delta_h : \mathbf{S} \rightarrow \mathbb{R}_+$ assigning to each aggregate shock $S_t \, \epsilon \, \mathbf{S}$ a deprecation rate $\delta_{kt} = \delta_k(S_t)$, respectively $\delta_{ht} = \delta_h(S_t)$. The term η_{it} denotes a household-specific shock to the stock of human capital and is defined by a function $\eta : \mathbf{s} \times \mathbf{S} \rightarrow \mathbb{R}$ assigning to each $(s, S) \, \epsilon \, \mathbf{s} \times \mathbf{S}$ a realization $\eta_{it} = \eta(s_{it}, S_t)$. Notice that we allow for the possibility that $\eta_{it} > 0$.[7] Since $\tilde{r}_{ht} \eta_{it}$ is labor

[7]Of course, we restrict the depreciation functions so that the depreciation rate of physical and human capital never exceeds 100 percent.

income of household i, the random variable η_{it} determines the nature of idiosyncratic labor income risk.

Assumption 3. The depreciation shocks are defined by $\delta_{kt} = \delta_k(S_t)$, $\delta_{ht} = \delta_h(S_t)$, and $\eta_{it} = \eta(s_{it}, S_t)$.

Some remarks on the formulation of the budget constraint (2) are in order.

Remark 1. The model does not distinguish between general and specific human capital. Similarly, the idiosyncratic shocks to human capital, η_{it}, could be either shocks to general or shocks to specific human capital. A negative human capital shock, $\eta_{it} < 0$, can occur when a displaced worker loses firm- or sector-specific human capital. Jovanovic (1979) and Ljungvist and Sargent (1998) analyze search models with specific human capital and idiosyncratic shocks to human capital, but they assume risk-neutral workers and do not model the accumulation of human capital. A decline in health (disability) provides a second example for a negative human capital shock. In this case, both general and specific human capital might be lost. Internal promotions and upward movement in the labor market provide two examples of positive human capital shocks ($\eta_{it} > 0$).

Remark 2. The budget constraint (2) assumes that the wage is paid in each period. Thus, if we interpret the $\eta-$shock as the skill loss of a displaced worker, we abstract from the foregone wage during the period of unemployment and focus on the (permanent) difference between the wage before job displacement and the wage after job displacement. Empirically, the permanent component of this wage differential is quite large (Jacobson, LaLonde, and Sullivan 1993, Neal 1995, and Topel 1991).

Remark 3. Investment in human capital is often modeled as time investment. This is equivalent to formulation (2) if $Y_{1t} = A_t F(K_{1t}, H_t L_t)$ is the quantity of goods produced (consumption plus physical capital) and $Y_{2t} = A_t F(K_{2t}, H_t(1 - L_t))$ is the quantity of human capital produced, where L_t is the time spent producing the good. Clearly, it is the joint assumption that the two production functions exhibit constant-returns-to-scale and are identical that drives the result.

Remark 4. So far, there is no labor-leisure choice. This extension is briefly discussed in Section V.

Remark 5. Equation (2) does not impose a non-negativity constraint on human capital investment ($x_{hit} \geq 0$). In equilibrium, this non-negativity constraint will not be violated if positive human capital shocks are not too large. This can immediately be inferred from the Corollary.

Remark 6. To simplify the analysis, we do not explicitly mention financial markets. However, the equilibrium allocation of the above economy in which households accumulate physical capital is also the equilibrium allocation of a stock market economy in which the firm is a stock company that makes the intertemporal investment decision.[8] If we normalize the number of outstanding shares to one, the stock price is $Q_t = K_{t+1}$, household $i's$ equity share is $\theta_{i,t+1} Q_t = k_{i,t+1}$, and the return to equity investment is $\tilde{r}_{kt} - \delta_{kt}$. Moreover, the equilibrium allocation is unchanged if households are given the opportunity to trade $j = 1, \ldots, J$ securities in zero net supply with payoffs $D_{jt} = D_j(S_t)$. In particular, the introduction of a

[8] In general, this type of market arrangement might lead to conceptual problems when markets are incomplete because shareholders (households) do not agree on the optimal investment policy (Magill and Quinzii, 1996). This, however, is not the case for the economy analyzed in this paper since here we have agreement among shareholders in the sense that the equilibrium investment policy maximizes the expected present discounted value of one-period profits using any household's intertemporal marginal rate of substitution to discount future profits.

risk-free asset in zero net supply (borrowing and lending at the risk-free rate) will not change the equilibrium allocation.

The budget constraint can be rewritten in a way that shows how the households's optimization problem is basically a standard intertemporal portfolio choice problem. To this end, define the following variables: $w_{it} \doteq k_{it} + h_{it}$ (total wealth) and $\tilde{k}_{it} \doteq k_{it}/h_{it}$ (the capital-to-labor ratio). With this new notation, the fraction of total wealth invested in physical capital is $\tilde{k}_{it}/(1+\tilde{k}_{it})$ and the fraction of total wealth invested in human capital is $1/(1+\tilde{k}_{it})$. Introduce further the following (average) rate of returns on the two investment opportunities: $r_{kt} \doteq \tilde{r}_{kt} - \delta_{kt}$ and $r_{ht} \doteq \tilde{r}_{ht} - \delta_{ht}$. Using this notation, the budget constraint reads:

$$
\begin{aligned}
w_{i,t+1} \;&=\; \left[1 + \frac{\tilde{k}_{it}}{1+\tilde{k}_{it}} r_{kt} + \frac{1}{1+\tilde{k}_{it}} \left(r_{ht} + \eta_{it}\right)\right] w_{it} \,-\, c_{it} \qquad (3) \\
& w_{it} \geq 0 \;,\; \tilde{k}_{it} \geq 0 \;, \\
& (w_{i0}, \tilde{k}_{i0}) \; given \;.
\end{aligned}
$$

Households have identical preferences over consumption plans $\{c_{it}\}$. These preferences allow for a time-additive expected utility representation:

$$
U(\{c_{it}\}) = E\left[\sum_{t=0}^{\infty} \beta^t u(c_{it})\right] \;. \qquad (4)
$$

Moreover, we assume that the one-period utility function, u, is given by $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma \neq 1$, or $u(c) = log\, c$, that is, preferences exhibit constant degree of relative risk aversion γ.

