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The Free Rider Problem: a Dynamic Analysis*

Abstract

We study the Markov equilibria of a model of free riding in which n infinitely lived agents choose between private consumption and contributions to a durable public good. We consider economies with reversibility, where investments can be positive or negative; and economies with irreversibility, where investments are non-negative and the public good can only be reduced by depreciation. With reversibility, there is a continuum of equilibrium steady states: the highest equilibrium steady state of g is increasing in n, and the lowest is decreasing. With irreversibility, the set of equilibrium steady states converges to a unique point as depreciation converges to zero: the highest steady state possible with reversibility. In both cases, the highest steady state converges to the efficient steady state as agents become increasingly patient.

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1 Introduction

Most free rider problems have a significant dynamic component. Public goods, for example, are often durable: it takes time to accumulate them and they depreciate slowly, projecting their benefits for many years. Similarly, environmental problems depend on variables that slowly evolve over time like capital goods. In all these examples what matters for the agents in the economy is the stock of the individual contributions accumulated over time. Although there is a large literature studying free-rider problems in static environments, much less is known about dynamic environments. A number of important questions still need to be fully answered. What determines the steady states of these problems and their welfare properties? Is the free rider problem better or worse as the number of agents increases? Can we achieve efficient steady states when agents are sufficiently patient?

In this paper, we present a simple model of free riding to address these questions. In the model, n infinitely lived agents allocate their income between private consumption and contributions to a public good in every period. The public good is durable and depreciates at a rate d. We consider two scenarios. First, we study economies with reversibility, in which in every period individual investments can either be positive or negative. Second, we study an economy where the investment is irreversible, so individual investments are non-negative and the public good can only be reduced by depreciation. Although there is a significant literature that has studied free riding in economies with reversibility, in the case of irreversibility progress has been made only in specific environments that do not fit the classical description of free rider problems.[1] To our knowledge this is the first paper that provides a comparative analysis of Markov equilibria in these environments with and without irreversibility.

We start the analysis by studying the set of equilibria in economies with reversibility. We show that there is a continuum of equilibria, each characterized by a different stable steady state. The set of equilibrium steady states has three notable features. First, it always includes in its interior the level of the public good that would be reached in equilibrium by an agent alone in autarky: the steady state in a community with n agents can be either larger or smaller than when an agent is alone. Second, the upper- and lower-bounds of the set of equilibrium steady states are, respectively, increasing and decreasing in n. This implies that as the number of agents increases, the set of equilibrium steady states expands, and the free rider problem can either improve or worsen with the rise of population. Finally, for any size of population n and any rate of depreciation, the highest (and best) steady state converges to the efficient level as the discount factor converges to one. When agents are sufficiently patient, therefore, the efficient steady state

[1] A more detailed discussion of the literature is presented at the end of this section.

can be achieved with simple Markovian strategies. This is perhaps remarkable since we have a non-cooperative dynamic free riding game, with arbitrarily large numbers of players, and the Markov assumption rules out reward or punishment strategies that are contingent on individual actions or complicated histories, as required in folk-theorem constructions supporting cooperation in repeated games.

In an economy with irreversibility the set of equilibrium steady states is smaller, and contained in the set of equilibrium steady states with reversibility. We show that as the rate of depreciation converges to zero, this set converges to a unique point corresponding to the highest equilibrium steady state that can be supported with irreversibility. An immediate implication is that, as the discount factor converges to one, *all* equilibrium steady states with irreversibility are approximately efficient if the rate of depreciation is small.

The fact that reversibility affects so much the equilibrium set may appear surprising. In a planner's solution the irreversibility constraint is irrelevant: it affects neither the steady state (that is unique), nor the convergence path.[2] The reason why irreversibility is so important in a dynamic free rider game is precisely the fact that investments are inefficiently low. The intuition is as follows. In the equilibria with reversibility, an agent holds back his/her contribution for fear that it will crowd out the contributions of other players, or even be appropriated by other agents in future periods. With irreversibility, however, at some point the equilibrium investment function with reversibility must fall so low that the irreversibility constraint is binding. Even if this happens out of the equilibrium path, this affects the entire equilibrium investment function. In states just below the point in which the constraint is binding, the agents know that the constraint will not allow the other agents to reduce the public good when it passes the threshold. These incentives induce higher investments and a higher value function, with a ripple effect on the entire investment function. This effect induces the agents to cooperate more and results in a unique (high) stable steady state when depreciation is sufficiently small.

From a purely methodological point of view, the paper develops a novel approach to characterize the Markov equilibria that can have more general applicability in the study of stochastic games with discrete time. The idea is to construct Markov strategies that induce a *weakly* concave value function: the flat regions in the value function allow additional freedom in choosing the players' reaction functions and in sustaining the equilibrium. This approach is essential to prove existence of a Markov equilibrium in the difficult case of an economy with irreversibility.

Our paper is related to three strands of literature. First, it is related to the literature on

[2] On the convergent path the stock of public good is never reduced: it keeps increasing until the steady state is reached, and then it stops; the irreversibility constraint is, thus, never binding on the equilibrium path. This of course is true if the initial state g_0 is smaller than the steady state, an assumption that we will maintain throughout this paper for simplicity of exposition.

Markov equilibria in differential public good games initiated by Fershtman and Nitzan [1991].[3] This literature has been the first to propose a framework to study dynamic free rider problems. It differs from our work in two respects. First, it focuses exclusively on the environment of linear quadratic differential games in which preferences are described by specific quadratic functions and strategies are assumed to be linear in the state variable.[4] Second, and most importantly, it restricts the analysis to the case of reversible investments. In our work we consider a standard game with discrete time and general utility functions; we also do not limit the analysis to linear strategies: this, as we will see, will be important to capture the full range of equilibrium phenomena. Finally, we propose a framework that allows the comparison of economies with reversibility and irreversibility.

The second strand of literature to which our paper is related is the research on monotone contribution games (Lockwood and Thomas [2002] and Matthews [2011]).[5] These papers assume that players' individual actions can only increase over time. They differ from our work in three important ways. First, the class of games studied in these papers does not include our (standard) free-rider game. Instead, the analysis is focused to environments in which the stage games have a "prisoner dilemma structure." Both papers assume that keeping the action constant (i.e., the most uncooperative action) is a dominant strategy for all players, independently from the level of the action or the level of other players' actions. As shown by Matthews [2011], this assumption is important for the characterization of the equilibria in these papers. In our free-rider environment agents may find it optimal (and indeed do find it optimal) to make a contribution even if the other players choose their minimal contributions.[6] Second, in our model the stock of the public good can either increase or decrease over time. This is obviously true when the investment is reversible, but it is also true with irreversibility because of depreciation. In the literature on monotone games, instead, players' individual contributions (and therefore the aggregate contributions as well) can only increase over time.[7] Third, these papers focus on subgame perfect equilibria supported by

[3] See also Dockner and Long [1993], Wirl [1996], Rubio and Casino [2002], Itaya and Shimomura [2001], and Fujiwara and Matsueda [2009].

[4] Non-linear strategies are discussed in Dockner and Long [1993]. Rubio and Casino [2002], however, highlight complications of considering non-linear strategies that arise in this and other related models.

[5] A number of significant papers in the monotone games literature are less directly related. These papers require additional assumptions that make their environments hard to compare to ours. Gale [2001] provides a general framework of monotone games with no discounting, and applies it to a contribution game in which agents care only about the limit contributions as $t \to \infty$. Admati and Perry [1991], Compte and Jehiel [2004] and Marx and Matthews [2000] consider environments in which the benefit of the contribution occurs at the end of the game if a threshold is reached and in which players receive either partial or no benefit from interim contributions. The first two of these papers, moreover, assume that players contribute sequentially, one at a time.

[6] As standard in public good games, we assume the individual benefit for a marginal contribution converges to infinity as the stock of g converges to zero.

[7] Indeed, the game we study is not in the class of monotone games.

trigger strategies, while we focus on Markov equilibria. In environments in which many players interact anonymously as in many natural free rider problems, Markov equilibria seem an important benchmark to understand equilibrium behavior. As we mentioned above, our paper is the first work to study and compare Markov equilibria with and without reversibility.

The final strand of literature is the more recent research on dynamic political economy. This literature studies the Markov equilibria of dynamic accumulation games similar to the game studied here, but with different collective decision processes: free riding with the possibility of writing incomplete contracts (Harstad [2012]), noncooperative bargaining (Battaglini and Coate [2007] and Battaglini, Nunnari, and Palfrey [2012b]), political agency models (Besley and Persson [2011] and Besley, Ilzetzki and Persson [2011]). All of these papers restrict their analysis to environments with reversibility.[8] We are confident that insights developed in our paper on irreversible economies can help understanding public investments even in these alternative models of public decision making in future research.

2 The model

Consider an economy with n agents. There are two goods: a private good x and a public good g. The level of consumption of the private good by agent i in period t is x_t^i, the level of the public good in period t is g_t. An allocation is an infinite nonnegative sequence $z = (x_\infty, g_\infty)$ where $x_\infty = (x_1^1, ..., x_1^n, ..., x_t^1, ..., x_t^n, ...)$ and $g_\infty = (g_1, ..., g_t, ...)$. We refer to $z_t = (x_t, g_t)$ as the allocation in period t. The utility U^j of agent j is a function of $z^j = (x_\infty^j, g_\infty)$, where $x_\infty^j = (x_1^j, ..., x_t^j, ...)$. We assume that U^j can be written as:

$$U^j(z^j) = \sum_{t=1}^{\infty} \delta^{t-1} \left[x_t^j + u(g_t) \right],$$

where $u(\cdot)$ is continuously twice differentiable, strictly increasing, and strictly concave on $[0, \infty)$, with $\lim_{g \to 0^+} u'(g) = \infty$ and $\lim_{g \to +\infty} u'(g) = 0$. The future is discounted at a rate δ.

There is a linear technology by which the private good can be used to produce public good, with a marginal rate of transformation $p = 1$. The private consumption good is nondurable, the public good is durable, and the stock of the public good depreciates at a rate $d \in [0, 1]$ between periods. Thus, if the level of public good at time $t - 1$ is g_{t-1} and the total investment in the public good is I_t, then the level of public good at time t will be

$$g_t = (1 - d)g_{t-1} + I_t.$$

[8] An exception is Battaglini, Nunnari, and Palfrey [2012a] that presents an experimental study based on the theoretical model of this paper.

We consider two alternative economic environments. In a *Reversible Investment Economy* (RIE) the public policy in period t is required to satisfy three feasibility conditions:

$$\begin{aligned} x_t^j &\geq 0 \ \forall j, \ \forall t \\ g_t &\geq 0 \ \forall t \\ I_t + \sum_{j=1}^{n} x_t^j &\leq W \ \ \forall t \end{aligned} \tag{1}$$

where W is the aggregate per period level of resources in the economy. The first two conditions guarantee that allocations are nonnegative. The third condition is simply the economy's resource constraint. In an *Irreversible Investment Economy* (IIE), the second condition is substituted with:

$$g_t \geq (1-d)g_{t-1} \ \ \forall t \tag{2}$$

The RIE corresponds to a situation in which I_t can be negative. The constraint that the state variable g_t is non negative in a RIE is natural when g_t is physical capital and it will maintained throughout the analysis. It should however be noted that it is not relevant for the results.

It is convenient to distinguish the state variable at t, g_{t-1}, from the policy choice g_t and to reformulate the budget condition. If we denote $y_t = (1-d)g_{t-1} + I_t$ as the new level of public good after investing I_t in the current period when the last period's level of the public good is g_{t-1}, then the public policy in period t can be represented by a vector $(y_t, x_t^1, ..., x_t^n)$. Substituting y_t, the budget balance constraint $I_t + \sum_{j=1}^{n} x_t^j \leq W$ can be rewritten as:

$$\sum_{j=1}^{n} x_t^j + [y_t - (1-d)g_{t-1}] \leq W,$$

With this notation, we must have $x_t \geq 0, y_t \geq 0$ in a RIE, and $x_t \geq 0, y_t \geq (1-d)g_{t-1}$ in a IIE.

The initial stock of public good is $g_0 \geq 0$, exogenously given. Public policies are chosen as in the classic free rider problem, modeled by a voluntary contribution game. In period t, each agent j is endowed with $w_t^j = W/n$ units of private good. We assume that each agent has full property rights over a share of the endowment (W/n) and in each period chooses on its own how to allocate its endowment between an individual contribution to the stock of public good (which is shared by all agents) and private consumption, taking as given the strategies of the other agents. The individual contribution by agent j at time t is denoted i_t^j (so $i_t^j = W/n - x_t^j$). In a RIE, the level of individual contribution can be negative, with the constraint that $i_t^j \in [-(1-d)g_t/n, W/n] \ \forall j$.[9] In a IIE, an agent's contribution must satisfy $i_t^j \in [0, W/n] \ \forall j$. The total economy-wide increase

[9] The constraint $i_t^j \geq -(1-d)g_t/n$ guarantees that the sum of reductions in g is never larger than the total stock

in the stock of the public good in any period is then given by the sum of the agents' individual contributions.

