A Multivariate Model of Strategic Asset Allocation

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Abstract

Much recent work has documented evidence for predictability of asset returns. We show how such predictability can affect the portfolio choices of long-lived investors who value wealth not for its own sake but for the consumption their wealth can support. We develop an approximate solution method for the optimal consumption and portfolio choice problem of an infinitely-lived investor with Epstein-Zin utility who faces a set of asset returns described by a vector autoregression in returns and state variables. Empirical estimates in long-run annual and postwar quarterly US data suggest that the predictability of stock returns greatly increases the optimal demand for stocks. Nominal bonds have only a small role in optimal long-term portfolios. We extend the analysis to consider long-term inflation-indexed bonds and find that extremely conservative investors should hold large positions in these bonds when they are available.

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1 Introduction

Academic finance has had a remarkable impact on many participants in the financial services industry, from mutual fund managers to corporate risk managers. Curiously, however, financial planners offering portfolio advice to long-term investors have received little guidance from academic financial economists.

The mean-variance analysis of Markowitz (1952) has provided a basic paradigm, and has usefully emphasized the ability of diversification to reduce risk, but this model ignores several critically important factors. Most notably, the analysis is static; it assumes that investors care only about risks to wealth one period ahead. In reality, however, many investors—both individuals and institutions such as charitable foundations or universities—seek to finance a stream of consumption over a long lifetime.

Financial economists have understood at least since the work of Samuelson (1969) and Merton (1969, 1971, 1973) that the solution to a multi-period portfolio choice problem can be very different from the solution to a static portfolio choice problem. In particular, if investment opportunities are varying over time, then long-term investors care about shocks to investment opportunities—the productivity of wealth—as well as shocks to wealth itself. They may seek to hedge their exposures to wealth productivity shocks, and this gives rise to intertemporal hedging demands for financial assets. Brennan, Schwartz, and Lagnado (1997) have coined the phrase "strategic asset allocation" to describe this far-sighted response to time-varying investment opportunities.

Unfortunately Merton's intertemporal model is hard to solve in closed form. For many years solutions to the model were only available in those trivial cases where it reduces to the static model. Therefore the Merton model has not become a usable empirical paradigm, has not displaced the Markowitz model, and has had minimal influence on financial planners and their clients.

Recently this situation has begun to change as a result of several related developments. First, computing power and numerical methods have advanced to the point at which realistic multi-period portfolio choice problems can be solved numerically using discrete-state approximations. Balduzzi and Lynch (1999), Barberis (1999), Brennan, Schwartz, and Lagnado (1996, 1997), Cocco, Gomes, and Maenhout (1998), and Lynch (1999) are important examples of this style of work. Second, financial theorists have discovered some new closed-form solutions to the Merton model. In a continuous-time model with a constant riskless interest rate and a single risky asset whose expected return follows a mean-reverting (Ornstein-Uhlenbeck) process, for example, the model can be solved if long-lived investors have power utility defined over terminal wealth (Kim and Omberg 1996), or if investors have power utility defined over consumption and the innovation to the expected asset return is perfectly correlated with the innovation to the unexpected return, making the asset market effectively complete (Wachter 1999), or if the investor has Epstein-Zin utility with intertemporal elasticity of substitution equal to one (Campbell and Viceira 1999, Schroder and Skiadas 1999). Similar results are available in affine models of the term structure (Brennan and Xia 1998, Campbell and Viceira 2001, Liu 1998, Wachter 1998). Third, approximate analytical solutions to the Merton model have been developed (Campbell and Viceira 1999, 2001). These solutions are based on perturbations of the known exact solutions for intertemporal elasticity of substitution equal to one, so they are accurate provided that the intertemporal elasticity is not too far from one. They offer analytical insights into investor behavior in models that fall outside the still limited class that can be solved exactly.

Despite this encouraging progress, it remains extremely hard to solve realistically complex cases of the Merton model. Discrete-state numerical algorithms become slow and unreliable in the presence of many assets and state variables, and approximate analytical methods seem to require a daunting quantity of algebra. Neither approach has been developed to the point at which one can specify a general vector autoregression (VAR) for asset returns and hope to solve the associated portfolio choice problem.

The purpose of this paper is to remedy this situation by extending the approximate analytical approach of Campbell and Viceira (1999, 2001). Specifically, we show that if asset returns are described by a VAR, if the investor is infinitely lived with Epstein-Zin utility, and if there are no borrowing or short-sales constraints on asset allocations, then the Campbell-Viceira approach implies a system of linear-quadratic equations for portfolio weights and consumption as functions of state variables. These equations are generally too cumbersome to solve analytically, but can be solved very rapidly by simple numerical methods. As the time interval of the model shrinks, the solutions become exact if the elasticity of intertemporal substitution equals one. They are accurate approximations for short time intervals and elasticities close to one.

We apply our method to a VAR for short-term real interest rates, excess stock returns, and excess bond returns. We also include variables that have been identified as return predictors by past empirical research: the short-term interest rate (Fama and Schwert 1977, Campbell 1987, Glosten, Jagannathan, and Runkle 1993); the dividend-price ratio (Campbell and Shiller 1988, Fama and French 1988a); and the yield spread between long-term and short-term bonds (Shiller, Campbell, and Schoenholtz 1983, Fama 1984, Fama and French 1989, Campbell and Shiller 1991). In a variant of the basic approach we construct data on hypothetical inflation-indexed bond returns, following the approach of Campbell and Shiller (1996), and study the allocation to stocks, inflation-indexed bonds, nominal bonds, and bills.

Two closely related papers are by Brennan, Schwartz, and Lagnado (1996) and Lynch (1999). Brennan, Schwartz, and Lagnado consider asset allocation among stocks, nominal bonds, bills, and interest-rate futures, using short- and long-term nominal interest rates and the dividend-price ratio as state variables. The investor is assumed to have power utility defined over wealth at a given horizon, and the stochastic optimization problem is solved using numerical dynamic programming imposing borrowing and short-sales constraints. Lynch considers asset allocation among portfolios of stocks sorted by size and book-to-market ratios, using the long-short yield spread and the dividend-price ratio as state variables, and assuming power utility defined over consumption. He solves the optimization problem with and without short-sales constraints, again using numerical dynamic programming. Our paper, by contrast, assumes recursive Epstein-Zin utility defined over an infinite stream of consumption and does not impose any portfolio constraints. The simplicity of our solution method allows us to consider an unrestricted VAR in which lagged returns are state variables along with the short-term nominal interest rate, dividend-price ratio, and yield spread. Our method also allows us to break intertemporal hedging demands into components associated with individual state variables.

The organization of the paper is as follows. Section 2 explains our basic setup, and Section 3 describes our approximate solution method. Section 4 presents empirical results for the case where stocks, nominal bonds, and bills are available. Section 5 considers portfolio allocation in the presence of inflation-indexed bonds. Section 6 concludes.

2 The Model

Our model is set in discrete time. We assume an infinitely-lived investor with Epstein-Zin (1989, 1991) recursive preferences defined over a stream of consumption. This contrasts with papers such as Brennan, Schwartz and Lagnado (1996, 1997), Kim and Omberg (1996), and Barberis (2000) that consider finite-horizon models with power utility defined over terminal wealth. We allow an arbitrary set of traded assets and state variables. Thus we do not make the assumption of Wachter (1998, 1999) that markets are complete, and we substantially extend the work of Campbell and Viceira (1999) in which there is a single risky asset with a single state variable.

2.1 Securities

There are n assets available for investment. The investor allocates her after-consumption wealth among these assets. The real return on her portfolio $R_{p,t+1}$ is given by

$$R_{p,t+1} = \sum_{i=2}^{n} \alpha_{i,t} \left(R_{i,t+1} - R_{1,t+1} \right) + R_{1,t+1}, \tag{1}$$

where $\alpha_{i,t}$ is the portfolio weight on asset i. The first asset is a short-term instrument whose real return is $R_{1,t+1}$. Although we use the short-term return as a benchmark and measure other returns relative to it, we do not assume that this return is riskless. In practice we use a nominal bill as the short-term asset; the nominal return on a nominal bill is riskless, but the real return is not because it is subject to short-term inflation risk. In most of our empirical analysis we consider two other assets: stocks and long-term nominal bonds. In Section 5 we also consider long-term inflation-indexed bonds.

2.2 Dynamics of state variables

We postulate that the dynamics of the relevant state variables are well captured by a first-order vector autoregressive process or VAR(1). This type of dynamic specification has been used by Kandel and Stambaugh (1987), Campbell (1991, 1996), Hodrick (1992), and Barberis (2000), among others. In principle the use of a VAR(1) is not restrictive since any vector autoregression can be rewritten as a VAR(1) through an expansion of the vector of state variables. For parsimony, however, in our empirical work we avoid additional lags that would require an expanded state vector with additional parameters to estimate. Specifically, we define

$$\mathbf{x}_{t+1} \equiv \begin{bmatrix} r_{2,t+1} - r_{1,t+1} \\ r_{3,t+1} - r_{1,t+1} \\ \vdots \\ r_{n,t+1} - r_{1,t+1} \end{bmatrix}, \tag{2}$$

where $r_{i,t+1} \equiv \log(R_{i,t+1})$ for all i, and $\mathbf{x}_{t+1}$ is the vector of log excess returns. In our empirical application, $r_{1,t+1}$ is the real short rate, $r_{2,t+1}$ refers to the real stock return and $r_{3,t+1}$ to the real return on nominal bonds.

We allow the system to include other state variables $\mathbf{s}_{t+1}$, such as the dividend-price ratio. Stacking $r_{1,t+1}, \mathbf{x}_{t+1}, \mathbf{s}_{t+1}$ into an $m \times 1$ vector $\mathbf{z}_{t+1}$, we have

$$\mathbf{z}_{t+1} \equiv \begin{bmatrix} r_{1,t+1} \\ \mathbf{x}_{t+1} \\ \mathbf{s}_{t+1} \end{bmatrix}. \tag{3}$$

We will call $\mathbf{z}_{t+1}$ the state vector and we assume a first order vector autoregression for $\mathbf{z}_{t+1}$:

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{z}_t + \mathbf{v}_{t+1}, \tag{4}$$

where $\mathbf{\Phi}_0$ is the $m \times 1$ vector of intercepts, $\mathbf{\Phi}_1$ is the $m \times m$ matrix of slope coefficients, and $\mathbf{v}_{t+1}$ are the shocks to the state variables satisfying the following distributional assumptions:

$$\begin{aligned} \mathbf{v}_{t+1} &\overset{i.i.d.}{\sim} \mathcal{N}(0, \mathbf{\Sigma}_v), \\ \Sigma_v \equiv \mathrm{Var}_t(\mathbf{v}_{t+1}) &= \begin{bmatrix} \sigma_1^2 & \mathbf{\Sigma}'_{1x} & \mathbf{\Sigma}'_{1s} \\ \mathbf{\Sigma}_{1x} & \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}'_{xs} \\ \mathbf{\Sigma}_{1s} & \mathbf{\Sigma}_{xs} & \mathbf{\Sigma}_{ss} \end{bmatrix}. \end{aligned} \tag{5}$$

Thus, we allow the shocks to be cross-sectionally correlated, but assume that they are homoskedastic and independently distributed over time. The VAR framework conveniently captures the dependence of expected returns of various assets on their past histories as well as on other predictive variables. The stochastic evolution of these other state variables $\mathbf{s}_{t+1}$ is also determined by the system.

The assumption of homoskedasticity is of course restrictive. It rules out the possibility that the state variables predict changes in risk; they can affect portfolio choice only by predicting changes in expected returns. Authors such as Campbell (1987), Harvey (1989, 1991), and Glosten, Jagannathan, and Runkle (1993) have explored the ability of the state variables used here to predict risk and have found only modest effects that seem to be dominated by the effects of the state variables on expected returns. Chacko and Viceira (1999) show how to include changing risk in a long-term portfolio choice problem, using a continuous-time extension of the methodology of Campbell and Viceira (1999); they find that changes in equity risk are not persistent enough to have large effects on the intertemporal hedging demand for equities. Aït-Sahalia and Brandt (2000) adopt a semiparametric methodology that accommodates both changing expected returns and changing risk.

Given our homoskedastic VAR formulation, the unconditional distribution of $\mathbf{z}_t$ is easily derived. The state vector $\mathbf{z}_t$ inherits the normality of the shocks $\mathbf{v}_{t+1}$. It has unconditional mean $\boldsymbol{\mu}_z$ and variance-covariance matrix $\boldsymbol{\Sigma}_{zz}$ given by

$$\begin{aligned} \boldsymbol{\mu}_z &= \left(\mathbf{I}_m - \boldsymbol{\Phi}_1\right)^{-1} \boldsymbol{\Phi}_0, \\ \text{vec}(\boldsymbol{\Sigma}_{zz}) &= \left(\mathbf{I}_{m^2} - \boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1\right)^{-1} \text{vec}\left(\boldsymbol{\Sigma}_v\right), \end{aligned} \tag{6}$$

where $\text{vec}(\cdot)$ is the vectorization operator and $\otimes$ is the kronecker product operator.

2.3 Preferences and optimality conditions

We assume that the investor has Epstein-Zin (1989, 1991) recursive preferences. This preference specification has the desirable property that the notion of risk aversion is separated from that of the elasticity of intertemporal substitution. Following Epstein-Zin, we let

$$U\left(C_t, \mathrm{E}_t\left(U_{t+1}\right)\right) = \left[\left(1-\delta\right) C_t^{\frac{1-\gamma}{\theta}} + \delta\left(\mathrm{E}_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{7}$$

where C_t is consumption at time t, $\gamma > 0$ is the relative risk aversion coefficient, $\psi > 0$ is the elasticity of intertemporal substitution, $0 < \delta < 1$ is the time discount factor, $\theta \equiv (1-\gamma)/(1-\psi^{-1})$, and $\mathrm{E}_t\left(\cdot\right)$ is the conditional expectation operator.

Epstein-Zin recursive utility nests as a special case the standard, time-separable power utility specification. Figure 1 shows graphically the relation between Epstein-Zin utility and power utility. The horizontal axis in the figure shows the intertemporal elasticity of substitution ψ, while the vertical axis shows the coefficient of relative risk aversion γ. The set of points with unit elasticity of intertemporal substitution is drawn as a vertical line, while the set of points with unit relative risk aversion is drawn as a horizontal line. For time-separable power utility, $\gamma = \psi^{-1}$ and hence $\theta = 1$. This corresponds to the hyperbola $\gamma = \psi^{-1}$ plotted in the figure. Log utility obtains when we impose the additional restriction $\gamma = \psi^{-1} = 1$. This is the point in the figure where all three lines cross.

At time t, the investor uses all relevant information to make optimal consumption and portfolio decisions. She faces the intertemporal budget constraint

$$W_{t+1} = (W_t - C_t)\, R_{p,t+1}, \tag{8}$$

where C_t is consumption and W_t is wealth at time t.

Epstein and Zin (1989, 1991) have shown that with this budget constraint, the Euler equation for consumption is

$$\mathrm{E}_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} R_{p,t+1}^{-(1-\theta)} R_{i,t+1} \right] = 1, \tag{9}$$

for any asset i, including the portfolio p itself. This first-order condition reduces to the standard one in the power utility case where $\gamma = \psi^{-1}$ and $\theta = 1$.

The investor's optimal consumption and portfolio policies must satisfy the Euler equation (9). When investment opportunities are constant, the optimal policies imply a constant consumption-wealth ratio and a myopic portfolio rule—that is, the investor chooses her portfolio as if her investment horizon was only one period. However, when investment opportunities are time-varying, there are no known exact analytical solutions to this equation except for some specific values of γ and ψ. Giovannini and Weil (1989) have shown that with $\gamma = 1$, it is optimal for the investor to follow a myopic portfolio rule. This case corresponds to the horizontal line plotted in Figure 1. They also show that with $\psi = 1$, the investor optimally chooses a constant consumption-wealth ratio. This corresponds to the vertical line in Figure 1. However, with $\gamma = 1$, the optimal consumption-wealth ratio is not constant unless $\psi = 1$ and, conversely, with $\psi = 1$ the optimal portfolio rule is not myopic unless $\gamma = 1$. This corresponds to the point where the vertical and horizontal lines in Figure 1 cross—i.e., the log utility case. To solve for the optimal rules in all other cases, we extend the approximate analytical solution method in Campbell and Viceira (1999, 2001) to a multivariate framework.

3 Solution Methodology

3.1 An approximate framework

The return on the portfolio in (1) is expressed in terms of the simple returns on the assets. Since it is more convenient to work with log returns in our framework, we first derive an expression for the log return on the portfolio. Following Campbell and Viceira (1999, 2001), we approximate the log return on the portfolio as

$$r_{p,t+1} = r_{1,t+1} + \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left(\sigma_x^2 - \Sigma_{xx}\boldsymbol{\alpha}_t\right), \tag{10}$$

where $\sigma_x^2 \equiv \text{diag}(\Sigma_{xx})$ is the vector consisting of the diagonal elements of Σ_{xx}, the variances of excess returns. This approximation holds exactly in continuous time and is highly accurate for short time intervals. Just as in a continuous-time model, (10) prevents bankruptcy even when asset positions are leveraged; Campbell and Viceira (2001) discuss the relation of this approach with continuous-time modelling. When there is only one risky asset, (10) collapses to the approximation derived in Campbell and Viceira (1999). Detailed derivations for this and other results in this section are provided in Appendix A.

The budget constraint in (8) is nonlinear. Following Campbell (1993, 1996), we log-linearize around the unconditional mean of the log consumption-wealth ratio to obtain

$$\Delta w_{t+1} \approx r_{p,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k, \tag{11}$$

where Δ is the difference operator, $\rho \equiv 1 - \exp\left(\mathrm{E}[c_t - w_t]\right)$ and $k \equiv \log(\rho) + (1-\rho)\log(1-\rho)/\rho$. When consumption is chosen optimally by the investor, ρ depends on the optimal level of c_t relative to w_t and in this sense is endogenous. This form of the budget constraint is exact if the elasticity of intertemporal substitution $\psi = 1$, in which case $c_t - w_t$ is constant and $\rho = \delta$.

Next, we apply a second-order Taylor expansion to the Euler equation (9) around the conditional means of $\Delta c_{t+1}, r_{p,t+1}, r_{i,t+1}$ to obtain

$$\begin{aligned} 0 &= \theta \log \delta - \frac{\theta}{\psi}\mathrm{E}_t \Delta c_{t+1} - (1-\theta)\,\mathrm{E}_t r_{p,t+1} + \mathrm{E}_t r_{i,t+1} \\ &\quad + \frac{1}{2}\mathrm{Var}_t\left[-\frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta)\, r_{p,t+1} + r_{i,t+1}\right]. \end{aligned} \tag{12}$$

This loglinearized Euler equation is exact if consumption and asset returns are jointly lognormally distributed, which is the case when the elasticity of intertemporal substitution $\psi = 1$. It can be usefully transformed as follows. Setting $i = 1$ in (12), subtracting from the general form of (12), and noting that $\Delta c_{t+1} = \Delta(c_{t+1} - w_{t+1}) + \Delta w_{t+1}$, we obtain, for asset $i = 2, ..., n$,

$$\begin{aligned} & \mathrm{E}_t(r_{i,t+1} - r_{1,t+1}) + \frac{1}{2}\mathrm{Var}_t(r_{i,t+1} - r_{1,t+1}) \qquad (13) \\ = \; & \frac{\theta}{\psi}\left(\sigma_{i,c-w,t} - \sigma_{1,c-w,t}\right) + \gamma\left(\sigma_{i,p,t} - \sigma_{1,p,t}\right) \\ & - \left(\sigma_{i,1,t} - \sigma_{1,1,t}\right), \end{aligned}$$

where $\sigma_{i,c-w,t} = \mathrm{Cov}_t(r_{i,t+1}, c_{t+1} - w_{t+1})$, $\sigma_{1,c-w,t} = \mathrm{Cov}_t(r_{1,t+1}, c_{t+1} - w_{t+1})$, $\sigma_{i,p,t} = \mathrm{Cov}_t(r_{i,t+1}, r_{p,t+1})$, $\sigma_{1,p,t} = \mathrm{Cov}_t(r_{1,t+1}, r_{p,t+1})$, $\sigma_{i,1,t} = \mathrm{Cov}_t(r_{i,t+1}, r_{1,t+1})$, and $\sigma_{1,1,t} = \mathrm{Var}_t(r_{1,t+1})$. The left hand side of this equation is the risk premium on asset i over asset 1, adjusted for Jensen's Inequality by adding one-half the variance of the excess return. The equation relates asset i's risk premium to its excess covariance with consumption growth, its excess covariance with the portfolio return, and the covariance of its excess return with the return on asset 1. (The last term drops out when asset 1 is riskless.) Of course, consumption growth and the portfolio return are endogenous so this is a first-order condition describing the optimal solution rather than a statement of the solution itself.