Assumption 4. Preference allow for a time-additive expected utility representation (4) with one-period utility function that displays constant relative risk aversion.

II.B. Equilibrium

In general, a sequential equilibrium is a process of prices (returns) and actions defined by a sequence of functions mapping histories (date-events),(s^t, S^t), into current prices and actions. In this paper, however, we are only interested in sequential equilibria with a recursive (Markov) structure. Indeed, in this paper we focus attention on recursive equilibria that are simple in a sense to be defined next.

Introduce the aggregate capital-to-labor ratio $\tilde{K}_t \doteq K_t/H_t$ and the production function $f = f(\tilde{K})$ with $f(\tilde{K}) \doteq F(\tilde{K}, 1)$. Using the definitions $r_{kt} \doteq \tilde{r}_{kt} - \delta_{kt}$ and $r_{ht} \doteq \tilde{r}_{ht} - \delta_{ht}$, the first-order conditions associated with the firm's static maximization problem define two functions $r_k : \mathbb{R}_+ \times \mathbf{S} \to \mathbb{R}_+$ and $r_h : \mathbb{R}_+ \times \mathbf{S} \to \mathbb{R}_+$ as follows

$$\begin{aligned} r_k(\tilde{K}_t, S_t) &= A(S_t) f'(\tilde{K}_t) - \delta_k(S_t) \\ r_h(\tilde{K}_t, S_t) &= A(S_t)\left[f(\tilde{K}_t) - \tilde{K}_t f'(\tilde{K}_t)\right] - \delta_h(S_t)\ . \end{aligned} \tag{5}$$

Below we show that there is an equilibrium in which the capital-to-labor ratio is determined by a function $\tilde{K} : \mathbf{S} \to \mathbb{R}_+$ assigning to each aggregate state S_{t-1} a capital-to-labor ratio $\tilde{K}_t = \tilde{K}(S_{t-1})$. Thus, we have $r_{kt} = r_k(\tilde{K}(S_{t-1}), S_t)$ and $r_{ht} = r_h(\tilde{K}(S_{t-1}), S_t)$, and endogenous returns (prices) in period t therefore depend on (S_{t-1}, S_t) only. The budget constraint (3) in conjunction with preferences (4) then imply that in this equilibrium any individually optimal plan is generated by a policy function $g : \mathbb{R}_+ \times \mathbf{s} \times \mathbf{S}^{\mathbf{2}} \to \mathbb{R}^3_+$ that assigns to each state $(w_{it}, s_{it}, S_{t-1}, S_t)$ an action $(c_{it}, \tilde{k}_{it}, w_{i,t+1})$. Notice that we do not index the policy function, g, by i, that is, we confine attention to symmetric recursive equilibria.

Definition *A simple recursive equilibrium is a list of functions* $\tilde{K} : \mathbf{S} \to \mathbb{R}_+$, $r_k : \mathbb{R}_+ \times \mathbf{S} \to \mathbb{R}_+$, $r_h : \mathbb{R}_+ \times \mathbf{S} \to \mathbb{R}_+$, *and* $g : \mathbb{R}_+ \times \mathbf{s} \times \mathbf{S}^2 \to \mathbb{R}^3_+$ *satisfying the following conditions.*

i) *Firms maximize: the functions* r_k *and* r_h *are defined by (5).*

ii) *Households maximize: the policy function* g *generates a plan* $\{c_{it}, w_{it}, \tilde{k}_{it}\}$ *that maximizes* *expected lifetime utility (4) subject to the budget constraint (3).*

iii) *Markets clear*:

$$\frac{\sum_i \frac{\tilde{k}_{it}}{1+\tilde{k}_{it}} w_{it}}{\sum_i \frac{1}{1+\tilde{k}_{it}} w_{it}} = \tilde{K}_t$$

Remark 7. The above market clearing condition simply states that the physical to human capital ratio chosen by households is equal to the ratio chosen by the firm: $\sum_i k_{it}/\sum_i h_{it} = K_t/H_t$. Equilibrium values of physical and human capital are determined by $k_{it} = \tilde{k}_{it} w_{it}/(1+\tilde{k}_{it})$ and $h_{it} = w_{it}/(1+\tilde{k}_{it})$, and these markets automatically clear because of the constant-returns-to-scale assumption. Similarly, summing over the individual budget constraints implies goods market clearing $Y_t = C_t + X_{kt} + X_{ht}$.

Remark 8. In general, one would expect the portfolio choice of individual households to depend on their wealth and idiosyncratic shock realization, that is, one would expect $\tilde{k}_{it} = \tilde{k}(w_{it}, s_{it}, S_{t-1}, S_t)$. In this case, inspection of the market clearing condition immediately shows that no simple recursive equilibrium exists. However, if it happens to be the case that $\tilde{k}_{it} = \tilde{k}(S_{t-1})$, then the market clearing condition is satisfied if $\tilde{K}(S_{t-1}) = \tilde{k}(S_{t-1})$.

