Marriage as a Rat Race: Noisy Pre-Marital Investments with Assortative Matching*

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March 2011. Preliminary

Abstract

We study the incentives of parents to invest in their children when these investments improve their marriage prospects, in a frictionless marriage market with non-transferable utility. Stochastic returns to investment eliminate the multiplicity of equilibria that plagues models with deterministic returns, and ensure that a unique equilibrium often exists. Equilibrium investment is efficient when there is complete symmetry between the sexes. However, when there is any asymmetry between the sexes, investments are generically excessively relative to Pareto-efficiency. Our model can be used for examine several implications of gender differences. For example, if shocks are more variable for boys than for girls, girls will invest more than boys. If there is an excess of boys, then there is parental over-investment in boys and under-investment in girls, and total investment will be excessive.

Keywords: marriage, ex ante investments, gender differences, assortative matching tournament, sex ratio.
JEL codes: C72, C78, D62, H31, J12.

*Thanks to Martin Cripps and Klaus Ritzberger for very helpful comments.
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1 Introduction

We study the incentives of parents to invest in their children when these investments also improve their marriage prospects, in a frictionless marriage market with non-transferable utility. It has usually been thought that ex ante investments suffer from the hold-up problem, since a parent will not internalize the effects of such investments in her own child upon the welfare of the child's future spouse. However, Peters and Siow (2002) argue that in large marriage markets where the quality of one's match depends on the level of investment, a parent has an incentive to invest more in order to improve the match of her offspring. They argue that the resulting outcome will be Pareto efficient. This is a remarkable result, since they assume a marriage market without transferable utility – with transferable utility, Cole et al. (2001) show that in large markets, prices can provide incentives for efficient investment decisions.

In this paper, we argue that the optimism of Peters and Siow (2002) must be somewhat tempered. When the return to investment is deterministic, we show that there is very large set of equilibria. These include efficient outcomes, but also a continuum of inefficient ones. In order to overcome this embarrassment of riches, we propose a model where the returns to investment are stochastic. This is also realistic – talent risk is an important fact of life. Recent studies of the inter-generational transmission of wealth, in the tradition of Becker and Tomes (1979), find an inter-generational wealth correlation of 0.4 in the United States, which is far from 1. Equilibrium in this model is unique and we are therefore able to make determinate predictions. The model also allows us to address several questions of normative importance and social relevance. Are investments efficient, in the absence of prices? What are the implications of biological or social differences between the sexes for investment decision? What are the implications of sex ratio imbalances in countries such as China – Wei and Zhang (2009) argue that marriage market competition for scarce women underlies the high savings rate in China.

Our paper is related to the literature on matching tournaments or contests. This literature typically models a situation where there is a fixed set of prizes, and agents on the one side of the market compete by making investments, with prizes being allocated to agents according the rank order of their investments (see for example, Cole et al., 1992 and Hopkins and Kornienko, 2004). If the "prizes" derive no utility from these investments, e.g. when the prize is social status, then an agent's investment exerts a negative positional externality on the other side of the market, so that there is over-investment. On the other hand, if the "prizes" derive utility from these investments – for example, if men compete for a set of women with fixed qualities, or students compete for university places – then either over-investment or under-investment is possible, depending on how much these investments are valued.

In our context, investments are two-sided – the investments of men are valued by women and symmetrically, the investments of women are valued by men. Men do not care directly about how their investments are valued by women, they care only about the consequent improvement in match quality that they get. Women are in a similar situation, since they care only about the improvement in the quality of men that they might get. One might expect therefore, that this could give rise to under-investment or over-investment, depending on parameter values.

Surprisingly, our model yields definitive conclusions. Under very special circumstances, when the sexes are completely symmetric, with identical distributions of shocks and a balanced sex ratio, investments will be efficient. However, if there are any differences between the sexes, investments are generically excessive, as compared to Pareto-efficient investments.

The rest of the paper is set out as follows. Section 2 discusses the problems of a model with deterministic returns, as in Peters and Siow (2002), and also discusses other related literature. Section 3 sets out a general framework and shows that investments are efficient in the absence of gender differences. Section 4 considers a model of gender differences in the distribution of talent shocks. Specializing to the case where there is ex ante homogeneity within each sex, we show that investments are generically excessive, considering in turn additive and multiplicative shocks. Section 5 examines the implications of gender differences more specifically, by considering the case where talent shocks are more dispersed for boys than for girls. Section 6 examines sex ratio imbalances and their effects on investments.

2 Motivation and related literature

2.1 On Peters and Siow (2002)

The fundamental problem is the following: investment in a child is assumed to benefit her spouse if she marries, but the benefit to the spouse is not considered by her parents. There is therefore a gap between the *privately optimal* or Nash investment in a child, which we denote $\bar{x}$, and some *socially optimal* level which is naturally greater. Peters and Siow (2002) (PS) argue that nonetheless there is an equilibrium where parents choose the socially efficient level of investment.

Let us consider the PS model in a particularly simple context which is illustrated in Figure 1. Suppose a boy is matched with a girl. The utility of the boy's parents is increasing unconditionally in the investment level of the girl x_G, but they have to bear the cost of investment x_B in their son. Thus, if they choose x_B purely to maximize their utility, they would choose only the privately optimal investment $\bar{x}_B$. But it is easy to see that the resulting investment levels $(\bar{x}_B, \bar{x}_G)$ are inefficient. Both families would be better off at the point E. Suppose in particular that the family of the boy believes that if they increased investment to x_B^{**} that their son would match better, specifically with a girl with investment x_G^{**}. Then, making that investment would be rational given that belief. Could that belief be consistent, and therefore investments (x_B^{**}, x_G^{**}) be a rational expectations equilibrium?

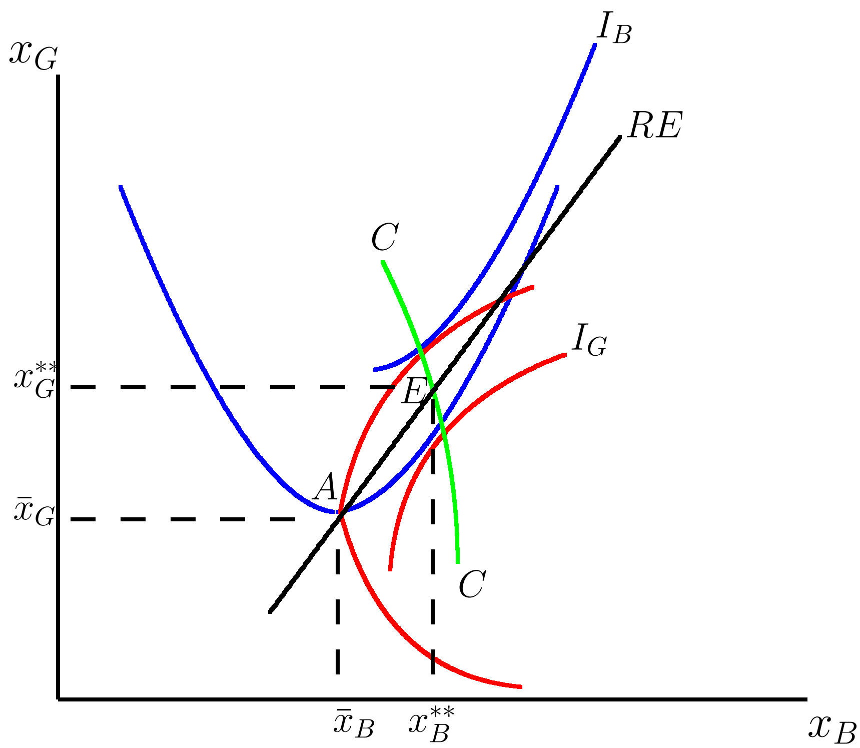

Figure 1: Illustration of basic concepts. I_B, I_G are indifference curves for the boys and the girl respectively. At point A, investments $(\bar{x}_B, \bar{x}_G)$ are privately optimal. Efficient investments are on the contract curve CC. The rational expectations equilibrium is point E.

Assume a unit measure of boys, all of whom are ex ante identical, and an equal measure of girls, who are similarly ex ante identical. Assume that the quality of the child, x, as assessed by the partner in marriage market, equals the level of parental investment. Suppose that all boys invest x_B^{**} and all girls invest x_G^{**}, where this profile is efficient. What prevents a boy from deviating and investing the individually optimal amount, $\bar{x}_B$? If his match prospects are unaffected by this deviation, this would be profitable. So, this deviation can be made unprofitable only if he is punished by being left unmatched. Since we have continuum populations on both sides of the market, this can be achieved without forcing any girl to remain single, and thus, the resulting allocation is stable and measure preserving. In other words, we can construct a matching in the game after investments are realized that is feasible, stable, and measure preserving, such that given this matching rule, a parent has an incentive to invest the efficient amount in a child.

However, despite these restrictions on matching even off the equilibrium path, in terms of equilibria there is an embarrassment of riches. Let (x_B, x_G) be a pair of investments that are weakly greater than the individually optimal investments $(\bar{x}_B, \bar{x}_G)$, and where the payoff of gender i from being matched with a partner with investment level x_j is weakly greater than the payoff from choosing the individually optimal investment level $\bar{x}_i$ and being unmatched. Any such pair can be supported as an equilibrium, by specifying that any agent who deviates to a lower investment level will be left unmatched. Furthermore, if the parent of a boy deviates upwards, and chooses a higher level of investment, his son cannot realize a higher match quality, since all the girls are choosing x_G. We therefore have a "folk theorem" – any pair of investments satisfying the above conditions is an equilibrium. In particular, the configuration where each parent chooses the individually optimal investments can also be supported as an equilibrium. Given that all girls are choosing $\bar{x}_G$, the parent of a boy has no incentive to increase the investment in a boy since this will not improve his match quality.

These arguments apply equally to the original PS environment where families differ in wealth. Even in this case we claim that there is a continuum of inefficient equilibria.[1] The simplest way to see this is to note that almost all of the continuum of equilibria we have constructed in the homogenous case are strict equilibria, in the sense that any individual who invests differently does strictly worse. If we perturb wealth levels slightly, and wealth affects payoffs continuously, then these equilibria will continue to be strict. More specifically, suppose that all families make the same level of investment $\hat{x}$, believing that if they invest less, their child will not marry. The only thing that is required is that the distribution of wealth is not too dispersed, so that there is a common level of investment that first, is not so low that it is below the richest family's privately optimal investment and not so high that the poorest family would prefer to deviate downwards and be unmatched. None of these equilibria are efficient. In fact, for all of them, a measure zero of agents make an efficient investment. If $\hat{x}$ is relatively low then all agents underinvest. If $\hat{x}$ is higher, some agents underinvest and some overinvest.