3.2 Solving the approximate model

To solve the model, we now guess that the optimal portfolio and consumption rules take the form

$$\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t, \qquad (14)$$

$$c_t - w_t = b_0 + \mathbf{B}_1'\mathbf{z}_t + \mathbf{B}_2'\mathbf{z}_t\mathbf{z}_t'\mathbf{B}_2 \qquad (15)$$

That is, the optimal portfolio rule is linear in the VAR state vector but the optimal consumption rule is quadratic. $\mathbf{A}_0, \mathbf{A}_1, b_0, \mathbf{B}_1$, and $\mathbf{B}_2$ are constant coefficient matrices to be determined, with dimensions $(n-1) \times 1, (n-1) \times m, 1 \times 1, m \times 1$, and $m \times 1$, respectively. This is a multivariate generalization of the solution obtained by Campbell and Viceira (1999).

To verify this guess and solve for the parameters of the solution, we write the conditional moments that appear in (13) as functions of the VAR parameters and the unknown parameters of (14) and (15). We then solve for the parameters that satisfy (13). Recalling that the vector of excess returns is written as $\mathbf{x}_t$, the conditional expectation on the left hand side of (13) is

$$\mathrm{E}_t\left(\mathbf{x}_{t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(\mathbf{x}_{t+1}\right)=\mathbf{H}_x\mathbf{\Phi}_0+\mathbf{H}_x\mathbf{\Phi}_1\mathbf{z}_t+\frac{1}{2}\sigma_x^2, \tag{16}$$

where $\mathbf{H}_x$ is a selection matrix that selects the vector of excess returns from the full state vector.

Appendix A shows that the three conditional covariances on the right hand side of (13) can all be written as linear functions of the state variables. In matrix notation,

$$\boldsymbol{\sigma}_{c-w,t}-\sigma_{1,c-w,t}\boldsymbol{\iota} \equiv \begin{bmatrix} \sigma_{2,c-w,t}-\sigma_{1,c-w,t} \\ \vdots \\ \sigma_{n,c-w,t}-\sigma_{1,c-w,t} \end{bmatrix} = \mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{z}_t, \tag{17}$$

$$\boldsymbol{\sigma}_{p,t}-\sigma_{1,p,t}\boldsymbol{\iota} \equiv \begin{bmatrix} \sigma_{2,p,t}-\sigma_{1,p,t} \\ \vdots \\ \sigma_{n,p,t}-\sigma_{1,p,t} \end{bmatrix} = \Sigma_{xx}\boldsymbol{\alpha}_t+\Sigma_{1x}, \tag{18}$$

$$\boldsymbol{\sigma}_{1,t}-\sigma_{1,1,t}\boldsymbol{\iota} \equiv \begin{bmatrix} \sigma_{2,1,t}-\sigma_{1,1,t} \\ \vdots \\ \sigma_{n,1,t}-\sigma_{1,1,t} \end{bmatrix} = \Sigma_{1x}, \tag{19}$$

where $\boldsymbol{\iota}$ is a vector of ones.

3.3 Optimal portfolio choice

Solving the Euler equation (13) for the portfolio rule we have

$$\begin{aligned} \boldsymbol{\alpha}_t &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left[\mathrm{E}_t\left(\mathbf{x}_{t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(\mathbf{x}_{t+1}\right)+\left(1-\gamma\right)\Sigma_{1x}\right] \\ &\quad +\frac{1}{\gamma}\Sigma_{xx}^{-1}\left[-\frac{\theta}{\psi}\left(\boldsymbol{\sigma}_{c-w,t}-\sigma_{1,c-w,t}\boldsymbol{\iota}\right)\right], \end{aligned} \tag{20}$$

where $\mathrm{E}_t(\mathbf{x}_{t+1}) + \mathrm{Var}_t(\mathbf{x}_{t+1})/2$ and $\boldsymbol{\sigma}_{c-w,t} - \sigma_{1,c-w,t}\boldsymbol{\iota}$ are the linear functions of $\mathbf{z}_t$ given in (16) and (17), respectively. This equation is a multiple-asset generalization of Restoy (1992) and Campbell and Viceira (1999). It expresses the optimal portfolio choice as the sum of two components.

The first term on the right hand side of (20) is the myopic component of asset demand. When the benchmark asset 1 is riskless ($\Sigma_{1x} = 0$), then the myopic allocation is the vector of Sharpe ratios on risky assets, scaled by the inverse of the variance-covariance matrix of risky asset returns and the reciprocal of the coefficient of relative risk aversion. Investors with $\gamma \neq 1$ adjust this allocation slightly by a term $(1-\gamma)\Sigma_{1x}$ when asset 1 is risky. Because of its myopic nature, this component does not depend on ψ, the elasticity of intertemporal substitution.

The second term on the right hand side of (20) is the intertemporal hedging demand. In our model, the investment opportunity set is time varying since expected returns on various assets are state-dependent. Merton (1969, 1971) shows that a rational investor who is more risk averse than a logarithmic investor will hedge against adverse changes in investment opportunities. For a logarithmic investor, the optimal portfolio rule is purely myopic and hence the hedging demand is identically equal to zero. This can be easily seen from (20) since when $\gamma = 1$, $\theta = 0$ and the hedging component vanishes. Also, when investment opportunities are constant over time, hedging demand is zero for any level of risk aversion. This case corresponds to having only the intercept term in our VAR specification. It is straightforward to verify that the coefficient matrices Λ_0 and Λ_1 in the hedging component are zero matrices in this case and thus there is no hedging component of asset demand.

Substituting (16) and (17) in (20) and rearranging the terms yields

$$\boldsymbol{\alpha}_t \equiv \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t, \tag{21}$$

where

$$\begin{aligned}
\mathbf{A}_0 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left[\mathbf{H}_x\Phi_0 + \frac{1}{2}\sigma_x^2 - \frac{\theta}{\psi}\Lambda_0 + (1-\gamma)\Sigma_{1x}\right], \\
\mathbf{A}_1 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left[\mathbf{H}_x\Phi_1 - \frac{\theta}{\psi}\Lambda_1\right].
\end{aligned}$$

Equation (21) verifies our initial guess for the form of the optimal portfolio rule and expresses the coefficient matrices $\mathbf{A}_0$, $\mathbf{A}_1$ as functions of the underlying parameters describing preferences and the dynamics of the state variables. $\mathbf{A}_0$ and $\mathbf{A}_1$ also depend on the parameters in the consumption-wealth ratio equation, $b_0, \mathbf{B}_1$ and $\mathbf{B}_2$, through the coefficient matrices Λ_0 and Λ_1.

Campbell and Viceira (1999) show that in their univariate model, given the loglinearization parameter ρ, the optimal portfolio rule does not depend on the intertemporal elasticity of substitution ψ. ψ only affects portfolio choice to the extent that it enters into the determination of ρ. Appendix A analyzes the coefficients in (21) and shows that this property carries over to our model with multiple assets and state variables.

3.4 Optimal consumption

Next, we solve for the optimal consumption-wealth ratio. Setting $i = p$ in (12) and rearranging,

$$\mathrm{E}_t(\Delta c_{t+1}) = \psi \log \delta + \chi_{p,t} + \psi \mathrm{E}_t(r_{p,t+1}), \tag{22}$$

where

$$\chi_{p,t} = \frac{1}{2}\left(\frac{\theta}{\psi}\right) \mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right). \tag{23}$$

This equation relates expected consumption growth to preferences and investment opportunities. A patient investor with high δ plans more rapid consumption growth. Similarly, when the return on the portfolio is expected to be higher, the investor increases planned consumption growth to take advantage of good investment opportunities. The sensitivity of planned consumption growth to both patience and returns is measured by the elasticity of intertemporal substitution ψ.

The term $\chi_{p,t}$ arises from the precautionary savings motive. Randomness in future consumption growth, relative to portfolio returns, increases precautionary savings and lowers current consumption if $\theta > 0$ (a condition satisfied by power utility for which $\theta = 1$), but reduces precautionary savings and increases current consumption if $\theta < 0$.

We show in Appendix A that combining equation (22) and the log-linearized budget constraint (11), we obtain a difference equation in $c_t - w_t$:

$$c_t - w_t = -\rho\psi \log \delta - \rho\chi_{p,t} + \rho(1-\psi)\mathrm{E}_t(r_{p,t+1}) + \rho k + \rho \mathrm{E}_t(c_{t+1} - w_{t+1}), \tag{24}$$

where both $\mathrm{E}_t(r_{p,t+1})$ and $\chi_{p,t}$ are quadratic functions of the VAR state variables. Given our conjectured quadratic form for the optimal consumption-wealth ratio, both sides of this equation are quadratic in the VAR state variables. This confirms our initial conjecture on the form of the consumption-wealth ratio and gives us a set of equations that solve for the coefficients of the optimal consumption policy, $b_0, \mathbf{B}_1$ and $\text{vec}(\mathbf{B}_2\mathbf{B}_2')$.

4 An Empirical Application: Stocks, Bonds, and Bills

Section 3 provides a general theoretical framework for strategic asset allocation. In this section, we use the framework to investigate how investors who differ in their consumption preferences and risk aversion allocate their portfolios among three assets: stocks, nominal bonds, and nominal Treasury bills. Investment opportunities are described by a VAR system that includes short-term ex-post real interest rates, excess stock returns, excess bond returns and variables that have been identified as return predictors by empirical research: the short-term nominal interest rate, the dividend-price ratio, and the yield spread between long-term bonds and Treasury bills.

The short-term nominal interest rate has been used to predict stock and bond returns by authors such as Fama and Schwert (1977), Campbell (1987), and Glosten, Jagannathan, and Runkle (1993). An alternative approach, suggested by Campbell (1991) and Hodrick (1992), is to stochastically detrend the short-term rate by subtracting a backwards moving average (usually measured over one year). For two reasons we do not adopt this alternative here. First, we emphasize a long-term annual data set in which we cannot measure a one-year moving average of short rates. Second, we want our model to capture inflation dynamics. If we include both the ex-post real interest rate and the nominal interest rate in the VAR system, we can easily calculate inflation by subtracting one from the other. This allows us to separate nominal from real variables, so that we can extend our model to include a hypothetical inflation-indexed bond in the menu of assets. We consider this extension in section 5.

We compute optimal portfolio rules for different values of γ, assuming either $\psi = 1$ or $\psi = \gamma^{-1}$. In both cases, we set $\delta = 0.92$ in annual terms. The first case gives the exact solution of Giovannini and Weil (1989), where the consumption-wealth ratio is constant and equal to $1 - \delta$. This implies that the loglinearization parameter $\rho \equiv 1 - \exp(\mathrm{E}[c_t - w_t])$ is equal to δ. The second case is the familiar power utility specification.

Section 4.1 describes the annual and quarterly data used in this exercise, and section 4.2 reports the estimates of the VAR system. The numerical procedure used to calculate optimal asset allocations is described in detail in Appendix B. Section 4.3 discusses our findings on asset allocation.

4.1 Data description

Our calibration exercise is based on annual and quarterly data for the US stock market. The annual dataset covers over a century from 1890 to 1995. Its source is the data used in Grossman and Shiller (1981), updated for the recent period by Campbell (1999).[2] This dataset contains data on prices and dividends on S&P 500 stocks as well as data on inflation and short-term interest rates. The equity price index is the end-of-December S&P 500 Index, and the price index is the Producer Price Index. The short rate is the return on 6-month commercial paper bought in January and rolled over July. We use this dataset to construct time series of short-term, nominal and ex-post real interest rates, excess returns on equities, and dividend-yields. Finally, we obtain data on long-term nominal bonds from the long yield series in Shiller (1989), which we have updated using the Moody's AAA corporate bond yield average. We construct the long bond return from this series using the loglinear approximation technique described in Chapter 10 of Campbell, Lo and MacKinlay (1997):

$$r_{n,t+1} \approx D_{n,t} y_{n,t} - (D_{n,t} - 1)\, y_{n-1,t+1},$$

where n is bond maturity, the bond yield is written Y_{nt}, the log bond yield $y_{n,t} = \log(1 + Y_{n,t})$, and $D_{n,t}$ is bond duration. We calculate duration at time t as

$$D_{n,t} \approx \frac{1 - (1 + Y_{n,t})^{-n}}{1 - (1 + Y_{n,t})^{-1}},$$

and we set n to 20 years. We also approximate $y_{n-1,t+1}$ by $y_{n,t+1}$.

The quarterly data begin in 1952:2, shortly after the Fed-Treasury Accord that fundamentally changed the stochastic process for nominal interest rates, and end in 1997:4. We obtain our quarterly data from the Center for Research in Security Prices (CRSP). We construct the ex post real Treasury bill rate as the difference of the log return (or yield) on a 90-day bill and log inflation, and the excess log stock return as the difference between the log return on a stock index and the log return on the 90-day bill. We use the value-weighted return, including dividends, on the NYSE, NASDAQ and AMEX markets. We construct the excess log bond return in a similar way, using the 5-year bond return from the US Treasury and Inflation Series (CTI) file in CRSP.

[2]See the Data Appendix to Campbell (1999), available on the author's website.

The nominal yield on Treasury bills is the log yield on a 90-day bill. To calculate the dividend-price ratio, we first construct the dividend payout series using the value-weighted return including dividends, and the price index series associated with the value-weighted return excluding dividends. Following the standard convention in the literature, we take the dividend series to be the sum of dividend payments over the past year. The dividend-price ratio is then the log dividend less the log price index. The yield spread is the difference between the 5-year zero-coupon bond yield from the CRSP Fama-Bliss data file (the longest yield available in the file) and the bill rate.

4.2 VAR estimation

Table 1 gives the first and second sample moments of the data. Except for the dividend-price ratio, the sample statistics are in annualized, percentage units. Mean excess log returns are adjusted by one-half their variance to account for Jensen's Inequality. For the postwar quarterly dataset, Treasury bills offer a low average real return (a mere 1.813% per year) along with low variability. Stocks have an excess return of 7.119% per year compared to 0.712% for the 5-year bond. Although volatility is much higher for stocks than for bonds (16.093% vs. 5.576%), the Sharpe ratio is almost three and a half times as high for stocks as for bonds. The average Treasury bill rate and yield spread are 5.867% and 0.616%, respectively.

Covering a century of data, the annual dataset gives a different description of the relative performance of each asset. The real return on short-term nominal debt is quite volatile, due to greater volatility in both real interest rates and inflation before World War II. Stocks offer a slightly lower excess return, and yet a higher standard deviation, than the postwar quarterly data. The Depression period is largely responsible for this result. The long-term bond also performs rather poorly, giving a Sharpe ratio of only 0.105 versus a Sharpe ratio of 0.345 for stocks. The bill rate has a lower mean in the annual dataset, but the yield spread has a higher mean. Both bill rates and yield spreads have higher standard deviations in the annual dataset. Figure 2 plots the history of the variables included in the annual VAR.

Table 2 reports the estimation results for the VAR system in the annual dataset (Panel A) and the quarterly dataset (Panel B). The top section of each panel reports coefficient estimates (with t-statistics in parentheses) and the R^2 statistic (with the p-value of the F test of joint significance in parentheses) for each equation in the system.[3] The bottom section of each panel shows the covariance structure of the innovations in VAR system. The entries above the main diagonal are correlation statistics, and the entries on the main diagonal are standard deviations multiplied by 100. All variables in the VAR are measured in natural units, so standard deviations are per year in panel A and per quarter in panel B.

The first row of each panel corresponds to the real bill rate equation. Only the lagged real bill rate and the lagged nominal bill rate have t-statistics above 2 in both sample periods. The rest of the variables are either not significant or only marginally significant in predicting real bill rates one period ahead.

The second row corresponds to the equation for the excess stock return. Predicting excess stock returns is difficult: This equation has the lowest R^2 in the annual sample, and the second lowest R^2 in the quarterly sample. The dividend-price ratio, with a positive coefficient, is the only variable with a t-statistic well above 2. The coefficient on the lagged nominal short-term interest rate is marginally significant in the quarterly sample, and it has a negative sign in both samples. Lagged excess bond returns and yield spreads both have positive coefficients, but they are not statistically significant.

The third row is the equation for the excess bond return. In the long annual dataset, lagged excess returns on stocks and bonds, real Treasury bill rates, and yield spreads help predict future excess bond returns. In the quarterly postwar data, only lagged excess returns on stocks help predict future excess bond returns. The fit of the equation is also much worse than the fit in the annual sample. In part, this difference in results may reflect approximation error in our procedure for constructing annual bond returns; the possibility of such error should be kept in mind when interpreting our annual results.

The last three rows report the estimation results for the remaining state variables, each of which are fairly well described by a univariate AR(1) process. The nominal bill rate in the fourth row is predicted by the lagged yield spread in the quarterly data set, but the main predictor is the lagged nominal yield, whose coefficient is above 0.9 in both samples, implying extremely persistent dynamics. The log dividend-price ratio in the fifth row also has persistent dynamics; the lagged dividend-price ratio

[3] We estimate the VAR imposing the restriction that the unconditional means of the variables implied by the VAR coefficient estimates equal their full-sample arithmetic counterparts. Standard, unconstrained least-squares fits exactly the mean of the variables in the VAR excluding the first observation. We use constrained least-squares to ensure that we fit the full-sample means.

has a coefficient of 0.78 in the annual data and 0.94 in the quarterly data. The yield spread in the sixth row also seems to follow an AR(1) process, but is considerably less persistent than the other variables, especially in the quarterly sample.

The bottom section of each panel describes the covariance structure of the innovations in the VAR system. Unexpected log excess stock returns are highly negatively correlated with shocks to the log dividend-price ratio in both samples. This result is consistent with previous empirical results in Campbell (1991), Campbell and Viceira (1999), Stambaugh (1999) and others. Unexpected log excess bond returns are negatively correlated with shocks to the nominal bill rate, but positively correlated with the yield spread. This positive correlation is 41% in the quarterly sample, and about 27% in the annual sample.

The signs of these correlations help to explain the contrasting results of recent studies that apply Monte Carlo analysis to judge the statistical evidence for predictability in stock and bond returns. Stock-market studies typically find that asymptotic tests overstate the evidence for predictability of stock returns (Hodrick 1992, Goetzmann and Jorion 1993, Nelson and Kim 1993). Bond-market studies, on the other hand, find that asymptotic procedures are actually conservative and understate the evidence for predictability of bond returns (Bekaert, Hodrick, and Marshall 1997). The reason for the discrepancy is that asymptotic results in the stock market are based on positive regression coefficients of stock returns on the dividend-price ratio, while asymptotic results in the bond market are based on positive regression coefficients of bond returns on the yield spread. Stambaugh (1999) shows that the small-sample bias in such regressions has the opposite sign to the sign of the correlation between innovations in returns and innovations in the predictive variable. In the stock market the log dividend-price ratio is negatively correlated with returns, leading to a positive small-sample bias which helps to explain some apparent predictability; in the bond market, on the other hand, the yield spread is positively correlated with returns, leading to a negative small-sample bias which cannot explain the positive regression coefficient found in the data.

The signs of these correlations also have important effects on the volatility of bond and stock returns over long holding periods. We now explore these effects in some detail as they are highly relevant for long-term investors.

