Money and prices under uncertainty[1]

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Review of Economic Studies, 72, 223-246 (2005) [4]

[1]We had very helpful conversations with Jayasri Dutta, Ronel Elul, John Geanakoplos, Demetre Tsomokos and Michael Woodford; the editor and referees made valuable comments and suggestions. The work of Polemarchakis would not have been possible without earlier joint work with Gaetano Bloise and Jacques Drèze.

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[4]Earlier versions appeared as "Money and prices under uncertainty," Working Paper No. 01-32, Department of Economics, Brown University, September, 2001 and "Monetary equilibria with monopolistic competition and sticky prices," Working Paper No. 02-25, Department of Economics, Brown University, 2002

Abstract

We study whether monetary economies display nominal indeterminacy: equivalently, whether monetary policy determines the path of prices under uncertainty. In a simple, stochastic, cash-in-advance economy, we find that indeterminacy arises and is characterized by the initial price level and a probability measure associated with state-contingent nominal bonds: equivalently, monetary policy determines an average, but not the distribution of inflation across realizations of uncertainty. The result does not derive from the stability of the deterministic steady state and is not affected essentially by price stickiness. Nominal indeterminacy may affect real allocations in cases we identify. Our characterization applies to stochastic monetary models in general, and it permits a unified treatment of the determinants of paths of inflation.

Key words: monetary policy; uncertainty; indeterminacy; fiscal policy; price rigidities.

JEL classification numbers: D50; D52; E31; E40; E50.

1 Introduction

Whether monetary policy can control the inflation rate is a question of theoretical interest and practical importance. The failure to control inflation can be the cause of suboptimal fluctuations, if indeterminacy is real. Optimal fiscal-monetary policy supports an optimal allocation of resources; if such a policy is also consistent with other, suboptimal, equilibrium allocations, then, it does not "implement" the targeted allocation.[1]

To address this question, we consider a stochastic cash-in-advance economy, where prices are either flexible or sticky, and show that indeterminacy is pervasive: monetary policy does not suffice to determine the stochastic path of inflation. This indeterminacy may affect real allocations even with flexible prices, depending on the conduct of monetary policy, the completeness of asset markets, and the timing of transactions in goods and asset markets.

In a deterministic monetary model, as is well known, the initial price level is indeterminate. Given the initial price level, the inflation rates in subsequent periods are determined by monetary policy that sets either the path of money supplies or the path of short-term nominal interest rates. This determination of inflation rates is understood by the Fisher equation, which describes the relationship between the nominal and real interest rates and the inflation rate. In a stochastic economy, there is a higher degree of indeterminacy, and monetary policy no longer suffices to determine the path of inflation rates. This is because the Fisher equation in a stochastic economy holds only in expected terms. In the case of flexible prices, such indeterminacy may be characterized by the indeterminacy of the "distribution" of inflation rates across states of nature.

However, a characterization of the indeterminacy in a stochastic economy by the distribution of inflation rates may fail to apply in cases of interest, such as economies with sticky-prices: for instance, if prices are set in advance, the distribution of inflation rates is a fortiori uniform. We argue that a robust characterization of the indeterminacy in stochastic monetary economies involves the prices of assets, not the prices of goods. Specifically, we consider the prices of state-contingent nominal elementary securities that pay off a unit of account if and only if particular states realize. We show that, regardless of whether commodity prices are sticky or flexible, the indeterminacy in a stochastic economy is described by the initial price level and the distribution of the prices of state-contingent nominal bonds.[2] By analogy with the terminology in the finance literature,[3] we call the distribution of the prices of state-contingent nominal bonds the nominal equivalent martingale measure. Although other variables may be used to characterize the indeterminacy, our characterization applies

[1]Chari and Kehoe (1999) survey the literature.

[2]When prices are set in advance, the initial price level is exogenously given. For such an economy, our claim should be understood as: there is indeterminacy regarding the second-period price level and the distribution of the prices of state-contingent bonds.

[3]See, for example, Duffie (2001).

to stochastic monetary economies very generally, and helps unify results.

We show that the degree of indeterminacy in an economy with imperfect competition and sticky prices is exactly the same as in the economy with perfect competition and flexible prices. The reason that the degree of indeterminacy remains the same is that, since monopolistic firms set prices and accommodate demand, predetermined prices effectively remove the same number of supply conditions and the degree of indeterminacy remains unaffected. That indeterminacy is real when prices are sticky is due to the fact that output is "demand-determined" (Carlstrom and Fuerst, 1998).

The fact that the initial price level and the nominal equivalent martingale measure are indeterminate implies that monetary policy leaves indeterminacy of degree equal to the number of the terminal nodes of the date-event tree in a finite-period model. This the reason that fiscal policy may matter for price level determination. Following Woodford (1996, 1999), Benhabib, Schmitt-Grohé and Uribe (2001, 2002), a fiscal policy rule is called "Ricardian" if it guarantees that the public debt vanishes at each terminal node for all possible, equilibrium or non-equilibrium, values of price levels and other endogenous variables. Such a fiscal policy rule does not matter for price level determination. If, however, the fiscal policy is not of this form, it may add additional restrictions on equilibrium price levels, because public debt at each terminal node must vanish in equilibrium, due to the transversality condition of households. If, for example, fiscal policy sets the composition of the debt portfolio and the level of real transfers at each date-event, then it is non-Ricardian, and the indeterminacy we discuss here vanishes.

It is worth emphasizing that the type of indeterminacy in our paper does not derive from the stability of a steady state or the infinity of the horizon.[4] For this reason, we state most of the results for finite-horizon economies. Nevertheless, one might want to restrict attention to equilibria that stay in a neighborhood of a steady state. In such a case, its stability would be important, which is related to recent discussions of the Taylor rule: although, as long as fiscal policy is Ricardian, the coefficient in the Taylor rule does not change the degree of indeterminacy, it affects the number of locally bounded equilibria.[5]

There is a vast and important literature on indeterminacy of monetary equilibria. Sargent and Wallace (1975) discussed the indeterminacy of the initial price level under interest rate policy; Lucas and Stokey (1987) derived the condition for the uniqueness of a recursive equilibrium with money supply policy; Woodford (1994) analyzed the dynamic paths of equilibria associated with the indeterminacy of the initial price level under money supply policy. In this paper, we give the exact characterization of the indeterminacy in stochastic economies in terms of the initial price level and the nominal equivalent martingale measure and extend

[4] Benhabib and Farmer (1999) is a useful survey of the literature on indeterminacy arising from the stability of a steady state.

[5] Woodford (1999) and Benhabib, Schmitt-Grohé and Uribe (2001); Benhabib, Schmitt-Grohé and Uribe (2002) examine how non-Ricardian fiscal policy interacts with the Taylor rule to obtain a unique equilibrium.

the argument to the sticky-price case. Also, we show that there is a continuum of recursive equilibria with interest rate policy. Carlstrom and Fuerst (1998) discussed the indeterminacy of sticky-price equilibria when the nominal interest rates are zero. Here, we discuss the indeterminacy in more general case. In closely related models, Dubey and Geanakoplos (1992, 2000) considered non-Ricardian fiscal policy with no transfers and Tsomokos (2001) extended their model to an open economy. Drèze and Polemarchakis (2000) and Bloise, Drèze and Polemarchakis (2003) studied the existence and indeterminacy of monetary equilibria with a particular Ricardian fiscal policy, seigniorage distributed contemporaneously as dividend to the private sector. Our paper is an application of these results to macroeconomic models.

The rest of the paper is organized as follows: In Section 2, we study a benchmark, two-period economy with flexible prices. In Section 3, we extend the indeterminacy results to sticky-price economies. In Section 4, we consider an infinite-horizon economy. Concluding remarks are in Section 5.

2 Flexible Prices

In this section, we describe the benchmark economy with flexible prices and characterize the set of equilibria . All markets are perfectly competitive. Money is valued through a cash-in-advance constraint, as in Lucas and Stokey (1987). We show that the indeterminacy is characterized by the initial price level and the probability measure associated with nominal state prices, which is referred to as the nominal equivalent martingale measure.

2.1 Households

There are three periods: $t = 0, 1, 2$. A stochastic shock, $s \in \mathcal{S} = \{1, \dots, S\}$, realizes at the beginning of the second period. Each state occurs with a probability $f(s) > 0$. Production and consumption occur in the first two periods. The last period is added for an accounting purpose, where households and the fiscal authority redeem their debt.[6]

There is a continuum of identical households, distributed uniformly over $[0, 1]$. At each date-event, households produce a single, homogeneous product. The output produced by a representative household in period 0 is y_0 and at state s in period 1, it is $y_1(s)$; consumption is c_0 and $c_1(s)$.

The preferences of the representative household are described by the lifetime expected utility

$$u\big[c_0, \overline{y}_0 - y_0\big] + \beta \sum_s u\big[c_1(s), \overline{y}_1(s) - y_1(s)\big] f(s). \tag{1}$$

[6]Section 4 extends the argument to the infinite horizon case.

Here, we interpret $\overline{y}$ as the endowment of time, and $\overline{y} - y$ as the consumption of leisure, l.[7] The flow utility function, $u(c, l)$, satisfies standard conditions:

Assumption 1. *The flow utility function, $u : \mathbb{R}^2_{++} \to \mathbb{R}$, is continuously differentiable, strictly increasing, and strictly concave. Both goods are normal:*

$$u_{11}u_2 - u_{12}u_1 < 0, \quad \textit{and} \quad u_{22}u_1 - u_{12}u_2 < 0.$$

The Inada conditions hold:

$$\lim_{c \to 0} u_1 = \lim_{l \to 0} u_2 = \infty.$$

In particular, this guarantees that $u_1(c, y-c)/u_2(c, y-c)$ is strictly decreasing in c.

We assume that a household cannot consume what it produces; instead, it has to purchase consumption goods with cash from other households.[8]

Concerning the timing of transactions we assume that at each date-event the asset market opens before the goods market. An important consequence of this assumption is that the cash the households obtain from sales of its output has to be carried over to the next period.

