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Abstract

Interactive epistemology in dynamic games studies forms of strategic reasoning like backward induction and forward induction by formally representing the players' beliefs about each other, conditional on each history. Work on this topic typically relies on epistemic models where states of the world specify both strategies and beliefs. In this literature, strategies are interpreted as objective descriptions of what the players would choose at each history. But the intuitive interpretation of strategy is that of (subjective) contingent plan of action. As players do not delegate their moves to devices that mechanically execute a strategy, plans cannot be anything but beliefs of players about their own behavior. In this paper we analyze strategic reasoning in dynamic games with perfect information by means of epistemic models where behavior is described only by the play path, and players' beliefs include their contingent plans. We define *rational planning*, a property of beliefs only, and *material consistency*, which connects plans with choices on the play path. *Material rationality* is the conjunction of rational planning and material consistency. In perfect information games of depth two, the simplest dynamic games, correct belief in material rationality only implies a Nash outcome, not the backward-induction one. We have to consider stronger assumptions of persistence of belief in material rationality in order to obtain backward induction and forward induction. We relate our work to the existing literature, and we discuss the extension of our analysis to games with imperfect information.

1 Introduction

Interactive epistemology in dynamic games studies forms of strategic reasoning like backward induction and forward induction by means of formal representations of the players' conditional beliefs about each other, given the histories that can occur in the game being played.[1] Most of the received literature relies on state-space models where a state specifies both the players' strategies and their conditional beliefs. A player's strategy associates to every history

[1] For surveys on this topic see Battigalli and Bonanno (1999), Perea (2001) and Brandenburger (2007).

where the player is active an action available at that history. This is interpreted *objectively*, as a description of what the player would *actually* do at every such history, even at those that do not happen at the state. In particular, a player's strategy is a primitive object, described separately from her beliefs. To be sure, it can be the *object* of the player's own beliefs—whereby, typically, she is certain of it—but it is not *expressed* in terms of beliefs.[2]

A more intuitive interpretation of a strategy—in particular, the correct one in contexts where players cannot delegate their choices to mechanical devices—is, however, that of a *subjective plan* describing what the player thinks she would choose, at each history of hers, should that history occur.[3] Under this interpretation, prescriptions of actions to be chosen at unreached histories can only reflect a player's *belief* about her own behavior, conditional on counterfactual events. Most existing epistemic models, therefore, bring about a language that is both implicit—behavior and beliefs about own behavior are conflated in a single mathematical object, the strategy—and possibly restrictive—such conflation hides some implied underlying assumptions, as we discuss in the last section of the paper. Thus, while we do not claim that including strategies in the description of a state is conceptually incorrect, we believe that a reformulation of the received models in terms of a more explicit language would allow for a better understanding of strategic reasoning.[4]

In this paper, we propose to study interactive epistemology in perfect information games by means of models where the primitive object describing behavior at a state is the play *path* rather than the strategies. This methodological move, first advanced by Samet (1996) in the context of conditional *knowledge*, presents interesting challenges in representing even the most elementary forms of strategic reasoning. The first challenge lies in the definition of sequential (Bayesian) rationality. In models where states specify strategies, assigning expected values to the actions available at a history, conditional on reaching that history, is immediate. In the framework of this paper, where behavior (paths) and beliefs about own behavior are not described by the same object, the analogous definition of sequential rationality is more articulate. We define a seemingly natural notion of *material rationality*,[5] made of two parts:

Rational Planning: this is a property of player i's system of conditional beliefs, and in

[2]Battigalli and Siniscalchi (1999) assume conditional probabilities over strategy profiles (including own strategies) and assume that a rational player assigns conditional probability one to her actual strategy, given each information set not precluded by that strategy. Thus, a player's strategy is dictated by her beliefs, and conversely, a strategy determines the player's belief about it.

[3]If the extensive form describing the dynamic game is taken seriously, a player cannot commit in advance (not even secretly) to play a strategy, as such commitment should appear as an explicit move in a larger, all-encompassing extensive form.

[4]See also the discussion in Battigalli and Dufwenberg (2009), who note that interpreting strategies as beliefs is crucial in the context of games with belief-dependent preferences (so called "psychological games").

[5]See also Aumann (1998).

particular her beliefs about behavior (including her own) at the various nodes; these conditional beliefs define a profile of behavior strategies (b_i, b_{-i}); the behavioral strategies of the opponents, b_{-i}, define a subjective decision tree, and rational planning of i requires that b_i is obtained by dynamic programming on this subjective decision tree.

Material Consistency: this property relates i's conditional beliefs with her actual behavior: on the actual path, i never takes actions excluded by her beliefs, that is, by her plan of action.

Let us test the power of these concepts in the simplest dynamic games: generic perfect information games of depth two, e.g., leader-follower games. First, we rehearse the standard argument, according to which rationality and *initial* (or, unconditional) *belief* in rationality are enough to obtain the backward-induction solution in such games. Rationality implies that the second mover best-responds to the first-mover choice, which pins down the strategy of the second mover. Initial belief in rationality implies that the first-mover *assigns probability one to this strategy*. Finally, rationality of the first mover implies that she best responds to it.

Such intuitive arguments can be given a formal representation when strategies are part of the state of the world. In this case, the statement "for each action a_1 of the leader, if a_1 were chosen then the follower would best-respond to a_1" is an event to which the leader can assign probability one. In our framework, however, we cannot afford the luxury of having players who assign probabilities to "objective" strategies, and we work with material rationality instead. What are the consequences of material rationality and belief in material rationality? Of course, this depends on how "belief" is defined. If we mean "initial belief," as above, then we cannot obtain the backward-induction outcome, even in leader-follower games. But we can recover the backward-induction outcome if we assume stronger forms of belief. Suppose that material rationality holds and that the leader believes in the material rationality of the follower conditional on each initial action. Such post-action beliefs are crucial because they define the subjective decision tree of the leader: the subjective value of each action a_1 is determined by what she believes about the follower *conditional on taking* a_1. If the leader believes in the material rationality of the follower, conditional on each initial action, then each such action has the backward-induction value. Thus, the materially rational leader's choice and the materially rational follower's response are on the backward-induction path.

This brief discussion suggests that assuming material rationality and some degree of persistence in beliefs may allow an interesting formal analysis of strategic thinking in dynamic games. It is worth noting that the concept of "strong belief" in an event E (whereby E is believed conditional on each history compatible with it) is at the heart of the epistemic analysis of forward-induction reasoning, and that forward induction yields the backward-induction path in generic games with perfect information (Battigalli and Siniscalchi, 2002). Such results are obtained in epistemic models where states of the world specify strategies. We show that similar results hold within the more parsimonious epistemic models considered here.

1.1 An illustrative example: the Stackelberg mini-game

Consider a quantity-setting duopoly where each of two firms, Ann and Bob, can choose either a low quantity or a high quantity, and Ann moves first. The extensive form of the game and the payoffs associated to the various combinations of outputs are given in Figure 1. Note that the backward-induction path is the Stackelberg sequence $(\mathtt{U}, \mathtt{L})$.

	low (left)	high (right)
high (up)	$3,1$	$0,0$
low (down)	$2,2$	$1,3$

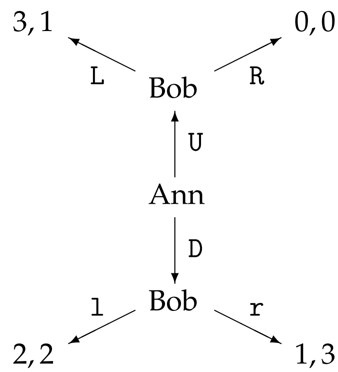

Figure 1: The Stackelberg mini-game.

Let us review the standard epistemic analysis of this game. A state of the world specifies a strategy and a type for each firm, where a type determines (conditional) beliefs about strategies and types of the other firm. In this context, rationality of Bob means that he *would* respond with $\mathtt{L}$ to $\mathtt{U}$, and with $\mathtt{r}$ to $\mathtt{D}$. If Ann believes in Bob's rationality, she assigns probability one to the strategy just described. Thus, if she is herself rational, she plays $\mathtt{U}$, and the backward-induction path obtains. Now consider the Nash equilibrium ($\mathtt{D}$,$\mathtt{Rr}$). In the standard framework with strategies in the states, path ($\mathtt{D}$,$\mathtt{r}$) is inconsistent with rationality and belief in rationality. In the framework we propose, however, this imperfect Nash equilibrium is consistent with material rationality and initial belief in material rationality. In order to restore the backward-induction solution, we have to add the assumption of Ann's *strong belief* in Bob's material rationality (or other assumptions that imply this, see Section 4).

In our framework, a state of the world (for this example) is a list (z, t_A, t_B), where Ann's (Bob's) type t_A (resp. t_B) specifies her beliefs on paths and Bob's (resp. Ann's) types, conditional on each decision node of the game. Thus, each type t_i of each player i has an unconditional belief $\beta_i(t_i)(\cdot|\varnothing)$, where $\varnothing$ denotes the initial node, and two conditional beliefs given Ann's actions, $\beta_i(t_i)(\cdot|\mathtt{U})$ and $\beta_i(t_i)(\cdot|\mathtt{D})$,[6] related to the unconditional belief via the chain rule of conditional probabilities. Ann's plan is determined by her unconditional belief, while

[6] In our analysis we allow also beliefs conditional on terminal nodes, but we do so mainly for notational convenience.

Bob's is determined by his conditional beliefs given $\mathtt{U}$ and $\mathtt{D}$. In every state (z, t_A, t_B) consistent with Bob's material rationality, MR_B, the path must be either $z = (\mathtt{U}, \mathtt{L})$ or $z = (\mathtt{D}, \mathtt{r})$. However, if $\beta_A(t_A)(\mathtt{U}|\varnothing) = 0$, then the assumption that $\beta_A(t_A)(MR_B|\varnothing) = 1$ implies that $\beta_A(t_A)(\mathtt{r}|\mathtt{D}) = 1$, but it does not also imply that $\beta_A(t_A)(\mathtt{L}|\mathtt{U}) = 1$. Thus, since Ann might not attach the backward-induction value to her action $\mathtt{U}$, it is indeed possible that at a state (z, t_A, t_B) the path is $z = (\mathtt{D}, \mathtt{r})$, while material rationality for both players and Ann's unconditional belief in Bob's material rationality are satisfied.

In general, we show that in generic perfect information games of depth two, material rationality and (unconditional) belief in material rationality only imply a Nash equilibrium path, not the backward-induction path (Proposition 1). Then we prove that in generic games of depth two, the backward-induction path does obtain, if we assume material rationality and strong belief in material rationality within a sufficiently rich epistemic structure (Proposition 2, Corollary 1). For more complex games, such as centipede games of depth three or more, we show that (correct) common belief in material rationality does not even guarantee a Nash equilibrium path. On the other hand, building on Battigalli and Siniscalchi (2002), we show that *common strong belief in the other players' material rationality* (an assumption that captures forward-induction reasoning) implies the backward-induction path in generic games of perfect information (of any depth), provided that the epistemic structure is sufficiently rich (Theorem 2). If instead the epistemic structure is not assumed to be rich, we can only show that common strong belief in material rationality implies a Nash path. These results are analogous to results obtained for epistemic structures with strategies in the state space. We discuss the relationship between such "strategy-based" approach and our "path-based" approach in Section 4.

