A Theory of Money with Market Places*

Akihiko Matsui† and Takashi Shimizu‡

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Abstract

This paper considers an infinitely repeated economy in which divisible fiat money is used to trade goods. The economy has many market places. In each period, each agent chooses a market place, randomly meets someone who comes to the same market place, and they trade their goods when both agree to do so. There exist various classes of stationary equilibria. In some equilibria, all the agents visit the same market place, while in others, market places are specialized, i.e., only one type of good is traded in each active market place. In some equilibria, each good is traded at a single price, while in others, every good is traded at two different prices. Each class itself consists of equilibria with infinitely many price and welfare levels. However, it is shown that only efficient single price equilibria with specialized market places are evolutionarily stable. An inefficient equilibrium is upset by the mutants who visit a new market place to establish a more efficient trading pattern than before. An extension to the case with multiple currencies is also examined.

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†Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: amatsui@e.u-tokyo.ac.jp)
‡Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: tshimizu@e.u-tokyo.ac.jp)
1 Introduction

In a transaction, one needs to have what his trade partner wants, and vice versa. However, it is hard for, say, an economist who wants to have her hair cut to find a hairdresser who wishes to learn economic theory. In order to mitigate this problem of a lack of double coincidence of wants, money is often used as a medium of exchange. If there is a generally acceptable good called money, then the economist can divide the process into two: first, she teaches economics students to obtain money, and then finds a hairdresser to exchange money for haircut. The hairdresser uses the money later to obtain what he wants, and so on. Money is accepted by many people as it is believed to be accepted by many.

Focusing on this function of money, Kiyotaki and Wright [7] formalize the process of monetary exchange. In their model, agents are randomly matched to form a pair and trade their goods when they both agree to do so. This and the subsequent models, called the search theoretic models of money, have provided us with the foundations of monetary economics. The purpose of the present paper is to further develop the microfoundations of monetary economics by introducing market places which agents choose to visit, looking for trade partners.

The introduction of market places is based on the following two observations. First, matching rarely occurs in a completely random fashion in the real economy. People go to a fish market as buyers to buy fish; many potential workers use particular channels for job openings rather than simply walking on streets to meet potential employers. In reality, there are market places where agents look for their trade partners, although they often meet with each other in a random fashion once they go to a specific market place.

The second observation, which is related to the first, is that agents, especially sellers, often differentiate themselves from their competitors when they try to attract customers. A discount store does not simply wait for one-time customers to find its prices cheap, but it tries to attract them by advertisement and to retain them as regular customers. In a sense, it tries to change the probability of matching with potential buyers. This feature is not incorporated in the search theoretic models of money, and therefore, the price adjustment mechanism therein does not work well. If we view market places as abstract places, then visiting a new market place may be interpreted as differentiating itself from those in an old place it used to visit. It turns out that this function of market places activates the price adjustment
mechanism in the present analysis.

In order to see how market places are functioning, let us now briefly explain the model and the results. Roughly speaking, the present model is described as follows. It has an infinite number of periods and a continuum of infinitely lived agents. There are infinitely many market places, each of which consists of two physically identical sides, A and B. In each period, all the agents simultaneously choose a market place and one of its sides. In each market place, the agents on side A are matched with those on side B in a random fashion to form pairs, with the long side rationed. When two agents are matched and find that one of them produces the good that the other wishes to consume, they negotiate the price, which is modeled as a simultaneous offer game.

We adopt two approaches for the main analysis, an equilibrium approach and an evolutionary approach. We first examine stationary equilibria of this model. There exist various classes of stationary equilibria. Equilibria can be classified based upon two characteristics, the degree of specialization of market places, i.e., the number of goods traded in one market place, and the degree of price dispersion, especially, whether each good is traded at a single price or not. In some equilibria, all the agents visit the same market place, while in others, market places are specialized, i.e., only one type of good is traded in each (active) market place. In some equilibria, each good is traded at a single price, while in others, each good is traded at two different prices. Each class itself consists of a continuum of equilibria which correspond to a continuum of price and welfare levels.

The equilibrium approach does not have sufficient predictive power. In the present model, the only way that a seller can increase the matching probability is to switch to a new market place to differentiate itself from other sellers, but no buyer visits the place if no seller is expected to visit there. Therefore, no unilateral deviation to an empty market place is profitable. In other words, the equilibrium approach cannot have agents utilize empty places to start a new transaction pattern, including the one with price cut.

The evolutionary approach overcomes this coordination problem by allowing a small group of agents, both sellers and buyers, to jointly visit a new location and start a new transaction pattern. It is shown that only efficient single price equilibria with complete specialization are stable against evolutionary pressure.

The present paper prepares sufficiently many market places that can be used for transactions, but do not specify which place is used for which trans-
action. Specialization of market places may emerge as a result of the analysis. In this sense, this paper serves a microfoundation for the trading post approaches à la Shapley and Shubik [14]. In a model with trading posts, people submit their possessions to certain trading posts in which prespecified pairs of goods are traded. The goods they submit to one side of a post are traded with the other type of goods submitted to the opposite side. Agents obtain the goods on the opposite side in proportion to the amount they submitted. Trading mechanism at trading posts is put in a black box. It is verified that the canonical evolutionarily stable outcome of the present paper corresponds to the stationary equilibrium examined in Hayashi and Matsui [4], who constructed a model of fiat money by using the trading post approach.

As we mentioned above, the search theoretic models of money lack a satisfactory price adjustment mechanism. In the beginning, these models assume indivisible goods including fiat money mainly due to a technical difficulty of tracking inventory. In a model of random matching, it is hard to keep track of inventory of each agent as different agents encounter different experiences. Trejos and Wright [16] introduced divisible goods with indivisible fiat money and addressed the problem of price levels in terms of purchasing power of one unit of fiat money. Shi [15] presented a model of divisible fiat money. He considered a situation in which each household can simultaneously engages in multiple transaction activities by assuming that each household consists of a continuum of agents. Green and Zhou [3] partially succeeded in solving a model with divisible fiat money, using the technique of Markov processes with infinitely many states.

A new problem arises in Green and Zhou: an existence of equilibria with a continuum of price levels. Green and Zhou enlarged the frontier of the search theoretic models of money, but at the same time it reveals a fundamental problem of indeterminacy associated with them.

A crucial reason for this indeterminacy is that the probability of matching is exogenous, and therefore, say, a seller cannot attract more customers by lowering its price even if there is excess supply in the market, i.e., the number of potential sellers exceeds that of potential buyers. Introducing market

\[\text{Iwai's trading zone model [6] is also related to the present paper. Given the number of commodities } n, \text{ each agent chooses one of } n(n+1)/2 \text{ trading zones in which random matching takes place. The matching probability in a certain zone is proportional to the number of agents visiting the zone. Each agent can hold one unit of commodity or fiat money, both of which are indivisible. It considers, among others, the problem of which markets one uses to obtain one's consumption good.}\]
places which agents choose to visit, the present paper allows the possibility of changes in the probability of matching so that the price adjustment mechanism works.

The rest of the paper is organized as follows. Section 2 presents our framework. Section 3 defines and characterizes stationary equilibria, which is followed by the welfare analysis and comments on the effects of short-run monetary policies. Section 4 identifies the essentially unique evolutionarily stable equilibrium. Section 5 extends the model to the one with multiple currencies and discusses some issues associated with it. Section 6 concludes the paper. Appendices for lengthy proofs are also attached.

2 Model

We consider an infinite repetition of an economy which is inhabited by a continuum of agents with measure one. Time is discrete and indexed as \( t = 1, 2, \ldots \). There are \( K \) types of agents, \( 1, \ldots, K \). Assume \( K \geq 3 \). The generic element is denoted by \( k \). The mass of each type is \( 1/K \). An agent of type \( k \) obtains utility \( u \) if he consumes one unit of good \( k \ (k = 1, \ldots, K) \). He can produce at most one unit of good \( k + 1 \ (\text{mod } K) \) in each period. We assume that its production cost is zero, but that agents strictly prefer doing nothing to producing \textit{ceteris paribus} \footnote{This assumption corresponds to the existence of infinitesimally small production cost. It excludes an equilibrium in which gift giving occurs among anonymous agents.}. Every good is perishable and indivisible except for good 0, or fiat money, which is non-perishable and divisible. Each agent can hold any amount of fiat money with no cost. \( M \) is the total nominal stock of fiat money.

There are countably many \textit{market places}, indexed as \( 1, 2, 3, \ldots \). Its generic element is denoted by \( z \). Each market place has two physically identical sides, \( A \) and \( B \).

Each period consists of the following four stages.

\textbf{Stage 1:} Agents simultaneously choose a market place and one of its sides.

\textbf{Stage 2:} At each market place \( z \), a random matching takes place. The matching technology is frictionless, though a long side is rationed. Also, the matching is uniform. Formally speaking, if, in \( z \), the measure of the agents visiting side \( A \) is \( n_A \), that of the agents visiting side \( B \) is
\( n_B \), and among those visiting side \( B \) of \( z \) are the agents who belong to set \( S \) with its measure being \( n_S \), then the probability that an agent visiting side \( A \) meets with someone in \( S \) is \( \min\{n_S/n_A, n_S/n_B\} \).

**Stage 3:** If a type \( k \) agent and a type \( k + 1 \) (mod \( K \)) agent are matched, the type \( k \) agent offers a price \( p_S \), and the type \( k + 1 \) (mod \( K \)) agent bids a price \( p_B \). Here is assumed that agent’s type is observable to his partner, but not his money holding. Other types of matching have no consequence in trade, and therefore, we assume that after these matchings they go back home without making any further move. Stage 4 is applicable only to the case when \( k \) and \( k + 1 \) (mod \( K \)) agents are matched.

**Stage 4:** If \( p_S \leq p_B \), then the type \( k \) agent produces his good and exchanges it for fiat money \( p_S \), at the same time, the type \( k + 1 \) (mod \( K \)) agent exchanges fiat money \( p_S \) for the good, consumes it, and obtains utility \( u \). They go back home and never meet with each other again.\(^3\)

An agent can observe public records consisting of measures on types in each side of each market place, amount traded in each market place, and the entire distribution on money holdings besides his private transaction records\(^4\).

While a full-fledged strategy of an agent is a function from the set of entire histories observable to the agent into his action space, the subsequent argument is restricted to the set of Markov strategies, according to which actions depend only on the current money holdings of the agent in question.\(^5\)

Formally, a *Markov strategy* is defined to be a triple \( \sigma = (\lambda, o, \beta) \) where

- \( \lambda : \mathbb{R}_+ \rightarrow \mathbb{N} \times \{A, B\} \): a location strategy;
- \( o : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \): an offer strategy; and

\(^3\)The subsequent analysis will not be affected at all even if we change the rule on which price to use as long as the price is between \( p_S \) and \( p_B \).

\(^4\)It should be noted here that the subsequent analysis would not be affected even if we modify the information structure so that no public record is observable. The only thing that matters is the observability of one’s own current money holding.

\(^5\)We use the word “Markov” more restrictively than used in some other contexts in the sense that Markov strategy in our definition is independent of the current location and the current money holding distribution of the entire economy. However, even if such an alternative definition is used, the subsequent results remain unchanged.
\[ \beta : \mathbb{R}_+ \to \mathbb{R}_+ : \text{a bidding strategy.} \]

For example, \( \lambda(\eta) = (z, s) \) implies that the agent who takes \( \lambda \) and holds \( \eta \) units of money choose side \( s \) of the \( z \)th market place. The set of all Markov strategies is denoted by \( \Sigma \), whereas the set of all strategies is denoted by \( \Sigma^\prime \).\(^6\)

Moreover, we impose symmetry between types on Markov strategies in the subsequent arguments unless otherwise mentioned. We call a Markov strategy profile \( \sigma = (\sigma^1, \ldots, \sigma^K) \) symmetric if, when all type \( k = 1, \ldots, K \) agents are according to \( \sigma^k \) and money holdings distribution for all types are identical, the probability that a type \( k \) agent is matched with a type \( k+i \mod K \) agent is the same as the probability that a type \( k+1 \mod K \) agent is matched with a type \( k+i+1 \mod K \) agent for any \( n = 1, \ldots, K-1 \) and offer prices and bid prices are common\(^7\). From now on, we say “an agent meets a buyer” to indicate the situation in which a type \( k \) agent \( (k = 1, \ldots, K) \) meets a type \( k+1 \) agent, and so forth. We allow different agents of the same type to take different strategies.