Even with a more dispersed distribution of wealth, one can construct more complex but still inefficient equilibria. For example, if wealth were more widely distributed then we divide families into two groups, rich and poor, each of which groups has a common level of investment. For example, the matching rule matches those families with sons who choose investment $\hat{x}_L$ to those families with daughters with the same investment, and matches those who choose $\hat{x}_H$ to daughters with the same investment. This suggests that we can construct equilibria even in very dispersed distributions of wealth provided using multiple levels of investment.[2]

[1] PS seem to assume that in a rational expectations equilibrium in this context matching must be strictly monotone in investment. The additional equilibria we consider are only weakly increasing. However, the expectations of the agents are met in equilibrium and in this sense they are rational expectations equilibria.

[2] In contrast, if one assumes that if a boy reduces his investment, he is still matched, but with the lowest type of girl, then all pooling equilibria would seem vulnerable. A deviation by the poorest male family down to the Nash level would not be punished by the poorest female family and so the equilibrium could not be maintained. However, this difficulty applies equally to PS's proposed efficient equilibrium as the efficient level of investment is above the privately optimal level. Thus, under such weak punishment, in the case where all families are ex ante homogenous, the unique equilibrium consists of the individually optimal investments. In the PS setting where families differ in wealth, there seems to be no pure strategy equilibrium. Peters (2007, 2009) acknowledges these difficulties: he finds that equilibria of a non-cooperative game with a finite number of participants do not approach the efficient equilibrium even as the number of participants goes to infinity.

We now show that these problems can be resolved. First, we will be explicit about the matching process that follows investment. Specifically, we shall assume that for any profile of investments, the resulting matching will be feasible, measure preserving and stable. In contrast, the PS notion of a rational expectations equilibrium allows parents to expect "returns" in the matching market that may not be feasible, in the event that a parent chooses an investment level that is off the equilibrium path. Our approach to deriving the payoffs from investment is similar to Cole et al. (2001), who construct these payoffs from stable matching in the assignment game following investments, with the difference that we assume non-transferable utility.

Second, we augment the model by adding an idiosyncratic element of match quality. That is, we assume that the quality of an individual on the marriage market is not entirely determined by investments, but also has a stochastic element. This seems realistic, since talent risk is an important fact of life. While the remainder of this paper sets out the model in full generality, let us suppose here that the match quality of a boy is given by $x_B + \varepsilon$, where ε is distributed on a connected support with a density function $f(\cdot)$ and cdf $F(\cdot)$. The match quality of a girl is given by $x_G + \eta$, where η is distributed with a density function $g(\cdot)$ and cdf $G(\cdot)$. Now suppose that the parent of every boy invests the same amount $\hat{x}_B$, while the parent of every girl invests the same amount $\hat{x}_G$. We claim that it is no longer an equilibrium to invest the individually optimal amounts. If the parent of a boy invests $\bar{x}_B$, the marginal private cost of investment must equal the marginal private benefit, i.e. the marginal net cost is zero. However, there is now a benefit to improving match quality, since there is a distribution of qualities on both sides of the market. By investing a little more in a boy, the quality of the boy increases for every realization of ε, and he ranks higher in the distribution of boy qualities, and will therefore be matched with a better quality girl. Thus, it will be optimal to deviate and increase investment beyond the individually optimal amount. An equilibrium must be characterized by the condition that for a boy, the marginal benefit from investment, in terms of improving match quality, is equal to the marginal net cost. An analogous condition holds for girls as well. In other words, the introduction of a random element of quality eliminates the multitude of equilibria, and ensures that equilibria are unique, under some regularity conditions.

Does this incentive to invest provide for efficient investments? We find that surprisingly if the distribution of shocks is symmetric across genders, the unique equilibrium coincides with the efficient investments. However, if the distribution of shocks is not symmetric, then investments will not in general be efficient. For example, if shocks are more variable for men than for women, then women will overinvest, while men will underinvest, relative to efficiency. Similarly, if the sex ratio is unbalanced so that there are more men than women, there is likely to be overinvestment by men and underinvestment by women.

2.2 Other literature

Matching tournaments with non-transferable utility were first examined by Cole et al. (1992) (see also, Peters (2007) and Hopkins (2010)). The essential idea is that investments are made prior to matching market (examples include labour and marriage markets and college admissions) taking place, and the level of one's investment influences one's attractiveness in this subsequent matching market. The desire to match well provides an additional incentive to invest and typically increases equilibrium investments. In Cole et al. (1992) only one side makes an investment. Two-sided investments were studied by Cole et al. (2001) and Felli and Roberts (2000) under transferable utility. Peters (2007) looks at the mixed strategy equilibria of finite matching tournaments under complete information. Peters (2009), Hopkins (2010) and Hoppe et al. (2009) analyze matching tournaments with incomplete information.

A recurrent question is whether the investments made will be efficient. Cole et al. (2001) find that with transferable utility, both efficient and inefficient investment is possible. As we have seen Peters and Siow (2002) argue that even under non-transferable utility, a rational expectations equilibrium investment will be efficient. However, Peters (2007, 2009) investigates matching tournaments with finite numbers of participants and finds that the resulting equilibria do not necessarily approach the efficient equilibrium possible with a continuum of agents, even as the number of participants goes to infinity.

Gall et al. (2009) also examine investments, matching and affirmative action in a non-transferable utility setting. Their focus differs from most of the above mentioned literature since they consider a situation where efficiency requires negative assortative matching. However, in the absence of transfers, stable matchings are positively assortative, providing a possible rationale for affirmative action. Their focus is on how inefficiencies in the matching process influence investment decisions, where the return on investment is stochastic. They also examine the effects of affirmative action, that constrains the matching directly, upon investment incentives.

The idea of perturbing games as a means to refine the number of equilibria is not new. However, in the context of matching tournaments, the approach taken up to now has been to employ incomplete information, see Hoppe et al. (2009), Peters (2009), Hopkins (2010). That is, agents are assumed to differ ex ante in terms of quality or wealth, and equilibrium investment is monotone in one's type. Here, as in Gall et al. (2009), the approach is somewhat simpler. Agents are ex ante identical. Yet still individuals face a smooth optimization problem which has a unique equilibrium.

Investment to improve marriage prospects has been the subject of a number of other studies. Using the transferable utility framework Iyigun and Walsh (2007) and Chiappori et al. (2009) study the distributional consequences of institutional and gender diffferences. Burdett and Coles (2001) analyze a non-transferable utility model, where matching is subject to search frictions.

3 A General Framework

In this section, we show at a very general level that if there is symmetry between the sexes, even if there is wide heterogeneity within each sex, that equilibrium investments will be efficient. Assume that there is a finite set of types, indexed by $i \in \{1, 2, .., n\}$. Type i has a measure μ_B^i of boys and a measure μ_G^i of girls. A boy of type i has an idiosyncratic component of quality, ε, that is distributed with a density function $f_i(\varepsilon)$ and a cumulative distribution function $F_i(\varepsilon)$. Let η denote the realization of the idiosyncratic quality component for a girl; for type i, this is distributed with a density function g_i and cdf G_i. We shall assume that the quality of a boy, as assessed by a girl, equals $q(x, \varepsilon)$, where q is continuous and strictly increasing in both arguments. Similarly, a girl's quality on the marriage market is given by $q(x, \eta)$. Finally, for simplicity, we shall assume that each individual strictly prefers to be matched rather than unmatched.

Assume that a parent of type i derives a direct private benefit $b_B(x)$ from an investment of x in the quality of a boy, and $b_G(x)$ in the quality of a girl and incurs a cost $\tilde{c}_{Bi}(x)$ and $\tilde{c}_{Gi}(x)$ respectively Define the net cost of investment in a boy as $c_{Bi}(x) = \tilde{c}_{Bi}(x) - b_j(x)$, and assume that this is convex and eventually increasing. Net costs for girls are similarly defined. Note that one's type affects one costs of investment but not one's attractiveness as a match. For example, PS originally assumed that families varied in wealth, so that rich families in effect had lower costs of investment. But having wealthier parents does not make one inherently more attractive.

Parents are altruistic and internalize the effects of their decisions on the utility of their own child, but not on the utility of their child's partner. Thus if a girl of type i with parental investment x_G is matched with a boy whose parent has invested x_B, her payoff and that of her parents equals

$$U_G^i(x_G, x_B) = q(x_B, \varepsilon) + b_G(x_G) - \tilde{c}_{Gi}(x_G) = q(x_B, \varepsilon) - c_{Gi}(x_G). \quad (1)$$

Similarly for a boy of type i with investment x_B matched with a girl with investment x_G, his utility would be

$$U_B^i(x_B, x_G) = q(x_G, \eta) + b_B(x_B) - \tilde{c}_{Bi}(x_B) = q(x_G, \eta) - c_{Bi}(x_B). \quad (2)$$

For each type i, let $(\bar{x}_{Bi}, \bar{x}_{Gi})$ denote the *individually optimal* or Nash investment for boys and girls respectively. That is, for boys it is the investment that minimizes $c_{Bi}(x) = \tilde{c}_{Bi}(x) - b_B(x)$, or equivalently the investment such that $c'_{Bi}(\bar{x}_{Bi}) = 0$. The privately optimal investment for girls $\bar{x}_{Gi}$ is defined similarly.

Such investments, however, are not efficient. To see this, assume a girl of type i matches with a boy of type j. Pareto efficient investments for this couple consist of a pair (x_B, x_G) that maximizes

$$W = \lambda \left[\int q(x_B, \varepsilon) f_j(\varepsilon)\, d\varepsilon - c_{Gi}(x_G) \right] + (1 - \lambda) \left[\int q(x_G, \eta) g_i(\eta)\, d\eta - c_{Bj}(x_B) \right], \quad (3)$$

for some $\lambda \in [0,1]$, where λ is the relative weight placed on the welfare of girls. Differentiating with respect to x_B and x_G, setting to zero and rearranging, we obtain as a first order condition for Pareto efficiency,

$$\frac{c'_{Bj}(x_B)}{\int q_x(x_B,\varepsilon)f_i(\varepsilon)\,d\varepsilon} = \frac{\lambda}{1-\lambda} = \frac{\int q_x(x_G,\eta)g_i(\eta)\,d\eta}{c'_{Gi}(x_G)}. \tag{4}$$

This has the interpretation that the investments by boy and girl are such that they must lie on the contract curve in (x_B, x_G) space. That is, the indifference curve of the boy and girl have the same slope equal to $\lambda/(1-\lambda)$. Of particular interest is the case where λ, the weight place on girls' welfare, is equal to their proportion in the population (for the moment this is one half; we go on to consider unbalanced sex ratios in Section 5). Let x^{**}_{Bi}, x^{**}_{Gi} denote the efficient investments in this case. We shall call these the *utilitarian efficient* investments.