2 Preliminaries

2.1 Games with perfect information

Throughout the paper we fix a finite game with perfect information, using the standard notation in Osborne and Rubinstein (1994). Thus we assume the following:

- a finite set N of *players* and a finite set A of *actions*;
- a finite set H of *histories*, that is, a finite set of finite sequences in A, containing the empty sequence $\varnothing$, which we call the *initial* history, and such that, for every $(a^1, \ldots, a^k) \in H$ and $\ell < k$, the corresponding subsequence is also a history, that is, $(a^1, \ldots, a^\ell) \in H$; for $h \in H$, $A(h) = \{a \in A : (h, a) \in H\}$ is the set of actions *available* at h; if $A(h) = \varnothing$, then h is said to be *terminal*, and the set of terminal histories, or *paths*, is denoted by Z; we let

δ denote the *depth* of the game, i.e. the number of elements in the longest sequence in H;

- a function $\iota : H \setminus Z \to N$; for each player i, we let $H_i = \iota^{-1}(i)$;
- for each player i, a *payoff function* $u_i : Z \to \mathbb{R}$.

The induced weak and strict *precedence* relations on H will be denoted by $\preceq$ and $\prec$, respectively. Thus, for $h, h' \in H$ we write $h \preceq h'$ whenever h is an initial subsequence (prefix) of h', and we write $h \prec h'$ if in addition $h \neq h'$. The terminal successors of a history h are those in the set $Z(h) = \{z \in Z : h \preceq z\}$. For $h \in H \setminus Z$ and $h' \in H$ with $h \prec h'$, we write $\alpha(h, h')$ for the unique $a \in A$ such that $(h, a) \preceq h'$.

To avoid discussions of relatively minor issues, we focus our attention on the case where for every two distinct terminal histories z, z', the player who moves at their last common predecessor is not indifferent between them:

Assumption (No relevant ties)**.** For every $i \in N$, $h \in H_i$ and $a, a' \in A(h)$,

$$a \neq a' \qquad \Rightarrow \qquad \{u_i(z) : z \in Z(h,a)\} \cap \{u_i(z) : z \in Z(h,a')\} = \varnothing.$$

2.2 Conditional probability systems

Given a compact metrizable space X, endowed with its Borel σ-algebra $\mathcal{F}$, and a finite family $\mathcal{C} \subset \mathcal{F}$ of clopen (closed and open) events, containing X itself, called *conditions*, a *conditional probability system* is a collection of probability measures $(\mu(\cdot|C))_{C \in \mathcal{C}}$ on X satisfying the following properties:

- $\mu(C|C) = 1$ for all $C \in \mathcal{C}$;
- $\mu(E|C) = \mu(E|C')\mu(C'|C)$ for all $E \in \mathcal{F}$ and $C, C' \in \mathcal{C}$ with $E \subseteq C' \subseteq C$.

The set of all conditional probability systems is denoted $\Delta^{\mathcal{C}}(X)$. Under the stated assumptions, $\Delta^{\mathcal{C}}(X)$ is a compact metrizable space—see Battigalli and Siniscalchi (1999). In our analysis, the family $\mathcal{C}$ corresponds to the collection of events that the players can *observe* in the game, namely, the histories.

2.3 Strategies and conditional beliefs

Given the game described above, the standard definition of a *strategy* for player i is that of a mapping from i's histories into available actions, that is,

$$s_i \in S_i = \mathop{\times}_{h \in H_i} A(h).$$

In this paper we keep the latter formal definition of strategy, but we depart from the received literature on interactive epistemology for dynamic games, in that we do not assume that players can reason about strategies as objective descriptions of behavior. Instead, we model a player's belief about behavior at a history (including her own), as her conditional belief on the set of paths, given the event that (she observes that) the path goes through that history.

Thus, in our framework a *state of the world* (or simply a *state*) specifies a path $z \in Z$ and a *type* $t_i \in T_i$ for each player i, where T_i is a compact metrizable space. Types encode the players' conditional beliefs, and conditional beliefs about each other's conditional beliefs, where for each player i the set of conditions $\mathcal{C}$ is the family of events of the form $Z(h) \times T_{-i}$,[7] with $h \in H$. This family of events is obviously isomorphic to H itself, and therefore in what follows, for every $h \in H$, we often write h instead of the more cumbersome $Z(h) \times T_{-i}$. Moreover, we write $[h]$ instead of $Z(h) \times T_i \times T_{-i}$ to denote the set of states where the path goes through h. Thus, we assume: for each player i, there is a continuous function $\beta_i : T_i \to \Delta^H(Z \times T_{-i})$. Note the obvious but important fact that histories are *uninformative* (to player i) about types (of the other players), since our state space has a product structure and the conditioning events concern only one "side" of the product, namely the paths. In what follows we call a tuple $(T_i, \beta_i)_{i \in N}$ as described above, a *path-based type structure*, or just *type structure*, and we say that it is *complete* if for every player i the mapping β_i is onto.[8]

For each $i \in N$ and $t_i \in T_i$, it is convenient to view the probability measures $\beta_i(t_i)(\cdot|h)$ as probability measures on the whole state space $Z \times T_i \times T_{-i}$. Thus, given an event $E \subseteq Z \times T_i \times T_{-i}$, we say that player *i believes in E conditional on h* at a state (z, t_i, t_{-i}), provided that

$$\beta_i(t_i)(\{(z', t'_{-i}) \in Z \times T_{-i} : (z', t_i, t'_{-i}) \in E\} \,|\, h) = 1,$$

and we denote by $B_i(E|h)$ the set of such states.[9] For $h = \varnothing$ we write simply $B_i(E)$, and we say that i believes in E. Finally, there is *(correct) common belief in E* at every state in the event

$$CB(E) = E \cap B(E) \cap B(B(E)) \cap \cdots,$$

where for every event E we write $B(E)$ as an abbreviation for $\cap_{i \in N} B_i(E)$.

[7] As usual, we define $T_{-i} = \times_{j \in N \setminus \{i\}} T_j$. Similar notations are used, without notice, throughout the paper.

[8] On completeness see Brandenburger (2003). Battigalli and Siniscalchi (1999) prove by construction that a complete type structure exists: it is the canonical structure where each T_i is the set of *hierarchies* of conditional probability systems satisfying collective coherency.

[9] A standard monotone class argument (see e.g. Proposition 3.4.24 in Srivastava, 1998) shows that the set $B_i(E|h)$ is measurable. Note that our definition of the operators $B_i(\cdot|h)$ implies that $Z \times \{t_i\} \times T_{-i} \subseteq B_i(Z \times \{t_i\} \times T_{-i}|h)$ for every history h, player i, and type t_i. Thus, player i is always (i.e. given every condition, and whatever her type is) certain of her true type and, therefore, of her belief-revision rule. This strong form of *introspection* is appropriate in the framework of this paper, because by definition a condition (history) conveys no information about a player's belief-revision rule, as we have already remarked earlier. Di Tillio, Halpern, and Samet (2011) discuss a different interpretation of conditional probability and a weaker notion of introspection.

3 Rational planning and material consistency

For every player i and every type t_i of hers, the probabilities $\beta_i(t_i)((h,a)|h)$ define a *behavior strategy* for every player j, as h varies in H_j and a varies in $A(h)$. In particular, for $j = i$, they define a behavior strategy for player i herself. However, we stress that despite this formal equivalence, such probabilities only represent i's beliefs, and nothing in our basic framework requires that i will *act* accordingly. A type structure can have a state where player i is sure, given a history that does obtain at the state, that she will not choose the action that she will indeed choose at that history. Formally, this is a state (z, t_i, t_{-i}) such that $\beta_i(t_i)((h, \alpha(h,z))|h) = 0$ for some history $h \in H_i$ with $h \prec z$.

Thus, we interpret those probabilities as the result of i's *planning*: starting from conditional beliefs about the other players' behavior, i solves the corresponding subjective decision tree by dynamic programming, breaking ties arbitrarily; while this does not imply any commitment, it does deliver a *rational plan*, that is, a belief by player i that at each history of hers, she would follow the (optimal) recommendation of the dynamic programming solution, should that history indeed occur. Once the plan is in i's mind, together with the beliefs about others that she started with, the entire profile of behavior strategies $((\beta_i(t_i)((h,\cdot)|h))_{h\in H_j})_{j\in N}$ is defined.

The continuation value for type t_i of player i, corresponding to action $a \in A(h)$ at history $h \in H_i$, is her expected payoff, conditional on history (h,a), namely

$$\sum_{z\in Z(h,a)} \beta_i(t_i)(z|(h,a))u_i(z).$$

Thus, for all $i \in N$, $t_i \in T_i$ and $h \in H_i$, we define the set of locally optimal actions at h as

$$A^*(h, \beta_i(t_i)) = \arg\max_{a\in A(h)} \sum_{z\in Z(h,a)} \beta_i(t_i)(z|(h,a))u_i(z). \tag{1}$$

If type t_i believes that she would indeed take an optimal action at h, should the latter be reached, then $\beta_i(t_i)((h,\cdot)|h)$ must be supported in $A^*(h, \beta_i(t_i))$. This motivates the following:

Definition 1. Player i satisfies *rational planning* at state (z, t_i, t_{-i}) if for all $h \in H_i$ and $a \in A(h)$,

$$a \notin A^*(h, \beta_i(t_i)) \qquad \Rightarrow \qquad \beta_i(t_i)((h,a)|h) = 0.$$ [10]

Let RP_i denote the set of such states, and let $RP = \cap_{i\in N} RP_i$.

As we have already argued, and as is indeed clear from its definition, rational planning is a property of beliefs, *per se* it has no implication about actual behavior. Hence, a player's

[10] By the one-shot deviation principle, if i plans rationally at (z, t_i, t_{-i}) then the behavioral strategy implied by t_i is dynamically optimal in the decision tree implied by t_i.

belief in rational planning of other players need not have any implication on her belief about their behavior. In order to obtain such implications, we have to add the belief that the other players's behavior is consistent with their plan. This leads to the following:

Definition 2. Player i is *materially consistent* at state (z, t_i, t_{-i}) if for every $h \in H_i$,

$$h \prec z \qquad \Rightarrow \qquad \beta_i(t_i)((h, \alpha(h,z))|h) > 0.$$

Let MC_i denote the set of such states, and let $MC = \cap_{i \in N} MC_i$.

In other words, i is not materially consistent if at some history along the realized path, she takes an action that she planned to exclude, conditional on that history being reached.[11] The conjunction of rational planning and material consistency plays an important role in our analysis, and therefore it deserves its own name:[12]

Definition 3. Player i is *materially rational* at each state in $MR_i = MC_i \cap RP_i$. $MR = \cap_{i \in N} MR_i$.