The state variable can have alternative interpretations.

Example 1 (Public capital). It is natural to assume that g is physical public capital. In this case it may seem natural to assume that the environment is irreversible. Once a bridge is constructed, it can not be decomposed and transformed back to consumption. Similarly, a painting donated to a public museum can not typically be withdrawn. The choice of the model to adopt (reversible or irreversible) should depend on the nature of the public good. If the public good is easily divisible and can be easily appropriated (as, for example, wood and other valuable resources from a forest) an agent may choose to appropriate part of the accumulated level. When withdrawals are possible (both because allowed, or because they can not be prevented), then the model may be described as RIE. The ability of the community to prevent agents from "privatizing" (or stealing) the stock of public good is a technological variable that may vary case to case.

Example 2 (Pollution). Suppose the state g is the level of global warming with the convention that the larger is g, the worse is global warming. The utility of an agent now is $u(x,g) = x - c(g)$, where $c(\cdot)$ is increasing, convex and differentiable. It is natural to assume that an agent can either increase or decrease global warming by choosing a "dirty" or a "clean" technology. This environment can be modelled as before if we assume that an increase in the "greenness" of the technology costs, at the margin, a dollar's worth of current consumption. Given this, we have, as before: $g_t = (1-d)g_{t-1} - \sum_j i_t^j$, where now i_t^j stands for the individual contribution to green technology (and it can be positive or negative). In this context it is, therefore, natural to assume the economy is reversible.

Example 3 ("Social Capital" or "Fiscal Capacity") A number of recent works have highlighted the importance of a variety of forms of intangible, or semi-tangible community assets, like social capital (Putnam [2000]) or fiscal capacity (Besley and Persson [2011]). For the case of social capital, it may seem natural to assume that agents take actions that can be either positive or negative for capital accumulation. Moreover, because social capital is an intangible asset, we may assume it takes values in $g \in (-\infty, +\infty)$. It follows that the accumulation of social capital can probably be modelled as a reversible investment economy as described in Example 1 (where it

of public good, naturally it must be satisfied in any symmatric equilibrium. The analysis is similar if we allow each player to withdraw up to $(1-d)g$ since no player finds it optimal to reduce g to zero (the marginal utility of g at zero is infinity). In this case, however, we have to assume a rationing rule in case the individuals withdraw more than $(1-d)g$. A simple rationing rule generating identical results is the following. At the beginning of each period player i can claim any amount $\omega_t^i \leq (1-d)g_{t-1}$ from the pool: if $\sum \omega_t^i \leq (1-d)g_{t-1}$, then i receives his demand ω_t^i; if $\sum \omega_t^i > (1-d)g_{t-1}$, then the public good is rationed pro quota, $\omega_t^i = (1-d)\frac{\omega_t^i}{\sum \omega_t^j} g_{t-1}$. The player can then consume x_t^i with $x_t^i \leq W/n + \omega_t^i$ and leave the rest of $W/n + \omega_t^i$ in the public good.

is assumed that there is a minimum level of capital, zero) or as in Example 2 (where capital is in the real line). The case of fiscal capacity is similar, and indeed Besley and Persson [2011] assume it is reversible. There may be however cases in which even this type of social investment is not reversible, or it is partially reversible. This is probably the case when fiscal capital is embodied in institutions that can not be easily undone. In these cases, an irreversible investment economy can be a more appropriate model.

To study the properties of the dynamic free rider problem described above, we study symmetric Markov perfect equilibria, where all agents use the same strategy, and these strategies are time-independent functions of the state, g. A strategy is a pair $(x(\cdot), i(\cdot))$: where $x(g)$ is an agent's level of consumption and $i(g)$ is an agent's contribution to the stock of public good in state g. Given these strategies, by symmetry, the stock of public good in state g is $y(g) = (1-d)g + ni(g)$. For the remainder of the paper we refer to $y(g)$ as the *investment function*. Associated with any Markov perfect equilibrium of the game is a value function, $v(g)$, which specifies the expected discounted future payoff to an agent when the state is g. An equilibrium is continuous if the investment function, $y(g)$, and the value function, $v(g)$, are both continuous in g. In the remaining of the paper we will focus on continuous equilibria. In the following we refer to equilibria with the properties described above simply as equilibria.

The focus on Markov equilibria seems particularly appropriate for this class of dynamic games. Free rider problems are often intended to represent situations in which a large number of agents autonomously and independently contribute to a public good (Olson [1965, Chapter 1.B]). In a large economy, it is natural to focus on an equilibrium that is anonymous and independent from the action of any single agent. The Markov perfect equilibrium respects this property, by making strategies contingent only on the payoff relevant economic state. The focus on continuous equilibria is also standard in the applied literature studying Markov equilibria of dynamic public accumulation games (Levhari and Mirman [1980], Battaglini and Coate [2007,2008], Besley and Persson [2011], Harstad [2012], and many others).

3 The planner's problem

As a benchmark with which to compare the equilibrium allocations, we first analyze the sequence of public policies that would be chosen by a benevolent planner who maximizes the sum of utilities of the agents. This is the welfare optimum because the private good enters linearly in each agent's utility function. The planner's solution is extremely simple in the environment described in the previous section: this feature will help highlighting the subtlety of the strategic interaction studied in the next two sections.

Consider first an economy with reversible investment. The planner's problem has a recursive representation in which g is the state variable, and $v_P(g)$, the planner's value function can be represented recursively as:

$$v_P(g) = \max_{y,x} \left\{ \begin{array}{c} \sum_{j=1}^{n} x^i + nu(y) + \delta v_P(y) \\ s.t \ \ \sum_{j=1}^{n} x^i + y - (1-d)g \leq W, \ x^i \geq 0 \ \forall i, \ y \geq 0 \end{array} \right\} \tag{3}$$

By standard methods (see Stokey, Lucas, and Prescott [1989]), we can show that a continuous, strictly concave and differentiable $v_P(g)$ that satisfies (3) exists and is unique. The optimal policies have an intuitive characterization. When the accumulated level of public good is low, the marginal benefit of increasing y is high, and the planner finds it optimal to spend as much as possible on building the stock of public good: in this region of the state space $y_P(g) = W + (1-d)g$ and $\sum_{j=1}^{n} x^i = 0$. When g is high, the planner will be able to reach the level of public good $y_P^*(\delta, d, n)$ that solves the planner's unconstrained problem: i.e.

$$nu'(y_P^*(\delta, d, n)) + \delta v_P'(y_P^*(\delta, d, n)) = 1. \tag{4}$$

Applying the envelope theorem, we can show that at the interior solution $y_P^*(\delta, d, n)$ we have $v_P'(y_P^*(\delta, d, n)) = 1 - d$. From (4), we therefore conclude that:

$$y_P^*(\delta, d, n) = [u']^{-1} \left(\frac{1 - \delta(1-d)}{n} \right) \tag{5}$$

The investment function, therefore, has the following simple structure. When g is lower than $\frac{y_P^*(\delta,d,n)-W}{1-d}$, $y_P^*(\delta, d, n)$ is not feasible: the planner spends W on the public good so $y_P(g) = (1-d)g + W$. When g is larger or equal than $\frac{y_P^*(\delta,d,n)-W}{1-d}$, instead, the planner does not invest beyond $y_P(g) = y_P^*(\delta, d, n)$. In this case, without loss of generality, we can set $x^i(g) = (W + (1-d)g - y(g))/n \ \forall i$.[10] Summarizing, we have:

$$y_P(g) = \min \{W + (1-d)g, y_P^*(\delta, d, n)\}. \tag{6}$$

This investment function implies that the planner's economy converges to one of two possible steady states (see Figure 1). If $W/d \leq y_P^*(\delta, d, n)$, then the rate of depreciation is so high that the planner cannot reach $y_P^*(\delta, d, n)$, (except temporarily if the initial state is sufficiently large). In this case the steady state is $y_P^o = W/d$, and the planner invests all resources in all states on the equilibrium path (Figure 1, Case 1). If $W/d > y_P^*(\delta, d, n)$, $y_P^*(\delta, d, n)$ is sustainable as a steady state. In this case, in the steady state $y_P^o = y_P^*(\delta, d, n)$, and the (per agent) level of private consumption is positive: $x^* = (W + (1-d)g - y)/n > 0$ (Figure 1, Case 2).

[10] Indeed, the planner is indifferent regarding the distribution of private consumption.

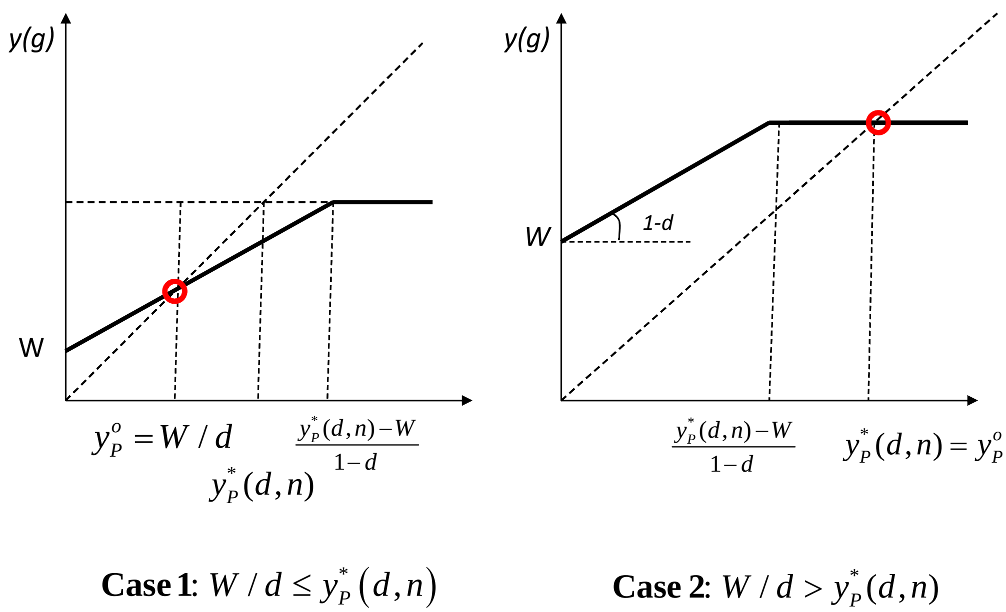

Figure 1: The Planner's Problem

An economy in which the planner's optimum can be feasibly sustained as a steady state is the most interesting case. With this in mind we define:

Definition 1. *An economy is said to be regular if* $W/d > y_P^*(\delta, d, n)$.

In the rest of the analysis we focus on regular economies.[11] This is done only for simplicity: extending the results presented below for economies with $W/d \leq y_P^*(\delta, d, n)$ can be done using the same techniques developed in this paper.

The planner's optimum for the IIE case is not very much different. The planner finds it optimal to invest all resources for $g \leq \frac{y_P^*(\delta,d,n)-W}{1-d}$. For $g \in \left(\frac{y_P^*(\delta,d,n)-W}{1-d}, \frac{y_P^*(\delta,d,n)}{1-d}\right)$, the planner finds it optimal to stop investing at $y_P^*(\delta, d, n)$, as before. For $g \geq \frac{y_P^*(\delta,d,n)}{1-d}$, $y_P^*(\delta, d, n)$ is not feasible, so it is optimal to invest 0, and to set $y_P(g) = (1-d)g$. This difference in the investment function for IIE, however, is essentially irrelevant for the optimal path and the steady state of the economy. Starting from any g_0 lower than the steady state y_P^*, levels of g larger or equal than $\frac{y_P^*(\delta,d,n)}{1-d}$ are impossible to reach, and the irreversibility constraint does not affect the optimal investment path.

We conclude this section noting that, both with reversibility and with irreversibility, the planner's solution has a very simple structure: the planner finds it optimal to invest as much as possible

[11] The limit case of $d = 0$ is also included as a regular economy.

in every period until the steady state is reached. This has two implications: first, gradualism in investment is never optimal for the planner; second, the irreversibility constraint is irrelevant. As we will see, neither of these two features holds in the free-rider games we study in the next two sections: gradualism is indeed a typical feature of the equilibrium investment function, both with and without irreversibility; and the irreversibility constraint plays an important role in determining the set of equilibrium steady states.