It is also possible to rearrange the first order condition for welfare maximization to obtain,

$$c'_{Bj}(x_B) \times c'_{Gi}(x_G) = \int q_x(x_B,\varepsilon)f_j(\varepsilon)\,d\varepsilon \times \int q_x(x_G,\eta)g_i(\eta)\,d\eta > 0. \tag{5}$$

This relation holds irrespective of the value of λ. Note that this implies that Pareto efficient investments always exceed the privately optimal level as the under the privately optimal investments, we have $c'_{Bj}(\bar{x}_B) = c'_{Gi}(\bar{x}_G) = 0$.

This argument also extends to a heterogeneous society, even if the social planner puts unequal welfare weights on parents of different types. Consider for example a society with a continuum of types (e.g. wealth), where each the qualities chosen are strictly increasing in type. With assortative matching, matching will be "within type". Thus, boys and girls who are matched have equal weight in the social welfare function, even if different types have different weights.

We shall focus upon pure strategy equilibria where all agents who are identical choose the same strategy. That is, if two parents belong to the same type i, and have a child of the same gender j, they choose the same level of investment x_{ji}. Thus an equilibrium consists of a profile $((x_{Bi})_{i=1}^n, (x_{Gi})_{i=1}^n)$, specifying investment levels for each type of each gender.

Suppose that a boy of type i chooses investment level x. If the shock realization is ε, the induced quality is $q(x,\varepsilon)$, and this has probability density $f_i(\varepsilon)$. Let $\tilde{f}_i(q|x_i)$ denote the induced density function over qualities given investment x_i.

Any strategy profile induces a distribution of qualities of boys and girls in the population. In particular, $\mathbf{x}_B \equiv (x_{Bi})_{i=1}^n$, in conjunction with the realization of idiosyncratic shocks, induces a distribution of qualities in the population of boys. Let $\tilde{F}$ denote the cumulative distribution function of boys qualities. Since ε is assumed to be atomless, and q is continuous, $\tilde{F}$ admits a density function $\tilde{f}$, although its support may not be connected if the investment levels of distinct types are sufficiently far apart (i.e. there may be gaps in the distribution of qualities). Similarly, let $\tilde{G}$ denote the cumulative distribution function of girl qualities.

We assume for now that there are equal measures of boys and girls, and normalize each of these to 1, i.e. $\sum_{i=1}^{n} \mu_B^i = 1$ and $\sum_{i=1}^{n} \mu_G^i = 1$. We require that a matching satisfies three conditions: it must be feasible, it must be stable (in the sense of Gale and Shapley, 1962), and it must be measure preserving. Stable matchings will be assortative, i.e. if a boy of type q is matched to a girl of type $\phi(q)$ if and only if

$$\tilde{F}(q) = \tilde{G}(\phi(q)) \tag{6}$$

Thus the distributions $\tilde{F}$ and $\tilde{G}$ define $\phi = \tilde{G}^{-1}(\tilde{F}(q))$ and the match payoffs associated with equilibrium investments for each type of boy and each type of girl. Specifically, suppose that almost all boys of type i invest x_{Bi}. If a particular individual has a shock realization ε, then his match quality is given by $\phi(q(x_{Bi}, \varepsilon))$. Thus the expected payoff from investment level x_{Bi} is given by

$$\int \phi(q(x_{Bi}, \varepsilon)) f_i(\varepsilon)\, d\varepsilon - c_{Bi}(x).$$

One delicate issue is what happens to match payoffs when a boy deviates from the investment level that all other agents of his type are choosing, and chooses $x_{Bi} + \Delta$. For some realizations of ε, it may be the case that the quality $q(x_{Bi} + \Delta, \varepsilon)$ does not belong to the support of the distribution $\tilde{F}(q)$ that is induced by the profile of equilibrium investments. Stability implies that the match quality of this boy is no worse than the match quality $\check{q} = \sup\{q : q < q(x_{Bi} + \Delta, \varepsilon) \text{ and } q \in C(\tilde{F})\}$, where $C(\tilde{F})$ denotes the support of $\tilde{F}$. It can also be no better than $\inf\{q : q > q(x_{Bi} + \Delta, \varepsilon) \text{ and } q \in C(\tilde{F})\}$. We shall assume throughout that the match quality of such a boy is given by $\check{q}$ if the set over which the supremum it taken is non-empty, and if not, the boy is left unmatched.

The following assumption will play an important role in our results:

Assumption 1 (Symmetry): Men and women are symmetric with regards to costs of investment and the idiosyncratic component of quality. Specifically, for any type i: i) $\mu_B^i = \mu_G^i$, ii) the investment cost functions do not differ across the sexes so that $c_{Bi}(\cdot) = c_{Gi}(\cdot)$, and iii) $f_i = g_i$, the idiosyncratic component of quality has the same distribution.

Assumption 1 is strong, but there are reasonable conditions under which it is satisfied. Suppose that investment costs or the idiosyncratic component depend upon the "type" of the parent (e.g. parental wealth, human capital or social status), but not directly upon gender. If the gender of the child is randomly assigned, with boys and girls having equal probability, then Assumption 1 will be satisfied. Assumption 1 will not be satisfied if there is parental sex selection, with different types having differential incentives to select for boys or girls. It will also fail if the distributions of shocks are gender specific.

We shall call an equilibrium *symmetric* if $x_{Bi} = x_{Gi} \forall i$, so each type of parent invests the same amount regardless of the gender of their child. Our notion of efficiency here is also more stringent than the Pareto criterion - specifically, it is the Pareto criterion with equal welfare weights for the two sexes.

Theorem 1 *If* $((x^*_{Bi})^n_{i=1}, (x^*_{Gi})^n_{i=1})$ *is an equilibrium where type* i *of gender* j *is matched with positive probability, then* $x^*_{ji} > \bar{x}_{Bi}$*, the privately optimal investment level.*

Proof. The profile $((x^*_{Bi})^n_{i=1}, (x^*_{Gi})^n_{i=1})$ induces probability distributions over qualities of boys and girls, q_B and q_G. Let $\tilde{F}$ (resp. $\tilde{G}$) denote the cumulative distribution function over the qualities of the boys (resp. girls) respectively. Since the idiosyncratic component of quality for each type does not have any atoms, $\tilde{F}$ and $\tilde{G}$ do not have any mass points, and admit density functions $\tilde{f}$ and $\tilde{g}$ respectively. If type i of a boy invests x^*_{Bi} and has a shock realization ε, his quality is equal to $q(x^*_{Bi}, \varepsilon)$. Since he is matched with positive probability, there is an interval of realizations of ε such that he is matched. Denote his match quality type by $\tilde{\phi}(q(x^*_{Bi}, \varepsilon))$. $\tilde{\phi}$ is defined by the equation $\tilde{F}(q(x^*_{Bi}, \varepsilon)) = \tilde{G}(\tilde{\phi}(q(x^*_{Bi}, \varepsilon))$, and is strictly increasing in ε for an interval of values of ε. Since q is strictly increasing in both arguments, this implies that $\tilde{\phi}(q(x^*_{Bi}, \varepsilon))$ is strictly increasing in x^*_{Bi} for ε in the interior of this interval. Thus the derivative of $c_{Bi}(.)$ must be strictly negative at x^*_{Bi}. ■

Theorem 2 *Suppose that Assumption 1 (symmetry) is satisfied. The efficient profile of investments is a symmetric equilibrium. A symmetric strategy profile is an equilibrium if and only if it locally maximizes ex ante payoffs.*

Proof. Suppose that the profile is symmetric so that for each type i, $x^*_{Bi} = x^*_{Gi}$. This implies that $\tilde{F} = \tilde{G}$, and so $\phi(q(x^*_{Bi}, \varepsilon)) = q(x^*_{Bi}, \varepsilon)$. Consider now the benefit for a boy of type i from investing $x^*_{Bi} + \Delta$. For any realization of the shock ε, his quality equals $q(x^*_{Bi} + \Delta, \varepsilon)$. If $q(x^*_{Bi} + \Delta, \varepsilon)$ belongs to the support of $\tilde{F}$, he will be matched with a type $\phi(q(x^*_{Bi} + \Delta, \varepsilon)) = q(x^*_{Bi} + \Delta, \varepsilon)$. Thus, if $q(x^*_{Bi} + \Delta, \varepsilon)$ belongs to the support of $\tilde{F}$ for almost all realizations of ε, given the distribution F_i, the payoff gain from investing $x^*_{Bi} + \Delta$ equals:

$$\int q(x^*_{Bi} + \Delta, \varepsilon) f_i(\varepsilon)\, d\varepsilon - c_{Bi}(x^*_{Bi} + \Delta). \tag{7}$$

If $q(x^*_{Bi} + \Delta, \varepsilon)$ does not belong to the support of $\tilde{F}$, then the deviating individual is matched with the partner of the next lowest type in the support of $\tilde{F}$, and is left unmatched if there is no such lower type. Thus for any Δ, the payoff from choosing $x^*_{Bi} + \Delta$ is less than or equal to the expression in (7). The payoff from the equilibrium investments is given by

$$\int q(x^*_{Bi}, \varepsilon) f_i(\varepsilon)\, d\varepsilon - c_{Bi}(x^*_{Bi}). \tag{8}$$

If the symmetric profile $(x^*_{Bi})^n_{i=1}$ is efficient, then the expression in (7) is less than that in (8) for all Δ, and so is also an equilibrium. This establishes "only if". To show "if", note that for $(x_{Bi})^n_{i=1}$ to be an equilibrium, (7) must be less than (8) for all Δ

such that $q(x^*_{Bi} + \Delta, \varepsilon)$ belongs to the support of $\tilde{F}$ for almost all realizations of ε. Taking the limit as $\Delta \to 0$, and noting that in this case the support condition satisfied, a necessary condition for equilibrium is that

$$\int q_x(x^*_{Bi}, \varepsilon) f_i(\varepsilon)\, d\varepsilon - c'_{Bi}(x^*_{Bi}) = 0. \tag{9}$$

This is the first order condition for efficient investments. Similarly, the second order condition must also be satisfied. In other words, if we have a symmetric equilibrium, then the investments for each type of boy must locally maximize the difference between his benefit in terms of expected quality and investment costs. ■

Our proof that the efficient profile is an equilibrium uses the rule that if $q(x^*_{Bi}+\Delta, \varepsilon)$ does not belong to the support of $\tilde{F}$, then the deviating individual is matched with the partner of the next lowest type in the support of $\tilde{F}$, and is left unmatched if there is no such lower type. Two remarks are in order here:

1. We can dispense with this assumption on the matching if the support of the idiosyncratic shocks is unbounded. In this case, for every type i and for any Δ, $q(x^*_{Bi} + \Delta, \varepsilon)$ belongs to the support of $\tilde{F}$ for all realizations of ε. Thus the efficient investment levels also satisfy the global condition for equilibrium. This is of course, not needed: if the support of the shocks is “large enough", this will suffice.