3.1 Common belief in material rationality

A standard result of the literature on interactive epistemology for simultaneous moves games states that an outcome is consistent with rationality and common belief in rationality if and only if it is rationalizable. It is therefore natural to consider similar epistemic assumptions in the present context. The assumptions of material rationality and common belief thereof are represented by the event

$$CB(MR) = MR \cap B(MR) \cap B(B(MR)) \cap \cdots .$$

The latter is not a vacuous assumption, since it holds in the *BI structure*, which is defined as follows. By no relevant ties, there is a unique backward-induction (henceforth BI) strategy profile s^{BI} which induces the unique BI path z^{BI}. Then, the BI structure is the type structure where each player has only one type, and for each nonterminal history $h \in H \backslash Z$, this type assigns probability one to action $s^{BI}(h)$ given h. In the BI structure, $CB(MR)$ holds.

Definition 4. A type structure $(T_i, \beta_i)_{i \in N}$ *contains the BI structure* if there is a profile of types $(t_i^{BI})_{i \in N} \in \times_{i \in N} T_i$ such that for all $i \in N$ and $h \in H \setminus Z$, $\beta_i(t_i^{BI})(Z(h, s^{BI}(h)) \times \{t_{-i}^{BI}\}|h) = 1$.

We now present a preliminary result. Despite its simplicity, the proof requires some care and it illustrates the features of the adopted framework.

[11] In logic, a *material implication* is an if-then statement that holds false if and only if its antecedent is true while the consequent is false. We use the term "material" because lack of i's material consistency occurs if the material implication "if h is reached, then i acts according to her plan" is false for some history h of i.

[12] Our definition of material rationality is similar to that of Aumann (1998).

Proposition 1. *Assume that $\delta = 2$. For every type structure $(T_i, \beta_i)_{i\in N}$ and every state $(z^*, t^*) \in MR \cap B(MR)$, z^* is a (mixed) Nash equilibrium path. Conversely, for every (mixed) Nash equilibrium path z^*, there exists a type structure $(T_i, \beta_i)_{i\in N}$ and a profile of types t^* in it, such that $(z^*, t^*) \in MR \cap B(MR)$, and in fact, $(z^*, t^*) \in CB(MR)$.*[13]

Proof. Since the game has depth two, we may assume without essential loss of generality that each state has the form $((a_1, a_2), t)$. Fix a type structure $(T_i, \beta_i)_{i\in N}$ and a state $((a_1^*, a_2^*), t^*) \in MR \cap B(MR)$. We will show that the behavioral strategy profile implied by the type of the first mover, $t^*_{\iota(\varnothing)}$, is a Nash equilibrium that gives positive probability to (a_1^*, a_2^*).[14]

Let $i = \iota(\varnothing)$ be the first mover. We first show that t_i^* assigns the BI value to each first-stage action that she plans to choose with positive probability, whether i moves or another player moves at the second stage. Indeed, we have

$$RP_i \subseteq B_i(\{((a_1, a_2), t) : \iota(a_1) = i \;\Rightarrow\; a_2 = s^{BI}(a_1)\}),$$
$$MR_{-i} \subseteq \{((a_1, a_2), t) : \iota(a_1) \neq i \;\Rightarrow\; a_2 = s^{BI}(a_1)\}.$$

Thus, since $((a_1^*, a_2^*), t^*) \in RP_i \cap B_i(MR_{-i})$,

$$\beta_i(t_i^*)(\{((a_1, a_2), t_{-i}) : a_2 = s^{BI}(a_1)\}|\varnothing) = 1,$$

and hence it follows from the chain rule that, for all $a_1 \in A(\varnothing)$,

$$\beta_i(t_i^*)(a_1|\varnothing) > 0 \quad \Longrightarrow \quad \beta_i(t_i^*)((a_1, s^{BI}(a_1))|a_1) = 1. \tag{2}$$

Since $((a_1^*, a_2^*), t^*) \in RP_i$, for all $a_1 \in A(\varnothing)$ such that $\beta_i(t_i^*)(a_1|\varnothing) > 0$ we have

$$a_1 \in \arg\max_{a_1' \in A(\varnothing)} \sum_{a_2' \in A(a_1')} \beta_i(t_i^*)((a_1', a_2')|a_1') u_i(a_1', a_2'). \tag{3}$$

Let $(b^*)_{j\in N}$ be the behavior strategy profile defined by $\beta_i(t_i^*)$. By (2), for each player $j \neq i$, b_j^* is a best reply to b^*_{-j}. By (2) and (3), b_i^* is a best reply to b^*_{-i}. Thus, $(b_j^*)_{j\in N}$ is a Nash equilibrium. Furthermore, $((a_1^*, a_2^*), t^*) \in MC_i$ implies $b_i^*(a_1^*) > 0$, and hence, by (2), $b^*_{\iota(a_1^*)}(a_2^*|a_1^*) = 1$. This proves that (a_1^*, a_2^*) occurs with positive probability under $(b^*)_{j\in N}$, as claimed.

For the second claim in the proposition, see Appendix A.1. ■

In games of depth three or more, however, even this weak result fails to hold. Consider, for example, the three-legged centipede game (Rosenthal, 1981) depicted in Figure 2. The BI solution is to go always down, and (D) is the unique Nash path. However, in Appendix A.2 we exhibit a type structure with a state in $CB(MR)$ where the path is (C, c, D'). Intuitively, at

[13] We can show that the converse holds for all perfect information games with no relevant ties.

[14] In effect, by no relevant ties, probability one. But this is immaterial for our argument.

this state Ann surprises Bob and tricks him into thinking that she is not materially rational; Bob's beliefs are incorrect, indeed he is initially certain that path D occurs, but he is also certain that material rationality holds, and that Ann believes in $CB(MR)$.

$$\begin{tikzcd}
\text{Ann} \arrow[r, "C"] \arrow[d, "D"'] & \text{Bob} \arrow[r, "c"] \arrow[d, "d"'] & \text{Ann} \arrow[r, "C'"] \arrow[d, "D'"'] & 2,4 \\
1,0 & 0,2 & 3,1 &
\end{tikzcd}$$

Figure 2: The three-legged centipede

3.2 Common strong belief in material rationality

The examples discussed in the introduction and in the previous section indicate that in order to obtain the BI path, or even a Nash outcome for games of depth three or more, some persistence of belief in material rationality is needed. In this section we introduce *strong belief* in material rationality, which captures precisely this idea.

Definition 5. Fix a type structure $(T_i, \beta_i)_{i \in N}$ and an event E in it. Player i has *strong belief* in E at state (z, t_i, t_{-i}) if for every $h \in H$,

$$[h] \cap E \neq \varnothing \qquad \Rightarrow \qquad \beta_i(t_i)(E|h) = 1.$$

Let $SB_i(E)$ denote the set of such states.

Strong belief in material rationality yields the BI outcome in simple games, provided that the type structure is sufficiently rich, as the following result shows.

Proposition 2. *Suppose that $\delta = 2$ and fix a type structure $(T_i, \beta_i)_{i \in N}$ that contains the BI structure. Then material rationality and strong belief of the first mover in the other players' material rationality imply the BI path: let $i = \iota(\varnothing)$, then for every $(z, t) \in MR \cap SB_i(MR_{-i})$, z is the BI path.*

Proof. Let $i = \iota(\varnothing)$ be the first mover and fix $(z, t) \in MR \cap SB_i(MR_{-i})$. Pick any $a_1 \in A(\varnothing)$ and let $j = \iota(a_1)$ be the player who moves after a_1 (if it is not terminal). Let s_i^{BI} and s_j^{BI} be the strategies that select the BI move at every history. As a preliminary observation, note that for every $(z', t') \in MR_j$, if $a_1 \prec z'$ then $z' = (a_1, s_j^{BI}(a_1))$ (see the proof of Proposition 1). Now consider the conditional belief $\beta_i(t_i)(\cdot|a_1)$. If $j = i$, that is, if i plays again after a_1, then $(z, t) \in RP_i$ implies $\beta_i(t_i)((a_1, s_i^{BI}(a_1))|a_1) = 1$. Otherwise, assume $j \neq i$, and let us show that $(z, t) \in SB_i(MR_{-i})$ implies $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$. Since the structure contains the BI structure, it contains the state $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI})$. It is easily checked that $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI}) \in MR_{-i}$, hence $[a_1] \cap MR_{-i} \neq \emptyset$. Thus, by definition of strong belief, $(z,t) \in SB_i(MR_{-i})$ implies $\beta_i(t_i)(MR_{-i}|a_1) = 1$ and hence $\beta_i(t_i)(MR_j|a_1)$. By the preliminary observation above, we then obtain $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$. This is true for every $a_1 \in A(\emptyset)$, so i assigns the BI value to every such action. Since $(z,t) \in MR$, this implies that z is the BI path. ■

Corollary 1. *Suppose that $\delta = 2$ and fix a complete type structure $(T_i, \beta_i)_{i\in N}$. Then material rationality and strong belief of the first mover in the other players' material rationality imply the BI path in this structure: let $i = \iota(\emptyset)$, then for every $(z,t) \in MR \cap SB_i(MR_{-i})$, z is the BI path.*

Proof. By Lemma 1 in Appendix A.3, each complete type structure contains the BI structure. The thesis then follows from Proposition 2. ■

Building on Battigalli and Siniscalchi (2002), the result can be extended to all perfect information games with no relevant ties, i.e. without restrictions on the depth of the game δ, by looking at material rationality and *common* strong belief in the *other* players' material rationality. In order to establish this result, we define strong belief and common strong belief for *profiles* of events, as follows. For $\underline{E} = (E_i)_{i\in N}$, where for every player i, $E_i \subseteq Z \times T_i \times T_{-i}$ is measurable, define $\cap\underline{E} = \cap_{i\in N} E_i$ and $\cap_{-i}\underline{E} = \cap_{j\neq i} E_j$, and let

$$\underline{CSB}(\underline{E}) = \big(E_i \cap SB_i(\cap_{-i}\underline{E})\big)_{i\in N}.$$

Note that with this notation, $\cap\underline{CSB}(\underline{E})$ is the event that each E_i obtains, and each player i strongly believes in $\cap_{j\neq i} E_j$. Then, recursively for all $m \geq 1$, letting $\underline{CSB}^0(\underline{E}) = \underline{E}$ by convention,

$$\underline{CSB}^m(\underline{E}) = \underline{CSB}(\underline{CSB}^{m-1}(\underline{E}))$$

Definition 6. There is *(correct) common strong belief in the other players' material rationality* at each state in

$$CSBMR = \bigcap_{m\geq 0} \cap\underline{CSB}^m\big((MR_i)_{i\in N}\big).$$

As a preliminary result, we show that in every type structure, there is a unique path consistent with common strong belief in material rationality; then we show that this path is, in fact, a Nash equilibrium path. Before we state the result, note that if player i strongly believes in the other players' material rationality, and the latter is compatible with a history h where i moves, then player i must be certain of the other players' material rationality, when conditioning on the event that h is reached *and* an action $a \in A(h)$ is taken. Moreover, since histories are uninformative about beliefs, the same is true when considering the event that, in addition, the other players' beliefs lie in a certain set E_{-i}. Formally, for all $i \in N$, $h \in H_i$, $a \in A(h)$ and measurable $E_{-i} \subseteq T_{-i}$, letting $E = Z \times T_i \times E_{-i}$,

$$[h] \cap E \cap MR_{-i} \neq \emptyset \qquad \Rightarrow \qquad SB_i(MR_{-i} \cap E) \subseteq B_i(MR_{-i} \cap E|(h,a)). \tag{4}$$

In the proof of the following proposition, this fact will be used inductively, to show that for every history h that is compatible with common strong belief in rationality, and for every action a available at h, the player moving at h expects a unique path following a, that is, her beliefs given (h,a) are concentrated on a single path.