The representative household enters the initial period 0 with nominal assets w_0. At the beginning of the period, the fiscal authority distributes nominal transfers (taxes if negative), τ_0, across households. Then, the asset market opens, in which cash and a complete set of contingent claims are traded. Let $q(s)$ be the price of the contingent claim that pays off one unit of currency if and only if state s occurs in the next period. The budget constraint for the household in the asset market is

$$\hat{m}_0 + \sum_s q(s) b_1(s) \leq w_0 + \tau_0, \tag{2}$$

where $\hat{m}_0$ is the amount of cash obtained by the household and $b_1(s)$ the portfolio of elementary securities.

Let r_0 be the nominal interest rate in period 0, and thus, $1/(1 + r_0)$ be the price of a nominally riskless bond that pays one unit in every state of nature in the next period. The no-arbitrage condition then implies that

$$\sum_s q(s) = \frac{1}{1 + r_0}. \tag{3}$$

The market for goods opens next. The purchase of consumption goods is subject to the cash-in-advance constraint

$$P_0 c_0 \leq \hat{m}_0. \tag{4}$$

[7] In the terminology of Lucas and Stokey (1987), $\overline{y}$ and $\overline{y} - y$ are the endowment and consumption of "credit goods," and c is consumption of "cash goods."

[8] In this, we follow Lucas and Stokey (1987).

The household also receives cash by selling its product, y_0. Hence, the amount of cash that it carries over to the next period, m_0, is

$$m_0 = P_0 y_0 + \hat{m}_0 - P_0 c_0. \tag{5}$$

Given (5), the cash-in-advance constraint (4) is equivalent to the constraint

$$m_0 \geq P_0 y_0. \tag{6}$$

It turns out that (6) is more convenient than (4) to describe the cash constraint in our economy. This is due to the assumption that the asset market precedes the goods market.

The household enters state s in the second period with nominal wealth

$$w_1(s) = m_0 + b_1(s). \tag{7}$$

Substituting for $\hat{m}_0$ and $b_1(s)$ from (5) and (7) into (2) yields the flow budget constraint in period 0:

$$\begin{aligned} P_0 c_0 + \frac{r_0}{1+r_0} m_0 + \sum_s q(s) w_1(s) & \\ \leq w_0 + \tau_0 + P_0 y_0. & \end{aligned} \tag{8}$$

The household's choice in the first period is subject to the flow budget constraint (8) and the cash constraint (6).

The transactions of the household in the second period are similar, except that it faces no uncertainty. The nominal interest rate in state $s \in \mathcal{S}$ is $r_1(s)$. The flow budget constraint and the cash constraint that the household faces at state s are

$$\begin{aligned} P_1(s) c_1(s) + \frac{r_1(s)}{1+r_1(s)} m_1(s) + \frac{1}{1+r_1(s)} w_2(s) & \\ \leq w_1(s) + \tau_1(s) + P_1(s) y_1(s), & \end{aligned} \tag{9}$$

and

$$m_1(s) \geq P_1(s) y_1(s), \tag{10}$$

where $w_2(s)$ is the nominal wealth of the household at the end of state s in the second period.

In the following period, the only economic activity the household conducts is the repayment of its debt:

$$w_2(s) \geq 0. \tag{11}$$

Given this debt constraint, the flow budget constraints (8) and (9) reduce to the single, lifetime budget constraint

$$\begin{aligned} P_0 c_0 + \frac{r_0}{1+r_0} m_0 + \sum_s q(s) \left\{ P_1(s) c_1(s) + \frac{r_1(s)}{1+r_1(s)} m_1(s) \right\} & \\ \leq w_0 + \tau_0 + P_0 y_0 + \sum_s q(s) \big\{ \tau_1(s) + P_1(s) y_1(s) \big\}. & \end{aligned} \tag{12}$$

The cash constraints (6) and (10) take the form

$$\frac{r_0}{1+r_0}m_0 = \frac{r_0}{1+r_0}P_0 y_0,$$
$$\frac{r_1(s)}{1+r_1(s)}m_1(s) = \frac{r_1(s)}{1+r_1(s)}P_1(s)y_1(s),$$

because, if $r > 0$, the cash constraint binds; if $r = 0$ both sides of the equation are zero. Substituting these into the lifetime budget constraint, we obtain

$$P_0 c_0 + \sum_s q(s)P_1(s)c_1(s) \tag{13}$$
$$\leq w_0 + \tau_0 + \frac{P_0}{1+r_0}y_0 + \sum_s q(s)\left\{\tau_1(s) + \frac{P_1(s)}{1+r_1(s)}y_1(s)\right\}.$$

Given prices, P_0, $P_1(s)$, r_0, $r_1(s)$, and $q(s)$, the household chooses c_0, $c_1(s)$, y_0, and $y_1(s)$ so as to maximize utility (1) subject to the life-time budget constraint (13). The lifetime budget constraint should bind at an optimum (the transversality condition); that is,

$$w_2(s) = 0. \tag{14}$$

The first-order necessary and sufficient conditions for a solution are

$$\frac{u_1\left[c_0, \overline{y}_0 - y_0\right]}{u_2\left[c_0, \overline{y}_0 - y_0\right]} = 1 + r_0, \tag{15}$$
$$\frac{u_1\left[c_1(s), \overline{y}(s) - y_1(s)\right]}{u_2\left[c_1(s), \overline{y}(s) - y_1(s)\right]} = 1 + r_1(s), \tag{16}$$
$$\frac{\beta u_1\left[c_1(s), \overline{y}(s) - y_1(s)\right] f(s)}{u_1\left[c_0, \overline{y}_0 - y_0\right]} = \frac{q(s)P_1(s)}{P_0}. \tag{17}$$

According to (15)-(16),the marginal rate of substitution between consumption, c, and leisure, $\overline{y} - y$, equals the gross nominal interest rate; this is because production, y, is taxed at the rate of nominal interest, due to the fact that the cash obtained from selling goods should be carried over to the next period. Equation (17) is the standard Euler equation.

2.2 The monetary-fiscal authority

With similar arguments, the flow budget constraints that the monetary-fiscal authority faces are

$$\frac{r_0}{1+r_0}M_0 + \sum_s q(s)W_1(s) = W_0 + T_0, \tag{18}$$
$$\frac{r_1(s)}{1+r_1(s)}M_1(s) + \frac{1}{1+r_1(s)}W_2(s) = W_1(s) + T_1(s), \tag{19}$$

where M_0 and $M_1(s)$ are money supplies, W_0, $W_1(s)$, $W_2(s)$ are the total liabilities of the monetary-fiscal authority, and T_0 and $T_1(s)$ are aggregate transfers to the households.

Monetary policy Monetary policy sets either nominal interest rates, $r_0 \geq 0$ and $r_1(s) \geq 0$ or, alternatively, money supplies, $M_0 > 0$ and $M_1(s) > 0$.

Fiscal policy We assume that fiscal policy is Ricardian.[9] It is convenient to define the "composition" of the debt portfolio of the public sector, $\overline{W}_1(s)$, $s \in \mathcal{S}$, as

$$\sum_s \overline{W}_1(s) = 1, \quad \text{and} \quad W_1(s) = d\overline{W}_1(s),$$

where a scalar d can be interpreted as the "scale" of the debt portfolio of the public sector.

The fiscal authority sets the repayment rate $\alpha \in (0, 1]$ and the composition of the debt portfolio, $\overline{W}_1(s)$. At date 0, given W_0, r_0, M_0 and α, the transfer, $T(0)$, is determined by

$$T(0) = \frac{r_0}{1 + r_0} M_0 - \alpha W_0,$$

and the scale of debt portfolio, d, is determined by the flow budget constraint

$$d = \frac{1}{\sum_s q(s)\overline{W}_1(s)} (1 - \alpha) W_0.$$

At each state s in period 1, $T_1(s)$ is set as

$$T_1(s) = \frac{r(s)}{1 + r(s)} M_1(s) - W_1(s).$$

Note that the revenue of the monetary-fiscal authority at date 0 is $[r_0/(1+r_0)]M_0 - T_0$; thus, α is the fraction of its initial debt, W_0, which is redeemed at date 0. This fiscal policy rule is Ricardian in that

$$W_2(s) = 0, \tag{20}$$

at all $s \in \mathcal{S}$, and for all possible, equilibrium or non-equilibrium, values of P, r, and M.

2.3 Equilibrium conditions

Since households are identical, the market clearing conditions are

$$c_0 = y_0, \quad c_1(s) = y_1(s),$$
$$m_0 = M_0, \quad m_1(s) = M_1(s),$$
$$w_1(s) = W_1(s), \quad w_2(s) = W_2(s).$$

Also, consistency requires that

$$\tau_0 = T_0, \quad \tau_1(s) = T_1(s), \quad w_0 = W_0.$$

[9]The fiscal policy rule we consider is a stochastic analogue of the policy considered by Benhabib, Schmitt-Grohé and Uribe (2001, 2002).