2. This matching rule does not allow us to support inefficient equilibria. For such equilibria must satisfy the local first order and second order conditions for equilibrium, where this matching rule has no bite.

3. Consider a single-population matching model, where quality is a one-dimensional scalar variable. An example is partnership formation, e.g. firms consisting of groups of lawyers. Theorem 2 implies that one has efficient ex ante investments, even absent transferable utility. While the formal proof restricts attention to pair-wise matching, the extension to matches consisting of more than two partners is immediate.

4. This efficiency result is strong in one sense, but weak in another sense. On the one hand, we obtain utilitarian efficiency, rather than just Pareto efficiency. However, it requires strong symmetry assumptions.

4 When Each Gender is Homogenous but the Two Genders Differ

We now depart from the general analysis of the previous section and assume that each gender is homogenous. Thus, for example, all girls are assumed to have the same initial wealth. However, this will allow us to investigate the question as to whether efficiency of investment still holds when there is an asymmetry *between* the sexes. In particular, in this section we will consider the possibility that the distributions of shocks differ across the sexes. For example, investment in a boy could have a more variable range of outcomes than investment in a girl due to differences in labour market opportunities. We find that when such asymmetries are present generically there is inefficient investment. That is, the result of the previous section is not robust.

Consider a society where all families are identical. Thus, we suppress the index on types in the utilities (1) and (2), so that they are now respectively $U_G = q(x_B, \varepsilon) - c_G(x_G)$ and $U_B = q(x_G, \eta) - c_B(x_B)$. Recall that $q(x_B, \varepsilon)$ is the quality of a boy on the marriage market, where x_B is the investment and ε is a random shock, which is distributed according to a single continuous cumulative distribution function F. Similarly, if the investment in a girl is x_G, then realized quality is given by $q(x_G, \eta)$, with η which has distribution function G. Formally, we assume

1. Assumptions on shocks: let $F(\varepsilon)$ and $G(\eta)$ be twice differentiable and be distributed on the bounded intervals $[0, \bar{\varepsilon}]$ and $[0, \bar{\eta}]$ respectively.

2. Assumptions on costs: let $c_B(x_B)$ and $c_G(x_G)$ be twice differentiable and strictly increasing. Further assume

   (a) Convexity: $c''_B(x_B) > 0$ and $c''_G(x_G) > 0$.

   (b) Let the privately optimal levels of investment $\bar{x}_B$, $\bar{x}_G$ that satisfy $c'_B(\bar{x}_B) = 0$ and $c'_G(\bar{x}_G) = 0$ be strictly positive.

   (c) Bounded maximum rational investment: there are investment levels $\tilde{x}_B$, $\tilde{x}_G$ such that $\lim_{x \to \tilde{x}_B} c'_B(x) = \infty$ and $\lim_{x \to \tilde{x}_G} c'_G(x) = \infty$.

3. Assumptions on quality: let $q(x_B, \varepsilon)$ and $q(x_G, \eta)$ be strictly increasing and three times differentiable, with $q_{xx}(x_B, \varepsilon) \leq 0$, $q_{\varepsilon\varepsilon}(x_B, \varepsilon) \leq 0$, $q_{xx}(x_G, \eta) \leq 0$, $q_{\eta\eta}(x_G, \eta) \leq 0$. Let either

   (a) $q(x_B, \varepsilon)$ and $q(x_G, \eta)$ be additive so that q_ε and q_η are constants.

   (b) If $q(x_B, \varepsilon)$ and $q(x_G, \eta)$ are not additive, then let $q_\varepsilon(0, \varepsilon) = 0$ and $q_\eta(0, \eta) = 0$ and $q_{x\varepsilon}(x_B, \varepsilon) > 0$, $q_{xx\varepsilon}(x_B, \varepsilon) \leq 0$ and $q_{x\eta}(x_G, \eta) > 0$, $q_{xx\eta}(x_G, \eta) \leq 0$.

   (c) The value of not being matched is fixed at $\bar{u}$ which satisfies $\bar{u} \leq q(0, 0)$.

Consider an equilibrium where all boys invest x_B^* and all girls invest x_G^*. The probability that a boy's quality is below some level q is denoted by $\tilde{F}(q)$. With assortative matching, we have the matching function given by (6). However, since q is strictly increasing in the idiosyncratic shock, and since all agents on the same side of the market choose the same investment level, this is equivalent to matching according to the idiosyncratic shocks alone. That is, in such an equilibrium, the matching ϕ satisfies

$$F(\varepsilon) = G(\phi(\varepsilon)) \tag{10}$$

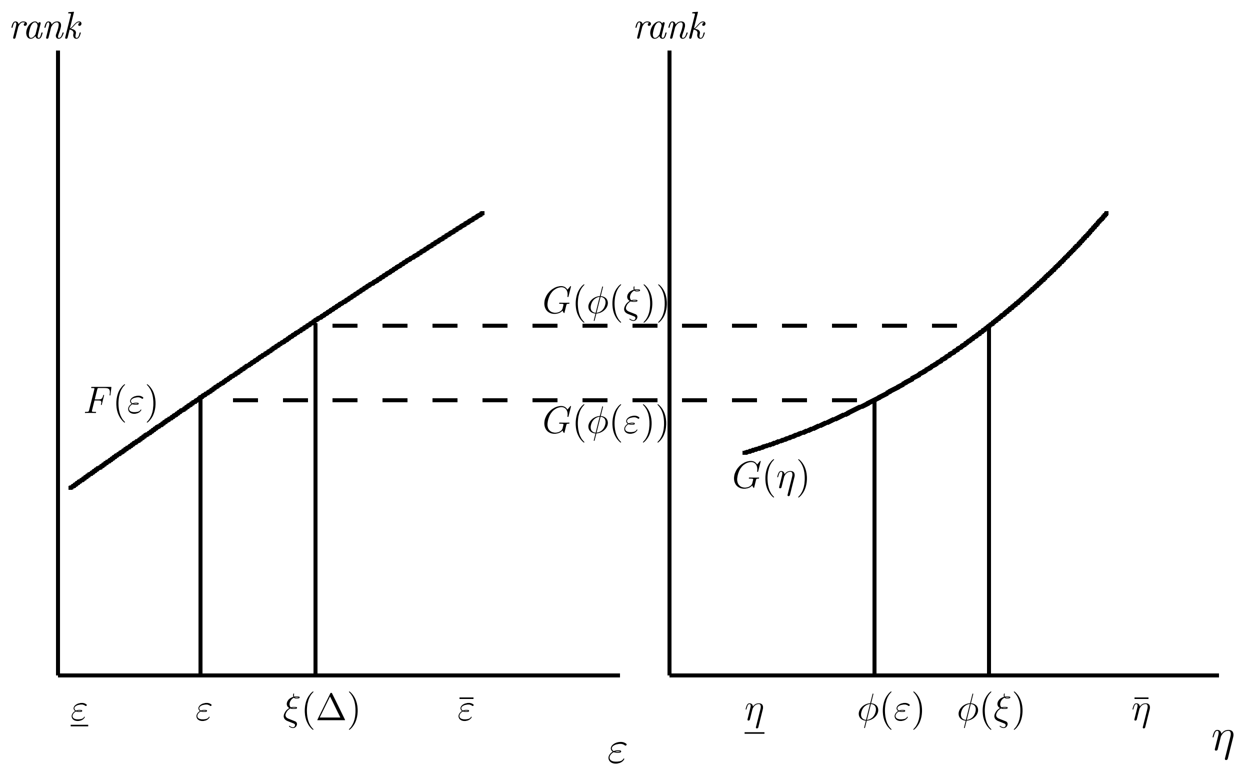

Figure 2: The parents of a boy with shock realisation ε considering increasing investment by an amount Δ can expect to overtake other boys with shock realisations between ε and $\xi(\Delta)$. The boy's match would improve from the girl with shock value $\phi(\varepsilon)$ to one at $\phi(\xi)$.

or $\phi(\varepsilon) = G^{-1}(F(\varepsilon))$.

Suppose that a parent of a boy deviates from this equilibrium and invests $x_B^* + \Delta$ in his son. If the realization of the shock for his son is ε, combined with the higher level of investment, the son will hold the same rank in the population of boys as a boy with a shock level ξ, where $\xi(\varepsilon, \Delta)$ is defined by the equation

$$q(x_B^* + \Delta, \varepsilon) = q(x_B^*, \xi(x_B^* + \Delta, \varepsilon)). \tag{11}$$

This idea is illustrated in Figure 2. Given this deviation, the boy now holds rank $F(\xi)$ in the population of boys and can expect a match with a girl holding rank $G(\phi(\xi))$ in the population of girls. She would be of quality $q(x_G^*, \phi(\xi))$.