Henceforth, we represent a symmetric strategy profile by a strategy for a type \( k \) agent. For example, for a location strategy \( \lambda, \lambda(\eta) = (1, A) \) means that the agents of all types visit a location \( (1, A) \), and \( \lambda(\eta) = (iK + k + j, A) \) means that type \( k \) agents visit market place \( iK + (k + j \mod K) \), type \( k+1 \) agents visit market place \( iK + (k + j + 1 \mod K) \), and so on.

We denote by \( \mu \) a distribution on money holdings and strategies: \( \mu(\{\eta\}; \{\sigma\}) \) is the fraction of the agents who take \( \sigma \) and hold \( \eta \) units of money, which we

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\(^6\)A full-fledged strategy is defined as follows. First, we denote by \( \mathcal{H}^t \) the set of histories up to period \( t \) (not including \( t \)). Let \( \mathcal{H} = \bigcup_{t \geq 1} \mathcal{H}^t \). Then We define a full-fledged strategy of an agent as a triple \( \hat{\sigma} = (\hat{\lambda}, \hat{\sigma}, \hat{\beta}) \) where:

- \( \hat{\lambda} : \mathcal{H} \to \mathbb{N} \times \{A, B\} \);
- \( \hat{\sigma} : \mathcal{H} \times \mathbb{N} \times \{A, B\} \to \mathbb{R}_+ \); and
- \( \hat{\beta} : \mathcal{H} \times \mathbb{N} \times \{A, B\} \to \mathbb{R}_+ \).

\(^7\)The formal definition is the following. A Markov strategy profile \( \sigma = (\sigma^1, \ldots, \sigma^K) \) where \( \sigma^k = (\lambda^k, \sigma^k, \beta^k) \) is symmetric if

- for any \( k \) and any \( i, j \), if \( \lambda^k(\eta) = (n, s), \lambda^{k+i \mod K}(\eta) = (n, s') \) with \( s \neq s' \), then there exists \( n' \) such that \( \lambda^{k+1 \mod K}(\eta) = (n', s''), \lambda^{k+i+1 \mod K}(\eta) = (n', s''') \) with \( s'' \neq s''' \).
- \( \sigma^1 = \sigma^2 = \cdots = \sigma^K \).
- \( \beta^1 = \beta^2 = \cdots = \beta^K \).
write $\mu(\eta; \sigma)$, i.e., without brackets; this should cause no confusion. Notice that we have extended the notion of symmetry, imposing it on distributions. Given $\mu$, $\mu_\Sigma$ is its marginal distribution on strategies, i.e., $\mu_\Sigma(\Sigma') \triangleq \mu(\mathbb{R}_+; \Sigma')$ is the fraction of the agents taking strategies in $\Sigma' \subset \Sigma$. Similarly, $\mu_H$ is its marginal distribution on money holdings.

The transition of an agent’s money holdings $\eta$ is simple. Suppose the agent takes $\sigma$. Then if he meets with a seller with $(\sigma', \eta') = ((\zeta', \omega', \rho'), \eta')$, and if $\rho(\eta) \geq \omega'(\eta')$, then he obtains $u$ and the money holdings become $\eta \rightarrow \omega'(\eta')$. If, on the other hand, the agent meets a buyer with $(\sigma', \eta') = ((\zeta', \omega', \rho'), \eta')$, then his money holdings become $\eta \rightarrow \omega'(\eta')$. Otherwise, $\eta$ remains unchanged.

Each agent tries to maximize the discounted sum of stage payoffs where $\delta \in (0, 1)$ is a common discount factor. We denote by $V(\sigma, \eta; \mu)$ an agent’s discounted average expected payoff when he chooses $\sigma$ with money holding $\eta$, and the other agents are distributed according to $\mu$.

## 3 Stationary Equilibria

### 3.1 Equilibrium Concept

To begin with, we define stationary distribution. A distribution $\mu$ is said to be stationary if $\mu$ is transformed into $\mu$ when almost all agents do not revise their strategies, and their money holdings follow the above transition rule. Now, we define our equilibrium concept.

**Definition 1** A stationary distribution $\mu$ is a stationary (symmetric Markov perfect) equilibrium iff

- only Markov strategies are taken, i.e., $\mu_\Sigma(\Sigma) = 1$,

- on any equilibrium path, at most finitely many market places are visited$^8$,

- no agent has an incentive to deviate with any money holding unless other agents are according to $\mu$; for any $\hat{\sigma} \in \hat{\Sigma}$, any $\eta \in \mathbb{R}_+$,

$$V(\sigma, \eta; \mu) \geq V(\hat{\sigma}, \eta; \mu).$$

$^8$This condition assures the existence of empty market places. Such empty places play a crucial role in eliminating inefficient equilibria by using the evolutionary approach. See the subsequent analysis on evolutionary stability.
Note this definition requires equilibrium should satisfy a requirement similar to subgame perfection with regard to money holdings.

3.2 Equilibria with No Specialization

Suppose that all the agents go to the same market place, and moreover, they are evenly distributed between sides A and B, and offer and bid a common price \( p \). Then no agent has an incentive to change his location strategy since a visit to another market place gives him no utility, and the situation is exactly the same if he visits the other market place. In this case, we effectively have the same situation as in a complete random matching model. It is also verified that they have no incentive to change their offer and bidding strategies if the price \( p \) is high enough. In other words, we obtain all the equilibria that were established in Green and Zhou.

In the class of equilibria discussed therein, each agent may end up in selling a good even if he has a sufficient amount of money to buy one. Consequently, the support of the money holding distribution is a countable set.\(^9\) There are a continuum of equilibria with different price and welfare levels.

3.3 Single Price Equilibria with Complete Specialization

In this subsection and the next, we consider equilibria with completely specialized market places, i.e., those in which only a single type of good is traded in each active market place.

Given a price level \( p \geq 0 \), a single price equilibrium with complete specialization and with \( p \) (henceforth, we call it a \( p\text{-SPE} \)) is a stationary equilibrium in which every good is traded at price \( p \). The canonical \( p\text{-SPE} \), \( \mu_p \), is defined as follows:

- \( \mu_p(0; \sigma_p) = 1 - m \)
- \( \mu_p(p; \sigma_p) = m \)

where \( m = \frac{M}{p} \) is the total “real” stock of fiat money, and \( \sigma_p = (\lambda_p, \sigma_p, \beta_p) \) is a Markov strategy such that

\(^9\)The reader should refer to Green and Zhou for the formal description of this class of equilibria.
\[ \lambda_p(\eta) = \begin{cases} (k, B) & \text{iff } \eta \geq p \\ (k + 1, A) & \text{iff } \eta < p, \end{cases} \]

\[ o_p(\eta) = p. \]

\[ \beta_p(\eta) = \begin{cases} p & \text{iff } \eta \geq p \\ \eta & \text{iff } \eta < p \end{cases} \]

In short, non-money holders go to side \( B \) of the place to meet the sellers of their consumption goods. While money holders with \( p \) go to side \( A \) of the place to meet the buyers of their production goods. We prove the canonical distribution constitutes a stationary equilibrium.

**Theorem 1** For any \( p \geq M \), and any \( \delta \), the canonical \( p \)-SPE \( \mu_p \) is a stationary equilibrium.

**Proof:** We denote \( V(\sigma_p, \ell; \mu_p) \) by \( V_{\ell} \) for \( \ell \in \mathbb{N}_+ \)\(^{10}\). We divide the proof into two cases:

**Case 1:** \( m \geq \frac{1}{2} \).

In this situation “sellers” are lying on the short side. Therefore, the buyers are rationed (except in the case of \( m = 1/2 \)), while the sellers are not. Let \( r = \frac{1-m}{m} \). Then we have the following value functions:

\[ V_0 = \delta V_1, \]
\[ V_{\ell} = r \left( (1 - \delta) u + \delta V_{\ell-1} \right) + (1 - r) \delta V_{\ell}, \quad \ell \geq 1. \]

Solving this system of equations, we obtain

\[ V_{\ell} = r \left( 1 - \left[ \frac{\delta r}{1 - \delta + \delta r} \right]^{\ell-1} \frac{\delta r}{1 + \delta r} \right) u, \quad \ell \geq 0. \]

The only incentive compatibility condition that we need to verify is the one under which money holders do not go to side \( A \) of an appropriate market place. It is given by

\[ V_{\ell} \geq \delta V_{\ell+1}, \quad \ell \geq 1, \]

\(^{10}\)For general \( \eta \), we set \( V(\sigma_p, \eta; \mu_p) = V(\sigma_p, \lfloor \frac{\eta}{p} \rfloor p; \mu_p) \) where \( \lfloor x \rfloor \) is the integer part of \( x \).
which is equivalent to

$$(1 - \delta) r \left( 1 - \left[ \frac{\delta r}{1 - \delta + \delta r} \right]^{\ell} \right) u \geq 0, \quad \ell \geq 1.$$  

Since $\delta < 1$, $r \leq 0$, this inequality holds.

**Case 2:** $m < \frac{1}{2}$.

In this situation “buyers” are lying on the short side. Therefore, the sellers are rationed, while the buyers are not. Let $r = \frac{m}{1 - m}$. Then, we have the following value functions:

$$V_0 = r \delta V_1 + (1 - r) \delta V_0,$$
$$V_\ell = (1 - \delta) u + \delta V_{\ell - 1}, \quad \ell \geq 1.$$  

Solving this system of equations, we obtain

$$V_\ell = \left( 1 - \delta^{\ell} \frac{1}{1 + \delta r} \right) u, \quad \ell \geq 0.$$  

The only incentive compatibility condition that we need to verify is the one under which money holders do not go to Side $A$ of an appropriate place. It is given by

$$V_\ell \geq r \delta V_{\ell + 1} + (1 - r) \delta V_\ell, \quad \ell \geq 1,$$

which is equivalent to

$$(1 - \delta) (1 - \delta^{\ell}) u \geq 0, \quad \ell \geq 1.$$  

Thus, this condition holds.

**3.4 Dual Price Equilibria with Complete Specialization**

In the present model, since there could be more than one market places for a specific transaction, there is no *a priori* reason that a single price prevails. In fact, there are equilibria in which the same goods are traded at different prices. The simplest class of equilibria of this type are given below.
A dual price equilibrium is a stationary equilibrium in which each good is traded at two different prices. In particular, we consider a dual price equilibrium with complete specialization and with \((p, np)\) (henceforth we call it \((p, np)\)-DPE) in which there exist two kinds of markets: “poor market” in which goods are traded at price \(p\), and “rich market” in which goods are traded at price \(np\) where \(n\) is an integer greater than one. The canonical \((p, np)\)-DPE, \(\mu_{(p, np)}\), is illustrated in Table 1.