4.3 Return volatility at short and long horizons

Our estimated VAR system implies that there are important horizon effects on the relative volatilities of different investment strategies. In Figure 3 we plot the annualized standard deviations of real returns on stocks and bills implied by our annual VAR for investment horizons up to 100 years (panel A) and implied by our quarterly VAR for investment horizons up to 100 quarters or 25 years (panel B).[4] We also consider standard deviations for two alternative investment strategies using nominal bonds. The "long bond rolled" strategy keeps the maturity of the long bond constant at 20 years, buying a 20-year bond each year and selling it the next year in order to invest in a new 20-year bond. This is the strategy implicitly assumed in virtually all time series of long-term bond returns. The "bond held to maturity" strategy assumes that an investor with horizon k buys a nominal bond with k years to maturity and holds it until maturity. The standard deviation of the real return on this strategy is just the standard deviation of cumulative inflation from time t to time $t+k$, since a nominal bond held to maturity is riskless in nominal terms.

Figure 3 shows that stocks are mean-reverting—their long-horizon returns are less volatile than their short-horizon returns—while bonds and bills are mean-averting—their long-horizon returns are actually more volatile than their short-horizon returns. Mean-aversion is particularly strong for bills in the annual dataset, where persistent variation in the real interest rate amplifies the volatility of returns over long horizons. Mean-aversion also affects the returns on rolling long bonds in the annual dataset (because of both variation in the real interest rate and predictability of bond returns from the yield spread), and the returns on holding bonds to maturity in the quarterly dataset (because of persistent movements in inflation). Mean-reversion in stock returns was pointed out by Fama and French (1988b) and Poterba and Summers (1988), and has been the subject of much subsequent research. Siegel (1998) has used long-term data to directly measure mean-aversion in fixed-income securities and has emphasized its importance for long-term investors, but this phenomenon has received relatively little attention in the academic literature.

The estimated horizon effects on volatility are large enough to alter the rankings of asset return volatilities across investment horizons. In the annual system, stocks

[4]Note that we are not looking directly at the long-horizon properties of returns, but at the long-horizon properties of returns imputed from our first-order VAR. Thus, provided that our VAR captures adequately the dynamics of the data, we can consistently estimate the moments of returns over any desired horizon.

are far more volatile than bonds and bills at short horizons, but safer than bills or rolling bonds at long horizons, a point stressed by Siegel (1998). In the quarterly system, stocks are the riskiest asset at all horizons, but their relative risk declines sharply with the horizon. Of the two bond strategies, rolling bonds is riskier at short horizons, but buying and holding is riskier at long horizons since it exposes investors to the persistent variation in inflation that has been characteristic of the postwar period.

These results suggest that long-horizon investors may have a perspective on risk that is very different from the perspective of myopic investors. We explore this issue in the sections that follow.

4.4 Strategic allocations to stocks, bonds, and bills

We have shown in section 3.3 that the optimal portfolio rule is linear in the vector of state variables. Thus the optimal portfolio allocation to stocks, bonds and bills changes over time. One way to characterize this rule is to examine its mean and volatility. To analyze level effects we compute the mean allocation to each asset as well as the mean hedging portfolio demand for different specifications of the vector of state variables. Specifically, we estimate a series of restricted VAR systems, in which the number of explanatory variables increases sequentially, and use them to calculate mean optimal portfolios for $\psi = 1$ or $1/\gamma$, $\delta = 0.92$ at an annual frequency, and $\gamma = 1$, 2, 5 or 20.

The first VAR system only has a constant term in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. The second system includes an intercept term, the ex-post real bill rate and log excess returns on stocks and bonds. We then add sequentially the nominal bill rate, the dividend yield and the yield spread. Thus we estimate five VAR systems in total.

Table 3 reports the results of this experiment for values of the coefficient of relative risk aversion γ equal to 1, 2, 5 and 20, with the intertemporal elasticity of substitution $\psi = 1$. Panel A considers the annual dataset, while Panel B considers the quarterly dataset. The entries in each column are mean portfolio demands in percentage points when the explanatory variables in the VAR system include the state variable in the column heading and those to the left of it. For instance, the "constant" column reports mean portfolio allocations when the explanatory variables include only a constant term, that is, when investment opportunities are constant. The right-hand “spread” column gives the case where all state variables are included in the VAR.

Table 3 reports results only for selected values of risk aversion, but we have also computed portfolio allocations for a continuum of values of risk aversion; Figure 4 plots these allocations and their myopic component using the annual VAR with all state variables included. In this figure the horizontal axis shows risk tolerance $1/\gamma$ rather than risk aversion γ, both in order to display the behavior of highly conservative investors more compactly, and because myopic portfolio demands are linear in risk tolerance.

Table 3 enables us to analyze two effects on the level of portfolio demands. By comparing numbers within any column, we can study how total asset allocation and intertemporal hedging demand vary with risk aversion. By comparing numbers within any row, we can examine the incremental effects of the state variables on asset allocation. Here we explore the first topic and leave the second for the next section. To simplify the discussion we focus only on the allocations implied by the full VAR, shown in the right-hand column of the table.

The first set of numbers in Table 3 reports the mean portfolio allocation to stocks, bonds and bills of a logarithmic investor. For this investor, the optimal portfolio rule is purely myopic. Equation (20) evaluated at $\gamma = 1$ shows that asset allocation depends only on the inverse of the variance-covariance matrix of unexpected excess returns and the mean excess return on stocks and bonds. This myopic allocation is long in stocks and bonds in both the annual dataset and the quarterly dataset. However, the ratio of stocks to bonds is close to one in the annual dataset and is about 50 in the quarterly dataset. The preference for stocks in the quarterly dataset is primarily due to the estimated large positive correlation between unexpected excess returns on stocks and bonds in the quarterly dataset. This shifts the optimal myopic allocation towards stocks—the asset with the largest Sharpe ratio. In the annual dataset the correlation between excess bond and stock returns is very low, implying that the optimal portfolio allocation to one asset is essentially independent of the optimal allocation to the other.

Conservative investors, with risk aversion $\gamma > 1$, have an intertemporal hedging demand for stocks. This demand is most easily understood by looking at Figure 4, which is based on the annual dataset. In Figure 4, the total demand for stocks is a nonlinear, hump-shaped function of risk tolerance $1/\gamma$, while the myopic portfolio demand is a linear function of $1/\gamma$.[5] Moreover, total stock demand is always larger than myopic portfolio demand for all $1/\gamma < 1$. This implies that intertemporal hedging demand must be a positive, nonlinear function of $1/\gamma$. We can verify this by looking at the hedging demands reported in Table 3. In both datasets, the hedging demand for stocks is always positive and exhibits a hump-shaped pattern as a function of $1/\gamma$. These patterns reflect the mean-reversion of stock returns illustrated in Figure 3, which is captured in our VAR model by the predictability of stock returns from the dividend-price ratio.

The hedging demand for stock is particularly large in the quarterly dataset (Table 3, panel B). In fact, in that dataset the hump-shaped hedging demand dominates the linear myopic demand so that total stock demand actually rises with risk aversion for intermediate levels of risk aversion. These results contrast with those of Campbell and Viceira (1999), which are closer to the results reported here for long-term annual data. The main reason for this contrast is that Campbell and Viceira (1999) made an error in estimating their model, understating the predictability of postwar quarterly stock returns and the absolute value of the correlation between innovations to stock returns and dividend yields. This error is explained in Campbell and Viceira (2000), which reports complete corrected results. In addition, the availability of nominal bonds—which are positively correlated with stocks in the quarterly dataset—tends to strengthen hedging demands by allowing investors to offset their equity risks with short positions in bonds.

Intertemporal hedging demands are just as striking for nominal bonds. Panel A in Table 3 shows that the portfolio hedging demand for bonds is negative and exhibits a U-shaped pattern across coefficients of relative risk aversion that eventually reverts to zero. Of course, this shape is also reflected in the total demand for nominal bonds, which is plotted in Figure 3. This pattern can be explained by the mean-aversion of bond returns illustrated in Figure 3. At a deeper level, it results from the effect of the yield spread on intertemporal hedging demand. We defer this discussion until the next section, where we analyze the effects of individual state variables on portfolio demands.

We have shown allocation results only for the case $\psi = 1$. In Appendix D we

[5] We can see this formally by looking at equation (20). This equation implies that myopic demand is a linear function of $1/\gamma$ with intercept given by $-\Sigma_{xx}^{-1}\Sigma_{1x}$. The intercept is zero only when shocks to the real interest rate on nominal Treasury bills are uncorrelated with excess returns on all other assets, that is, when Σ_{1x} is zero.

reproduce Table 3 for the power utility case ($\psi = 1/\gamma$). We have already noted in Section 3 that optimal portfolio demands do not depend on the elasticity of intertemporal substitution except through the loglinearization parameter ρ. Appendix D shows that this indirect effect is quantitatively insignificant. The allocations for power utility are almost indistinguishable from those in Table 3.

We turn now to the analysis of the variability of asset demands. From equation (20), we can express the optimal portfolio rule as

$$\alpha_{i,t} = \alpha_{i,t}^{m} + \alpha_{i,t}^{h}, \tag{25}$$

where $i = 2, 3$ denotes stocks and bonds respectively, m denotes myopic and h denotes hedging. Thus,

$$\mathrm{Var}\left(\alpha_{i,t}\right) = \mathrm{Var}\left(\alpha_{i,t}^{m}\right) + \mathrm{Var}\left(\alpha_{i,t}^{h}\right) + 2\,\mathrm{Cov}\left(\alpha_{i,t}^{m}, \alpha_{i,t}^{h}\right). \tag{26}$$

Table 4 reports this variance decomposition for the case $\gamma = 5$ and $\psi = 1$. Panel A refers to the annual sample, and panel B to the quarterly sample. The top rows in each panel report the contribution of $\mathrm{Var}\left(\alpha_{i,t}^{m}\right)$, $\mathrm{Var}\left(\alpha_{i,t}^{h}\right)$ and $\mathrm{Cov}\left(\alpha_{i,t}^{m}, \alpha_{i,t}^{h}\right)$ to the total variance of the demand for each asset. The hedging component explains at most 23% of the total variation in portfolio demand for both stocks and bonds in the annual dataset, and 14% in the quarterly dataset. Thus hedging portfolio demand is much more stable than total portfolio demand. Kim and Omberg (1996) and Campbell and Viceira (1999) give an intuitive explanation for this result, showing that hedging demand can change sign only in extreme circumstances where investors have replaced their normal long positions with short positions in risky assets. To a first approximation, intertemporal hedging shifts the intercept of risky asset demand rather than the slope with respect to state variables; put another way, long-term investors should "market-time" just as aggressively as short-term investors.

Figure 5 illustrates the result graphically. The figure plots time series of total portfolio allocations based on the annual dataset for $\gamma = 5$, along with the myopic and hedging components of total portfolio demand. For all three assets, hedging demands are considerably less volatile than myopic demands. The allocation to long-term nominal bonds is the most volatile, since this is the asset with the highest degree of predictability in the annual dataset. It is possible that the predictability of bond returns, and hence the volatility of bond allocations, is overstated by approximation error in the annual data; this deserves more careful investigation.

4.5 Which state variables matter?

The analysis so far has focused on the shape of asset demands and their hedging components. It is equally important to understand the effects of various state variables on the level and variability of asset demands. To analyze the level effects of state variables, we can compare average portfolio demands across rows in Table 3.

Panel A shows that there are important changes in the magnitude of hedging demands as we consider new state variables in the investor information set. In the case of stocks, hedging demand is very small when only lagged Treasury bill rates (either real or nominal) and excess returns on bonds and stocks are included in the VAR. It shoots up dramatically when the dividend-price ratio is introduced into the VAR as a regressor. The inclusion of the yield spread has mixed effects in the annual dataset, and negative effects in the quarterly dataset.

The correlation structure shown in Table 2 helps explain these results. In the full annual VAR system, there is a strong negative correlation between unexpected excess returns on stocks and shocks to the dividend-price ratio, while the magnitude of all other correlations in the table is much smaller. These correlations are not sensitive to the inclusion or exclusion of state variables in the VAR. The presence of the dividend-price ratio in the investor information set increases the hedging demand for stocks because negative shocks to the dividend-price ratio, which drive down expected returns on stocks, tend to coincide with positive realized excess returns on stocks. This negative correlation is even stronger in the quarterly dataset, which makes the pattern for hedging demands more pronounced in this dataset.

In the case of bonds, the yield spread has a tremendous negative impact on hedging demand. In fact, it changes the sign of hedging demand from positive to negative. Table 2 again explains this result. In both datasets, the yield spread is the most important forecasting variable for bond returns, and its innovations are positively correlated with excess bond returns. This correlation produces a negative hedging demand for bonds, since negative shocks to expected future bond returns tend to coincide with negative current bond returns. The magnitude of the correlation is larger in the quarterly dataset, which is why hedging demand is more negative in this dataset.[6]

[6] The positive correlation of bond and stock returns in the quarterly dataset also means that positive hedging demand for stocks tends to produce negative hedging demand for bonds in that dataset. Note also that shocks to nominal bill yields are highly negatively correlated with unexpected

Hedging demands can also be understood by reference to Figure 2, which shows that stock returns are mean-reverting, while nominal bond returns are mean-averting. A univariate representation of excess stock returns will have a negative correlation between expected and unexpected excess returns on stocks, while a univariate representation of excess bond returns will have a positive correlation between expected and unexpected excess returns on bonds. This makes stocks an attractive asset for conservative investors who seek to hedge intertemporally, while it makes nominal bonds a fundamentally unattractive asset.

We can also analyze the importance of each state variable for the variability of asset demands by looking at Table 5. The lower section of Table 5 report the contribution of each state variable to total asset demand variation Var $(\alpha_{i,t})$. For each block, the diagonal elements are the own variance components and the lower off-diagonal elements are the covariance components, normalized to sum to 100%. The dividend-price ratio explains 80% of the variance of total demand for stocks in the annual sample, and 91% in the quarterly sample. The dividend-price ratio plays a much less important role in explaining the variability of the portfolio demand for bonds, which is driven primarily by the lagged excess stock return and the yield spread.

In summary, our results indicate that the most important state variable determining the mean and volatility of stock demand is the dividend yield, while the yield spread is more important for bonds. The dividend yield generates a large positive intertemporal hedging demand for stocks, while the yield spread generates a large negative intertemporal hedging demand for bonds. Aït-Sahalia and Brandt (2000) also find that these variables are important determinants of optimal portfolio choice, though they find that the role of the dividend yield weakens if the late 1990's are included in the sample.

excess bond returns in both samples, but the coefficient on the nominal bill rate in the excess bond return equation in the VAR is small and not statistically significant. This lack of predictive power means that the inclusion of the nominal bill rate has a relatively small effect on the hedging demand for bonds.

5 Strategic Asset Allocation with Inflation-Indexed Bonds

Our results so far imply that the intertemporal hedging demand for long-term bonds is negative. This contrasts with conventional investment advice that conservative long-term investors should hold bonds to obtain a stable stream of income, disregarding short-run fluctuations in capital value. There are two possible reasons for the discrepancy between our results and conventional wisdom. First, the conventional wisdom disregards the distinction between nominal and inflation-indexed bonds. In the presence of significant inflation risk, long-term nominal bonds are not suitable assets for conservative long-term investors. Campbell and Viceira (2001) use a term structure model with time-varying real interest rates and inflation rates, but constant risk premia, and find that inflation uncertainty drastically reduces the intertemporal hedging demand for long-term nominal bonds in the postwar period.[7] Second, our model has a general dynamic structure in which either stocks or bonds might be good hedges for predictable variation in stock and bond returns. Conventional investment advice may be based on the presumption that bonds are the best hedges for predictable variation in returns on all risky assets; the model of Campbell and Viceira (2001) explicitly assumes this.

To determine which of these explanations is correct, we now extend our model to include an inflation-indexed perpetuity in the menu of available assets. This requires us to construct hypothetical real bond returns, because indexed bonds have only been recently issued by the US Treasury and thus data on indexed bonds are very limited. The VAR framework is well suited for this purpose, provided that we make the assumption that expected real returns on real bonds of all maturities and the expected real return on short-term nominal bills differ only by a constant. This amounts to assuming that the inflation risk premium on nominal bills is constant. We now briefly describe the construction procedure, which is adapted from the work of Campbell and Shiller (1996). Appendix C provides full details.

We first use the estimates of the coefficient matrices in the VAR to construct returns on hypothetical real perpetuities according to the procedure outlined in Appendix C. The procedure assumes a zero inflation risk premium. As noted in Campbell

[7] Campbell and Viceira also look at recent data since 1983, and find much smaller inflation uncertainty. We do not examine this period here as our VAR system is not sufficiently parsimonious for such a short sample period.

and Shiller (1996), if the inflation risk premium is not zero but constant, the procedure will miss the average level of the yield curve, but will still capture the dynamics of the curve. This is important, because intertemporal hedging demand depends sensitively on the dynamics of asset returns. With the correct dynamics in hand, we adjust the mean return by setting the Sharpe ratio of the real consol bond to the Sharpe ratio of nominal bonds.[8] Finally, we include the imputed excess return on real perpetuities in two new VAR systems. In the first VAR we replace the excess return on nominal bonds with the excess return on real perpetuities, while in the second system we include both variables. Appendix D shows the estimation results.

Table 5 reports the resulting mean asset demands for values of γ equal to 1, 2, 5, 20 and 2000 and $\psi = 1$. We include the case $\gamma = 2000$ because we want to study asset demand for infinitely risk averse investors, which we proxy using this large value of γ. We also report mean asset allocations under constant investment opportunities. To simplify the discussion and to save space, we include results only for the annual dataset. Figures 7 and 8 plot the allocations implied by the full VAR for a continuum of values of γ.

We start by looking at the optimal portfolio of a myopic logarithmic investor. This investor should hold a short position in the inflation-indexed perpetuity, despite the fact that the mean excess return on this asset is positive by construction. This allocation is the result of a large, positive correlation between excess returns on stocks and excess returns on the real perpetuity (shown in Appendix D), which makes it optimal for a logarithmic investor to short the real perpetuity to increase her allocation to stocks, the asset with the largest Sharpe ratio.

We can learn about the myopic allocations of non-logarithmic investors by looking at the allocations under constant investment opportunities shown in the "constant" column. Investors with $\gamma > 1$ have a myopic demand for real perpetuities that is not proportional to the optimal allocation of the logarithmic investor. In fact, it even changes sign and becomes positive for moderately risk averse investors. This is driven by the fact that the short-term bill is risky in real terms, so the portfolio with the smallest short-term risk is a combination, with roughly equal positive weights, of the short-term bill and the real perpetuity.[9]

[8] We have also considered setting the Sharpe ratio of the real consol bond equal to zero and setting it equal to the Sharpe ratio of stocks. These choices affect myopic asset demands, but do not have noticeable effects on intertemporal hedging demands. Results are available from the authors upon request.

[9] Equation (20) shows that the myopic demand of non-logarithmic investors is not proportional

The "spread" column in Table 5 shows total portfolio demands with time-varying investment opportunities. The total portfolio demand for the real consol bond is increasing in risk aversion, approaching 100% of the portfolio as the investor becomes infinitely conservative. By contrast, the total portfolio demand for stocks, the nominal bill and the nominal bond are decreasing in γ, approaching 0% as the investor becomes infinitely conservative. Thus inflation-indexed bonds drive out cash from the portfolios of conservative investors. In their model with time-varying interest rates but constant risk premia, Campbell and Viceira (2001) show that an infinitely risk-averse long-horizon investor with zero elasticity of intertemporal substitution would choose to be fully invested in a real perpetuity.[10] Table 5 shows that this result extends to a world in which both interest rates and expected excess returns are time-varying.[11]

to $1/\gamma$; instead it is a linear function of $1/\gamma$ whose intercept depends on the covariance between shocks to the real interest rate on nominal bills and unexpected excess returns on all other assets (Σ_{1x}). Appendix D shows that the correlation between shocks to the real interest rate on nominal bills and unexpected excess returns on the real consol bond is negative and large (about -85% in the annual dataset), which in turn translates into a large, positive intercept for the real perpetuity.

