Career Patterns and Career Concerns*

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Abstract

This paper develops a model of career concerns. As in Holmström (1999), a worker's productive abilities are revealed over time through output, and wages are based on expected output, and so on assessed ability. Specifically, work increases the probability that a skilled worker achieves an observable breakthrough. It is shown that effort at different times are strategic substitutes. As as result, effort (and, in case marginal cost is convex, the wage) is single-peaked with seniority. The agent works too little, too late. Both the delay and the underprovision of effort worsen if effort is observable. Finally, if the employer can commit to a wage path but faces competition by firms offering similar contracts, employers offer simple contract with piecewise constant wages, in addition to severance payments.

1 Introduction

Promotion policies in professional service firms are typically based on an "up-or-out-system" (law firms, accounting firms, consulting firms, etc.). Employees are expected to obtain promotion to

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partner in a certain time period; if not, they are expected to quit, when they are not dismissed forthright. While alternative theories have been put forth (e.g., tournament models), agency theory provides an appealing framework to analyze such systems (see Fama, 1980, or Fama and Jensen, 1983). This paper investigates the incentives and compensation of employees, how they evolve over time, how they depend on the work performance measurement as well as on the other provisions on the labor contract.

Our model borrows its key ingredients from Holmström (1999).[1] There are no explicit output-contingent contracts. The firm, or market, must pay the worker, or agent, a competitive wage, given his expected output, which in turn is based on his assessed ability. Information about ability is symmetric at the start. We depart from Holmström in the specification of the learning process. Skill and output are binary, and only a skilled agent can achieve a high output. The time at which this output arrives –the *breakthrough*– follows an exponential distribution, whose instantaneous intensity increases with the worker's effort. If the agent ever succeeds, and so proves himself, he is promoted and gets a constant compensation. While in some respects more stylized than Holmström's, this specification implies that effort increases not only expected output, but also the speed of learning, unlike in the Gaussian set-up. We view it as a plausible alternative modeling for labor markets in which frequently revised effort decisions provide highly informative signals infrequently.

Fundamental to the dynamics of incentives and wages is the *strategic substitutability* between current and future effort, via the worker's compensation: if career concerns are effective in providing incentives for high effort at some point in the worker's career, wages will reflect this increased productivity; in turn, this depresses incentives to exert high effort earlier in the worker's career, as higher future wages makes staying on the current job relatively more attractive.

Substitutability shapes the pattern of effort and compensation: because career concerns cannot work at the end of the worker's tenure, effort is single-peaked, with mid-career incentives depressing early incentives. (Though this does not rule out, as special cases, monotone effort paths.) When marginal cost is convex, wages are single-peaked as well. This stands in contrast with Holmström's model, in which effort (and wages) stochastically decrease over time. Because compensation does not only reflect effort, but also ability, and prolonged failure necessarily in-

[1]See Gilson and Mnookin (1989) for a vivid account of associate career patterns in law firms, and the relevance of Holmström's model as a possible explanation.

creases pessimism regarding this ability, wage dynamics can be slightly more complicated when the marginal cost is not convex, with an initial phase of decreasing wages preceding a single-peaked pattern.

As mentioned, substitutability does not arise in Holmström's model. It does not arise in Dewatripont, Jewitt and Tirole (1999a,b)'s analysis either, which is not surprising, as it cannot be picked up by a two-period model (career concerns cannot arise in the last period). Their analysis focuses instead on the strategic complementarity between expected effort and current effort, which generates, among others, equilibrium multiplicity. The same complementarity exists in our model. Nevertheless, the equilibrium is unique, under mild conditions.

As we show, career concerns provide insufficient incentives for effort. Furthermore, despite the single-peaked pattern of effort, whatever effort is provided is done so too late: a social planner constrained to the same total amount of effort would apply it earlier. That is, strategic substitutability induces backloading of effort. This contrasts with the inefficient frontloading of effort that arises in Holmström's model.

Although our model is at least consistent with wages that are not decreasing over time, it leaves open the question why non-decreasing compensation is such a prevalent phenomenon in practice. To investigate the sensitivity of our findings to labor market arrangements, we then consider three variations of the baseline model. First, it might be that firms have more commitment than is typically assumed, and while workers cannot commit not to leave their employer if a competing firm offers a better contract at any point, firms can nevertheless commit to contracts that specify an entire wage path. In that case, the optimal contract is strikingly simple: it is either a one-step or (if the horizon is long enough) two-step wage, followed by a lump-sum "severance" payment at the end of tenure, if the worker never succeeded. Effort is constant over each step.

Second, we examine how the quality of monitoring affects our conclusion: what if effort is observable, if not contractible, after all? In any of the Markov equilibria, effort provision is even lower than under non-observability, and it is further delayed; as a result, effort increases over time (which pushes wages up over time). While this means that, in line with earlier findings in this literature, imperfect observability helps generate incentives, it also points to the fact that empirical patterns might be better explained by models with better monitoring.

Finally, we endogenize the deadline, by letting workers leave whenever they consider it best, though employers rationally anticipate this. In that case, effort is not only single-peaked, it must be decreasing at the deadline, and so must the wage. The worker quits too late, relative to what would be optimal, but if he could commit to a deadline, he might choose a longer, or a shorter one than without commitment, depending on the circumstances.

We then turn to contract design, and ask whether having a deadline, corresponding to a rigid apprenticeship period, is useful.

The most closely related paper is Holmström, as discussed. See also Jovanovic (1979) and Murphy (1986) for related model. Our paper shares with Gibbons and Murphy (1989) the interplay of implicit incentives (career concerns) and explicit incentives (termination penalty). It shares with Prendergast and Stole (1996) the existence of a finite horizon, and thus, of complex dynamics related to seniority. See also Bar-Isaac for reputational incentives in a model in which survival depends on reputation. The binary set-up is reminiscent of Bergemann and Hege (2005), Mailath and Samuelson (2005), and Board and Meyer-ter-Vehn (2011). A theory of up-or-out contracts, based on asymmetric learning and promotion incentives, is investigated in Ghosh and Waldman (2010), while Chevalier and Ellison (1999) provide evidence of the sensitivity of termination to performance. Ferrer (2011) studies how lawyers' career concerns impacts litigation.

There is a growing literature on reputation in teams, which is certainly relevant for professional service firms, in which associates routinely engage in joint projects with partners. See Bar-Isaac (2007), Jeon (1996), Landers, Rebitzer and Taylor (1996), Levin and Tadelis (2005), Morrison and Wilhelm (2004), and Tirole (1996). Extending our set-up to allow for team work is subject to ongoing research.

2 The Model

2.1 Set-up

We shall consider the incentives of a single agent (or *worker*) to exert effort (or *work*). Time is continuous, and the horizon finite: $t \in [0, T]$, $T > 0$. Most results carry over to the case $T = \infty$, as shall be discussed, and the case of endogenous deadlines T will be studied in detail in Section 5.3.

The game (or *project*) can end before $t = T$, in case the agent's effort is successful. Specifically, we assume that there is a binary state of the world $\omega = 0, 1$ that is interpreted as the underlying skill, or ability of the agent. If the state is $\omega = 0$, the agent is bound to fail, no matter how much effort he exerts. If the state is $\omega = 1$, a success (or *breakthrough*) arrives at a time that is exponentially distributed, with an intensity that increases in the instantaneous level of effort exerted by the agent. The state can be interpreted as the agent's ability, but we will refer to the agent as a high- (resp., low-) ability agent in case the state is 1 (resp. 0). The prior probability of state 1 is $p^0 \in (0, 1)$.

Effort is a (measurable) function from time to the interval $[0, \bar{u}]$, where $\bar{u} \in \bar{\mathbb{R}}$ represents an upper bound to the instantaneous effort that the agent can exert. If the agent exerts effort u_t over the time interval $[t, t + \mathrm{d}t)$, the probability of a success over that time interval is $(\lambda + u_t)\mathrm{d}t$, where $\lambda \geq 0$ can be viewed as luck. Formally, the instantaneous arrival rate of a breakthrough at time t is given by $\omega \cdot (\lambda + u_t)$.

While the game has not ended, the agent receives a flow wage w_t. For now, let us think of this wage as an exogenous (integrable, non-negative) function of time only that accrues to the agent as long as the game has not ended, though equilibrium constraints will be imposed on this function, as this wage will reflect the market's expectations of the agent's effort and ability, given that the market values a success. This value is normalized to one.

In addition to receiving this wage, the agent incurs a cost of effort: exerting effort level u_t over the time interval $[t, t + \mathrm{d}t)$ entails a flow cost $c(u_t)\mathrm{d}t$. We shall consider two cases: in the convex case, we assume that $\bar{u} = \infty$, c is increasing, thrice differentiable and convex, with $c(0) = 0$, $\lim_{u \to 0} c'(u) = 0$, $\lim_{u \to \infty} c'(u) = \infty$, $c'' > 0$ and $c''' \geq 0$. In the linear case, $\bar{u} = \infty$ and $c(u) = \alpha \cdot u$, where $\alpha > 0$. Plainly, the linear case is not a special case of what is called the convex one, but it yields similar results, while allowing for simple illustrations and sharper characterizations.

Achieving a success is desirable on two accounts: first, a known high-ability agent can expect a flow outside wage of $v \geq 0$, so that this outside option v is a (flow) opportunity cost for him that is incurred as long as no success has been achieved.[2] The outside option of the low-ability

[2]A natural case is the one in which v equals the flow value of success given that the agent has established that $\omega = 1$. This would be his payoff in the Markov equilibrium of the complete information game. Since successes arrive at rate λ and are worth 1, $v = \lambda$ in that case.

agent is normalized to 0. Second, reaching the deadline (without achieving a success) entails a fixed penalty of $k \geq 0$, representing diminished career opportunities to workers with such poor records. This penalty might be an adjustment cost, or the difference between the wage he could have hoped for had he succeeded, and the wage he will receive until retirement. In the linear cost case, we assume $k > \alpha$, for the penalty plays essentially no role otherwise. There is no discounting. At the beginning of the appendix, we explain how to derive the objective function from its discounted version as discounting vanishes.

Thus, the worker chooses $u : [0, T] \to [0, \bar{u}]$, measurable, to maximize his expected sum of rewards, net of the outside wage v:

$$\mathbb{E}_u \left[\int_0^{T \wedge \tau} \left[w_t - \tilde{v} - c\left(u_t\right) \right] \mathrm{d}t - \chi_{\tau \geq T} k \right],$$

where $\mathbb{E}_u$ is the expectation conditional on the worker's strategy u, $\tilde{v}$ is the random outside option of the agent (0 or v depending on his ability) and τ is the time at which a success occurs –a random variable that is exponentially distributed, with instantaneous intensity at time t equal to 0 if the state is 0, and to $\lambda + u_t$ if the state is 1, and χ_A is the indicator of event A.

Of course, at time t effort is only exerted, and the wage collected, conditional on the event that no success has been achieved. We shall omit to say so explicitly, as those histories are the only nontrivial ones. Given his past effort choices, the agent can compute his belief p_t that he is of high ability by using Bayes' rule. It is standard to show that, in this continuous-time environment, Bayes' rule reduces to the ordinary differential equation (O.D.E.)

$$\dot{p}_t = -p_t \left(1 - p_t\right) \left(\lambda + u_t\right), \; p_0 = p^0. \tag{1}$$

By the law of iterated expectations, we can then rewrite our objective as

$$\int_0^T e^{-\int_0^t p_s(\lambda + u_s)\mathrm{d}s} \left[w_t - p_t v - c\left(u_t\right) \right] \mathrm{d}t - k e^{-\int_0^T p_t(\lambda+u_t)\mathrm{d}t}.$$ [3]

The exponential term captures the possibility that time t is never reached. Using integration by parts,

$$e^{-\int_0^t p_s(\lambda+u_s)\mathrm{d}s} = \frac{1 - p_0}{1 - p_t}. \tag{2}$$

[3] To see this, note that the probability that no success has occurred by time t is given by $\exp[-\int_0^t p_s(\lambda+u_s)\mathrm{d}s]$.

Alternatively, observe that

$$\mathbb{P}\left[\tau \geq t\right] = \frac{\mathbb{P}\left[\omega = 0 \cap \tau \geq t\right]}{\mathbb{P}\left[\omega = 0 | \tau \geq t\right]} = \frac{\mathbb{P}\left[\omega = 0\right]}{\mathbb{P}\left[\omega = 0 | \tau \geq t\right]} = \frac{1 - p_0}{1 - p_t}.$$

Hence, the problem simplifies to the maximization of

$$\int_0^T \frac{1 - p_0}{1 - p_t}\left[w_t - c\left(u_t\right) - v\right] \mathrm{d}t - \frac{1 - p_0}{1 - p_T} k, \quad (3)$$

[4]

given w, over all measurable $u : [0, T] \to [0, \bar{u}]$, given (1). Before solving this program, we start by analyzing the simpler problem faced by a social planner.