4 Reversible investment economies

4.1 The equilibrium

We first study equilibrium behavior when the investment in the public good is reversible. Differently from the planner's case, in equilibrium no agent can directly choose the stock of public good y: an agent (say j) chooses only his own level of private consumption x and the level of its own contribution to the stock of public good. The agent realizes that in any period, given g and the other agents' level of private consumption, his/her contribution ultimately determines y. It is therefore as if agent j chooses x and y, subject to three feasibility constraints. The first constraint is a resource constraint that specifies the level of the public good:

$$y = W + (1-d)g - [x + (n-1)x_R(g)]$$

This constraint requires that stock of public good y equals total resources, $W+(1-d)g$, minus the sum of private consumptions, $x+(n-1)x_R(g)$. The function $x_R(g)$ is the *equilibrium* per capita level; naturally, the agent takes the equilibrium level of the other players, $(n-1)x_R(g)$, as given. The second constraint requires that private consumption x is non negative. The third requires total consumption nx to be no larger than total resources $(1-d)g+W$. Agent j's problem can therefore be written as:

$$\max_{y,x} \left\{ \begin{array}{c} x + u(y) + \delta v_R(y) \\ \\ s.t \;\; x + y - (1-d)g = W - (n-1)x_R(g) \\ \\ W - (n-1)x_R(g) + (1-d)g - y \geq 0 \\ \\ x \leq (1-d)g/n + W/n \end{array} \right\} \tag{7}$$

where $v_R(g)$ is his equilibrium value function.

In a symmetric equilibrium, all agents consume the same fraction of resources, so agent j can assume that in state g the other agents each consume:

$$x_R(g) = \frac{W + (1-d)g - y_R(g)}{n},$$

where $y_R(g)$ is the equilibrium investment function. Substituting the first constraint of (7) in the objective function, recognizing that agent j takes the strategies of the other agents as given, and ignoring irrelevant constants, the agent's problem can be written as:

$$\max_y \left\{ \begin{array}{c} u(y) - y + \delta v_R(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_R(g),\ y \geq \frac{n-1}{n} y_R(g) \end{array} \right\} \tag{8}$$

where it should be noted that agent j takes $y_R(g)$ as given.[12] The objective function shows that an agent has a clear trade off: a dollar in contribution produces an individual marginal benefit $u'(y) + \delta v'_R(y)$; the marginal cost of the contribution is -1, a dollar less in private consumption.[13]

The two constraints define the maximal and minimal feasible level of public good given the other players' investments.

A symmetric Markov equilibrium is therefore fully described in this environment by two functions: an aggregate investment function $y_R(g)$, and an associated value function $v_R(g)$. Two conditions must be satisfied. First, for all values of $g \geq 0$, $y_R(g)$ must solve (8) given $v_R(g)$. The second condition for an equilibrium requires that the value function $v_R(g)$ to be consistent with the agents' strategies, and hence consistent with the equilibrium investment function, $y_R(g)$. Each agent receives the same benefit for the expected investment in the public good, and consumes the same share of the remaining resources, $(W + (1-d)g - y_R(g))/n$. This implies:

$$v_R(g) = \frac{W + (1-d)g - y_R(g)}{n} + u(y_R(g)) + \delta v_R(y_R(g)) \tag{9}$$

We can therefore define:

Definition 2. *An equilibrium in a Reversible Investment Economy is a pair of functions, $y_R(\cdot)$ and $v_R(\cdot)$, such that for all $g \geq 0$, $y_R(g)$ solves (8) given the value function $v_R(\cdot)$; and for all $g \geq 0$, $v_R(g)$ solves (9) given $y_R(\cdot)$.*

For a given value function, if an equilibrium exists, the problem faced by an agent looks similar to the problem of the planner, but with two important differences. First, in the objective function the agent does not internalize the effect of the public good on the other agents. This is the classic free rider problem, present in static models as well: it induces a suboptimal investment in g. The second difference with respect to the planner's problem is that the agent takes the contributions of the other agents as given. The incentives to invest depend on the agent's expectations about the

[12] Since $y_R(g)$ is the equilibrium investment function, in a symmetric equilibrium $(n-1)y_R(g)/n$ is the level of investment that agent j expects from all the other agents, and that he/she takes as given in equilibrium.

[13] For simplicity of exposition we assume here that $v_R(g)$ is differentiable. We refer to the proofs in the appendix for the details.

other agents' current and future contributions, which are captured implicitly by the investment function $y_R(g)$. This radically changes the nature of the equilibria. Thus an agent may be willing to invest more or less today, depending on the exact shape of the investment function, which depends on how other agents plan to invest in the future at different levels of g. The relevant question is: Does this make the static free rider problem worse or better in a dynamic environment?

4.2 Characterization

To characterize the properties of equilibrium behavior, we first study a particular class of equilibria, the class of *weakly concave equilibria*. An equilibrium is said to be weakly concave if $v(y;g)$ is weakly concave on y for any state g, where $v(y;g)$ is the expected value of investing up to a level of public good, y:

$$v(y\,|g\,) = \frac{W + (1-d)g - y}{n} + u(y) + \delta v(y)$$

We show that this class of equilibria is nonempty and we characterize its key properties. We then prove that there is no loss of generality in focusing on this particular class in order to study the set of equilibrium steady states. We therefore use the class of weakly concave equilibria as a tool to gain insight on the more general equilibrium properties of the game.

In a weakly concave equilibrium, the agent's problem (8) is a standard concave programming problem similar to (3). Because the objective function may have a flat region, however, the investment function typically takes a more general form than the planner's solution (6). Figure 2 represents a typical equilibrium. The equilibrium investment function will generally take the following form:

$$y_R(g) = \begin{cases} \min\left\{W + (1-d)g, y\left(g^2\right)\right\} & g < g^2 \\ y\left(g\right) & g \in \left[g^2, g^3\right] \\ y\left(g^3\right) & g > g^3 \end{cases} \tag{10}$$

where g^2, g^3 are two critical levels of g, and $y(g)$ is a non decreasing function with values in $[g, W + (1-d)g]$. To see why $y_R(g)$ may take the form of (10), consider Figure 2. The top panel of the figure illustrates a canonical equilibrium investment function. The steady state is labeled y_R^o in the figure, the point at which the (bold) investment function intersects the (dotted) diagonal. The bottom panel of the figure graphs the corresponding objective function, $u(y) - y + \delta v_R(y)$. For $g < g^2$, the objective function of (8) is strictly increasing in y: either resources are sufficient to reach the level that maximizes the unconstrained objective function and so $y(g) \in \left[g^2, g^3\right]$ (in Figure 2, $y(g) = y\left(g^2\right)$ in $g^1 \leq g \leq g^2$); or it is optimal to invest all resources (in Figure 2,

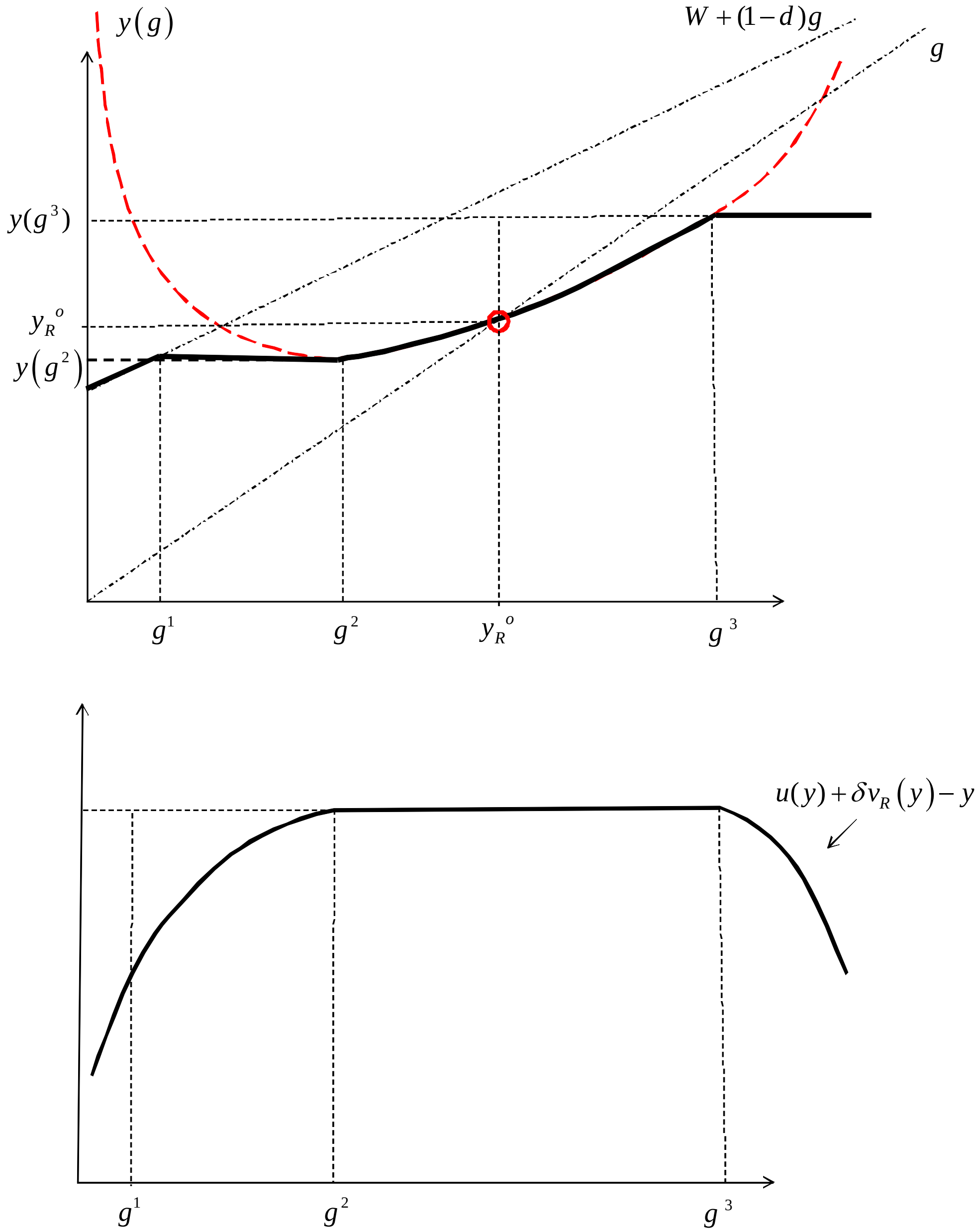

Figure 2: The equilibrium in an economy with reversibility

$y(g) = W + (1-d)g$ in $g \leq g^1$).[14] For $g > g^3$, the objective function is decreasing: the investment level is so high that the agents do not wish to increase g over $y(g^3)$. For intermediate levels of $g \in [g^2, g^3]$, an interior level of investment $y \in (g^2, g^3)$ is chosen. This is possible because the objective function is flat in this region: an agent is indifferent between any $y \in [g^2, g^3]$. The key observation here is that since the objective function has a flat region, the agents find it optimal to choose an *increasing* investment function in $[g^2, g^3]$: a weakly concave objective function, therefore, gives us more freedom in choosing the equilibrium investment function and even a higher level of investment. As we will see, this additional freedom allows to sustain a larger set of equilibria and a higher steady state level of g.

The open questions are whether the flat region in Figure 2 is a general equilibrium phenomenon or just an intellectual curiosity, and what degrees of freedom we have in choosing investment functions that are consistent with an equilibrium. For an investment curve as in Figure 2 to be an equilibrium, agents must be indifferent between investing and consuming for all states in $[g^2, g^3]$. If this condition does not hold, the agents do not find it optimal to choose an interior level $y(g)$. The marginal utility of investments is zero if and only if:

$$u'(g) + \delta v'(g) - 1 = 0 \ \forall g \in [g^2, g^3] \tag{11}$$

Since the expected value function is (9), we have:

$$v'(g) = \frac{1 - d - y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{12}$$

Substituting this formula in (11), we see that the investment function $y(g)$ must solve the following differential equation:

$$\frac{1 - u'(g)}{\delta} = \frac{1 - d - y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{13}$$

This condition is useful only if we eliminate the last (endogenous) term: $\delta v'(y(g))y'(g)$. To see why this is possible, note that $y(g)$ is in $[g^2, g^3]$ for any $g \in (g^2, g^3)$ in the example of Figure 2. In this case, (11) implies $\delta v'(y(g)) = 1 - u'(y(g))$. Substituting this condition in (13) we obtain the following necessary condition:

$$y'(g) = \frac{1 - d - \frac{n(1-u'(g))}{\delta}}{1 - n} \tag{14}$$

Condition (14) shows that there is a unique way to specify the shape of the investment function that is consistent with a "flat" objective function in equilibrium. This necessary condition, however,

[14] In Figure 2 it is assumed that we have $W + (1-d)g > g^2$ for for $g \geq g^1$, so the agent can afford to choose a level of y that maximizes the objective function (i.e. $y \in [g^2, g^3]$) if and only if $g \geq g^1$.

leaves considerable freedom to construct multiple equilibria: (14) defines a simple differential equation with a solution $y(g)$ unique up to a constant. To have a steady state at y_R^o we need a second condition: $y(y_R^o) = y_R^o$. This equality provides the initial condition for (14), and so uniquely defines $y(g|y_R^o)$ in $\left[g^2, g^3\right]$ (see the dashed curve in Figure 2).