[10] Wachter (1998) has extended this result to more general models with complete markets.

[11] Note that the allocation to the real consol bond for an investor with an extremely large coefficient of relative risk aversion does not equal 100% exactly. This is due primarily to the fact that the investor we consider in Table 5 has unit, not zero, elasticity of intertemporal substitution. There is also a small effect caused by the fact that the VAR system in Table 5 does not exactly capture the information set we used to construct the long-term real bond yield.

6 Conclusion

This paper has explored the implications for long-term investors of the empirical evidence on the predictability of asset returns. Dividend yields, interest rates, yield spreads, inflation, and other variables that predict asset returns in previous empirical research have substantial effects on optimal portfolio allocations among bills, stocks, and nominal and indexed bonds. These effects are strategic, working through intertemporal hedging demands, rather than merely tactical effects on myopic optimal portfolios.

Strategic effects on asset demands arise because shocks to the forecasting variables are correlated with the unexpected returns on stocks and bonds. The correlation is strongest for the dividend-price ratio, and thus we find that this variable is the most important determinant of both the level and the variability of optimal portfolio demand. Predictability of stock returns from the dividend-price ratio tilts the optimal portfolio holdings of moderately conservative investors towards stocks and away from bonds and cash. We find that the intertemporal hedging demand for long-term nominal bonds is negative, both in postwar monthly and long-term annual data; however results are quite different for long-term inflation-indexed bonds. We find that it is optimal for extremely conservative investors to hold most of their wealth in long-term inflation-indexed bonds when these assets are available.

These findings can be understood heuristically using the relations between investment horizon and return volatility illustrated in Figure 3 of this paper. Time-varying real interest rates make short-term investments unsafe for long-term investors, since they must reinvest short-term assets at unknown future rates. Mean-reversion in stock returns makes stocks appear safer to conservative investors with a long investment horizon. Long-term inflation-indexed bonds are riskless for long-term investors, but nominal bonds are poor substitutes for inflation-indexed bonds in the presence of substantial inflation risk.

Our research has several limitations that should be kept in mind when interpreting the results. First, we consider a long-term investor who has financial wealth but no labor income. We hope to remedy this serious omission in future work by extending the approach of Viceira (2001). Second, we do not impose borrowing or short-sales constraints; to do so would take us outside the tractable linear-quadratic approximate framework and would require a fully numerical solution method of the sort used by Brennan, Schwartz, and Lagnado (1996, 1997) and Lynch (1999). Third, our solutions are approximate for investors with elasticity of intertemporal substitution not equal to one. Campbell, Cocco, Gomes, Maenhout, and Viceira (1998) have checked the accuracy of the approximation in the simpler model of Campbell and Viceira (1999) with only one risky asset and one state variable, and have explored the effects of portfolio constraints in that context, but further work is needed within the richer dynamic framework used here. Fourth, we ignore the differential tax treatment of interest or dividend income and capital gains. Dammon, Spatt, and Zhang (1999) have recently argued that tax effects can be particularly important for long-term investors. Finally, we assume that investors know all the parameters of the model. We have found that these parameters, including not only the means and covariances of asset returns but also the parameters governing the dynamics of asset returns and state variables, can have enormous effects on optimal portfolio demands. Given this, it is not surprising that parameter uncertainty and learning can have a large effect on optimal long-term investment strategies as shown by Barberis (2000), Brennan (1998), Brennan and Xia (1999), and others. A challenging task for future research will be to integrate all these effects into a single empirically implementable framework.

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TABLE 1

Sample Statistics

	Sample Moment	1890 - 1995	1952Q2 - 1997Q4
(1)	$E[r^{\$}_{1,t} - \pi_t] + \sigma^2(r^{\$}_{1,t} - \pi_t)/2$	2.112	1.813
(2)	$\sigma(r^{\$}_{1,t} - \pi_t)$	8.891	1.460
(3)	$E[r^{\$}_{e,t} - r^{\$}_{1,t}] + \sigma^2(r^{\$}_{e,t} - r^{\$}_{1,t})/2$	6.242	7.119
(4)	$\sigma(r^{\$}_{e,t} - r^{\$}_{1,t})$	18.107	16.093
(5)	$SR = (3)/(4)$	0.345	0.442
(6)	$E[r^{\$}_{n,t} - r^{\$}_{1,t}] + \sigma^2(r^{\$}_{n,t} - r^{\$}_{1,t})/2$	0.661	0.712
(7)	$\sigma(r^{\$}_{n,t} - r^{\$}_{1,t})$	6.299	5.576
(8)	$SR = (6)/(7)$	0.105	0.128
(9)	$E[y^{\$}_t]$	4.321	5.867
(10)	$\sigma(y^{\$}_t)$	2.611	1.555
(11)	$E[d_t - p_t]$	-3.079	-3.371
(12)	$\sigma(d_t - p_t)$	0.275	0.244
(13)	$E[y^{\$}_{n,t} - y^{\$}_{1,t}]$	0.876	0.616
(14)	$\sigma(y^{\$}_{n,t} - y^{\$}_{1,t})$	1.459	0.588

Note: $r^{\$}_{1,t}$ = log return on T-bills, π_t = log inflation rate, $r^{\$}_{e,t}$ = log return on equities, $r^{\$}_{n,t}$ = log return on nominal bond, $(d-p)_t$ = log dividend-price ratio, rb_t = relative bill rate, $y^{\$}_{n,t}$ = log yield on the nominal bond, and $y^{\$}_{1,t}$ is the short yield. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset.

TABLE 2
VAR Estimation Results

A: Annual Sample (1890 - 1995)

Dependent Variable	rtb_t (t)	xr_t (t)	xb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.305	-0.056	0.147	0.685	-0.004	-0.869	0.239
	(2.456)	(-1.391)	(0.973)	(2.271)	(-0.138)	(-1.320)	(0.000)
xr_{t+1}	0.111	0.087	-0.219	-0.134	0.187	1.199	0.086
	(0.420)	(0.703)	(-0.763)	(-0.191)	(3.449)	(0.908)	(0.117)
xb_{t+1}	0.200	0.095	-0.091	-0.083	0.009	2.573	0.421
	(3.153)	(2.658)	(-0.652)	(-0.238)	(0.474)	(5.070)	(0.000)
y_{t+1}	-0.042	-0.011	0.029	0.918	-0.005	-0.029	0.783
	(-1.928)	(-1.571)	(0.907)	(12.378)	(-1.027)	(-0.228)	(0.000)
$(d-p)_{t+1}$	-0.562	-0.134	0.529	-0.509	0.779	-1.686	0.677
	(-2.243)	(-1.228)	(1.937)	(-0.795)	(13.666)	(-1.255)	(0.000)
spr_{t+1}	0.019	0.002	-0.016	0.085	0.004	0.838	0.542
	(1.109)	(0.357)	(-0.709)	(1.633)	(1.093)	(8.655)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	xb	y	$(d-p)$	spr
rtb	7.753	-0.194	-0.029	0.130	0.131	-0.167
xr	-	17.303	0.027	-0.175	-0.713	0.210
xb	-	-	4.794	-0.636	-0.115	0.266
y	-	-	-	1.217	0.221	-0.903
$(d-p)$	-	-	-	-	15.624	-0.192
spr	-	-	-	-	-	0.987

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, xb_t = excess bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE 2 (ctd.)

VAR Estimation Results

B: Quarterly Sample (1952Q2 - 1997Q4)

Dependent Variable	rtb_t (t)	xr_t (t)	xb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.445	0.006	-0.018	0.287	-0.001	0.034	0.324
	(6.142)	(1.040)	(-0.822)	(3.695)	(-0.519)	(0.207)	(0.000)
xr_{t+1}	0.256	0.061	0.355	-1.852	0.077	2.072	0.109
	(0.275)	(0.769)	(1.492)	(-2.097)	(3.011)	(0.857)	(0.001)
xb_{t+1}	0.159	-0.054	-0.050	0.310	0.003	0.995	0.043
	(0.447)	(-2.594)	(-0.424)	(0.771)	(0.424)	(0.949)	(0.179)
y_{t+1}	0.001	0.003	-0.008	0.962	0.000	0.490	0.796
	(0.016)	(0.990)	(-0.418)	(16.990)	(0.361)	(4.432)	(0.000)
$(d-p)_{t+1}$	-0.626	-0.064	-0.334	1.047	0.939	-1.926	0.892
	(-0.631)	(-0.744)	(-1.280)	(1.144)	(34.309)	(-0.783)	(0.000)
spr_{t+1}	-0.015	0.000	0.012	0.019	-0.001	0.497	0.277
	(-0.369)	(0.072)	(0.961)	(0.491)	(-0.876)	(6.460)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	xb	y	$(d-p)$	spr
rtb	0.599	0.229	0.451	-0.511	-0.241	0.404
xr	-	7.586	0.274	-0.211	-0.969	0.108
xb	-	-	2.725	-0.766	-0.321	0.413
y	-	-	-	0.350	0.253	-0.899
$(d-p)$	-	-	-	-	7.961	-0.137
spr	-	-	-	-	-	0.250

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, xb_t = excess bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE 3
Mean Asset Demands ($\psi = 1$ Case)

A: Annual Sample (1890 - 1995)						
	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$						
Stock	Total Demand	188.44	187.46	189.20	199.43	201.97
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	127.57	146.72	155.55	155.29	231.73
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-216.01	-234.18	-244.75	-254.72	-333.70
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1, \rho = 0.92$						
Stock	Total Demand	98.66	100.81	101.80	132.29	132.47
	Hedging Demand	0.00	2.54	3.02	28.13	27.15
Bond	Total Demand	70.78	89.58	95.75	89.71	53.49
	Hedging Demand	0.00	9.12	12.47	6.58	-64.28
Cash	Total Demand	-69.44	-90.39	-97.55	-122.00	-85.96
	Hedging Demand	0.00	-11.66	-15.49	-34.71	37.13
$\gamma = 5, \psi = 1, \rho = 0.92$						
Stock	Total Demand	44.79	52.42	52.30	74.29	81.38
	Hedging Demand	0.00	7.66	7.79	27.30	34.05
Bond	Total Demand	36.71	53.74	63.75	64.28	-18.87
	Hedging Demand	0.00	13.04	23.82	24.44	-68.29
Cash	Total Demand	18.50	-6.16	-16.05	-38.57	37.49
	Hedging Demand	0.00	-20.70	-31.61	-51.74	34.24
$\gamma = 20, \psi = 1, \rho = 0.92$						
Stock	Total Demand	17.86	29.22	27.93	35.99	40.85
	Hedging Demand	0.00	11.22	10.54	17.59	22.52
Bond	Total Demand	19.67	35.62	48.30	46.95	15.60
	Hedging Demand	0.00	14.80	30.06	28.75	0.40
Cash	Total Demand	62.47	35.15	23.77	17.06	43.55
	Hedging Demand	0.00	-26.02	-40.60	-46.34	-22.92

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE 3 (ctd.)

Mean Asset Demands ($\psi = 1$ Case)

	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
B: Quarterly Sample (1952Q2 - 1997Q4)						
$\gamma = 1, \psi = 1, \rho = 0.92^{1/4}$						
Stock	Total Demand	272.65	285.19	289.75	301.76	302.41
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	42.80	15.90	8.20	2.88	6.54
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-215.45	-201.09	-197.95	-204.64	-208.95
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1, \rho = 0.92^{1/4}$						
Stock	Total Demand	136.07	138.76	139.35	313.75	241.40
	Hedging Demand	0.00	-3.63	-5.14	163.32	90.64
Bond	Total Demand	16.28	-36.69	-68.17	-415.78	-465.80
	Hedging Demand	0.00	-39.75	-67.64	-412.63	-464.46
Cash	Total Demand	-52.36	-2.07	28.82	202.03	324.40
	Hedging Demand	0.00	43.37	72.79	249.31	373.82
$\gamma = 5, \psi = 1, \rho = 0.92^{1/4}$						
Stock	Total Demand	54.13	53.07	50.50	578.45	566.02
	Hedging Demand	0.00	-3.63	-6.84	518.86	506.30
Bond	Total Demand	0.38	-30.52	-30.40	-677.15	-1090.92
	Hedging Demand	0.00	-25.87	-24.64	-670.39	-1084.90
Cash	Total Demand	45.49	77.45	79.90	198.69	624.89
	Hedging Demand	0.00	29.51	31.48	151.53	578.60
$\gamma = 20, \psi = 1, \rho = 0.92^{1/4}$						
Stock	Total Demand	13.16	11.49	9.34	358.21	502.51
	Hedging Demand	0.00	-2.37	-4.42	344.09	488.00
Bond	Total Demand	-7.58	-17.53	7.91	-369.26	-799.10
	Hedging Demand	0.00	-9.02	16.28	-360.74	-789.46
Cash	Total Demand	94.42	106.03	82.75	111.05	396.59
	Hedging Demand	0.00	11.40	-11.86	16.66	301.45

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE 4
Variability of Asset Demands

A: Annual Sample (1890 - 1995)

$\gamma = 5, \psi = 1, \rho = 0.92^{1/4}$

	$\text{Var}(\alpha^m)/\text{Var}(\alpha)$ (%)	$\text{Var}(\alpha^h)/\text{Var}(\alpha)$ (%)	$\text{Cov}(\alpha^m, \alpha^h)/\text{Var}(\alpha)$ (%)
Stocks	90.04	13.73	-1.88
Bond	101.78	0.29	-1.03

Percentage of Total Variation Explained By:

		rtb_t	xr_t	xb_t	y_t	$(d-p)_t$	spr_t
Stocks							
	rtb_t	0.10	0.00	0.00	0.00	0.00	0.00
	xr_t	0.30	6.63	0.00	0.00	0.00	0.00
	xb_t	-0.14	-0.67	3.80	0.00	0.00	0.00
	y_t	0.32	1.35	-2.70	2.68	0.00	0.00
	$(d-p)_t$	0.07	-8.36	6.04	4.22	79.99	0.00
	spr_t	-0.19	-1.32	3.30	-1.89	4.19	2.28
Bonds							
	rtb_t	18.24	0.00	0.00	0.00	0.00	0.00
	xr_t	-6.64	17.29	0.00	0.00	0.00	0.00
	xb_t	1.31	-0.76	1.87	0.00	0.00	0.00
	y_t	0.03	-0.02	0.01	0.00	0.00	0.00
	$(d-p)_t$	-0.04	-0.57	0.18	-0.00	0.14	0.00
	spr_t	-16.22	13.19	-14.34	-0.08	-1.09	87.49

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, xb_t = excess bond return, $(d-p)_t$ = log dividend-price ratio, rb_t = relative bill rate, spr_t = yield spread. The bond is a 5-year nominal bond in the monthly dataset and a 20-year in the annual dataset.

TABLE 4 (ctd.)
Variability of Asset Demands

B: Quarterly Sample (1952Q2 - 1997Q4)

$\gamma = 5, \psi = 1, \rho = 0.92^{1/4}$

	$\text{Var}(\alpha^m)/\text{Var}(\alpha)$ (%)	$\text{Var}(\alpha^h)/\text{Var}(\alpha)$ (%)	$\text{Cov}(\alpha^m, \alpha^h)/\text{Var}(\alpha)$ (%)
Stock	47.19	17.48	17.66
Bond	53.31	22.72	11.98

Percentage of Total Variation Explained By:

		rtb_t	xr_t	xb_t	y_t	$(d-p)_t$	spr_t
Stocks							
	rtb_t	0.04	0.00	0.00	0.00	0.00	0.00
	xr_t	0.13	3.02	0.00	0.00	0.00	0.00
	xb_t	0.41	2.04	5.87	0.00	0.00	0.00
	y_t	-0.29	4.79	10.17	56.11	0.00	0.00
	$(d-p)_t$	0.15	-3.70	-0.77	-65.68	91.72	0.00
	spr_t	-0.07	-0.39	-1.07	-5.33	2.45	0.41
Bonds							
	rtb_t	1.05	0.00	0.00	0.00	0.00	0.00
	xr_t	-1.79	25.01	0.00	0.00	0.00	0.00
	xb_t	-2.06	6.09	6.30	0.00	0.00	0.00
	y_t	1.72	16.76	12.81	83.03	0.00	0.00
	$(d-p)_t$	-0.47	-6.77	-0.51	-50.74	36.99	0.00
	spr_t	2.38	-7.67	-7.47	-43.78	10.48	18.63

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, xb_t = excess bond return, $(d-p)_t$ = log dividend-price ratio, rb_t = relative bill rate, spr_t = yield spread. The bond is a 5-year nominal bond in the monthly dataset and a 20-year in the annual dataset.

TABLE 5

Mean Asset Demands with Hypothetical Real Bonds (Annual Sample: 1890 - 1995)

A: Nominal Bills, Stocks, and Real Consol Bond

State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	191.41	218.85
Real Consol Bond	-65.43	-75.96
Cash	-25.98	-42.89
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	95.46	130.88
Real Consol Bond	-6.04	-3.18
Cash	10.58	-27.70
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	37.89	58.58
Real Consol Bond	29.59	55.90
Cash	32.52	-14.49
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	9.10	15.86
Real Consol Bond	47.41	86.79
Cash	43.49	-2.66
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	-0.39	0.26
Real Consol Bond	53.29	97.28
Cash	47.11	2.46

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE 5 (ctd.)