2.2 The Social Planner

What is the expected value of a breakthrough? Recall that the value of a realized breakthrough is normalized to one. But a breakthrough only arrives with instantaneous probability $p_t\left(\lambda + u_t\right)$, as it occurs at rate $\lambda + u_t$ only if $\omega = 1$. Therefore, the planner maximizes

$$\int_0^T \frac{1 - p_0}{1 - p_t}\left[p_t\left(\lambda + u_t\right) - v - c\left(u_t\right)\right] \mathrm{d}t - k\frac{1 - p_0}{1 - p_T}, \quad (4)$$

over all measurable $u : [0, T] \to [0, \bar{u}]$, given (1). As for most of the optimization programs considered in this paper, we apply Pontryagin's maximum principle to get a characterization. The proof of the next lemma and of all formal results can be found in appendix. A strategy u is *extremal* if it only takes extreme values: $u_t \in \{0, \bar{u}\}$, for all t.

Lemma 2.1 *At any optimum,*

1. *Effort u is monotone (in t); it is non-increasing if and only if the deadline exceeds some finite length. If it is non-increasing, so is the marginal product $p\left(\lambda + u\right)$; if it is non-decreasing, then the marginal product is non-decreasing in at most one interval;*

[4] Note that we have replaced $p_t v$ by the simpler v in the bracketed term inside the integrand. This is because

$$\int_0^T \frac{p_t}{1 - p_t} v \mathrm{d}t = \int_0^T \frac{v}{1 - p_t} \mathrm{d}t - vT,$$

and we can ignore the constant vT, at least until Section 5.3, where the deadline is endogenized.

2. *In addition, in the case of linear cost, the optimal strategy is extremal and maximum effort precedes zero effort if and only if $v > \alpha\lambda$.*

Because the belief p is decreasing over time, note that the marginal product is decreasing whenever effort is decreasing, but the converse need not hold (as the product $p(\lambda + u)$ might vary in either direction). The interval over which the marginal product is non-decreasing can be empty, or the entire horizon. Conversely, it is straightforward to construct examples in which effort is increasing, and the marginal product is first decreasing, then increasing, then decreasing again. Note that, for the critical deadline mentioned in the first part of the lemma, effort is constant.

With linear cost, whether effort is non-increasing or non-decreasing depends only on the sign of $v - \alpha\lambda$. This does not contradict the first part of the lemma, as effort is constant (and 0) for long enough deadlines. Note that neither the initial belief (p^0), nor the terminal cost (k) affect whether maximum effort is exerted first or last. Of course, they affect the total amount of effort, but given this amount, they do not affect its timing. The role of the sign $\alpha\lambda - v$ in the ordering of these intervals can be seen as follows: consider exerting some bit of effort now or at the next instant (thus, keeping the total amount of planned effort fixed); by waiting, a loss $v\mathrm{d}t$ is incurred; on the other hand, with probability $\lambda\mathrm{d}t$, the marginal cost of this effort, α, will be saved. Therefore, if

$$v > \alpha\lambda,$$

it is better to work early than late, if at all. From now on, we shall focus on the case $v > \alpha\lambda$.

Assumption 2.2 *In the linear cost case, the parameters α, v and λ are such that*

$$v > \alpha\lambda.$$

Under this assumption, effort can be efficient even far from the deadline. An example of such a path is given by the left panel in Figure 1. The right panel gives the corresponding path for the value of output (i.e., $p_t(\lambda + \bar{u}_t)$).

Whether effort is still exerted at the deadline depends on how pessimistic the social planner is at that point. By standard arguments (see appendix), full effort is exerted at the deadline if and only if

$$p_T(1 + k) \geq \alpha. \tag{5}$$

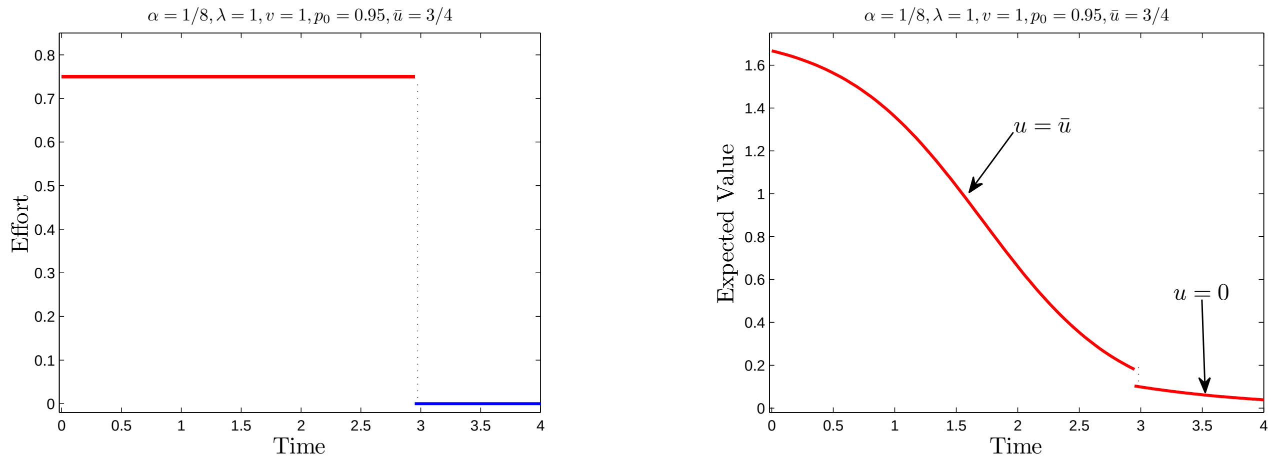

Figure 1: Effort and expected value at the social optimum

This states that the expected marginal social gains from effort (success and penalty avoidance) should exceed the marginal cost. If the social planner becomes too pessimistic, he "gives up" before the end. Note that the flow loss v no longer plays a role at that time, as the terminal (lump-sum) penalty overshadows any such flow cost.

It is straightforward to solve for the switching time, or switching belief in the linear case. This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted. More generally, we have:

Lemma 2.3

1. *Both in the convex and linear case, the final belief decreases with the deadline;*

2. *Total effort exerted increases with the deadline*

   (a) *in the linear case, if and only if* $\lambda (1 + k) < v$*;*

   (b) *in the convex case, if*

$$\max_u [(\lambda + u)(1 + k) - c(u)] < v.$$

Hence, total effort need not increase with the deadline; the sufficient condition given in the convex case (which implies $\lambda (1 + k) < v$) is not necessary; weaker, but less concise conditions can be given for the convex case, as well as examples in which total effort decreases with the deadline.

3 The agent's problem: The role of wages

Before solving for an equilibrium in which wages are determined by the market, consider the worker's optimal effort path given an *exogenous* (integrable) wage path $w : [0, T] \to \mathbb{R}_+$. The agent's problem differs from the social planner's in two respects: the agent fails to take into account the expected value of a success (in particular, at the deadline), a value that increases in the effort; instead, he takes into account the exogenous wages, which are less likely to be pocketed if more effort is exerted.

Recall that the worker's problem is given by (1)–(3). Let us start with a "technical" result.

Lemma 3.1 *A solution to* (1)–(3) *exists. For convex cost functions, the trajectory* x *is unique; for linear cost, if* x_1 *and* x_2 *are optimal trajectories, and* $x_{1,t} \neq x_{2,t}$ *over some interval* $[a, b] \in [0, T]$, *then* $w_t = v - \alpha\lambda$ *(a.e.) on* $[a, b]$.

That is, there is a unique solution (in terms of trajectories and hence control) for convex cost functions, and multiplicity in case of linear cost is confined to time intervals over which the wage is equal to a specific value. While this last case might appear non-generic, we shall see that it plays an important role in the equilibrium analysis nonetheless.

Transversality implies that the agent works at the deadline an amount that solves

$$p_T k = c'(u_T).^5$$

This is similar to the social planner's trade-off at the deadline, except that the worker does not take into account the lump-sum value of success (compare with (5)), and his effort level is consequently smaller.

[5] In the linear case, this must be understood as: the agent chooses $u = \bar{u}$ if and only if $p_T k \geq \alpha$, and chooses $u = 0$ otherwise.

3.1 Level of effort

What determines the instantaneous *level* of effort? It follows from Pontryagin's theorem that the amount of effort put in at time t solves

$$c'(u_t) = -\int_t^T (1-p_t)\frac{p_s}{1-p_s}[w_s - c(u_s) - v]\,\mathrm{d}s + (1-p_t)\frac{p_T}{1-p_T}k. \tag{6}$$

The left-hand side is the instantaneous marginal cost of effort. The marginal benefit (right-hand side) can be understood as follows. Conditioning throughout on reaching time t, the expected revenue over some interval $\mathrm{d}s$ at time $s \in (t,T)$ is

$$\mathbb{P}[\tau \geq s](w_s - c(u_s) - v)\,\mathrm{d}s.$$

From (2), recall that

$$\mathbb{P}[\tau \geq s] = \frac{1-p_t}{1-p_s} = (1-p_t)\left(1+\frac{p_s}{1-p_s}\right);$$

that is, effort at time t affects the probability that time s is reached only through the likelihood ratio $p_s/(1-p_s)$. From (1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_t}{1-p_t} = -\frac{p_t}{1-p_t}(\lambda + u_t),$$

and so a slight increase in u_t decreases the likelihood ratio at time s precisely by $-p_s/(1-p_s)$. Combining, such an increase changes expected revenue from time s by an amount

$$-(1-p_t)\frac{p_s}{1-p_s}[w_s - c(u_s) - v]\,\mathrm{d}s,$$

and integrating over s (including $s = T$) yields the result.

The trade-off captured by eqn. (6) illustrates a key difference between career concerns in this model and Holmström's: future compensation does affect incentives to put in effort.[6] Information is very coarse here. Either a success is observed or not. Hence, there is no scope for the wage to adjust linearly in the effort, so as to provide incentives that would be independent of the wage level itself. Although the log-likelihood ratio is linear in effort, as is the principal's posterior belief in Holmström's model, the coarse signal structure only allows the agent to affect the probability

[6] The reason why future compensation does not affect incentives in Holmström's model is that effort and talent affect output independently, effort affects the posterior belief linearly, and wage is itself linear in belief.

that the relationship terminates, unlike in Holmström's model in which signals and posterior beliefs are one-to-one.

As is intuitive, increasing the wedge between the future rewards from success and failure $(v - w_s)$ encourages high effort, *ceteris paribus.* Higher wages in the future depress incentives to exert effort today, as they reduce the premium from success $v - w_s$.

Similarly, a higher penalty for termination or a lower cost of effort provide stronger incentives.

3.2 Timing of effort

Differentiating eqn. (6) yields an arbitrage equation that determines how effort is allocated over time. (See the proof of Lemma 3.3.) Effort dynamics are governed by the following differential equation:

$$p_t \cdot \underbrace{c\left(u_{t+\mathrm{d}t}\right)}_{\text{cost saved}} + \underbrace{p_t\left(v - w_t\right)}_{\text{wage premium}} + \underbrace{c''\left(u_t\right)\dot{u}_t}_{\text{cost smoothing}} = \underbrace{p_t\left(\lambda + u_t\right)}_{\text{Pr. of success at } t} \cdot c'\left(u_t\right) \tag{7}$$

By shifting an effort increment $\mathrm{d}u$ from the time interval $[t, t + \mathrm{d}t)$ to $[t + \mathrm{d}t, t + 2\mathrm{d}t)$, the agent saves the marginal cost of this effort increment $c'(u_t)\mathrm{d}u$ with instantaneous probability $p_t(\lambda + u_t)\mathrm{d}t$ –the probability with which this additional effort won't have to be carried out. By exerting this additional effort early instead, the agent increases by $p_t \mathrm{d}u$ the probability that the entire cost of tomorrow's effort $c(u_{t+\mathrm{d}t})\mathrm{d}t$ will be saved. He also increases at the same rate the probability that he gets the "premium" $(v - w_t)\mathrm{d}t$ an instant earlier. Finally, if effort is increasing at time t, exerting the effort increment earlier improves the workload balance, which is worth $c''(u)\mathrm{d}u\mathrm{d}t$. This yields eqn. (7).[7]

With linear cost, cost-smoothing is irrelevant, and since this is the only term that is not proportional to belief p_t, the condition simplifies: frontloading effort is preferred if the wage premium exceeds the value of luck in cost units:

$$v - w_t \geq \alpha\lambda. \tag{8}$$

That the belief is irrelevant to the timing of effort (absent cost-smoothing motive) is intuitive: if the state is 0, the cost of the effort increment will be incurred either way, so that the comparison can be conditioned on the event that the state is 1.

[7] Note that all these terms are "second order" terms. Indeed, to the first order, it does not matter whether effort is slightly higher over $[t, t + \mathrm{d}t)$ or $[t + \mathrm{d}t, t + 2\mathrm{d}t)$.

3.3 Comparison with the social planner

Note that eqn. (8) reduces to the corresponding condition for the social planner when $w_t = 0$. Unlike the agent, the social planner internalizes future wages, which simply represent the value of possible success at these times. Hence, his arbitrage condition coincides with the agent's if the latter were to ignore the wages altogether. The same holds for the case of a convex cost function. To see this formally, note that the flow revenue term from eqn. (4) can be re-arranged as

$$\int_0^T \frac{1-p_0}{1-p_t} p_t \left(\lambda + u_t\right) \mathrm{d}t = -(1-p_0) \int_0^T \frac{\dot{p}_t}{(1-p_t)^2} \mathrm{d}t = (1-p_0) \ln \frac{1-p_T}{1-p_0},$$

and so this term only appears through the terminal belief, and hence the transversality condition. Note, however, that the transversality conditions do not coincide even if we set $w_s = 0$. As mentioned, the agent fails to take into account the value of a success at the last instant, so that his incentives then, and hence his strategy for the entire horizon, fails to coincide with the social planner's. The agent works too little, too late.