His expected payoff from the deviation will depend on whether he deviates upwards or downwards. While the cost of investment $c_B(x_B^* + \Delta)$ will vary smoothly with Δ, this is not necessarily true for the benefit from matching. For example, when a boy's investment is changed by $\Delta > 0$ (for $\Delta < 0$ see the proof to the theorem below), his expected match after the deviation is given by

$$B(\Delta) = \int_0^{\hat{\varepsilon}} q(x_G^*, \phi(\xi(x_B^* + \Delta, \varepsilon)))f(\varepsilon)\, d\varepsilon + (1 - F(\hat{\varepsilon}))q(x_G^*, \phi(\bar{\varepsilon})) \tag{12}$$

where $\hat{\varepsilon}$ is defined by $\xi(x_B^* + \Delta, \hat{\varepsilon}) = \bar{\varepsilon}$. That is, if the deviating boy has a high shock realization, specifically on the interval $[\hat{\varepsilon}, \bar{\varepsilon}]$, he will match with the highest ranking girl who has quality $q(x_G^*, \phi(\bar{\varepsilon}))$. The derivative of the expected match or benefit with respect to Δ is then given by

$$B_0'^{\hat{\varepsilon}} q_\eta(x_G^*, \phi(\xi(x_B^* + \Delta, \varepsilon)))\phi'(\xi(x_B^* + \Delta, \varepsilon))\xi_x(x_B^* + \Delta, \varepsilon) f(\varepsilon)\, d\varepsilon. \tag{13}$$

Given that $\xi(x_B^* + \Delta, \varepsilon)$ evaluated at $\Delta = 0$ is simply ε, the derivative of the expected payoff with respect to Δ, evaluated at $\Delta = 0$, is given by

$$B_0'^{\bar{\varepsilon}} q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_x(x_B^*, \varepsilon) f(\varepsilon)\, d\varepsilon. \tag{14}$$

Note that ϕ' and ξ_x are given by

$$\phi'(\varepsilon) = \frac{f(\varepsilon)}{g(\phi(\varepsilon))}.$$

and

$$\xi_x(x_B^*, \varepsilon) = \frac{q_x(x_B^*, \varepsilon)}{q_\varepsilon(x_B^*, \varepsilon)}.$$

Thus the first order condition for investment in boys can be re-written as

$$\int_0^{\bar{\varepsilon}} q_\eta(x_G^*, \phi(\varepsilon)) \frac{f(\varepsilon)}{g(\phi(\varepsilon))} \frac{q_x(x_B^*, \varepsilon)}{q_\varepsilon(x_B^*, \varepsilon)} f(\varepsilon)\, d\varepsilon = c_B'(x_B^*). \tag{15}$$

Similarly, the first order condition for investment in girls is given by

$$\int_0^{\bar{\eta}} q_\varepsilon(x_B^*, \phi^{-1}(\eta)) \frac{g(\eta)}{f(\phi^{-1}(\eta))} \frac{q_x(x_G^*, \eta)}{q_\eta(x_G^*, \eta)} g(\eta)\, d\eta = c_G'(x_G^*). \tag{16}$$

We show a general result on existence and uniqueness. An equilibrium is *quasi-symmetric* if all boys make the same investment x_B^* and all girls invest x_G^*, where x_B^* need not be equal to x_G^*.

Theorem 3 *There exists investments (x_B^*, x_G^*) that are the unique solution of the first order conditions (15) and (16), and so quasi-symmetric equilibrium is always unique. Suppose either (a) F and G are distributions of the same type (that is, $G(x) = F(ax + b)$), or (b) $f(\varepsilon)$ and $g(\eta)$ are weakly increasing functions. Then there is a $\bar{u}$ sufficiently low such that (x_B^*, x_G^*) is a strict Nash equilibrium. Thus there exists an open set of distributions such that there is a unique quasi-symmetric Nash equilibrium of the matching tournament.*

Proof. We first show that the first order conditions (15) and (16) represent maxima. We then go on to show that there exist unique values satisfying these conditions. Thus, the equilibrium exists and is unique.

We first consider upward deviations. Given the strict convexity of costs, we need to show that $B'(\Delta) \leq B'(0)$ for any $\Delta > 0$. A sufficient condition is that $B(\Delta)$ as given in (12) is concave in Δ. We have

$$\xi_{xx} = \frac{d}{dx_B}\frac{q_x(x_B,\varepsilon)}{q_\varepsilon(x_B,\varepsilon)} = \frac{q_{xx}q_\varepsilon - q_{\varepsilon x}q_x}{q_\varepsilon^2} \leq 0,\ \ \xi_{x\varepsilon} = \frac{d}{d\varepsilon}\frac{q_x(x_B,\varepsilon)}{q_\varepsilon(x_B,\varepsilon)} = \frac{q_{x\varepsilon}q_\varepsilon - q_{\varepsilon\varepsilon}q_x}{q_\varepsilon^2} \geq 0$$

given our assumptions that $q_{xx} \leq 0$, $q_{\varepsilon\varepsilon} \leq 0$ and $q_{\varepsilon x} \geq 0$. Further, q_η will be decreasing in Δ given the assumption that $q_{\eta\eta} \leq 0$. This leaves $\phi(\cdot)$. If a) F and G are identical or if they are of the same type (that is, $G(x) = F(ax + b)$), then $\phi(\varepsilon)$ is linear and therefore weakly concave. Thus, $\phi(\varepsilon)$ and hence $B(\Delta)$ is concave for an open set of distributions. If b), $f'(\varepsilon) \geq 0$ and $g'(\eta) \geq 0$, then note that (13) can be written as

$$B'^{\bar{\varepsilon}}_{\tilde{\varepsilon}} q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_x(x_B^* + \Delta, \gamma(x_B^* + \Delta, \varepsilon))f(\gamma(x_B^* + \Delta, \varepsilon))\,d\varepsilon. \tag{17}$$

where $\tilde{\varepsilon}$ satisfies $q(x_B^*, \tilde{\varepsilon}) = q(x_B^* + \Delta, \underline{\varepsilon})$ and $\gamma(x_B^* + \Delta, \varepsilon)$ is defined by the relation $q(x_B^*, \varepsilon) = q(x_B^* + \Delta, \gamma)$, which therefore implies that $\gamma_x < 0$. We then have

$$\begin{aligned} B''(\Delta) &= -\frac{\partial\tilde{\varepsilon}}{\partial\Delta} q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_x(x_B^* + \Delta, \gamma(x_B^* + \Delta, \varepsilon))f(\gamma(x_B^* + \Delta, \varepsilon)) \quad (18)\\ &+ \int_{\tilde{\varepsilon}}^{\bar{\varepsilon}} [q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_x(x_B^* + \Delta, \gamma(x_B^* + \Delta, \varepsilon))f'^*_B + \Delta, \varepsilon))\gamma_x(x_B^* + \Delta, \varepsilon) \\ &+ q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_{xx}(x_B^* + \Delta, \gamma(x_B^* + \Delta, \varepsilon))f(\gamma(x_B^* + \Delta, \varepsilon)) \\ &+ q_\eta(x_G^*, \phi(\varepsilon))\phi'(\varepsilon)\xi_{x\varepsilon}(x_B^* + \Delta, \gamma(x_B^* + \Delta, \varepsilon))f(\gamma(x_B^* + \Delta, \varepsilon))]\,d\varepsilon \end{aligned}$$

which is negative as $f' \geq 0$, $\xi_{xx} \leq 0$, $\xi_{x\varepsilon} \geq 0$, $\gamma_x < 0$ and $\partial\tilde{\varepsilon}/\partial\Delta > 0$.

Turning to downward deviations, when $\Delta < 0$ the boy's expected match quality will be

$$B(\Delta) = \int_{\tilde{\varepsilon}}^{\bar{\varepsilon}} q(x_G^*, \phi(\xi(x_B^* + \Delta, \varepsilon)))f(\varepsilon)\,d\varepsilon + F(\tilde{\varepsilon})\bar{u} \tag{19}$$

where $\tilde{\varepsilon}$ is defined by $\xi(x_B^* + \Delta, \tilde{\varepsilon}) = 0$. That is, if the deviating boy has a low shock realization, specifically on the interval $[0, \tilde{\varepsilon}]$, his overall quality will be so low that he will not match and will have utility $\bar{u}$. We can therefore calculate the derivative with respect to a downward deviation $\Delta < 0$ as

$$B'^{\bar{\varepsilon}}_{\tilde{\varepsilon}} q_\eta(x_G^*, \phi(\xi(\cdot)))\phi'(\xi(\cdot))\xi_x(x_B^* + \Delta, \varepsilon)f(\varepsilon)\,d\varepsilon + f(\tilde{\varepsilon})(q(x_G^*, 0) - \bar{u}) \tag{20}$$

Comparing with $B'(0)$ as given in (14), we can see that $\lim_{\Delta\uparrow 0} B'(\Delta) > B'(0)$ as long as $\bar{u} < q(x_G^*, 0)$. More generally, we can find $\bar{u}$ sufficiently low such that $B'(\Delta) > B'(0)$ for $\Delta < 0$. Thus, given the convexity of costs, downward deviations are not profitable.

The first order conditions (15) and (16) implicitly define reaction functions with $R_B(x_G)$ and $R_G(x_B)$ being the reaction functions implied by (15) and (16) respectively. Crossing of these functions will represent equilibrium points. We now prove that, under the above assumptions that we have made, the crossing must be unique and thus there is a unique equilibrium. We first show that that the reaction functions are both increasing. Differentiating (15) we have

$$\left.\frac{dx_B}{dx_G}\right|_{R_B} = \frac{\int q_{x\eta}(x_G, \phi(\varepsilon))\xi_x(x_B, \varepsilon)\theta_B(\varepsilon)\,d\varepsilon}{-\int q_\eta(x_G, \phi(\varepsilon))\xi_{xx}(x_B, \varepsilon)\theta_B(\varepsilon)\,d\varepsilon + c''_B(x_B)} \geq 0 \tag{21}$$

as $\xi_{xx} \leq 0$ as established above. Similarly, from (16), we have

$$\left.\frac{dx_G}{dx_B}\right|_{R_G} = \frac{\int q_{x\varepsilon}(x_B, \phi^{-1}(\eta))\xi_x(x_G, \eta)\theta_G(\eta)\, d\eta}{-\int q_\varepsilon(x_B, \phi^{-1}(\eta))\xi_{xx}(x_G, \eta)\theta_G(\eta)\, d\varepsilon + c''_G(x_G)} \geq 0. \tag{22}$$

Now, we show that $R_B(x_G)$ and $R_G(x_B)$ are both concave. We differentiate (22) to obtain, suppressing arguments,

$$\left.\frac{d^2x_G}{dx_B^2}\right|_{R_G} = \frac{(\int q_{xx\varepsilon}\xi_x\theta_G\, d\eta)(-\int q_\varepsilon\xi_{xx}\theta_G\, d\varepsilon + c''_G) - (-\int q_{x\varepsilon}\xi_{xx}\theta_G\, d\varepsilon)(\int q_{x\varepsilon}\xi_x\theta_G\, d\eta)}{(-\int q_\varepsilon\xi_{xx}\theta_G\, d\varepsilon + c''^2_G} \leq 0. \tag{23}$$

We can make a similar calculation for R_B.

We now examine where these reaction functions might cross. First, we address the case where quality is additive in the shock so that $q_{x\varepsilon}(x_B, \varepsilon) = 0$ and $q_{x\eta}(x_G, \eta) = 0$. Then, we can see from inspection of (21) that $R_B(x_G) = x_B^*$, a constant. Further, it is clear that $x_B^* > \bar{x}_B$. Similarly, $R_G(x_B) = x_G^*$. Thus, the equilibrium exists and is unique.