Mean Asset Demands with Hypothetical Real Bonds

(Annual Sample: 1890 - 1995)

B: Nominal Bills, Stocks, Real Consol Bond and Nominal Bond		
State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	188.49	221.00
Real Consol Bond	-76.96	-103.51
Nominal Bond	144.71	290.94
Cash	-156.25	-308.43
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	94.05	134.25
Real Consol Bond	-11.63	-21.99
Nominal Bond	70.13	73.28
Cash	-52.54	-85.54
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	37.38	70.75
Real Consol Bond	27.57	46.71
Nominal Bond	25.38	-36.43
Cash	9.68	18.97
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	9.04	21.95
Real Consol Bond	47.17	84.40
Nominal Bond	3.00	-20.03
Cash	40.79	13.68
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	-0.31	0.09
Real Consol Bond	53.63	96.92
Nominal Bond	-4.38	6.61
Cash	51.05	-3.62

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

Figure 1. Epstein-Zin Utility

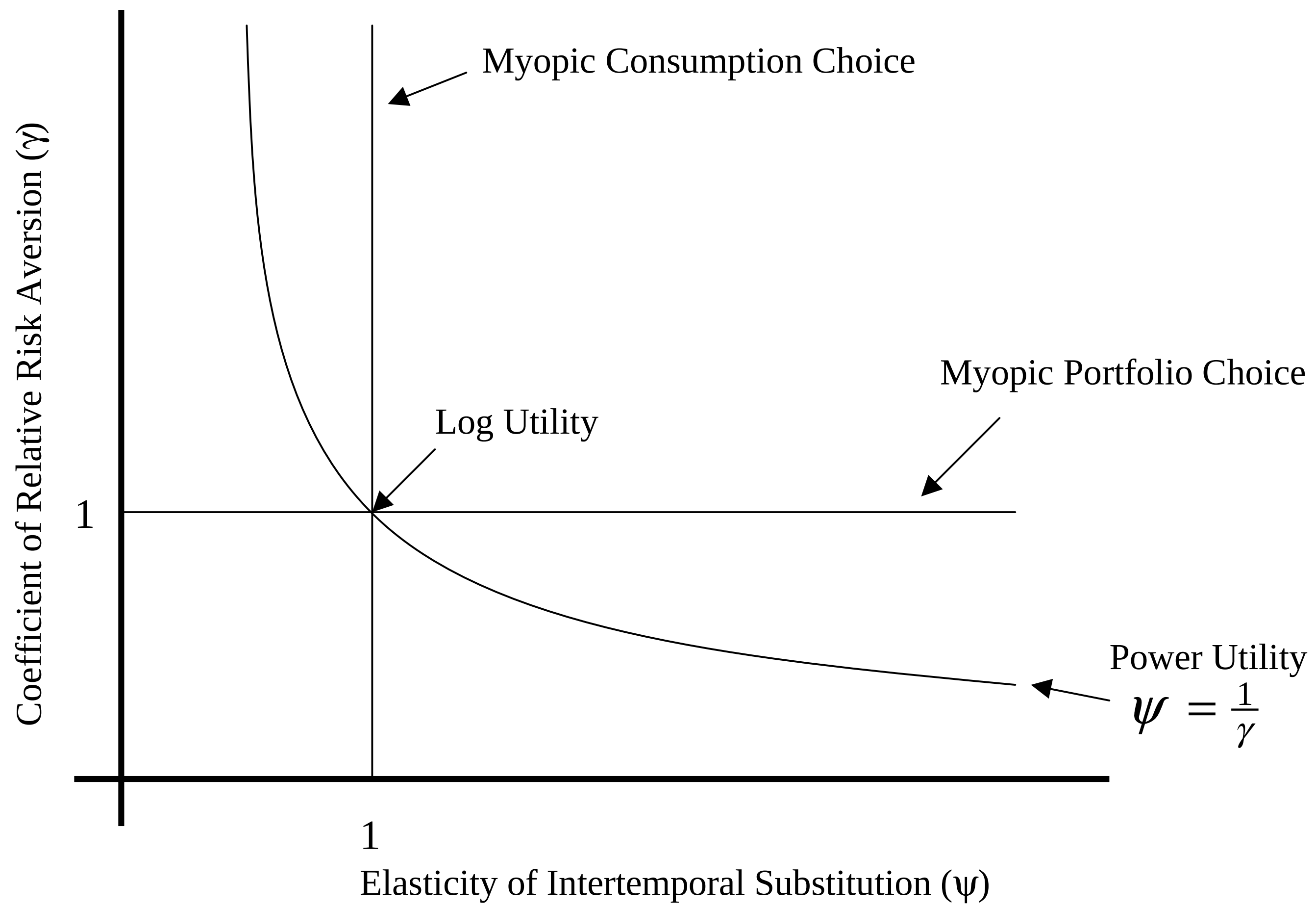


Figure 2. History of State Variables (1890 - 1995)

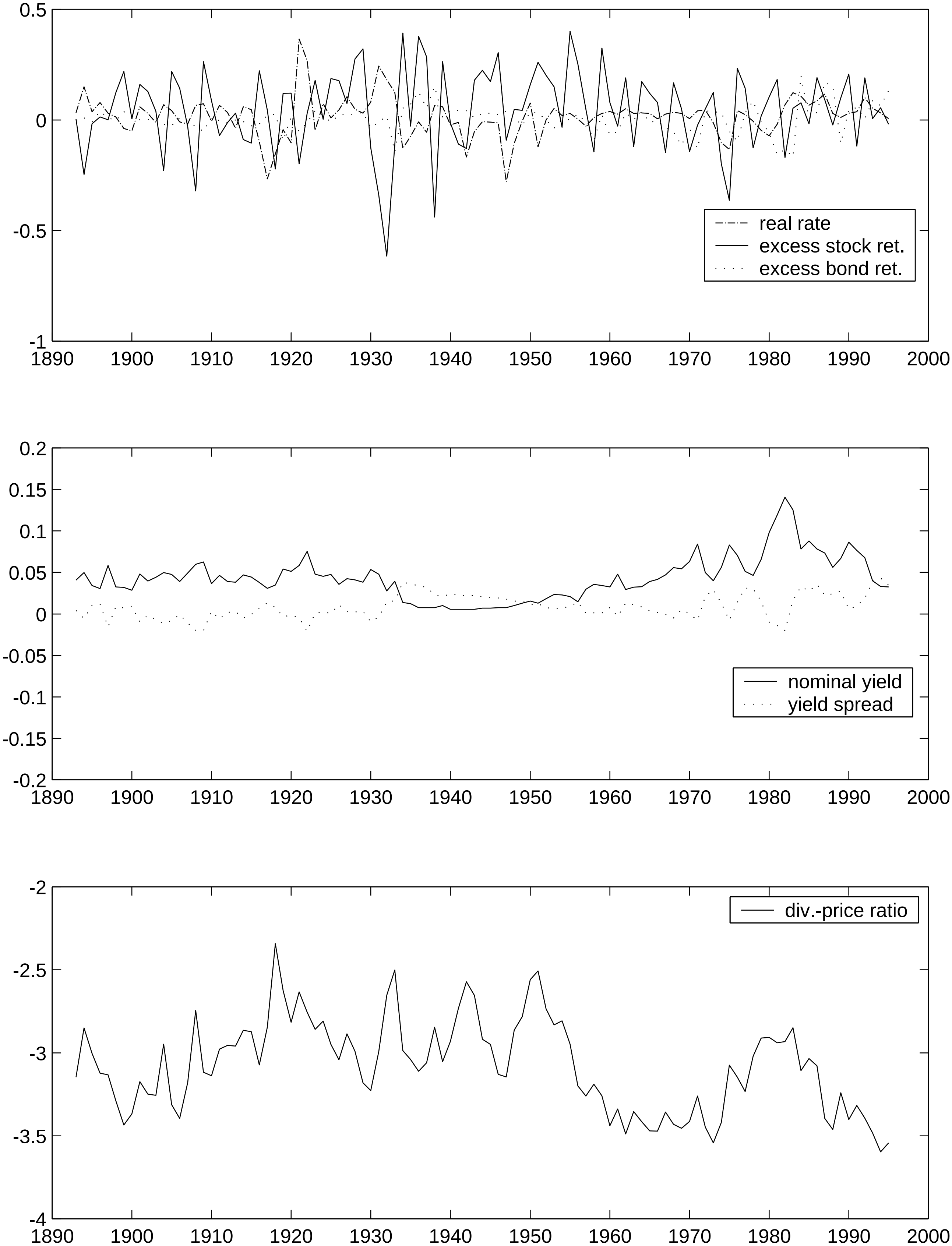
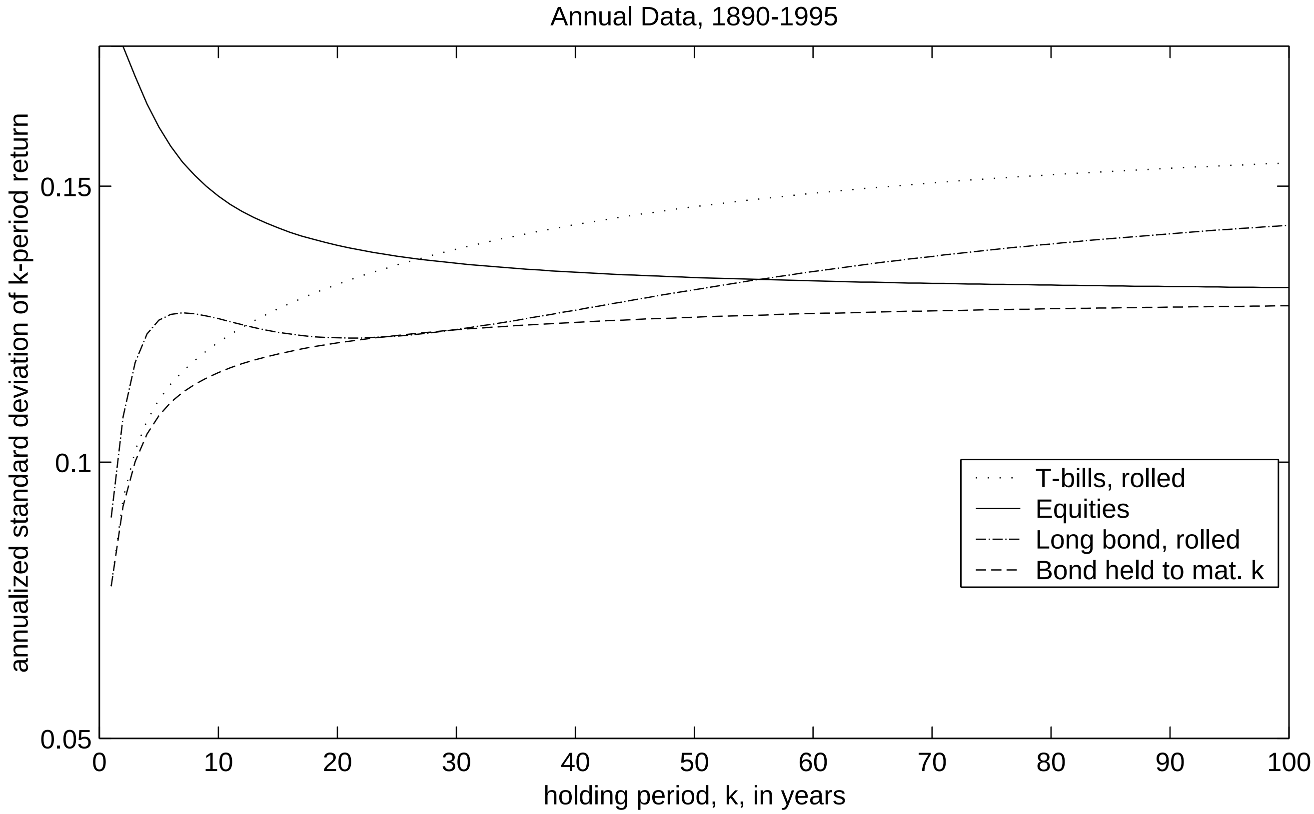


Figure 3. Variability of Multiperiod Asset Returns



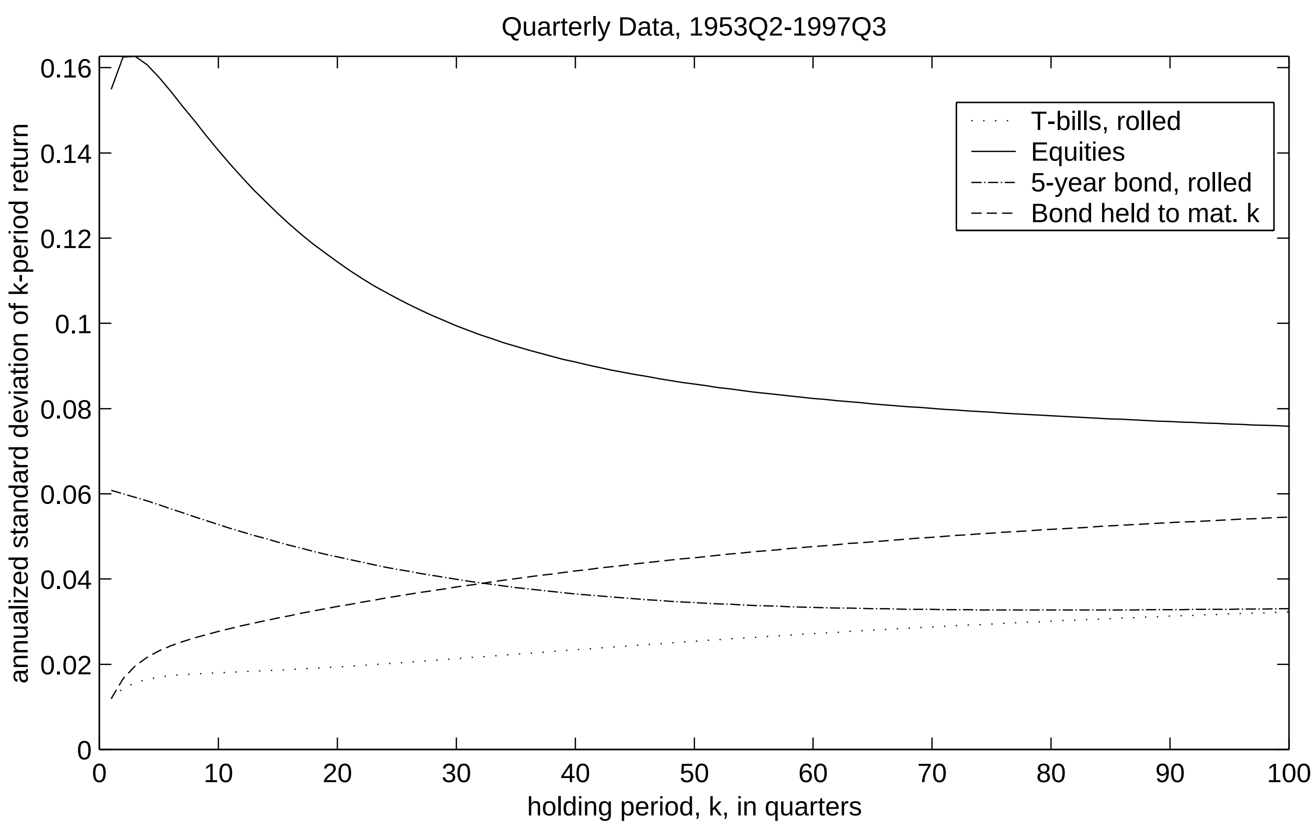


Figure 4. Optimal Asset Allocations to Stocks and Nominal Bond (Annual Sample)

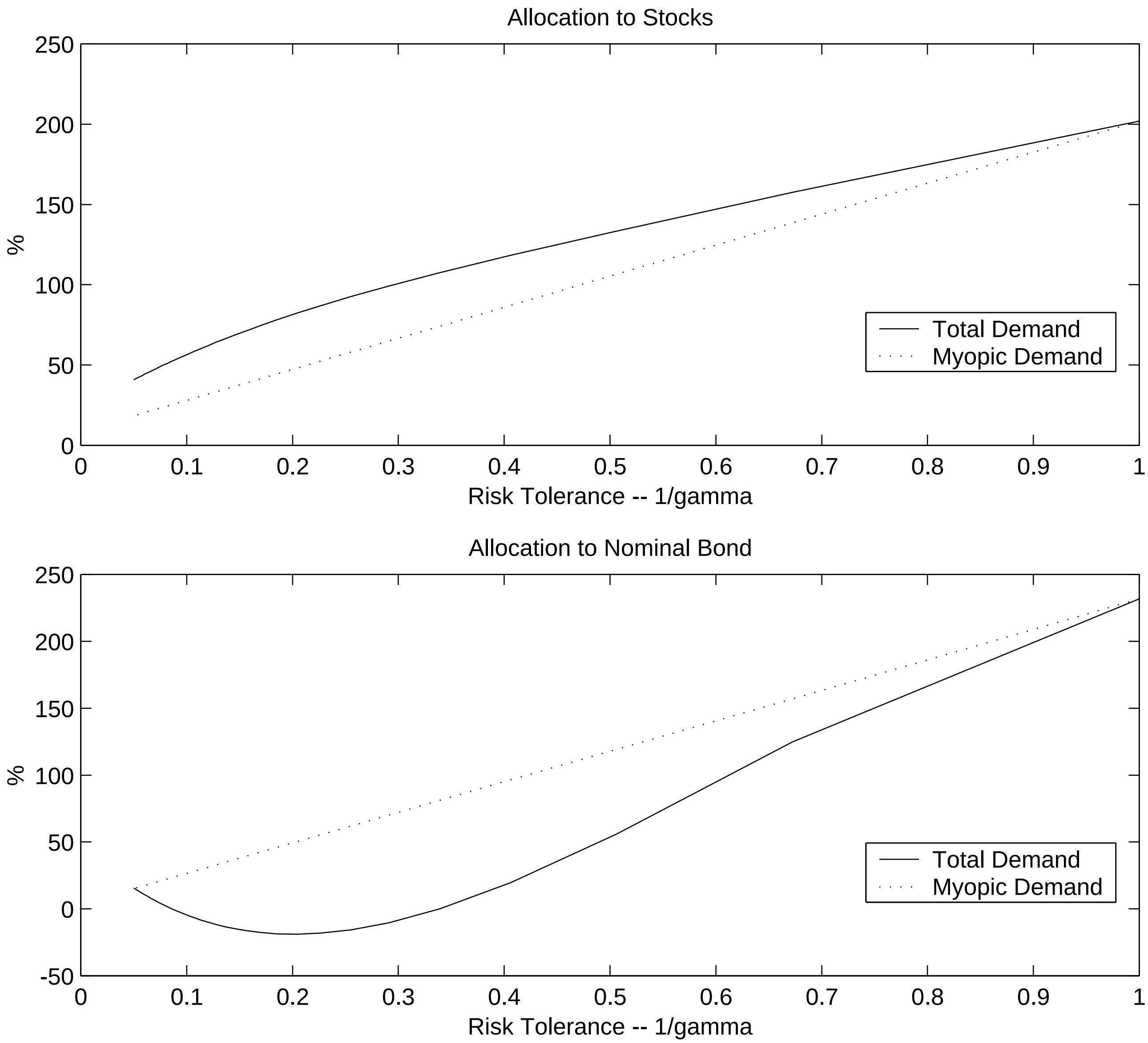


Figure 5. History of Asset Allocations (1890 - 1995)

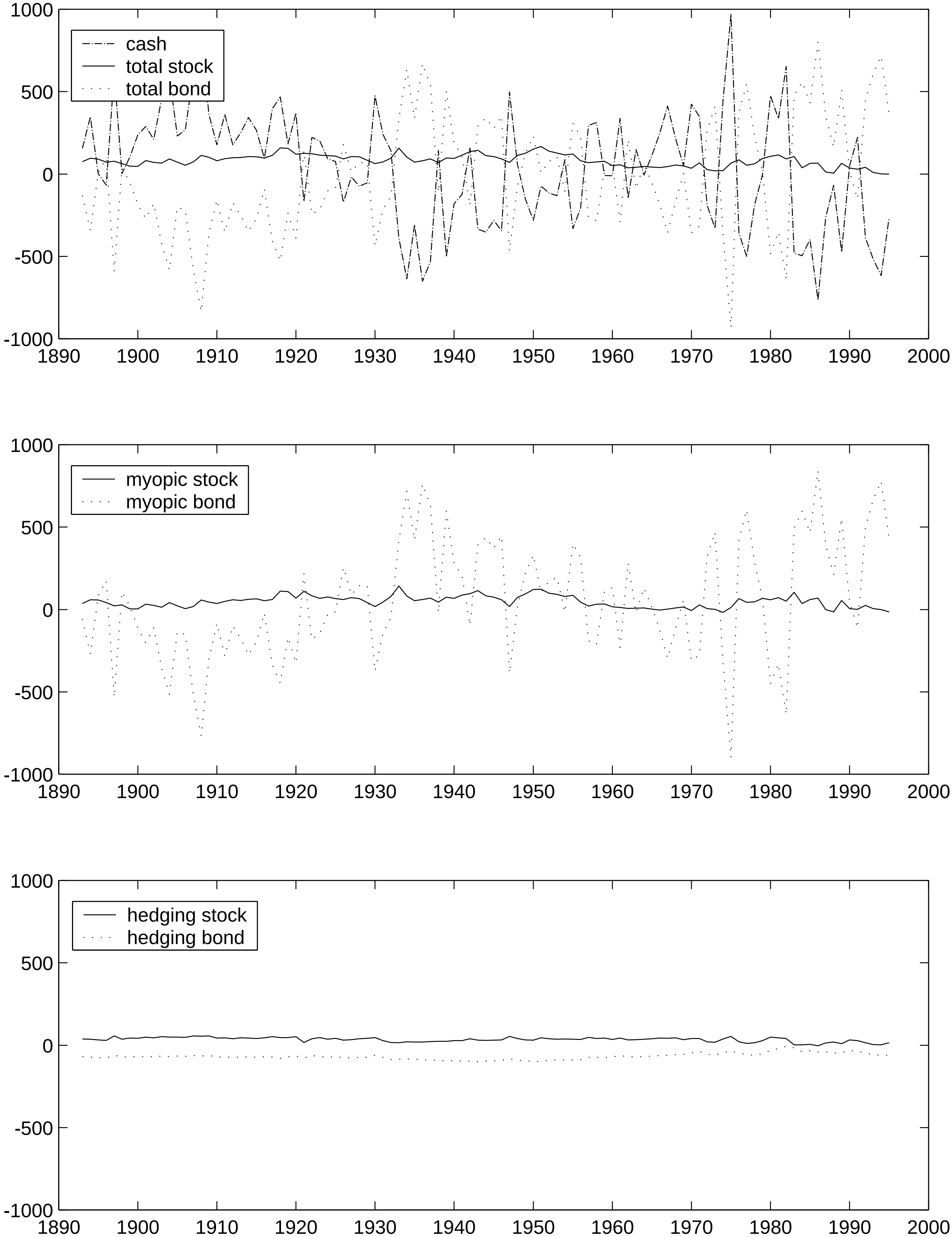


Figure 6. Optimal Allocation to Stocks and Real Consol Bond (Annual Sample)