<table>
<thead>
<tr>
<th>(\sigma) (\eta)</th>
<th>(h_{01})</th>
<th>(h_{1})</th>
<th>(0)</th>
<th>(np)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>(h_{01})</td>
<td>(h_{1})</td>
<td>(0)</td>
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</tr>
<tr>
<td>(\sigma_n)</td>
<td>(h_{0n})</td>
<td>(0)</td>
<td>(h_{n})</td>
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</tbody>
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Table 1:

In this table, \(\sigma_i = (\lambda_i, o_i, \beta)\) for \(i = 1, n\) are given by

\[
\lambda_1(\eta) = \begin{cases} 
(K + k, B) & \text{if } \eta \geq np, \\
(k + 1, A) & \text{if } np > \eta > \ell^*p, \\
(k, B) & \text{if } \ell^*p \geq \eta \geq p, \\
(k + 1, A) & \text{if } \eta < p,
\end{cases}
\]

\[
\lambda_n(\eta) = \begin{cases} 
(K + k, B) & \text{if } \eta \geq np, \\
(k + 1, A) & \text{if } np > \eta > \ell^*p, \\
(k, B) & \text{if } \ell^*p \geq \eta \geq p, \\
(K + k + 1, A) & \text{if } \eta < p,
\end{cases}
\]

\[
o_1(\eta) = p,
\]

\[
o_n(\eta) = \begin{cases} 
np & \text{if } \eta \leq \ell^*p, \\
p & \text{if } \eta > \ell^*p,
\end{cases}
\]

\[
\beta(\eta) = \begin{cases} 
np & \text{if } \eta \geq np, \\
p & \text{if } np > \eta \geq p, \\
\eta & \text{if } \eta < p.
\end{cases}
\]

In other words, market places 1, \ldots, \(K\) correspond to the “poor markets”, while \(K + 1, \ldots, 2K\) correspond to the “rich markets”. On the equilibrium path we consider, there are only four types of agents in the market, two types
of non-money holders who go to either of rich market or poor market in the position of the seller, money holders with $p$ who go to the poor market in the position of the buyer, and money holders with $np$ who go to the rich market in the position of the buyer.

The description of the strategies looks complicated partly because the strategies specify agents’ behavior off the equilibrium path as well. On the equilibrium path, we have

$$
\sigma_1(\eta) = \begin{cases} 
((k + 1, A), p, \cdot) & \text{if } \eta = 0, \\
((k, B), \cdot, p) & \text{if } \eta = p,
\end{cases}
$$

and

$$
\sigma_n(\eta) = \begin{cases} 
((K + k + 1, A), np, \cdot) & \text{if } \eta = 0, \\
((K + k, B), \cdot, np) & \text{if } \eta = np,
\end{cases}
$$

First of all, this canonical distribution constitutes a stationary equilibrium only if $h_{01} < h_1$, $h_{0n} > h_n$. To see this, suppose first that $h_{01} \geq h_1$ holds. Then the agents holding $np$ can get the good in the “poor” market without being rationed. If this is the case, then they never go to the “rich” market to buy the same good at the higher price. Suppose next that $h_{0n} \leq h_n$ holds. Then non-money holders can sell their goods in the “rich” market without being rationed. Therefore, they all go to the “rich” market to sell their goods at the higher price of $np$.

Let $\hat{r} = h_{01}/h_1$. Then let

$$
\hat{V}_0 = \delta \hat{V}_1, \\
\hat{V}_k = \hat{r} \left( (1 - \delta)u + \delta \hat{V}_{k-1} \right) + (1 - \hat{r})\delta \hat{V}_k, \quad k \geq 1.
$$

This is an auxiliary value function. It corresponds to the auxiliary strategy according to which an agent uses the “poor” market only, irrespective of the agent’s money holding. Also, let $\breve{r} = h_n/h_{0n}$. Then let

$$
\breve{V}_0 = \breve{r} \delta \breve{V}_n + (1 - \breve{r})\delta \breve{V}_0, \\
\breve{V}_k = \delta^{n-k}\breve{V}_n, \quad 1 \leq k < n, \\
\breve{V}_n = (1 - \delta)u + \delta \breve{V}_0.
$$

This is another auxiliary value function. It corresponds to the strategy according to which an agent whose money holding is greater than or equivalent
to \( p \) but less than \( np \) goes to the “poor” market as a seller to save money until it hits \( np \), with which he can go to the “rich” market as a buyer.

Using \( \hat{V}'s \) and \( \hat{V}'s \), we can write the value function in this equilibrium as follows:

\[
V^*_\ell = \begin{cases} 
\hat{V}_0 & \text{if } \ell = 0, \\
\hat{V}_\ell & \text{if } 1 \leq \ell \leq \ell^*, \\
\hat{V}_{\ell^*} & \text{if } \ell^* < \ell \leq n, \\
(1 - \delta)u + \delta V^*_{\ell-n} & \text{if } \ell > n.
\end{cases}
\]

We choose \( \ell^*, \hat{r}, \text{ and } \hat{r} \) so that the following conditions are satisfied:

**C0** \( \hat{V}_0 = \hat{V}_0 \).

**C1** \( \hat{V}_\ell \geq \hat{V}_\ell \) if \( 1 \leq \ell \leq \ell^* \).

**C2** \( \hat{V}_\ell \leq \hat{V}_\ell \) if \( \ell^* < \ell \leq n \).

From the description of \( \hat{V}'s \) and [C1], we obtain

\[
\hat{V}_0 = \frac{\delta \hat{r}}{1 + \delta \hat{r}} u = \frac{\delta \hat{r}}{1 + \delta \hat{r}} u = \hat{V}_0, \tag{1}
\]

Then we obtain the condition that \( \hat{r} = \hat{r} \). From now on, this common ratio is denoted by \( r \). Then sequentially applying (1) to \( \hat{V}'s \), we obtain

\[
\hat{V}_\ell = ru - \left[ \frac{\delta r}{1 - \delta + \delta r} \right]^{\ell-1} \frac{\delta r}{1 + \delta r} ru. \tag{2}
\]

Similarly, using the description of \( \hat{V}'s \) and (1), we obtain

\[
\hat{V}_\ell = \delta^{\ell - \ell^*} \frac{1 - \delta + \delta r}{1 + \delta r} u, \quad 1 \leq \ell \leq n. \tag{3}
\]

It is verified that \( \hat{V}_\ell \) is concave in \( \ell \), and that \( \hat{V}_\ell \) is convex in \( \ell \). Therefore, if we prove that \( \hat{V}_1 \geq \hat{V}_1 \) and \( \hat{V}_n \leq \hat{V}_n \) hold for some \( r \), then there exists \( \ell^* \) between 1 and \( n \) such that [C1] and [C2] hold. Indeed, it is easily verified that \( \hat{V}_1 \geq \hat{V}_1 \) holds if \( r \geq 1/n \). After tedious calculation, it is also verified that \( \hat{V}_n \leq \hat{V}_n \) holds if \( r = 1/n \). Thus, if \( r = 1/n \), then there exists \( \ell^* \) between 1 and \( n \) such that [C1] and [C2] hold. Moreover, if \( \delta \) is close to one, i.e.,
agents are sufficiently patient, then $\tilde{V}_1 \geq \tilde{V}_1$ is almost equivalent to $r \geq 1/n$, while $\tilde{V}_n \leq \tilde{V}_n$ is almost equivalent to $r \leq 1/n$ (still, both inequalities hold if $r = 1/n$). Thus, in the limit of $\delta$ going to one, the only canonical equilibrium is the one in which $r = 1/n$ holds.

In the case of $r = 1/n$, we show all the incentive compatibility conditions hold whenever [C0],[C1], and [C2] hold. See Appendix A for details.

Note that we can find appropriate $h_{01}, h_{0n}, h_1, h_n$ whenever $0 < r < 1$. Moreover, if $r = 1/n$, we have

$$\frac{M}{p} = h_1 + nh_n = \frac{n}{n + 1},$$

so $(\frac{1}{n} (n + 1) M, (n + 1) M)$-DPE always exists. Moreover, the fractions of poor market users and rich market users are uniquely determined as $\delta$ goes to 1.

**Theorem 2** For any integer $n \geq 2$, and any $\delta$, the canonical $(\frac{1}{n} (n + 1) M, (n + 1) M)$-DPE $\mu_{(\frac{1}{n} (n + 1) M, (n + 1) M)}$ is a stationary equilibrium.

### 3.5 Welfare Analysis

In this section, we investigate the welfare of various stationary equilibria. We define

$$V(\mu, \hat{\mu}) \overset{\text{def}}{=} \int_{\Sigma \times R_+} V(\sigma, \eta; \hat{\mu}) d\mu(\sigma, \eta),$$

and

$$W(\mu) \overset{\text{def}}{=} V(\mu, \mu).$$

We regard $W$ as the welfare of the economy. In other words, welfare is assumed to be measured by the average value of all the agents. Moreover, $W$ is used as the criterion of efficiency. Formally, we call a stationary equilibrium $\mu$ efficient if $\mu$ maximizes $W(\mu)$. Note that the maximum value of $W(\mu)$ is $\frac{1}{2} u$ due to the assumption of production and matching technology, i.e., one cannot produce and consume in the same period, and therefore, in each period, at most a half of the entire population obtain $u$.

Note first that if only one market place is used as in the case of no specialization, then the probability of a type $k$ agent being matched with an
agent of either type $k - 1$ or $k + 1$ is $2/K$. Furthermore, at most only a half of the matched agents can obtain $u$. Therefore, the welfare for this case is at most $\frac{1}{K}u$.

Next, we calculate the welfare for canonical single and dual price equilibria with complete specialization.

(i) $p$-SPE:

(a) Case 1: $m \geq \frac{1}{2}$, i.e. $p \leq 2M$.

$$W(\mu_p) = (1 - m)V_0 + mV_1$$

$$= (1 - m)u.$$  

(b) Case 2: $m \leq \frac{1}{2}$, i.e. $p \geq 2M$.

$$W(\mu_p) = mu.$$  

In particular, $W(\mu_p)$ attains the largest value when $p = 2M$, in other words,

$$W(\mu_{2M}) = \frac{1}{2}u.$$  

(ii) $(\frac{1}{n}(n + 1)M, (n + 1)M)$-DPE:

$$W\left(\mu_{\left(\frac{1}{n}(n+1)M, (n+1)M\right)}\right) = (h_{01} + h_{0n})V_0 + h_1V_1 + h_nV_n$$

$$= \frac{1}{n+1}u.$$  

In short, the canonical $2M$-SPE is efficient, while the canonical single price equilibria with other prices and all canonical dual price equilibria are inefficient.

**Theorem 3** The canonical $2M$-SPE is efficient.

Is the reverse of the above statement true? That is, can we say that only something like $2M$-SPE is efficient? Without any qualification, the answer is No. False is even the fact that any efficient equilibrium is one with complete specialization. The following equilibrium with partially specialized market places serves a counter example.

Let $\mu$ be defined as follows:
\begin{itemize}
  \item \( \mu(0; \sigma) = \frac{1}{2} r \),
  \item \( \mu(p; \sigma) = \frac{1}{2} \),
  \item \( \mu(3p; \sigma) = \frac{1}{2} (1 - r) \),
\end{itemize}

where \( \sigma = (\lambda, o, \beta) \) is a Markov strategy such that

\[
\lambda(\eta) = \begin{cases} 
  (k - 1, A) & \text{if } \eta \geq 2p, \\
  (k, B) & \text{if } 2p > \eta \geq p, \\
  (k + 1, A) & \text{if } \eta < p.
\end{cases}
\]

\[
o(\eta) = \begin{cases} 
  2p & \text{if } \eta \geq p, \\
  p & \text{if } \eta < p.
\end{cases}
\]

\[
\beta(\eta) = \begin{cases} 
  2p & \text{if } \eta \geq 2p, \\
  p & \text{if } 2p > \eta \geq p, \\
  \eta & \text{if } \eta < p.
\end{cases}
\]

A picture of market place \( k \) at some instance is illustrated in Table 2 (Vectors are visiting agents, the first argument is the type, and the second argument is money holding).