The no-arbitrage condition (3) implies that the prices of elementary securities, $q(s)$, $s \in \mathcal{S}$, can be written as

$$q(s) = \frac{\mu(s)}{1 + r_0}, \tag{21}$$

for some $\mu(s)$, $s \in \mathcal{S}$, satisfying

$$\sum_s \mu(s) = 1.$$

It follows that μ is viewed as a probability measure over $\mathcal{S}$, and called the *nominal equivalent martingale measure*. We shall see that there are no equilibrium conditions that determine μ, regardless of whether monetary policy sets interest rates or money supplies. A competitive equilibrium with interest rate policy is defined as follows:

Definition. Given initial nominal wealth, $w_0 = W_0$, interest rate policy, $\{r_0, r_1(s)\}$, and fiscal policy, $\{\alpha, \overline{W}_1(s)\}$, a competitive equilibrium consists of an allocation, $\{c_0, c_1(s), y_0, y_1(s)\}$, a portfolio of households, $\{m_0, m_1(s), w_1(s), w_2(s)\}$, a portfolio of the monetary-fiscal authority, $\{M_0, M_1(s), W_1(s), W_2(s)\}$, transfers, $\{T_0, T_1(s)\}$, spot-market prices, $\{P_0, P_1(s)\}$ and a nominal equivalent martingale measure, μ, such that

1. given W_0 and $\{r_0, r_1(s), M_0, M_1(s)\}$, fiscal policy $\{\alpha, \overline{W}_1(s)\}$ determines transfers $\tau_0 = T_0$ and $\tau_1(s) = T_1(s)$, $s \in \mathcal{S}$, and debt portfolio $\{W_1(s), W_2(s)\}$;

2. the monetary authority accommodates the money demand, $M_0 = m_0$ and $M_1(s) = m_1(s)$;

3. given interest rates, r_0, $r_1(s)$, spot-market prices, $p_0(j) = P(0)$, $p_1(s, j) = P_1(s)$, all j, nominal equivalent martingale measure, μ, and transfers, τ_0, $\tau_1(s)$, the household's problem is solved by c_0, $c_1(s)$, y_0, $y_1(s)$, m_0, $m_1(s)$, $w_1(s)$, and $w_2(s)$;

4. all markets clear.

We restrict attention to symmetric equilibria. A competitive equilibrium with money supply policy is similarly defined.

2.4 Equilibria with interest rate policy

The existence of equilibrium with interest rate policy requires further restrictions on the flow utility function.

Assumption 2. *The flow utility function, u, satisfies*

$$\lim_{c \to 0} \frac{u_1(c, y - c)}{u_2(c, y - c)} = \infty,$$

for each $y > 0$.

The following proposition shows that P_0 and μ are not determined, and hence, there is S-dimensional indeterminacy.

Proposition 1. *Given initial nominal wealth, $w_0 = W_0$, interest rate policy, $\{r_0, r_1(s)\}$, and fiscal policy, $\{\alpha, \overline{W}_1(s)\}$,*

1. *a competitive equilibrium exists;*
2. *the equilibrium allocation $\{c_0, c_1(s), y_0, y_1(s)\}$ is unique;*
3. *the initial price, P_0, and the nominal equivalent martingale measure, μ, are indeterminate: for any $P_0 > 0$ and for any strictly positive probability measure μ, any prices and portfolio $\{P_1(s), M_0, M_1(s), W_1(s)\}$ satisfying*

$$\frac{P_1(s)}{P_0} = \frac{\beta u_1[c_1(s), \overline{y}_1(s) - y_1(s)] f(s)}{u_1[c_0, \overline{y}_0 - y_0]} \frac{1 + r_0}{\mu(s)},$$

$$M_0 \geq P_0 c_0, \quad M_1(s) \geq P_1(s) c_1(s), \qquad (\textit{equality if } r_0, r_1(s) > 0),$$

$$W_1(s) = (1 - \alpha)(1 + r_0) W_0 \frac{\overline{W}_1(s)}{\sum_s \mu(s) \overline{W}_1(s)}$$

support the allocation $\{c_0, c_1(s), y_0, y_1(s)\}$.

Proof Given interest rates r_0 and $r_1(s)$, $s \in \mathcal{S}$, the first-order conditions (15)-(16) determines the allocation of resources at each date-event:

$$\frac{u_1\left[c_0, \overline{y}_0 - c_0\right]}{u_2\left[c_0, \overline{y}_0 - c_0\right]} = 1 + r_0$$

$$\frac{u_1\left[c_1(s), \overline{y}_1(s) - c_1(s)\right]}{u_2\left[c_1(s), \overline{y}_1(s) - c_1(s)\right]} = 1 + r_1(s).$$

Our assumptions on u guarantees the existence and uniqueness of the solutions to these equations. The equilibrium output at each date-event, y_0 and $y_1(s)$, $s \in \mathcal{S}$, is given by

$$y_0 = c_0, \quad \text{and} \quad y_1(s) = c_1(s).$$

Fiscal policy sets transfers so that $w_2(s) = W_2(s) = 0$, all s. Hence, the allocation is uniquely determined. It is straightforward to see that given any $P_0 > 0$, and μ, the prices and portfolio constructed as in the proposition support the equilibrium allocation. $\square$

The indeterminacy of μ implies that the inflation rate, $\pi_1(s) \equiv P_1(s)/P_0$, is indeterminate. Thus, interest rate policy does not determine the stochastic path of inflation. The indeterminacy, which is purely nominal here, becomes real if, for example, prices are sticky (section 3-4); the asset market is incomplete (Nakajima and Polemarchakis, 2001); or the timing of the markets is different. Also shocks could be purely extrinsic. If $r_1(s)$ and $\overline{y}_1(s)$

are identical for all s, there is no uncertainty in "fundamentals;" nevertheless, there are equilibria in which the inflation rate, $P_1(s)/P_0$, varies across states.

The reason that $P(0)$ and μ are indeterminate is simple, and closely related to the well known fact that only relative prices are determined in equilibrium. As discussed in Introduction, in an economy where money only serves as a unit of account (without monetary policy), the initial price level P_0 and the nominal state prices $q(s)$ are indeterminate. Now consider our economy in which monetary policy sets nominal interest rates. Interest rate policy does two things: (i) it adds one restriction on the nominal state prices as shown in the no-arbitrage condition (3); (ii) it determines the relative prices of consumption goods and real balances: $r_0/[1+r_0]$, $r_1(s)/[1+r_1(s)]$, as shown in equation (12). The latter determines the equilibrium quantities of real money balances, but does not reduce the indeterminacy; the former determines the sum of the nominal state prices, $q(s)$, but their distribution, μ, remains indeterminate. This is why in an stochastic economy with active monetary policy indeterminacy is characterized by P_0 and μ.

Note that the degree of indeterminacy here, S, equals the number of terminal nodes. It is straightforward to show that this is true in a general finite-horizon economy. The fact that monetary policy leaves the indeterminacy whose degree equals the number of terminal nodes is the key to understand why certain forms of fiscal policy may lead to determinacy. Write the debt the public sector leaves at the end of the second period as a function of prices: $W_2(s; P_0, \mu(s), P(s), s \in \mathcal{S})$. If the policy is Ricardian this is identically zero for any value of P_0, $\mu(s)$, $P(s)$. But if it is not, the equilibrium requirement that

$$W_2(s; P_0, \mu(s), P(s), s \in \mathcal{S}) = 0$$

may introduce S additional, nontrivial conditions on equilibrium prices. An example of such non-Ricardian policy considered by Woodford (1994, 1996, 1999) and Benhabib, Schmitt-Grohé and Uribe (2001, 2002), among others, is to set real transfers, $H \equiv T/P$, and the composition of the portfolio, $\overline{W}$. Such fiscal policy, if feasible, eliminates the indeterminacy discussed here. To see this, note that the relative prices, $q(s)P_1(s)/P_0 = \beta u_1(s)f(s)/u_1(0)$, and real balances, $M/P = c$, are determined by the real allocation. Thus, given the initial public debt, W_0, the intertemporal budget constraint of the public sector:

$$\frac{W_0}{P_0} = \frac{r_0}{1+r_0}c_0 - H_0 + \sum_{s \in \mathcal{S}} \frac{q(s)P_1(s)}{P_0}\left(\frac{r_1(s)}{1+r_1(s)}c_1(s) - H_1(s)\right)$$

determines the initial price level P_0. After some algebra, we can show that, given P_0, the scale of the debt portfolio, d, is determined by

$$d = P_0(1+r_0)\sum_{s \in \mathcal{S}} \frac{1}{\overline{W}_1(s)} \frac{q(s)P_1(s)}{P_0}\left(\frac{r_1(s)}{1+r_1(s)}c_1(s) - H_1(s)\right).$$

Given d, the public debt in state s is $W_1(s) = d\overline{W}_1(s)$, and the price level in that state,

$P_1(s)$, is determined by

$$\frac{W_1(s)}{P_1(s)} = \frac{r_1(s)}{1 + r_1(s)} c_1(s) - H_1(s).$$

Thus, the nominal indeterminacy vanishes with non-Ricardian fiscal policy, $\{H_0,\ H_1(s),\ \overline{W}(s)\}$.

Concerning the indeterminacy that obtains, further remarks are in order:

1. The absence of real effects derives from the completeness of the asset market: with nominal assets and less than a full set of elementary securities or equivalent, different distributions of the rate of inflation across states of the world affect the attainable reallocations of revenue and, as a consequence, the allocation of resources in the presence of heterogeneous households. This is precisely the logic of the argument in the literature on real indeterminacy with incomplete markets following Balasko and Cass (1989), Cass (1985) and Geanakoplos and Mas-Colell (1989).

2. When nominal indeterminacy has real effects, as is the case with an incomplete asset market, it is of interest to distinguish between real and nominal indeterminacy. It is here that the characterization of indeterminacy by the initial price level an a nominal equivalent martingale measure is advantageous: variations in the measure have real effects, while variations in the price level do not.

3. The portfolio policy of the public sector, the composition of the portfolio $\{\overline{W}_1(s)\}$, does not affect the allocation of resources at equilibrium; this is an instance of Ricardian equivalence.

2.5 Equilibria with money supply policy

Consider money supply policy, $\{M_0, M_1(s)\}$. Define c_0^* and $c_1^*(s)$, implicitly by

$$\frac{u_1\left[c^*, \overline{y}_0 - c_0^*\right]}{u_2\left[c^*, \overline{y}_0 - c_0^*\right]} = \frac{u_1\left[c^*, \overline{y}_1(s) - c_1^*(s)\right]}{u_2\left[c^*, \overline{y}_1(s) - c_1^*(s)\right]} = 1.$$

Such a c^* exists and is unique under Assumption 1; indeed, c^* is the level of consumption when the nominal interest rate is zero. With money supply policy, the following assumption simplifies the argument.

Assumption 3. *The flow utility function, u, has the property that for all $y > 0$,*

$$\lim_{c \to 0} c u_1(c, y - c) = 0,$$

and the function $c u_1(c, y - c)$ is monotonically increasing in the interval $(0, c^*(y))$.