The next proposition formalizes this discussion. Given the wage path w, denote by p^* the (belief) trajectory given the solution to the agent's problem, and p^{FB} the corresponding trajectory for the social planner.

Proposition 3.2 *For convex cost functions, given the deadline T, if $w > 0$,*

1. *The agent's effort is lower than the social planner's, i.e. $p_T^* > p_T^{FB}$. Furthermore, instantaneous effort at any time t is always lower than the social planner's, given the current belief p_t^*.*

2. *Suppose that the social planner is constrained to set $p_T = p_T^*$. Then the optimal trajectory p is below the agent's belief trajectory, i.e. for all $t \in (0, T)$, $p_t^* > p_t$.*

Note that the first part states that both aggregate effort is too low, but also instantaneous effort, *given* the agent's belief. Nevertheless, as a function of calendar time, effort might be higher for the agent at some dates, because the agent might be more optimistic than the social planner at that point. The next example (Figure 2) will illustrate this phenomenon in the case of equilibrium wages.

The second part of this proposition implies that, for the fixed aggregate effort chosen by the agent, this effort is exerted too late relative to what would be optimal: the prospect of collecting future wages encourages procrastination.

The same result holds in the linear case, up to the strictness of the inequalities: of course, if the agent's optimum effort is maximum throughout, he is working just as much as in the social planner's solution.

3.4 Effort Dynamics

What do we learn from eqn. (8) regarding the dynamics of effort in the linear case? First, note that, unless $w = v - \alpha\lambda$ holds identically over some interval, effort is extremal. Second, suppose that w is increasing. Then the left-hand side decreases over time, and the agent prefers frontloading up to some critical time, after which backloading becomes optimal (the critical time might be 0 or T). This does not quite imply that his effort is non-increasing; rather, if he puts in low effort, he must do so in some intermediate time interval. If he starts with high effort, his marginal product $p(\lambda + u)$ must decrease, at least over some initial phase. This would be inconsistent with increasing wages in equilibrium.

Similarly, if wages decrease over time, the agent first backloads, then frontloads effort. That is, if he ever puts in high effort, he will do so in some intermediate phase.

The same observations can be made by considering (7) for the convex case, though effort will not be extremal. We summarize this discussion with the following lemma.

Lemma 3.3

1. *If w is decreasing, u is a quasi-concave function of time; if w is increasing, it is quasi-convex; if w is constant, u is monotone.*

2. *With linear cost and strictly monotone wages, the optimal strategy is extremal.*

To conclude, even when wages are monotone, the worker's incentives need not be so over time. While the equilibrium wage path of the next section fails to be monotone, the trade-off laid out in (7) remains decisive.

4 Equilibrium

Suppose now that the wage is set by a principal (or *market*) without any commitment power. The principal does not observe the agent's past effort, but only that the worker has not succeeded so far. Non-commitment motivates the assumption that wage equals expected marginal product, i.e.

$$w_t = \mathbb{E}_t[p_t(\lambda + u_t)],$$

where p_t and u_t are the agent's belief and effort, respectively, at time t, given his private history of past effort (of course, it is assumed that he has had no successes so far), and the expectation reflects the principal's beliefs regarding the agent's history (in case the agent mixes).[8] However, given Lemma 3.1, the agent will not use a chattering control (i.e., a distribution over measurable functions (u_t)), but rather a single function (unless the cost is linear and $w = v - \alpha\lambda$ over some interval, but even then the multiplicity is limited to the distribution of effort over this interval).[9] Therefore, we may write

$$w_t = \hat{p}_t(\lambda_t + \hat{u}_t), \tag{9}$$

where $\hat{p}_t$ and $\hat{u}_t$ denote the belief and anticipated effort at time t, as viewed from the principal.

In equilibrium, expected effort must coincide with actual effort.

Definition 4.1 *An equilibrium is a measurable function u and a wage path w such that: (i) u is a best-reply to w_t given the agent's private belief p, which he updates according to (1); (ii) the wage equals the marginal product, i.e. (9) holds for all t, and (iii) beliefs are correct, that, is, for every t,*

$$\hat{u}_t = u_t,$$

and therefore, also, $\hat{p}_t = p_t$ at all $t \in [0, T]$.

[8] In discrete time, if $T < \infty$, and under assumptions that guarantee uniqueness of the equilibrium (see below), non-commitment *implies* that wage is equal to marginal product in equilibrium, by a backward induction argument. We shall follow the literature by directly *assuming* that wage is equal to marginal product.

[9] If there are such time intervals (as equilibrium existence will require for many parameter values), the multiplicity of best-replies over this interval is of no importance: the expected effort at any time during this interval, as well as the aggregate effort over this interval will be uniquely determined, and the agent is indifferent over all effort levels over this time interval; the multiplicity does not affect wages, effort or belief before or after such an interval.

Note that, if the agent deviates, the market will typically hold incorrect beliefs.

To understand the structure of equilibria, consider the following example, illustrated in Figure 2. Suppose that the principal expects the agent to put in the efficient amount of effort, which in this example decreases over time. Accordingly, the wage paid by the firm decreases over time as well. The agent's best-reply, then, is quasi-concave in general: effort first increases, and then decreases (see left panel). This means that the agent puts in little effort at the start, as the agent has no incentive "to kill the golden goose" by exerting effort too early. Once wages come down, effort becomes more attractive, so that the agent increases his effort level, before fading out as pessimism sets in. The principal's expectation does not bear out, then: the actual marginal product is single-peaked (in fact, it would decrease at the beginning if effort was sufficiently flat).

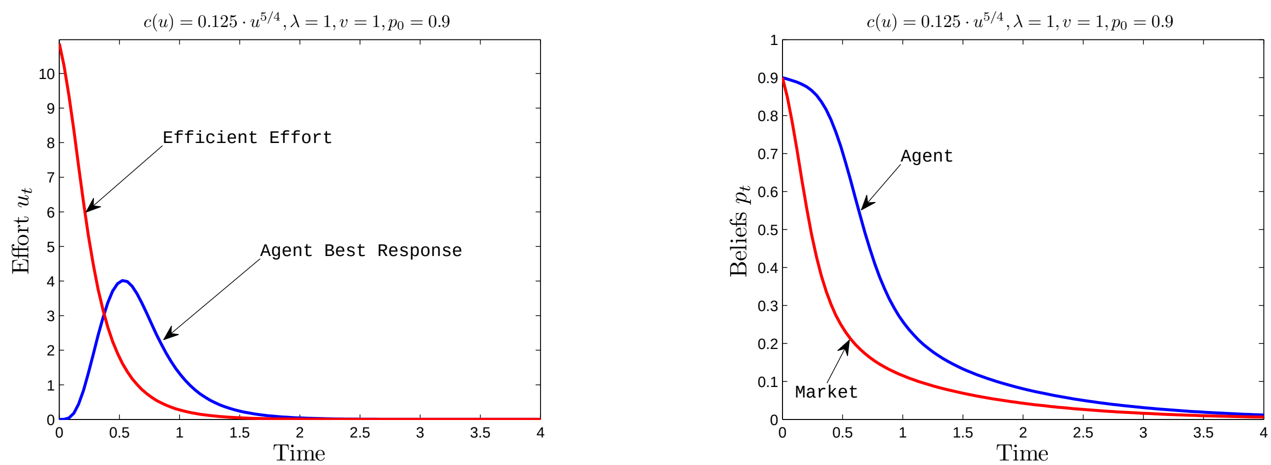

Figure 2: Agent's best-reply and beliefs to the efficient wage scheme

Note that eventually the agent exerts more (instantaneous) effort than would be socially optimal at that time. (See right panel). This is due to the fact that the agent is quite sanguine about the project at that time, having worked less than the social planner recommends. As it turns out, effort is always too low given the actual belief of the agent, but not necessarily given calendar time.

As this example makes clear, effort, let alone wages, should not be expected to be monotone in general. It turns out, however, that equilibrium cannot be more complicated than this example suggests.

Theorem 4.2 *An equilibrium exists. It is unique in the linear case if $\alpha < k$, and in the convex case if*

$$c''(0) \geq \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right)\frac{p^0}{1-p^0}.$$

In every equilibrium, effort is single-peaked, and the wage is non-decreasing in at most one interval. In the convex case, it is single-peaked.

Wages are not single-peaked in general for the linear case, and single-peakedness in the convex case relies on our assumption that the marginal cost is convex (i.e. $c''' > 0$). Figure 3 illustrates that this is not quite true otherwise (note that the cost is convex, but not the marginal cost). The mode of the wage lies to the left of the mode of effort: if the wage is increasing over time, it must be that effort is increasing, but not conversely.

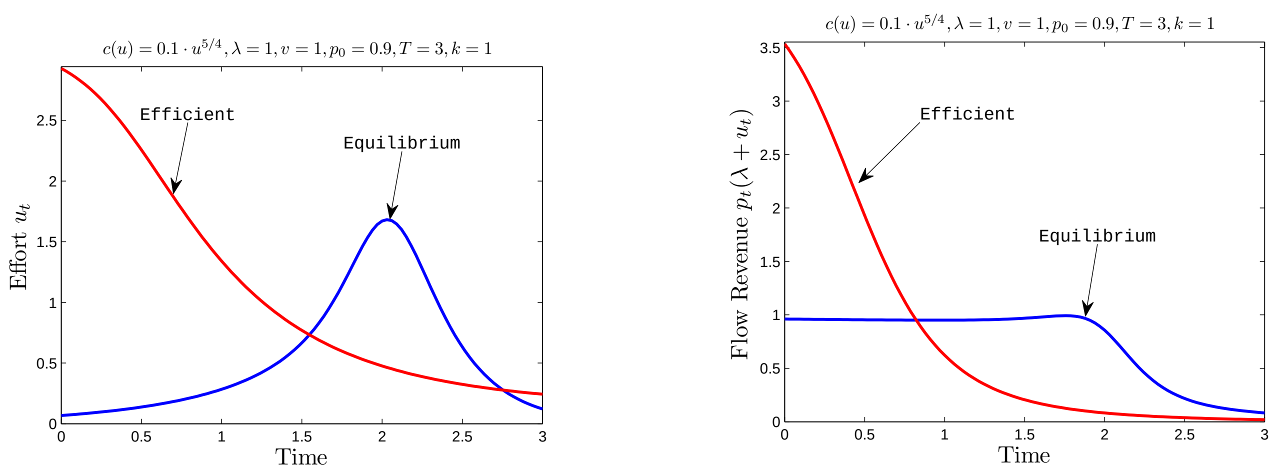

Figure 3: Effort and wages with convex costs

As stated in the theorem, there are simple sufficient conditions that guarantee equilibrium uniqueness, which boil down to assuming that the penalty k is large enough. It does not imply that there are multiple equilibria otherwise: we have been unable to construct any example of multiple equilibria.

A more precise description can be given in the case of linear cost.

Lemma 4.3 *With linear cost, any equilibrium path consists of at most four phases, for some* $0 \leq t_1 \leq t_2 \leq t_3 \leq T$*:*

1. *during* $[0, t_1]$*, no effort is exerted;*
2. *during* $(t_1, t_2]$*, effort is interior, i.e.* $u_t \in (0, \bar{u})$*;*
3. *during* $(t_2, t_3]$*, effort is maximal;*
4. *during* $(t_3, T]$*, no effort is exerted.*

Any of these intervals might be empty.[10]

Lemma 4.3 describes the overall structure of the equilibrium. As stated, any of the intervals might be empty, and it is easy to compute instances of each of the different possibilities.[11] Nevertheless, there is a certain ordering to this structure, depending on the deadline. If the deadline is very short, effort is first zero, then maximum. For deadlines of intermediate lengths, an intermediate phase kicks in, in which effort is interior. Finally, for long deadlines, a final phase appears, in which no effort is exerted. When a phase with interior effort exists, effort grows, so as to keep the wage constant at $v - \alpha\lambda$, which guarantees that the agent is indifferent between all effort levels. It is continuous at t_1 (i.e., $\lim_{t \downarrow t_1} u(t) = 0$), but jumps up at time t_2 (assuming the third interval is non-empty). See Figure 4 for an example of effort (left panel) and corresponding wage dynamics (right panel). (The parameters are the same as those used in Figure 1 above.)

Note that we have not specified the equilibrium strategy of the worker, because we have not derived his behavior following his own (unobservable) deviations. Yet it is not difficult to describe the worker's optimal behavior off-path, as it is the solution of the optimization problem studied before, for the belief that results from the agent's history, given the wage path.

The linear-cost case provides a simple way to understand what drives incentives. Given the deadline, on-path equilibrium effort is a function of the (equilibrium) belief and the time $t \leq T$.

[10]Here and elsewhere, the choices at the extremities of the intervals are irrelevant, and our specification is arbitrary in this respect.

[11]In particular, if $k < \alpha$, the agent never works at the deadline; if $1 + \alpha < v/\lambda$, and no effort is exerted at some point, it is then exerted until the end; if, contrary to our maintained assumption, $v/\lambda < \alpha$, the characterization simplifies to at most two intervals, with zero effort being followed by maximum effort.