<table>
<thead>
<tr>
<th>(k - 1)</th>
<th>\hline</th>
<th>(k + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>\hline</td>
<td>(0)</td>
<td>(3p)</td>
</tr>
<tr>
<td>\hline</td>
<td>(k)</td>
<td>\hline</td>
</tr>
</tbody>
</table>

Table 2: Market Place \( k \)

We can show that, given \( \delta, \sigma \) is a stationary equilibrium if \( r \) is small enough. Moreover, note that \( \mu \) is efficient since every agent consume or produce at every period.

However, this equilibrium is not robust in the following sense: Consider a Markov strategy \( \tilde{\sigma} = (\tilde{\lambda}, o, \tilde{\beta}) \) defined as follows:

\[
\tilde{\lambda}(\eta) = \begin{cases} 
  (k - 1, A) & \text{if } \eta \geq 4p, \\
  (k, B) & \text{if } 4p > \eta \geq p, \\
  (k + 1, A) & \text{if } \eta < p.
\end{cases}
\]
\[ \tilde{\beta}(\eta) = \begin{cases} 
2p & \text{iff } \eta \geq 4p, \\
p & \text{iff } 4p > \eta \geq p, \\
\eta & \text{iff } \eta < p.
\end{cases} \]

Note that \( \tilde{\sigma} \) gives the frequency of consuming strictly more than a half of all the time, while \( \sigma \) gives the frequency just equal to a half. Then, given \( r \), if \( \delta \) is large enough, it is more profitable to play \( \tilde{\sigma} \) from to play \( \sigma \).

By generalizing this idea, we obtain the following result:

**Theorem 4** If

(i) there exists \( \hat{\delta} \) such that \( \mu \) is a stationary equilibrium for any \( \delta \in (\hat{\delta}, 1) \),

and

(ii) \( \mu \) is efficient,

then \( \mu \) is \( p \)-SPE for some \( \frac{2}{3}M < p \leq 2M \).

For the proof, see Appendix B.

No matter how patient agents may be, there remain inefficient equilibria, including inefficient single and dual price equilibria with complete specialization. In order to drive out such inefficient outcomes, another force is necessary.

### 3.6 Monetary Policies: Short-run

The effects of monetary policies are one of the central issues in macroeconomics. This subsection, together with discussion in the subsequent section, presents some results related to this issue. In the short-run where we have a multitude of equilibria, the effects of monetary policies depend on the behavioral assumptions of the private agents. We examine two situations.

Consider first the situation in which a change in money supply does not cause any reaction of the private agents, i.e., their strategies remain the same. To make the analysis simple, consider the canonical \( p \)-SPE defined in the previous subsection, and assume that the government increases the money supply from \( M \) to \( M' > M \) by giving \( p \) units of currency each to some of the non-money holders. This means that the fraction of money holders increases from \( M/p \) to \( M'/p \). If they follow the strategies prescribed in the \( p \)-SPE, the real stock of money is also increased to \( m' = M'/p \). The welfare changes
from \(\min\{mu, (1 - m)u\}\) to \(\min\{m'u, (1 - m')u\}\). Thus, the monetary policy is effective. An increase in the real stock increases the welfare up to \(m = 1/2\) and decreases it beyond \(m = 1/2\).

Next, if the agents adjust their strategies to keep the real stock of money constant, then the monetary policy is completely neutralized if money supply is increased without causing a disequilibrium dynamic. To see this point, suppose that the government (the central bank) gives \(p \frac{M' - M}{M}\) units of money to the money-holders. Then all the money-holders have now \(p' = pM'/M\) units of money. If all the agents notice it and change their strategies to \(\sigma_{p'}\), the strategy used in the canonical \(p'-\text{SPE}\), then no real variables are affected, and the monetary policy is completely neutralized.\(^{11}\)

In the present framework, there is no economic force that makes prices, or the strategies of the agents, react to a change in money supply, provided that we stick to the equilibrium analysis. If we consider the process of price adjustment, we have a totally different story. In order to express it, we now turn to an evolutionary approach.

4 Evolutionary Stability

4.1 Stability Concept

This section examines the evolutionary stability of equilibrium distributions. In order for a distribution to be evolutionarily stable, we require that the original population fares at least as good as any small group of mutants in the long-run provided that the agents are sufficiently patient. The formal definition is given below.

**Definition 2** A distribution \(\mu\) is said to be *evolutionarily stable* if

(i) there exists \(\tilde{\delta} \in (0,1)\) such that \(\mu\) is a stationary (symmetric Markov perfect) equilibrium under any \(\delta \in (\tilde{\delta}, 1)\), and

(ii) for any \(\gamma > 0\), there exists \(\tilde{\delta} \in (0,1)\) such that for any \(\delta \in (\tilde{\delta}, 1)\), there exists \(\tilde{\epsilon} > 0\) such that for any \(\epsilon \in (0, \tilde{\epsilon})\), and any \(\tilde{\nu}\) such that \(\tilde{\nu}_H = \mu_H\), the following equation holds:

\[
V(\mu, (1 - \epsilon)\mu + \epsilon \tilde{\nu}) + \gamma > V(\tilde{\mu}, (1 - \epsilon)\tilde{\mu} + \epsilon \tilde{\nu}).
\]

\(^{11}\)Note that the way the money is distributed is not a culprit for the difference in the above results.
Unlike the standard analysis of evolution, agents are patient in the present model. Therefore, the future value is used to compare the payoffs between the original population and the mutants. In calculating these values, it is assumed that the fraction of the mutants remains “small”. Incorporating a possibility of growing population of mutants complicates the analysis, which we do not deal with in the present paper. Also, we assume that the original population survives even if it is “a little” worse than the mutants. In fact, in the present definition, a mutant cannot invade the population unless it fares better than the original population in the long-run. This reflects the idea that a “small” one-shot gain is considered negligible, and only constant gains over time would be counted as a threat to the original population. Note also that mutants defined by this may include “dummy”, i.e., those who do not actually mutate. This way we save some cumbersome notation.

4.2 Characterizing Evolutionarily Stable Equilibria

The following theorem shows that an inefficient equilibrium cannot be evolutionarily stable.

**Theorem 5** Any evolutionarily stable equilibrium is efficient, i.e., if $\mu$ is evolutionarily stable equilibrium, then $W(\mu) = \frac{1}{2}u$ holds.

For the detail of the proof, see Appendix C.

To see the intuition of the proof, consider the following example. Suppose the economy is trapped in the canonical 4M-SPE. Then there are too many sellers and too few buyers due to a high price. Suppose now that a small fraction $\epsilon$ of each type and each money holders mutate to play the strategy $\tilde{\sigma} = (\tilde{\lambda}, \tilde{\delta}, \tilde{\beta})$ such that

- $\tilde{\lambda}(\eta) = \begin{cases} (K + k, B) & \text{iff } \eta \geq 2M, \\ (K + k + 1, A) & \text{otherwise,} \end{cases}$
- $\tilde{\delta}(\eta) = 2M,$
- $\tilde{\beta}(\eta) = \begin{cases} 2M & \text{iff } \eta \geq 2M, \\ \eta & \text{otherwise.} \end{cases}$
We define $\hat{\mu}_1 = (1 - \epsilon)\mu_{4M} + \epsilon\hat{\mu}$ where $\hat{\mu}$ is the group of mutants. Right after the mutation occurs, $\hat{\mu}$ is given by

$$\hat{\mu}(4M; \hat{\sigma}) = \frac{1}{4},$$

$$\hat{\mu}(0; \hat{\sigma}) = \frac{3}{4}.$$ 

Since $\sigma_{4M}$ uses different market places from those used by $\hat{\sigma}$, the distribution of the money holdings of the mutants evolves independently of the original population. Note that the original population remains stationary. When $\hat{\mu}_1 = \hat{\mu}$,

$$\hat{\mu}_2(2M; \hat{\sigma}) = \frac{1}{2},$$

$$\hat{\mu}_2(0; \hat{\sigma}) = \frac{1}{2},$$

and $\hat{\mu}_2$ is stationary. And therefore, we have

$$\hat{\mu}_t = (1 - \epsilon)\mu_{4M} + \epsilon\hat{\mu}_2 \quad t \geq 2.$$ 

Then we have

$$V(\mu_{4M}, \hat{\mu}) = \frac{1}{4}u,$$

$$V(\hat{\mu}, \hat{\mu}) = \frac{1 + \delta}{4}u.$$ 

In other words, the canonical $4M$-SPE is not immune to the invasion of these mutants.

Moreover, we obtain the following corollary. It states that any evolutionarily stable outcome is not only efficient but also a single price equilibrium.

**Corollary 1** If $\mu$ is evolutionarily stable, then $\mu$ is $p$-SPE for some $\frac{3}{2}M < p \leq 2M$.

**Proof:** Straightforward from Theorems 4 and 5.

The price level is not uniquely determined since some agents may hold extra units of money less than the price level. These extra money holdings are completely useless since, for the sake of simplicity, we have not introduced a mechanism by which agents pool their extra holdings.

Our next result is that the canonical $2M$-SPE is evolutionarily stable. Also this result implies the existence of evolutionarily stable equilibrium.
Theorem 6 The canonical 2M-SPE is evolutionarily stable.

For the detail of the proof, see Appendix D. Here, we draw a sketch of the proof.

It is verified, using Theorem 1 that the canonical 2M-SPE satisfies the condition (i) in Definition 2.

Note that, without mutation, every agent in the original distribution alternates between production and consumption deterministically, and obtains $\frac{1}{2}u$ on average. We show that, as long as the size of mutants $\epsilon$ is low enough, he succeeds to do so for a long period with a probability near to 1, then he obtains value near to $\frac{1}{2}u$ at least (Lemma 2).

On the other hand, unless agents in a mutation distribution exploit outside markets in which agents of the original distribution trade, they obtain $\frac{1}{2}u$ at most by inside trading. In order to exploit outside markets infinitely often, some agents have to procure fiat money from outside markets infinitely often, in which they abandon opportunities for inside trading. In other words, mutants cannot exploit infinitely often without abandoning inside trading opportunities. We show that if the discount factor $\delta$ is high enough, mutants obtain value near to $\frac{1}{2}u$ at most (Lemma 3). It follows that the canonical 2M-SPE satisfies the condition (ii) in Definition 2.

5 Multiple Currencies

This section introduces an additional medium of exchange and considers some problems associated with multiple currencies. Its main purpose is to show a possibility of the present work to analyze broad issues on money, and therefore, the analysis of this section is illustrative rather than comprehensive.

Suppose that there is another divisible good, called good 0* or fiat money 0*, which is of the same nature as, but distinguishable from, good 0. Such an economy has various types of equilibria. We characterize only one of them and analyze the effect of changes in money supply. The class of equilibria we focus on is the one in which both goods 0 and 0* are used for transactions, but in different market places. One may think of cash for conventional transaction and electronic money for e-commerce as an example. Another example would be two currencies coexisting for transaction such as dollarization where dollar is used for specific transactions, while domestic currency is used for other transactions.
Let $M$ and $M^*$ be nominal stocks of fiat monies 0 and $0^*$, respectively. Consider the following canonical equilibrium:

- $\mu(0; \sigma) = n - m$
- $\mu(p; \sigma) = m$
- $\mu(0; \sigma^*) = n^* - m^*$
- $\mu(p^*; \sigma^*) = m^*$

where $m = \frac{M}{p}$ and $m^* = \frac{M^*}{p^*}$ are the total “real” stocks of fiat monies 0 and $0^*$, respectively.