According to the proposition that follows, with money supply policy there is the same degree of indeterminacy as with interest rate policy, but indeterminacy is real.

Proposition 2. *Given initial nominal wealth, $w_0 = W_0$, money supply policy, $\{M_0, M_1(s)\}$ and fiscal policy, $\{\alpha, \overline{W}_1(s)\}$,*

1. *a competitive equilibrium exists;*
2. *the initial price, P_0, and the nominal equivalent martingale measure, μ, are indeterminate. For any strictly positive P_0 and μ, there exists a unique competitive equilibrium corresponding to them.*
3. *the indeterminacy regarding P_0 and μ is real: different P_0 or different μ are associated with different allocations as well as different inflation rates.*

Proof Let the initial price P_0 and the strictly positive probability measure μ be arbitrarily given. Let M_0 and $M_1(s)$ be the money supplies chosen by the policy. Given M_0 and P_0, c_0, y_0, and r_0 are determined by

$$c_0 = \min\left\{\frac{M_0}{P_0}, c_0^*\right\}, \quad 1 + r_0 = \frac{u_1[c_0, \overline{y}_0 - c_0]}{u_2[c_0, \overline{y}_0 - c_0]},$$

and $y_0 = \overline{y}_0 - c_0$. At state s in the second period, if

$$M_1(s) > \frac{\beta c_1^*(s) u_1[c_1^*(s), \overline{y}_1(s) - c_1^*(s)]}{u_2[c_0, \overline{y}_0 - c_0]} \frac{f(s)}{\mu(s)} P_0,$$

then let $c_1(s) = c_1^*(s)$. Otherwise, $c_1(s)$ is a solution to

$$M_1(s) = \frac{\beta c_1(s) u_1[c_1(s), \overline{y}_1(s) - c_1(s)]}{u_2[c_0, \overline{y}_0 - c_0]} \frac{f(s)}{\mu(s)} P_0.$$

The unique existence of a solution is guaranteed by Assumption 3. Given $c_1(s)$, $y_1(s) = \overline{y}_1(s) - c_1(s)$,

$$P_1(s) = \frac{\beta u_1[c_1(s), \overline{y}_1(s) - c_1(s)]}{u_2[c_0, \overline{y}_0 - c_0]} \frac{f(s)}{\mu(s)} P_0,$$

and

$$1 + r_1(s) = \frac{u_1[c_1(s), \overline{y}_1(s) - c_1(s)]}{u_2[c_1(s), \overline{y}_1(s) - c_1(s)]}.$$

Given the path of nominal interest rates, $\{r_0, r_1(s)\}$, the debt portfolio, $\{W_1(s)\}$, is determined as in the proof of the previous proposition. □

That money supply policy leaves the same degree of indeterminacy as interest rate policy is intuitive, because both types of monetary policy set the same number of variables (the former sets the supplies, M, and the latter sets the prices, r). The indeterminacy with money supply policy is real because the cash constraint implies that $Pc \leq M$ in equilibrium: different P would be associated with different c.

Assumption 3 is made to simplify the proof, but from the construction of equilibria in the proof it is clear that the claim holds more generally. For example, define the set $A(s)$ by

$$A(s) = \{cu_1[c, \overline{y}(s) - c] \,:\, 0 < c \leq c^*(s)\}. \tag{22}$$

If $A(s)$ has non-empty interior for all s, then the claim of the proposition holds with a slight modification. It is worth noting, however, that the log utility does not satisfy the non-empty interior condition and our characterization does not apply. In the log utility case, P_0 and μ are determined, but $P_1(s)$ are not: different $P_1(s)$ are associated with different allocations in the second period, $c_1(s)$.

Given recent discussions on the "liquidity trap," the following corollary of the proposition would be of some interest.[10] It says that, as long as money supply does not decrease too much in the second period, there always exists an equilibrium in which the nominal interest rate equals zero at all date-events.

Corollary 3. *Given initial nominal wealth, $w_0 = W_0$, interest rate policy, $\{r_0, r_1(s)\}$, and fiscal policy, $\{\alpha, \overline{W}_1(s)\}$, if*

$$\frac{M_1(s)}{M_0} \geq \max_s \frac{\beta u_1[c_1^*(s), \overline{y}_1(s) - c_1^*(s)]}{u_2[c_0^*, \overline{y}_0 - c_0^*]},$$

then there exists a competitive equilibrium in which the nominal interest rate is identically zero, $r_0 = r_1(s) = 0$, all s.

Proof Choose any $P_0 \leq M_0/c_0^*$ and let $\mu = f$. Then, it is straightforward to see that the following allocation and price system constitute an equilibrium.

$$c_0 = c_0^*, \qquad c_1(s) = c_1^*(s),$$

$$y_0 = \overline{y}_0 - c_0^*, \qquad y_1(s) = \overline{y}_1(s) - c_1^*(s),$$

$$P_1 = P_0 \frac{\beta u_1[c_1^*(s), \overline{y}_1(s) - c_1^*(s)]}{u_2[c_0^*, \overline{y}_0 - c_0^*]},$$

and

$$r_0 = r_1(s) = 0.$$

$\square$

[10] For example, Benhabib, Schmitt-Grohé and Uribe (2002) and Woodford (1999). Note, however, that in this model, a zero interest rate equilibrium is (constrained) optimal. A related fact that when the nominal interest rate is zero the money supply is indeterminate is discussed by Carlstrom and Fuerst (1998) and Adao, Correia and Teles (2001).

3 Sticky Prices

We have seen that in a flexible price economy there is indeterminacy, which is indexed by the initial price level, P_0, and the nominal equivalent martingale measure, μ. The question addressed in this section is whether or not introducing price stickiness modifies the result. Specifically, we consider an economy where prices are set in advance: the initial price level P_0 is exogenously given, and the second period prices are set in the first period and independent of the realization of states, $P_1(s) = P_1$, all $s \in \mathcal{S}$.

It might appear that such price stickiness would affect the determinacy of equilibria, because it introduces S restrictions on equilibrium prices: the price in the initial period is exogenously given and the prices in the second period must be identical across states. Indeed, if those restrictions are simply added to the flexible-price economy, equilibrium becomes determinate. To see this, consider interest rate policy, and let P_1 be the price level in the second period, which does not depend on states s. Remember that interest rate policy determines an allocation uniquely. Then the inflation rate, P_1/P_0, is uniquely determined by the condition that

$$\frac{P_1}{P_0} = \sum_s \frac{\beta u_1\left[c_1(s), \overline{y}_1(s) - c_1(s)\right] f(s)}{u_2\left[c_0, \overline{y}_0 - c_0\right]}.$$

Given P_1/P_0, $\mu(s)$ is, in turn, determined by

$$\mu(s) = \frac{\beta u_1\left[c_1(s), \overline{y}_1(s) - c_1(s)\right] f(s)}{u_2\left[c_0, \overline{y}_0 - c_0\right]} \frac{P_0}{P_1}.$$

However, this only captures the uniqueness of a "flexible-price equilibrium" with the property that the second-period prices are identical across states. It does not imply the uniqueness of a "sticky-price equilibrium" in which all price-setters explicitly take into account the constraint that the second-period price must be set in advance. Indeed, we shall see that the sticky-price equilibrium is indeterminate, that the degree of indeterminacy is exactly the same as the one associated with flexible-price equilibrium, and that the indeterminacy is real even with interest rate policy.

As in the previous section, there is a continuum of households, which are distributed uniformly over $[0, 1]$. However, they differ in what they produce: at each date-event, household $j \in [0, 1]$ produces a differentiated product j. Let $y_0(j)$ and $y_1(s, j)$ denote the amounts of output produced by household j in period 0 and at state $s \in \mathcal{S}$ in period 1, respectively. The amounts of commodity $i \in [0, 1]$ consumed by household j are denoted by $c_0^j(i)$ and $c_1^j(s, i)$.

The lifetime utility of household j is modified as

$$u\left[c_0^j, \overline{y}_0 - y_0(j)\right] + \beta \sum_s u\left[c_1^j(s), \overline{y}_1(s) - y_1(s, j)\right] f(s), \tag{23}$$

where c_0^j and $c_1^j(s)$ are consumption of the "composite" goods defined by

$$c_0^j = \left\{ \int_0^1 \left[c_0^j(i)\right]^{\frac{\theta-1}{\theta}} di \right\}^{\frac{\theta}{\theta-1}},$$

$$c_1^j(s) = \left\{ \int_0^1 \left[c_1^j(s,i)\right]^{\frac{\theta-1}{\theta}} di \right\}^{\frac{\theta}{\theta-1}}.$$

Let $p_0(i)$ and $p_1(s,i)$ be the spot prices of good i in period 0 and at state s in period 1, respectively. Then, the prices of the composite goods, P_0 and $P_1(s)$, are given by

$$P_0 = \left\{ \int_0^1 \left[p_0(i)\right]^{1-\theta} di \right\}^{\frac{1}{1-\theta}},$$

$$P_1(s) = \left\{ \int_0^1 \left[p_1(s,i)\right]^{1-\theta} di \right\}^{\frac{1}{1-\theta}}.$$

The household's cost minimization leads to

$$P_0 c_0^j = \int_0^1 p_0(i) c_0^j(i)\, di,$$

$$P_1(s) c_1^j(s) = \int_0^1 p_1(s,i) c_1^j(s,i)\, di.$$

Let c_0 and $c_1(s)$ be the aggregate consumption at date 0 and state s, that is,

$$c_0 = \int_0^1 c_0^j\, dj, \quad \text{and} \quad c_1(s) = \int_0^1 c_1^j(s)\, dj.$$