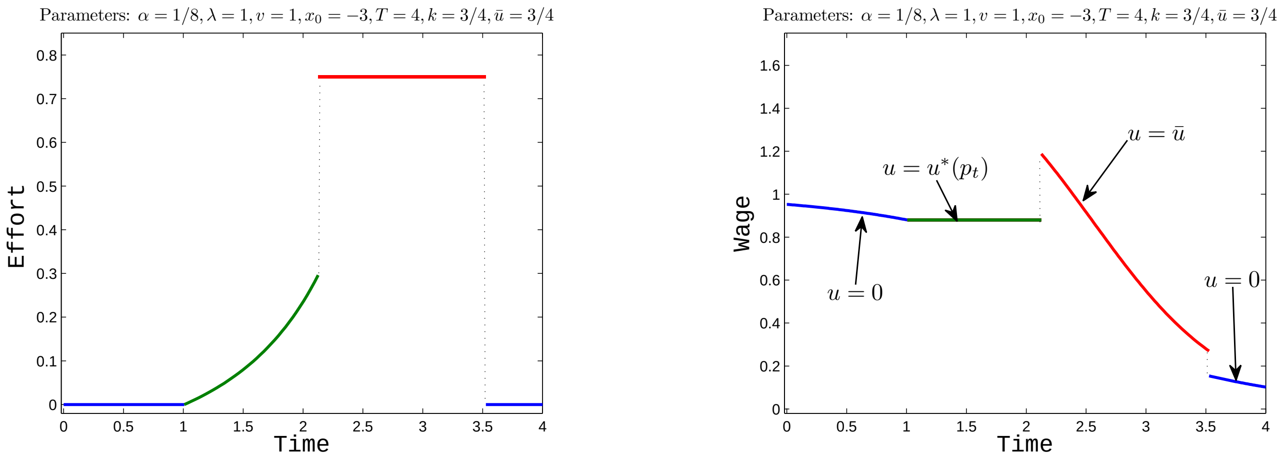

Figure 4: Effort and wages in the non-observable case

We can then define the boundaries, or frontiers, $p_k : [0, T] \to [0, 1]$ that divide the state space into regions according to equilibrium effort: p_3 is the boundary below which all effort stops; $p_2 > p_3$ the boundary at which maximum effort starts (that is, maximum effort is exerted between those two curves); and $p_1 > p_2$ the boundary below which mixing starts. By Lemma 4.3, the boundaries are each crossed at most once on the equilibrium path. It turns out that p_1 is independent of t: if mixing is on the equilibrium path, it begins at a belief that is independent of the specific path. As for p_2 and p_3, their structure hinges on the specific parameters. As the following figures illustrate, there are two distinct circumstances in which high effort is exerted: either effort is exerted because the belief is "right," given the remaining time, or because there is very little time left. See Figure 6 and compare with Figure 5. These figures use as parameters $\bar{u} = 1/2$, $\alpha = 1/5$, $v = \lambda = 1$, $x_0 = -4$, $T = 5$ and, depending on the figure, $k \in \{.3, .4, .6\}$.

Lemma 4.4 *For all* $t \leq T$,

1. *The no effort frontier* $p_3(t)$ *is decreasing in* k *and* v*. It is increasing in* α *and* λ*.*

2. *The full effort frontier* $p_2(t)$ *is decreasing in* α*,* λ *and* $\bar{u}$*. It is increasing in* k *and* v*.*

This result holds regardless of whether the full effort region is connected. It confirms the intuition that (in terms of beliefs) the agent works longer when the prize and the penalty are higher,

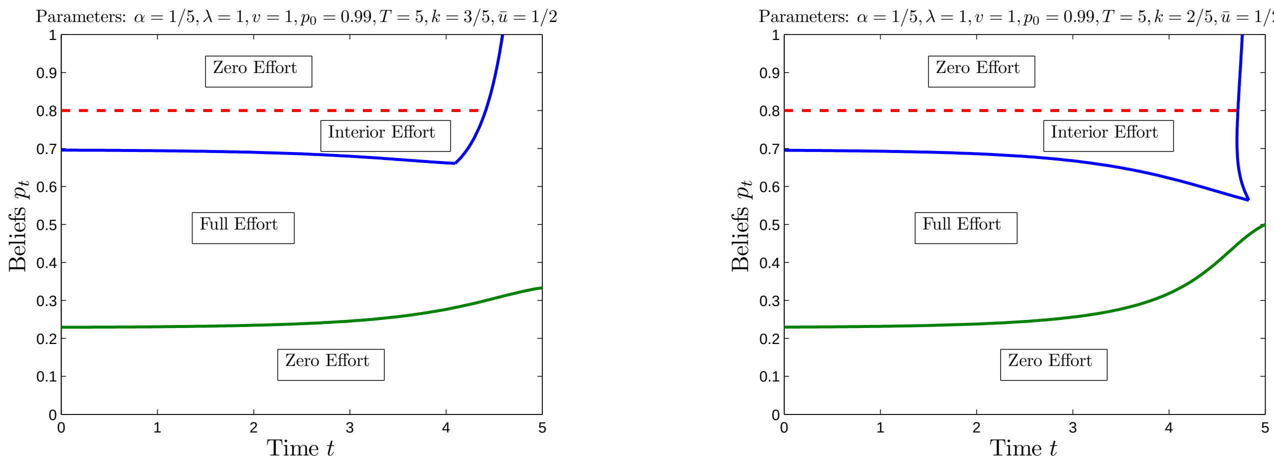

Figure 5: High k ($k = .6$) and medium k ($k = .4$)

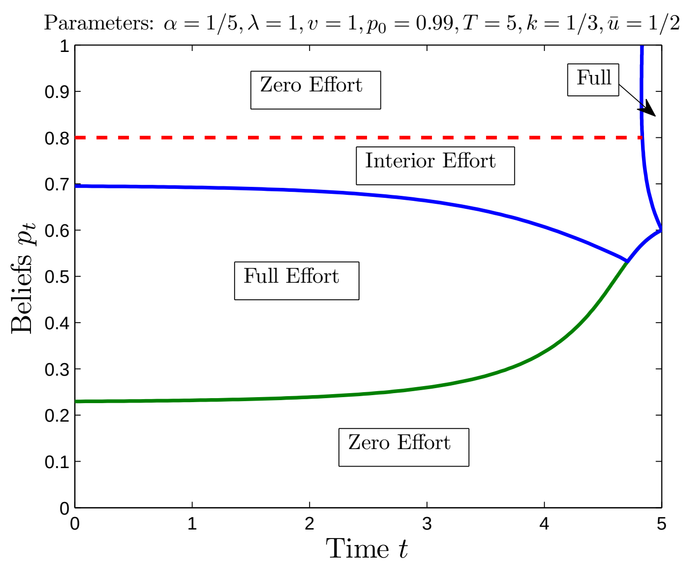

Figure 6: Low k ($k = .3$)

and works less when the marginal cost of effort and the luck component are more significant.

One might wonder whether the penalty k is really hurting the worker. After all, it endows him with some commitment to work. In the linear-cost case, simple algebra shows that increasing k increases the amount of work performed; furthermore, if parameters are such that working at some point is optimal, then the optimal (i.e. payoff-maximizing) termination penalty is strictly positive.

4.1 Discussion

The key driver behind the structure of equilibrium, as described in Theorem 4.2 is the strategic substitutability between effort at different dates. If more effort is expected "tomorrow," wages tomorrow will be higher in equilibrium, which depresses incentives, and hence effort, "today." There is substitutability between effort at different dates for the social planner as well, as higher planned effort tomorrow makes effort today less necessary, but wages provide an additional channel.

This substitutability appears to be new to the literature on career concerns. As we have mentioned, in the model of Holmström, the optimal choice of effort today and tomorrow are entirely independent, and because the variance of posterior beliefs is deterministic with Gaussian signals, the optimal choice of effort is deterministic as well. Dewatripont, Jewitt and Tirole emphasize the complementarity between expected effort and incentives for effort (at the same date): if the agent is expected to work hard, failure to achieve a high signal will be particularly detrimental to tomorrow's reputation, which provides a boost to incentives today. Substitutability between effort today and tomorrow does not appear in their model, because it is primarily focused on two periods, and at least three are required for this effect to appear. With two periods only, there are no incentives to exert effort in the second (and final) period anyhow.

Conversely, complementarity between expected and actual effort at a given time is not discernible in our model, in which time is continuous. But this complementarity appears in discrete time versions of our model, and three-period examples can be constructed that illustrate this point.

As a result of this novel effect, dynamics display new features. In Holmström's model, the wage is a supermartingale; in Dewatripont, Jewitt and Tirole, it is necessarily monotone. Here instead, effort can be first increasing, then decreasing, and wages can be decreasing first, increasing then, and decreasing again. These dynamics are not driven by the deadline.[12]. They are not driven either by the fact that, with two types, the variance of the public belief need not be monotone.[13] The same pattern emerges in examples with an infinite horizon, and a prior $p^0 < 1/2$ that guarantees that this variance only decreases over time, see Figure 7. As eqn. (6) makes clear, the provision of effort is tied to the capital gain that the agent obtains if he breaks through. Viewed as an integral, this capital gain is too low early on, it decreases over time, and then declines again, for a completely different reason. Indeed, this wedge depends on two components: the wage gap, and the impact of effort on the (expected) arrival rate of a success. Therefore, high initial wages would depress the first component, and hence kill incentives to exert effort early on. The latter component declines over time, so that eventually effort fades out again.

Similarly, one might wonder whether the possibility of non-increasing wages in this model is driven by the fact that the effort and wage paths under consideration are truly conditional paths, inasmuch as they assume that the agent has not succeeded so far. Yet it is not hard to provide numerical examples which illustrate that the same phenomenon arises for the unconditional flow payoff (v in case of a past success), though the increasing cumulative probability that a success has occurred by a given time, leading to higher payoffs (at least if $w_t < v$) dampens the downward tendency.

Yet such general dynamics are rarely observed: while it is difficult to ascertain effort patterns, wages do typically go up over time. See Abowd, Kramarz and Margolis (1999), Murphy (1986) and Topel (1991) among others, and Hart and Holmström (1987) and Lazear (1998) for surveys. Lazear (1981) obtains a positive impact of wages on seniority by (among others) assuming that the worker's outside option is increasing over time, and also derives the optimal deadline, or retirement age (Lazear, 1979).

Our model provides a benchmark to examine what labor market arrangements are likely to explain this. In the next section, we shall consider three such possibilities: what if the principal

[12]This is unlike for the social planner, for which we have seen that effort is non-increasing with an infinite horizon, while it is monotone (and possibly increasing) with a finite horizon.

[13]Recall that, in Holmström's model, this variance decreases (deterministically) over time, which plays an important role in his results.

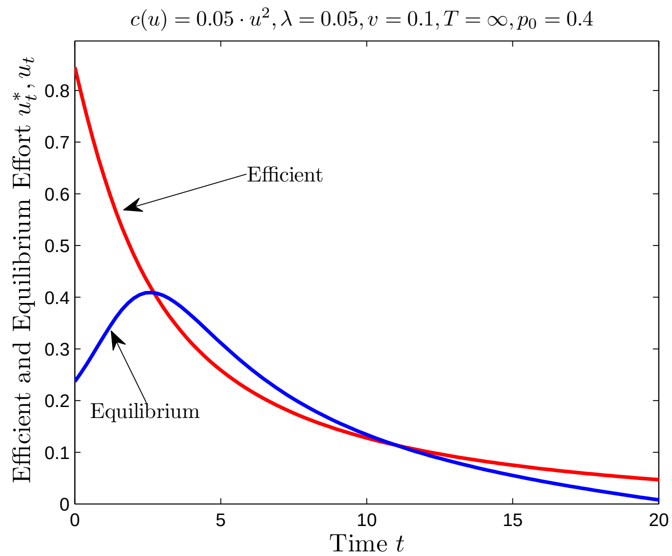

Figure 7: The same pattern in the case of $T = \infty$, $p^0 < 1/2$

has more commitment power than is typically assumed in career concerns model? How about if he has even less, so that there is no commitment to a specific deadline? Finally, how about if the monitoring is better than has been assumed?

Before considering such alternative arrangements, we conclude this section by arguing that our findings are robust to some of our specific modelling assumptions.

4.2 Robustness

Undoubtedly, our model has very stylized features: in particular, all uncertainty is resolved after only one breakthrough, and the quality of the project cannot change over time. We argue here that neither of these features is critical to our main findings. Throughout, attention is restricted to the case of linear cost.

4.2.1 Multiple breakthroughs

Suppose that one breakthrough does not resolve all uncertainty. More specifically, assume that there are three states of the world, $\omega = 0, 1, 2$, and two consecutive projects. The first one can be completed iff the state is not 0 ; assume (instantaneous) arrival rates of $\lambda_1 + u_t$ and $\lambda_2 + u_t$, respectively, conditional on $\omega = 1$ or $\omega = 2$; if the first project is completed, the agent tackles the second one, which in turn can only be completed if $\omega = 2$; assume again an arrival rate of $\lambda_2 + u_t$ if $\omega = 2$. Suppose that the horizon is infinite for both projects.