Second, if $q_{x\varepsilon}(x_B, \varepsilon) > 0$ and $q_{x\eta}(x_G, \eta) > 0$ note that now $R_B(x_G)$ and $R_G(x_B)$ are both strictly increasing and therefore invertible, and also both are strictly concave. Let $Q_B(x_B) = R_B^{-1}(x_B)$ be the inverse reaction function for boys. Further, given $q_\eta(0, \eta) = 0$ by assumption 2, then when $x_G = 0$ the first order condition (15) reduces to $c'(x_B) = 0$. That is, we have $R_B(0) = \bar{x}_B > 0$, and $Q_B(\bar{x}_B) = 0$. Similarly, $R_G(0) = \bar{x}_G > 0$. Thus, if we consider $Q_B(x_B)$ and $R_G(x_B)$ on the (x_B, x_G) plane, the first crossing point if any (and an equilibrium if it occurs) will be Q_B crossing R_G from below. Now, as $R_B(x_G)$ is strictly increasing and strictly concave, it follows that $Q_B(x_B)$ is strictly increasing and strictly convex. Clearly, as at such a first point of crossing we have $Q'_B \geq R'_G$ and as Q_B is convex and R_G is concave, to the right of the first crossing, it must hold that $Q'_B > R'_G$. Thus, there can be no further crossing as at a further crossing, it would have to hold that $Q'_B \leq R'_G$. So, any crossing is unique. Lastly, there must be at least one crossing, as by assumption 2c on costs and examination of the first order conditions we have $\lim_{x \to \tilde{x}_B} Q_G(x) = \infty$ and $\lim_{x \to \infty} R_G(x) = \tilde{x}_G$. ■

We can now consider some special cases of the quality function $q(.)$ to shed further light on these results.

4.1 Additive shocks

Consider first the case where $q(x, \varepsilon) = x + \varepsilon$. One interpretation is that investment or bequest are in the form of financial assets or real estate, while the shocks are to (permanent) labor income of the child. The interpretation is that total household income is like a public good (as in Peters-Siow), which both partners share. In this case, q_x, q_ε and q_η are all equal to one, and so the first order conditions (15) and (16) for investment reduce to

$$\int \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon = c'_B(x_B^*) \tag{24}$$

and

$$\int \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\, d\eta = c_G'(x_G^*). \tag{25}$$

Note that under additive shocks the condition for efficiency (5) reduces to $c_B'(x_B) \times c_G'(x_G) = 1$ and the condition for utilitarian efficiency is $c_B'(x_B) = c_G'(x_G) = 1$. Consider the case where $F = G$, i.e. the distribution of shocks is the same. Thus, $\frac{f(\varepsilon)}{g(\phi(\varepsilon))} = 1$ for all values of ε, so that $c_B'(x_B^*) = c_G'(x_G^*) = 1$. Thus investments are utilitarian efficient even if the investment cost functions are different for the two sexes.

One special case is where the shocks are uniform, with the ratio of the densities $\frac{f(\varepsilon)}{g(\phi(\varepsilon))} = k < 1$. In this case, $c_B'(x_B^*) = k < 1$, and $c_G'(x_G^*) = \frac{1}{k} > 1$. Thus the investments are not utilitarian efficient. Nonetheless they are efficient but for unequal Pareto weights.

We shall see shortly that the double uniform case is unusual, since even if the distributions are different, one is a linear transformation of the other. For generic distributions, we will get inefficiency, and in fact over-investment relative to Pareto efficiency. To provide some economic intuition for this result, consider an example where the distribution of shocks is uniform on $[0, 1]$ for men and where the density function for women, $g(\eta) = 2\eta$ on $[0, 1]$. Recall that the incentive for investment for a man at any value of ε depends upon the ratio of the densities, $\frac{f(\varepsilon)}{g(\phi(\varepsilon)}$. This ratio exceeds one for low values of ε, but is less than one for high values of ε. Conversely, for women, the incentive to invest depends upon the inverse of this ratio, $\frac{g(\eta)}{f(\phi^{-1}(\eta))}$, which is low at low values of η but high at high values of η. In other words, the ratio of the densities plays opposite roles for the two sexes. However, the weights with which these ratios are aggregated differs between the sexes; high values of η are given relatively large weight in the case of women, since $g(\eta)$ is large in this case, while they are given relatively less weight in the case of men. The following proposition provides a formal proof of this result.

Proposition 4 *When noise is additive, investments are generically excessive relative to Pareto efficiency.*

Proof. The first order condition for investment for boys and girls are given by (24) and (25). It is useful to make the following change in variables. Since $\eta = \phi(\varepsilon)$,

$$d\eta = \phi'(\varepsilon)\, d\varepsilon = \frac{f(\varepsilon)}{g(\phi(\varepsilon))}\, d\varepsilon. \tag{26}$$

Thus the first order condition for girls is rewritten as

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon = c'(x_G^*).$$

Consider the product

$$c'(x_B^*) \times c'(x_G^*) \geq \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon \right) \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon \right).$$

By the Cauchy-Schwarz inequality,

$$\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon\right)\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon\right) \geq \left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \left(\frac{f(\varepsilon)}{(g(\phi(\varepsilon)))^{1/2}}\right)(g(\phi(\varepsilon)))^{1/2}\, d\varepsilon\right]^2$$
$$\geq 1.$$

Note that Cauchy-Schwarz states that the inequality is strict if the two terms (i.e. distributions) are linearly independent. Thus $c'(x_B^*) \times c'(x_G^*) \geq 1$ with the inequality being strict if f and g are linearly independent.

Under the assumption of additive noise, we have $q_x = 1$. Thus, the Pareto efficiency condition (5) reduces to $c'(x_B^*) \times c'(x_G^*) = 1$. Thus, we have overinvestment generically if the distributions f and g differ. ■

4.1.1 An Example

Let us assume that $F(\varepsilon) = \varepsilon$ on $[0, 1]$, i.e. ε is uniformly distributed. Assume that $G(\eta) = \eta^n$ on $[0, 1]$. $F(\varepsilon) = G(\phi(\varepsilon))$ implies $\phi(\varepsilon) = \varepsilon^{\frac{1}{n}}, g(\phi(\varepsilon)) = n\varepsilon^{\frac{n-1}{n}}$. The equilibrium conditions are:

$$\begin{aligned} c'(x_B^*) &= \int_0^1 \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon = \frac{1}{n}\int_0^1 \varepsilon^{\frac{1-n}{n}}\, d\varepsilon \\ &= 1. \end{aligned} \tag{27}$$

and

$$\begin{aligned} c'(x_G^*) &= \int_0^1 \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\, d\eta = n^2 \int_0^1 \eta^{2n-2}\, d\eta \\ &= \frac{n^2}{2n-1}. \end{aligned} \tag{28}$$

The product of the marginal costs equals $\frac{n^2}{2n-1}$ which is positive and strictly greater than one for $n > 0.5$. Efficiency requires that the product equals 1.

4.2 Talent Shocks and Complementarities with Investment

Consider next the case where uncertainty about child quality is talent risk. It is plausible that the return to investment depends upon the talent of the child. To model this, we suppose that quality is given by a Cobb-Douglas production function, $q(x, \varepsilon) = x^\alpha \varepsilon$. All our results apply to a more general Cobb-Douglas form, $q(x, \varepsilon) = x^\alpha \varepsilon^\beta$; we can redefine a new random variable $\tilde{\varepsilon} = \varepsilon^\beta$, and the results that follow will apply. With this production function, $q_x = \alpha x^{\alpha-1}\varepsilon$ and $q_\varepsilon = x^\alpha$, so that the first order condition for equilibrium reduces to

$$\int (x_G^*)^\alpha \, \frac{f(\varepsilon)}{g(\phi(\varepsilon))} \frac{\alpha\varepsilon}{x_B^*} f(\varepsilon)\, d\varepsilon = c_B'(x_B^*), \tag{29}$$

and

$$\int (x_B^*)^\alpha \, \frac{g(\eta)}{f(\phi^{-1}(\eta))} \frac{\alpha\eta}{x_G^*} g(\eta)\, d\eta = c_G'(x_G^*). \tag{30}$$

Consider the case where $f = g$. In this case, the equilibrium investments satisfy the conditions

$$\alpha \frac{(x_G^*)^\alpha}{x_B^*} \mathbf{E}(\varepsilon) = c_B'(x_B^*)$$

and

$$\alpha \frac{(x_B^*)^\alpha}{x_G^*} \mathbf{E}(\eta) = c_G'(x_G^*).$$

Clearly, there is a unique solution to these equations, so that equilibrium is unique. If the cost functions are also identical, then $x_B^* = x_G^* = x^*$. This implies that $c'(x_B^*) = c'(x_G^*) = \alpha\,(x^*)^{\alpha-1}\, \mathbf{E}(\varepsilon)$, which is the same condition as for the first best.

Now let us assume that the cost functions are the same but allow the densities to be different. Suppose that f is a translation of g,i.e. $f(\varepsilon) = g(\varepsilon + k)$ for some k. If $k > 0$, this corresponds to the case where women are uniformly better than men. From the above conditions, we see that there must be underinvestment by women relative to the utilitarian efficient amount, and overinvestment by men, since $c'(x_G^*) < \alpha\,(x_G^*)^{\alpha-1}\, \mathbf{E}(\eta)$ while $c'(x_B^*) > \alpha\,(x_B^*)^{\alpha-1}\, \mathbf{E}(\varepsilon)$.

Making the change of variables used in (26), from the equilibrium conditions (29) and (30) we get

$$c'(x_B^*) \times c'(x_G^*) = \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon\, d\varepsilon \right) \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon) g(\phi(\varepsilon))\, d\varepsilon \right) \left(\alpha^2 \left(x_B^* x_G^* \right)^{\alpha-1} \right). \tag{31}$$

The efficiency condition requires

$$c'(x_B^*) \times c'(x_G^*) = \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} f(\varepsilon)\varepsilon d\varepsilon \right) \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon) f(\varepsilon)) d\varepsilon \right) \left(\alpha^2 \left(x_B^* x_G^* \right)^{\alpha-1} \right). \tag{32}$$

Writing $h(\varepsilon) = \frac{f(\varepsilon)}{g(\varepsilon)}$, we see that the integrals in the equilibrium condition can be rewritten

$$\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} f(\varepsilon)\varepsilon h(\varepsilon) d\varepsilon \right) \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon) f(\varepsilon) \frac{1}{h(\varepsilon)}\, d\varepsilon \right).$$

This implies that the Cauchy-Schwarz bounds for (31) and (32) are identical.