The demand for product j is then

$$y_0(j) = \left(\frac{p_0(j)}{P_0}\right)^{-\theta} c_0, \tag{24}$$

$$y_1(s,j) = \left(\frac{p_1(s,j)}{P_1(s)}\right)^{-\theta} c_1(s). \tag{25}$$

Suppose that the initial prices $p_0(j)$, $j \in [0,1]$, are given and identical for all j:

$$p_0(j) = \overline{p}, \qquad j \in [0,1].$$

It follows that $P_0 = \overline{p}$. In the first period, each household $j \in [0,1]$ chooses the second-period price, $p_1(s,j)$, before observing the shock s. It follows that the second-period price is identical across state, so that it is written as

$$p_1(s,j) = p_1(j), \qquad j \in [0,1],$$

for some $p_1(j)$. Given $\overline{p}$ and $p_1(j)$, the household must supply product j by the amount equal to the demand:

$$y_0(j) = \left(\frac{\overline{p}}{P_0}\right)^{-\theta} c_0, \tag{26}$$

$$y_1(s,j) = \left(\frac{p_1(j)}{P_1(s)}\right)^{-\theta} c_1(s). \tag{27}$$

Given prices, P_0, $P_1(s)$, r_0, $r_1(s)$, $\mu(s)$, and $\overline{p}$, household j chooses c_0^j, $c_1^j(s)$, and $p_1(j)$ so as to maximize the lifetime expected utility (1) subject to the demand functions (26)-(27) and the life-time budget constraint:

$$\begin{aligned}
P_0 c_0^j + \sum_s \frac{\mu(s)}{1+r_0} P_1(s) c_1^j(s) & \qquad (28)\\
\leq w_0 + \tau_0 + \frac{1}{1+r_0}\overline{p} y_0(j) & \\
+ \sum_s \frac{\mu(s)}{1+r_0}\left\{\tau_1(s) + \frac{1}{1+r_1(s)} p_1(j) y_1(s,j)\right\}. &
\end{aligned}$$

The first-order conditions with respect to c_0 and $c_1(s)$ lead to

$$\frac{\beta u_1\left[c_1^j(s), \overline{y}_1(s) - y_1(s,j)\right] f(s)}{u_1\left[c_0^j, \overline{y}_0(s) - y_0(j)\right]} = \frac{P_1(s)}{P_0}\frac{\mu(s)}{1+r_0}. \tag{29}$$

The first-order condition with respect to $p_1(j)$ is given by

$$\frac{\sum_s u_1\left[c_1^j(s), \overline{y}_1(s) - y_1(s,j)\right] f(s) / \left[1 + r_1(s)\right]}{\sum_s u_2\left[c_1^j(s), \overline{y}_1(s) - y_1(s,j)\right] f(s)} = \frac{\theta}{\theta - 1}, \tag{30}$$

where we have used the (symmetric) equilibrium condition that $p_1(j) = P_1$, all $j \in [0,1]$.

The next proposition shows that the sticky-price economy has exactly the same degree of indeterminacy as the flexible-price economy, and that the indeterminacy is indexed by P_1 and μ. Here, however, indeterminacy is real even with interest rate policy.

Proposition 4. *If second-period prices are set in advance, given initial nominal wealth, $w_0 = W_0$ and price level, $P_0 = \overline{p}$, interest rate policy, $\{r_0, r_1(s), s \in \mathcal{S}\}$, and fiscal policy, $\{\alpha \in (0,1], \overline{W}_1(s)\}$,*

1. *a competitive equilibrium exists;*
2. *the price level in the second period, P_1, and the nominal equivalent martingale measure, μ, are indeterminate;*
3. *the indeterminacy is real: different P_1 or different μ are associated with different allocations.*

Proof Let P_1 and μ be given. Then the first-order conditions (29) imply that equilibrium consumption, c_0 and $c_1(s)$ should satisfy

$$\frac{\beta u_1\left[c_1(s), \overline{y}_1(s) - c_1(s)\right] f(s)}{u_1\left[c_0, \overline{y}_0 - c_0\right]} = \frac{P_1}{P_0}\frac{\mu(s)}{1+r_0}. \tag{31}$$

Under our assumptions, these equations can be solved for $c_1(s)$ as strictly increasing functions of c_0. Write them as

$$c_1(s) = \phi_s(c_0), \qquad s \in \mathcal{S},$$

where $\lim_{c\to 0}\phi_s(c) = 0$ and $\lim_{c\to\overline{y}_0}\phi_s(c) = \overline{y}_1(s)$. The first-order condition (30) then implies that

$$\frac{\sum_s u_1\big[\phi_s(c_0), \overline{y}_1(s) - \phi_s(c_0)\big] f(s)/\big[1 + r_1(s)\big]}{\sum_s u_2\big[\phi_s(c_0), \overline{y}_1(s) - \phi_s(c_0)\big]} = \frac{\theta}{\theta - 1}.$$

Under our assumptions, there is a unique c_0 that satisfies this equation. This completes the proof. □

The reason why the degree of indeterminacy remains the same is simple. As is often stated, output is "demand-determined" with sticky prices. That is to because fixing a certain number of prices removes the same number of supply conditions. To see this in our model, note that if households could choose their prices freely, then household j would set their prices, $p_0(j)$ and $p_1(s,j)$, following the simple markup formula:

$$\frac{p_0(j)}{P_0}\frac{1}{1+r_0} = \frac{u_2[c_0^j, \overline{y}_0 - y_0(j)]}{u_1[c_0^j, \overline{y}_0 - y_0(j)]}, \tag{32}$$

$$\frac{p_1(s,j)}{P_1(s)}\frac{1}{1+r_1(s)} = \frac{\theta}{\theta-1}\frac{u_2[c_1^j(s), \overline{y}_1(s) - y_1(s,j)]}{u_1[c_0^j, \overline{y}_1(s) - y_1(s,j)]}. \tag{33}$$

Here, $p(j)/(1+r)$ is the "after-tax" price of product j, because cash obtained by selling consumption goods must be carried over to the next period; $\theta/(\theta-1)$ is the markup; and u_2/u_1 is the cost of labor. Those $S+1$ equations are the supply conditions in the flexible-price economy with monopolistic competition. Now turn back to our sticky-price economy where prices are set in advance. The initial price is exogenously given: $p_0(j) = \overline{p}$, which assumes that each household should supply its product by the exact amount demanded by other households at the predetermined price $\overline{p}$. Thus, equation (32) is no longer used to determine the level of output, $y_0(j)$. The second-period prices should be identical across states: $p_1(s,j) = p_1(j)$, all s, for some $p_1(j)$. This requires that conditions (33) be replaced by the single equation (30), which says that the markup formula (33) only holds in expectation. Thus, prices being set in advance introduces S restrictions on equilibrium prices, but it removes S supply conditions at the same time. Therefore, the degree of indeterminacy is unchanged.

The fact that different prices are associated with different allocations is clearly seen in the demand equations (31). As in the previous section, considering money supply policy instead of interest rate policy does not change the degree of indeterminacy. In the T-period economy, P_1 and μ are not determined. The degree of indeterminacy is therefore equal to the number of terminal nodes, just as in the two-period economy.

3.1 Staggered Price-Setting

To see that the above results are robust regarding the form of price stickiness, we consider another version of sticky prices: prices set in a staggered manner.

Suppose that at the beginning of the initial period each household is allocated into one of two groups. Households in the first group must set the first-period price of its product, $p_0(j)$, at $\overline{p}$, but they can charge the second-period price, $p_1(s,j)$, freely. Households in the second group, on the other hand, can charge the first-period price freely, but they must charge the same price in the second period, thus $p_0(j) = p_1(s,j)$, all $s \in \mathcal{S}$. The allocation of households into these two groups is done stochastically, and the probability that each household is allocated to each group is 1/2. We assume that there is perfect risk sharing among households.

A further restriction of the flow utility function simplifies the argument.

Assumption 4. *The flow utility function is additively separable:* $u(c) + v(\overline{y} - y)$*, with the property that*

$$\lim_{c \to 0} cu'(c) > 0.$$

The lifetime expected utility of household j is then written as

$$\frac{1}{2}\Big\{u\big[c_0^1\big] + v\big[\overline{y}_0 - y_0^1(j)\big] + \beta\sum_s\Big(u\big[c_1^1(s)\big] + v\big[\overline{y}_1(s) - y_1^1(s,j)\big]\Big)\Big\}$$
$$+ \frac{1}{2}\Big\{u\big[c_0^2\big] + v\big[\overline{y}_0 - y_0^2(j)\big] + \beta\sum_s\Big(u\big[c_1^2(s)\big] + v\big[\overline{y}_1(s) - y_1^2(s,j)\big]\Big)\Big\}$$

where c_0^i and $c_1^i(s)$, $i = 1, 2$, are consumption when the household is allocated to group i, and $y_0^i(j)$ and $y_1^i(s,j)$ are production of product j. Let $p_1(s,j)$ be the price charged in the second period at state s if the household is allocated to the first group; $p_0(j)$ be the price charged in both periods if the household is allocated to the second group. It follows that $y_0^i(j)$ and $y_1^i(s,j)$ are given by

$$y_0^1(j) = \left(\frac{\overline{p}}{P_0}\right)^{-\theta} c_0, \qquad y_1^1(s,j) = \left(\frac{p_1(s,j)}{P_1(s)}\right)^{-\theta} c_1(s), \quad s \in \mathcal{S},$$
$$y_0^2(j) = \left(\frac{p_0(j)}{P_0}\right)^{-\theta} c_0, \qquad y_1^2(s,j) = \left(\frac{p_0(j)}{P_1(s)}\right)^{-\theta} c_1(s), \quad s \in \mathcal{S},$$

where c_0 and $c_1(s)$ denote aggregate consumption.