Such an extension can be solved by "backward induction." Once the first project is completed, the continuation reduces to the equilibrium of Section 4. The value function of this problem then serves as continuation payoff for the first stage. While this value function cannot be solved in closed-form, it is easy to derive the solution numerically. The following example illustrates the structure of the solution. The parameters are $v = 1, \alpha = 1/2, \mathbb{P}[\omega > 0] = 0.85, \mathbb{P}[\omega = 2 \mid \omega > 0] = 0.75, \lambda_2 = 1, \lambda_1 = 0.6, c(u) = u^2/8$.

See Figure 8. The left panel shows effort and wages during the first stage. As is clear, the same pattern as in our model emerges: effort is single-peaked, and as a result, wages can be first decreasing, then single-peaked.

The right panel shows how efforts and beliefs evolve before and after the first success. The green curves represent the equilibrium belief that $\omega = 2$, before and after the success (the light green curve is the belief as long as no success has occurred, and the dark green one the belief right after a success has occurred); the blue curves are equilibrium effort (the light blue curve is effort as long as no success has occurred, the dark blue one is the effort right after a success). Note that effort at the start of the second project is also single-peaked as a function of the time at which this project is started (the later it is started, the more pessimistic the agent at that stage, though his belief has obviously jumped up given the success).

4.2.2 Changing state

Suppose finally that, unbeknownst to the agent and the principal, the state of the world is reset at random times, exponentially distributed at rate $\rho > 0$; whenever it is reset, the state is reset to 1 with probability $p^* \in (0, 1)$.[14] In our environment, this is "equivalent" to the stationary version developed by Holmström in the Gaussian case, though we do not restrict attention to steady states. Specifically, suppose that with instantaneous probability $\rho > 0$ the ability is reset,

[14] This specification bears a close similarity to Board and Meyer-ter-Vehn (2011), though it also differs in some key respects.

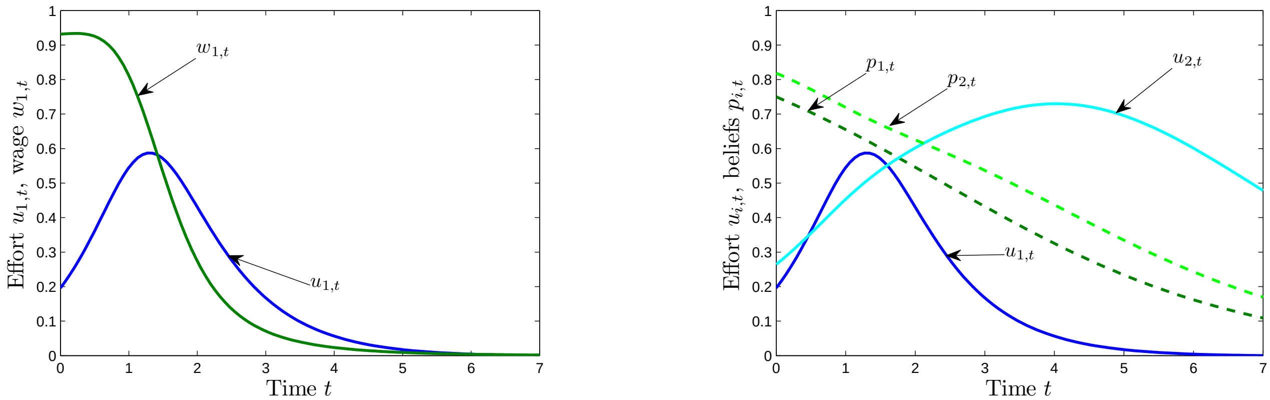

Figure 8: Efforts and beliefs with two breakthroughs

in which case it is high with probability p^*. Such an event remains unobserved by all parties. As before, a breakthrough ends the game, and the environment remains the same as before, with linear cost (and $v > \alpha\lambda$, as before) and an infinite horizon. (Thus, the baseline model with linear cost and $T = \infty$ is a special case in which $\rho = 0$.)

By Bayes' rule, the (agent's) belief p obeys

$$\dot{p}_t = \rho\,(p^* - p_t) - p_t\,(1 - p_t)\,(\lambda + u_t)\,, \quad p_0 = p^0,$$

and this is the same as the principal's belief in equilibrium. The equilibrium is unique, and effort and belief converges to some limiting value $p(u)$, which is independent of the prior, and decreasing in the eventual effort level u, as follows. (The proof of the following is available upon request.)

Lemma 4.5 *There exists $\alpha\lambda < \underline{v} < \bar{v}$ and $0 < \underline{p} < \bar{p} < p^*$ such that, if*

1. *$v > \bar{v}$, effort is eventually maximum, and p tends to a limit below $\underline{p}$;*
2. *$v \in (\underline{v}, \bar{v})$, effort is eventually interior, with p tending to a limit in $(\underline{p}, \bar{p})$;*
3. *$v < \underline{v}$, effort is eventually zero, and p tends to a limit above $\bar{p}$;*

The higher the value, the more effort is exerted, the lower is the asymptotic belief. This eventual belief is non-decreasing in p^* and ρ, not surprisingly, and non-increasing in $\bar{u}$. Finally, it is decreasing in λ when effort is extremal, but increasing otherwise.

Note that, if the prior belief is below the limiting value, effort, and hence the wage, is increasing over time (and it is easy to construct examples in which wage is increasing throughout). It would be interesting to consider the case in which the game does not end with a success, but rather continues with a value reset at the prior of 1 (which immediately starts declining towards p^*), but we have not pursued this here.

5 Alternative labor market arrangements

Throughout the new two subsections, attention is restricted to the case of linear cost.

5.1 Commitment by the principal

The assumption that the flow wage must equal the worker's marginal product is sometimes motivated by the presence of competition for the agent, rather than lack of commitment by the principal. Our model does not substantiate such a motivation: if the principal can commit, the outcome looks substantially different, even if there is competition for the agent.

Suppose that the agent cannot be forced to stay with the principal (so that he can leave at any time), but the principal can commit to a wage path that is conditioned to the absence of a breakthrough. Other principals, who are symmetrically informed (that is, they observe the wages paid by the principals who have employed the agent in the past), compete by making similar offers of wage paths (at all times). The same deadline T applies to all of them (the deadline could be the worker's retirement age, for instance, so that switching principals does not extend the work horizon).

Clearly, stronger forms of commitment can be thought of. If the principal could commit to an arbitrary, breakthrough-contingent wage scheme, the moral hazard problem would be solved entirely: under competition, the principal would could do no better than offer the value of a breakthrough, 1, to the agent, in case of a success, and nothing otherwise.

If the principal could at least commit to a wage scheme that were contingent on time (not on the occurrence of a breakthrough), but possibly involved payments after a breakthrough (as long as they are not condition on it), the moral hazard would also be mitigated, if not solved. Whatever is promised at time t in case of no breakthrough should also be promised in case of a breakthrough, so as to eliminate all disincentives wages exert on effort.

Here, wages can only be paid in the continued absence of a breakthrough. Think of the agent moving on once a breakthrough occurs, with the principal being unable to retain him in this event.

Because of competition, we can write the principal's problem as of maximizing the agent's welfare subject to constraints. Formally, we solve the following optimization problem $\mathcal{P}$. The principal chooses $u : [0,T] \to [0,\bar{u}]$ and $w : [0,T] \to \mathbb{R}_+$, integrable, to maximize $W(0,p^0)$, where, for any $t \in [0,T]$,

$$W(t,p_t) := \max_{w,u} \int_t^T \frac{1-p_t}{1-p_s}(w_s - v - \alpha u_s)\,\mathrm{d}s - k\frac{1-p_t}{1-p_T},$$

such that, given w, the agent's effort (strategy) is optimal:

$$u = \arg\max_u \int_t^T \frac{1-p_t}{1-p_s}(w_s - v - \alpha u_s)\,\mathrm{d}s - k\frac{1-p_t}{1-p_T},$$

and the principal offers as much to the agent at later times than the competition could offer at best, given the equilibrium belief:

$$\forall \tau \geq t : \int_\tau^T \frac{1-p_\tau}{1-p_s}(w_s - v - \alpha u_s)\,\mathrm{d}s - k\frac{1-p_\tau}{1-p_T} \geq W(\tau,p_\tau)\,;^{15} \tag{10}$$

finally, profits at later times must be non-positive:

$$0 \geq \int_t^T \frac{1-p_t}{1-p_s}(p_s(\lambda+u_s) - w_s)\mathrm{d}s.$$

Note that competing principals are subject to the same constraints as the principal under consideration: because the agent might ultimately leave them as well, they can offer no better than $W(\tau,p_\tau)$ at time τ, given belief p_τ. This leads to an "infinite regress" of constraints, with the value function appearing in the constraints themselves. To be clear, $W(\tau,p_\tau)$ is not, in general,

[15] Note that we are not considering the actual "game," in which the agent could deviate in his effort scheme, leave the firm, and competing firms would have to form beliefs about the agent's past effort choices.

the continuation payoff that results from the solution to the optimization problem, but the value of the optimization problem if it were to start at time τ. Because of the constraints, the solution is not time-consistent, and dynamic programming is of little help. Fortunately, this problem can be solved, as shown in appendix –at least as long as $\bar{u}$ is large enough. Formally, we assume that

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1\right) v - \lambda.$$

Before describing its solution, let us provide some intuition.

Recall the first-order condition (6) that determines the agent's effort. Clearly, the lower the future total wage bill, the stronger the agent's incentives to exert effort, which is inefficiently low in general. Therefore, considering two times $t < t'$, to provide strong incentives at time t', it is best to frontload any promised payment to the agent to times before t', as such payments will no longer matter at that time. Ideally, the principal would pay what he owes the agent upfront, as a "signing bonus." This, however, is not possible given the constraint (10), as an agent left with no future payments would leave the principal right after cashing in the signing bonus.

But from the perspective of incentives at time t, backloading promised payments is better. To see this, note that the coefficient of the wage w_s, $s > t$, in (6) is (up to the factor $(1 - p_t)$) the likelihood ratio $p_s / (1 - p_s)$, as explained before (6). Alternatively, note that

$$(1 - p_t) \frac{p_s}{1 - p_s} = \mathbb{P}[\omega = 1 | \tau \geq s] \, \mathbb{P}[\omega = 1] = \mathbb{P}[\omega = 1 \cap \tau \geq s];$$

that is, effort at time t is affected by wage at time $s > t$ inasmuch as time s is reached and the state is 1: otherwise effort plays no role anyhow.

In terms of the principal's profit –or the agent's payoff–, the coefficient placed on the wage at time s (see 3) is

$$\mathbb{P}[\tau \geq s],$$

i.e., whether this wage is paid (or collected). Because players grow more pessimistic over time, the former coefficient decreases faster than the latter one: backloading payments is good for incentives at time t. Of course, to provide incentives with later payments, those payments must be increased, as a breakthrough might occur until then, which would void them; but it also decreases the probability that these payments must be paid in the same proportion. So what matters is not the probability that time s is reached as much as the fact that later payments depress incentives less, as reaching those later times is indicative of state 0, which is less relevant (indeed, irrelevant with linear cost, see the discussion after (8)) for incentives.

To sum up: from the perspective of time t incentives, backloading payments is useful; from the point of view of time $t' > t$, it is detrimental, but frontloading is constrained by (10).

Note finally that, given the focus on linear cost, there is no benefit in giving the agent any "slack" in his incentive constraint at time t; otherwise, by frontloading slightly future payments, incentives at time t would not be affected, while incentives at later times would be enhanced.

The following result, then, should come as no surprise.

Theorem 5.1 *The following is a solution to the optimization problem $\mathcal{P}$, for some $t \in [0, T]$. Maximum effort is exerted up to time t, and zero effort is exerted afterwards. The wage is equal to $v - \alpha\lambda$ up to time t, so that the agent is indifferent between all levels of effort up to then, and it is 0 for all times $s \in (t, T)$; a lump-sum is paid at time T.*[16]

It is possible that high effort is exerted throughout. In fact, this is what happens if T is short enough. If, and only if, the deadline is long enough is there a phase in which no effort is exerted.

The lump-sum at time T can be interpreted as severance pay. As time proceeds, the agent produces revenue that exceeds the flow wage collected: the liability recorded by the principal grows over time, shielding it from the threat of competition, as this liability will eventually be settled via this severance pay.

5.2 Observable effort

5.2.1 Set-up

To what extent are dynamics driven by the assumption that effort is non-observable? Consider the case in which effort is observable, while still uncontractible. That is, the principal cannot commit, and as a result must pay upfront the expected value of the agent's effort, but the actual effort is observed as soon as it is exerted. Therefore, the belief of the principal coincides with the agent's at all times, on and off the equilibrium path, and the payment flow is given by

$$w_t = p_t(\lambda + \hat{u}_t),$$

[16]The wage path that solves the problem is not unique in general.

where p_t is the common belief, and $\hat{u}_t$ is the effort level that the market expects the agent to exert in the next instant. The agent then maximizes

$$\int_0^T \frac{1-p_0}{1-p_t} \left[p_t(\lambda + \hat{u}_t) - \alpha u_t - v\right] \mathrm{d}t - k\frac{1-p_0}{1-p_T}.$$

In contrast to (3), the revenue is no longer a function of time only, as chosen effort affects future beliefs, hence future wages.