We specify $\sigma, \sigma^*$ such that market places $1, \ldots, K$ are the market for fiat money 0, while $K + 1, \ldots, 2K$ are those for $0^*$. $\sigma = (\lambda, o, \beta)$ is the same as $\sigma_p$ defined in the canonical $p$-single price equilibrium when mone holding of $0^*$ is zero. To be precise it is defined as follows:

- $\lambda(\eta, \eta^*) = \begin{cases} (k, B) & \text{iff } \eta \geq p \\ (K + k, B) & \text{iff } \eta < p, \eta^* \geq p^* \\ (k + 1, A) & \text{iff } \eta < p, \eta^* < p^* \end{cases}$

- $o(\eta, \eta^*) = p$

- $\beta(\eta, \eta^*) = \begin{cases} p & \text{iff } \eta \geq p \\ \eta & \text{iff } \eta < p \end{cases}$

Similarly, $\sigma^* = (\lambda^*, o^*, \beta^*)$ is defined as follows:

- $\lambda^*(\eta, \eta^*) = \begin{cases} (K + k, B) & \text{iff } \eta^* \geq p^* \\ (k, B) & \text{iff } \eta^* < p^*, \eta \geq p \\ (K + k + 1, A) & \text{iff } \eta^* < p^*, \eta < p \end{cases}$

- $o(\eta, \eta^*) = p^*$

- $\beta(\eta, \eta^*) = \begin{cases} p^* & \text{iff } \eta^* \geq p^* \\ \eta^* & \text{iff } \eta^* < p^* \end{cases}$

To preclude degenerate cases from the analysis, we consider the cases in which $m, m^*, n - m$, and $n^* - m^*$ are all nonzero.
Theorem 7 For any $p$ and $p^*$ satisfying $\frac{M}{p} + \frac{M^*}{p^*} < 1$, and any $\delta \in (0, 1)$, the canonical distribution $\mu$ is a stationary equilibrium if and only if either
\[
\frac{m}{n} = \frac{m^*}{n^*} \text{ or } \frac{n-m}{n} = \frac{m^*}{n^*} \quad \text{(or both)} \text{ holds.}
\]

Proof: The only incentive constraint we have to check is whether or not the agents are willing to switch from one currency to the other for their transactions. The rest of the proof is omitted since it is essentially the same as that of Theorem 1. Furthermore, for the if-part, we check only the case of $\frac{m}{n} = \frac{m^*}{n^*} > \frac{1}{2}$. Other cases can be proven in a similar manner.

Consider this case. Let $r = \frac{n-m}{n}$ and $r^* = \frac{n^*-m^*}{n^*}$. Then the value functions are calculated as before to obtain\(^\text{12}\)
\[
V(\sigma, \ell p; \mu) = r \left( 1 - \left[ \frac{\delta r}{1 - \delta + \delta r} \right]^{\ell-1} \frac{\delta r}{1 + \delta r} \right) u \quad \ell \geq 0,
\]
and
\[
V(\sigma^*, \ell p^*; \mu) = r^* \left( 1 - \left[ \frac{\delta r^*}{1 - \delta + \delta r^*} \right]^{\ell-1} \frac{\delta r^*}{1 + \delta r^*} \right) u \quad \ell \geq 0,
\]
respectively. Since $r = r^*$ holds, $V(\sigma, \ell p; \mu) = V(\sigma^*, \ell p^*; \mu)$ holds, too. Therefore, a non-money holder is indifferent between two currencies to obtain. Money holders are strictly better off using what they possess first. Therefore, no agent has an incentive to deviate.

To show the only-if-part, suppose the contrary, i.e., that neither $\frac{m}{n} = \frac{m^*}{n^*}$ nor $\frac{n-m}{n} = \frac{m^*}{n^*}$ holds. We show the case of $\frac{m}{n} < \frac{m^*}{n^*} < \frac{1}{2}$ only. Other cases are proven in a similar manner. In this case, a non-money holder taking $\sigma$ has an incentive to switch to $\sigma^*$. For if he follows $\sigma$, his value is
\[
V_0 = \left( 1 - \frac{1}{1 + \delta \frac{n-m}{n}} \right) u,
\]
\(^\text{12}\)In this proof, we denote $V(\sigma, \eta; \mu)$ and $V(\sigma^*, \eta^*; \mu)$ for $V(\sigma, \eta, 0; \mu)$ and $V(\sigma^*, 0, \eta^*; \mu)$, respectively. Moreover, we do not check the incentives for $\sigma$ (resp. $\sigma^*$) when $\eta^* > 0$ (resp. $\eta > 0$), but it is easy to do so.
while if he takes $\sigma^*$, his value would be
\[ V_0^* = \left(1 - \frac{1}{1 + \delta \frac{m^*}{n^* - m^*}}\right) u, \]
and therefore, $\frac{m}{n} < \frac{m^*}{n^*} < \frac{1}{2}$ implies $V_0^* > V_0$.

If the money supply of one currency changes, then there are basically two ways to adjust strategies: one is the price adjustment we discussed elsewhere, and the other is a switch of some agents from one currency to the other. In particular, the switch necessarily occurs if the price adjustment is not swift since one currency is more attractive than the other until the price adjustment is completed. Therefore, it is worthwhile to see how the fraction $n$ of the agents using good 0 is adjusted once the supply of fiat money 0 is increased. We keep the price level constant in this analysis.

Suppose $\frac{m}{n} = \frac{m^*}{n^*} < \frac{1}{2}$ in the first place, and suppose that the issuer of good 0 increases its money supply a little to $M' > M$ so that $\frac{m^*}{n^*} < \frac{m'}{n'} < \frac{1}{2}$ holds where $m' = M'/p$. Good 0 becomes more attractive than good 0* as shown in the proof of Theorem 7. As a result, people, especially non-money holders taking $\sigma$, start switching to good 0 from good 0*. The fraction $n$ increases. It increases until the ratios of money holders and non-money holders become equal between the two currencies. A new equilibrium is reached. In this equilibrium, the level of welfare is higher than before since the new ratio, denoted $m'/n'$, is closer to the optimal ratio, 1/2, than the old one.

If, on the other hand, $\frac{m}{n} = \frac{m^*}{n^*} > \frac{1}{2}$ holds, then we will have a different result. In this case, money is over-issued compared to the optimal level. Therefore, it may be beneficial for the money issuer to reduce its money supply. Suppose the issuer of good 0 reduces it. Then good 0 becomes more attractive than good 0*, and $n$ increases as in the previous case. But this time, the more $n$ becomes, the more attractive good 0 becomes than good 0*. In particular, when $n > n^*$, this process continues until no agent uses good 0* for transaction$^{13}$. Over-issuance makes the currency itself vulnerable. Therefore, the money issuers may have an incentive to restrain themselves

$^{13}$When $n$ is very small, this process may converge to an asymmetric equilibrium since an increase in $n$ has a large effect on $\frac{m}{n}$. 

from collecting seigniorage too much. In those countries which experience hyper inflation, monopolizing the sole medium of exchange is detrimental to the economy since this self-discipline works only when there is a competitor.

The above result implies that competition between currencies imposes discipline on the money issuers as argued by Hayek [5]. Note that Gresham’s law does not apply here since people have right to refuse the use of over-issued currency. As Hayek argued, if, by law, agents have to accept either money at the fixed rate, then we will have the problem of over-issuance by the issuers who try to collect seigniorage. Deepening this analysis may help us understanding the implication of emerging electronic money such as e-gold.

6 Concluding Remarks

We have introduced market places to a search theoretic model of money and conducted some analyses. We have adopted two solution concepts in the main analysis, equilibrium and evolutionary stability.

We have viewed the equilibrium approach as a proxy for the short-run situation. In some equilibria, one market place is used for all transactions, while in others, markets are specialized; only one commodity good is traded in each active market place. There are a continuum of equilibria with different price and welfare levels. There also exist dual price equilibria, in which the same good is traded at two different prices. We have also analyzed some effects of monetary policies. In some cases, money supply can be changed without causing a change in price level by assuming that agents do not alter their offer and bidding strategies. This expresses the case of price-rigidity, which is often considered as a ground for Keynesian economies. In such a case, the monetary policy is effective in that it reduces excess demand or supply and increases the level of welfare.

We have adopted the evolutionary approach as a proxy for a long-run situation. In the long-run, the price adjustment mechanism works, and only efficient single price equilibria prevails. In this time span, those who fare better than others survive, while those who do not do well shift from one strategy to a better one, or disappear. This evolutionary view of the markets is found in, among others, Alchian [1] and Friedman [2].

\[14\] It should be noted that the assumption of simultaneous trials is made for the sake of technical simplicity rather than realistic description of changes in behavior. In reality, even if a seller cuts its price, she has to wait for a while to attract new customers as is
When destroying some equilibrium by mutation or experimentation, mutants utilize some unused places to create a new market. Theoretically, this idea of using unused places for deviation is similar to the ideas of the secret handshake in Robson [13] and the cheap-talk in Matsui [10]. In Matsui [10], an unused message is used to signal others that one is a new type. They take cooperative actions only if they both send this new message. In a similar manner, successful mutants of the present model choose a new market place to meet with each other and take “cooperative” actions to increase their average payoffs.\footnote{See also Mailath, Samuelson, and Shaked [9].}

To conclude the paper, four remarks are in order. First, restriction to Markov perfect equilibria plays a crucial role in the proof of the (essential) uniqueness result of the evolutionarily stable outcome. Markov perfection excludes, for example, the possibility of discouraging sellers’ price-cut. Without the restriction, agents may take a strategy according to which they follow the mutants who go to a new place and behave as sellers so that the mutant cannot increase the probability of matching with buyers even if they cut the price. We do not think that this change in the results undermines our analysis. Rather, the lack of retaliation and punishment against price-cut is an essential feature that makes the price adjustment mechanism work, and Markov perfection expresses it in a simple form.

Second, the matching technology used in the present model involves no “volume” effect, i.e., the matching probability depends only on the relative size of agents visiting side A to those visiting side B. If there is a “volume” effect, i.e., if the larger the absolute size of a market place, the greater is the matching probability, then some result, especially, the one in Section 4 on evolution may be modified since creating a small new market may not pay off: Indeed, such a new market never appears if the “volume” effect is too large as in Iwai [6]. If, on the other hand, this exists but not too large, then the further the price is away from $2M$, the more likely the corresponding equilibrium is destroyed by mutants creating a new market since a gain from a better seller-buyer ratio exceeds a loss due to the “volume” effect.

Third, we have analyzed the effects of evolutionary pressure without any
specification of explicit evolutionary dynamics. It is true that in such an analysis, we have succeeded in obtaining an important result that only efficient equilibria are stable in the long-run. But it is obvious that analyses with explicit dynamics would deepen our understanding upon the mechanism of the monetary economy.\(^6\)

Fourth, Section 5 has extended our model to a situation in which multiple currencies exist. The importance of topics about multiple currencies cannot be emphasized too much. To deepen such an analysis, we need to enrich our analysis, examining the incentives of “issuers” more rigorously. Examples are their incentives to issue fiat money, how they interact with other issuers. We leave it for future research.

References


Appendices

A Incentive Compatibility Conditions for Canonical Dual Price Equilibria

The incentive compatibility conditions that we need to verify are the following ones:

- With regard to non money holders:
  
  C0’ Non money holders are indifferent between selling at the “poor market” and selling at the “rich market”.

- With regard to money holders with \( \ell p (0 < \ell \leq \ell^*) \):
  
  C1SR they never want to sell at the “rich market”.
  
  C1SP they never want to sell at the “poor market”.

- With regard to money holders with \( \ell p (\ell^* < \ell \leq n) \):
  
  C2SR they never want to sell at the “rich market”.
  
  C2BP they never want to buy at the “poor market”.

- With regard to money holders with \( \ell p (\ell \geq n) \):
  
  C3SR they never want to sell at the “rich market”.
  
  C3BP they never want to buy at the “poor market”.

In the case that \( r = 1/n \), we show all these conditions hold whenever [C0],[01], and [02] hold.

Proof: To start with, we prove the following fact:

Fact 1 If \( \ell^*(\delta) = \ell^* (<n) \), then \( \delta^{n-\ell^*-1} - r \left( 1 + \delta + \ldots + \delta^{n-\ell^*-1} \right) \geq 0 \).