Proposition 1, presented below, shows that the degrees of freedom allowed by (14) are sufficient to characterize all the stable steady states we can have in equilibrium, weakly-concave or not. A steady state y_R^o is said to be stable if there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o such that for any $N_{\varepsilon'}(y_R^o) \subseteq N_\varepsilon(y_R^o)$, $g \in N_{\varepsilon'}(y_R^o)$ implies $y_R(g) \in N_{\varepsilon'}(y_R^o)$. Intuitively, starting in a neighborhood of a stable steady state, g remains in a neighborhood of a stable steady state for all following periods.[15] Define the two thresholds:

$$y_R^*(\delta, d, n) = [u']^{-1} (1 - \delta(1-d)/n) \text{, and } y_R^{**}(\delta, d, n) = [u']^{-1} (1 - \delta (1 - d/n)) \tag{15}$$

We say that an equilibrium steady state y_R^o is supported by a concave equilibrium if there is a concave equilibrium $y_R(g), v_R(g)$ such that $y_R(y_R^o) = y_R^o$. An equilibrium is monotonic if the investment function, $y(g)$, is non decreasing in g. The following Proposition shows that the set of equilibrium steady states of monotonic equilibria can be easily characterized in closed form. The details about the equilibrium strategies are in the appendix.

Proposition 1. *A public good level y_R^o is a stable steady state of a monotonic equilibrium in a RIE if and only if $y_R^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$. Each y_R^o in this set is supported by a concave equilibrium with investment function as illustrated in Figure* 2.

We may obtain an intuitive explanation of why the steady state must be in $[y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$ by making three observations. First, an equilibrium steady state must be in the interior of the feasibility region, that is $y_R^o \in (0, W/d)$:[16] intuitively, $y_R^o > 0$, since at 0 the marginal utility of the public good is infinite; and $y_R^o < W/d$ since even in the planner's solution we have this property. Second, in a stable steady state we must have $y_R'(y_R^o) \in (0,1)$.[17] The highest and the lowest steady states, moreover, correspond to the equilibria with the highest and, respectively, the lowest $y_R'(g)$: so $y_R'(y_R^{**}(\delta, d, n)) = 1$ and, respectively , $y'(y_R^*(\delta, d, n)) = 0$. Third, since the solution is interior and the agents can choose the investment they like in a neighborhood of y_R^o,

[15] An alternative stability concept that has been used in the litetature is achievability (Matthews [2011]). A steady state is achievable if it is the limit of an equilibrium path. Our concept of stability is weaker: this allows us to have a stronger characterization of the equilibrium set in our environment. It is easy to see that if an equilibrium is not stable, then it is not reachable in a monotonic Markov equilibrium. On the other hand, all steady states characterized in Proposition 1 are achievable (as in the equilibrium illustrated by Figure 2).

[16] The feasibility set is given by $y \geq 0$ and $y \leq W + (1-d)g$, so a steady state must satisfy $y_R^o \geq 0$ and $y_R^o \leq W + (1-d)y_R^o$. The second inequality implies $y_R^o \leq W/d$.

[17] We are assuming that $y_R(g)$ is differentiable for the sake of the argument here, details are provided in the appendix.

$y_R(g)$ can have positive slope at y_R^o only if the agents's objective function is flat in the neighborhood (otherwise the agents would choose the same optimum point irrespective of g). By the argument presented above, this implies (14). Using (14) and $y'_R(y_R^o) \leq 1$, we obtain the upper bound, $y_R^{**}(\delta, d, n)$; similarly, using (14) and $y'_R(y_R^o) \geq 0$, we obtain the lower bound, $y_R^*(\delta, d, n)$. Proposition 1 formalizes this argument, and it uses the construction described above to prove that $y_R^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$ is sufficient as well as necessary for y_R^o to be a stable steady state.

In the working paper version of the paper (Battaglini, Nunnari and Palfrey [2012c]) we show that non-monotonic equilibria always exist in reversible economies. When we consider non-monotonic equilibria the maximal steady state remains the same as in Proposition 1, the minimal however can be lower than $y_R^*(\delta, d, n)$.[18] The consideration of non-monotonic equilibria, however, is not particularly relevant for the comparison of irreversible versus reversible economies since except when depreciation is high, all the steady states that can be achieved in irreversible economies can be achieved with monotonic equilibria as well. To keep the presentation focused, the analysis of non-monotonic equilibria is left in the working paper version (Battaglini, Nunnari and Palfrey [2012c]).

4.3 Efficiency

Proposition 1 shows that, as in the static model, an equilibrium allocation is always inefficient, even in the best equilibrium: $y_P^*(\delta, d, n) > y_R^{**}(\delta, d, n)$ for any $n > 1$ and $\delta < 1$. Let $\overline{y}(\delta, d)$ be the steady state that would be achieved by an agent alone in autarky: $y_P^*(\delta, d, 1) = [u']^{-1}(1 - \delta(1-d))$. We can make three observation regarding the magnitude of the inefficiency.

Corollary 1. *In a RIE we have:*

- *For any $n > 1$ we have $\overline{y}(\delta, d) \in (y_R^*(\delta, d, n), y_R^{**}(\delta, d, n))$;*
- *The highest equilibrium steady state increases in n; the smallest steady state decreases in n. As $n \to \infty$, $y_R^*(\delta, d, n) \to [u']^{-1}(1)$ and $y_R^{**}(\delta, d, n) \to [u']^{-1}(1-\delta)$;*
- *For any n and d, $|y_R^{**}(\delta, d, n) - y_P^*(\delta, d, n)| \to 0$ as $\delta \to 1$.*

The first point in Corollary 1 shows that the accumulated level of g in a community with n players may be *either* higher *or* lower than the level that an agent alone in autarky would accumulate. This is in contrast to the static case (when $\delta = 0$), where the level of accumulation is independent of n. The second point shows that, in terms of the steady state level of g, the

[18] In Battaglini at al. [2012] we show that the lowest possible steady state with non monotonic strategies is $[u']^{-1}(1 + \delta(n + d - 2)/n) < y_R^*(\delta, d, n)$.

common pool problem may become better or worse as the size of the community increases. The multiplicity of equilibria, moreover, is not an artifact of the assumption of a finite population. Finally, the last point highlights the fact that the best equilibrium steady state converges to the efficient level as $\delta \to 1$. What is remarkable in this result is the fact that the efficient steady state can be achieved with an extremely simple equilibrium (Markov) in which agents focus exclusively on the state g.[19]

To interpret Proposition 1 it is useful to start from the special case in which $\delta = 0$ and so the free rider problem is static. In this case there is a unique equilibrium "steady state" at $y_R^o = [u']^{-1}(1)$, independent of n. In addition, the agents' actions are pure strategic substitutes. If agent j is forced to invest $1/n + \Delta$, then all the other agents find it optimal to reduce their investment exactly by $\Delta/(n-1)$. Previous research on dynamic public good games has stressed this aspect of the games. Fershtman and Nitzan [1991], in particular, show an equilibrium in which the substitutability effect is so strong that the steady state is even lower than the equilibrium of a static game. In general, however, the strategic interaction in dynamic games is much richer. In the working paper version of this work, we formally prove that all the equilibria in Proposition 1 with steady state lower than $\overline{y}(\delta, d)$ the players' contributions are *strategic substitutes*. But in the equilibria with steady states larger than $\overline{y}(\delta, d)$ the players' contributions *must* be *strategic complements* on the equilibrium path. Strategic complementarity is necessary in these equilibria because an agent is willing to keep investing until $y_R^o > \overline{y}(\delta, d)$ only if he expects the other agents to react to his investment by increasing their own investments. This complementarity allows the agents to mitigate the free rider problem and partially "internalize" the public good externality. In these equilibria, the agents accumulate more than what would be reached by an agent in perfect autarky. As Corollary 1 proves, this complementarity effect may be extremely powerful, allowing to achieve an efficient steady as $\delta \to 1$ with simple Markov strategies.

5 Irreversible economies

We now turn to irreversible investment economies. When the agents cannot directly reduce the stock of the public good, the optimization problem of an agent can be written like (7), but with an additional constraint: the individual level of investment cannot be negative; the only way to reduce the stock of g, is to wait for the work of depreciation. Following similar steps as before,

[19] The literature on quadratic differential games has shown examples in which the efficient steady state can be reached in a Markov equilibrium if the economy starts from specific initial steady states (see Rubio and Casino [2002] who qualify an important result by Dockner and Long [1993]). To our knowledge, Corollary 1 presented above is the first result showing limit efficiency of the best steady state of a Markov equilibrium for generic initial steady state, generic utility (and in discrete time).

we can write the maximization problem faced by an agent as:

$$\max_{y} \left\{ \begin{array}{c} u(y) - y + \delta v_{IR}(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_{IR}(g), \; y \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y_{IR}(g) \end{array} \right\} \tag{16}$$

where the only difference with respect to (8) is the second constraint. To interpret it, note that it can be written as $y \geq (1-d)g + \frac{n-1}{n}\left[y_{IR}(g) - (1-d)g\right]$: the new level of public good cannot be lower than $(1-d)g$ plus the investments from all the other agents (in a symmetric equilibrium, an individual investment is $\left[y_{IR}(g) - (1-d)g\right]/n$).

As in the reversible case, a continuous symmetric Markov equilibrium is fully described in this environment by two functions: an aggregate investment function $y_{IR}(g)$, and an associated value function $v_{IR}(g)$. The aggregate investment function $y_{IR}(g)$ must solve (16) given $v_{IR}(g)$. The value function $v_{IR}(g)$ must be consistent with the agents' strategies. Similarly, as in the reversible case, we must have:

$$v_{IR}(g) = \frac{W + (1-d)g - y_{IR}(g)}{n} + u\left(y_{IR}(g)\right) + \delta v_{IR}(y_{IR}(g)) \tag{17}$$

We can therefore define:

Definition 3. *An equilibrium in a Irreversible Investment Economy is a pair of functions, $y_{IR}(\cdot)$ and $v_{IR}(\cdot)$, such that for all $g \geq 0$, $y_{IR}(g)$ solves (16) given the value function $v_{IR}(\cdot)$, and for all $g \geq 0$, $v_{IR}(g)$ solves (17) given $y_{IR}(g)$.*

As pointed out in Section 3, when public investments are efficient, irreversibility is irrelevant for the equilibrium allocation. The investment path chosen by the planner is unaffected because the planner's choice is *time consistent*: he never finds it optimal to increase g if he plans to reduce it later. In the monotone equilibria characterized in the previous section, the investment function may be inefficient, but it is weakly increasing in the state. Agents invest until they reach a steady state, and then they stop. It may seem intuitive, therefore, that irreversibility is irrelevant in this case too. In this section we show that, to the contrary, irreversibility changes the equilibrium set: it induces the agents to significantly increase their investment and it leads to significantly higher steady states when depreciation is small.