At the very least, then, we must describe wages, and behavior, as a function of time t and current belief p. In fact, we shall restrict attention to equilibria in Markov strategies

$$u : [0,1] \times [0,T] \to [0,\bar{u}]\,,$$

such that u is upper semicontinuous in (p,t), and such that the value function

$$V\left(p,t\right) = \sup_u \left\{ \int_t^T \frac{1-p_t}{1-p_s} \left[p_s\left(\lambda + u\left(p_s,s\right)\right) - \alpha u(p_s,s) - v\right] \mathrm{d}s - k\frac{1-p_t}{1-p_T} \right\},$$

with $p_t = p$, is piecewise differentiable.[17] We shall prove that such equilibria (*Markov* equilibria) exist.

5.2.2 Equilibrium structure

We first argue that if the agent ever exerts low effort, he has always done so before.

Lemma 5.2 *Fix a Markov equilibrium. If* $u = 0$ *on some open set* $\Omega \subset [0,1] \times [0,T]$*, then also* $u\left(p',t'\right) = 0$ *if the equilibrium trajectory that starts at* $\left(p',t'\right)$ *intersects* Ω*.*

This lemma implies that every equilibrium has a relatively simple structure: if the agent is ever willing to exert high effort, he keeps being willing to do so at any later time, at least on the equilibrium path. In any equilibrium involving extremal effort levels only, there are at most two phases: first, the worker exerts no effort, and then full effort. This is precisely the opposite of the

[17]That is, there exists a finite partition of $[0,1] \times [0,T]$ into closed subsets S_i with non-empty interior, such that V is differentiable on the interior of S_i, and the intersection of any pair S_i, S_j is either empty or a smooth 1-dimensional manifold.

optimal policy for the social planner (under our assumption $v > \alpha\lambda$), in which high effort comes first (see lemma 2.1). The agent can only be trusted by the market to put in high effort if he is "back to the wall," so that maximum effort will remain optimal at any later time, no matter what he does until then; if the market paid the worker for high effort, yet he was supposed to let up his effort later on, then the worker would gain by deviating to low effort, pocketing the high wage in the process; because the observable deviation to no effort would make everyone more optimistic, it would only increase his incentives to exert high effort later and thus increase his wage at later times.

This, of course, relies heavily on the Markovian assumption. As the next theorem states, there are multiple Markov equilibria.

Theorem 5.3 *Given T, there exists continuous, non-increasing $\underline{p}, \bar{p} : [0, T] \to [0, 1]$, $\underline{p}_t \leq \bar{p}_t$, $\underline{p}_T = \bar{p}_T$ such that:*

i All Markov equilibria involve maximum effort above $\bar{p}$:

$$p_t > \bar{p}_t \Rightarrow u(p, t) = \bar{u};$$

ii All Markov equilibria involve no effort below $\underline{p}$:

$$p_t \leq \underline{p}_t \Rightarrow u(p, t) = 0;$$

iii These bounds are tight: there exists a Markov equilibrium $\underline{\sigma}$ (resp. $\bar{\sigma}$) in which effort is either 0 or $\bar{u}$ if and only if p is below or above $\underline{p}$ (resp. $\bar{p}$).

The proof of Lemma 5.3, in appendix, provides an explicit description of these boundaries. Given Lemma 5.2, these boundaries are crossed at most once, from below, along any equilibrium trajectory. Note that these functions might take one as value, in which case effort is never exerted at that time: indeed, there exists t^* (independent of T) such that effort is zero at all times $t < T - t^*$ (if $T > t^*$). The threshold $\underline{p}$ is decreasing in the cost of effort α, and increasing in the outside option v and penalty k. Considering the equilibrium with maximum effort, the agent works more, the more desirable success is.

These results are illustrated in Figure 9 for the same parameters as in Figure 4 in the unobservable case.

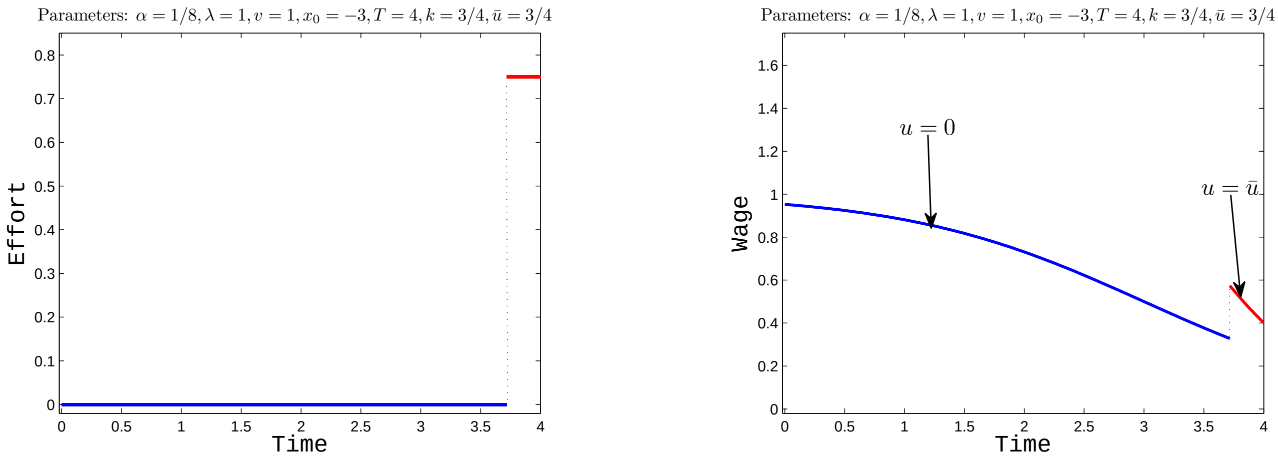

Figure 9: Effort and wages in the observable case

It is worth noting that, while $\underline{\sigma}$ and $\bar{\sigma}$ provide upper and lower bounds on the equilibrium effort exerted in an equilibrium (in the sense of (i)–(ii)), these equilibria are not the only ones. There exist other Markov equilibria involving only extremal effort levels, whose switching boundary lies between $\underline{p}$ and $\bar{p}$, as well as equilibria in which interior effort levels are exerted at some states. In particular, the proof builds an equilibrium in which the agent exerts an amount of effort in $(0, \bar{u})$ at all times t for all values of p in $\left[\underline{p}_t, \bar{p}_t\right]$. This effort is equal to $\bar{u}$ once the curve $\underline{p}$ is reached, decreases continuously along the equilibrium trajectory from that point on, until the upper boundary is reached (which, unless a breakthrough occurs, necessarily happens before time T, as $\underline{p}_T = \bar{p}_T$), at which point the effort level jumps up to $\bar{u}$.[18]

In the extremal equilibria, wages are decreasing over time, except for an upward jump at the point at which effort jumps up to $\bar{u}$. In the interior-effort equilibrium described in the proof (in which effort is interior everywhere between $\underline{p}$ and $\bar{p}$), wages decreases continuously over time at all times.

Equilibrium multiplicity can be understood as follows. Because the principal only expects high effort if the belief is high and the deadline close, such states (belief and times) are desirable

[18]It is not possible to strengthen (4) further to the statement that, once maximum effort is exerted, it is exerted throughout: there is considerable leeway in specifying equilibrium strategies between $\bar{p}$ and $\underline{p}$, and nothing precludes maximum effort to be followed by interior effort. (Of course, if $\bar{p}$ is crossed, effort remains maximal.)

for the agent, as the higher wage more than outweighs the effort cost. Yet low effort is the best way to reach those states, as effort depresses beliefs: hence, if the principal expects the agent to shirk until a high boundary is reached (in (p,t)-space), the agent has strong incentives to shirk to reach this boundary; if the principal expected the agent to shirk until an even higher boundary, this would only reinforce this incentive –up to some point.

This intuition foreshadows already what is stated in the next subsection: observability further depresses incentives and reduces effort, relative to non-observability. But as explained, it is also more in consonance with increasing wages: as effort is non-decreasing over time, the only force that pushes down wages is growing pessimism, not declining work.

For the purpose of comparative statics, we focus on the equilibrium that involves the largest region of effort.

Lemma 5.4 *The boundary of the maximal effort equilibrium $\underline{p}(t)$ is non-increasing in k and v and non-decreasing in α and λ.*

The effect of the maximum effort level $\bar{u}$ is ambiguous. Finally, one might wonder whether increasing the termination penalty k can increase welfare, for some parameters, as it might help resolve the commitment problem. Unlike in the non-observable case, this turns out never to occur, at least in the maximum-effort equilibrium: increasing the penalty decreases welfare, though it unambiguously increases total effort. The proof is in Appendix **??**. Similarly, increasing v, the value of succeeding, increases effort (in the maximum-effort equilibrium), though it decreases the worker's payoff.

5.2.3 Comparison with the non-observable case

Along the equilibrium path, the dynamics of effort look very different when one compares the social planner, the agent when effort is unobservable, and the agent when effort is observable. Yet it turns out that effort can easily be ranked across those cases. To do so, the key is to describe effort in terms of the state (p,t), i.e., the public belief and calendar time.

For the observable case, it is enough to focus on the region (i.e., subset of the (p,t)-space) defined by the frontier $\underline{p}$, as this characterizes the maximum effort equilibrium, and it will turn out that even in this equilibrium, the agent works less than under non-observability.

Lemma 5.5 *The maximal effort region for the observable case is contained in the full effort region(s) for the non-observable case.*

Lemma 5.5 confirms the intuition that observability of effort reduces the incentives to work. In particular, the highest effort equilibrium in the observable case involves unambiguously lower effort levels than the (unique) equilibrium in the unobservable case. Recall also from Lemma 3.2.(1) that the (interior or full) effort region in the non-observable case is in turn contained in the full effort region for the social planner.

How about non-Markov equilibria? Defining such equilibria formally in our continuous-time environment is problematic, but it is clear that, threatening the agent with reversal to the Markov equilibrium $\bar{\sigma}$ provides incentives for high effort that extend beyond the high-effort region defined by $\underline{\sigma}$ –in fact, beyond the high-effort region in the unobservable case. The social planner's solution, however, remains out of reach, since the punishment is restricted to beliefs below $\underline{p}$.

5.3 Endogenous Deadlines

The last two subsections have shown how more commitment power or better monitoring drastically affect the effort and wage pattern. We argue here that endogenizing the deadline does not.

By an endogenous deadline, we mean that the worker decides when to quit optimally. We assume (for now) that he has no commitment power. The principal anticipates the quitting decision, and takes this into account while determining the agent's equilibrium effort, and therefore, the wage he should be paid.

More specifically, in each interval $[t, t+\mathrm{d}t)$ such that the agent has not quit yet, the principal pays a wage $w_t \mathrm{d}t$, the agent then decides how much effort to exert over this interval, and at the end of it, whether to stay or leave, which is an observable choice.

This raises the issue of the principal's beliefs if the agent were to deviate and stay beyond what the equilibrium specifies. For simplicity, we adopt passive beliefs. That is, if the agent is supposed to drop out at some time but fails to do so, the principal does not revise his belief regarding the past effort choices, ascribing the failure to quit to a mistake, and anticipates equilibrium play in the continuation (which means, as it turns out, that he anticipates the agent quitting at the next opportunity).[19]

We have argued above that endogenous deadlines do not affect the possible effort and wage patterns. In fact, we show in appendix that, with convex cost, effort is *always* decreasing at the equilibrium deadline. This implies, in particular, that the wage is decreasing at that stage. Furthermore, it is simple to construct examples in which effort is not decreasing throughout.

Hence, effort is single-peaked, and wages are first decreasing, and then single-peaked (both might be decreasing throughout).

Furthermore, we show in appendix that the deadline is always too long relative to the deadline chosen by the social planner. Of course, effort (and hence the worker's marginal product) are decreasing throughout in the first-best solution.

How about if the worker could commit to the deadline, (but not to effort levels)? We show that the optimal deadline with commitment can be either shorter or longer than without commitment. In either case, however, the deadline is set so as to increase aggregate effort, and so increase wages. But sometimes this means increasing the deadline –so as to increase the duration over which higher effort levels are sustained, even if that means quitting at a point where staying in is unprofitable– or decreasing the deadline –so as to make high effort levels credible. Figure 10 below illustrates.

Finally, having the worker quit when it is best for him to do so (without commitment to the deadline) reinforces our comparison between observable and non-observable effort. As we show, the deadline chosen is shorter, and the total effort exerted is lower, when effort is observed by the principal (in the linear cost case).

These results are summarized in the following lemma. Exact characterizations are provided in the proof (See Appendix D.3).

Lemma 5.6 *With convex cost,*

[19]In the linear cost case, this means that we fix the off-equilibrium beliefs to specify $\hat{u}_t = \bar{u}$ if $p_t > p^*$, where p^* is the lowest belief at which it would be optimal for the agent to exert maximum effort if he anticipated quitting at the end of the interval $[t, t + \mathrm{d}t)$ (see appendix for p^* in closed-form), and $\hat{u}_t = 0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $p = p_T$ were the terminal belief.