Proof of Fact 1: By [C1], we have \( \hat{V}_{\ell^*} \geq \hat{V}_{\ell^*} \), while, by [C2], we have \( \hat{V}_{\ell^*+1} \leq \hat{V}_{\ell^*+1} \). Then we have

\[
\frac{\delta r}{1 - \delta + \delta r} \left( \hat{V}_{\ell^*} - \hat{V}_{\ell^*} \right) + \hat{V}_{\ell^*+1} - \hat{V}_{\ell^*+1} \geq 0.
\]
It follows that we have
\[ \delta^{n-\ell^* - 1} - r \left( 1 + \delta + \ldots + \delta^{n-\ell^* - 1} \right) \geq 0. \]

\[ \square \]

**C0’**: This condition is equivalent to [C0].

**C1SR**: This condition is given by
\[ V_{\ell}^* \geq r \delta V_{\ell+n}^* + (1 - r) \delta V_{\ell}^* \quad 0 < \ell \leq \ell^*, \]
which is equivalent to
\[ (1 - \delta) \left( 1 - \delta + \delta r \right) r - \left( \frac{\delta r}{1 - \delta + \delta r} \right)^{\ell - 1} \delta r^2 \geq 0. \tag{4} \]

Since (4) holds if \( \ell = 1 \), and the left hand side of (4) is increasing in \( \ell \), it always holds.

**C1SP**: This condition is given by
\[ V_{\ell}^* \geq \delta V_{\ell+1}^* \quad 0 < \ell \leq \ell^*. \tag{5} \]

- If \( \ell < \ell^* \), (5) is equivalent to
\[ (1 - \delta) r \left( 1 - \left( \frac{\delta r}{1 - \delta + \delta r} \right)^{\ell} \right) \geq 0, \]
which always holds.
- If \( \ell = \ell^* \), (5) holds since
\[ V_{\ell^*}^* = \tilde{V}_{\ell^*}^* \]
\[ \geq \tilde{V}_{\ell^*}^* \quad ([C1]) \]
\[ = \delta \tilde{V}_{\ell^*+1}^* \]
\[ = \delta V_{\ell^*+1}^*. \]
**C2SR** : This condition is given by

\[ V_\ell^* \geq r \delta V_{\ell+n}^* + (1 - r) \delta V_\ell^* \quad \ell^* < \ell \leq n, \]

which is equivalent

\[ \frac{(1 - \delta)^2}{\delta} \left( \delta^{n-\ell-1} - r \left( 1 + \ldots + \delta^{n-\ell-1} \right) \right) u \geq 0. \]

This holds since, by Fact 1,

\[ \left( \delta^{n-\ell-1} - r \left( 1 + \ldots + \delta^{n-\ell-1} \right) \right) \geq \delta^{n-\ell-1} - \delta^{n-\ell^*+1} + r \left( \delta^{n-\ell} + \ldots + \delta^{n-\ell^*+1} \right) \]

\[ = \delta^{n-\ell-1} (1 - \delta^{\ell^*+1}) + r \left( \delta^{n-\ell} + \ldots + \delta^{n-\ell^*+1} \right) \geq 0. \]

**C2BP** : This condition is given by

\[ V_\ell^* \geq r \left( (1 - \delta) u + \delta V_{\ell-1}^* \right) + (1 - r) \delta V_\ell^* \quad \ell^* < \ell \leq n, \]

which is equivalent to

\[ (1 - \delta + \delta r) V_\ell^* - \delta r V_{\ell-1}^* - (1 - \delta) ru \geq 0. \]  \hspace{1cm} (6)

- If \( \ell = \ell^* + 1 \), (6) holds since

  I.h.s. of (6) = \( (1 - \delta + \delta r) \hat{V}_{\ell^*+1} - \delta r \hat{V}_{\ell^*} - (1 - \delta) ru \)

  \[ \geq (1 - \delta + \delta r) \hat{V}_{\ell^*+1} - \delta r \hat{V}_{\ell^*} - (1 - \delta) ru \quad ([C2]) \]

  = 0.

- If \( \ell > \ell^* + 1 \), (6) is equivalent to

  \[ (\delta^{n-\ell} - r \left( 1 + \ldots + \delta^{n-\ell} \right)) u \geq 0. \]

  This holds since, by Fact 1,

  \[ \delta^{n-\ell} - r \left( 1 + \ldots + \delta^{n-\ell} \right) \geq \delta^{n-\ell} - \delta^{n-\ell^*+1} + r \left( \delta^{n-\ell+1} + \ldots + \delta^{n-\ell^*+1} \right) \]

  \[ = \delta^{n-\ell} (1 - \delta^{\ell^*+1}) + r \left( \delta^{n-\ell+1} + \ldots + \delta^{n-\ell^*+1} \right) \geq 0. \]
C3SR : This condition is given by

\[ V^*_\ell \geq r \delta V^*_\ell + (1 - r) \delta V^*_\ell \quad \ell \geq n, \]

which is equivalent to

\[ (1 - \delta) ((1 + \delta) V^*_\ell - \delta ru) \geq 0. \]

This holds since, by the monotonicity of \( V_\ell \),

\[ \text{l.h.s.} \geq (1 - \delta) ((1 + \delta) V_0 - \delta ru) = 0. \]

C3SP : This condition is given by

\[ V^*_\ell \geq \delta V^*_\ell+1 \quad \ell \geq n. \tag{7} \]

We decompose \( \ell \geq n \) into \( in + j \) where \( i \) is an integer greater than or equal to 1, and \( j \) is an integer between 0 and \( n - 1 \), then we have

\[ V^*_{in+j} = (1 - \delta^i) u + \delta^i V^*_j. \tag{8} \]

- If \( j < n - 1 \), by (8), (7) is equivalent to

\[ (1 - \delta) (1 - \delta^i) u + \delta^i (V^*_j - \delta V^*_{j+1}) \geq 0. \]

This holds since
- if \( j \leq \ell^* \), it is followed by [C1SP],
- if \( j > \ell^* \), it holds that \( \hat{V}_j = \delta \hat{V}_{j+1} \).

- If \( j = n - 1 \), by (8), (7) is equivalent to

\[ (1 - \delta) (1 - \delta^i) u + \delta^i (V^*_n - \delta V^*_n) \geq 0. \]

This holds since
- if \( \ell^* < n - 1 \), it holds \( \hat{V}_{n-1} = \delta \hat{V}_n \).
- if \( \ell^* = n - 1 \), it holds that

\[ \text{l.h.s} = (1 - \delta) (1 - \delta^i) u + \delta^i \left( \hat{V}_{n-1} - \delta \hat{V}_n \right) \]

\[ \geq (1 - \delta) (1 - \delta^i) u + \delta^i \left( \hat{V}_{n-1} - \delta \hat{V}_n \right) \tag{[C1]} \]

\[ = 0. \]
C3BP: This condition is given by

\[ V^*_{\ell} \geq r \left( (1 - \delta) u + \delta V^*_{\ell-1} \right) + (1 - r) \delta V^*_{\ell} \quad \ell \geq n. \]  \hspace{1cm} (9)

- If \( j = 0 \) (recall the decomposition of \( \ell \)), by (8), (9) is equivalent to

\[ (1 - \delta) \left( 1 - \delta^{i-1} \right) \left( 1 - r \right) u + \delta^{i-1} \left( V^*_n - r \left( (1 - \delta) u + \delta V^*_n \right) - (1 - r) \delta V^*_n \right) \geq 0. \]

This holds since

- if \( \ell^* < n - 1 \), it holds that

\[
\begin{align*}
\text{l.h.s.} &= (1 - \delta) \left( 1 - \delta^{i-1} \right) \left( 1 - r \right) u + \delta^{i-1} \left( 1 - \delta \right) \left( 1 + \delta r \right) \hat{V}_n - ru \\
&= (1 - \delta) \left( 1 - \delta^{i} \right) \left( 1 - r \right) u \\
&\geq 0,
\end{align*}
\]

- if \( \ell^* = n - 1 \), it holds that

\[
\begin{align*}
V^*_n - r \left( (1 - \delta) u + \delta V^*_n \right) - (1 - r) \delta V^*_n \\
&= (1 - \delta + \delta r) \hat{V}_n - r \left( (1 - \delta) u + \delta \hat{V}_{n-1} \right) \\
&\geq (1 - \delta + \delta r) \hat{V}_n - r \left( (1 - \delta) u + \delta \hat{V}_{n-1} \right) \hspace{1cm} ([C2]) \\
&= 0.
\end{align*}
\]

- If \( j > 0 \), by (8), (9) is equivalent to

\[ (1 - \delta) \left( 1 - \delta^{i} \right) \left( 1 - r \right) u + \delta^{i} \left( V^*_j - r \left( (1 - \delta) u + \delta V^*_j \right) - (1 - r) \delta V^*_j \right) \geq 0. \]

This holds since

- if \( j \leq \ell^* \), it holds that

\[ \hat{V}_j = r \left( (1 - \delta) u + \delta \hat{V}_{j-1} \right) - (1 - r) \delta \hat{V}_j, \]

- if \( j > \ell^* \), it is derived from [C2BP].
B Proof of Theorem 4

The proof is proceeded by stating the following claims. All the claims are concerned with realizations in any equilibrium that satisfies the assumptions of the theorem.

Claim 1: Almost all the agents either produce or consume in every period.
Proof. Otherwise, some welfare loss is incurred.

Claim 2: In every market place, the fraction of the agents visiting Side A is the same as the fraction of those visiting Side B.
Proof. Otherwise, some fraction of agents fail to be matched, which contradicts Claim 1.

Claim 3: In every market place, each type of good is traded at a single price, if traded at all.
Proof. Suppose the contrary, i.e., that some goods are traded at $p$ and $p' > p$ in some market place. It follows that there are two types of sellers, one offers $p$, the other offers $p'$. Moreover, from Claim 1, both sellers should be able to sell goods without fail. Then sellers offering $p$ has an incentive to deviate and offer $p'$. Repeating this deviation sufficiently many times, the seller can accumulate enough (additional) money to buy an extra good, which is a contradiction.

Claim 4: There exists no market place which type $k$ agents visit and agents of some type but $k-1$ or $k+1 \pmod K$ visit, i.e., each market place must belong to one of the following three categories:

(i) “one-good” market places, i.e., those of which type $k$ agents visit one side, and only $k-1$ type agents or only type $k+1 \pmod K$ agents visit the other side;

(ii) “two-good” market places, i.e., those of which type $k$ agents visit one side, and only both type $k-1$ agents and type $k+1 \pmod K$ agents visit the other side;

(iii) “inactive” market places, i.e., those places that no agent visits.

Proof. Otherwise, some fraction of agents fail to be matched.

Claim 5: There exists no market place in which two types of goods are traded.
We put the proof at the end of this appendix since it is cumbersome.
Claim 6: Every good is traded at the same price.

Proof. From Claim 5, a buyer can buy his consumption good without fail at every market place in which it is sold. Therefore, he has an incentive to buy at a cheaper market place.

Claim 7: The price is less than or equal to $2M$.

Proof. Otherwise, the proportion of agents with money holding more than or equal to the price is strictly less than a half, which contradicts Claims 1 and 2.

Claim 8: The price is more than $2/3M$.

Proof. Otherwise, there exist some buyers with money holding more than or equal to $2p$ or some sellers with money holding more than or equal to $p$. Then the sellers have an incentive to switch to buyers since the earlier consumption is made, the better it is for the agents.

Proof of Claim 5: Suppose the contrary, i.e., that there exists a market place in which two types of goods are traded. We assume, due to symmetry, that market places $1,\ldots,K$ are such places. From Claim 4, we may assume, without loss of generality, that some of type $k - 1$ agents and some of type $k + 1$ agents visit Side $A$ of market place $k$ and some type $k$ agents visit Side $B$ of market place $k$. Let the proportion of type $k - 1$ agents in Side $A$ be $r$ ($0 < r < 1$). We denote the price at which trades are made between type $k - 1$ agents and type $k$ agents by $p$, and the price between type $k + 1$ agents and type $k$ agents by $p'$.