Proposition 3. *Fix a type structure. Every state in CSBMR specifies the same path.*

Proof. Let $\underline{MR} = (MR_i)_{i\in N}$. For every path z and history $h \prec z$, let $d(h,z)$ denote the length of the subpath from h to z.[15] We prove by induction in $m \geq 0$, that for every $i \in N$, every $h \in H_i$ with $\max_{z\in Z(h)} d(h,z) \leq m+1$ and $[h] \cap \cap \underline{CSB}^m(\underline{MR}) \neq \varnothing$, and every $a \in A(h)$ with $[(h,a)] \cap MR_i \neq \varnothing$, there exists $z_{h,a} \in Z$ such that

$$MR_i \cap \bigcap_{k=0}^{m} SB_i\big(\cap_{-i}\, \underline{CSB}^k(\underline{MR})\big) \;\subseteq\; B_i([z_{h,a}]|(h,a)).$$

Note that this implies that at every state (z, t_i, t_{-i}) in the event on the left-hand side, if $h \prec z$, then $\alpha(h,z)$ must be, by no relevant ties, the unique locally optimal action for t_i at h. In other words, there exists $a_h^* \in A(h)$ such that

$$[h] \cap MR_i \cap \bigcap_{k=0}^{m} SB_i\big(\cap_{-i}\, \underline{CSB}^k(\underline{MR})\big) \;\subseteq\; [(h,a_h^*)].$$

For $m = 0$, our claim is trivially true. Let $n \geq 1$, assume the claim holds for all $0 \leq m \leq n-1$, and fix $i \in N$, $h \in H_i$ with $\max_{z\in Z(h)} d(h,z) \leq n+1$ and $[h] \cap \cap \underline{CSB}^n(\underline{MR}) \neq \varnothing$, and $a \in A(h)$ with $[(h,a)] \cap MR_i \neq \varnothing$. Note that the induction hypothesis implies

$$MR_i \cap B_i\Big(\bigcap_{j\neq i}\big([(h,a)] \cap MR_j \cap \bigcap_{k=0}^{n-1} SB_j\big(\cap_{-j}\, \underline{CSB}^k(\underline{MR})\big)\big)\Big|(h,a)\Big) \;\subseteq\; MR_i \cap B_i([z_{h,a}]|(h,a)). \quad (5)$$

By our definitions,

$$\cap_{-i}\, \underline{CSB}^n(\underline{MR}) = \textstyle\bigcap_{j\neq i}\Big(MR_j \cap \displaystyle\bigcap_{k=0}^{n-1} SB_j\big(\cap_{-j}\, \underline{CSB}^k(\underline{MR})\big)\Big). \quad (6)$$

From $[h] \cap \cap \underline{CSB}^n(\underline{MR}) \neq \varnothing$ it follows that $[h] \cap \cap_{-i} \underline{CSB}^k(\underline{MR}) \neq \varnothing$ for all $0 \leq k \leq n-1$ and hence, using (4) and the induction hypothesis,

$$\begin{aligned}
MR_i \cap \bigcap_{k=0}^{n} SB_i\big(\cap_{-i}\, \underline{CSB}^k(\underline{MR})\big) \;&\subseteq\; MR_i \cap \bigcap_{k=0}^{n} B_i\big(\cap_{-i}\, \underline{CSB}^k(\underline{MR})\big|(h,a)\big) \\
&=\; MR_i \cap B_i\big(\cap_{-i}\, \underline{CSB}^n(\underline{MR})\big|(h,a)\big) \\
&=\; MR_i \cap B_i\Big(\bigcap_{j\neq i}\big(MR_j \cap \bigcap_{k=0}^{n-1} SB_j\big(\cap_{-j}\, \underline{CSB}^k(\underline{MR})\big)\big)\Big|(h,a)\Big) \quad \text{(by (6))} \\
&=\; MR_i \cap B_i\Big(\bigcap_{j\neq i}\big([(h,a)] \cap MR_j \cap \bigcap_{k=0}^{n-1} SB_j\big(\cap_{-j}\, \underline{CSB}^k(\underline{MR})\big)\big)\Big|(h,a)\Big) \\
&\subseteq\; MR_i \cap B_i([z_{h,a}]|(h,a)) \quad \text{(by (5))}.
\end{aligned}$$

[15] If $z = (h, \alpha(h,z))$ then $d(h,z) = 1$, if $z = (h, \alpha(h,z), \alpha((h,\alpha(h,z)),z))$ then $d(h,z) = 2$, and so on.

■

Using the latter proposition, we then obtain the following result,[16] whose proof is in Appendix A.4.

Theorem 1. *Fix a type structure. Each state in CSBMR specifies a (mixed) Nash equilibrium path.*

In Appendix A.5 we prove that in a complete type structure, a path is consistent with common strong belief in rationality if and only if it is consistent with Pearce's (1984) extensive form (correlated) rationalizability, a non-empty solution concept. The proof of the following theorem, in the same appendix, then follows from Battigalli's (1997) result that in perfect information games with no relevant ties, all extensive form rationalizable profiles induce the BI path.

Theorem 2. *Fix a complete type structure. Every state in CSBMR specifies the BI path.*

4 Discussion and extensions

In this paper we use epistemic structures where states specify paths rather than strategies, and we study how assumptions about rationality and beliefs are related to standard solution concepts for games with perfect information. Some of our results are similar to those obtained in the literature by means of structures where states specify strategies rather than paths. In this section we discuss further this relationship, and we hint about an extension of our framework to games with imperfect information.

4.1 From strategy-based to path-based structures

Most papers on dynamic epistemic game theory consider type structures where states of the world specify strategies, not just paths. We now record their formal definition. Recall that S_i denotes player i's strategy set. For every strategy profile $s \in \times_{i\in N} S_i$, denote by $\pi(s)$ the path induced by s. For each player i and history h, let $S_{-i}(h)$ be the set of strategies of player $-i$ that allow history h, that is, the set of all $s_{-i} \in S_{-i}$ such that $h \preceq \pi(s_i, s_{-i})$ for some $s_i \in S_i$. A *strategy-based type structure* is a tuple $(\bar{T}_i, \bar{\beta}_i)_{i\in N}$ where each $\bar{T}_i$ is a compact metrizable space, each $\bar{\beta}_i : \bar{T}_i \to \Delta^{\mathcal{H}_i}(S_{-i} \times \bar{T}_{-i})$ is a continuous function, and $\mathcal{H}_i$ denotes the family of subsets of $S_{-i} \times \bar{T}_{-i}$ of the form $S_{-i}(h) \times \bar{T}_{-i}$, with $h \in H$.[17]

[16] See Battigalli and Friedenberg (2010, Corollary 8.1, Proposition 8.2).

[17] As we argued earlier, in path-based structures the family of conditioning events, those of the form $Z(h) \times T_{-i}$, is isomorphic to H. In strategy-based structures this isomorphism is lost, as there can be distinct histories h, h' with $S_{-i}(h) = S_{-i}(h')$.

A strategy-based type structure naturally induces a path-based type structure. For each player i, let $T_i = S_i \times \bar{T}_i$ and define the mapping $\beta_i : T_i \to \Delta^H(Z \times T_{-i})$ as follows: for each $t_i = (s_i, \bar{t}_i) \in T_i$, $h \in H$, $z \in Z$ and measurable $E_{-i} \subseteq \bar{T}_{-i}$, letting s_i^h denote the strategy of player i that allows h and prescribes the same actions as s_i at nodes that do not precede h,

$$\beta_i(t_i)\big(\{z\}\times\{s_{-i}\}\times E_{-i}\big|h\big) = \begin{cases} \bar{\beta}_i(\bar{t}_i)\big(\{s_{-i}\}\times E_{-i}\big|S_{-i}(h)\times\bar{T}_{-i}\big) & \text{if } (s_i^h, s_{-i}) \in S_i(z)\times S_{-i}(z), \\ 0 & \text{otherwise.} \end{cases}$$

In words, $(s_i, \bar{t}_i)$ represents an epistemic state of player i in which he plans to follow s_i (even after deviations made "by mistake") and has conditional beliefs about the opponents given by $\bar{t}_i$. Thus, the beliefs of type $t_i = (s_i, \bar{t}_i)$ about his own behavior, given a history h, are simply those dictated by s_i at h, while his beliefs about the other players' behavior and beliefs are inherited from those specified for $\bar{t}_i$ in the original strategy-based structure. We leave it to the reader to check that $(T_i, \beta_i)_{i\in N}$ is indeed a path-based type structure. It is also routine to see that $(z, (s_i, \bar{t}_i), (s_{-i}, \bar{t}_{-i})) \in RP_i$ if and only if the pair $(s_i, \bar{t}_i)$ satisfies rationality in the standard sense of the literature on strategy-based structures,[18] while $(z, (s_i, \bar{t}_i), (s_{-i}, \bar{t}_{-i})) \in MR_i$ if and only if, in addition, $s_i(h) = \alpha(h, z)$ for all $h \in H_i$ such that $h \prec z$.

As we anticipated in the introduction, strategy-based structures embody some implicit assumptions on the players' reasoning. The more basic language of path-based structures allows us to make these assumptions explicit. Indeed, it can be verified that the path-based structure induced by a strategy-based structure automatically satisfies the following properties. First, since in the strategy-based structure players choose pure strategies, at each state of the induced path-based structure each player has *deterministic beliefs about own behavior*, that is, for each history $h \in H_i$ the measure $\beta_i(\cdot|h)$ puts probability one on a single element of $A(h)$. Second, since in the strategy-based structure every strategy of every player appears, in the induced path-based structure there is *uncontradictable material consistency*, that is, $[h] \cap MC \neq \varnothing$ for every history h. Furthermore, there is *strong belief in the other players' material consistency* at every state, i.e. $SB_i(MC_{-i}) = Z \times T_i \times T_{-i}$ for every player i. Finally, at each state of the induced path-based structure, each player's beliefs about the other players' types satisfy *own-action independence*, that is, for every player i, state (z, t_i, t_{-i}), history $h \in H_i$, action $a \in A(h)$, and measurable set $E_{-i} \subseteq T_{-i}$,

$$\beta_i(t_i)(Z \times E_{-i}|h) = \beta_i(t_i)(Z \times E_{-i}|(h, a)). \tag{7}$$

In the framework of this paper, none of these properties holds automatically, and neither

[18] That is, s_i is a best response to $\bar{t}_i$ in every subgame. Some papers require s_i to be optimal only in subtrees not excluded by s_i itself, but such difference turns out to be immaterial.

did we use any of them in our results.[19] Since they do hold when the assumed path-based structure is induced by a strategy-based structure, it is natural to ask whether they are necessary or sufficient to translate existing results in the strategy-based literature into the path-based framework. While developing a full formal relationship between the two families of models is beyond the scope of this work, we remark that the results in the preceding sections do already provide some answers. Theorems 1 and 2 show that common strong belief in material rationality alone, without any of the properties above, already delivers analogues of existing results—in Battigalli and Friedenberg (2010) and Battigalli and Siniscalchi (2002), respectively.[20]

By contrast, as discussed in the introduction and formally shown in Proposition 1, we know that another existing result—the result that in leader-follower games rationality and (first mover's) belief in rationality imply the BI path—fails to hold when translated into our framework. In effect, this is due to the lack of the properties discussed above, as imposing those properties does restore (the analogue of) that classic result. This is the content of the next proposition, proved in Appendix A.6. Given a type structure $(T_i, \beta_i)_{i \in N}$, for each player i let $I_i = Z \times \bar{I}_i \times T_{-i}$, where $\bar{I}_i$ is the set of types t_i satisfying (7).