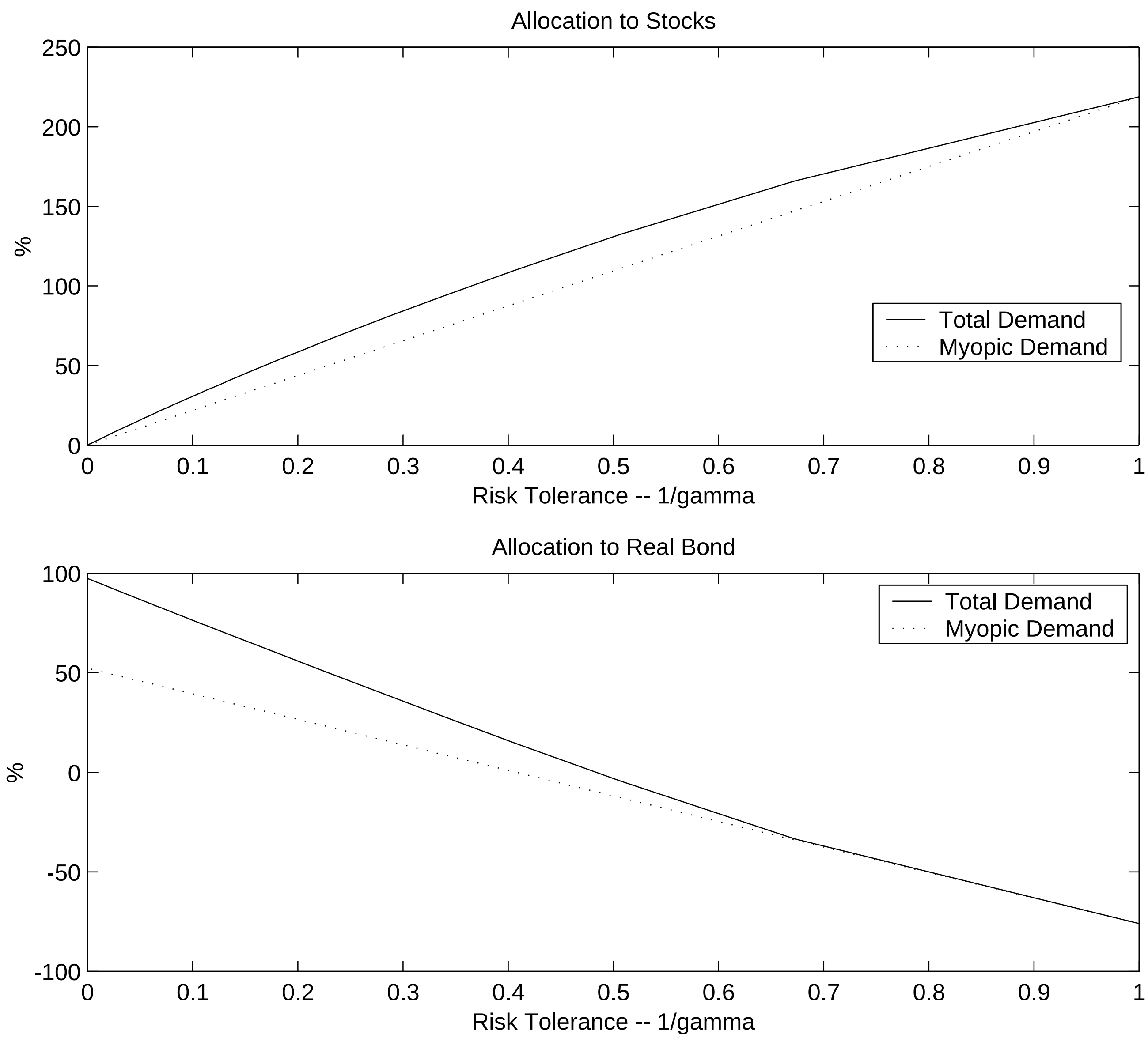
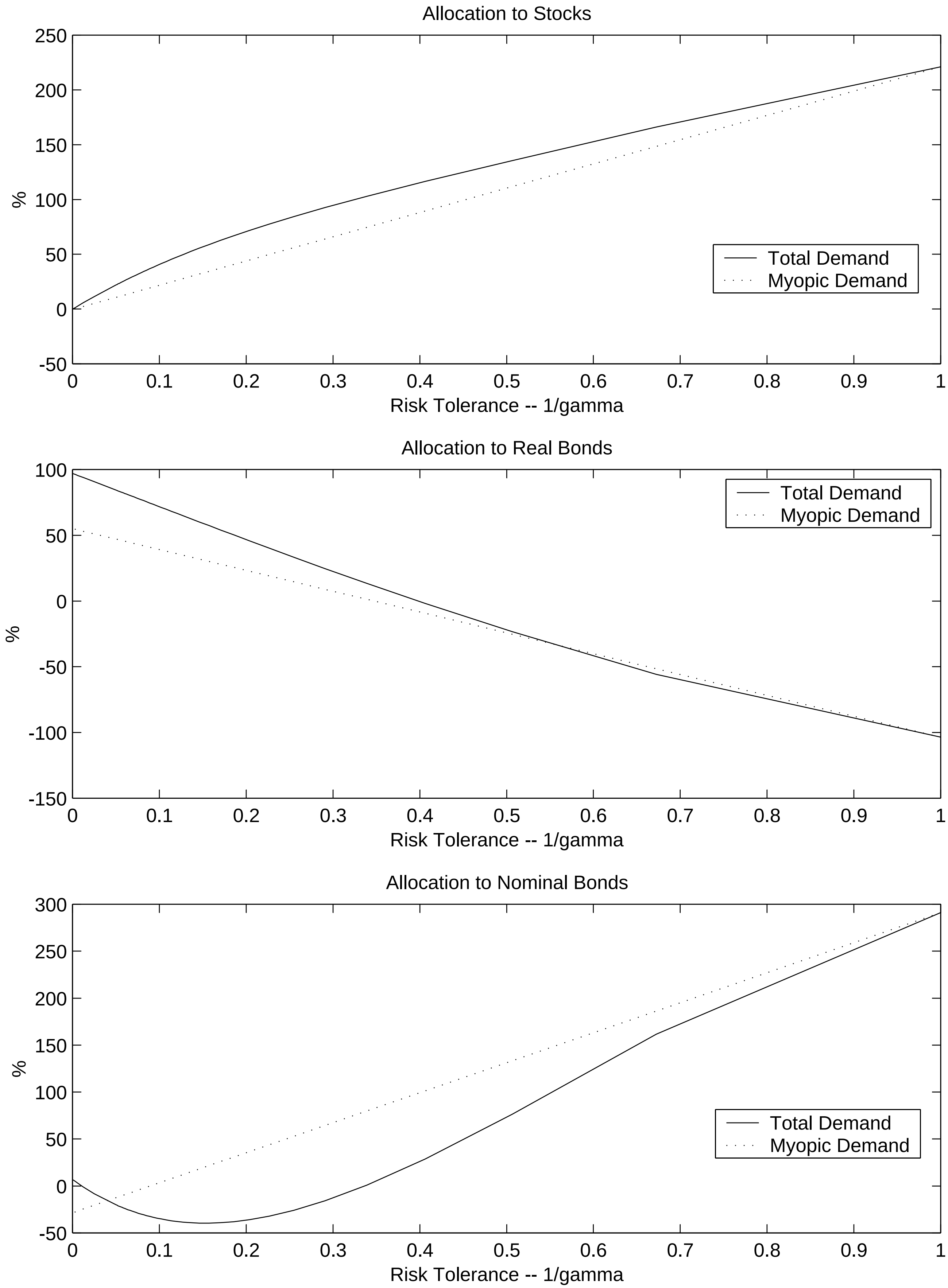


Figure 7. Optimal Allocation to Stocks, Real Consol Bond and Nominal Bond (Annual Sample)



Appendix to A Multivariate Model of Strategic Asset Allocation

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A Appendix A: Derivation of Main Equations in Text

We first summarize three results on matrix algebra that will be convenient in deriving the expressions given in the text.

Result 1.

$$\begin{aligned}
& \mathbf{z}_{t+1}\mathbf{z}_{t+1}' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) \\
= & \left(\Phi_0 + \Phi_1\mathbf{z}_t + \mathbf{v}_{t+1}\right)\left(\Phi_0 + \Phi_1 z_t + \mathbf{v}_{t+1}\right)' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) \\
= & \Phi_0\Phi_0' + \Phi_1\mathbf{z}_t\Phi_0' + \mathbf{v}_{t+1}\Phi_0' + \Phi_0\mathbf{z}_t'\Phi_1' + \Phi_1\mathbf{z}_t\mathbf{z}_t'\Phi_1' \\
& + \mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1' + \Phi_0\mathbf{v}_{t+1}' + \Phi_1\mathbf{z}_t\mathbf{v}_{t+1}' + \mathbf{v}_{t+1}\mathbf{v}_{t+1}' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) \\
= & \mathbf{v}_{t+1}\Phi_0' + \mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1' + \Phi_0\mathbf{v}_{t+1}' + \Phi_1\mathbf{z}_t\mathbf{v}_{t+1}' + \mathbf{v}_{t+1}\mathbf{v}_{t+1}' - \Sigma_v.
\end{aligned}$$

■

Result 2.

$$\begin{aligned}
& r_{i,t+1} - \mathrm{E}_t\left(r_{i,t+1}\right) \\
= & \mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1} - \mathrm{E}_t\left(\mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1}\right) \\
= & \mathbf{v}_{t+1}^{(i)} + \mathbf{v}_{t+1}^{(1)}
\end{aligned}$$

where $\mathbf{x}_{t+1}^{(i-1)}$ denotes the $(i-1)$th element of the excess return vector $\mathbf{x}_{t+1}$ and likewise with $\mathbf{v}_{t+1}$.■

Result 3. (Muirhead, 1982, pp.518)

$$\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right) = \left(\mathbf{I}_{m^2} + \sum_{i,j}^{m}\left(\mathbf{Q}_{ij} \otimes \mathbf{Q}_{ij}'\right)\right)\left(\Sigma_v \otimes \Sigma_v\right),$$

where $\mathbf{Q}_{ij}$ is a $m \times m$ zero matrix except for the (i,j)th element which is equal to 1.■

Derivation of Equation (10)

The log return on the portfolio $r_{p,t+1}$ is a discrete-time approximation to its continuous-time counterpart. We begin by specifying the return processes for the short-term instrument B_t and other risky assets $\mathbf{P}_t$ in continuous time:

$$\frac{dB_t}{B_t} = \mu_{b,t}dt + \boldsymbol{\sigma}_b d\mathbf{W}_t, \tag{27}$$

$$\frac{d\mathbf{P}_t}{\mathbf{P}_t} = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{W}_t, \tag{28}$$

where $\mu_{b,t}$ and $\boldsymbol{\mu}_t$ are the drifts, $\boldsymbol{\sigma}_b$ and $\boldsymbol{\sigma}$ are the diffusion, and $\mathbf{W}_t$ is a m-dimensional standard Brownian motion.[12] We allow the drifts to depend on other state variables, but for notational

[12]The dimensions of $\mu_b, \boldsymbol{\mu}, \boldsymbol{\sigma}_b, \boldsymbol{\sigma}$ are $1 \times 1, (n-1) \times 1, 1 \times m, (n-1) \times m$, respectively.

simplicity, we suppress this dependency and simply use the time subscript. Moreover, note that the same $\mathbf{W}_t$ appears in the two equations.

We can obtain the log return on each asset using Ito's Lemma:

$$d \log B_t = \left(\frac{dB_t}{B_t}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_b \boldsymbol{\sigma}_b'\right) dt, \tag{29}$$

$$d \log P_{i,t} = \left(\frac{dP_{i,t}}{P_{i,t}}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'\right) dt, \tag{30}$$

where $\boldsymbol{\sigma}_i$ is the ith row of the diffusion matrix $\boldsymbol{\sigma}$, and $i = 1, .., n-1$.

Let V_t be the value of the portfolio at time t. We will use $d \log V_t$ to approximate $r_{p,t+1}$. By Ito's Lemma,

$$d \log V_t = \left(\frac{dV_t}{V_t}\right) - \frac{1}{2}\left(\frac{dV_t}{V_t}\right)^2. \tag{31}$$

We will now derive these two terms in order:

$$\begin{aligned}
\frac{dV_t}{V_t} &= \boldsymbol{\alpha}_t' \left(\frac{d\mathbf{P}_t}{\mathbf{P}_t}\right) + \left(1 - \boldsymbol{\alpha}_t' \boldsymbol{\iota}\right) \frac{dB_t}{B_t} \\
&= \boldsymbol{\alpha}_t' \left(d \log \mathbf{P}_t + \frac{1}{2}\left[\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'\right] dt\right) + \left(1 - \boldsymbol{\alpha}_t' \boldsymbol{\iota}\right)\left(d \log B_t + \frac{1}{2}\left(\boldsymbol{\sigma}_b \boldsymbol{\sigma}_b'\right) dt\right) \\
&= \boldsymbol{\alpha}_t' \left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right) + d \log B_t \\
&\quad + \frac{1}{2} \boldsymbol{\alpha}_t' \left(\left[\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'\right] - \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota}\right) dt + \frac{1}{2} \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' dt,
\end{aligned}$$

where $\boldsymbol{\iota}$ is a $n \times 1$ vector of ones and the bracket $[\cdot]$ denotes a vector with $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'$ the ith entry. Next,

$$\begin{aligned}
\left(\frac{dV_t}{V_t}\right)^2 &= \boldsymbol{\alpha}_t' \left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)\left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)' \boldsymbol{\alpha}_t + \left(d \log B_t\right)^2 \\
&\quad + 2 \boldsymbol{\alpha}_t' \left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)\left(d \log B_t\right) + o\left(dt\right),
\end{aligned}$$

where the $o(dt)$ terms vanish because they involve either $(dt)^2$ or $(dt)(d\mathbf{W}_t)$.

Now, from equation (27)–(29) and ignoring dt terms,

$$d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota} = \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b\right) d\mathbf{W}_t.$$

Thus,

$$\begin{aligned}
\left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)\left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)' &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_b\right)', \\
\left(d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}\right)\left(d \log B_t\right) &= \left(\boldsymbol{\sigma} - \boldsymbol{\iota} \cdot \boldsymbol{\sigma}_1\right) \cdot \boldsymbol{\sigma}_b'.
\end{aligned}$$

Collecting these results and using our notation for excess returns: $\mathbf{x}_{t+1} = d \log \mathbf{P}_t - d \log B_t \cdot \boldsymbol{\iota}$,

$r_{1,t+1} = d\log(B_t)$ and $dt = 1$,

$$\begin{aligned} & r_{p,t+1} \\ = \;& d\log V_t \\ = \;& \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left([\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'] - \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' \cdot \boldsymbol{\iota} \right) \\ & -\frac{1}{2}\left[\boldsymbol{\alpha}_t' \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)' \boldsymbol{\alpha}_t + 2\boldsymbol{\alpha}_t' \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\boldsymbol{\sigma}_b' \right] \end{aligned}$$

Using the notation in the VAR system with the Cholesky decomposition for $\Sigma_v = \mathbf{G}\mathbf{G}'$, $\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_b$ is equal to the ith row of $\mathbf{G}, \mathbf{G}_i$. Hence,

$$\begin{aligned} \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)' &= \mathbf{G}_{2:n}\mathbf{G}_{2:n}' = \Sigma_{xx}, \\ \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b' &= \mathbf{G}_1 \mathbf{G}_1' = \sigma_1^2, \\ \boldsymbol{\sigma}_i \boldsymbol{\sigma}_i' &= \mathbf{G}_i \mathbf{G}_i' + \boldsymbol{\sigma}_b \mathbf{G}_i' + \mathbf{G}_i \boldsymbol{\sigma}_b' + \boldsymbol{\sigma}_b \boldsymbol{\sigma}_b', \\ [\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'] &= \sigma_x^2 + 2\Sigma_{1x} + \sigma_1^2 \boldsymbol{\iota}, \\ \left(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b\right)\boldsymbol{\sigma}_b' &= \mathbf{G}_{2:n}\mathbf{G}_1' = \Sigma_{1x}, \end{aligned}$$

where $\mathbf{G}_{2:n}$ denotes the submatrix formed by taking the 2nd to nth rows of $\mathbf{G}$.

With these terms, the return on the portfolio is

$$\begin{aligned} r_{p,t+1} &= \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left(\sigma_x^2 + 2\Sigma_{1x}\right) - \frac{1}{2}\boldsymbol{\alpha}_t' \Sigma_{xx} \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t' \Sigma_{1x}, \\ &= \boldsymbol{\alpha}_t' \mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t' \left(\sigma_x^2 - \Sigma_{xx}\boldsymbol{\alpha}_t\right). \end{aligned}$$

■

Solving for the Optimal Portfolio Rule.

Subtracting the log Euler equation (12) with $i = 1$ from (12), we obtain

$$\begin{aligned} & \mathrm{E}_t\left(r_{i,t+1} - r_{1,t+1}\right) + \frac{1}{2}\mathrm{Var}_t\left(r_{i,t+1} - r_{1,t+1}\right) \qquad (32) \\ = \;& \mathrm{Cov}_t\left(\frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1}, r_{i,t+1}\right) - \mathrm{Cov}_t\left(\frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1}, r_{1,t+1}\right) \\ & -\frac{1}{2}\left(\mathrm{Var}_t\left(r_{i,t+1}\right) - \mathrm{Var}_t\left(r_{1,t+1}\right) - \mathrm{Var}_t(r_{i,t+1} - r_{1,t+1})\right). \end{aligned}$$

Using the budget constraint (11) and the trivial identity $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$,

$$\begin{aligned} & \frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1} \\ = \;& \frac{\theta}{\psi}\left(c_{t+1} - w_{t+1}\right) + \gamma r_{p,t+1} + \text{ time } t \text{ terms and constants.} \end{aligned}$$

Thus, equation (32) can be written as

$$
\begin{aligned}
& \mathrm{E}_t\left(r_{i,t+1}-r_{1,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{i,t+1}-r_{1,t+1}\right) \\
= \;& \frac{\theta}{\psi}\left[\sigma_{i,c-w,t}-\sigma_{1,c-w,t}\right]+\gamma\left[\sigma_{i,p,t}-\sigma_{1,p,t}\right] \\
& -\frac{1}{2}\left(\mathrm{Var}_t\left(r_{i,t+1}\right)-\mathrm{Var}_t\left(r_{1,t+1}\right)-\mathrm{Var}_t(r_{i,t+1}-r_{1,t+1})\right).
\end{aligned}
$$

We will derive these terms now.