Part 1: We show that $p < p'$. First, $p \leq p'$ holds, for if not, due to symmetry, a type $k$ agent has an incentive to go to market place $k - 1$ as a buyer, buying at $p'$, and $k + 1$ as a seller, selling at $p > p'$. By this deviation, the agent not only reduces the uncertainty but saves some money (note that the uncertainty comes from the fact that he does not know whether he meets a seller or a buyer). Suppose next that $p = p'$. Then every good is traded at price $p$ in every market place (including the market places in which only one type of good is traded). Some agents with money holding $\eta' \geq p$ visit Side $B$ on the equilibrium path. Thus, we have\textsuperscript{17}

$$V(\eta') = r((1 - \delta) u + \delta V(\eta' - p)) + (1 - r) \delta V(\eta' + p).$$

\textsuperscript{17}We omit $\sigma, \mu$ in the subsequent expressions.
Their incentive compatibility conditions are
\[ V(\eta') \geq r((1 - \delta)u + \delta V(\eta' - p)) , \]
\[ V(\eta') \geq (1 - r)\delta V(\eta' + p) . \]
Then we have
\[ (1 - \delta)u + \delta V(\eta' - p) = V(\eta') = \delta V(\eta' + p) . \]  \(10\)
One of the incentive compatibility conditions for an agent with \(\eta' - p\) is
\[ V(\eta' - p) \geq \delta V(\eta') . \]
Then we have
\[ V(\eta') \geq \frac{1}{1 + \delta}u, \]  \(11\)
by (10). On the other hand, an agent with \(\eta' + p\) has the following condition:
\[ V(\eta' + p) = \frac{1}{\delta}V(\eta') \]
\[ = \frac{1 - \delta^2}{\delta}V(\eta') + \delta V(\eta') \]
\[ \geq \frac{1 - \delta}{\delta}u + \delta V(\eta') \]
\[ > (1 - \delta)u + \delta V(\eta') , \]
by (10) and (11). Then we have
\[ V(\eta' + p) = \delta V(\eta' + 2p) . \]
Inductively, we obtain
\[ V(\eta' + np) = \frac{1}{\delta^n (1 + \delta)}u, \quad \forall n \in \mathbb{N} . \]
However, we have
\[ \lim_{n \to \infty} V(\eta' + np) = \infty , \]
which contradicts

\[ V(\eta) \leq u \quad \forall \eta \in \mathbb{R}_+. \]

**Part 2:**

We choose \( \bar{p} \in \mathbb{R}_+, m, n \in \mathbb{N} \) such that \( m\bar{p} = p, n\bar{p} = p' \), and \( m \) and \( n \) have no common divisor but 1 if we can find such numbers; otherwise, we choose \( \bar{p}, \varepsilon \in \mathbb{R}_+, m, n \in \mathbb{N} \) such that \( p = m\bar{p}, p' = n\bar{p} + \varepsilon \), \( m \) and \( n \) have no common divisor but 1, and \( \varepsilon \) is small enough.

For some \( \bar{\eta} \), we let the Markov strategy \( \sigma = (\lambda, o, \beta) \) be defined as follows:

\[
\lambda(\eta) = \begin{cases} 
(k - 1, A) & \text{iff } \eta \geq \bar{\eta} + \bar{\ell}\bar{p}, \\
(k, B) & \text{iff } \bar{\eta} + \bar{\ell}\bar{p} > \eta \geq \bar{\eta} + m\bar{p}, \\
(k + 1, A) & \text{iff } \eta < \bar{\eta} + m\bar{p}.
\end{cases}
\]

\[
o(\eta) = \begin{cases} 
n\bar{p} & \text{iff } \eta \geq \bar{\eta} + m\bar{p}, \\
p & \text{iff } \eta < \bar{\eta} + m\bar{p}.
\end{cases}
\]

\[
\beta(\eta) = \begin{cases} 
p' & \text{iff } \eta \geq \bar{\eta} + \bar{\ell}\bar{p}, \\
p & \text{iff } \bar{\eta} + \bar{\ell}\bar{p} > \eta \geq \bar{\eta} + m\bar{p}, \\
\eta & \text{iff } \eta < \bar{\eta} + m\bar{p}.
\end{cases}
\]

where \( \bar{\ell} = 3mn + m + n \). There is a room for arbitrage, and this strategy enables an agent to buy at the lower price \( \bar{p} \) and sell at the higher price \( \bar{p}' \). We show that every agent can obtain more than \( \frac{1}{2}u \) by taking \( \sigma \), which will be a contradiction.

Combining this strategy, an initial money holding \( \eta \in [\bar{\eta}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p}) \), and the equilibrium distribution, we construct a probability space \( P^n \) which governs the stochastic process of money holding that some agent will face. Moreover, this stochastic process is a time-homogeneous Markov process with the state space \([\bar{\eta}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p})\). We let \( L \equiv [\bar{\eta}, \bar{\eta} + m\bar{p}) \) and \( H \equiv [\bar{\eta} + \bar{\ell}\bar{p}, \bar{\eta} + \bar{\ell}\bar{p} + n\bar{p}) \).

We define various stochastic processes. First, a stochastic process \((S_t)_{t \geq 0}\) on \( P^0 \) is recursively defined as follows:

\[
S_0 = 0.
\]
For $t \geq 1$,

$$S^t = \begin{cases} 0 & \text{iff } S^{t-1} = 1, \eta_t \in L, \text{ or } S^{t-1} = 0, \eta_t \notin H, \\ 1 & \text{iff } S^{t-1} = 1, \eta_t \notin L, \text{ or } S^{t-1} = 0, \eta_t \in H. \end{cases}$$

This stochastic process switches to 1 when $\eta$ enters the region of $H$, and to 0 when it enters $L$.

Next, stochastic processes $(N^t_{LH})_{t \geq 0}$ and $(N^t_{HL})_{t \geq 0}$ on $\mathcal{P}^\eta$ are recursively defined as follows:

- $N^0_{LH} = N^0_{HL} = 0$.

- For $t \geq 1$,

$$N^t_{LH} = \begin{cases} N^{t-1}_{LH} + 1 & \text{iff } S^{t-1} = 0, \eta_t \in H, \\ N^{t-1}_{LH} & \text{otherwise.} \end{cases}$$

$$N^t_{HL} = \begin{cases} N^{t-1}_{HL} + 1 & \text{iff } S^{t-1} = 1, \eta_t \in L, \\ N^{t-1}_{HL} & \text{otherwise.} \end{cases}$$

$(N^t_{LH})$ (resp. $(N^t_{HL})$) counts the number of times that $\eta$ moves from $L$ to $H$, (resp. $H$ to $L$).

Finally, random variables $N^t_B$ and $N^t_S$ on $\mathcal{P}^\eta$ are recursively defined as follows:

- $N^0_B = N^0_S = 0$.

- For $t \geq 1$,

$$N^t_B = \begin{cases} N^{t-1}_B + 1 & \text{iff he buys his consumption goods at period } t, \\ N^{t-1}_B & \text{otherwise.} \end{cases}$$

$$N^t_S = \begin{cases} N^{t-1}_S + 1 & \text{iff he sells his production goods at period } t, \\ N^{t-1}_S & \text{otherwise.} \end{cases}$$

$(N^t_B)$ (resp. $(N^t_S)$) counts the number of the periods at which the agent acts as a buyer (resp. seller).
Lemma 1  There exists a positive constant $N$ such that
\[
\lim_{t \to \infty} \frac{N_{LH}^t}{t} = \lim_{t \to \infty} \frac{N_{HL}^t}{t} \geq N \text{ a.s.}
\]

Proof:  We define random variables $\tau^n$’s on $\mathcal{P}^n$ as follows:
\[
[\tau^n = t] \overset{\text{def}}{=} \bigcup_{i=1}^{t-1} [\eta_1, \ldots, \eta_{i-1} \notin L, \eta_i \in L, \eta_{i+1}, \ldots, \eta_{t-1} \notin H, \eta_t \in H].
\]
Moreover, we define $D, \ P$ as\(^ {18}\)
\[
D = 2 + \left[ \frac{\tilde{t} - m}{m} \right] + \left[ \frac{\tilde{t} - m}{n} \right],
\]
\[
P = r^{\left[ \frac{\tilde{t} - m}{m} \right]} \times (1 - r)^{\left[ \frac{\tilde{t} - m}{n} \right]}.
\]
Then we have
\[
\sup_{\eta \in [\eta_{\min} + \tilde{t} + m \eta]} E^n[\tau^n] \leq \sum_{i=1}^{\infty} i D (1 - P)^{i-1} P = \frac{D}{P},
\]
where $E^n$ is the expectation operator under $\mathcal{P}^n$. By a Markov property, we have
\[
\lim_{t \to \infty} \frac{N_{LH}^t}{t} \geq \frac{1}{\sup E^n[\tau^n]} \text{ a.s.}
\]
\[
\geq \frac{P}{D} > 0.
\]
On the other hand, we have
\[
N_{LH}^t \leq N_{HL}^t \leq N_{LH}^t + 1.
\]
Combining these inequalities, we obtain
\[
\lim_{t \to \infty} \frac{N_{HL}^t}{t} = \lim_{t \to \infty} \frac{N_{LH}^t}{t} \geq N \equiv \frac{P}{D} > 0, \text{ a.s.}
\]
\(^ {18}\lfloor x \rfloor\) is the least integer more than or equal to $x$.  

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This completes the proof of the lemma.

Using Lemma 1, we have

\[
N^t_B - N^t_S \geq \left(1 + \left\lfloor \frac{\bar{t} - n - m}{m} \right\rfloor \right) N^t_{HL} - \left(1 + \left\lfloor \frac{\bar{t} + n - m}{n} \right\rfloor \right) N^t_{LH} \\
= (3n + 1) N^t_{HL} - (3m + 3) N^t_{LH}.
\]

Since \( N^t_B + N^t_S = t \) holds, we have

\[
N^t_B \geq \frac{t}{2} + \frac{1}{2} (3n + 1) N^t_{HL} - \frac{1}{2} (3m + 3) N^t_{LH}.
\]

After some calculation, we obtain

\[
\liminf_{t \to \infty} \frac{N^t_B}{t} \geq \frac{1}{2} + \frac{1}{2} (3(n - m) - 2) N \quad \text{a.s.}
\]

\[
> \frac{1}{2},
\]

that is to say, any agent taking this strategy can obtain goods at more than a half of the entire periods. Thus, if \( \delta \) is sufficiently close to 1, this strategy attains more than \( 1/2u \) on average. Since this is true for every agent, it leads to a contradiction.

\[ \]

C Proof of Theorem 5

Note first that, in upsetting the original population, we do not necessarily use the most “plausible” mutants. Which mutants are plausible are often situation-dependent, and the proof becomes too complicated to handle if we start addressing the plausibility of mutation. One example which we think is plausible is described in the main text.

Suppose that \( \mu \) is an evolutionarily stable equilibrium and \( W(\mu) < \frac{1}{2}u \). We denote by \( \bar{K} \) the upper bound of market places used by strategies in the support of \( \mu_\Sigma \).

We construct a mutant distribution \( \tilde{\mu} \) such that any strategy in the support of \( \tilde{\mu}_\Sigma \) never visits the market places \( 1, \ldots, \bar{K} \). Then the payoff of the original population is not affected by the mutants, and vice versa, i.e., we
have
\[ V(\mu, (1 - \epsilon) \mu \oplus \epsilon \mu) = W(\mu) < \frac{1}{2} u, \]
\[ V(\tilde{\mu}, (1 - \epsilon) \mu \oplus \epsilon \mu) = W(\tilde{\mu}). \]

Thus, it suffices to show that for any small \( \gamma \), if \( \delta \) is large enough and \( \epsilon \) is small enough, then \( W(\tilde{\mu}) \geq \frac{1}{2} u - \gamma \).