III. Existence and Characterization of Equilibrium

Consider an economy in which households live in autarky. That is, consider the decision problem of a household i who has direct access to the production technology F, but no access to financial markets. In this case, household i chooses a plan $\{c_{it}, k_{it}, h_{it}, x_{kit}, x_{hit}\}$ solving

$$\begin{aligned} max \sum_{t=0}^{\infty} E\left[\beta^t u(c_{it})\right] \\ s.t. \; : \; c_{it} + x_{kit} + x_{hit} &= A_t F(k_{it}, h_{it}) \\ k_{i,t+1} &= (1-\delta_{kt})k_{it} + x_{kit} \;\; , \;\; k_{it} \geq 0 \end{aligned} \tag{6}$$

$$h_{i,t+1} = (1 - \delta_{ht} + \eta_{it})h_{it} + x_{iht} \ , \ \ h_{it} \geq 0$$

$$(k_{i0}, h_{i0}) \, given \ ,$$

with $u(c_{it}) = \frac{c_{it}^{1-\gamma}}{1-\gamma}$, $\gamma \neq 1$, or $u(c_{it}) = log \, c_{it}$. The stochastic productivity and depreciation parameters in (6) are again defined by functions $A_t = A(S_t)$, $\delta_{kt} = \delta_k(S_t)$, $\delta_{ht} = \delta_h(S_t)$, and $\eta_{it} = \eta(s_{it}, S_t)$. The transition probabilities $\pi(s_{i,t+1}, S_{t+1}|s_{it}, S_t) = \pi(s_{i,t+1}, S_{t+1}|S_t)$ of the Markov process $\{s_{it}, S_t\}$ are given by the formula $\pi(s_{i,t+1}, S_{t+1}|S_t) = \sum_{-i} \pi(s_{1,t+1}, \ldots, s_{I,t+1}, S_{t+1}|S_t)$. Because of our previous assumption that the transition probabilities are symmetric with respect to households, these marginal transition probabilities are the same for all households.

Let w_{it} and $\tilde{k}_{it}$ be defined as before. Because of the constant-returns-to-scale assumption, the maximization problem (6) can be rewritten as

$$\begin{aligned} & maxE \left[\sum_{t=0}^{\infty} \beta^t u(c_{it}) \right] \qquad (7) \\ & s.t. \ : \ \ w_{i,t+1} = \left[1 + r(\tilde{k}_{it}, s_{it}, S_t)\right] w_{it} - c_{it} \\ & \qquad w_{it} \geq 0 \ , \ \tilde{k}_{it} \geq 0 \ , \\ & \qquad (w_{i0}, \tilde{k}_{i0}) \, given \ , \end{aligned}$$

where we introduced the total return on investment (in physical and human capital)

$$r(\tilde{k}_{it}, s_{it}, S_t) = \frac{\tilde{k}_{it}}{1 + \tilde{k}_{it}} r_k(\tilde{k}_{it}, S_t) + \frac{1}{1 + \tilde{k}_{it}} \left(r_h(\tilde{k}_{it}, S_t) + \eta(s_{it}, S_t) \right) \ . \qquad (8)$$

In (8) the investment return functions r_k and r_h are defined as in (5).

It follows from the structure of the decision problem (7) that any plan solving (7) is generated by a policy function $h : \mathbb{R}_+ \times \mathbf{s} \times \mathbf{S} \to \mathbb{R}_+^3$ assigning to each state (w_{it}, s_{it}, S_t) an action $(c_{it}, \tilde{k}_{it}, w_{i,t+1})$. Define the consumption-to-wealth ratio , $\tilde{c}_{it} = \frac{c_{it}}{[1+r(\tilde{k}_{it}, s_{it}, S_t)]w_{it}}$. We have the following result:

Proposition 1. *Suppose a solution to the one-agent decision problem (7) exists and has the property that* $\tilde{k}_{it} = \tilde{k}(S_{t-1})$, $\tilde{k} : \mathbf{S} \rightarrow \mathbb{R}_+$, *and* $\tilde{c}_{it} = \tilde{c}(S_t)$, $\tilde{c} : \mathbf{S} \rightarrow [\epsilon, 1-\epsilon]$, *for some* $\epsilon > 0$. *Then a simple recursive equilibrium exists. In this equilibrium the firm chooses* $\tilde{K}(S_{t-1}) = \tilde{k}(S_{t-1})$ *and households choose* $\tilde{k}_{it} = \tilde{k}(S_{t-1})$ *and* $\tilde{c}(S_t)$. *Moreover, if households have the opportunity to trade short-lived assets* $j = 1, \ldots, J$ *in zero net supply with payoffs* $D_{jt} = D_j(S_t)$, *then this allocation remains an equilibrium allocation and no trading of the* $j = 1, \ldots, J$ *short-lived assets is an equilibrium outcome.*

Proof: See appendix.

Remark 9. The property that the ratio $\tilde{k}$ is independent of s_{it} and w_{it}, and therefore the same for all households, is essential for the result that joint production (market economy of previous section) and autarky production (economy of current section) lead to the same allocation: if all households in the market economy choose the same $\tilde{k}$, then the production function is basically linear, and the scale of production is irrelevant.