To illustrate the impact of irreversibility on equilibrium behavior, suppose for simplicity that $d = 0$ and consider Figure 3, where the red dashed line represents some arbitrary monotone equilibrium with steady state y^o in the model with reversibility. Next, suppose we ignore the irreversibility constraint where it is not binding, so we keep the same investment function for $g \leq y^o$ where $y_R(g) \geq g$ and then set the investment function equal to g when $y_R(g) < g$. This gives us the modified investment function $\widetilde{y}_R(g)$, represented by the green solid line. This

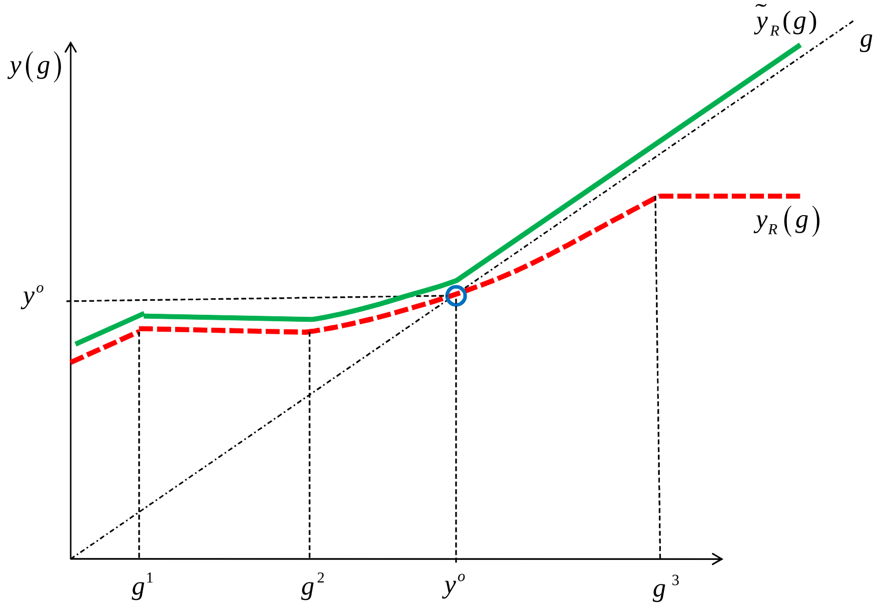

Figure 3: The irreversibility constraint and the reversible equilibrium.

investment function induces essentially the same allocation: the same steady state y^o and the same convergent path for any initial $g_0 \leq y^o$. Unfortunately, $\widetilde{y}_R(g)$ is no longer an equilibrium. On the left of y^o the objective function, $u(y) - y + \delta v_{IR}(y)$, is flat. On the right of y^o, the objective function would remain flat if the investment were the red dashed line as with reversibility; with irreversibility, however, the constraint $y \geq g$ forces the investment to increase at a faster rate than $y_R(g)$. Because $y_R(g)$ is ex ante suboptimal, the "forced" increase in investment makes the objective function increase on the right of y^o. But then choosing y^o would no longer be optimal in state y^o, so it cannot be a steady state.[20]

Does an equilibrium exist? What does it look like? Let $\widehat{y}(g)$ be the unique solution of (14) that is tangent to the line $y = (1-d)g$ (see Figure 4 for an example). As it can be easily verified from (14), the point at which $\widehat{y}(g)$ is tangent to $y = (1-d)g$ is $\overline{y}(\delta, d)$.[21] Define $y^o_{IR}(\delta, d, n)$ as

[20] This problem does not arise with the planner's solution because the planner's solution is time consistent. After the planner's steady state y^*_P is reached the planner would keep g at y^*_P. If the planner's is forced to increase y on the right of y^*_P, we would have a kink at y^*_P, but it would be a "downward" kink. Such a kink makes the objective function fall at a faster rate on the right of the steady state, so it preserves concavity and it does not disturb the optimal solution. The kink is "upward" in the equilibrium with irreversibility because the steady state is not optimal, so the irreversibility constraint, $y \geq g$, increases expected welfare. This creates a sort of "commitment device" for the future; the agents know that g can not be reduced by the others (or their future selves).

[21] Formally, $\widehat{y}(g)$ is the solution of (14) with the initial condition $\widehat{y}(\overline{y}(\delta, d)) = (1-d)\overline{y}(\delta, d)$.

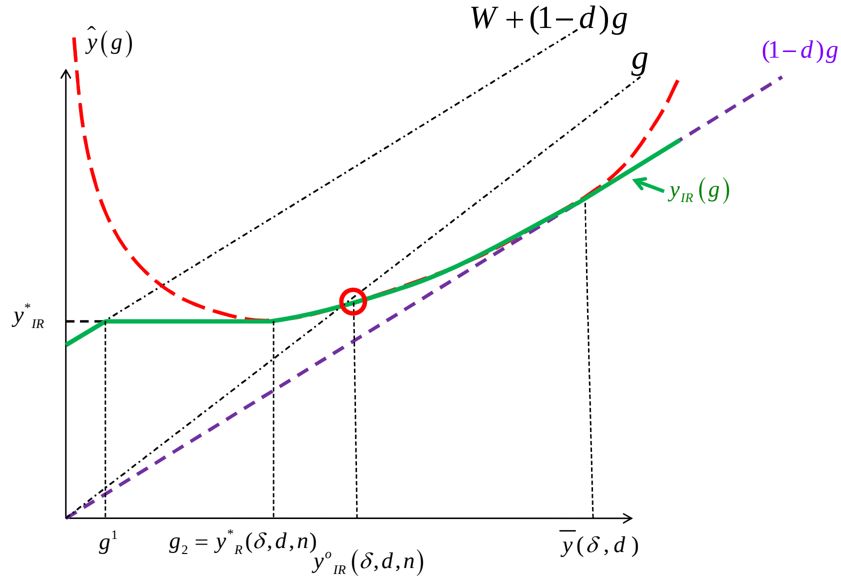

Figure 4: The irreversible equilibrium as $d \to 0$.

the fixed point of this function:[22]

$$\widehat{y}(y^o_{IR}(\delta,d,n)) = y^o_{IR}(\delta,d,n). \tag{18}$$

The following Proposition states the existence result. In the appendix we provide a detailed description of the equilibrium strategies.

Proposition 2. *In any IIE there is a monotonic equilibrium with an investment function as illustrated in Figure 4 and steady state $y^o_{IR}(\delta,d,n)$ as defined in (18). This equilibrium is weakly concave.*

Proposition 4 establishes that the dynamic free rider game with irreversibility admits an equilibrium with standard concavity properties. Figure 4 shows the investment function associated with the equilibrium. In equilibrium the investment function merges smoothly with the irreversibility constraint: at the point of the merger (i.e. $\overline{y}(\delta,d)$, where the constraint becomes binding), the investment function has slope $1-d$. This feature is essential to avoid the problems discussed above.

Proposition 4 does not establish that the there is a unique equilibrium steady state. The following result establishes bounds for the set of stable steady states in a IIE and shows that when depreciation is not too high all stable steady states must be close precisely to $y^o_{IR}(\delta,d,n)$:

[22] Note that $y^o_{IR}(\delta,d,n)$ is a function of δ, d and n since $\widehat{y}(g)$ depends on these variables.

Proposition 3. *There is lower bound $y^*_{IR}(\delta, d, n) \geq y^*_R(\delta, d, n)$ such that y_{IR} is a stable steady state of a monotonic equilibrium only if $y_{IR} \in [y^*_{IR}(\delta, d, n), y^{**}_R(\delta, d, n)]$. Moreover, as $d \to 0$ $y^*_{IR}(\delta, d, n)$, $y^{**}_R(\delta, d, n)$* and *$y^o_{IR}(\delta, d, n)$ all converge to $[u']^{-1}(1-\delta)$.*

There is an intuitive explanation for Proposition 5. Because of decreasing returns, the investment in g declines over time, so the constraint $y \geq (1-d)g$ must be binding when g is high enough. When this happens the agents are forced to keep the investment higher than what they would like. Since the equilibrium is inefficiently low (because the agents do not fully internalize the social benefit of g), the constraint $y \geq (1-d)g$ increases expected welfare in these states. The states where the constraint $y \geq (1-d)g$ is binding are typically out of equilibrium (that is on the right of the steady state): in the equilibrium illustrated in Figure 4, for example, the constraint is binding for $g > y^o_{IR}(\delta, d, n)$. The irreversibility constraint, however, has a ripple effect on the entire investment function. In a left neighborhood of $\overline{y}(\delta, d)$, the constraint is not binding; still, the agents expect that the other agents will preserve their investment, so the strategic substitutability will not be too strong. Steady states lower than $y^o_{IR}(\delta, d)$ can occur with reversibility because the agents expect high levels of "strategic substitutability." Proposition 5 shows that when d is sufficiently low, the irreversibility constraint makes these expectations impossible in equilibrium, inducing an equilibrium steady state close to the maximal steady state of the reversible case, $y^{**}_R(\delta, d, n)$. Thus, as $d \to 0$, there is a unique equilibrium steady state in the irreversible case, i.e., $y^o_{IR} = y^{**}_R(\delta, d, n)$.

An immediate implication of Propositions 4 and 5 is the following result:

Corollary 2. *As $\delta \to 1$ the highest stable steady state in a IIE converges to the efficient level. As $\delta \to 1$ and $d \to 0$, every stable steady state in a IIE converges to the efficient level.*

Results proving the efficiency of the best steady state in monotone games as $\delta \to 1$ have been previously presented in the literature by Lockwood and Thomas [2002] and more recently by Matthews [2011]. We have already explained in Section 1.1. that these results do not apply to our environment because they rely on assumptions that are not verified in our environment. We emphasize here three additional novel aspects of Corollary 2. First, the result shows that the community can achieve efficiency using a very simple equilibrium (Markov) that requires minimal coordination among the players (in previous results the efficient steady states are supported by subgame perfect equilibria where behavior depends on the entire history of the game). Second, we do not need $d = 0$ to have the result: when d is small, all equilibria must be approximately efficient. Finally, here irreversibility is not necessary for efficiency, the same equilibrium exists in a RIE: irreversibility only guarantees its uniqueness as $d \to 0$.

Propositions 1 and 2 show that both with reversibility and with irreversibility the agents' contributions are gradual and aggregate investment is inefficiently slow: indeed, as it can immediately be seen from the equilibria represented in Figures 2 and 4, the steady state is typically reached only in the limit.[23] This is a purely strategic phenomenon since, as we have discussed in Section 3, a planner never finds gradualism in investment optimal. Previous to our work, a number of authors have highlighted how gradualism is a necessary feature of contribution games with irreversibility and no depreciation (see, in particular, Lockwood and Thomas [2002] and Matthews [2011]). We are, however, not aware of previous results that have shown that gradualism is a feature of Markov equilibria in irreversible economies with depreciation, or in economies with reversibility.

6 Conclusions

In this paper we have studied a simple model of free riding in which n infinitely lived agents choose between private consumption and contributions to a durable public good. We have considered two possible cases: economies with reversible investments, in which in every period individual investments can either be positive or negative; and economies with irreversible investments, in which the public good can only be reduced by depreciation. For both cases we have characterized the set of steady states that can be supported by symmetric Markov equilibria in continuous strategies.

We have highlighted three main results. First, we have shown that economies with reversible investments have typically a continuum of equilibria. In the best equilibrium the steady state is higher in a community with n agents than in autarky, and it is increasing in n; in the worst equilibrium, the steady state is lower in autarky, and it decreases in n. While in a static free rider's problem the players' contributions are strategic substitutes, in a dynamic model they may be strategic complements. Second, we have shown that in economies with irreversible investments, the set of equilibrium steady states is much smaller: indeed, as depreciation converges to zero, the set of equilibrium steady states converges to the best equilibrium that can be reached in economies with reversible investments. Irreversibility, therefore, helps the agents removing the coordination problem that plagues most of the equilibria in the reversible case, and so it necessarily induces higher investment. Third, as agents become increasingly patient, the best steady state in both economies with reversibility and irreversibility converges to the efficient level. As patience increases and depreciation decreases, all equilibrium steady states in an irreversible economy converge to the efficient level.

Although in this paper we have focused on a free rider problem in which agents act independently and there is no institution to coordinate their actions, the approach we have developed to

[23] As it can be formally proven, of the equilibria constructed in Proposition 1, the steady state is reached in finite time on in the equilibrium correspondent to the minimal steady state, $y^*_R(\delta, d, n)$.

characterize the Markov equilibria has a wider applicability and can be used to study dynamic games in other environments as well. In future work, it would be interesting to investigate economies with irreversible investments when public decisions are taken by legislative bargaining or other types of centralized political processes.

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Appendix 1: Proofs

6.1 Proof of Proposition 1

Let $y_R^*(\delta, d, n)$ and $y_R^{**}(\delta, d, n)$ be defined by (15). Since we are in a regular economy, we have $W/d > y_R^{**}(\delta, d, n)$. We first prove here that for any $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$, there is Markov equilibrium with steady state equal to y^o. Each y^o is supported by a concave equilibrium with investment function $y_R(g\,|y^o\,)$ described by (10), where

$$g^2 = \max\left\{\min_{g\geq 0}\left\{g\,|y\left(g\,|y^o\,\right) \leq W + (1-d)g\,\right\}, y_R^*(\delta, d, n)\right\}, \tag{19}$$

g_3 is defined by $y\left(g^3\,|y^o\,\right) = y_R^{**}(d, n)$, and $y(g) = y\left(g\,|y^o\,\right)$ is the the unique solution of (14) with initial condition $y\left(y_R^o\,|y_R^o\,\right) = y^o$. This proves the "sufficiency" part of the statement. Then we prove that the steady state must be in $[y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$. This proves the "necessity" part of the statement.