Proposition 4. *Suppose that $\delta = 2$ and fix a type structure $(T_i, \beta_i)_{i \in N}$ that contains the BI structure. Then material rationality, own-action independence for the first mover, belief of the first mover in the other players' rational planning, and strong belief of the first mover in the other players' material consistency imply the BI path: let $i = \iota(\varnothing)$, then for every $(z, t) \in MR \cap I_i \cap B_i(RP_{-i}) \cap SB_i(MC_{-i})$, z is the BI path.*

For perfect-information games of any depth, Ben-Porath (1997) shows that (correct) common belief in rationality does not rule out any outcome that survives one round of weak dominance, followed by iterated strict dominance, a procedure first proposed by Dekel and Fudenberg (1990). (An example of such an outcome is the path (C, c, D') in Figure 2.) We can prove the analogous result in our framework: no path allowed by the Dekel-Fudenberg procedure is excluded by the assumption that players are materially consistent and, furthermore, there is common belief in (i) rational planning, (ii) own-action independence and (iii) strong belief in material consistency.

[19] In particular, while we used strong belief in material rationality, the latter does not imply strong belief in material consistency, as strong belief is not monotonic. Indeed, in general, $SB_i(MC_{-i} \cap RP_{-i}) \not\subseteq SB_i(MC_{-i})$.

[20] In turn, such results are similar to those obtained for strategy-based lexicographic probability structures, where "strong belief" is replaced by "assumption": common assumption of lexicographic rationality yields iterated admissibility in rich structures (Brandenburger, Friedenberg, and Kiesler, 2008; Kiesler and Lee, 2011) and a Nash equilibrium outcome in any structure (Brandenburger and Friedenberg, 2010). On the comparison between extensive-form and normal-form epistemic game theory, see Brandenburger (2007).

4.2 Imperfect Information

Rational planning, an essential ingredient of our analysis, is based on the possibility to assign a continuation value to an action a available at a history h using beliefs conditional on (h, a). In a game with imperfect information, the player moving at h might be unable to distinguish history h from another history h' where, of course, action a is also available. In this case, the relevant condition when evaluating action a is the *set* of histories comprising (h, a) and every other history of the form (h', a), where h' belongs to the same information set as h. Thus, with imperfect information, in order to be consistent with the analysis of perfect information games given earlier, it becomes necessary to extend the set of conditions to include more than just histories, and indeed more than just information sets.

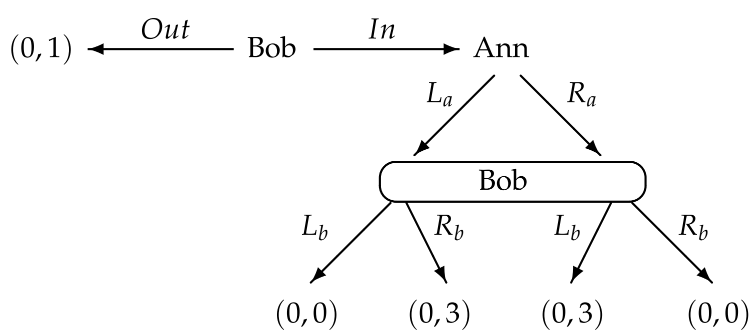

Figure 3: A game with imperfect information

Consider the game depicted in Figure 3. Let z^{LL} denote the path (In, L_a, L_b), and define z^{LR}, z^{RL} and z^{RR} analogously. What is Bob's expected utility from action L_b, given that he chose In? In order to give a formal answer, the set of paths $\{z^{LL}, z^{RL}\}$, which we cannot identify with a standard information set of Bob, must be added as a condition, and Bob's expected utility must be computed using his beliefs given this condition. Similarly, to evaluate action R_b, the set of paths $\{z^{LR}, z^{RR}\}$ must be added as a condition.

Another, more serious and subtler issue is brought about by imperfect information, as we now demonstrate using again the game in Figure 3. Absent further assumptions, a rationally planning and materially consistent Bob can choose Out based on the calculation that, conditional on In, the expected utility of both L_b and R_b is strictly less than 1. Such a calculation contradicts the traditional expected utility argument—by which a maximizing Bob cannot expect less than 3/2 whatever his beliefs about Ann may be—but is compatible with Bob's rational planning and material consistency in our framework. Indeed, all we need is a type of Bob who, conditional on In, attaches probability 1 to $\{z^{LL}, z^{RL}\}$ (i.e. plans to choose L_b), probability $p < 1/3$ to z^{RL} given $\{z^{LL}, z^{RL}\}$ (L_a given L_b), and probability $q < p$ to z^{LR}

given $\{z^{LR}, z^{RR}\}$ (L_a given R_b). Note that, although the sets of paths $\{z^{LL}, z^{RL}\}$ and $\{z^{LR}, z^{RR}\}$ only differ because of Bob's own action, such a type of Bob holds different beliefs about Ann conditional on such events. In effect, if these beliefs were the same, whatever they were, the conditional (on In) expected payoffs of L_b and R_b would sum to 3, as per the traditional expected utility argument.

Interestingly, this desirable invariance of Bob's beliefs about Ann's behavior obtains as a result when *all* properties inherited from a strategy-based structure, generalized in the obvious way to the imperfect information setup, are imposed. To see why, let T_a^L (resp. T_a^R) denote the set of types of Ann that plan to choose L_a (resp. R_a) with probability one. If Ann's material consistency is uncontradictable, the sets T_a^L and T_a^R are both nonempty, and if every type of Ann has deterministic beliefs about her own behavior, they form a partition of Ann's space of types. Thus,

$$MC_a \cap [In, L_a] = MC_a \cap [In] \cap [T_a^L] \quad \text{and} \quad MC_a \cap [In, R_a] = MC_a \cap [In] \cap [T_a^R],$$

where we use square brackets to denote events. If Bob strongly believes that Ann is materially consistent, then he assigns probability one to MC_a conditional on $[In]$ and also conditional on $[In, L_b]$ and $[In, R_b]$. Moreover, if Bob satisfies own action independence, the probabilities he ascribes to $[T_a^L]$ and $[T_a^R]$ are the same, whether they are conditional on $[In]$, $[In, L_b]$ or $[In, R_b]$. Then, by the equalities above, $SB_b(MC_a) \cap I_b$ implies that Bob's belief about Ann's action is also independent of Bob's own action. In words, if the proper subgame is reached, under deterministic plans a materially consistent Ann takes a given action if and only if she plans to choose it with probability one. Thus, if Bob's belief about Ann's type (and hence her plan) is independent of his action, and conditional on his move he still believes that Ann is materially consistent, then also Bob's belief about Ann's action is independent of his action.

A Appendices

A.1 Proof of the second claim in Proposition 1

Fix a Nash equilibrium in behavioral strategies $b^* = (b_i^*)_{i \in N}$. Since the game has perfect information and no relevant ties, b^* yields some path $z^* = (a_1^*, a_2^*)$ with probability one. We construct a simple type structure with a state $(z^*, t^*) \in CB(MR)$.

For each player there is only one type with beliefs determined by b^* as follows: for all

$i \in N$, $a_1 \in A(\varnothing)\backslash\{a_1^*\}$, $a_2 \in A(a_1)$,

$$\begin{aligned} T_i &= \{t_i^*\}, \\ \beta_i(t_i^*)((a_1^*, a_2^*), t_{-i}^*|\varnothing) &= \beta_i(t_i^*)((a_1^*, a_2^*), t_{-i}^*|a_1^*) = 1, \\ \beta_i(t_i)(t_i^*)((a_1, a_2), t_{-i}^*|a_1) &= \begin{cases} b_{\iota(a_1)}^*(a_2|a_1) & \text{if } i \neq \iota(a_1), \\ 1 & \text{if } i = \iota(a_1) \text{ and } a_2 = s^{BI}(a_1), \\ 0 & \text{if } i = \iota(a_1) \text{ and } a_2 \neq s^{BI}(a_1). \end{cases} \end{aligned}$$

In words, players initial beliefs are correct, each player plans to use her optimal action in the second stage *off* the z^* path and has beliefs about the opponents' behavior conditional on *off*-path actions determined by b^*. Since b^* is an equilibrium, $a_2^* = s^{BI}(a_1^*)$. Thus, all players different from the first mover $\iota(\varnothing)$ are materially rational at (z^*, t^*). To see that also the first mover is materially rational at this state, consider that, since b^* is a Nash equilibrium, if $\iota(a_1) = \iota(\varnothing)$ then $u_{\iota(\varnothing)}(z^*) \geq u_{\iota(\varnothing)}(a_1, s^{BI}(a_1))$, if $\iota(\varnothing) \neq \iota(a_1)$ then $u_{\iota(\varnothing)}(z^*) \geq \sum_{a_2 \in A(a_1)} u_{\iota(\varnothing)}(a_1, a_2) b_{\iota(a_1)}^*(a_2|a_1)$. Thus $(z^*, t^*) \in MR$, furthermore, by construction, at (z^*, t^*) every player unconditionally believes (z^*, t^*). Since each player i has only one type, t_i^*, this implies $(z^*, t^*) \in CB(MR)$.