Remark 10. There is a simple intuition for the result that households choose the same ratio $\tilde{k}$ in the equilibrium of the market economy. Because of the joint assumption of homothetic preferences and no exogenous source of income (labor income is generated through human capital accumulation), the relative share of wealth invested in any asset (physical and human capital) is independent of the wealth level. Further, these portfolio shares do not depend on s^t because idiosyncratic shocks have no predictive power. In short, portfolio shares, and therefore the ratio $\tilde{k}$, are the same for all households. A similar intuition shows that the (relative) excess demand for any security j whose payoffs do not depend on s_{t+1} is the same for all households (independent of w_{it} and s_{it}), and the only way to clear markets is to have zero excess demand for each household. An alternative intuition for the no-trade result is provided by (12), which shows that idiosyncratic income shocks are permanent in the sense that individual income (approximately)follows a logarithmic random walk.

Proposition 1 assumes that a solution to the one-agent decision problem (7) exists. If $\gamma < 1$ and capital returns are too high or if $\gamma > 1$ and capital returns are too low (too negative), then a solution to (7) will not exist. However, if the condition

$$sup_{S,\tilde{k}} \;\; \beta E\left[\left(1 + r(\tilde{k}, s_i', S')\right)^{1-\gamma} \mid S\right] < 1 \tag{9}$$

is satisfied, then a solution exists (proposition 2 below). Notice that for $\gamma = 1$ (log-utility), (9) reduces to $\beta < 1$. Condition (9) extends the condition appearing in Jones and Manuelli (1990) to the case of uncertainty. Jones, Manuelli, and Stachetti (1999) consider an economy with uncertainty similar to the one analyzed here. They, however, confine attention to the linear Markov case with Cobb-Douglas production function and no depreciation shocks, but allow for random variables with uncountable support. For linear Markov processes with Cobb-Douglas production function and no depreciation shocks, condition (9) is the finite-state-space analog of the existence condition in Jones, Manuelli, and Stachetti (1999).

Proposition 2. *Suppose condition (9) is satisfied. Then there exists a solution to the maximization problem (7) with* $\tilde{k}_{it} = \tilde{k}(S_{t-1})$ *and* $\tilde{c}_{it} = \tilde{c}(S_t)$, *where* $\tilde{k} : \mathbf{S} \to \mathbb{R}_+$ *and* $\tilde{c} : \mathbf{S} \to [\epsilon, 1-\epsilon]$. *The functions (Euclidian vectors)* $\tilde{k}$ *and* $\tilde{c}$ *are the unique solution to the following equation system:*

$$\forall S : \tilde{c}(S) \;=\; \frac{1}{1 + \left(\beta E\left[\left(1 + r(\tilde{k}(S), s_i', S')\right)^{1-\gamma} \tilde{c}^{-\gamma}(S')|S\right]\right)^{1/\gamma}} \tag{10}$$

$$E\left[\frac{r_h(\tilde{k}(S), S') + \eta(s_i', S') - r_k(\tilde{k}(S), S')}{\left(1 + r(\tilde{k}(S), s_i', S')\right)^{\gamma} \tilde{c}(S')^{\gamma}}|S\right] \;=\; 0\,.$$

In particular, if $\gamma = 1$ *(log-utility), we have* $\tilde{c} = 1 - \beta$.

Proof: See appendix.

Remark 11. The equation system (10) determining the ratio variables $\tilde{k}$ and $\tilde{c}$ are the modified version of the Euler equations (A1) that are associated with the maximization problem (7). The first equation in (A1) says that the utility cost of investing (saving) one more unit of the good must be equal to the expected discounted utility gain of doing so, and the second equation in (A2) states the equality of expected (marginal utility weighted) returns on the two investment opportunities. The equation system (10) results from (A1) using the definition $c_{it} = \tilde{c}_{it}(1 + r_{it})w_{it}$ and the budget constraint $w_{i,t+1} = (1 + r_{it})w_{it} - c_{it}$.

Combining proposition 1 and 2, we have:

Corollary *Suppose condition (9) is satisfied. Then there exists a simple recursive equilibrium with equilibrium allocation*

$$\tilde{k}_{it} = \tilde{k}(S_{t-1}) \quad ; \quad \tilde{c}_{it} = \tilde{c}(S_t) \quad ; \quad c_{it} = \tilde{c}(S_t)\left[1 + r(\tilde{k}(S_{t-1}), s_{it}, S_t)\right] w_{it}$$

$$k_{it} = \frac{\tilde{k}(S_{t-1})}{1 + \tilde{k}(S_{t-1})} w_{it} \quad ; \quad h_{it} = \frac{1}{1+\tilde{k}(S_{t-1})} w_{it} \quad ; \quad w_{i,t+1} = [1 - \tilde{c}(S_t)]\left[1 + r(\tilde{k}(S_{t-1}), s_{it}, S_t)\right] w_{it}$$

and aggregate asset returns

$$r_{kt} = r_k(\tilde{k}(S_{t-1}), S_t) \ ; \ r_{ht} = r_h(\tilde{k}(S_{t-1}), S_t) \ ,$$

where $\tilde{k}$ and $\tilde{c}$ are the solution to the equation system (10) and r, r_k, and r_h are the functions defined in (5) and (8). Moreover, the above allocation and asset returns are also the equilibrium allocation and asset returns for an economy in which households have the opportunity to trade short-lived securities $j = 1, \cdots, J$ in zero net supply with payoffs $D_{jt} = D_j(S_t)$. In this recursive equilibrium, there is no trade of securities, and security prices are

$$Q_j(S_t) = E\left[M(S_t, S_{t+1})D_j(S_{t+1})|S_t\right] \ ,$$

where the pricing kernel, M, is given by

$$M(S_t, S_{t+1}) = \beta E\left[\left(\frac{(1 + r(\tilde{k}(S_t), s_{i,t+1}, S_{t+1})\tilde{c}(S_{t+1})(1 - \tilde{c}(S_t))}{\tilde{c}(S_t)}\right)^{-\gamma} |S_t, S_{t+1}\right] \ .$$

Remark 12. Although the ratio variables $\tilde{k}_t$ and $\tilde{c}_t$ are the same for all households, the variables k_{it}, h_{it}, and c_{it} of course differ across households. That is, idiosyncratic risk matters. Note also that the equilibrium values of the ratios $\tilde{k}_t$ and $\tilde{c}_t$ are in general different from the values that obtain when markets are complete.