6.1.1 Sufficiency

To construct the equilibrium we proceed in 3 steps.

Step 1. We first construct the strategies for a generic y^o and prove their key properties. Let $y\left(g\,|y^o\,\right)$ be the solution of the differential equation when we require the initial condition: $y\left(y^o\,|y^o\,\right) = y^o$, for $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$. Let $g^2(y)$ be defined by (19). This, essentially, is the largest point between the point at which $y\left(g\,|y^o\,\right)$ crosses from below $W + (1-d)g$, and $y_R^*(\delta, d, n)$. Let $g^3(y^o)$ be defined by $y\left(g^3(y^o)\,|y^o\,\right) = y_R^{**}(\delta, d, n)$.

Lemma A.1. *$y'\left(g\,|y^o\,\right) \in (0, 1)$ in $\left[g^2(y^o), y_R^{**}(\delta, d, n)\right]$ and $y''\left(g\,|y^o\,\right) \geq 0$.*

Proof. From (14), $y'\left(g\,|y^o\,\right) \geq 0$ for $g \geq y_R^*(\delta, d, n)$, *and* $y'\left(g\,|y^o\,\right) \leq 1$ for $g \leq y_R^{**}(\delta, d, n)$. Since $y''\left(g\,|y^o\,\right) = \frac{n}{1-n}\left[\frac{u''(g)}{\delta}\right]$, $y''(g) > 0$. ■

Lemma A.2. *For any $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$, $g^3(y^o) \geq y_R^{**}(\delta, d, n)$.*

Proof. Note that $y\left(y_R^{**}(\delta, d, n)\,|y^o\,\right)$ is increasing in y^o. Moreover $y\left(y_R^{**}(\delta, d, n)\,|y_R^{**}(\delta, d, n)\,\right) = y_R^{**}(\delta, d, n)$. So $y\left(y_R^{**}(\delta, d, n)\,|y^o\,\right) < y_R^{**}(\delta, d, n)$ for $y^o < y_R^{**}(\delta, d, n)$. It follows that $g^3(y^o) \geq y_R^{**}(\delta, d, n)$ for any $y^o \leq y_R^{**}(\delta, d, n)$. ■

We have:

Lemma A.3. *$y\left(g\,|y^o\,\right) \in (0, W + (1-d)g)$ in $\left(g^2(y^o), g^3(y^o)\right)$.*

Proof. First note that $y\left(g^2(y^o)|y^o\right) \leq W + (1-d)g^2(y^o)$. Since $y'\left(g\,|y^o\,\right) < 1$ for $g < y_R^{**}(\delta, d, n)$ we must have $y\left(g\,|y^o\,\right) < W + (1-d)g$ for $g \in \left(g^2(y^o), y_R^{**}(\delta, d, n)\right)$. For $g > y_R^{**}(\delta, d, n)$, we have $W + (1-d)g > W + (1-d)y_R^{**}(\delta, d, n)$. Since $y\left(g\,|y^o\,\right) < y_R^{**}(\delta, d, n)$ in $\left(g^2(y^o), g^3(y^o)\right)$, We have

$y\left(g\left|y^o\right.\right) < y_R^{**}(\delta, d, n) < W + (1-d)y_R^{**}(\delta, d, n) < W + (1-d)g$ in $\left[y_R^{**}(\delta, d, n), g^3(y^o)\right)$ as well. Similarly, since $y'\left(g\left|y^o\right.\right) \geq 0$ for $g > g^2(y^o)$ and $y\left(g^2(y^o)|y^o\right) \geq 0$, we must have $y\left(g\left|y^o\right.\right) > 0$ for $g > g^2(y^o)$. Note that $y\left(g^2(y^o)\left|y^o\right.\right) \geq 0$ since $y'\left(g\left|y^o\right.\right) \in (0, 1-d)$ in $[y_R^*(\delta, d, n), y^o]$ implies that $y\left(g\left|y^o\right.\right) > g$ for all $g \in [y_R^*(\delta, d, n), y^o]$. ■

For any $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$, we now define the investment function:

$$y_R(g\left|y^o\right.) = \begin{cases} \min\left\{W + (1-d)g, y\left(g^2(y^o)\left|y^o\right.\right)\right\} & g \leq g^2(y^o) \\ y\left(g\left|y^o\right.\right) & g^2(y^o) < g \leq g^3(y^o) \\ y_R^{**}(\delta, d, n) & g \geq g^3(y^o) \end{cases}$$

For future reference, define $g^1(y^o) = \max\left\{0, \left(y\left(g^2(y)\left|y^o\right.\right) - W\right)/(1-d)\right\}$. This is the point at which $W + (1-d)g^2(y^o) = y\left(g^2(y^o)\left|y^o\right.\right)$, if positive. Clearly, we have $g^1(y^o) \in \left[0, g^2(y^o)\right]$. We have:

Lemma A.4. *For any* $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$, $y\left(g\left|y^o\right.\right) \in \left[g^2(y^o), g^3(y^o)\right]$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$.

Proof. Since $y\left(g\left|y^o\right.\right)$ it is monotonic non-decreasing in $g \in \left[g^2(y^o), g^3(y^o)\right]$,

$$y\left(g\left|y^o\right.\right) \in \left[y\left(g^2(y^o)\left|y^o\right.\right), y\left(g^3(y^o)\left|y^o\right.\right)\right] \quad \forall g \in \left[g^2(y^o), g^3(y^o)\right].$$

Since $y\left(g\left|y^o\right.\right)$ has slope lower than one in $\left[g^2(y^o), g^3(y^o)\right]$ and $y\left(y^o\left|y^o\right.\right) = y^o$ for $y^o \geq g^2(y^o)$, we must have $y\left(g^2(y^o)\left|y^o\right.\right) \geq g^2(y^o)$, so $y\left(g\left|y^o\right.\right) \geq g^2(y^o)$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$. Similarly, $y\left(g^3(y^o)\left|y^o\right.\right) \leq g^3(y^o)$, so $y\left(g\left|y^o\right.\right) \leq g^3(y^o)$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$. ■

Step 2. We now construct the value functions corresponding to each steady state $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$. For $g \in \left[g^2(y^o), g^3(y^o)\right]$ define the value function recursively as

$$v(g|y^o) = \frac{W + (1-d)g - y\left(g\left|y^o\right.\right)}{n} + u(y\left(g\left|y^o\right.\right)) + \delta v(y\left(g\left|y^o\right.\right)). \tag{20}$$

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (20) is a contraction: it defines a unique, continuous and differentiable value function $v_0(g\left|y^o\right.)$ for this interval of g. (differentiability follows from the differentiability of $y\left(g\left|y^o\right.\right)$). We have

Lemma A.5. For any $y^o \in [y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$ and any $g \in \left[g^2(y^o), g^3(y^o)\right]$, $u'(g) + \delta v_0'(g; y^o) = 1$.

Proof. Note that by Lemma A.4, for $g \in \left[g^2(y^o), g^3(y^o)\right]$, we have $y\left(g\left|y^o\right.\right) \in \left[g^2(y^o), g^3(y^o)\right]$. From (14) we can write (for simplicity we write $y'(g\left|y^o\right.) = y'(g)$):

$$\frac{1 - u'(g)}{\delta} = \frac{1 - y'(g)}{n} + u'(y(g))y'(g) + \left[1 - u'(y(g))\right]y'(g)$$

for any $g \in \left[g^2(y^o), g^3(y^o)\right]$. But then using (14) again allows to substitute $1-u'(y(g))$ to obtain:

$$\begin{aligned} \frac{1-u'(g)}{\delta} &= \frac{1-y'(g)}{n} + u'(y(g))y'(g) \\ &+\delta\left[\frac{1-y'(y(g))}{n} + u'(y^2(g))y'((y(g)) + \left[1-u'(y^2(g))\right]y'(y(g))\right]y'(g) \end{aligned}$$

where $y^0(g) = g$, $y^1(g) = y(g)$, $y^m(g) = y(y^{m-1}(g))$, and $[y']^0(g) = 1$, $[y']^1(g) = y'(g)$, and $[y']^m(g) = y'([y']^{m-1}(g))$. Iterating we have:

$$\begin{aligned} \frac{1-u'(g)}{\delta} &= \lim_{n\to\infty}\sum_{j=0}^{n}\delta^j\left[\frac{1-y'(y^j(g\,|y^o\,)\,|y^o\,)}{n} + u'(y^{j+1}(g))y'(y^j(g\,|y^o\,)\,|y^o\,)\right]\prod_{i=0}^{j}[y']^i\,(y^{i-1}(g)) \\ &= v'(g\,|y^o\,) \end{aligned}$$

This implies $u'(g) + \delta v_0'(g; y^o) = 1$. ■

In the rest of the state space we define the value function recursively. In $\left[g^1(y^o), g^2(y^o)\right]$, if $g^1(y^o) < g^2(y^o)$, the value function is defined as:

$$v_0(g\,|y^o\,) = \frac{W + (1-d)g - y\left(g^2(y^o)\,|y^o\,\right)}{n} + u(y\left(g^2(y^o)\,|y^o\,\right)) + \delta v_0(y\left(g^2(y^o)\,|y^o\,\right)) \tag{21}$$

for $y\left(g^2(y^o)\,|y^o\,\right) \in \left[g^2(y^o), g^3(y^o)\right]$.

Lemma A.6. *For any $g \in \left[g^1(y^o), g^3(y^o)\right]$, $u(g) + \delta v(g\,|y^o\,)$ is concave and has slope larger or equal than 1.*

Proof. If $g^1(y^o) = g^2(y^o)$, the result follows from the previous lemma. Assume therefore, $g^1(y^o) < g^2(y^o)$. In this case $g^2(y^o) = y_R^*(\delta, d, n)$. For any $g \in \left[g^1(y^o), g^2(y^o)\right]$, $y(g; y^o) = y\left(y_R^*(\delta, d, n)\,|y^o\,\right)$. So we have $v_0'(g\,|y^o\,) = (1-d)/n$ implying: $u'(g) + \delta v_0'(g\,|y^o\,) = u'(g) + \delta(1-d)/n > 1$ since $g \leq g^2(y^o) = y_R^*(\delta, d, n)$. The statement then follows from this fact and Lemma A.5. ■

Consider $g < g^1(y^o)$. In $\left[g_{-1}, g^1(y^o)\right]$ the value function is defined as:

$$v_{-1}(g\,|y^o\,) = u(W + (1-d)g) + \delta v_0(W + (1-d)g\,|y^o\,)$$

where $g_{-1} = \max\left\{0, \frac{g^1(y^o)-W}{1-d}\right\}$. Assume that we have defined the value function in $g \in \left[g_{-t}, g_{-(t-1)}\right]$ as v_{-t}, for all t such that $g_{-(t-1)} > 0$. Then we can define $v_{-(t+1)}$ as:

$$v_{-(t+1)}(g\,|y^o\,) = u(W + (1-d)g) + \delta v_{-t}(W + (1-d)g\,|y^o\,),$$

in $\left[g_{-(t+1)}, g_{-t}\right]$ with $g_{-(t+1)} = \frac{g_{-t}-W}{1-d}$.

Lemma A.7. *For any $g \in [0, g^3(y^o)]$, $u(g) + \delta v(g\,|y^o\,)$ is concave and it has slope greater than or equal than 1.*

Proof. We prove this by induction on t. Consider now the interval $\left[\frac{g^1(y^o)-W}{1-d}, g^1(y^o)\right]$. In this range we have

$$v'_{-1}(g\,|y^o\,) = \left[u'(W + (1-d)g) + \delta v'_0(W + (1-d)g\,|y^o\,)\right](1-d) \geq 1-d$$

since $W + (1-d)g \in \left[g^1(y^o), g^3(y^o)\right]$. It follows that for $g \in \left[\frac{g^1(y^o)-W}{1-d}, g^1(y^o)\right]$:

$$u'(g) + \delta v'_{-1}(g\,|y^o\,) \geq u'(g) + \delta(1-d) > 1 \tag{22}$$

Where the last inequality follows from the fact that $g \leq g^2(y^o) < y_R^{**}(\delta, d, n)$. Note, moreover, that the right and left derivative of $v(g\,|y^o\,)$ at $g^1(y^o)$ are the same. To see this note that by the argument above, the left derivative is $(1-d)/n$; by Lemma A.5, however, the right derivative is $(1 - u'(y_R^*(\delta, d, n)))\,/\delta = (1-d)/n$ as well. We conclude that $u'(g)+\delta v'_{-1}(g\,|y^o\,)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in \left[g_{-t}, g^3(y^o)\right]$, $u(g)+\delta v_{-t}(g\,|y^o\,)$ is concave and $u'(g) + \delta v'_{-t}(g\,|y^o\,) > 1$. Consider in $g \in \left[g_{-(t+1)}, g_{-t}\right]$. We have:

$$v'_{-(t+1)}(g\,|y^o\,) = \left[u'(W + (1-d)g) + \delta v'_{-t}(W + (1-d)g\,|y^o\,)\right](1-d) \geq 1-d$$

since $W + (1-d)g \geq \left[g_{-t}, g^3(y^o)\right]$. So $u'(g) + \delta v'_{-(t+1)}(g\,|y^o\,) \geq u'(g) + \delta(1-d) \geq 1$. By the same argument as above, moreover, v is concave at g_{-t}. We conclude that for any $g \leq g^1$, $u(g) + \delta v(g\,|y^o\,)$ is concave and it has slope larger than 1. ■

We can define the value function for $g \geq g^3(y^o)$ as:

$$v_1(g\,|y^o\,) = \frac{W + (1-d)g - y_R^{**}(\delta, d, n)}{n} + u(y_R^{**}(\delta, d, n)) + \delta v_0(y_R^{**}(\delta, d, n)\,|y^o\,)$$

since, by Lemma A.2, $g^3(y^o) \geq y_R^{**}(\delta, d, n)$.