Using the equation for log return on the portfolio and ignoring time t terms and constants,

$$
\begin{aligned}
\sigma_{i,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1}+r_{1,t+1},r_{i,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\left(\Sigma_{xx}^{(i-1)}+\Sigma_{1x}\right)+\Sigma_{1x}^{(i-1)}+\sigma_1^2, \\
\sigma_{1,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1}+r_{1,t+1},r_{1,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\Sigma_{1x}+\sigma_1^2.
\end{aligned}
$$

To evaluate the conditional covariances $\sigma_{i,c-w,t}$ and $\sigma_{1,c-w,t}$, we use the conjectured policy rule for the consumption-wealth ratio.

$$
\begin{aligned}
& \sigma_{i,c-w,t} \\
= \;& \mathrm{Cov}_t\left(c_{t+1}-w_{t+1}-\mathrm{E}_t\left(c_{t+1}-w_{t+1}\right),r_{i,t+1}-\mathrm{E}_t(r_{i,t+1})\right) \\
= \;& \mathrm{Cov}_t\left(\mathbf{B}_1'v_{t+1}+\mathbf{B}_2'\left(\mathbf{v}_{t+1}\Phi_0'+\mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1'+\Phi_0\mathbf{v}_{t+1}'+\Phi_1\mathbf{z}_t\mathbf{v}_{t+1}'+\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\mathbf{B}_2,\,\mathbf{v}_{t+1}^{(i)}+\mathbf{v}_{t+1}^{(1)}\right) \\
= \;& \mathbf{B}_1'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)+2\mathbf{B}_2'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)\Phi_0'\mathbf{B}_2+2\mathbf{B}_2'\left(\Sigma_v^{(i)}+\Sigma_v^{(1)}\right)\mathbf{z}_t'\Phi_1\mathbf{B}_2,
\end{aligned}
$$

where the second equality follows from using Result 1 and 2, and $\Sigma_v^{(i)}$ denotes the ith column of Σ_v. Similarly,

$$
\sigma_{1,c-w,t}=\mathbf{B}_1'\Sigma_v^{(1)}+2\mathbf{B}_2'\Sigma_v^{(1)}\Phi_0'\mathbf{B}_2+2\mathbf{B}_2'\Sigma_v^{(1)}\mathbf{z}_t'\Phi_1\mathbf{B}_2.
$$

With these expressions,

$$
\begin{aligned}
& \boldsymbol{\sigma}_{c-w,t}-\sigma_{1,c-w,t}\boldsymbol{\iota} \\
= \;& \left(\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_1+2\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_2\mathbf{B}_2'\Phi_0\right)+\left(2\begin{bmatrix}\Sigma_v^{(2)\prime}\\ \vdots \\ \Sigma_v^{(n)\prime}\end{bmatrix}\mathbf{B}_2\mathbf{B}_2'\Phi_1\right)\mathbf{z}_t \\
= \;& \left[\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_1+2\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_2\mathbf{B}_2'\Phi_0\right]+\left[2\left(\Sigma_v\mathbf{H}_x'\right)'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right]\mathbf{z}_t \\
= \;& \Lambda_0+\Lambda_1\mathbf{z}_t,
\end{aligned}
$$

as claimed in equation (17).■

Solving for the Optimal Consumption Rule

We derive first equation (24). To derive this equation, note that log consumption growth verifies the following trivial identity: $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$. Substituting the log-linearized budget constraint (11) into this equation and taking expectations we obtain

$$\begin{aligned} \mathrm{E}_t(\Delta c_{t+1}) &= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t\left(\Delta w_{t+1}\right) \\ &= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t(r_{p,t+1}) + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k. \end{aligned} \tag{33}$$

Combining the two equations (22) and (33), we obtain a difference equation in $c_t - w_t$, given in (24).

Next we show that both the expected log return on the wealth portfolio $\mathrm{E}_t r_{p,t+1}$ and the variance term $\chi_{p,t}$ in equation (22) for expected log consumption growth are quadratic functions of the vector of state variables.

Taking conditional expectations of equation (10) and substituting the portfolio policy rule $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t$,

$$\begin{aligned} & \mathrm{E}_t\left(r_{p,t+1}\right) \\ = \; & \boldsymbol{\alpha}_t'\mathrm{E}_t\left(\mathbf{x}_{t+1}\right) + \mathrm{E}_t\left(r_{1,t+1}\right) + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\sigma_x^2 - \Sigma_{xx}\boldsymbol{\alpha}_t\right) \\ = \; & \left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\mathbf{H}_x\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) + \mathbf{H}_1\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) \\ & + \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\sigma_x^2 - \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\Sigma_{xx}\left(\mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t\right) \\ = \; & \Gamma_0 + \Gamma_1\mathbf{z}_t + \Gamma_2\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right), \end{aligned}$$

where

$$\begin{aligned} \Gamma_0 &\equiv \mathbf{A}_0'\mathbf{H}_x\Phi_0 + \mathbf{H}_1\Phi_0 + \frac{1}{2}\mathbf{A}_0'\sigma_x^2 - \frac{1}{2}\mathbf{A}_0'\Sigma_{xx}\mathbf{A}_0, \\ \Gamma_1 &\equiv \Phi_0'\mathbf{H}_x'\mathbf{A}_1 + \mathbf{A}_0'\mathbf{H}_x\Phi_1 + \mathbf{H}_1\Phi_1 + \frac{1}{2}\sigma_x^2\mathbf{A}_1 - \mathbf{A}_0'\Sigma_{xx}\mathbf{A}_1, \\ \Gamma_2 &\equiv \mathrm{vec}\left(\mathbf{A}_1'\mathbf{H}_x\Phi_1\right)' - \frac{1}{2}\mathrm{vec}\left(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1\right)', \end{aligned}$$

and $\mathbf{H}_1$ and $\mathbf{H}_x$ are selection matrices that select the short-term real interest rate and the vector of excess returns from the full state vector.

We now evaluate the variance term

$$\chi_{p,t} = \frac{1}{2}\left(\frac{\theta}{\psi}\right)\mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right).$$

Using the trivial identity for Δc_{t+1} and the budget constraint (11), substituting the conjecture for the consumption rule

$$\begin{aligned} c_t - w_t &= b_0 + \mathbf{B}_1'\mathbf{z}_t + \mathbf{B}_2'\mathbf{z}_t\mathbf{z}_t'\mathbf{B}_2 \\ &= b_0 + \mathbf{B}_1'\mathbf{z}_t + vec\left(\mathbf{B}_2\mathbf{B}_2'\right)' vec\left(\mathbf{z}_t\mathbf{z}_t'\right) \end{aligned}$$

and $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t$, and ignoring time t terms and constants, we can write the argument of the variance as:

$$\begin{aligned}
& \Delta c_{t+1} - \psi r_{p,t+1} \\
= & \left[\mathbf{B}_1' + 2\mathbf{z}_t'\Phi_1\mathbf{B}_2\mathbf{B}_2' + (1-\psi)\,\boldsymbol{\alpha}_t'\mathbf{H}_x + (1-\psi)\,\mathbf{H}_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2'\right]\mathbf{v}_{t+1} \\
& + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right) \\
= & \left[\Pi_1 + \mathbf{z}_t'\Pi_2\right]\mathbf{v}_{t+1} + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right),
\end{aligned}$$

where

$$\begin{aligned}
\Pi_1 &\equiv \mathbf{B}_1' + (1-\psi)\,\mathbf{A}_0'\mathbf{H}_x + (1-\psi)\,\mathbf{H}_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2', \\
\Pi_2 &\equiv 2\left(\Phi_1\mathbf{B}_2\mathbf{B}_2'\right) + (1-\psi)\,\mathbf{A}_1'\mathbf{H}_x.
\end{aligned}$$

Since $\mathbf{v}_{t+1}$ is conditionally normally distributed, all third moments are zero. Thus,

$$\begin{aligned}
& \text{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right) \\
= & \Pi_1\Sigma_v\Pi_1' + \left[2\Pi_1\Sigma_v\Pi_2'\right]\mathbf{z}_t + \left[\text{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\right]\text{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right) \\
& + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{Var}_t\left(\text{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right),
\end{aligned}$$

and $\text{Var}_t\left(\text{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)$ is given by the expression in Result 3 above. Putting these pieces together, we have

$$\chi_{p,t} = V_0 + \mathbf{V}_1\mathbf{z}_t + \mathbf{V}_2\text{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right),$$

where

$$\begin{aligned}
V_0 &\equiv \frac{\theta}{2\psi}\left[\Pi_1\Sigma_v\Pi_1' + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{Var}_t\left(\text{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)\right], \\
\mathbf{V}_1 &\equiv \frac{\theta}{2\psi}\left[2\Pi_1\Sigma_v\Pi_2'\right], \\
\mathbf{V}_2 &\equiv \frac{\theta}{2\psi}\left[\text{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\right].
\end{aligned}$$

Derivation of Equation (34)

Simple substitution of the expressions for $\text{E}_t r_{p,t+1}$ and $\chi_{p,t}$, and the expression for the conditional expectation of $(c_{t+1} - w_{t+1})$ into the RHS of (24) yields

$$c_t - w_t = \Xi_0 + \Xi_1\mathbf{z}_t + \Xi_2\text{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right), \tag{34}$$

where

$$\begin{aligned}
\Xi_0 &\equiv \rho[-\psi\log\delta + k - V_0 + (1-\psi)\,\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 \\
&\quad + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\text{vec}(\Phi_0\Phi_0') + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)'\text{vec}(\Sigma_v)], \\
\Xi_1 &\equiv \rho\left[-\mathbf{V}_1 + (1-\psi)\,\Gamma_1 + \mathbf{B}_1'\Phi_1 + 2\Phi_0'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right], \\
\Xi_2 &\equiv \rho\left[-\mathbf{V}_2 + (1-\psi)\,\Gamma_2 + \text{vec}\left(\Phi_1'\mathbf{B}_2\mathbf{B}_2'\Phi_1\right)'\right].
\end{aligned}$$

Equation (34) confirms our initial conjecture on the form of the consumption-wealth ratio. Notice that Ξ_0, Ξ_1, Ξ_2 depend on $b_0, \mathbf{B}_1$ and $\text{vec}(\mathbf{B}_2\mathbf{B}_2')$. Therefore, for the solution to be consistent, $\{b_0, \mathbf{B}_1, \text{ vec}(\mathbf{B}_2\mathbf{B}_2')\}$ must solve the following set of equations:

$$
\begin{aligned}
b_0 &= \Xi_0, \\
\mathbf{B}_1 &= \Xi_1', \\
\text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right) &= \Xi_2'.
\end{aligned} \tag{35}
$$

The resulting set of values for $b_0, \mathbf{B}_1$ and $\text{vec}(\mathbf{B}_2\mathbf{B}_2')$ determines the optimal consumption rule.■

Verification that $\mathbf{A}_0$ and $\mathbf{A}_1$ do not depend on ψ

¿From (21), $\mathbf{A}_0$ and $\mathbf{A}_1$ can be expressed as:

$$
\begin{aligned}
\mathbf{A}_0 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\sigma_x^2 + (1-\gamma)\Sigma_{1x}\right) + \frac{1}{\gamma}\frac{1-\gamma}{1-\psi}\Sigma_{xx}^{-1}\Lambda_0 \\
&= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\sigma_x^2 + (1-\gamma)\Sigma_{1x}\right) + \frac{1-\gamma}{\gamma}\Sigma_{xx}^{-1}\left[\left(\Sigma_v\mathbf{H}_x'\right)'\frac{\mathbf{B}_1}{1-\psi} + 2\left(\Sigma_v\mathbf{H}_x'\right)'\frac{\mathbf{B}_2\mathbf{B}_2'}{1-\psi}\Phi_0\right] \\
\mathbf{A}_1 &= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_1\right) + \left(\frac{1}{\gamma}\Sigma_{xx}^{-1}\right)\frac{1-\gamma}{1-\psi}\Lambda_1 \\
&= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_1\right) + \left(\frac{1-\gamma}{\gamma}\Sigma_{xx}^{-1}\right)\left[2\left(\Sigma_v\mathbf{H}_x'\right)'\frac{\mathbf{B}_2\mathbf{B}_2'}{1-\psi}\Phi_1\right].
\end{aligned}
$$

Thus, showing that $\mathbf{A}_0$ and $\mathbf{A}_1$ are independent of ψ is equivalent to showing $\mathcal{B}_1 \equiv \mathbf{B}_1/(1-\psi)$ and $\mathcal{B}_2 \equiv \mathbf{B}_2\mathbf{B}_2'/(1-\psi)$ are independent of ψ.

First consider $\mathcal{B}_2$. From (35), we have

$$
(1-\psi)\text{vec}\left(\mathcal{B}_2\right) = \rho\left[-\mathbf{V}_2' + (1-\psi)\Gamma_2' + (1-\psi)\text{vec}\left(\Phi_1'\mathcal{B}_2\Phi_1\right)\right]. \tag{36}
$$

Using the definition of $\mathbf{V}_2$,

$$
\begin{aligned}
-\mathbf{V}_2' &= \frac{1-\gamma}{2(1-\psi)}\text{vec}\left[(1-\psi)^2\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)\Sigma_v\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)'\right] \\
&= \frac{1-\gamma}{2}(1-\psi)\text{vec}\left[\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)\Sigma_v\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)'\right] \\
&\equiv (1-\psi)\overline{\mathbf{V}}_2'
\end{aligned}
$$

Note that $\mathbf{A}_1$, and hence $\overline{\mathbf{V}}_2'$, do not depend on ψ, given $\mathcal{B}_2$. Moreover,

$$
(1-\psi)\Gamma_2' = (1-\psi)\left[\text{vec}\left(\mathbf{A}_1'\mathbf{H}_x\Phi_1\right) - \frac{1}{2}\text{vec}\left(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1\right)\right],
$$

where the expression in brackets depends only on $\mathcal{B}_2$. Thus, (36) reduces to

$$
\text{vec}\left(\mathcal{B}_2\right) = \rho\left[\overline{\mathbf{V}}_2' + \Gamma_2' + \text{vec}\left(\Phi_1'\mathcal{B}_2\Phi_1\right)\right].
$$

This is a quadratic equation in $\mathcal{B}_2$, with coefficients independent of ψ. Consequently, the solution for $\mathcal{B}_2$ will also be independent of ψ.

Now, using the same logic, we can show that $\mathcal{B}_1$ is independent of ψ. From (35),

$$(1-\psi)\mathcal{B}_1 = \rho\left[-\mathbf{V}_1' + (1-\psi)\,\Gamma_1' + (1-\psi)\Phi_1'\mathcal{B}_1 + (1-\psi)2\left(\Phi_0'\mathcal{B}_2\Phi_1\right)'\right]. \tag{37}$$

Note that given $\mathcal{B}_1$, A_0 (given above) is independent of ψ. Now,

$$\begin{aligned}
-\mathbf{V}_1' &= \frac{1-\gamma}{(1-\psi)}\left[\left((1-\psi)\overline{\Pi}_1\right)\Sigma_v\left((1-\psi)\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)\right)'\right]' \\
&= (1-\gamma)(1-\psi)\left[\overline{\Pi}_1\Sigma_v\left(2\Phi_1\mathcal{B}_2 + \mathbf{A}_1'\mathbf{H}_x\right)'\right]' \\
&\equiv (1-\psi)\overline{V}_1
\end{aligned}$$

where

$$\overline{\Pi}_1 \equiv \mathcal{B}_1' + \mathbf{A}_0'\mathbf{H}_x + \mathbf{H}_1 + 2\Phi_0'\mathcal{B}_2.$$

Also, Γ_1 is only a function of $\mathcal{B}_1$ and $\mathcal{B}_2$ via its dependence on $\mathbf{A}_0$ and $\mathbf{A}_1$, not of ψ. Therefore, (37) becomes

$$\mathcal{B}_1 = \rho\left[\overline{\mathbf{V}}_1 + \Gamma_1' + \Phi_1'\mathcal{B}_1 + 2\left(\Phi_0'\mathcal{B}_2\Phi_1\right)'\right],$$

which again implies that the solution for $\mathcal{B}_1$ does not depend on ψ. This completes our proof.■

B Appendix B: Numerical Procedure

Equations (21) and (35) show that the coefficients $\{\mathbf{A}_0, \mathbf{A}_1\}, \{b_0, \mathbf{B}_1, \text{ vec}(\mathbf{B}_2\mathbf{B}_2')\}$ in the optimal policy rules are functions of the underlying parameters. When there is one state variable as in Campbell and Viceira (1999), solving explicitly for these coefficients is manageable. However, with multiple state variables, such an exercise is practically impossible. Therefore, we employ a numerical procedure to find these coefficients instead.

We describe the numerical procedure in steps:

1. For a given set of values of $\{\gamma, \psi, \rho\}$, we start with some initial values for δ and $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$—denote these by $\delta^{(1)}$ and $\{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$. Through equation (21), this implies a set of values for $\{\mathbf{A}_0, \mathbf{A}_1\}$—denote by $\{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}$.
2. With $\rho, \delta^{(1)}, \{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}, \{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$, the coefficients $\{\Xi_0, \Xi_1, \Xi_2\}$ in the $c - w$ difference equation (34) can be calculated. By equating these coefficients with the $\{b_0, \mathbf{B}_1, \text{ vec}(\mathbf{B}_2\mathbf{B}_2')\}$ in the conjectured policy function, we have a new set of values for $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$—call them $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$. Since the initial values are arbitrary, $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$ will be different from $\{b_0^{(1)}, \mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(1)}\}$ in general. Thus, we recompute $\{\mathbf{A}_0, \mathbf{A}_1\}$ using $\rho, \delta^{(1)}$ and $\{b_0^{(2)}, \mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2\mathbf{B}_2')^{(2)}\}$ to get $\{\mathbf{A}_0^{(2)}, \mathbf{A}_1^{(2)}\}$. A new set of $\{\Xi_0, \Xi_1, \Xi_2\}$ can then be obtained. We continue until values of $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$ and hence $\{\mathbf{A}_0, \mathbf{A}_1\}$ converge, for a given ρ and $\delta^{(1)}$. Call these new values $\{\mathbf{A}_0^{\delta^{(1)}}, \mathbf{A}_1^{\delta^{(1)}}\}$, $\{b_0^{\delta^{(1)}}, \mathbf{B}_1^{\delta^{(1)}}, \text{ vec}(\mathbf{B}_2^{\delta^{(1)}}\mathbf{B}_2^{\delta^{(1)}\prime})\}$, where the superscript emphasizes the fact that these values are based on the initial value $\delta^{(1)}$.
3. The convergence criterion for the $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2\mathbf{B}_2')\}$ is rather stringent. We first calculate the maximum of the squared deviations of all elements from 2 consecutive iterations. We then require for parameter convergence that the sum of 20 such consecutive maxima be less than 0.00001.
4. We now describe how to calculate the implied δ from the converged values of $\mathbf{A}_0$, $\mathbf{A}_1$, b_0, $\mathbf{B}_1$,vec$(\mathbf{B}_2\mathbf{B}_2')$. Using the fact that $\rho = 1 - \exp(\text{E}[c_t - w_t])$ and that

$$\begin{aligned} \text{E}(c_t - w_t) &= b_0 + \mathbf{B}_1' E(\mathbf{z}_t) + \text{ vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\text{E}(\mathbf{z}_t\mathbf{z}_t')) \\ &= b_0 + \mathbf{B}_1'\boldsymbol{\mu}_z + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z'), \end{aligned}$$

   we have

$$b_0 = \log(1-\rho) - \mathbf{B}_1'\boldsymbol{\mu}_z - \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z').$$

   On the other hand, from the definition of Ξ_0,

$$\begin{aligned} \Xi_0 &= \rho[-\psi \log\delta + k - V_0 + (1-\psi)\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 \\ &\quad + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Phi_0\Phi_0') + \text{vec}(\mathbf{B}_2\mathbf{B}_2')' \text{vec}(\Sigma_v)], \end{aligned}$$

   which depends on δ. Equating these two equations and solve for the new δ, we have

$$\delta^{(2)} = \exp\left[\frac{-1}{\rho\psi}(\mathcal{F}_2 - \rho\mathcal{F}_1)\right],$$

where

$$\begin{aligned}\mathcal{F}_1 &\equiv k - V_0 + (1-\psi)\,\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{vec}(\Phi_0\Phi_0') \\ &\quad + \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{vec}(\Sigma_v), \\ \mathcal{F}_2 &\equiv \log\left(1-\rho\right) - \mathbf{B}_1'\boldsymbol{\mu}_z - \text{vec}\left(\mathbf{B}_2\mathbf{B}_2'\right)' \text{vec}\left(\Sigma_{zz} + \boldsymbol{\mu}_z\boldsymbol{\mu}_z'\right),\end{aligned}$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are evaluated at $\{\mathbf{A}_0^{\delta^{(1)}}, \mathbf{A}_1^{\delta^{(1)}}\}$, $\{b_0^{\delta^{(1)}}, \mathbf{B}_1^{\delta^{(1)}}, \text{vec}(\mathbf{B}_2^{\delta^{(1)}}\mathbf{B}_2^{\delta^{(1)}}{}')\}$ and the VAR estimates.