We can gain additional insight into the structure of equilibrium by considering the implications for individual income and consumption. The corollary implies that

$$\begin{aligned} \frac{y_{hi,t+1}}{y_{hit}} &= \varphi_y(S_{t-1}, S_t, S_{t+1}) \left[1 + r(\tilde{k}(S_{t-1}), s_{it}, S_t)\right] \\ \frac{c_{i,t+1}}{c_{it}} &= \varphi_c(S_t, S_{t+1}) \left[1 + r(\tilde{k}(S_t), s_{i,t+1}, S_{t+1})\right] \ , \end{aligned} \tag{11}$$

where $y_{hit} = \tilde{r}_{ht} h_{it}$ is labor income of household i in period t. Thus, conditional on the history of aggregate shocks, the growth rates of labor income and consumption are unpredictable (recall that $s_{i,t+1}$ is unpredictable). Taking logs and using the approximation $log(1+r) \approx r$, we find

$$log y_{i,t+1} \approx \tilde{\varphi}_y(S_{t-1}, S_t, S_{t+1}) + log y_{it} + \tilde{\eta}_{it} \ , \tag{12}$$

where $\tilde{\varphi}_{yt} = log\varphi_{yt} + \frac{\tilde{k}_t}{1+\tilde{k}_t} r_{kt} + \frac{1}{1+\tilde{k}_t} r_{ht}$ and $\tilde{\eta}_{it} = \frac{\eta_{it}}{1+\tilde{k}_t}$. In other words, conditional on the history of aggregate shocks, individual labor income therefore follows approximately a logarithmic random walk. In this sense, income shocks are permanent, which provides yet another intuition for the no-trade result and relates the current production economy to the exchange economy studied by Constantinides and Duffie (1996).

The model has interesting implications for aggregate consumption if aggregate shocks are unpredictable: $\pi(S'|S) = \pi(S)$. In this case, per capita consumption growth is

$$\begin{aligned} \frac{C_{t+1}}{C_t} &= E\left[\frac{c_{i,t+1}}{c_{it}} | S^{t+1}\right] \\ &= \beta \left(1 + E[r(\tilde{k}, s_{i,t+1}, S_{t+1}) | S_{t+1}]\right) \ . \end{aligned} \tag{13}$$

Thus, per capita consumption growth rates are i.i.d., that is, it follows (approximately) a logarithmic random walk and the risk-free rate is constant. Annual data on real short-term interest rates and consumption show only small deviations from these two properties (Campbell and Cochrane, 1999).

IV. Extension: Labor-Leisure Choice

Suppose now that output is produced according to $Y_t = A_t F(K_t, H_t L_t)$, where A_t and F have the same properties as before and L_t is total number of hours households spent working.[9] Preferences are no given by

$$U(\{c_{it}\}) = E\left[\sum_{t=0}^{\infty} \beta^t u(c_{it}) v(1 - l_{it})\right] , \tag{14}$$

where u is again a CRRA-utility function and v is twice continuously differentiable, strictly increasing, and strictly concave function satisfying the appropriate boundary conditions to ensure the interiority of the optimal labor choice. *Mutatis Mutandis*, the a simple recursive equilibrium is defined as in section II.

Define again total welath as $w_{it} = k_{it} + h_{it}$ and the ratio variables $\tilde{k}_{it} = k_{it}/h_{it}$ and $\tilde{c}_{it} = c_{it}/[(1 + r_{it})w_{it}]$. A straightforward extension of the arguments made in the proof of propositions 1 and 2 shows that there is a simple recursive equilibrium in which households choose $\tilde{k}_{it} = \tilde{K}(S_{t-1})$, $\tilde{c}_{it} = \tilde{c}(S_t)$, and $l_{it} = l(S_{t-1}, S_t)$ if the following equation system has a solution:

$$\forall S : \tilde{c}(S) \ = \ \frac{1}{1 + \left(\beta E\left[\left(1 + r(\tilde{k}(S), l(S, S'), s_i', S')\right)^{1-\gamma} \tilde{c}^{-\gamma}(S')|S\right]\right)^{1/\gamma}} \tag{15}$$

$$E\left[\frac{r_h(\tilde{k}(S), l(S, S'), S') + \eta(s_i', S') - r_k(\tilde{k}(S), l(S, S'), S')}{\left(1 + r(\tilde{k}(S), l(S, S'), s_i', S')\right)^{\gamma} \tilde{c}(S')^{\gamma}}|S\right] \ = \ 0$$

[9] A more realistic assumption might be that $\sum_i h_{it} l_{it}$ enters as an argument into the production function. However, since in equilibrium $l_{it} = l_t$ (see below), this assumption leads to the same result.

$$l(S,S') \;=\; \frac{(1-\gamma)\left[r_h(\tilde{k}(S),l(S,S'),S')+\delta_h(S')\right]v(1-l(S,S'))}{\tilde{c}(S')\left(1+\tilde{k}(S)\right)v(1-l(S,S'))}$$

where r_k and r_h are the modified return functions (5).