Lemma A.8. *For any $g \geq 0$, $u(g) + \delta v(g\,|y^o\,)$ is concave and it has slope less than or equal than 1.*

Proof. For $g > g^3(y^o)$, $v'(g\,|y^o\,) = (1-d)\,/n$. Since, by Lemma A.2, $g \geq y_R^{**}(\delta, d, n) \geq y_R^*(\delta, d, n)$, we have $u'(g) + \delta v'(g\,|y^o\,) < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal than 1 for $g \leq g^3(y^o)$. This establishes the result. ■

Step 3. Define

$$x(g\,|y^o\,) = \frac{W + (1-d)g - y(g\,|y^o\,)}{n}, \text{ and } i(g\,|y^o\,) = \frac{y(g\,|y^o\,) - (1-d)g}{n}$$

as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g\,|y^o\,) \in [0, W/n]$. We now establish that $y(g\,|y^o\,)$, $x(g\,|y^o\,)$ and the associated value function $v(g\,|y^o)$ defined in the previous steps constitute an equilibrium. We first show that given $y(g\,|y^o)$, $v(g\,|y^o)$ describes the expected continuation value to an agent, starting at state g. Since $y(g\,|y^o) \in \left[g^2(y^o), g^3(y^o)\right]$ for $g \in \left[g^2(y^o), g^3(y^o)\right]$, $v(g\,|y^o)$ must be described by (20) for $g \in \left[g^2(y^o), g^3(y^o)\right]$. By construction, moreover, $v(g\,|y^o)$ is the expected continuation value to an agent in all states $g \geq g^3(y^o)$, and $g \leq g^2(y^o)$. We now show that $y(g\,|y^o)$ is an optimal reaction function given $v(g\,|y^o)$. An agent solves the problem (8), where $y_R(g) = y(g\,|y^o)$. Note that $y(g\,|y^o)$ satisfies the constraints of this problem if $y(g\,|y^o) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g\,|y^o)$, so if $y(g\,|y^o) \leq W + (1-d)g$; and if $y(g\,|y^o) \geq \frac{n-1}{n}y(g\,|y^o)$, so if $y(g\,|y^o) \geq 0$. Both conditions are automatically satisfied by construction. If $g < g^1(y^o)$, we have $u'(y) + \delta v'(y) \geq 1$ for all $y \in [0, W + (1-d)g]$, so $y(g\,|y^o) = W + (1-d)g$ is optimal. If $g \geq g^1(y^o)$, then $y(g\,|y^o)$ is an unconstrained optimum, so again it is an optimal reaction function.

6.1.2 Necessity

We now prove that any stable steady state of an equilibrium must be in $[y_R^*(\delta, d, n), y_R^{**}(\delta, d, n)]$. We proceed in two steps.

Step 1. We first prove that $y_R^o \leq y_R^{**}(\delta, d, n)$. Suppose to the contrary that there is stable steady state at $y_R^o > y_R^{**}(\delta, d, n)$. We must have $y_R^o \in (y_R^{**}(\delta, d, n), W/d]$, since it is not feasible for a steady state to be larger than W/d. Consider a left neighborhood of y_R^o, $N_\varepsilon(y_R^o) = (y_R^o - \varepsilon, y_R^o)$. The value function can be written in $g \in N_\varepsilon(y_R^o)$ as:

$$\begin{aligned} v_R(g) &= \frac{W + (1-d)g - y_R(g)}{n} + u(y_R(g)) + \delta v_R(y_R(g)) \\ &= u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g) + \frac{W + (1-d)g}{n} + (1 - 1/n)\, y_R(g) \end{aligned}$$

In $N_\varepsilon(y_R^o)$ the constraint $y \geq \frac{n-1}{n} y_R(g)$ cannot be binding, else we would have $y_R(g) = (1 - 1/n)y_R(g)$, so $y_R(g) = 0$: but this is not possible in a neighborhood of $y_R^o > 0$. We consider two cases.

Case 1. Suppose first that $y_R^o < W/d$. We must therefore have that $y_R(g) < W + (1-d)g$ in $N_\varepsilon(y_R^o)$, so the constraint $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y$ is not binding. The solution is in the interior of the constraint set of (8), and the objective function $u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g)$ is constant for $g \in N_\varepsilon(y_R^o)$.

Lemma A.9. *There is a neighborhood $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is strictly increasing.*

Proof. Suppose to the contrary that, for any $N_\varepsilon(y_R^o)$, there is an interval in $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is constant. Using the expression for $v_R(g)$ presented above, we must have $v_R'(g) = (1-d)/n$ for any g in this interval. Since $N_\varepsilon(y_R^o)$ is arbitrary, then we must have a sequence $g^m \to y_R^o$ such that $v'_R(g^m) = (1-d)/n \ \forall m$. We can therefore write:

$$\begin{aligned} v_R^-(y_R^o) &= \lim_{\Delta\to 0}\frac{v_R(y_R^o)-v_R(y_R^o-\Delta)}{\Delta} = \lim_{\Delta\to 0}\lim_{m\to\infty}\frac{v_R(g^m)-v_R(g^m-\Delta)}{\Delta} \\ &= \lim_{m\to\infty}\lim_{\Delta\to 0}\frac{v_R(g^m)-v_R(g^m-\Delta)}{\Delta} = \frac{1-d}{n} \end{aligned}$$

where $v_R^-(y_R^o)$ is the left derivative of $v_R(g)$ at y_R^o, and the second equality follows from the continuity of $v_R(g)$. Consider now a marginal reduction of g at y_R^o. The change in utility is (as $\Delta \to 0$):

$$\begin{aligned} \Delta U\left(y_R^o\right) &= u(y_R^o-\Delta)-u(y_R^o)+\delta\left[v_R(y_R^o-\Delta)-v_R(y_R^o)\right]+\Delta \\ &= \left[1-\left(u'(y_R^o)+\delta\frac{1-d}{n}\right)\right]\Delta \end{aligned}$$

In order to have $\Delta U\left(y_R^o\right) \le 0$, we must have $u'(y_R^o)+\delta(1-d)/n \ge 1$. This implies $y_R^o \le y_R^*(\delta,d,n) < y_R^{**}(\delta,d,n)$, a contradiction. Therefore, if there is stable steady state at $y_R^o > y_R^{**}(\delta,d,n)$, then $y_R(g)$ is strictly increasing in a neighborhood $N_\varepsilon(y_R^o)$. ■

Lemma A.9 implies that there is a neighborhood $N_\varepsilon(y_R^o)$ in which $u(g)+\delta v_R(g)-g$ is constant. Since y_R^o is a stable steady state and $y_R(g)$ is strictly increasing. Moreover, for any open left neighborhood $N_{\varepsilon'}(y_R^o)=(y_R^o-\varepsilon', y_R^o)\subset N_\varepsilon(y_R^o)$, $g\in N_{\varepsilon'}(y_R^o)$ implies $y_R(g)\in N_{\varepsilon'}(y_R^o))$. These observations imply:

Lemma A.10. *There is a neighborhood $N_\varepsilon(y_R^o)$ in which*

$$y'_R(g) = \frac{n}{n-1}\left(\frac{1-u'(g)}{\delta}-\frac{1-d}{n}\right) \tag{23}$$

Proof. There is a $N_\varepsilon(y_R^o)$ and a constant K such that $\delta v_R\left(g\right) = K+g-u(g)$ for $g\in N_\varepsilon(y_R^o))$. Hence $v_R\left(g\right)$ is differentiable in $N_\varepsilon(y_R^o)$. Moreover, $y_R(g)\in N_\varepsilon(y_R^o)$ for all $g\in N_\varepsilon(y_R^o)$. Hence $u(y_R(g))+\delta v\left(y_R(g)\right)-y_R(g)$ is constant in $g\in N_\varepsilon(y_R^o)$ as well. These observations and the definition of $v_R(g)$ imply that $v'_R(g) = \frac{1-d}{n}+\left(1-\frac{1}{n}\right)y'_R(g)$ in $N_\varepsilon(y_R^o)$ (where $y_R(g)$ must be differentiable otherwise $v_R(g)$ would not be differentiable). Given that $u'(g)+\delta v'_R(g) = 1$ in $g\in N_\varepsilon(y_R^o)$, we must have:

$$u'(g)+\delta v'_R(g) = u'(g)+\delta\left[\frac{1-d}{n}+\left(1-\frac{1}{n}\right)y'_R(g)\right] = 1$$

which implies (23) for any $g\in N_\varepsilon(y_R^o)$. ■

Let g^m be a sequence in $N_\varepsilon(y_R^o)$ such that $g^m\to y_R^o$. We must have

$$\begin{aligned} y_R^-(y_R^o) &= \lim_{\Delta\to 0}\frac{y_R(y_R^o)-y_R(y_R^o-\Delta)}{\Delta} = \lim_{\Delta\to 0}\lim_{m\to\infty}\frac{y_R(g^m)-y_R(g^m-\Delta)}{\Delta} \\ &= \lim_{m\to\infty}\lim_{\Delta\to 0}\frac{y_R(g^m)-y_R(g^m-\Delta)}{\Delta} = \frac{n}{n-1}\left(\frac{1-u'(y_R^o)}{\delta}-\frac{1-d}{n}\right) \end{aligned}$$

where $y_R^-(y_R^o)$ is the left derivative of $y_R(y_R^o)$, and the second equality follows from continuity. Consider a state $(y_R^o - \Delta)$. For y_R^o to be stable we need that for any small Δ:

$$y_R(y_R^o - \Delta) \geq y_R^o - \Delta = y_R(y_R^o) + (y_R^o - \Delta) - y_R^o$$

where the equality follows from the fact that $y_R(y_R^o) = y_R^o$. As $\Delta \to 0$, this implies $y_R^-(y_R^o) \leq 1$ in $N_\varepsilon(y_R^o)$. By (24), we must therefore have:

$$\frac{n}{n-1}\left(\frac{1-u'(y_R^o)}{\delta} - \frac{1-d}{n}\right) \leq 1$$

This implies: $y_R^o \leq y_R^{**}(\delta, d, n)$, a contradiction.

Case 2. Assume now that $y_R^o = W/d$ and it is a strict local maximum of the objective function $u(y) + \delta v_R(y) - y$. In this case in a left neighborhood $N_\varepsilon(y_R^o)$, we have that the upperbound $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_R(g)$ is binding: implying $y_R(g) = W+(1-d)g$ in $N_\varepsilon(y_R^o)$. We must therefore have a sequence of points $g^m \to y_R^o$ such that $g^m = y_R(g^{m-1})$ and $y_R(g^m) = W + (1-d)g^m$ $\forall m$. Given this, we can write:

$$\begin{aligned} v_R(g^m) &= u(g^{m+1}) + \delta v_R(g^{m+1}) = u(g^{m+1}) + \delta\left[u(g^{m+2}) + \delta v_R(g^{m+2})\right] \\ &= \sum\nolimits_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j}) \end{aligned}$$

note that since $g^{m+1} = W + (1-d)g^m$, the derivative of g^{m+1} with respect to g^m is $\left[g^{m+1}\right]' = (1-d)$. By an inductive argument, it is easy to see that $\left[g^{m+j}\right]' = (1-d)^j$. So $v_R(g^m)$ is differentiable and:

$$\delta v_R'(g^m) = \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j}).$$

Since $u'(g^m) + \delta v_R'(g^m) \geq 1$, we have:

$$u'(g^m) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j}) \geq 1$$

for all m. Consider the limit as $m \to \infty$. Since $u'(g)$ is continuous and $g^m \to y_R^o$, we have:

$$\begin{aligned} 1 &\leq \lim_{m\to\infty}\left[u'(g^m) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j})\right] \\ &= u'(y_R^o) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(y_R^o) = \frac{u'(y_R^o)}{1-\delta(1-d)} \end{aligned}$$

This implies $y_R^o \leq \left[u'\right]^{-1}(1-\delta(1-d)) < y_R^{**}(\delta, d, n)$, a contradiction.