5. Naturally, one might use $\delta^{(2)}$ as input again and reiterate through Step (1)–(4) until δ converges. Graphically, we attempt to find the intersection point between the implied δ curve with the 45°-line. Unfortunately, a plot of the implied δ calculated as described above against a range of input δ indicates that the implied δ curve cuts the 45°-line from above. This immediately implies that a standard iterative procedure over δ will not work in general. Therefore, we resort to a grid-search method over a range of δ.

6. The grid-search method proceeds as follows: we start with a wide range of δ and form a coarse mesh over this range. In our implementation, we choose to evaluate Step (1)–(4) over 21 points over this range. From these calculations, 21 new implied δ are obtained. We locate the inputting $\delta^{(1)}$ whose implied $\delta^{(2)}$ gives the minimum positive deviation from $\delta^{(1)}$ among these 21 pairs. Call this δ^{+}. Similarly, we locate the inputting $\delta^{(1)}$ whose implied $\delta^{(2)}$ gives the minimum negative deviation from $\delta^{(1)}$. Call this δ^{-}. Because of the downward-sloping implied δ curve, existence of this pair $\left(\delta^{+}, \delta^{-}\right)$ is guaranteed as long as the initial range of δ is wide enough. Note that $\delta^{+} < \delta^{-}$.

7. We then form a 21-point mesh again over $\left(\delta^{+}, \delta^{-}\right)$ and repeat Step (6) until both the minimum positive deviation and negative deviation are less than 0.00001, or $\delta^{-} - \delta^{+} < 0.00001$. When the first of the convergence criteria for δ is met, the converged δ is computed as the average of the $\delta^{(2)}$ with the minimum positive deviation and the $\delta^{(2)}$ with the minimum negative deviation. When the second criterion is met, a simple average of $\left(\delta^{+}, \delta^{-}\right)$ is used.

C Appendix C: Construction of Hypothetical Real Bonds

Recall that the first element of our VAR system is the ex post real bill return. Therefore, the ex ante log real bill return at time $t+1$ is the first element of $\mathrm{E}_t\left(\mathbf{z}_{t+1}\right) = \Phi_0 + \Phi_1 \mathbf{z}_t$. In other words, the log real yield at time t is given by

$$\widehat{y}_{1t} = \mathbf{H}_1 \cdot \mathrm{E}_t\left(\mathbf{z}_{t+1}\right) \equiv \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+1},$$

where $\mathbf{H}_1 \equiv (1,0,...,0)$ and $\widehat{\mathbf{z}}_{t,t+1} \equiv \mathrm{E}_t\left(\mathbf{z}_{t+1}\right)$.

The next step is to assume that the log expectations hypothesis holds for the real term structure; that is,

$$y_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{E}_t\left(y_{1,t+i}\right),$$

where $y_{n,t}$ is the log yield on a real bond with maturity n. Note that we have implicitly assume that inflation risk premium is zero. An estimate of $y_{n,t}$ can be easily constructed as follows:

$$\widehat{y}_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} \widehat{y}_{1,t+i} = \frac{1}{n}\sum_{i=0}^{n-1} \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+i+1}.$$

To compute $\widehat{\mathbf{z}}_{t,t+i+1}$, we can iterate the VAR(1) system forward to get

$$\widehat{\mathbf{z}}_{t,t+k} = \left(\sum_{j=0}^{k-1} \Phi_1^j\right)\Phi_0 + \Phi_1^k \mathbf{z}_t.$$

Using this result, log yield can be expressed as a function of current state variables:

$$\begin{aligned}
\widehat{y}_{n,t} &= \tfrac{1}{n}\mathbf{H}_1 \sum_{i=1}^{n} \widehat{\mathbf{z}}_{t,t+i} \\
&= \frac{1}{n}\mathbf{H}_1 \sum_{i=1}^{n}\left[\left(\sum_{j=0}^{i-1}\Phi_1^j\right)\Phi_0 + \Phi_1^i \mathbf{z}_t\right] \\
&\equiv \frac{1}{n}\mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n \mathbf{z}_t\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Q}_n &\equiv \sum_{i=1}^{n}\Phi_1^i = \Phi_1\left(\mathbf{I}_m - \Phi_1\right)^{-1}\left(\mathbf{I}_m - \Phi_1^n\right), \\
\mathbf{Q}_c &\equiv \left(\mathbf{I}_m - \Phi_1\right)^{-1}\left(\mathbf{I}_m - \mathbf{Q}_n\right)\Phi_0,
\end{aligned}$$

and $\mathbf{I}_m$ is the identity matrix, $m = \dim(\mathbf{z}_t)$.

Finally, the 1-period return on a hypothetical real n-period bond is calculated as

$$\begin{aligned} r_{n,t+1} &= n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n-1,t+1} \\ &\approx n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n,t+1} \\ &= \mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_t\right) - \frac{n-1}{n}\mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_{t+1}\right) \end{aligned}$$

The next step is to construct a real consol bond from these zero-coupon bonds. Campbell, Lo and MacKinlay (1997) show how to use a loglinearization framework to construct real consol bond returns. Specifically,their equations (10.1.16) and (10.1.17) show that the log yield on a real consol $y_{c,\infty,t}$ is given by

$$y_{c,\infty,t} = (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i r_{c,\infty,t+1+i},$$

where $r_{c,\infty,t+i}$ is the one-period log return on a consol bond at time $t+i$ and $\rho_c = 1-\exp\left(\mathrm{E}[-p_{c,t}]\right)$, where $p_{c,t}$ is the log "cum-dividend" price of the consol bond including its current coupon payout.

Taking conditional expectations at time t and imposing the expectations hypothesis,

$$\begin{aligned} y_{c,\infty,t} &= (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i\mathbf{H}_1\widehat{\mathbf{z}}_{t,t+i+1} \\ &= \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j\right)\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1}\right)\mathbf{z}_t. \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} \sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j &= \frac{1}{1-\rho_c}\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}, \\ \sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1} &= \left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1. \end{aligned}$$

Thus, the log yield can be expressed as function of the VAR parameters, current state variables and the loglinearization constant ρ_c:

$$y_{c,\infty,t} = \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1\mathbf{z}_t.$$

We can now write an expression for the return on the consol bond:

$$\begin{aligned} r_{c,\infty,t+1} &\approx D_{c,\infty}y_{c,\infty,t} - (D_{c,\infty}-1)\,y_{c,\infty,t+1} \\ &= \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_0 + \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1\left(\mathbf{z}_t - \rho_c\mathbf{z}_{t+1}\right). \end{aligned}$$

D Appendix D: Tables

TABLE A

Mean Asset Demands ($\psi = 1/\gamma$ Case)

	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
A: Annual Sample (1890 - 1995)						
$\gamma = 1, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	188.44	187.46	189.20	199.43	201.97
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	127.57	146.72	155.55	155.29	231.73
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-216.01	-234.18	-244.75	-254.72	-333.70
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	98.66	100.81	101.80	132.29	132.47
	Hedging Demand	0.00	2.54	3.02	28.13	27.15
Bond	Total Demand	70.78	89.58	95.75	89.71	53.41
	Hedging Demand	0.00	9.12	12.47	6.58	-64.35
Cash	Total Demand	-69.44	-90.39	-97.55	-122.00	-85.88
	Hedging Demand	0.00	-11.66	-15.49	-34.71	37.20
$\gamma = 5, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	44.79	52.42	52.30	74.29	81.40
	Hedging Demand	0.00	7.66	7.79	27.30	34.06
Bond	Total Demand	36.71	53.74	63.75	64.28	-19.38
	Hedging Demand	0.00	13.04	23.82	24.44	-68.76
Cash	Total Demand	18.50	-6.16	-16.05	-38.57	37.99
	Hedging Demand	0.00	-20.70	-31.61	-51.74	34.69
$\gamma = 20, \psi = 1/\gamma, \rho = 0.92$						
Stock	Total Demand	17.86	29.22	27.93	35.99	40.86
	Hedging Demand	0.00	11.22	10.54	17.59	22.53
Bond	Total Demand	19.67	35.62	48.30	46.95	15.39
	Hedging Demand	0.00	14.80	30.06	28.75	0.21
Cash	Total Demand	62.47	35.15	23.77	17.06	43.75
	Hedging Demand	0.00	-26.02	-40.60	-46.34	-22.73

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE A (ctd.)
Mean Asset Demands ($\psi = 1/\gamma$ Case)

B: Quarterly Sample (1952Q2 - 1997Q4)						
	State Variables:	Constant	AR_t	y_t	$(d-p)_t$	spr_t
$\gamma = 1, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	272.65	285.19	289.75	301.76	302.41
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Bond	Total Demand	42.80	15.90	8.20	2.88	6.54
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
Cash	Total Demand	-215.45	-201.09	-197.95	-204.64	-208.95
	Hedging Demand	0.00	0.00	0.00	0.00	0.00
$\gamma = 2, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	136.07	138.76	139.35	313.78	241.39
	Hedging Demand	0.00	-3.63	-5.14	163.34	90.64
Bond	Total Demand	16.28	-36.69	-68.17	-415.77	-465.79
	Hedging Demand	0.00	-39.75	-67.64	-412.62	-464.45
Cash	Total Demand	-52.36	-2.07	28.82	202.00	324.39
	Hedging Demand	0.00	43.37	72.79	249.28	373.81
$\gamma = 5, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	54.13	53.07	50.50	579.78	566.99
	Hedging Demand	0.00	-3.63	-6.84	520.15	507.23
Bond	Total Demand	0.38	-30.52	-30.40	-677.46	-1092.46
	Hedging Demand	0.00	-25.87	-24.64	-670.69	-1086.39
Cash	Total Demand	45.49	77.45	79.90	197.68	625.47
	Hedging Demand	0.00	29.51	31.48	150.55	579.16
$\gamma = 20, \psi = 1/\gamma, \rho = 0.92^{1/4}$						
Stock	Total Demand	13.16	11.49	9.34	362.38	-
	Hedging Demand	0.00	-2.37	-4.42	348.14	-
Bond	Total Demand	-7.58	-17.53	7.91	-371.48	-
	Hedging Demand	0.00	-9.02	16.28	-362.91	-
Cash	Total Demand	94.42	106.03	82.75	109.11	-
	Hedging Demand	0.00	11.40	-11.86	14.77	-

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. AR_t column reports mean asset demands when the VAR ssytem includes a constant, the ex-post real return on T-Bills, the excess return on stocks, and the excess return on bonds. The rest of the columns add sequentially the nominal T-Bill rate (y_t column), the dividend yield ($(d-p)_t$ column) and the yield spread (spr_t column). The bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset. " - " indicates that the recursion for δ failed to converge

TABLE B
VAR Estimation Results
Nominal Bills, Stocks and Real Consol Bond

A: Annual Sample (1890 - 1995)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.320	-0.057	0.012	0.593	-0.007	-0.592	0.235
	(2.367)	(-1.476)	(0.131)	(2.500)	(-0.226)	(-1.101)	(0.000)
xr_{t+1}	-0.008	0.091	-0.117	0.016	0.188	1.030	0.084
	(-0.021)	(0.745)	(-0.574)	(0.024)	(3.440)	(0.742)	(0.132)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.060	-0.011	-0.019	0.904	-0.006	0.078	0.784
	(-2.173)	(-1.555)	(-1.102)	(11.336)	(-1.204)	(0.787)	(0.000)
$(d-p)_{t+1}$	-0.467	-0.144	0.080	-0.875	0.767	-0.844	0.677
	(-1.514)	(-1.360)	(0.426)	(-1.396)	(12.730)	(-0.632)	(0.000)
spr_{t+1}	0.030	0.002	0.011	0.092	0.004	0.776	0.543
	(1.538)	(0.356)	(0.966)	(1.648)	(1.205)	(9.670)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	7.726	-0.178	-0.845	0.132	0.119	-0.161
xr	-	17.185	0.211	-0.180	-0.703	0.206
$xrcb$	-	-	12.489	-0.586	-0.077	0.627
y	-	-	-	1.220	0.234	-0.902
$(d-p)$	-	-	-	-	15.650	-0.191
spr	-	-	-	-	-	0.986

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE B (ctd.)

VAR Estimation Results

Nominal Bills, Stocks and Real Consol Bond

B: Quarterly Sample (1952Q2 - 1997Q4)							
Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates							
rtb_{t+1}	0.413	0.005	-0.015	0.308	-0.001	0.032	0.323
	(6.119)	(0.817)	(-0.340)	(3.918)	(-0.719)	(0.191)	(0.000)
xr_{t+1}	0.994	0.085	0.556	-2.188	0.084	1.920	0.111
	(1.004)	(1.080)	(1.214)	(-2.480)	(3.196)	(0.788)	(0.001)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.015	0.002	-0.013	0.961	0.000	0.487	0.794
	(-0.286)	(0.699)	(-0.409)	(16.967)	(0.448)	(4.394)	(0.000)
$(d-p)_{t+1}$	-1.305	-0.086	-0.479	1.358	0.932	-1.830	0.889
	(-1.298)	(-1.013)	(-0.980)	(1.489)	(34.093)	(-0.734)	(0.000)
spr_{t+1}	0.011	0.001	0.021	0.016	-0.001	0.497	0.276
	(0.294)	(0.488)	(0.954)	(0.395)	(-0.903)	(6.481)	(0.000)

Cross-Correlation of Residuals						
	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	0.601	0.229	-0.246	-0.507	-0.239	0.397
xr	-	7.594	-0.127	-0.217	-0.969	0.114
$xrcb$	-	-	1.327	-0.447	0.079	0.156
y	-	-	-	0.351	0.258	-0.899
$(d-p)$	-	-	-	-	7.949	-0.143
spr	-	-	-	-	-	0.251

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE C

VAR Estimation Results

Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

A: Annual Sample (1890 - 1995)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	xnb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates								
rtb_{t+1}	0.323	-0.054	0.020	0.153	0.704	-0.002	-0.899	0.242
	(2.419)	(-1.362)	(0.234)	(1.014)	(2.371)	(-0.060)	(-1.243)	(0.000)
xr_{t+1}	-0.012	0.085	-0.131	-0.245	-0.160	0.180	1.520	0.088
	(-0.033)	(0.693)	(-0.645)	(-0.860)	(-0.229)	(3.302)	(0.990)	(0.180)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
xnb_{t+1}	0.228	0.094	0.029	-0.086	-0.085	0.010	2.490	0.423
	(1.904)	(2.545)	(0.352)	(-0.622)	(-0.245)	(0.506)	(5.426)	(0.000)
y_{t+1}	-0.059	-0.010	-0.017	0.027	0.923	-0.005	0.025	0.786
	(-2.201)	(-1.461)	(-1.042)	(0.841)	(12.631)	(-1.077)	(0.203)	(0.000)
$(d-p)_{t+1}$	-0.457	-0.131	0.113	0.553	-0.477	0.786	-1.951	0.686
	(-1.530)	(-1.219)	(0.604)	(2.073)	(-0.736)	(13.711)	(-1.311)	(0.000)
spr_{t+1}	0.030	0.002	0.010	-0.015	0.082	0.004	0.805	0.546
	(1.543)	(0.292)	(0.895)	(-0.654)	(1.546)	(1.111)	(7.973)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	xnb	y	$(d-p)$	spr
rtb	7.690	-0.173	-0.849	-0.036	0.123	0.104	-0.156
xr	-	17.143	0.211	0.036	-0.174	-0.703	0.202
$xrcb$	-	-	12.489	0.240	-0.589	-0.078	0.629
xnb	-	-	-	4.810	-0.640	-0.126	0.269
y	-	-	-	-	1.213	0.220	-0.902
$(d-p)$	-	-	-	-	-	15.415	-0.182
spr	-	-	-	-	-	-	0.983

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE C (ctd.)
VAR Estimation Results
Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

B: Quarterly Sample (1952Q2 - 1997Q4)								
Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	xnb_t (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
Coefficient Estimates								
rtb_{t+1}	0.537	0.010	0.085	-0.058	0.280	-0.001	0.023	0.330
	(4.940)	(1.625)	(1.097)	(-1.460)	(3.529)	(-0.606)	(0.138)	(0.000)
xr_{t+1}	0.091	0.050	-0.168	0.423	-1.987	0.083	1.984	0.113
	(0.056)	(0.562)	(-0.146)	(0.707)	(-2.128)	(3.169)	(0.819)	(0.002)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
xnb_{t+1}	0.228	-0.050	0.067	-0.080	0.329	0.002	1.004	0.043
	(0.400)	(-2.034)	(0.172)	(-0.344)	(0.805)	(0.299)	(0.942)	(0.272)
y_{t+1}	0.007	0.003	0.005	-0.010	0.956	0.001	0.486	0.794
	(0.096)	(0.914)	(0.109)	(-0.326)	(16.614)	(0.492)	(4.317)	(0.000)
$(d-p)_{t+1}$	-0.188	-0.043	0.416	-0.523	1.109	0.935	-1.910	0.889
	(-0.111)	(-0.457)	(0.348)	(-0.845)	(1.138)	(34.632)	(-0.771)	(0.000)
spr_{t+1}	-0.018	0.000	-0.002	0.014	0.022	-0.001	0.499	0.278
	(-0.352)	(0.047)	(-0.062)	(0.616)	(0.562)	(-0.978)	(6.400)	(0.000)

Cross-Correlation of Residuals	rtb	xr	$xrcb$	xnb	y	$(d-p)$	spr
rtb	0.598	0.236	-0.242	0.451	-0.512	-0.247	0.405
xr	-	7.582	-0.131	0.279	-0.215	-0.969	0.112
$xrcb$	-	-	1.327	0.712	-0.446	0.083	0.153
xnb	-	-	-	2.731	-0.766	-0.325	0.413
y	-	-	-	-	0.350	0.257	-0.899
$(d-p)$	-	-	-	-	-	7.932	-0.140
spr	-	-	-	-	-	-	0.250

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset. The numbers in the main diagonal of the lower panel are standard deviations multiplied by 100.

TABLE D

Mean Asset Demands with Hypothetical Real Bonds
(Quarterly Sample: 1952Q2 - 1997Q4)

A: Nominal Bills, Stocks, and Real Consol Bond

State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	277.57	313.71
Real Consol Bond	228.90	246.49
Cash	-406.47	-460.20
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	137.74	243.58
Real Consol Bond	119.24	-507.49
Cash	-156.98	363.91
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	53.84	329.71
Real Consol Bond	53.44	-977.11
Cash	-7.28	747.40
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	11.89	212.08
Real Consol Bond	20.54	-480.32
Cash	67.57	368.24
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	-1.95	4.41
Real Consol Bond	9.68	89.18
Cash	92.27	6.41

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE D (ctd.)

Mean Asset Demands with Hypothetical Real Bonds

(Quarterly Sample: 1952Q2 - 1997Q4)

B: Nominal Bills, Stocks, Real Consol Bond and Nominal Bond

State Variables:	Constant	spr_t
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	286.88	347.48
Real Consol Bond	355.95	628.27
Nominal Bond	-81.99	-239.64
Cash	-460.85	-636.11
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	144.20	281.46
Real Consol Bond	207.46	637.71
Nominal Bond	-56.93	-797.37
Cash	-194.74	-21.81
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	58.60	597.31
Real Consol Bond	118.37	496.39
Nominal Bond	-41.90	-1273.07
Cash	-35.07	279.38
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	15.79	-
Real Consol Bond	73.82	-
Nominal Bond	-34.38	-
Cash	44.77	-
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	1.67	-
Real Consol Bond	59.12	-
Nominal Bond	-31.90	-
Cash	71.11	-

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. *spr* column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset. " - " indicates that the recursion for δ failed to converge