V. Conclusion

This paper developed a tractable macroeconomic model with incomplete markets and showed that there are simple recursive equilibria. Because of space limitations, this paper did not discuss any applications of the framework to macroeconomic policy analysis. However, the model has already been used to study the growth and welfare effects of social insurance (Krebs 2001) and the welfare cost of business cycles (Krebs, 2002).

The current paper does not address the question why certain insurance markets for idiosyncratic human capital risk are missing. One possible explanation for this lack of insurance might be the asymmetry of information with respect to idiosyncratic human capital shocks. An interesting question for future research is to investigate under what conditions the equilibrium allocation of the incomplete-markets economy is also the constrained efficient allocation of an economy with asymmetric/private information. The work by Atkeson and Lucas (1992) suggest that this equivalence does not necessarily hold. However, Cole and Kocherlakota (2001) have shown that the incomplete-market equilibria lead to constrained efficient allocations if information about both individual income and wealth are private. Extending the analysis of Cole and Kocherlakota (2001) to the current model is an interesting topic for future research.

Appendix 1: Proof of Proposition 1

The proof of proposition 1 splits into two parts. First, it is shown that the solution to the one-agent decision problem (7) is also the equilibrium allocation of a one-agent market economy with supporting prices given by (5) (second welfare theorem for the one-agent economy). The proof offered here uses Euler equations and transversality condition, but a dynamic programming approach along the lines of Prescott and Mehra (1980) would also work.[10] Second, it is argued that equilibrium choices and prices for the one-agent market economy are also the equilibrium choices and prices for the I-agent economy.

Suppose that a solution $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ to the one-agent decision problem (7) exists. Then it must satisfy the Euler equations associated with (7), which read

$$(c_{it})^{-\gamma} = \beta E\left[(1 + r_{i,t+1})\,(c_{i,t+1})^{-\gamma}\,|s^t, S^t\right] \qquad \text{(A1)}$$
$$E\left[(c_{i,t+1})^{-\gamma}\,(r_{h,t+1} + \eta_{i,t+1} - r_{k,t+1})\,|s^t, S^t\right] \;=\; 0\;.$$

We next show that the existence of a solution to (7) in conjunction with the condition that $\tilde{k}_{it}$ and $\tilde{c}_{it}$ are bounded away from zero imply that the transversality condition

$$\beta^t E\left[(c_{it})^{-\gamma}\,(1 + r_{it})w_{it}\right] \to 0\;. \qquad \text{(A2)}$$

holds (the transversality condition is necessary).

Since $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ is a solution to a maximization problem, expected lifetime utility associated with $\{c_{it}\}$ exists and is finite. Thus, for $\gamma \neq 1$ the series $\sum_{t=0}^{\infty} a_t \doteq \lim_{T\to\infty}\sum_{t=0}^{T} a_t$ with $a_t = \beta^t E[(c_{it})^{1-\gamma}/(1-\gamma)]$ must converge, which implies that we must have $a_T \to 0$. In other words,

$$\beta^t E\left[\frac{(c_{iT})^{1-\gamma}}{1-\gamma}\right] \to 0\;. \qquad \text{(A3)}$$

[10] The results in Prescott and Mehra (1980) are not directly applicable because of the unboundedness of decision variables.

Using $w_{it} = \frac{c_{it}}{(1+r_{it})\tilde{c}_{it}}$ and $\epsilon \leq \tilde{c}_{it} \leq 1-\epsilon$ and $r_{min} \leq r_{it} \leq r_{max}$ for some $\epsilon > 0$, $r_{min} > -1$, and $r_{max} < \infty$, we find that the transversality condition (A2) holds iff (A3) holds. The existence of r_{min} and r_{max} follows from the maintained assumptions on the production process. For $\gamma = 1$ (log-utility), an analogous argument shows that (A2) holds.

We can think of $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ as the solution to a social planner problem in a one-agent economy. There is a market problem corresponding to this social planner problem, which is to maximize (4) subject to the budget constraint (3) with (given) market returns $r_{kt} = r_k(\tilde{k}_{it}, S_t)$ and $r_{ht} = r_h(\tilde{k}_{it}, S_t)$. Equations (A1) and (A2) are also the Euler equations and transversality condition associated with this market problem (straightforward calculation). Since Euler equations and transversality condition together are sufficient conditions for utility maximization[11] and because $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ is budget-feasible, this plan is also the solution to the market problem. Thus, we have shown that if the one-agent decision problem has a solution, then this solution is also the competitive equilibrium of the one-agent market economy.

Consider now the I-agent market economy. From the above argument we conclude that the policy $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ is also individually optimal in the I-agent market economy when returns are given by $r_{kt} = r_k(\tilde{k}_{it}, S_t)$ and $r_{ht} = r_h(\tilde{k}_{it}, S_t)$. Thus, it suffices to show that market clearing holds. But with a common capital-to-labor ratio, $\tilde{k}_{it} = \tilde{k}(S_{t-1})$, market clearing automatically holds (for any possible wealth distribution).[12]

So far, we have not mentioned recursivity. Suppose that the plan $\{w^{it}, \tilde{k}_{it}, c_{it}\}$ is generated by a policy function $h : \mathbb{R}_+ \times \mathbf{s} \times \mathbf{S} \to \mathbb{R}^3_+$ that assigns to each state (w_{it}, s_{it}, S_t) an action $(c_{it}, \tilde{k}_{it}, w_{i,t+1})$ and has the additional property $\tilde{k}_{it} = \tilde{k}(S_{t-1})$. Then the corresponding market

[11] See, for example, Stokey and Lucas (1989). With our finite-state-space assumption, their proof of sufficiency extends to the uncertainty case in a straightforward way.