Case 3. Assume now that $y_R^o = W/d$, but it is not a strict maximum of $u(y) + \delta v_R(y) - y$ in any left neighborhood. It must be that $u(y) + \delta v_R(y) - y$ is constant in some left neighborhood $N_\varepsilon(y_R^o)$. If this were not the case, then in any left neighborhood we would have an interval in which $y_R(g)$ is constant, but this is impossible by Lemma A.9. But then if $u(y) + \delta v_R(y) - y$ is constant in some $N_\varepsilon(y_R^o)$, the same argument as in Case 1 of Step 1 implies a contradiction.

Step 2. We now prove that $y_R^o \geq y_R^*(\delta, d, n)$. Assume there is stable steady state at $y_R^o < y_R^*(\delta, d, n)$. Since $\lim_{g \to 0} u'(g) = \infty$, $y_R^o > 0$. There is therefore a neighborhood $N_\varepsilon(y_R^o) = (y_R^o, y_R^o + \varepsilon)$ in which $y_R(g)$ satisfies all the constraints of (8) and it maximizes $u(y) + \delta v_R(y) - y$ We conclude that the objective function $u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g)$ is constant in $N_\varepsilon(y_R^o)$. By the same argument as in Lemma A.9 it follows that there is a neighborhood $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is strictly increasing. Since y_R^o is a stable steady state and $y_R(g)$ is strictly increasing in $N_\varepsilon(y_R^o)$, there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o such that for any open right neighborhood $N_{\varepsilon'}(y_R^o) = (y_R^o, y_R^o + \varepsilon') \subset N_\varepsilon(y_R^o)$, $g \in N_{\varepsilon'}(y_R^o))$ implies $y_R(g) \in N_{\varepsilon'}(y_R^o)$. By the same argument as in Lemma A.10, it follows that there is a $N_{\varepsilon'}(y_R^o)$ in which $y_R'(g)$ is given by (23). Equation (23), however, implies that $y_R'(g) \geq 0$ only for states $g \geq y_R^*(\delta, d, n)$. This implies that $y_R(g)$ is non-monotonic, a contradiction. ■

6.2 Proof of Proposition 2

Since we are in a regular economy, we have $W/d > y_R^{**}(\delta, d, n)$. We construct here a concave and monotonic equilibrium with steady state is $y_{IR}^o(d, n)$ as defined in (18). We proceed in two steps.

Step 1. We first construct the strategies. Remember that $\overline{y}(\delta, d) \equiv y_P^*(\delta, d, 1) = [u']^{-1}(1 - \delta(1-d))$. This is the point at which the solution of the differential equation (14) has slope $(1-d)$. Define g_{IR}^2 as:

$$g_{IR}^2 = \max \left\{ \min_{g \geq 0} \{ g \, | \widehat{y}(g) \leq W + (1-d)g \}, y_R^*(\delta, d, n) \right\}. \tag{24}$$

The investment function is defined as:

$$y_{IR}(g) = \begin{cases} \min \left\{ W + (1-d)g, \widehat{y}\left(g_{IR}^2\right) \right\} & g \leq g_{IR}^2 \\ \widehat{y}(g) & g_{IR}^2 < g \leq \overline{y}(\delta, d) \\ (1-d)g & g \geq \overline{y}(\delta, d) \end{cases}$$

Using the same argument as in the proof of Proposition 1, we can prove that $y_{IR}(g)$ is continuous and almost everywhere differentiable with right and left derivative at any point, and $y_{IR}(g) \in [(1-d)g, W + (1-d)g]$ for any g. Finally, it is easy to see that $y_{IR}(g)$ has a unique fixed-point y_{IR}^o such that $y_{IR}(y_{IR}^o) = y_{IR}^o \in \left[g_{IR}^2, \overline{y}(\delta, d)\right]$.

Step 2. We now construct the value function $v_{IR}(g)$ associated to $y_{IR}(g)$, and prove that $y_{IR}(g), v_{IR}(g)$ is an equilibrium. For $g \leq \overline{y}(\delta, d)$, we define the value function exactly as in

Step 2 of Section 7.1.1. For $g \geq \overline{y}(\delta,d)$, note that $y_{IR}(g) < g$, so we can define the value function recursively as:

$$v_{IR}(g) = \frac{W}{n} + u((1-d)g) + \delta v_{IR}((1-d)g). \tag{25}$$

The value function defined above is continuous in g. Using the same argument as in Step 2 of Section 7.1.1 we can show that $u(g) + \delta v(g; y^o_{IR}) - y$ is weakly concave in g for $g \leq \overline{y}(\delta,d)$; it is strictly increasing in $\left[0, g^2_{IR}\right]$, and flat in $\left[g^2_{IR}, \overline{y}(\delta,d)\right]$. Consider now states $g > \overline{y}(\delta,d)$. Let $g^4 = \frac{\overline{y}(\delta,d)}{1-d}$. In $\left[\overline{y}(\delta,d), g^4\right]$, we must have $(1-d)g \in \left[g^2_{IR}, \overline{y}(\delta,d)\right]$. Note that $u'(g) + \delta v'_{IR}(g) = 1$ for $g \in \left[g^2_{IR}, \overline{y}(\delta,d)\right]$, so by (25) we have

$$v'_{IR}(g) = (1-d)\left[u'((1-d)g) + \delta v'_{IR}((1-d)g)\right] = 1-d$$

for $g \in \left[\overline{y}(\delta,d), g^4\right]$. This fact implies that $u'(g) + \delta v'_{IR}(g) = u'(g) + \delta(1-d)$ for any $g \in \left[\overline{y}(\delta,d), g^4\right]$, and hence it is concave in this interval. It follows that $v_{IR}(g)$ is concave in $g \leq g^4$ because $u'(g) + \delta v'_{IR}(g) \leq 1$ for any $g \in \left[\overline{y}(\delta,d), g^4\right]$. Using a similar approach we can prove that $v_{IR}(g)$ is concave for all g, and we have $u'(g) + \delta v'_{IR}(g) \leq 1$ for $g \geq \overline{y}(\delta,d)$. To prove that $y_{IR}(g), v_{IR}(g)$ is an equilibrium, we proceed exactly as in Step 3 of Section 7.1.1 to establish that $y_{IR}(g)$ is optimal given $v_{IR}(g)$, and that $v_{IR}(g)$ satisfied (17) given $y_{IR}(g)$. ■

6.3 Proof of Proposition 3

We proceed in 2 steps.

Step 1. The same argument used in Step 1 of Section 7.1.2 shows that no equilibrium stable steady state can be greater than $y^{**}_R(\delta,d,n) = [u']^{-1}(1-\delta(1-d/n))$. The same argument used in Step 2 in Section 7.1.2 we can show that no equilibrium can be less than $y^*_R(\delta,d,n)$, so $y^*_{IR}(\delta,d,n) \geq y^*_R(\delta,d,n)$.

Step 2. Consider a sequence $d^m \to 0$. For each d^m there is at least an associated equilibrium $y_m(g)$, $v_m(g)$ with steady state y^o_m. If follows trivially that $\lim_{m\to\infty} y^{**}_R(d^m,n) = [u']^{-1}(1-\delta) = \overline{y}(0)$.

What remains to be shown is that $\lim_{m\to\infty} y^*_{IR}(d^m,n) = [u']^{-1}(1-\delta) = \overline{y}(0)$. Let Γ_m be the set of equilibrium steady states when the rate of depreciation is d^m. We now show by contradiction that for any $\xi > 0$, there is a $\widetilde{m}$ such that for $m > \widetilde{m}$, $\inf_y \Gamma_m \geq \overline{y}(0) - \xi$. Since $\inf_y \Gamma_m \leq y^{**}_R(\delta,d,n)$, this will immediately imply that $y^*_{IR}(\delta,d,n) \to \overline{y}(0)$. Suppose to the contrary there is a sequence of steady states y^0_m, with associated equilibrium investment and value functions $y_m(g)$, $v_m(g)$, and an $\xi > 0$ such that $y^0_m < \overline{y}(0) - \xi$ for any arbitrarily large m. Define $y^0_m(g) = y_m(g)$, and $y^j_m(g) = y_m(y^{j-1}_m(g))$ and consider a marginal deviation from the steady state from y_m^0 to $y_m^0+\Delta$. By the irreversibility constraint we have $y_m(g) \geq (1-d^m)\, g$. Using this property and the fact that y_m^0 is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0+\Delta) - y_m(y_m^0) \geq (1-d^m)\left(y_m^0+\Delta\right) - y_m^0 = (1-d^m)\,\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given Δ:

$$\frac{y_m(y_m^0+\Delta) - y_m^0}{\Delta} \geq 1 + o_1\left(d^m\right)$$

where $o_1\left(d^m\right) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that:

$$\frac{y_m^{j-1}(y_m^0+\Delta) - y_m^0}{\Delta} \geq 1 + o_{j-1}\left(d^m\right)$$

where $o_{j-1}\left(d^m\right) \to 0$ as $m \to 0$. We must have:

$$y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^j(y_m^0) \geq (1-d^m)\, y_m^{j-1}(y_m^0+\Delta) - y_m^0$$

We therefore have:

$$y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0+\Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0+\Delta)$$

so we have:

$$\begin{aligned}\frac{y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^0}{\Delta} &\geq \frac{y_m^{j-1}(y_m^0+\Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta} \\ &\geq 1 + o_j\left(d^m\right) \end{aligned} \tag{26}$$

where $o_j\left(d^m\right) = o_{j-1}\left(d^m\right) - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta}$, so $o_j\left(d^m\right) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0+\Delta$ as:

$$V(y_m^0+\Delta) = \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\frac{W + (1-d^m)\, y_m^{j-1}(y_m^0+\Delta) - y_m^j(y_m^0+\Delta)}{n} + u(y_m^j(y_m^0+\Delta))\right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x+\Delta) - f(x)$. We can write:

$$\begin{aligned}\Delta V(y_m^0)/\Delta &= \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\begin{array}{c}\frac{(1-d^m)\Delta y_m^{j-1}(y_m^0)/\Delta - \Delta y_m^j(y_m^0)/\Delta}{n} \\ +\left[u'(y_m^0) + o\left(\Delta\right)\right]\Delta y_m^j(y_m^0)/\Delta\end{array}\right] \\ &\geq \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\begin{array}{c}\frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} \\ +\left[u'(y_m^0) + o\left(\Delta\right)\right](1+o_j\left(d^m\right))\end{array}\right]\end{aligned} \tag{27}$$

where $o\left(\Delta\right) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose Δ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\left(u(y_m^j(y_m^0+\Delta)) - u(y_m^j(y_m^0))\right) / \left[y_m^j(y_m^0+\Delta) - y_m^j(y_m^0)\right]$$

converges to $u'(y_m^j(y_m^0))$ as $\Delta \to 0$. The inequality in 27 follows from (26). Given Δ, as $m \to \infty$, we therefore have $\lim_{m\to\infty} \Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)+o(\Delta)}{1-\delta}$. We conclude that for any $\varepsilon > 0$, there must be a Δ_ε such that for any $\Delta \in (0, \Delta_\varepsilon)$ there is a m_Δ guaranteeing that $\Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1-\delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent's objective function is:

$$u'(y_m^0) + \delta\Delta V(y_m^0)/\Delta - 1 \geq \frac{u'(y_m^0)}{1-\delta} - \delta\varepsilon - 1$$

for m sufficiently large. A necessary condition for the un-profitability of a deviation from y_m^0 to $y_m^0 + \Delta$ is therefore:

$$y_m^0 \geq [u']^{-1} (1 - \delta + \delta\varepsilon (1 - \delta)). \tag{28}$$

Since ε can be taken to be arbitrarily small, for an arbitrarily large m, (28) implies $y_m^0 \geq \overline{y}(0) - \xi/2$, which contradicts $y_m^0 < \overline{y}(0) - \xi$. We conclude that $y_{IR}^*(\delta, d, n) \to \overline{y}(0)$ as $d \to 0$. ■