A.2 The type structure for the three-legged centipede

Ann's set of types is $T_a = \{t_a^*, t_a^{BI}, \hat{t}_a\}$ and Bob's is $T_b = \{t_b^*, t_b^{BI}\}$. The beliefs of each type of Ann are described by a table where each row corresponds to a path z, each column corresponds to a conditioning event (a non-terminal history h), and for each (z, h) the corresponding cell describes the type's beliefs over $\{z\} \times \{t_b^*, t_b^{BI}\}$ conditional on h (for example, looking at row (C, d) and column C of matrix t_a^{BI}, we see that t_a^{BI} assigns probability one to path (C, d) and type t_b^{BI} conditional on C). Analogous matrices represent, for each type of Bob, his beliefs over $\{z\} \times \{t_a^*, t_a^{BI}, \hat{t}_a\}$.

t_a^*	$\varnothing$	C	C,c
D	0,0	0,0	0,0
C,d	0,0	0,0	0,0
C,c,D'	1,0	1,0	1,0
C,c,C'	0,0	0,0	0,0

t_a^{BI}	$\varnothing$	C	C,c
D	0,1	0,0	0,0
C,d	0,0	0,1	0,0
C,c,D'	0,0	0,0	0,1
C,c,C'	0,0	0,0	0,0

$\hat{t}_a$	$\varnothing$	C	C,c
D	0,0	0,0	0,0
C,d	0,0	0,0	0,0
C,c,D'	0,0	0,0	0,0
C,c,C'	1,0	1,0	1,0

t_b^*	$\varnothing$	C	C,c
D	0,1,0	0,0,0	0,0,0
C,d	0,0,0	0,0,0	0,0,0
C,c,D'	0,0,0	0,0,0	0,0,0
C,c,C'	0,0,0	0,0,1	0,0,1

t_b^{BI}	$\varnothing$	C	C,c
D	0,1,0	0,0,0	0,0,0
C,d	0,0,0	0,1,0	0,0,0
C,c,D'	0,0,0	0,0,0	0,1,0
C,c,C'	0,0,0	0,0,0	0,0,0

The beliefs of types t_a^{BI} and t_b^{BI} correspond to the backward-induction plans and path. The backward-induction state is $((D), t_a^{BI}, t_b^{BI})$, whereas $((C, c, D'), t_a^*, t_b^*)$ is a state where Ann surprises Bob and tricks him into thinking that she is not materially rational. It is clear that t_a^{BI} and t_b^{BI} satisfy rational planning, and it can be checked that t_a^* and t_b^* also do. Indeed, at state $((C, c, D'), t_a^*, t_b^*)$ there is (correct, unconditional) common belief in material rationality, and a non-Nash outcome occurs. To see this, first note that

$$((D), t_a^{BI}, t_b^{BI}) \in MR \cap B(MR) \cap B(B(MR)) \cap \cdots \subseteq B_i(MR \cap B(MR) \cap B(B(MR)) \cap \cdots).$$

It is easily checked that at $((C, c, D'), t_a^*, t_b^*)$ both players are materially consistent and plan rationally and that Ann's beliefs (conditional and unconditional) are correct. Bob's beliefs are incorrect, but he unconditionally assigns probability one to state $((D), t_a^{BI}, t_b^*)$, hence Bob (unconditionally) believes MR and that Ann unconditionally believes $MR \cap B(MR) \cap B(B(MR)) \cap \cdots$. This implies that $((C, c, D'), t_a^*, t_b^*) \in MR \cap B(MR) \cap B(B(MR)) \cap \cdots$.

A.3 Complete structures contain the BI structure

Lemma 1. *A complete type structure contains the BI structure.*

Proof. For every h, let $z^{BI}(h)$ and $s^{BI}(h)$ denote, respectively, the BI path in the subgame starting at h, and the BI action at h (if h is nonterminal). We construct by recursion a sequence of type profiles $(t^n)_{n=0}^{\infty}$ that converges to some t^{BI} satisfying

$$\forall i \in N, \forall h \in H, \quad \beta_i(t_i^{BI})(\{(z^{BI}(h), t_{-i}^{BI})\}|h) = 1 \tag{8}$$

Fix a type profile t^0 arbitrarily. For the inductive step of the recursive construction, suppose we have constructed a type profile t^{n-1}, $n = 1, 2, ...$; then for each player i there is a unique conditional probability system $\mu_i^n \in \Delta^H(Z \times T_{-i})$ such that

$$\forall h \in H, \quad \mu_i^n(\{(z^{BI}(h), t_{-i}^{n-1}\}|h) = 1.$$

By completeness, there is some type t_i^n such that $\beta_i(t_i^n) = \mu_i^n$. Suppose we have constructed types t_i^n, $i \in N$, $n = 1, 2, ...$ By compactness, the sequence $(t^n)_{n=0}^{\infty}$ has a convergent subsequence $(t^{n_k})_{k=1}^{\infty}$. Call t^{BI} the limit of this subsequence. We must show that t^{BI} satisfies (8). By continuity of β_i, $\beta_i(t_i^{BI}) = \lim_{k\to\infty} \beta_i(t_i^{n_k})$. All the conditional probability systems $\beta_i(t_i^{n_k})$ are arrays of Dirac measures $\delta_{i,h}^k$ ($h \in H$) where each $\delta_{i,h}^k$ is concentrated on point $(z^{BI}(h), t_{-i}^{n_{k-1}})$. Since $(z^{BI}(h), t_{-i}^{n_{k-1}})$ converges to $(z^{BI}(h), t_{-i}^{BI})$, $\delta_{i,h}^k$ converges to δ_i^{BI}, the Dirac measure concentrated on $(z^{BI}(h), t_{-i}^{BI})$. Therefore (8) is satisfied.

A.4 Proof of Theorem 1

Fix a state $(z^*, t^*) \in CSBMR$ and a player i. At this state, i has beliefs about his own behavior represented by the behavioral strategy b_i^* with $b_i^*(a|h) = \beta_i(t_i^*)((h,a)|h)$ for each $h \in H_i$, and he has beliefs about the opponents' behavior represented by the unique behavioral strategy profile $b_{i,-i}^*$ with $b_{i,-i}^*(a|h) = \beta_i(t_i^*)((h,a)|h)$ for each $h \in H_{-i}$. Since $(z^*, t^*) \in RP_i$, b_i^* is a best response to $b_{i,-i}^*$, therefore every pure strategy s_i in the support of b_i^* is also a best response to $b_{i,-i}^*$. By inspection of the proof of Proposition 3, for each $h \prec z^*$, $\beta_i(t_i)((h, \alpha(h, z^*))|h) = 1$, which implies that i's beliefs are confirmed by path z^*. Therefore the behavior strategy profile $(b_i^*)_{i \in N}$ is a self-confirming equilibrium with independent, unitary beliefs, as defined in Fudenberg and Levine (1993),[21] and z^* is the path resulting from $(b_i^*)_{i \in N}$ with probability one. By the Corollary in Kamada (2010), in a perfect information game every self-confirming equilibrium with independent, unitary beliefs is realization-equivalent to a (mixed) Nash equilibrium. Hence z^* is a Nash equilibrium path.

A.5 Proof of Theorem 2

Fix a complete structure $(T_i, \beta_i)_{i \in N}$. The idea of the proof is that CSBMR in such a structure yields first-order beliefs consistent with extensive form rationalizability, henceforth abbreviated as EFR (see Pearce, 1984; Battigalli, 1997). By this we mean that for every $(z,t) \in CSBMR$ and $i \in N$ there is a corresponding conditional probability system $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$, where the set of conditions is the family of sets of the form $S_{-i}(h)$ with $h \in H$, and $S_{-i}(h)$ is the set of strategy profiles of players $-i$ that allow h, satisfying the following: for every $h \in H_{-i}$, $a \in A(h)$ and $n \geq 0$, $\beta_i(t_i)(a|h) = \nu_{-i}(t_i)(S_{-i}(h,a)|S_{-i}(h))$ and if $S_{-i}(h) \cap S_{-i}^n \neq \varnothing$, then $\nu_{-i}(t_i)(S_{-i}^n|S_{-i}(h)) = 1$, where S_i^n is the nonempty set of strategies that survive n steps of the EFR procedure, and $S_{-i}^n = \times_{j \neq i} S_j^n$ (see (13), (14) and Theorem 3 below). Given such beliefs, the unique local best reply at each node on the BI path is the BI action, this follows from Battigalli (1997).

As a preliminary observation, note that for each $\nu_{-i} \in \Delta^H(S_{-i})$ we obtain a corresponding decision tree $\Gamma_i(\nu_{-i})$ for player i by assigning to each action $a \in A(h)$ of the other players ($h \in H_{-i}$) the conditional probability $\nu_{-i}(S_{-i}(h,a)|S_{-i}(h))$. Then we can determine by backward-induction on $\Gamma_i(\nu_{-i})$ the set of optimal actions $A^*(h, \nu_{-i})$ for each $h \in H_i$. We will map types to conditional probability systems on S_{-i} and *vice versa*, so that both determine the same decision tree. The following lemma shows how to associate each t_i with a corresponding $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$. Recall that for every $i \in N$ and $t_i \in T_i$, the probabilities $\beta_i(t_i)((h,a)|h)$ describe a behavior strategy profile as j varies in N and h in H_j.

[21] See Definitions 1, 4, 5 and note that every perfect information game has observed deviators.

Lemma 2. *Fix $i \in N$ and $t_i \in T_i$. For all $h \in H$ and $s_{-i} \in S_{-i}$, define*

$$\nu_{-i}(t_i)(s_{-i}|S_{-i}(h)) = \begin{cases} 0 & \text{if } s_{-i} \notin S_{-i}(h), \\ \prod_{h' \in H_{-i}: h' \not\prec h} \beta_i(t_i)((h', s'_{-i}(h'))|h') & \text{if } s_{-i} \in S_{-i}(h). \end{cases} \tag{9}$$

Then $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$, and for all $h \in H_{-i}$ and $a \in A(h)$,

$$\nu_{-i}(t_i)(S_{-i}(h,a)|S_{-i}(h)) = \beta_i(t_i)((h,a)|h). \tag{10}$$

Proof. Regard $-i$ as a coalition. By perfect information, $-i$ has perfect recall.[22] Consider the following behavior strategy b^h_{-i} of $-i$: for all $h' \in H_{-i}$ and $a \in A(h')$,

$$b^h_{-i}(a|h') = \begin{cases} 1 & \text{if } h' \prec h \text{ and } a = \alpha(h', h), \\ 0 & \text{if } h' \prec h \text{ and } a \neq \alpha(h', h), \\ \beta_i(t_i)((h', a)|h') & \text{if } h' \not\prec h. \end{cases}$$

By Kuhn's theorem, b^h_{-i} induces a realization-equivalent mixed strategy $\nu^h_{-i} \in \Delta(S_{-i})$ of $-i$, with

$$\begin{aligned} \nu^h_{-i}(s_{-i}) &= \prod_{h' \in H_{-i}} b^h_{-i}(s_{-i}(h')|h) \\ &= \begin{cases} \prod_{h' \in H_{-i}: h' \not\prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') & \text{if } s_{-i}(h') = \alpha(h', h) \\ & \text{for all } h' \in H_{-i} \text{ with } h' \prec h, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the second equality follows from the definition of b^h_{-i}. Thus, $\nu^h_{-i}(s_{-i}) = 0$ for every $s_{-i} \notin S_{-i}(h)$. It follows that $\nu_{-i}(t_i)(\cdot|S_{-i}(h)) = \nu^h_{-i}(\cdot) \in \Delta(S_{-i})$ and $\nu_{-i}(t_i)(S_{-i}(h)|S_{-i}(h)) = 1$. Then (10) follows from the realization-equivalence of b^h_{-i} and $\nu_{-i}(t_i)(\cdot|S_{-i}(h))$. To show that $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$, we only have to verify the chain rule. Fix non terminal histories $g \prec h$, so that $S_{-i}(h) \subseteq S_{-i}(g)$ and pick any $s_{-i} \in S_{-i}(h)$, so that s_{-i} selects every action of $-i$ in h and hence also in g, then

$$\begin{aligned} \nu_{-i}(t_i)(s_{-i}|S_{-i}(g)) &= \prod_{h' \in H_{-i}: h' \not\prec g} \beta_i(t_i)((h', s_{-i}(h'))|h') \\ &= \prod_{h' \in H_{-i}: h' \not\prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') \prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') \\ &= \nu_{-i}(t_i)(s_{-i}|S_{-i}(h)) \nu_{-i}(t_i)(S_{-i}(h)|S_{-i}(g)), \end{aligned}$$

where the second equality follows from the fact that $g \prec h$ implies that the two sets $\{\bar{h} \in H_{-i} : \bar{h} \not\prec h\}$ and $\{\bar{h} \in H_{-i} : g \preceq \bar{h} \prec h\}$ partition $\{\bar{h} \in H_{-i} : \bar{h} \not\prec g\}$, and the third