[12] Put differently, with a common $\tilde{k}$, the technology is basically linear, and joint production in one firm is equivalent to production in I individual firms (one firm per household).

equilibrium is clearly a simple recursive equilibrium.

Finally, suppose that households have the opportunity to trade $j = 1, \ldots, J$ securities in zero net supply with payoffs $D_{jt} = D_j(S_t)$. To render the individual optimization problem well-defined (to rule out Ponzi-schemes), let us impose the constraints $\theta_{ijt} \geq -B$ for some $B > 0$. Here η_{ijt} denotes the portfolio holding of security j by households i. Fix the the plan $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ that is the equilibrium allocation of the market economy without the $j = 1, \ldots, J$ securities, and define a security price function $Q_t = Q(S_t)$ by

$$Q_j(S_t) = E\left[M(s_{i,t+1}, S_t, S_{t+1}) D_j(S_{t+1}) | S_t\right] \qquad \text{(A4)}$$

$$M(s_{i,t+1}, S_t, S_{t+1}) = \beta \left(\frac{(1 + r(\tilde{k}(S_t), s_{i,t+1}, S_{t+1})\tilde{c}(S_{t+1})(1 - \tilde{c}(S_t))}{\tilde{c}(S_t)} \right)^{-\gamma} .$$

The pricing kernel M is simply the intertemporal marginal rate of substitution, $\beta \left(c_{i,t+1}/c_{it}\right)^{-\gamma}$. Clearly, the assumption that $\tilde{k}_{i,t+1} = \tilde{k}(S_t)$ and $\tilde{c}_{it} = \tilde{c}(S_t)$ is essential for ensuring that the pricing kernel in period $t+1$, $M_{t+1} = M(\tilde{k}(S_t), s_{i,t+1}, S_{t+1})$, does not depend on s_{it}, which in turn ensures that the expression on the right-hand-side of (A4) is the same for all households $i = 1, \ldots, I$. Moreover, the unpredictability of $s_{i,t+1}$ implies that the right-hand-side of (A4) is unchanged if we include s_{it} or $s_t = (s_{1t}, \ldots, s_{It})$ in the information set when calculating the conditional expectation. Thus, if household i is given the opportunity to trade the securities at prices (A4), then his Euler equations regarding security trade will be satisfied at $\theta_{it} = 0$ (no trade). Since an extended version of the transversality condition still holds, the choice of $\{w_{it}, \tilde{k}_{it}, c_{it}\}$ together with $\theta_{it} = 0$ is individually optimal. Since by construction all markets clear, we have found a (recursive) equilibrium for the extended economy in which households have the opportunity to trade the securities $j = 1, \ldots, J$.

Appendix 2: Proof of Proposition 2

The proof runs as follows. First, we show that there is a solution to the Euler equations which has the stated properties. Second, we show that any solution to the Euler equations also satisfies a transversality condition. Since in our case Euler equations and transversality condition are sufficient conditions for an optimum, we have proved that a solution to the maximization problem (7) with the stated properties exists. Uniqueness of the solution immediately follows from the strict concavity of the objective function in conjunction with the convexity of the choice set.[13]

We will prove the proposition for $\gamma \neq 1$. For $\gamma = 1$ (log-utility) the proof follows similar lines. Using $c_{it} = \tilde{c}_{it}(1+r_{it})w_{it}$ and $w_{i,t+1} = (1+r_{it})w_{it} - c_{it}$, we find that the Euler equations (A1) are satisfied if the equation system (10) in proposition 2 has a solution $0 < \tilde{c}(S) \leq 1$ and $\tilde{k}(S) \geq 0$. Denote the number of elements of $\mathbf{S}$ by $|\mathbf{S}|$. Since the state space, $\mathbf{S}$, is finite, the functions $\tilde{c}$ and $\tilde{k}$ can be identified with finite-dimensional vectors $\tilde{c} \in \mathbb{R}_+^{|\mathbf{S}|}$ and $\tilde{k} \in \mathbb{R}_+^{|\mathbf{S}|}$. Let $x = (\tilde{c}, \tilde{k}) \in \mathbb{R}_+^{2|\mathbf{S}|}$. Finding a solution to the equation system (10) amounts to finding a fixed point, $x = Tx$, for the operator $T : \mathbf{X} \to \mathbf{X}$, $\mathbf{X} \subset \mathbb{R}_+^{2|\mathbf{S}|}$, defined as follows. If $x' = Tx$ with $x' = (\tilde{c}', \tilde{k}')$, then $\tilde{c}'$ is given by the right-hand-side of the first set of Euler equations in (10) and $\tilde{k}'$ is determined as the solution of the second set of Euler equations. Notice that for any $\tilde{c} >> 0$, the solution, $\tilde{k}'$, to the second set of Euler equations exists and is unique. This immediately follows from the properties of r_h, r_k, r that are an implication of the assumption of a standard neoclasscial production function. To prove the existence of a solution to (10), we apply Brower's fixed point theorem. Thus, we need to show the

[13] There is an alternative way of proving proposition 2. First, extend the argument in Becker and Boyd (1997) and Jones and Manuelli (1990) to show that a solution to (7) exists, that is, show that the objective function is semi-continuous and the choice set is compact in the product topology. Since the solution to (7) is unique (strict concavity of the utility function in conjunction with convexity of the choice set) and Euler equations are necessary, it then suffices to show that a unique solution to the Euler equations exists (contraction mapping theorem). Jones, Manuelli, and Staccetti (1999) provide a proof along those lines for economies with Cobb-Douglas production function, linear Markov shocks, and no depreciation shocks.

existence of a non-empty, convex, and compact set $\mathbf{X}$ for which T is continuous.