Since there is perfect risk sharing among households, consumption is identical between the two groups:

$$c_0^1 = c_0^2 = c_0, \quad \text{and} \quad c_1^1(s) = c_1^2(s) = c_1(s), \quad s \in \mathcal{S}.$$

The first-order conditions with respect to c_0 and $c_1(s)$ lead to

$$\frac{\beta u'\big[c_1(s)\big]f(s)}{u'\big[c_0\big]} = \frac{P_1(s)}{P_0}\frac{\mu(s)}{1+r_0}. \tag{34}$$

The first-order condition with respect to the second-period price charged by the household in the first group, $p_1(s,j)$, is given as: for each $s \in \mathcal{S}$,

$$v'\big[\overline{y}_1(s) - y_1^1(s,j)\big] = u'\big[c_1(s)\big]\frac{p_1(s,j)}{P_1(s)}\frac{\theta-1}{\theta}\frac{1}{1+r_1(s)}. \tag{35}$$

The first-order condition with respect to the price charged in both periods by the second group of households, $p_0(j)$, is

$$\begin{aligned} & y_0^2(j)\left(v'\left[\overline{y}_0 - y_0^2\right] - u'\left[c_0\right]\frac{p_0(j)}{P_0}\frac{\theta-1}{\theta}\frac{1}{1+r_0}\right) \\ & \quad + \beta\sum_s y_1^2(s,j)\left(v'\left[\overline{y}_1(s) - y_1^2(s)\right] - u'\left[c_1(s)\right]\frac{p_0(j)}{P_1(s)}\frac{\theta-1}{\theta}\frac{1}{1+r_1(s)}\right)f(s) = 0. \end{aligned} \tag{36}$$

In a symmetric equilibrium, households in the same group choose the same prices, so that we can write

$$\begin{aligned} p_0(j) = p_0, \qquad & p_1(s,j) = p_1(s), \\ y_0^1(j) = y_0^1, \qquad & y_0^2(j) = y_0^2, \\ y_1^1(s,j) = y_1^1(s), \qquad & y_1^2(s,j) = y_1^2(s). \end{aligned}$$

By definition, the price levels, P_0 and $P_1(s)$, are given by

$$P_0 = \left[\frac{1}{2}\overline{p}^{1-\theta} + \frac{1}{2}p_0^{1-\theta}\right]^{\frac{1}{1-\theta}}, \tag{37}$$

$$P_1(s) = \left[\frac{1}{2}p_0^{1-\theta} + \frac{1}{2}p_1(s)^{1-\theta}\right]^{\frac{1}{1-\theta}}. \tag{38}$$

Note that $P_1(s)/P_0$ is an increasing function of $p_1(s)/P_1(s)$, and $p_0/P_1(s)$ is a decreasing function of $p_1(s)/P_1(s)$. Production of differentiated products is given by

$$y_0^1 = \left(\frac{\overline{p}}{P_0}\right)^{-\theta} c_0, \qquad y_1^1(s) = \left(\frac{p_1(s)}{P_1(s)}\right)^{-\theta} c_1(s), \tag{39}$$

$$y_0^2 = \left(\frac{p_0}{P_0}\right)^{-\theta} c_0, \qquad y_1^2(s,j) = \left(\frac{p_0}{P_1(s)}\right)^{-\theta} c_1(s). \tag{40}$$

Consider interest rate policy, $\{r_0, r_1(s), s \in \mathcal{S}\}$. The next proposition shows that this economy, once again, has S-dimensional indeterminacy, indexed by the initial price P_0 and the nominal equivalent martingale measure μ.

Proposition 5. *If second-period prices are set in a staggered manner, given initial nominal wealth, $w_0 = W_0$, and price level for the constrained households, $\overline{p}$, interest rate policy, $\{r_0, r_1(s), s \in \mathcal{S}\}$, and fiscal policy, $\{\alpha \in (0,1], \overline{W}_1(s)\}$,*

1. *a competitive equilibrium exists;*
2. *the initial price level, P_0, and the nominal equivalent martingale measure, μ, are indeterminate;*
3. *the indeterminacy is real: different P_0 or different μ are associated with different allocations.*

Proof Let P_0 and μ be given. Note that P_0 determines p_0 by (37). Consider the first-order conditions (35):

$$v'\left(\overline{y}_1(s) - \left(\frac{p_1(s)}{P_1(s)}\right)^{-\theta}\right) c_1(s) = u'\left[c_1(s)\right] \frac{p_1(s)}{P_1(s)} \frac{\theta - 1}{\theta} \frac{1}{1 + r_1(s)},$$

for each $s \in \mathcal{S}$. Given our assumption, these equations imply that $c_1(s)$ is a strictly increasing function of $p_1(s)/P_1(s)$. Given the fact that $P_1(s)/P_0$ is a strictly increasing function of $p_1(s)/P_1(s)$, the first-order conditions (34) determines $p_1(s)/P_1(s)$, as a strictly increasing function of c_0. Then, consider (36), and note that the left-hand side of this equation is strictly increasing in c_0 and becomes negative as $c_0 \to 0$. Hence, there is a unique c_0. Note that $c_1(s)$, $p_1(s)/P_1(s)$, and $P_1(s)/P_0$ are functions of c_0 and derived above. This completes the proof. $\square$

The intuition why the degree of indeterminacy remains the same is the same as in the case of prices set in advance: fixing a certain number of prices removes the same number of supply conditions.

4 Infinite Horizon

It is straightforward to see that results in the previous sections extend to the infinite-horizon economies. A general argument would be to construct an equilibrium in the infinite-horizon economy as the limit of a sequence of equilibria of finite-horizon economies. Given that most macro models are set in infinite horizon, however, we provide more detailed analysis on the infinite-horizon economy. For simplicity, we consider the case of flexible prices, and show that results in Section 2 generalize to the infinite-horizon economy. We also discuss recursive equilibria and contrast our result with the one obtained by Lucas and Stokey (1987).

Suppose that shocks follow a Markov chain with transition probabilities $f(s'|s) > 0$. The history of shocks up through date t is denoted by $s^t = (s_0, \ldots, s_t)$, and called a date-event. The initial shock, s_0, is given, and the initial date-event is denoted by 0. The probability of date-event s^t is $f(s^t)$. Successors of date-event s^t is $s^{t+j}|s^t$. For $s^{t+j|s^t}$, the probability that s^{t+j} occurs, conditional on s^t, is $f(s^{t+j}|s^t)$.

Let μ denote the nominal equivalent martingale measure. It is a probability measure over the date-event tree with $\mu(s^t) > 0$, all s^t. Let $q(s^{t+1}|s^t)$ denote the price at s^t of the elementary security that pays off one unit of currency if and only if the date-event s^{t+1} is reached. Then,

$$q(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)},$$

where $r(s^t)$ is the nominal interest rate at s^t. More generally,

$$q(s^j|s^t) = \frac{1}{1 + r(s^t)} \cdots \frac{1}{1 + r(s^{j-1})} \mu(s^j|s^t),$$

where $q(s^j|s^t)$ is the price at s^t of the contingent claim that pays off one unit of currency if and only if s^j is reached ($q(s^t|s^t) = 1$).

For simplicity, consider the flexible-price economy. The representative household has preferences given by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\big[c(s^t), \overline{y}(s_t) - y(s^t)\big] f(s^t), \tag{41}$$

where $c(s^t)$ is consumption at s^t; $\overline{y}(s_t)$ is the endowment; $y(s^t)$ is production.[11] At each date-event, the asset market opens first, followed by the goods market. As in Section 2, the constraints the household faces are summarized by: (i) the flow budget constraints:

$$\begin{aligned} P(s^t)c(s^t) + \frac{r(s^t)}{1+r(s^t)} m(s^t) + \frac{1}{1+r(s^t)} \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t) w(s^{t+1}) \\ \leq w(s^t) + \tau(s^t) + P(s^t)y(s^t), \end{aligned} \tag{42}$$

(ii) the cash constraints:

$$m(s^t) \geq P(s^t)y(s^t), \tag{43}$$

and (iii) the natural debt limit (Ljungqvist and Sargent, 2000):

$$w(s^t) \geq -\sum_{j=t}^{\infty} \sum_{s^j|s^t} q(s^j|s^t)\big[P(s^j)\overline{y}(s_j) + \tau(s^j)\big]. \tag{44}$$

Here, $P(s^t)$ is the price level at s^t; $m(s^t)$ is the nominal balances carried over from s^t to the next period; $w(s^{t+1})$ is the nominal value of the financial asset at the beginning of s^{t+1}; $\tau(s^t)$ is the nominal transfer from the fiscal authority at s^t. The natural debt limit (44) says that the amount the agent can borrow at a given date-event, $-w(s^t)$, is bounded by the present discounted value of his future earnings, $P(s^j)\overline{y}(s_j) + \tau(s^j)$. It is equivalent to

$$\lim_{j\to\infty} \sum_{s^j|s^t} q(s^j|s^t) w(s^j) \geq 0. \tag{45}$$

Remember that when the goods market follow the asset market, the cash-in-advance constraint is equivalent to (43).

The first-order conditions are given by

$$\frac{u_1[c(s^t), \overline{y}(s_t) - y(s^t)]}{u_2[c(s^t), \overline{y}(s_t) - y(s^t)]} = 1 + r(s^t), \tag{46}$$

$$\frac{\beta u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - y(s^{t+1})] f(s^{t+1}|s^t)}{u_1[c(s^t), \overline{y}(s_t) - y(s^t)]} = \frac{P(s^{t+1})}{P(s^t)} \frac{\mu(s^{t+1}|s^t)}{1+r(s^t)}; \tag{47}$$

and the transversality condition is

$$\lim_{j\to\infty} \sum_{s^j|s^t} q(s^j|s^t) w(s^j) = 0. \tag{48}$$

[11] We have assumed that the endowment, $\overline{y}$, depends only on the current shock.