[22] In games with imperfect information $-i$ does not have perfect recall and $\nu_{-i}(t_i)(\cdot|S_{-i}(h))$ is a correlated strategy of $-i$. The proof still works.

equality follows from the construction of $\nu_{-i}(t_i)(\cdot|S_{-i}(g))$ and its realization-equivalence with b^g_{-i}. $\nu_{-i}(t_i)(S_{-i}(h)|S_{-i}(g))$ is the probability of reaching h from g given that each action of i on this path has conditional probability one; since s_{-i} allows h, this transition probability is

$$\prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') = \prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', \alpha(h', h))|h').$$

■

The next lemma shows that, given some map $\tau_{-i} : S_{-i} \to T_{-i}$, we can associate to each pair $(s_i, \nu_{-i}) \in S_i \times \Delta^H(S_{-i})$ a conditional probability system in $\Delta^H(Z \times T_{-i})$ corresponding to (s_i, ν_{-i}) in the following sense. The marginal probabilities on Z (which are equivalent to a behavior strategy profile) agree with s_i, when the latter is viewed as i's conditional belief about her own actions, whereas the conditional belief about the actions of others are derived from ν_{-i} in the spirit of Kuhn's transformation from mixed to behavioral strategies. Finally, the marginal on T_{-i} is obtained from τ_{-i}. An important feature of this conditional probability system is that it always assigns probability one to the material consistency of the other players. Let $\pi(s|h)$ denote the path induced by strategy profile s starting from history h, writing just $\pi(s)$ for $h = \varnothing$.

Lemma 3. *Fix $(s_i, \nu_{-i}) \in S_i \times \Delta^H(S_{-i})$ and a mapping $\tau_{-i} : S_{-i} \to T_{-i}$. There exists $\psi_i \in \Delta^H(Z \times T_{-i})$ such that for all $h \in H$ and $s_{-i} \in S_{-i}(h)$,*

$$\psi_i(\{(\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})\}|h) = \nu_{-i}(s_{-i}|S_{-i}(h)).$$

Proof. For every $h \in H$, let $\psi_i(\cdot|h)$ be the finite-support probability measure defined as follows: for every $h \in H$ and $E \subseteq Z(h) \times T_{-i}$,

$$\psi_i(E|h) = \nu_{-i}\left(\{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E\}|S_{-i}(h)\right).$$

We must show that ψ_i satisfies the chain rule: for all $g \prec h$ and $E \subseteq Z(h) \times T_{-i}$,

$$\psi_i(E|g) = \psi_i(E|h)\psi_i(h|g).$$

There are two cases: either s_i makes h unreachable from g, so that both $\psi_i(E|g)$ and $\psi_i(h|g)$ are zero, or s_i always selects the action in h at each $h' \in H_i$ with $g \preceq h' \prec h$. In the first case the equality is trivially satisfied. In the second case, since $E \subseteq Z(h) \times T_{-i}$, we have

$$\{s_{-i} \in S_{-i}(g) : (\pi(s_i, s_{-i}|g), \tau_{-i}(s_{-i})) \in E\} = \{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E\},$$

and hence

$$
\begin{aligned}
\psi_i(E|g) &= \sum_{s_{-i}\in S_{-i}(g):(\pi(s_i,s_{-i}|g),\tau_{-i}(s_{-i}))\in E} \nu_{-i}(s_{-i}|S_{-i}(g)) \\
&= \sum_{s_{-i}\in S_{-i}(h):(\pi(s_i,s_{-i}|h),\tau_{-i}(s_{-i}))\in E} \nu_{-i}(s_{-i}|S_{-i}(g)) \\
&= \nu_{-i}(S_{-i}(h)|S_{-i}(g)) \sum_{s_{-i}\in S_{-i}(h):(\pi(s_i,s_{-i}|h),\tau_{-i}(s_{-i}))\in E} \nu_{-i}(s_{-i}|S_{-i}(h)) \\
&= \psi_i(E|h)\psi_i(h|g),
\end{aligned}
$$

where the first and last equalities follow from the definition of ψ_i and the third is implied by the chain rule for ν_{-i}. ■

Note that by construction ψ_i is such that $\psi_i((h, s_i(h))|h) = 1$ for every $h \in H_i$. In words, s_i is the plan of i entailed by ψ_i. In what follows, write $\psi_i(s_i, \nu_{-i}; \tau_{-i})$ instead of ψ_i to emphasize the dependence on s_i, ν_{-i} and τ_{-i}.

To make the proof of the main result a bit shorter we adapt the recursive definition of EFR from the characterization given by Battigalli (1997). Let $S_i^0 = S_i$; given S_i^n and $S_{-i}^n = \prod_{j\neq i} S_j^n$, the set of $(n+1)$-rationalizable strategies for i is defined as[23]

$$
S_i^{n+1} = \left\{ \begin{array}{c} s_i : \exists \nu_{-i} \in \Delta^H(S_{-i}), \forall m \in \{1, ..., n\}, \forall h \in H, \\ S_{-i}^m \cap S_{-i}(h) \neq \varnothing \Rightarrow \nu_{-i}(S_{-i}^m|S_{-i}(h)) = 1, \forall h \in H_i, s_i(h) \in A^*(h, \nu_{-i}) \end{array} \right\}. \tag{11}
$$

(By convention, the condition $\forall m \in \{1, ..., n\}$, $\forall h \in H$, $S_{-i}^m \cap S_{-i}(h) \neq \varnothing \Rightarrow \nu_{-i}(S_{-i}^m|S_{-i}(h)) = 1$ is trivially satisfied for $n = 0$.) It can be shown that each set S_i^n is nonempty and that $s_i \in S_i^n$ if and only if every strategy s_i' realization-equivalent to s_i survives n steps of the definition of EFR found in the literature.

We now use completeness to recursively define, for each $n \geq 0$, a profile of functions $\tau^n = (\tau_i^n)_{i\in N}$, where $\tau_i^n : S_i \to T_i$ for each player i, such that for each strategy profile $s \in S^{n+1}$, the profile of types $\tau^{n+1}(s)$ satisfies $(\pi(s), \tau^{n+1}(s)) \in \underline{CSB}^n\big((MC_i \cap RP_i)_{i\in N}\big)$. For each player i, define τ_i^0 arbitrarily. Suppose $(\tau_j^n)_{j\in N}$ has been defined and let $\tau_{-i}^n(s_{-i}) = (\tau_j^n(s_j))_{j\neq i}$. Now, for each $s_i \in S_i \setminus S_i^{n+1}$, let $\tau_i^{n+1}(s_i) = \tau_i^n(s_i)$. For each $s_i \in S_i^{n+1}$ there is some $\nu_{-i}(s_i)(\cdot|\cdot) \in \Delta^H(S_{-i})$ such that:

$$
\begin{aligned}
\forall h \in H,\ \forall m \in \{0, ..., n\}, &\qquad S_{-i}^m \cap S_{-i}(h) \neq \varnothing \Rightarrow \nu_{-i}(s_i)(S_{-i}^m|S_{-i}(h)) = 1; \\
\forall h \in H_i, &\qquad s_i(h) \in A^*(h, \nu_{-i}(s_i));
\end{aligned} \tag{12}
$$

[23] In this definition we are assuming that s_i prescribes a local best response to the justifying belief ν_{-i} at each $h \in H_i$. This is realization-equivalent to assuming that s_i yields a "global" best response in each subtree $\Gamma_i(h, \nu_{-i})$ with root $h \in H_i$ allowed by s_i, which is the best response property required in the papers about extensive form rationalizability.

then pick

$$\tau_i^{n+1}(s_i) \in \beta_i^{-1}(\psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^n))$$

arbitrarily, where $\psi_i(\cdot)$ is given by Lemma 3 and $\beta_i^{-1}(\psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^n))$ is nonempty because β_i is onto (completeness). Given $s = (s_i)_{i \in N}$, let $\tau^{n+1}(s) = (\tau_i^{n+1}(s_i))_{i \in N}$.

The following lemma decomposes event $CSBMR$ into separate events about each player's material consistency, rational planning, and strong belief in different "degrees of strategic sophistication" of the other players.

Lemma 4. *Let $RP_i^0 = RP_i$ and $RP_i^{m+1} = RP_i^m \cap SB_i(\cap_{j \neq i}(MC_j \cap RP_j^m))$ for $m \geq 0$. Then*

$$RP_i^{m+1} = RP_i \cap \big(\cap_{k=0}^m SB_i(\cap_{j \neq i} MC_j \cap RP_j^k)\big) \qquad \forall m \geq 0,$$

and

$$\underline{CSB}^{m+1}\big((MC_i \cap RP_i)_{i \in N}\big) = (MC_i \cap RP_i^{m+1})_{i \in N} \qquad \forall m \geq 0.$$

Therefore,

$$CSBMR = \cap_{m=0}^{\infty} \cap_{i \in N} \big(MC_i \cap RP_i \cap (\cap_{k=0}^m SB_i(\cap_{j \neq i} MC_j \cap RP_j^k))\big).$$

Proof. The proof is by induction in m. The results are obvious for $m = 0$. Now let $m \geq 0$ and suppose that

$$RP_i^m = RP_i \cap \big(\cap_{k=0}^{m-1} SB_i(\cap_{j \neq i} MC_j \cap RP_j^k)\big).$$

Then

$$\begin{aligned}
RP_i^{m+1} &= RP_i^m \cap SB_i\big(\cap_{j \neq i} MC_j \cap RP_j^m\big) \\
&= RP_i \cap \big(\cap_{k=0}^{m-1} SB_i(\cap_{j \neq i} MC_j \cap RP_j^k)\big) \cap SB_i\big(\cap_{j \neq i} MC_j \cap RP_j^m\big) \\
&= RP_i \cap \big(\cap_{k=0}^{m} SB_i(\cap_{j \neq i} MC_j \cap RP_j^k)\big),
\end{aligned}$$

where the first equality holds by definition of RP_i^{m+1} and the second follows from the induction hypothesis. Now suppose that

$$\underline{CSB}^m\big((MC_i \cap RP_i)_{i \in N}\big) = (MC_i \cap RP_i^m)_{i \in N}.$$

Then

$$\begin{aligned}
\underline{CSB}^{m+1}\big((MC_i \cap RP_i)_{i \in N}\big) &= \underline{CSB}\big(\underline{CSB}^m((MC_i \cap RP_i)_{i \in N})\big) \\
&= \underline{CSB}\big((MC_i \cap RP_i^m)_{i \in N}\big) \\
&= \big(MC_i \cap RP_i^m \cap SB_i(\cap_{j \neq i} MC_j \cap RP_j^m))\big)_{i \in N} \\
&= \big((MC_i \cap RP_i^{m+1})_{i \in N}\big),
\end{aligned}$$

where the first equality holds by definition of $\underline{CSB}^{m+1}$, the second by the induction hypothesis, the third by definition of $\underline{CSB}$, and the fourth by definition of RP_i^{m+1}. ■

In what follows, for every $i \in N$ and $n \geq 0$ write $\overline{RP}_i^n$ for the set of all $t_i \in T_i$ such that $(z, t_i, t_{-i}) \in RP_i^n$ for some (and hence for all) $(z, t_{-i}) \in Z \times T_{-i}$. Similarly, write $\overline{MC}_{-i}$ for the set of all $(z, t_{-i}) \in Z \times T_{-i}$ such that $(z, t_i, t_{-i}) \in MC_{-i}$ for some (and hence for all) $t_i \in T_i$.