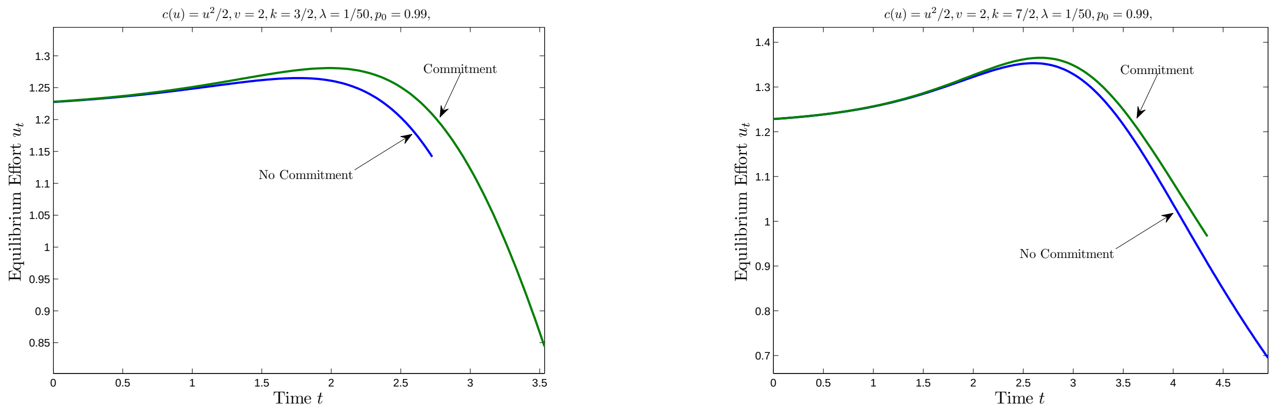

Figure 10: Setting the deadline with commitment can push it higher or lower than without.

1. *Effort is always decreasing at the optimal deadline without commitment;*

2. *This optimal deadline is longer than the optimal deadline for the social planner;*

3. *The deadline with commitment can be shorter or longer than without;*

Furthermore, with linear cost, total effort is lower, and the deadline chosen shorter, when effort is observable than when it is not.

One might also wonder how optimal deadlines affect the structure of the optimal contract with commitment but competition developed in Section 5.1. A complete analysis (for the linear cost case) is provided in appendix D.3.3. By an optimal deadline, we mean the deadline that society would like to impose to maximize social welfare, that would apply to all competing firms simultaneously. In the absence of such external enforcement, it is not hard to see that, if the deadline were part of the contract, firms might as well offer contracts with an infinite deadline. With external enforcement, however, the deadline can be finite (depending on parameters). For all parameters, it is such that the second phase –in which effort and wages are zero– is non-empty: the value to extend the deadline beyond the point at which the worker would start shirking is always optimal. This is unlike what the social planner would impose in terms of effort: it would be optimal to choose a deadline and an effort path that specifies full effort until the deadline.

We conclude this section by comparing the performance of a deadline with a finishing line. A deadline T is a time at which the game stops. A finishing line, instead, is a value of the belief, $\hat{x}$, at which the game stops, and the penalty k is incurred. Given some finishing line, what is the optimal strategy of the worker? As a consequence, what is the optimal finishing line, and is setting a finishing line preferable to a deadline? A finishing line makes more sense when effort is observable than not, and so we assume it is. Attention is restricted to Markov strategies, which, given the absence of deadline, reduce to measurable functions $u(\cdot)$ of the (public) belief only. As usual, equilibrium requires that the expected effort that determines the wage coincides with optimal effort.

Lemma 5.7

1. *Given the finishing line $\hat{x}$, the optimal strategy involves first full effort, then interior and decreasing effort, then zero effort;*[20]
2. *The optimal finishing line involves the same belief as the optimal deadline without commitment and unobservable effort.*

6 Concluding Remarks

Rather than summarize our findings, let us point out what we view as the most promising extensions of this agenda.

We have discussed when the worker would choose to quit, not when the firm would like to lay off the worker. To examine this issue, it is necessary to introduce some friction in the model: as the firm is paying a fair wage at all times in the current model, it has nothing to lose nor to gain by firing the worker. Yet this is an important question, in light of the rigid tenure policies adopted by many professional service firms. Why not keep the employee past the probationary period, adjusting the wage for the diminished incentives and lower assessed ability?[21] Firms have a cost of hiring (or firing) workers –possibly due to the delay in filling a vacancy– but derive a surplus from the worker in excess of the competitive wage they have to pay. Studying the

[20]Of course, depending on the finishing line, the project might stop before effort drops from maximum effort.
[21]See Gilson and Mnookin (1989) for a discussion of this puzzle for the case of law firms.

efficiency properties and the characteristics of the resulting labor market (composition of the working force, duration of unemployment) seems to be an interesting undertaking.

Despite the richness of the model and the absence of closed-form solutions, this model appears rather tractable, as the characterization, comparative statics and extensions illustrate. It is then natural to apply this framework to the analysis of partnerships. After all, in law or consulting firms, projects are often assigned to a team of employees that combine partners with junior associates. This raises several issues. The team must achieve several possibly conflicting objectives: incentivizing both the partner and the associate, and eliciting information about the associate's ability. How should profits be shared in the team to do so? When should the project be terminated, or the junior associate replaced? Is it indeed optimal to combine workers whose assessed ability differs, as opposed to workers about whom information is symmetric? Analyzing such questions raises a new modeling challenge, as results are likely to be sensitive to the technology that combines the agents' abilities and effort.

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Appendix

Throughout this appendix, we shall use the log-likelihood ratio

$$x_t := \ln \frac{1 - p_t}{p_t}$$

of state $\omega = 0$ vs. $\omega = 1$. We set $x^0 := \ln(1 - p^0)/p^0$. Note that x increases over time and, given u, follows the O.D.E.

$$\dot{x}_t = \lambda + u_t,$$

with $x_0 = x^0$. We shall also refer to x_t as the "belief," hoping that this will create no confusion.

We start by explaining how the objective function can be derived as the limit of a discounted version of our problem. Suppose that V is the value of a success and $V_L = V - k$ is the value of failure.Given discount rate r, the agent's payoff is given by

$$\left(1 + e^{-x_0}\right) V_0 = \int_0^T e^{-rt} \left(1 + e^{-x_t}\right) \left(\frac{\dot{x}_t}{1 + e^{x_t}} V + w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT} \left(1 + e^{-x_T}\right) V_L$$

where V_0 is his *ex ante* payoff.
Integrating by parts we obtain

$$\begin{aligned}
\left(1 + e^{-x_0}\right) V_0 &= \int_0^T e^{-rt} e^{-x_t} \dot{x}_t V \mathrm{d}t + \int_0^T e^{-rt} \left(1 + e^{-x_t}\right) \left(w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT} \left(1 + e^{-x_T}\right) (V - k) \\
&= -e^{-rt} e^{-x_t} V\Big|_0^T + \int_0^T e^{-rt} \left(\left(1 + e^{-x_t}\right) \left(w_t - c\left(u_t\right)\right) - e^{-x_t} rV\right) \mathrm{d}t + e^{-rT} \left(1 + e^{-x_T}\right) (V - k) \\
&= e^{-x_0} V - e^{-rT} e^{-x_T} V + \int_0^T e^{-rt} \left(1 + e^{-x_t}\right) \left(w_t - c\left(u_t\right) - \frac{rV}{1 + e^{x_t}}\right) \mathrm{d}t + e^{-rT} \left(1 + e^{-x_T}\right) (V - k),
\end{aligned}$$

so that as $r \to 0$ (and defining v as $rV \to v$) we obtain

$$\left(1 + e^{-x_0}\right) (V_0 - V) = \int_0^T \left(1 + e^{-x_t}\right) \left(w_t - c\left(u_t\right) - \frac{v}{1 + e^{x_t}}\right) \mathrm{d}t - k \left(1 + e^{-x_T}\right).$$

Similarly, one can show the social planner's payoff is given by

$$\left(1 + e^{-x_0}\right) (V_0 - V) - e^{-x_0} + k = -\int_0^T \left(1 + e^{-x_t}\right) \left(c\left(u_t\right) + \frac{v}{1 + e^{x_t}}\right) \mathrm{d}t - (1 + k)\, e^{-x_T}.$$

A Proofs for Section 2

Proof of Lemma 2.1. In both the linear and convex cases, existence and uniqueness of a solution follows as special case of Lemma 3.1, when $w = 0$ identically (the transversality condition must be adjusted). To see that the social planner's problem is equivalent to this, note that (whether the cost is convex or linear), the "revenue" term of the social planner's objective satisfies

$$\int_0^T (1+e^{-x_t})\frac{\lambda+u_t}{1+e^{x_t}}\mathrm{d}t = \int_0^T \dot{x}_t e^{-x_t}\mathrm{d}t = e^{-x^0} - e^{-x_T},$$

and so this revenue only affects the necessary conditions through the transversality condition at T.

Let us start with the linear case. The social planner maximizes

$$\int_0^T (1+e^{-x_t})\left(\frac{\lambda+u_t}{1+e^{x_t}} - \alpha u_t - v\right)\mathrm{d}t - ke^{-x_T}, \text{ s.t. } \dot{x}_t = \lambda + u_t.$$

We note that the maximization problem cannot be abnormal, since there is no restriction on the terminal value of the state variable. See Note 5, Ch. 2, Seierstad and Sydsæter (1987). The same holds for all later optimization problems.

It will be understood from now on that statements about derivatives only hold almost everywhere.

Let γ_t be the costate variable. The Hamiltonian for this problem is

$$H(x,u,\gamma,t) = e^{-x_t}(\lambda+u_t) - (1+e^{-x_t})(v+\alpha u_t) + \gamma_t(\lambda+u_t).$$

Define $\phi_t := \partial H/\partial u_t = (1-\alpha)e^{-x_t} - \alpha + \gamma_t$. Note that given x_t and γ_t, the value of ϕ_t does not depend on u_t. Pontryagin's principle applies, and yields

$$u_t = \bar{u} \ (u_t = 0) \Leftrightarrow \phi_t := \frac{\partial H}{\partial u_t} = (1-\alpha)e^{-x_t} - \alpha + \gamma_t > (<) 0,$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(\lambda - v + (1-\alpha)u_t), \gamma_T = ke^{-x_T}.$$

Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda - v), \phi_T = (1+k-\alpha)e^{-x_T} - \alpha.$$

Note that ϕ is either increasing or decreasing depending on the sign of $\alpha\lambda - v$. Therefore, the equilibrium is either maximum effort–no effort, or no effort–maximum effort, depending on the sign of this expression.

Consider now the convex case. Applying Pontryagin's theorem (and replacing the revenue term by its expression in terms of x_t and x^0, as explained above) yields as necessary conditions

$$\dot{\gamma}_t = -e^{-x}(c(u)+v), \gamma_t = (1+e^{-x_t})c'(u_t),$$

where γ_t is the co-state variable, as before. Differentiate the second expression with respect to time, and use the first one to obtain

$$\dot{u} \quad = \quad \frac{(\lambda + u)\, c'(u) - c(u) - v}{c''(u)\,(1 + e^{x})},$$

in addition to $\dot{x} = \lambda + u$ (time subscripts will often be dropped for brevity). Let

$$\phi(u) := (\lambda + u)\, c'(u) - c(u) - v.$$

Note that $\phi(0) = -v < 0$, and $\phi'(u) = (\lambda + u)\, c''(u) > 0$, and so ϕ is strictly increasing and convex. Let $u^* \geq 0$ be the unique solution to

$$\phi(u^*) = 0,$$

and so ϕ is negative on $[0, u^*]$ and positive on $[u^*, \infty)$. Accordingly, $u < u^* \implies \dot{u} < 0$, $u = u^* \implies \dot{u} = 0$ and $u > u^* \implies \dot{u} > 0$. Given the transversality condition

$$(1 + e^{x_T})\, c'(u_T) = 1 + k,$$

we can then define $x_T\left(x^0\right)$ by

$$x_T(x^0) = \frac{1}{\lambda + u^*}\left[\ln\left(\frac{1+k}{c'(u^*)} - 1\right) - x^0\right],$$

and so effort is decreasing throughout if $x_T > x_T(x^0)$, increasing throughout if $x_T < x_T(x^0)$, and equal to u^* throughout otherwise. The conclusion then follows from the proof of Lemma 2.3, which establishes that the belief at the deadline is increasing in the deadline. □

Proof of Lemma 2.3. We shall use the necessary conditions obtained in the previous proof. Part (1) is almost immediate. Note that in both the linear and convex case, the necessary conditions define a vector field $(\dot{u}, \dot{x})$, with trajectories that only define on the time left before the deadline and the current belief. Because trajectories do not cross (in the plane $(-\tau, x)$, where τ is time-to-go and x is the belief), and belief x can only increase with time, if we compare two trajectories starting at the same level x^0, the one that involves a longer deadline must necessarily involve as high a terminal belief x as the other (as the deadline expires).

(2) In the linear case, it is straightforward to solve for the switching time (or switching belief) under Assumption 2.2. For all terminal beliefs $x_T > x^*$, for which no effort is exerted at the deadline, the switching belief between equilibrium phases is determined by

$$(1 + k - \alpha)e^{-x_T} - \alpha = \int_x^{x_T} e^{-s}\frac{\alpha\lambda - v}{\lambda}\mathrm{d}s,$$

which gives as value of x (as a function of t)

$$x(t) = \ln\Big((1 + k - v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\Big) - \ln\alpha.$$

This represents a frontier in (t, x) space that the equilibrium path will cross from below for sufficiently long deadlines. Consistent with the fact that, in the optimum, a switch to zero effort is irreversible, when $u_t = 0$ and $\dot{x}_t = \lambda$, the path leaves this locus (i.e., it holds that $x'(t) < \lambda$).

The switching belief $x(t)$ decreases in T: the longer the deadline, the longer maximum effort will be exerted (recall that x measures pessimism). This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted. These are the comparative statics mentioned in the text before Lemma 2.3.