The flow budget constraint of the monetary-fiscal authority is

$$\frac{r(s^t)}{1+r(s^t)}M(s^t) + \frac{1}{1+r(s^t)} \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t)W(s^{t+1}) = W(s^t) + T(s^t), \tag{49}$$

where $W(0) = w(0)$ is given. Monetary policy sets either a path of nominal interest rates, $\{r(s^t)\}$, or a path of money supplies, $\{M(s^t)\}$. Fiscal policy sets a path of transfers, $\{T(s^t)\}$, and a path of liabilities, $\{W(s^t)\}$, that satisfy the flow budget constraint. In the infinite-horizon economy, fiscal policy is said to be Ricardian if it guarantees that the present discounted value of the public liability converges to zero:

$$\lim_{j\to\infty} \sum_{s^j|s^t} q(s^j|s^t)W(s^j) = 0, \tag{50}$$

for any path of q, P, r, M, in or out of equilibrium. Of course, (50) is satisfied at equilibrium because of the transversality condition (48) of the household. As in the two-period case, let $\{\overline{W}(s^{t+1})\}_{s^{t+1}|s^t}$ denote the composition of the debt portfolio and $d(s^t)$ the scale of the debt. Let $H(s^t) \equiv T(s^t)/P(s^t)$ denote the real amount of the transfer from the fiscal authority. We consider the following specific forms of fiscal policy.

Ricardian policy Given the initial liability, $W(0)$, the fiscal authority sets sequences $\{\alpha(s^t)\}$, $\underline{\alpha} \leq \alpha(s^t) \leq 1$, all s^t, with $0 < \underline{\alpha} < 1$, and $\{\overline{W}(s^{t+1})\}$, $\overline{W}(s^{t+1}) \neq 0$. At each date-event, the nominal transfer, $T(s^t)$, and the scale of the debt, $d(s^t)$, are determined by the flow budget constraint:

$$T(s^t) = \frac{r(s^t)}{1+r(s^t)}M(s^t) - \alpha(s^t)W(s^t),$$

$$d(s^t) \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1+r(s^t)}\overline{W}(s^{t+1}) = [1-\alpha(s^t)]W(s^t).$$

Non-Ricardian policy Given $W(0) > 0$, the fiscal authority sets a path of real transfers, $H(s^t)$, $H(s^t) \leq 0$, and a path of compositions of debt, $\{\overline{W}(s^{t+1})\}$, $\overline{W}(s^{t+1}) > 0$. At each date-event, the scale of debt, $d(s^t)$, is determined by the flow budget constraint:

$$d(s^t) \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1+r(s^t)}\overline{W}(s^{t+1}) = W(s^t) - \frac{r(s^t)}{1+r(s^t)}M(s^t) + P(s^t)H(s^t).$$

In a deterministic setup, Woodford (1994) and Benhabib, Schmitt-Grohé and Uribe (2001) considered the non-Ricardian fiscal policy $H_t = H$. In a stochastic setup, Dubey and Geanakoplos (2000) considered the non-Ricardian policy $H(s^t) = 0$; Bloise, Drèze and Polemarchakis (2002) and Drèze and Polemarchakis (2000) considered the Ricardian fiscal policy $\alpha(s^t) = 1$.

Note, however, that the following form of fiscal policy does not fit into any of the two regimes considered here:

$$M(s^t) = M(s^{t-1}) + T(s^t).$$

This form of fiscal policy is considered, among others, in Woodford (1994). This is not Ricardian because it may violate (50) when the nominal interest rate becomes very low. To see this, suppose that the money supply grows at a constant rate, g: $M(s^t) = (1+g)^{t+1}\overline{M}$, $\overline{M} > 0$. Since $W(s^t) = M(s^{t-1})$ with such fiscal policy,

$$\begin{aligned}\sum_{s^t} q(s^t|0)W(s^t) &= \sum_{s^t} q(s^t|0)M(s^{t-1}) \\ &= \left(\sum_{s^t} q(s^t|0)\right)(1+g)^t\overline{M} \\ &= \left(\sum_{s^t} \frac{1}{1+r(0)}\cdots\frac{1}{1+r(s^{t-1})}\mu(s^t|0)\right)(1+g)^t\overline{M}.\end{aligned}$$

This does not converge to zero if the nominal interest rate becomes less than g almost surely. This is why "deflationary equilibria" are ruled out when $g \geq 0$ in Woodford (1994).

4.1 Equilibria with interest rate policy

The following proposition summarizes the result for the infinite-horizon economy with interest rate policy. The proof is in Appendix.

Proposition 6. *Given interest rate policy, $\{r(s^t)\}$,*

1. *with Ricardian fiscal policy, $\{\alpha(s^t), \overline{W}(s^{t+1})\}$, there is nominal indeterminacy indexed by the initial price level P_0 and the nominal equivalent martingale measure μ;*

2. *with non-Ricardian fiscal policy, $\{H(s^t), \overline{W}(s^{t+1})\}$, the equilibrium is unique.*

The uniqueness of equilibrium with the non-Ricardian policy considered here follows from the fact that there is only a unique path of price levels that is consistent with the transversality condition (48).

4.2 Recursive equilibria

Lucas and Stokey (1987) obtained conditions under which there exists a unique recursive equilibrium with money supply policy. Here, we provide a corresponding result with interest rate policy.

Suppose that fiscal policy is Ricardian and interest rate policy is $r(s^t) = r(s_t)$: the one-period rate depends only on the contemporaneous shock. In this case, the allocation also depends only on the current shock: $c(s^t) = c(s_t)$ and $y(s^t) = y(s_t)$. Of course, the stationarity of real allocations does not imply that the equilibrium inflation rate process, $\pi(s^t) = P(s^t)/P(s^{t-1})$, is Markov, because μ can be any probability measure over date-events. Let $g(s^t)$ denote the growth rates of money supply: $g(s^t) \equiv M(s^t)/M(s^{t-1})$. A recursive equilibrium is defined as an equilibrium in which the inflation rate and the money growth rate depend only on the shocks in the current and the previous periods:

$$\pi(s^t) = \pi(s_{t-1}, s_t), \quad \text{and} \quad g(s^t) = g(s_{t-1}, s_t).$$

The inflation rate follows such a process if and only if the nominal equivalent martingale measure is Markovian: $\mu(s_{t+1}|s^t) = \mu(s_{t+1}|s_t)$. Hence, a recursive equilibrium is an equilibrium of the form $\{c(s),\ y(s),\ \pi(s,s'),\ g(s,s'),\ \mu(s'|s)\}$. We assume that the transition matrix of the shock process, $\left[f(s'|s)\right]$, is invertible.

Proposition 7. *Given a stationary interest rate policy, $\{r(s)\}$, and Ricardian fiscal policy, $\{\alpha(s), \overline{W}(s)\}$,*

1. *the nominal equivalent martingale measure, $\{\mu(s'|s)\}$, is indeterminate. A different measure, $\{\mu(s'|s)\}$, corresponds to a different inflation process, $\{\pi(s,s')\}$, and a different money growth process, $\{g(s,s')\}$;*
2. *there exists a unique measure, $\{\mu(s'|s)\}$, that leads to an equilibrium in which the money growth process depends only on the current shock: $g(s,s') = g(s')$, all $s, s' \in \mathcal{S}$.*

The proof is in Appendix. Note that the latter claim in this proposition corresponds to the result in Lucas and Stokey (1987), for they define a recursive equilibrium of the form $\{c(s),\ y(s),\ \pi(s,s'),\ g(s),\ r(s),\ \mu(s'|s)\}$.

4.3 Equilibria with money supply policy

Now consider money supply policy $\{M(s^t)\}$. Define $c^*(s_t)$ by

$$c^*(s_t) = \arg\max_c u[c, \overline{y}(s_t) - c].$$

The following proposition summarizes the result for Ricardian policy.

Proposition 8. *Given initial nominal wealth, $w_0 = W_0$, money supply policy, $\{M(s^t)\}$ and Ricardian fiscal policy, $\{\alpha(s^t),\ \overline{W}(s^{t+1})\}$, there is indeterminacy indexed by the initial price level P_0 and the nominal equivalent martingale measure μ. The indeterminacy is real: different P_0 and μ are associated with different allocations.*

The proof is in Appendix. Note that Assumption 3 plays a crucial role here. If $cu_1(c, y-c)$ does not converge to zero as $c \to 0$, then there are no equilibria in which real balances, M/P, converge to zero, as discussed in Woodford (1994). In such a case, therefore, there may still exist indeterminacy but P_0 or μ may not be freely chosen. As in the finite-horizon economy, our characterization fails if the set $A(s)$ in (22) has empty interior. Furthermore, unlike the finite-horizon economy, if the flow utility function is of the log form, an equilibrium with strictly positive nominal interest rates is unique. To see this, suppose that the flow utility function is $u(c, l) = \ln c + \ln l$. Then, the first-order conditions with strictly positive nominal interest rates are given as

$$M(s^{t+1}) = \beta \big[\overline{y}(s_t) - c(s^t)\big] P(s^t) \frac{f(s^{t+1}|s^t)}{\mu(s^{t+1}|s^t)},$$

which leads to

$$\frac{1}{P(s^t)\big[\overline{y}(s_t) - c(s^t)\big]} = \sum_{s^{t+1}|s^t} \frac{\beta f(s^{t+1}|s^t)}{M(s^{t+1})}.$$

Together with the binding cash constraint, $P(s^t)c(s^t) = M(s^t)$, this equation determines $P(s^t)$ and $c(s^t)$.

As in the two-period economy, as long as the money supply does not converge to zero too fast, there always exists an equilibrium in which the nominal interest rate equals zero at all date-events. A sufficient condition for the existence of such equilibria is that for all s^t, $s^{t+1}|s^t$, and t,

$$\frac{M(s^{t+1})}{M(s^t)} \geq \max_{s,s'} \frac{\beta u_1[c^*(s'), \overline{y}(s') - c^*(s')]}{u_2[c^*(s), \overline{y}(s) - c^*(s)]}.$$

5 Concluding Remarks

We have argued that, in a stochastic monetary economy, indeterminacy is pervasive and is characterized by the initial price level and a nominal equivalent martingale measure. An intuition is that the Fisher equation, which relates the inflation rate and the nominal and real interest rates, holds only in expected terms: monetary policy restricts only the sum of the prices of state-contingent bonds, and it leaves their distribution indeterminate.

We have shown that our characterization of indeterminacy applies to stochastic monetary models very generally, and thus, unifies the results obtained previously. In particular, the degree of indeterminacy remains the same regardless of whether prices are flexible or sticky. The source of indeterminacy we discuss is closely related to the Walrasian claim that only relative prices are determined in equilibrium, and it does not derive from the stability of the non-stochastic steady state.