Claim. *For every $n \geq 0$, $i \in N$ and $s_i \in S_i$,*

$$s_i \in S_i^{n+1} \Rightarrow \tau_i^{n+1}(s_i) \in \overline{RP}_i^n, \tag{13}$$

$$(t_i \in \overline{RP}_i^n, \forall h \in H_i, \beta_i(t_i)((h, s_i(h))|h) > 0) \Rightarrow s_i \in S_i^{n+1} \tag{14}$$

Proof. The proof is by induction in n. For the case $n = 0$, suppose that $s_i \in S_i^1$. Then $\tau_i^1(s_i) = \psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^0) \in \overline{RP}_i^0$. Conversely, pick $t_i \in \overline{RP}_i^0$ with $\beta_i(t_i)((h, s_i(h))|h) > 0$ for all $h \in H_i$. Derive $\nu_{-i}(t_i)(\cdot|\cdot)$ from $\beta_i(t_i)$ as in (9). By Lemma 2, $\nu_{-i}(t_i)$ satisfies (10), therefore $\beta_i(t_i)$ and $\nu_{-i}(t_i)$ determine the same decision tree. Since $t_i \in \overline{RP}_i^0$ and $\beta_i(t_i)((h, s_i(h))|h) > 0$ for every $h \in H_i$, s_i must be optimal in this decision tree. Hence $s_i \in S_i^1$.

Now suppose by way of induction that (13) and (14) hold for every natural number smaller than $n > 0$. First we show that this induction hypothesis implies that, for each $m = 0, ...n - 1$,

$$MC_{-i} \cap [h] \cap RP_{-i}^m \neq \varnothing \quad \Longleftrightarrow \quad S_{-i}(h) \cap S_{-i}^{m+1} \neq \varnothing. \tag{15}$$

Indeed, fix any state (z, t_i, t_{-i}) in the intersection on the left-hand side above, so that $h \prec z$. Then there is some s'_{-i} such that for every $j \neq i$ and $h' \in H_j$, $\beta_j(t_j)((h', s_j(h')|h') > 0$, and $s'_j(h') = \alpha(h', z)$ whenever $h' \prec z$. Since $h \prec z$, $s'_{-i} \in S_{-i}(h)$. By the induction hypothesis, $s'_{-i} \in S_{-i}^{m+1}$. Thus, $S_{-i}(h) \cap S_{-i}^{m+1} \neq \varnothing$. Conversely, pick $s'_{-i} \in S_{-i}(h) \cap S_{-i}^{m+1}$. By the induction hypothesis, $s'_{-i} \in S_{-i}^{m+1}$ implies $\tau_j^{m+1}(s'_j) \in \overline{RP}_j^m$ for each $j \neq i$. Since $s'_{-i} \in S_{-i}(h)$, there is some $z' \in Z(h)$ such that $s'_{-i} = \alpha(h', z')$ for each $h' \in H_{-i}$ with $h' \prec z'$. The construction of τ_{-i}^{m+1} implies that $\{(z', \tau_{-i}^{m+1}(s'_{-i}))\} \times T_i \subseteq MC_{-i}$. Hence, $\{(z', \tau_{-i}^{m+1}(s'_{-i}))\} \times T_i \subseteq MC_{-i} \cap [h] \cap RP_{-i}^m$, and (15) follows.

To prove (13), let $s_i \in S_i^{n+1}$. We claim that $\tau_i^{n+1}(s_i) \in \overline{RP}_i^n$. By definition of $\overline{RP}_i^n$ and Lemma 4, we have to show that $\tau_i^{n+1}(s_i) \in \overline{RP}_i^0$ and that type $\tau_i^{n+1}(s_i) = \psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^n)$ strongly believes each event $MC_{-i} \cap RP_{-i}^m$ with $m = 0, ..., n - 1$. The former is true by construction. Now, fix $m = 0, ..., n - 1$ and h with $[h] \cap MC_{-i} \cap RP_{-i}^m \neq \varnothing$. By (15), $S_{-i}(h) \cap S_{-i}^{m+1} \neq \varnothing$, and by (12), $\nu_{-i}(s_{-i})(S_{-i}^{m+1}|S_{-i}(h)) = 1$. This implies that

$$\beta_i(\tau_i^{n+1}(s_i))(\{(\pi(s_i, s'_{-i}|h), \tau_{-i}^n(s'_{-i})) : s'_{-i} \in S_{-i}^m \cap S_{-i}(h)\}|h) = 1.$$

By construction of τ_{-i}^n and the induction hypothesis, this implies

$$\beta_i(\tau_i^{n+1}(s_i))(\overline{MC}_{-i} \cap (Z(h) \times \overline{RP}_{-i}^m)|h) = 1.$$

Thus, $\tau_i^{n+1}(s_i) \in \overline{RP}_i^n$.

Finally, to prove (14), suppose that $t_i \in \overline{RP}_i^n$ and $\beta_i(t_i)((h, s_i(h))|h) > 0$ for all $h \in H_i$. Define $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$ as in (9) of Lemma 2. Since $\beta_i(t_i)$ and $\nu_{-i}(t_i)$ determine the same decision tree and $t_i \in \overline{RP}_i$, s_i is optimal in this decision tree. We only have to check that $\nu_{-i}(t_i)$ satisfies the conditions in (11). Fix $h \in H$ and $m < n$. Suppose $S_{-i}^{m+1} \cap S_{-i}(h) \neq \varnothing$. Then $MC_{-i} \cap [h] \cap RP_{-i}^m \neq \varnothing$ by (15), and by definition of RP_i^n,

$$\beta_i(t_i)(\overline{MC}_{-i} \cap (Z(h) \times \overline{RP}_{-i}^m)|h) = 1.$$

The definition of $\nu_{-i}(t_i)$ implies $\nu_{-i}(t_{-i})(S_{-i}^{m+1}|S_{-i}(h)) = 1$. This proves that $s_i \in S_i^{n+1}$. ■

Theorem 3. *Fix a complete type structure and a terminal history $z \in Z$. The following are equivalent:*

(i) There exists a profile of types t such that CSBMR holds at state (z, t).

(ii) There exists a rationalizable strategy profile s such that $z = \pi(s)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $CSBMR$ holds at state (z, t), so that $(z, t) \in \cap_{i \in N}(MC_i \cap RP_i^n)$ for every n. For each player i, pick any strategy s_i such that $\beta_i(t_i)((h, s_i(h))|h) > 0$ for all $h \in H_i$. By (14), s_i is rationalizable, whereas material consistency implies that $\pi(s) = z$.

$(ii) \Rightarrow (i)$. Suppose that s is rationalizable and $\pi(s) = z$. Let K be the first integer n such that $S^{n+1} = S^n$. By construction $\tau^K = \tau^n$ for every $n \geq K$. By (13), $Z \times \{\tau_i^{n+1}(s_i)\} \times T_{-i} \subseteq RP_i^n$ for every i and n. By construction, there is material consistency at state $(z, t) = \big(\pi(s), (\tau^K(s)\big)$. Thus, $CSBMR$ holds at (z, t). ■

Corollary 2. *Fix a complete type structure; then CSBMR holds at some state (z, t), and $z = z^{BI}$ for every such state.*

Proof. The definition of rationalizability in (11) is realization-equivalent to the one analyzed by Battigalli (1997), who shows that in a perfect information game with no relevant ties, every rationalizable strategy profile induces the backward-induction path. As the set of rationalizable profiles is nonempty, $z = z^{BI}$ if and only if $z = \pi(s)$ for some rationalizable strategy profile s. Thus, by Theorem 3, $z = z^{BI}$ if and only if there exists a profile of types t such that $CSBMR$ holds at (z, t). ■

A.6 Proof of Proposition 4

Let $i = \iota(\varnothing)$ be the first mover and fix $(z, t) \in MR \cap I_i \cap B_i(RP_{-i}) \cap SB_i(MC_{-i})$. Pick any $a_1 \in A(\varnothing)$ and let $j = \iota(a_1)$ be the player who moves after a_1 (if it is not terminal). Let s_i^{BI} and s_j^{BI} be the strategies that select the BI move at every history. As a preliminary observation, note that for every $(z', t') \in MR_j$, if $a_1 \prec z'$ then $z' = (a_1, s_j^{BI}(a_1))$ (see the proofs of Propositions 1 and 2). Now consider the conditional belief $\beta_i(t_i)(\cdot|a_1)$. If $j = i$, that is, if i plays again after a_1, then $(z, t) \in RP_i$ implies $\beta_i(t_i)((a_1, s_i^{BI}(a_1))|a_1) = 1$. Otherwise, assume $j \neq i$, and let us show that $(z, t) \in I_i \cap B_i(RP_{-i}) \cap SB_i(MC_{-i})$ implies $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$. Since the given structure contains the BI structure, it contains the state $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI})$. It is easily checked that $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI}) \in MC_{-i}$. Thus, $(z, t) \in SB_i(MC_{-i})$ implies $\beta_i(t_i)(MC_{-i}|a_1) = 1$ and hence $\beta_i(t_i)(MC_j|a_1) = 1$. Since $(z, t) \in B_i(RP_{-i})$, $\beta_i(t_i)(RP_{-i}|\varnothing) = 1$. Recall that rational planning is a property of types: $RP_{-i} = Z \times T_i \times \overline{RP}_{-i}$, where $\overline{RP}_{-i} \subseteq T_{-i}$. Therefore $(z, t) \in B_i(RP_{-i}) \cap I_i$ implies $\beta_i(t_i)(RP_{-i}|a_1) = 1$ hence $\beta_i(t_i)(RP_j|a_1) = 1$. This and $\beta_i(t_i)(MC_j|a_1) = 1$ imply $\beta_i(t_i)(MR_j|a_1) = 1$. By the preliminary observation above, we then obtain $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$. This is true for every $a_1 \in A(\varnothing)$, so i assigns the BI value to every such action. Since $(z, t) \in MR$, this implies that z is the BI path.

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