We define \( \tilde{\eta} \) as follows:
\[ \tilde{\eta} \overset{\text{def}}{=} \inf \{ \eta' | \mu H (\{ \eta \leq \eta' \}) \geq \frac{1}{2} \}, \]
and divide the proof into two cases, \( \tilde{\eta} > 0 \) and \( \tilde{\eta} = 0 \). In both cases, the mutants choose new market places and start a new transaction pattern with a different price \( p^* \).

**Case 1: \( \tilde{\eta} > 0 \).**

In this case, we let \( p^* = \tilde{\eta} \). We then partition the mutants into two sets \( S_1 \) and \( S_2 \) of equal sizes so that a mutant with money holding \( \eta \) at the time of mutation belongs to \( S_1 \) if \( \eta < p^* \), and \( S_2 \) if \( \eta > p^* \), respectively (if there is a mass at \( \eta = p^* \), then we divide them so that the sizes of the two sets are equal). Such two sets can be found by way of the definition of \( p^* = \tilde{\eta} \).

Suppose now that an agent possesses \( \eta_0 \) units of money at the time of mutation and belongs to \( S_1 \). Then he ignores \( \eta_0 \), starts with producing his production good, and alternates production and consumption, trading goods at the price of \( p^* \). On the other hand, if he belongs to \( S_2 \), which implies \( \eta_0 > p^* \), then he ignores \( \eta_0 - p^* \), and starts with consuming his consumption good with the rest of the behavior being the same as those in set \( S_1 \).

Formally, we define Markov strategies \( \tilde{x}^{k_{1n_0}} = (\tilde{\lambda}_{1n_0}, \tilde{\delta}, \tilde{\beta}) \), \( \tilde{x}^{k_{2n_0}} = (\tilde{\lambda}_{2n_0}, \tilde{\delta}, \tilde{\beta}) \) as follows:

- \( \tilde{\lambda}_{1n_0} (\eta) = \begin{cases} (K + k, B) & \text{iff } \eta \geq \eta_0 + p^*, \\ (K + k + 1, A) & \text{otherwise.} \end{cases} \)
- \( \tilde{\lambda}_{2n_0} (\eta) = \begin{cases} (K + k, B) & \text{iff } \eta \geq \eta_0, \\ (K + k + 1, A) & \text{otherwise.} \end{cases} \)
- \( \tilde{\delta} (\eta) = p^*. \)

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\[ \tilde{\beta}(\eta) = \begin{cases} p^* & \text{iff } \eta \geq p^*; \\ \eta & \text{otherwise.} \end{cases} \]

A mutant in \( S_\ell (\ell = 1, 2) \) takes \( \sigma_{\ell m} \) if his money holding is \( \eta_0 \) at the time of mutation.

In this mutant distribution, agents in \( S_1 \) and \( S_2 \) alternate their moves, and a mutant in \( S_1 \) is always matched with another in \( S_2 \) for transaction, and vice versa. Therefore, we have

\[ W(\tilde{\mu}) = \frac{1}{2} \eta > W(\mu). \]

\[ \square \]

**Case 2:** \( \tilde{\eta} = 0 \).

This case is equivalent to \( \mu_H(\eta = 0) > \frac{1}{2} \). In order to construct a distribution in which the buyer-seller ratio is one-to-one, we need to distribute money from the “rich” to the “poor”. Let \( N \) and \( \tilde{\eta} \) be a pair of a positive integer and a positive number such that \( \frac{1}{2N} \)-fraction of agents have at least \( \tilde{\eta} \) units of money. We can find such a pair since \( M > 0 \) holds. Among mutants, let these agents constitute set \( T_1 \), and let the rest of the mutants be in set \( T_2 \). Let \( p^* = \tilde{\eta}/N \).

Take an agent in \( T_1 \) with the money holding of \( \eta_0 \) at the time of mutation. Note \( \eta_0 \geq Np^* \). His location strategy is

\[ \tilde{\lambda}_{1\eta_0}(\eta) = \begin{cases} (\tilde{K} + k, B) & \text{if } \eta \geq \eta_0 - (N - 1)p^*, \\ (\tilde{K} + k + 1, A) & \text{otherwise.} \end{cases} \]

In other words, he acts as a buyer \( N \)-times at the beginning as if his initial money holding were \( Np^* \).

Next, take an agent in \( T_2 \). she ignores her initial money holding \( \eta_0 \), and starts her new life as a seller. To be precise, her location strategy is

\[ \tilde{\lambda}_{2\eta_0}(\eta) = \begin{cases} (\tilde{K} + k, B) & \text{if } \eta \geq \eta_0 + p^*, \\ (\tilde{K} + k + 1, A) & \text{otherwise.} \end{cases} \]

Both agents in \( T_1 \) and \( T_2 \) offer \( p^* \) and bid \( p^* \) if they have more than or equal to \( p^* \) units of money.

If these mutants take the abovementioned strategies, then in \( N \) periods, the fraction of buyers becomes a half since the agents in \( T_1 \) repeat buying
goods for \( N \) consecutive times, distributing money to those in \( T_2 \). From the \( N \)th period on, they alternate between sellers and buyers. Thus, the average value of the mutants satisfies

\[
V(\hat{\mu}) \geq \delta^N \frac{1}{2} u.
\]

The right hand side of the above inequality tends to \( \frac{1}{2} u > W(\mu) \) as \( \delta \) goes to one (note that \( N \) does not depend on \( \delta \)). Hence, the mutant distribution constructed above upsets the original population.

\[\square\]

## D Proof of Theorem 6

It is obvious from Theorem 1 that the canonical 2M-SPE satisfies the condition (i) of Definition 2.

We denote the canonical 2M-SPE by \( \mu \), the associated strategy by \( \sigma_{2M} \), and a candidate mutant distribution by \( \hat{\mu} \). To simplify notation, we denote \( \hat{\mu} \overset{\text{def}}{=} (1 - \epsilon) \mu \oplus \epsilon \hat{\mu} \). In the following, when we say “the \( t \)th period”, we mean the \( t \)th period after mutation occurred.

**Lemma 2** For any \( \gamma > 0 \), and any \( \delta \in (0, 1) \), there exists \( \bar{\epsilon}(\gamma, \delta) > 0 \) such that for any \( \epsilon \in (0, \bar{\epsilon}(\gamma, \delta)) \), the following equation holds:

\[
V(\mu, \hat{\mu}) > \frac{1}{2} u - \gamma.
\]

**Proof of Lemma 2:** We first classify all the agents into two, infected and normal agents. *Infected agents* at the \( t \)th period \( (t = 1, 2, \ldots) \) are either mutants or those who have met infected agents at least once before or in the \( t \)th period. Those who are not infected agents are called *normal agents* at a given period. Note that, due to the matching technology, normal agents have succeeded in alternating between production and consumption until that period. Note also that a normal agent at period \( t \) is still normal in the next period if he meets another normal agent at that period. Let \( I_t \) \((t = 1, 2, \ldots)\) be an upper bound of the measure of the infected agents at period \( t \), and let \( N_t \) be a lower bound of the measure of normal agents at
period $t$. Then we have the following inequalities:

$$N_1 \geq 1 - \frac{2K\epsilon}{1 - \epsilon};$$  \hspace{1cm} (12)

$$I_1 \leq \frac{2K\epsilon}{1 - \epsilon};$$  \hspace{1cm} (13)

$$N_{t+1} \geq \frac{N_t^2}{1 + I_t}; \text{ and}$$  \hspace{1cm} (14)

$$I_{t+1} \leq 1 - N_{t+1}.$$  \hspace{1cm} (15)

$N_{t+1}$ is calculated under the worst scenario, i.e., the one according to which all the infected agents go to one side of existing active market to maximize the number of infection. Now, let $r_t$ be a lower bound of the probability that one is matched with a normal agent. We have

$$r_t \geq \frac{N_t}{1 + I_t}.$$  \hspace{1cm} (16)

Next, we define $V_i^T (i = 1, 2)$ as follows:

$$V_0^T = r_t \delta V_1^{T+1},$$

$$V_1^T = r_t \left( (1 - \delta) u + \delta V_0^{T+1} \right),$$

where $V_i^T$ is a lower bound of the value that a normal agent with money holding $2M_i$ can obtain. If $\delta$ is sufficiently close to one, then

$$V(\mu, \hat{\mu}) \geq \frac{1}{2} (V_0^1 + V_1^1).$$

From (12), (13), (14), (15), and (16), we have

$$r_{t+1} \geq \frac{2}{2 - \Pi_{\tau=1}^t r_\tau} - 1.$$

It is verified that for each $t = 1, 2, \ldots$,

$$r_t \rightarrow 1.$$

For any $T \geq 1$, we have

$$\frac{1}{2} (V_0^T + V_1^T) \geq \frac{1 - \delta}{2} \sum_{i=1}^T \delta^{T-1} \Pi_{\tau=1}^T r_\tau u$$

$$> \frac{1}{2} r_T (1 - \delta) \left( 1 - \delta^T r_T^T \right) u,$$

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where we make use of $r_{t+1} \leq r_t$. Since the last term converges to $\frac{1}{2} (1 - \delta^T) u$ as $r_T$ approaches 1, we can take a sufficiently large $T$, and find $\epsilon(\gamma, \delta) > 0$ such that for any $\epsilon \in (0, \epsilon(\gamma, \delta))$,

$$V(\mu, \hat{\mu}) > \frac{1}{2} u - \gamma.$$ 

\[ \]  

Lemma 3

$$V(\hat{\mu}, \hat{\mu}) \leq \left( \frac{1}{2} + \frac{1 - \delta}{4} \right) u.$$ 

**Proof of Lemma 3:** We consider the following maximization problem:

$$\max_{(b_t, s_t, i_t)_{t=1}^{\infty}} (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_t$$

\[ \text{s.t. } \begin{align*}
    u_t &= \left( b_t + \frac{1}{2} i_t \right) u \\
    b_t + s_t + i_t &= 1 \\
    m_0 &= M \\
    m_t &= m_{t-1} + 2M (s_t - b_t) \\
    b_t &\leq \frac{m_{t-1}}{2M} \\
    b_t, s_t, i_t, m_t &\geq 0,
\end{align*} \]

where $b_t$ (resp. $s_t$) is the fraction of the mutants who are matched with those in the original population as buyers (resp. sellers), and $i_t$ is the fraction of the mutants who are matched with each other. This is a social planner’s problem for the group of mutants, ignoring the strategic feasibility and the friction such as the possibility of not being matched. Thus, if we let $u^*$ denote the maximum value of the problem, then $V(\hat{\mu}, \hat{\mu}) \leq u^*$ holds.

Solving this problem, we obtain

$$u^* = \left( \frac{1}{2} + \frac{1 - \delta}{4} \right) u.$$ 

This concludes the proof of the lemma.
Given $\gamma$, let $\bar{\delta}$ be a positive number in $(1 - 4\frac{2}{u}, 1)$. Next, given $\gamma$ and $\delta \in (\bar{\delta}, 1)$, define $\bar{\epsilon}(\gamma, \delta)$ as in the proof of Lemma 2. Then, by Lemma 2 and Lemma 3, we have

\[
\mathcal{V}(\mu, \hat{\mu}) - \mathcal{V}(\bar{\mu}, \hat{\mu}) + \gamma \\
> \left( \frac{1}{2}u - \frac{1}{2}\gamma \right) - \left( \frac{1}{2}u + \frac{1}{2}\gamma \right) + \gamma \\
= 0
\]

for any $\epsilon \in (0, \bar{\epsilon}(\gamma, \delta))$, and any $\hat{\mu}$ satisfying the equation of Condition (ii) of Definition 2, which concludes the proof of the Theorem.