Proposition 5 *Suppose that quality is multiplicative and that the distributions f and g are symmetric. Then investments are generically excessive relative to Pareto efficiency.*

Proof. Let f and g be symmetric functions around their means, $\tilde{\varepsilon}$ and $\tilde{\eta}$. Symmetry implies that $f(\tilde{\varepsilon}-\Delta)=f(\tilde{\varepsilon}+\Delta)$ for any Δ. If f and g are both symmetric, then $g(\phi(\varepsilon))$ is also symmetric around $\tilde{\varepsilon}$. Further, $\frac{f(\varepsilon)}{g(\phi(\varepsilon))}$ is also symmetric around $\frac{\tilde{\varepsilon}}{\phi(\tilde{\varepsilon})}$. Finally, symmetry implies that $\phi(\tilde{\varepsilon}+\Delta)+\phi(\tilde{\varepsilon}-\Delta)=2\phi(\tilde{\varepsilon})$. Using these facts,

$$\begin{aligned}
\int \phi(\varepsilon)g(\phi(\varepsilon))\,d\varepsilon &= \int^{\tilde{\varepsilon}} \phi(\varepsilon)g(\phi(\varepsilon))\,d\varepsilon + \int_{\tilde{\varepsilon}} \phi(\varepsilon)g(\phi(\varepsilon))\,d\varepsilon \\
&= \int^{0} \phi(\tilde{\varepsilon}-\Delta)g(\phi(\tilde{\varepsilon}-\Delta))\,d\Delta + \int_{0} \phi(\tilde{\varepsilon}+\Delta)g(\phi(\tilde{\varepsilon}+\Delta))\,d\Delta \\
&= \left(\int^{0} \phi(\tilde{\varepsilon}-\Delta)+\phi(\tilde{\varepsilon}+\Delta)\right) g(\phi(\tilde{\varepsilon}-\Delta))\,d\Delta \\
&= 2\phi(\tilde{\varepsilon})\int^{0} g(\phi(\tilde{\varepsilon}-\Delta))\,d\Delta \\
&= 2\phi(\tilde{\varepsilon})\int^{\tilde{\varepsilon}} g(\phi(\varepsilon))\,d\varepsilon.
\end{aligned}$$

Similarly,

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon d\varepsilon = 2\tilde{\varepsilon}\int^{\tilde{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\,d\varepsilon.$$

Thus we may write the product of the marginal costs as

$$c'(x_B^*)\times c'(x_G^*) = \left(2\tilde{\varepsilon}\int^{\tilde{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon d\varepsilon.\right)\left(2\tilde{\eta}\int^{\tilde{\varepsilon}} g(\phi(\varepsilon))\,d\varepsilon.\right)\left(\alpha^2\left(x_B^* x_G^*\right)^{\alpha-1}\right). \quad (33)$$

By the Cauchy-Scwharz inequality,

$$\begin{aligned}
c'(x_B^*)\times c'(x_G^*) &\geq 4\tilde{\varepsilon}\tilde{\eta}\left(\int^{\tilde{\varepsilon}}\left(\frac{f(\varepsilon)}{(g(\phi(\varepsilon)))^{1/2}}\right)(g(\phi(\varepsilon)))^{1/2}\,d\varepsilon.\right)^2\left(\alpha^2\left(x_B^* x_G^*\right)^{\alpha-1}\right) && (34)\\
&\geq 4\tilde{\varepsilon}\tilde{\eta}\left(\frac{1}{2}\right)^2\left(\alpha^2\left(x_B^* x_G^*\right)^{\alpha-1}\right). && (35)
\end{aligned}$$

Thus the product of marginal costs exceeds $\tilde{\varepsilon}\tilde{\eta}\left(\alpha^2\left(x_B^* x_G^*\right)^{\alpha-1}\right)$, and investments are excessive relative to Pareto efficiency. ■

This result is a robust one, in the following sense. Suppose that f and g are symmetric and linearly independent. The Cauchy-Schwarz inequality implies that the inequality in (34) will be strict. Now if we perturb the distributions so the $\tilde{f}$ is close to f and $\tilde{g}$ to g, then $c'(x_B^*)\times c'(x_G^*)$ will still be greater than 1, since the integrals defining this are continuous in the distributions. In other words, we will have excessive investments even with asymmetric distributions as long as the asymmetries are not too large.

Why does the result require that the asymmetry not be too large? To provide some intuition for this, let us return to the example where the distribution of shocks is uniform on $[0,1]$ for men and where the density function for women, $g(\eta)=2\eta$ on $[0,1]$. Here again, the ratio of the densities that is relevant for men, $\frac{f(\varepsilon)}{g(\phi(\varepsilon))}$, is relatively large when ε is low. While these values of ε still have large weight (since $f(\varepsilon)$ is constant in ε), in the multiplicative case, the payoff to investment is low when ε is small. Under symmetry, neither particularly low values nor particularly high values of ε have any special weight and thus the inefficiency result applies.

4.2.1 The Example Re-interpreted

Let us assume that f and g are symmetric around 1 on the interval $[0, 2]$. It will suffice therefore to specify the functions on $[0, 1]$. Let f be uniform, so that $F(\varepsilon) = \frac{\varepsilon}{2}$ on $[0, 1]$, and let $G(\eta) = \frac{\eta^n}{2}$ on $[0, 1]$. Let $\alpha = 1$, so that quality is multiplicative, $q = x\varepsilon$. It may be useful to note that the shock distributions are "symmetrized extensions" of the shocks in example 1. That is, the function g in the current example is constructed by taking the function g in example 1, and "reflecting it" in a mirror situated at 1 (the resulting function has to be halved, so as to ensure that it represents a probability distribution).

The equilibrium conditions are now given by

$$\begin{aligned} c'(x_B^*) &= \left(2\int_0^1 \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))}d\varepsilon\right)\frac{x_G^*}{x_B^*} = \frac{1}{2n}\int_0^1 \varepsilon^{\frac{1}{n}}d\varepsilon. \\ &= \frac{1}{n}\frac{x_G^*}{x_B^*}. \end{aligned} \tag{36}$$

Note that in Example 1, where shocks where additive, marginal costs were equal to $1/n$.

$$\begin{aligned} c'(x_G^*) &= \left(2\int_0^1 \frac{g(\eta)}{f(\phi^{-1}(\eta))}g(\eta)\eta d\eta\right)\frac{x_B^*}{x_G^*} = n^2\int_0^1 \eta^{2n-1}d\eta \\ &= \frac{n^2}{2n-1}\frac{x_B^*}{x_G^*}. \end{aligned} \tag{37}$$

Here again, the expression is identical to the one in Example 1, except for the term in the investment levels. Thus,

$$c'(x_B^*) \times c'(x_G^*) = \frac{n}{2n-1}.$$

For arbitrary Pareto weights λ, the efficiency condition is

$$c'(x_B^{**}) \times c'(x_G^{**}) = \left[\frac{\lambda}{1-\lambda}\mathbf{E}(\varepsilon)\right]\left[\frac{1-\lambda}{\lambda}\mathbf{E}(\eta)\right] = 1.$$

We therefore see that the equilibrium investments are always excessive, just as in the additive case. Indeed, the extent of inefficiency is same as in example one, provided that one uses the symmetrized extensions of the original density functions for the multiplicative case.

This example illustrates a more general duality between the additive and multiplicative model. Suppose that we have an additive model with density functions for the noise, f and g, on compact supports. Now consider a multiplicative model with densities $\tilde{f}$ and $\tilde{g}$ that are "symmetrized extensions" of f and g respectively. That is $\tilde{f} = \frac{f}{2}$ on its original support $[a, b]$ and $\tilde{f}$ is symmetric around b. $\tilde{g}$ is defined analogously. Then the overinvestment in the additive model is, up to a constant, the same as in the multiplicative model.

For another example, take $g(\eta)$ to be uniform on $[0, 1]$ and $f(\varepsilon) = 1/2 + 3\varepsilon(1 - \varepsilon)$ also on $[0, 1]$, which is a unimodal distribution with mean 1/2. The efficiency condition becomes $c'(x_B^*) \times c'(x_G^*) = 1/4$. In equilibrium, however,

$$\left(\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon \, d\varepsilon\right) \left(\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \phi(\varepsilon) g(\phi(\varepsilon)) \, d\varepsilon\right) = \left(\int_0^1 f^2(\varepsilon)\varepsilon \, d\varepsilon\right) \left(\int_0^1 F(\varepsilon) \, d\varepsilon\right) = \frac{21}{80} > \frac{1}{4}.$$

Thus, again equilibrium investment is excessive.

4.3 A Mixed Model

Suppose that shocks are additive for women but multiplicative for women. One example is a traditional society, where women do not work, and so investment in them takes the form of a dowry; while men invest in human capital.

$$\begin{aligned} c'(x_B^*) \times c'(x_G^*) &= \left(\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon \, d\varepsilon\right) \left(\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} g(\phi(\varepsilon)) \, d\varepsilon\right), \\ &\geq \left(\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} f(\varepsilon)\sqrt{\varepsilon} \, d\varepsilon\right)^2 = \left(\mathbf{E}(\sqrt{\varepsilon})\right)^2. \end{aligned}$$

Pareto efficient investments satisfy

$$c'(x_B^{**}) \times c'(x_G^{**}) = \mathbf{E}(\varepsilon).$$

Since $\left(\mathbf{E}(\sqrt{\varepsilon})\right)^2 < \mathbf{E}(\varepsilon)$, the Cauchy-Schwarz lower bound is uninformative. However, if we assume that both f and g are symmetric, we get generic overinvestment.

Proposition 6 *Suppose that quality is multiplicative for men and additive for women. If the distributions f and g are symmetric, investments are generically excessive relative to Pareto efficiency.*

Proof. Using the same notation as for the multiplicative model, we may write the product of the marginal costs as

$$\begin{aligned} c'(x_B^*) \times c'(x_G^*) &= \left(\int \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\varepsilon \, d\varepsilon\right) \left(\int g(\phi(\varepsilon)) \, d\varepsilon\right), \\ & \left(2\tilde{\varepsilon} \int^{\tilde{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon) d\varepsilon.\right) \left(2 \int^{\tilde{\varepsilon}} g(\phi(\varepsilon)) \, d\varepsilon.\right) \\ &\geq \tilde{\varepsilon}. \end{aligned} \tag{38}$$

■

Here again, we can see an equivalence between symmetric extensions of examples in the additive case and the mixed case.

5 Gender differences

Our analysis has highlighted the importance of the distribution of shocks for investment decisions by the two sides of the market. A key interpretation of these shocks is talent shocks. One issue, that excites great controversy, is whether the distributions differ for men and women. For example, Baron-Cohen (2004) and Pinker (2003) argue that there are intrinsic gender differences that are rooted in biology, while Fine (2010) has attacked this view. We now explore the implications of differences in variability between the sexes – it has been established that boys show greater variance in school performance than girls.