Whether or not indeterminacy is real depends on the completeness of the asset market, the flexibility of prices, and the instruments of monetary policy controls, among others. In addition, the timing of the asset and good markets matters: if the goods market opens before the asset market in each period, indeterminacy has real effects even when prices are flexible and monetary policy sets nominal interest rates. [12] This is related to the finding of Carlstrom and Fuerst (2001), who considered the money-in-the-utility model and compared

[12] See Nakajima and Polemarchakis (2002).

the case in which the real balances at the beginning of each period are in the flow utility function and the case in which those at the end of each period are.

Further work should consider the consequences of the timing of transactions and, more generally, the temporal structure of the model. A continuous time specification would be of interest.

Appendix

Proof of Proposition 6

Consider Ricardian fiscal policy, $\{\alpha(s^t), \overline{W}(s^{t+1})\}$. Pick $P(0) > 0$ and μ arbitrarily. With interest rates, $\{r(s^t)\}$, the first-order condition (46), together with the market clearing condition, $c(s^t) = y(s^t)$, determines the allocation, $\{c(s^t), y(s^t)\}$:

$$\frac{u_1[c(s^t), \overline{y}(s_t) - c(s^t)]}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} = 1 + r(s^t).$$

Given $P(0)$, price levels, $P(s^{t+1})$, $t \geq 0$, are determined by the first-order condition (47):

$$\frac{P(s^{t+1})}{P(s^t)} = \frac{\beta u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - c(s^{t+1})] f(s^{t+1}|s^t)}{u_1[c(s^t), \overline{y}(s_t) - c(s^t)]} \frac{1 + r(s^t)}{\mu(s^{t+1}|s^t)}.$$

The transversality condition (48) is guaranteed to hold by the fiscal policy.

With non-Ricardian policy, $\{H(s^t), \overline{W}(s^{t+1})\}$, the unique equilibrium is constructed as follows. As in the case of Ricardian policy, interest rate policy determines the equilibrium allocation, $\{c(s^t), y(s^t)\}$, uniquely. The individual optimization implies that

$$\begin{aligned}
\frac{W(0)}{P(0)} &= \sum_{t=0}^{\infty} \sum_{s^t} \frac{q(s^t|0)P(s^t)}{P(0)} \left\{ \frac{r(s^t)}{1 + r(s^t)} \frac{M(s^t)}{P(s^t)} - H(s^t) \right\} \\
&= \frac{\beta^t u_1[c(s^t), \overline{y}(s_t) - y(s^t)] f(s^t)}{u_1[c(0), \overline{y}(0) - y(0)]} \left\{ \frac{r(s^t)}{1 + r(s^t)} c(s^t) - H(s^t) \right\} \\
&> 0
\end{aligned}$$

Since the equilibrium allocation is unique, the right hand side of the second equation is unique. Since $W(0) > 0$, there is a unique $P(0) > 0$ that solves the equation. Given $P(0)$, $q(s^t|0)P(s^t)$ are determined uniquely by

$$q(s^t|0)P(s^t) = P(0) \frac{\beta^t u_1[c(s^t), \overline{y}(s_t) - y(s^t)] f(s^t)}{u_1[c(0), \overline{y}(0) - y(0)]}. \tag{51}$$

The paths of debt and prices, $\{W(s^{t+1}), P(s^t), q(s^t|0)\}$ are, then, determined inductively as follows. Suppose that $P(s^t)$, $q(s^t|0)$, and $W(s^t)$ have been determined. At each date-event s^{t+1} we have

$$W(s^{t+1}) = \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|s^{t+1}) P(s^{t+j}) \left\{ \frac{r(s^{t+j})}{1 + r(s^{t+j})} c(s^{t+j}) - H(s^{t+j}) \right\}. \tag{52}$$

Using $W(s^{t+1}) = d(s^t)\overline{W}(s^{t+1})$,

$$d(s^t) = \frac{1}{\overline{W}(s^{t+1})} \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|s^{t+1}) P(s^{t+j}) \left\{ \frac{r(s^{t+j})c(s^{t+j})}{1+r(s^{t+j})} - H(s^{t+j}) \right\}.$$

Taking the summation over $s^{t+1}|s^t$, and rearranging terms,

$$d(s^t) = \frac{1+r(s^t)}{q(s^t|0)} \sum_{s^{t+1}|s^t} \frac{1}{\overline{W}(s^{t+1})}$$
$$\times \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^{t+1}} q(s^{t+j}|0) P(s^{t+j}) \left\{ \frac{r(s^{t+j})c(s^{t+j})}{1+r(s^{t+j})} - H(s^{t+j}) \right\}.$$

This determines $d(s^t)$, and hence $W(s^{t+1}) = d(s^t)\overline{W}(s^{t+1})$. Given $W(s^{t+1})$, $P(s^{t+1})$ are determined by (52); given $P(s^{t+1})$, $q(s^{t+1}|0)$ are determined by (51), which determine $\mu(s^{t+1}|s^t)$. Note that, since $\overline{W}(s^t) > 0$, $H(s^t) < 0$, all s^t and t, $W(s^t) > 0$, all s^t and t. Hence, the resulting $P(s^t)$ and $\mu(s^t)$ are all positive. The path of money supplies are given by $M(s^t) \geq p(s^t)c(s^t)$ (equality when $r(s^t) > 0$).

Proof of Proposition 7

The indeterminacy is shown as follows. Let $\mu(s, s')$ be any strictly positive Markov transition probabilities. The allocation, $\{c(s), y(s)\}$, is determined by the interest rate policy, $\{r(s)\}$. The equilibrium inflation and money growth processes are then given by

$$\pi(s, s') = \frac{\beta u_1[c(s'), \overline{y}(s') - c(s')]}{u_2[c(s), \overline{y}(s) - c(s)]} \frac{f(s'|s)}{\mu(s'|s)},$$

and, if $r(s) > 0$, all $s \in \mathcal{S}$,

$$g(s, s') = \pi(s, s') \frac{c(s)}{c(s')},$$

for all $s, s' \in \mathcal{S}$.

If we further require that the money growth process is of the form: $g(s)$,

$$g(s') = \frac{\beta u_1[c(s'), y(s') - c(s')]}{u_2[c(s), y(s) - c(s)]} \frac{c(s')}{c(s)} \frac{f(s'|s)}{\mu(s'|s)},$$

for all $s, s' \in \mathcal{S}$. Let A, B and G be the diagonal matrices defined by

$$A = \text{diag}\big(\ldots, u_2[c(s), y(s) - c(s)]c(s), \ldots\big),$$

$$B = \text{diag}\big(\ldots, \beta u_1[c(s'), y(s') - c(s')]c(s'), \ldots\big),$$

$$G = \text{diag}\bigg(\ldots, \frac{1}{g(s')}, \ldots\bigg).$$

Let

$$F = \big[f(s'|s)\big], \quad M = \big[\mu(s'|s)\big].$$

Then, if $g(s, s') = g(s')$, all $s, s' \in \mathcal{S}$, $\mu(s'|s)$ are given by

$$M = A^{-1}FBG.$$

Proof of Proposition 8

Given Ricardian fiscal policy, $\{\alpha(s^t), \overline{W}(s^{t+1})\}$, choose strictly positive $P(0) > 0$ and μ arbitrarily. Then, a unique equilibrium corresponding to them is constructed as follows. Given $M(0)$ and $P(0)$, $c(0)$, $y(0)$, and $r(0)$ are determined by

$$c(0) = \min\left\{\frac{M(0)}{P(0)}, c^*(0)\right\}, \quad 1 + r(0) = \frac{u_1[c(0), \overline{y}(0) - c(0)]}{u_2[c(0), \overline{y}(0) - c(0)]},$$

and $y(0) = c(0)$. The rest of the equilibrium paths of c, y, P, and r are determined inductively. If $c(s^t)$, $y(s^t)$, $P(s^t)$, and $r(s^t)$ have been determined, for a successor date-event $s^{t+1}|s^t$, if

$$M(s^{t+1}) > \frac{\beta c^*(s_{t+1}) u_1[c^*(s_{t+1}), \overline{y}(s_{t+1}) - c^*(s_{t+1})]}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \frac{f(s^{t+1}|s^t)}{\mu(s^{t+1}|s^t)} P(s^t),$$

then $c(s^{t+1}) = c^*(s_{t+1})$; otherwise, $c(s^{t+1})$ is a solution to

$$M(s^{t+1}) = \frac{\beta c(s^{t+1}) u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - c(s^{t+1})]}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \frac{f(s^{t+1}|s^t)}{\mu(s^{t+1}|s^t)} P(s^t).$$

The unique existence of a solution is guaranteed by Assumption 3. Given $c(s^{t+1})$, $y(s^{t+1}) = c(s^{t+1})$,

$$P(s^{t+1}) = \frac{\beta u_1[c(s^{t+1}), \overline{y}(s_{t+1}) - c(s^{t+1})]}{u_2[c(s^t), \overline{y}(s_t) - c(s^t)]} \frac{f(s^{t+1}|s^t)}{\mu(s^{t+1}|s^t)} P(s^t),$$

and

$$1 + r(s^{t+1}) = \frac{u_1[c(s^{t+1}), \overline{y}(s^{t+1}) - c(s^{t+1})]}{u_2[c(s^{t+1}), \overline{y}(s^{t+1}) - c(s^{t+1})]}.$$

The path of debt portfolio, $\{W(s^{t+1})\}$, is

$$W(s^{t+1}) = w(s^{t+1}) = [1 - \alpha(s^t)]q(s^t)W(s^t) \frac{\overline{W}(s^{t+1})}{\sum_{\tilde{s}^{t+1}|s^t} q(\tilde{s}^{t+1})\overline{W}(\tilde{s}^{t+1})}$$

It is straightforward to see that the constructed paths satisfy the first-order conditions (46)-(47) and the transversality condition (48). Clearly, the indeterminacy is real.

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