We choose $\mathbf{X} \equiv ([\epsilon,1])^{|\mathbf{S}|} \times ([0,B])^{|\mathbf{S}|}$ for some $0 < \epsilon < 1$ and $B < \infty$ (below we show that we can bound $\tilde{c}$ away from one). Clearly, this set is non-empty, convex, and compact. Moreover, it is straightforward to show the continuity of T on $\mathbf{X}$. Thus, it is left to show that the two numbers B and ϵ exist. Notice the importance of bounding $\tilde{c}$ away from zero, since T is not even defined if $\tilde{c}(S) = 0$ for some $S \in \mathbf{S}$.

We begin with the existence of a strictly positive number ϵ. We want to show that if $\tilde{c} \in ([\epsilon,1])^{|\mathbf{S}|}$, then $\tilde{c} = T\tilde{c} \in ([\epsilon,1])^{|\mathbf{S}|}$. Since $T\tilde{c}(S) \leq 1$ obviously holds, we only need to show that $\forall S : T\tilde{c}(S) \geq \epsilon$ if $\epsilon \leq \tilde{c} \leq 1$. Suppose therefore that $\tilde{c}(S) \geq \epsilon$ for all S. In this case the inequality $T\tilde{c}(S) \geq \epsilon$ holds for all S if

$$\forall S \, : \, \frac{1}{1 + \epsilon^{-1} \left(\beta E \left[\left(1 + r(\tilde{k}(S), s_i', S') \right)^{1-\gamma} |S \right] \right)^{1/\gamma}} \;\geq\; \epsilon \, . \qquad \text{(A5)}$$

Condition (9) ensures that the term in the brackets is strictly less than one, which implies that for each S we can find a small enough $\epsilon(S) > 0$ so that (A5) holds. Since there are only finitely many states S, we can choose $\epsilon \doteq min_S \; \epsilon(S)$. Notice that in general $B = B(\epsilon)$, but that the number ϵ can be found independently of $\tilde{k}$ and therefore B.

Finally, we show that the transversality condition (A2) holds. Using $c_{iT} = \tilde{c}_{iT}(1+r_{iT})w_{iT}$ and the fact that $\epsilon \leq \tilde{c}_{iT} \leq 1-\epsilon$ and $r_{min} \leq r_{iT} \leq r_{max}$ for some $\epsilon > 0$, $r_{min} > -1$, and $r_{max} < \infty$, we find that (A2) holds iff the following holds

$$\beta^T E \left[w_{iT}^{1-\gamma} \right] \to 0 \, . \qquad \text{(A6)}$$

Repeated substitution of $w_{i,t+1} = (1 - \tilde{c}_{it})(1 + r_{it})w_{it}$, $t = 0, 1, \ldots, T-1$, shows that (A6) holds if

$$\beta E \left[(1 - \tilde{c}_{i,t+1})^{1-\gamma} (1 + r_{i,t+1})^{1-\gamma} |S_t \right] \;<\; 0 \, . \qquad \text{(A7)}$$

In (A7) we condition the expectation on S_t only because of the Markov assumption and the properties of $c_{i,t+1}$ and $r_{i,t+1}$. Clearly, if $(1-\tilde{c}_{it})^{1-\gamma} < 1$, then condition (9) ensures that (A7) holds. Since $0 < (1-\tilde{c}_{it}) < 1$, for $\gamma < 1$ the inequality $(1-\tilde{c}_{it})^{1-\gamma} < 1$ is satisfied. It is left to show that (A7) also holds for the case $\gamma > 1$.

Let $\tilde{c}_{max} = max_S \tilde{c}(S)$. The Euler equations (10) imply

$$\forall S: \ \tilde{c}(S) \leq \frac{1}{1+\tilde{c}_{max}^{-1}\left(\beta E\left[(1+r_{i,t+1})^{1-\gamma}|S\right]\right)^{1/\gamma}} \ .$$

Taking the maximum and rearranging terms yields

$$\tilde{c}_{max} \leq 1 - \left(\beta E\left[(1+r_{i,t+1})^{1-\gamma}|S\right]\right)^{1/\gamma} \ . \qquad \text{(A8)}$$

Inequality (A8) establishes an upper bound on $\tilde{c}$ that is strictly smaller than one. Using this inequality, condition (9) in the main text, and $\gamma > 1$, we find

$$\begin{aligned}
\beta E\left[(1-\tilde{c}_{i,t+1})^{1-\gamma}(1+r_{i,t+1})^{1-\gamma}|S_t\right] &\leq (1-\tilde{c}_{max})^{1-\gamma}\beta E\left[(1+r_{i,t+1})^{1-\gamma}|S_t\right] \\
&\leq \left(\beta E\left[(1+r_{i,t+1})^{1-\gamma}|S_t\right]\right)^{\frac{1-\gamma}{\gamma}} \beta E\left[(1+r_{i,t+1})^{1-\gamma}|S_t\right] \\
&< 1 \ .
\end{aligned}$$

This completes the proof of proposition 2.

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