Suppose that the shocks are more variable for men then for women. A distribution F is larger in the dispersive order than a distribution F, or $F \geq_d G$ if

$$g(G^{-1}(z)) \geq f(F^{-1}(z)) \text{ for all } z \in (0,1), \tag{39}$$

with the inequality being strict on a set of z values with positive measure (see Shaked and Shanthikumar (2007, pp148-9)). An example of a distribution satisfying this condition would be two uniform distributions where one distribution has support on a strictly longer interval than the other. A second example is of two normal distributions with different variances. These measures of dispersive order do not rely upon an equality of means (see Hopkins and Kornienko (2010) for further examples and discussion).

Proposition 7 *Assume shocks are additive. If the distribution of shocks for boys is more dispersed than that for girls, that is $F \geq_d G$, then there is under-investment in boys and there is over-investment in girls relative to the utilitarian efficient level.*

Proof. With equal measures of men and women, if $\eta = \phi(\varepsilon)$, then ε and η have the same rank z in the two distributions. So, if $F \geq_d G$, then, by the definition (39), we have $\frac{f(\varepsilon)}{g(\phi(\varepsilon))} \leq 1$ for all values of ε, and is strictly less on a set values of ε of positive measure. Thus, the integral on the left-hand side of equation (24) is strictly less than one, and the integral on the left-hand side of equation (25) is strictly greater than one. If the investment cost functions are the same for the sexes, then girls invest more than boys. But in any case as utilitarian efficiency requires $c'_B(x_B) = 1 = c'_G(x_G)$, boys' investment is lower and girls' investment is higher than the utilitarian levels. ■

It has been observed that frequently the average performance of girls in school is better than that of boys. Our model provides a possible partial explanation for this – the incentives to invest for girls are greater, from marriage market matching considerations.

6 Sex Ratio Imbalances

Sex ratio imbalances are an important phenomenon in countries such as China and India. How do these imbalances affect parental incentives to invest in their children? Suppose as in the previous section, that each gender is homogeneous. However, the two genders differ in size so that the relative measure of girls is $r < 1$. Thus, only some of the boys will be matched. In these circumstances, the utilitarian measure of welfare (3), with Pareto weights given by population shares, is now given by

$$\begin{aligned} W &= \frac{r}{1+r}\left(\int q(x_G,\eta)g(\eta)\,d\eta - c_B(x_B)\right) + \frac{r}{1+r}\left(\int_{\hat{\varepsilon}} q(x_B,\varepsilon)f(\varepsilon)\,d\varepsilon - c_G(x_G)\right) \\ &+ \frac{1-r}{1+r}\left(-c_B(x_B)\right). \end{aligned} \tag{40}$$

The first term is the utility from having a boy who is matched, the second from a girl, and the last term is that from an unmatched boy.

Let us first consider efficiency in investments, conditional on a given sex ratio r. We specialize to the model with additive shocks. Thus, the first order conditions for maximizing the utilitarian measure of welfare are

$$c_B'(x_B^{**}) = r,\ c_G'(x_G^{**}) = 1. \tag{41}$$

At the matching stage, since $r \leq 1$, all girls should be matched, and the highest quality boys should be matched. Since every girl is matched, the investment in her generates benefits for herself as well as for her partner (for sure). Thus the first best investment level in a girl, x_G^{**}, satisfies $c'(x_G^{**}) = 1$. Now consider investment in a boy. If we assume that the idiosyncratic component of match values is sufficiently small, then welfare optimality requires that only a fraction r of boys invest, and that their investments also satisfy $c'(.) = 1$. However, if we restrict attention to symmetric investment strategies, then investment will take place in all boys, and since investment occurs before ε is realized, each boy has a probability r of being matched, and thus the first best efficient level of investment in a boy, x_B^{**}, must satisfy $c'(x_B^{**}) = r$, i.e. the marginal cost must equal the expected marginal benefit. Note that the condition for Pareto efficiency here, with arbitrary weights on the welfare of boys and girls is

$$c_B'(x_B^{**}) \times c_G'(x_G^{**}) = r. \tag{42}$$

We now turn to equilibrium. Suppose that all boys invest the same amount x_B^* and all girls invest the same amount x_G^*. Since the top r fraction of boys will only be matched, this corresponds to those having a realization of $\varepsilon \geq \hat{\varepsilon}$ where $F(\hat{\varepsilon}) = 1 - r$. In this case, a boy of type $\varepsilon \geq \hat{\varepsilon}$ will be matched with a girl of type $\phi(\varepsilon)$, where

$$1 - F(\varepsilon) = r[1 - G(\phi(\varepsilon))]. \tag{43}$$

Notice now that the derivative of the matching function is given by

$$\phi'(\varepsilon) = \frac{f(\varepsilon)}{rg(\varepsilon)}. \tag{44}$$

That is, an increase in ε increases a boy's match quality relatively more quickly, since the distribution of girls is relatively thinner now (since $r < 1$). The first order condition for boys in an equilibrium where all boys invest the same amount x_B^*, while all girls invest the same amount x_G^* is given by

$$\frac{1}{r}\int_{\hat{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\,d\varepsilon + f(\hat{\varepsilon})(\underline{\eta} + x_G^*) = c'(x_B^*). \tag{45}$$

As compared to our previous analysis, where r was equal to one, we notice two differences. The first term is the improvement in match quality, and the sparseness of girls increases the investment incentives, due to the term in $1/r$. Additionally, an increment in investment raises the probability of my boy getting matched, at a rate $f(\hat{\varepsilon})$, and in this event, my payoff equals the absolute payoff of the worst quality girl, $\underline{\eta} + x_G$. Let us call this last term the "misery effect" – it reflects the disutility (misery) from remaining single. As Hajnal (1982) has noted, in Asian societies such as China and India, marriage rates have historically been extremely high (over 99%, as compared to the traditional "European marriage pattern" with marriage rates around 90%). Thus the misery effect is likely to be large in Asian societies.

We see that if $r < 1$, this tends to amplify investments in boys, for two reasons. First, a given increment in investment pushes boys more quickly up the distribution of girls, and second, there is an incentive to invest in order to increase the probability of match taking place at all, since there is discontinuous payoff loss from not being matched at $\underline{\varepsilon}$, due to the misery effect.

Similarly, the first order condition for investment in girls is given by

$$r\int_{\underline{\eta}}^{\bar{\eta}} \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\,d\eta = c'(x_G^*). \tag{46}$$

Notice here that the role of $r < 1$ is to reduce investment incentives, since an increment in investment pushes a girl more slowly up the distribution of boy qualities. Furthermore, there is no counterpart to the misery effect for the scarcer sex, and the only reason to invest arises from the consequent improvement in match quality.

Proposition 8 *If $r < 1$ and the noise is additive, then investment is excessive relative to Pareto efficiency, for any distributions of noise.*

Proof. If $r < 1$, then $f(\hat{\varepsilon})(\underline{\eta} + x_G^*) > 0$ since $\underline{\eta} > 0$, reflecting our assumption that the misery effect is strictly positive, and so

$$c'(x_B^*) \times c'(x_G^*) > \left(\frac{1}{r}\int_{\hat{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\,d\varepsilon\right)\left(r\int \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\,d\eta\right).$$

Making a change of variables, from η to ε,

$$c'(x_B^*) \times c'(x_G^*) > \frac{1}{r}\left(\int_{\hat{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\,d\varepsilon\right)\left(\int_{\hat{\varepsilon}} g(\phi(\varepsilon))\,d\varepsilon\right).$$

By the Cauchy-Schwarz inequality,

$$\frac{1}{r}\left(\int_{\hat{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon\right)\left(\int_{\hat{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon\right) \geq \frac{1}{r}\left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \left(\frac{f(\varepsilon)}{(g(\phi(\varepsilon)))^{1/2}}\right)(g(\phi(\varepsilon)))^{1/2}\ d\varepsilon\right]^2 = \frac{1}{r}\left(r^2\right). \tag{47}$$

Thus $c'(x_B^*) \times c'(x_G^*) > r$. ■

It is worth pointing out that even absent the misery effect, there will be strictly excessive investments, even if the noise distributions are identical, unless they both happen to be uniform. When $r < 1$, $g(\phi(\varepsilon))$ is not a linear transformation of $f(\varepsilon)$ unless both distributions are uniform. Thus the inequality in (47) will be strict, and excessive investment follows.

As we have already noted, if F and G have the same distributions, and $r = 1$, investments are utilitarian efficient. Thus if the social planner can ensure a balanced sex ratio, investments need not be regulated since they will coincide with the first best level.

As an example, consider the case where the noise distributions are both uniform. Let k be the ratio of the densities, i.e. $k = \frac{f(\varepsilon)}{g(\eta)}$. The equilibrium investment levels are defined by

$$c'(x_B^*) = k + f(\hat{\varepsilon})(x_G^* + \underline{\eta}).$$

and

$$c'(x_G^*) = \frac{r}{k}.$$

If $k = 1$ so that the two distributions have the same dispersions, we see that there are excessive investment in boys relative to utilitarian levels – the marginal cost of investment exceeds one, while efficiency at the investment stage requires $c'(x_B^*) = r < 1$. There are insufficient investments in girls, since $c'(x_G^*) = r$ rather than the efficient level, one. More generally, for any k, if we compare to (42), total investment is excessive – the product of the marginal costs exceeds r due the second term in the investment condition for boys, arising from the incentive to raise investments in order to ensure a match.

Our results are relevant in the context of recent empirical work by Wei and Zhang (2009), arguing that the high savings rate in China is attributable to the sex ratio imbalance. They argue that parents of boys feel compelled to invest more, thus raising the overall savings rate. Our analysis shows that while this is indeed the case, this is counterbalanced by the reduced investment incentive for parents of girls. If we assume that the costs of investment are quadratic (so that marginal costs are linear), aggregate investment in the economy, $X^*(r)$, will be proportional to the weighted sum of the right hand side of the optimality conditions, i.e.

$$X^*(r) = \frac{r}{1+r} x_G^* + \frac{1}{1+r} x_B^* \propto \frac{1+r^2}{1+r} + \frac{x_G^*}{\bar{\varepsilon}(1+r)}.$$

Note that the first term, on the right hand side, $\frac{1+r^2}{1+r}$, is increasing in r, while the derivative of sign of the second term is ambiguous, since both the numerator and denominator are increasing in r. Thus it theoretical ambiguous whether a sex ratio imbalance gives rise to an overall increase in parental investment and savings rates. However, if the incentive for boys to ensure a match is large enough, total investment will increase with sex ratio imbalances.

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