Furthermore, by differentiating, the boundary $x(\cdot)$ satisfies $x'(t) < 0$ (resp. > 0) if and only if $1 + k < v/\lambda$. In that case, total effort increases with T: considering the plane $(-\tau, x)$, where τ is time-to-go and x is the belief, increasing the deadline is equivalent to increasing τ, i.e. shifting the initial point to the left; if $x' < 0$, it means that the range of beliefs over which high effort is exerted (which is 1-to-1 with time spent exerting maximum effort, given that $\dot{x} = \lambda + \bar{u}$) increases. If instead x' is positive, total effort decreases with T, by the same argument.

Consider now the convex case. Note that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u)\, c'(u) - c(u) - v}{c''(u)\,(1 + e^{x})\,(\lambda + u)},$$

along with

$$(1 + e^{x_T})\, c'(u_T) = 1 + k,$$

which is the transversality condition, can be integrated to

$$\phi(u) = \frac{(k+1)\phi(u_T)}{1 + k - c'(u_T)} \frac{1}{1 + e^{-x}}.$$

Note also that, defining $g(u) := \frac{\phi(u)}{1+k-c'(u)}$,

$$g'(u) = \frac{(\lambda + u)\, c''(u)}{1 + k - c'(u)} + \frac{\phi(u)}{(1 + k - c'(u))^2} c''(u),$$

which is of the sign of

$$\psi(u) := (\lambda + u)(1 + k) - c(u) - v,$$

which is strictly concave, negative at ∞, and positive for u small enough if and only if $1 + k > v/\lambda$.

Note that increasing T is equivalent to increasing x_T, which in turn is equivalent to decreasing u_T, because the transversality condition yields

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'(u)\, e^{x}}{(1 + e^{x})\, c''(u)} < 0.$$

Because ϕ is increasing, u increases when u_T decreases if and only if ψ is decreasing at u.

So if $\max_u[(\lambda + u)(1 + k) - c(u)] < v$, ψ is negative for all u, and it follows that u increases for fixed x; in addition to all values of x that are visited in the interval $[x^0, x_T]$, as T increases, additional effort accrues at time T; overall, it is then unambiguous: total effort increases.

On the other hand, if $1 + k > v/\lambda$, then if the deadline is long enough for effort to be small throughout, effort at $x < x_T$ decreases as T increases, but since an additional increment of effort is produced at time T, it is unclear. A simple numerical example shows that total effort can then decrease. □

B Proofs for Section 3

Proof of Lemma 3.1. We address the two claims in turn.

Existence: Note that the state equation is linear in the control u, while the objective's integrand is concave in u. Hence the set $N(x, U, t)$ is convex (see Thm. 8, Ch. 2 of Seierstad and Sydsæter, 1987). Therefore, the Filippov-Cesari existence theorem applies.

Uniqueness: We can write the objective as, up to constant terms,

$$\int_0^T (1+e^{-x_t})(w_t - v - c(u_t))\mathrm{d}t - ke^{-x_T},$$

or, using the likelihood ratio $l_t := p_t / (1-p_t) > 0$,

$$J(l) := \int_0^T (1+l_t)(w_t - v - c(u_t))\ \mathrm{d}t - kl_T.$$

Consider the linear case. Letting $g_t := w_t - v + \alpha\lambda$, we rewrite the objective in terms of the likelihood ratio as

$$\int_0^T l_t g_t \mathrm{d}t - (k-\alpha)l_T + \alpha \ln l_T + \text{ Constant}.$$

Because the first two terms are linear in l while the last is strictly concave, it follows that there exists a unique optimal terminal odds ratio $l_T^* := l_T$. Suppose that there exists two optimal trajectories l_1, l_2 that differ. Because $l_{1,0} = l_{2,0} = p^0 / \left(1-p^0\right)$ and $l_{1,T} = l_{2,T} = l_T^*$, yet the objective is linear in l_t, it follows that every feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ is optimal as well.[22] Consider any interval $[a,b] \subset [0,T]$ for which $t \in [a,b] \implies \min\{l_{1,t}, l_{2,t}\} < \max\{l_{1,t}, l_{2,t}\}$. Consider any feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ for all t, $l_t \in (\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\})$ for $t \in [a,b]$ and associated control such that $u_t \in (0, \bar{u})$ for $t \in [a,b]$. Because there is an open set of variations of u that must be optimal in $[a,b]$, it follows from Lemma 2.4.ii of Cesari (1983) that $g_t = 0$ (a.e.) on $[a,b]$.

Consider now the convex case. Suppose that there are two distinct optimal trajectories l_1 and l_2, with associated controls u_1 and u_2. Assume without loss of generality that

$$l_{1,t} < l_{2,t} \text{ for all } t \in (0,T].$$

We analyze the modified objective function

$$\tilde{J}(l) := \int_0^T (1+l_t)(w_t - v - \tilde{c}_t(u_t))\mathrm{d}t - kl_T,$$

in which we replace the cost function $c(u_t)$ with

$$\tilde{c}_t(u) := \begin{cases} \alpha_t u & \text{if} \quad u \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}] \\ c(u) & \text{if} \quad u \notin [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}], \end{cases}$$

[22] Feasibility means that $\dot{l}_t \in [l_t\lambda, l_t(\lambda + \bar{u})]$ for all t.

where

$$\alpha_t := \frac{\max\{c(u_{1,t}), c(u_{2,t})\} - \min\{c(u_{1,t}), c(u_{2,t})\}}{\max\{u_{1,t}, u_{2,t}\} - \min\{u_{1,t}, u_{2,t}\}}.$$

(If $u_{1,t} = u_{2,t} =: u_t$ for some t, set α_t equal to $c'(u_t)$). Because $\tilde{c}_t(u) \geq c(u)$ for all t, u, the two optimal trajectories l_1 and l_2, with associated controls u_1 and u_2, are optimal for the modified objective $\tilde{J}(l)$ as well. Furthermore, $\tilde{J}(l_1) = J(l_1)$ and $\tilde{J}(l_2) = J(l_2)$.

We will construct a feasible path l_t and its associated control $u_t \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ which attains a higher payoff $\tilde{J}(l)$ and therefore a strictly higher payoff $J(l)$. Suppose $u_t \in [u_{1,t}, u_{2,t}]$ for all t. Letting $g_t := w_t - v + \alpha\lambda - \dot{\alpha}_t$, we rewrite the modified objective as

$$\int_0^T l_t g_t \mathrm{d}t - \int_0^T \dot{\alpha}_t \ln l_t \mathrm{d}t - (k - \alpha_T) l_T + \alpha_T \ln l_T + \text{ Constant.}$$

We now consider a continuous function $\varepsilon_t \geq 0$ and two associated variations on the paths l_1 and l_2,

$$\begin{aligned} l'_{1,t} &:= (1-\varepsilon_t) l_{1,t} + \varepsilon_t l_{2,t} \\ l'_{2,t} &:= (1-\varepsilon_t) l_{2,t} + \varepsilon_t l_{1,t}. \end{aligned}$$

Because l_1 and l_2 are optimal, for any ε_t it must be the case that

$$\begin{aligned} \tilde{J}(l_1) - \tilde{J}(l'_1) &\geq 0 \\ \tilde{J}(l_2) - \tilde{J}(l'_2) &\geq 0. \end{aligned}$$

We can write these payoff differences as

$$\begin{aligned} \int_0^T \varepsilon_t (l_{1,t} - l_{2,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{2,t} - l_{1,t}}{l_{1,t}} \mathrm{d}t - (k-\alpha_T)\varepsilon_T (l_{1,T} - l_{2,T}) - \alpha_T \varepsilon_T \frac{l_{2,T} - l_{1,T}}{l_{1,T}} + o(\|\varepsilon\|) &\geq 0 \\ \int_0^T \varepsilon_t (l_{2,t} - l_{1,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{1,t} - l_{2,t}}{l_{2,t}} \mathrm{d}t - (k-\alpha_T)\varepsilon_T (l_{2,T} - l_{1,T}) - \alpha_T \varepsilon_T \frac{l_{1,T} - l_{2,T}}{l_{2,T}} + o(\|\varepsilon\|) &\geq 0. \end{aligned}$$

Letting

$$\rho_t : l_{1,t}/l_{2,t} < 1 \text{ for all } t > 0,$$

we can sum the previous two conditions (up to the second order term). Finally, integrating by parts, we obtain the following condition,

$$\int_0^T \left[\frac{\dot{\varepsilon}_t}{\varepsilon_t}\left(2 - \rho_t - \frac{1}{\rho_t}\right) + \dot{\rho}_t \frac{1-\rho_t^2}{\rho_t^2}\right] \alpha_t \varepsilon_t \mathrm{d}t \geq 0,$$

which must hold for all ε_t. Using the fact that $\dot{\rho} = \rho(u_2 - u_1)$ we have

$$\int_0^T \left[-\frac{\dot{\varepsilon}_t}{\varepsilon_t}(1-\rho_t) + (u_{2,t} - u_{1,t})(1+\rho_t)\right] \alpha_t \varepsilon_t \frac{1-\rho_t}{\rho_t} \mathrm{d}t \geq 0. \tag{11}$$

We now identify bounds on the function ε_t so that both variations l'_1 and l'_2 are feasible and their associated controls lie in $[\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ for all t. Consider the following identities

$$\begin{aligned} \dot{l}'_1 &= -l'_{1,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{2,t} - l_{1,t}) - \lambda l'_{1,t} - (1-\varepsilon_t) u_{1,t} l_{1,t} - \varepsilon_t u_{2,t} l_{2,t} \\ \dot{l}'_2 &= -l'_{2,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{1,t} - l_{2,t}) - \lambda l'_{2,t} - \varepsilon_t u_{1,t} l_{1,t} - (1-\varepsilon_t) u_{2,t} l_{2,t}. \end{aligned}$$

We therefore have the following expressions for the function $\dot{\varepsilon}/\varepsilon$ associated with each variation

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t} - u_t)\frac{1-\varepsilon_t}{\varepsilon_t} l_{1,t} + l_{2,t}(u_{2,t} - u_t)}{l_{2,t} - l_{1,t}}, \tag{12}$$

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t} - u_t) l_{1,t} + \frac{1-\varepsilon_t}{\varepsilon_t} l_{2,t}(u_{2,t} - u_t)}{l_{1,t} - l_{2,t}}. \tag{13}$$

In particular, whenever $u_{2,t} > u_{1,t}$ the condition

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}, \frac{l_{1,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}\right]$$

ensures the existence of two effort levels $u_t \in [u_{1,t}, u_{2,t}]$ that satisfy conditions (12) and (13) above. Similarly, whenever $u_{1,t} > u_{2,t}$ we have the bound

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{l_{1,t}(u_{1,t} - u_{2,t})}{l_{2,t} - l_{1,t}}, \frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}\right].$$

Note that $\dot{\varepsilon}_t/\varepsilon_t = 0$ is always contained in both intervals.

Finally, because $\rho_0 = 1$ and $\rho_t < 1$ for all $t > 0$, we must have $u_{1,t} > u_{2,t}$ for $t \in [0, t^*)$ with $t^* > 0$. Therefore, we can construct a path ε_t that satisfies

$$(u_{2,t} - u_{1,t})\frac{1+\rho_t}{1-\rho_t} < \frac{\dot{\varepsilon}_t}{\varepsilon_t} < 0 \quad \forall t \in [0, t^*),$$

with $\varepsilon_0 > 0$, and $\varepsilon_t \equiv 0$ for all $t \geq t^*$. Substituting into condition (11) immediately yields a contradiction. □

Proof of Proposition 3.2. Recall from the proof of Lemma 2.3 that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u)c'(u) - c(u) - v}{c''(u)(1 + e^x)(\lambda + u)}$$

must hold for the optimal trajectory (in the (x, u)-plane) for the social planner. Denote this trajectory x^{FB}. The corresponding law of motion for the agent's optimum trajectory x^* given w is

$$\frac{1}{x'(u)} = \frac{(\lambda + u)c'(u) - c(u) + w_t - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

(Note that, not surprisingly, time matters). This implies that (in the (x, u)-plane) the trajectories x^{FB} and x^* can only cross one way, if at all, with x^* being the flatter one. Yet the (decreasing) transversality curve of the social planner, implicitly given by

$$(1 + e^{x_T})c'(u_T) = 1 + k,$$

lies above the (decreasing) transversality curve of the agent, which is defined by

$$(1 + e^{x_T})c'(u_T) = k.$$

Suppose now that the trajectory x^{FB} ends (on the transversality curve) at a lower belief x_T^{FB} than x^*: then it must be that effort u was higher throughout along that trajectory than along x^* (since the latter is flatter, x^{FB}

must have remained above x^* throughout). But since the end value of the belief x is simply $x^0 + \int_0^T u_s \mathrm{d}s$, this contradicts $x_T^{FB} < x_T^*$.

It follows that for a given x, the effort level u is higher for the social planner.

The same reasoning implies the second conclusion: if $x_T^{FB} = x_T^*$, so that total effort is the same, yet the trajectories can only cross one way (with x^* being flatter), it follows that x^* involves lower effort first, and then larger effort, i.e. the agent backloads effort. □

Proof of Lemma 3.3.

Consider the convex case. Applying Pontryagin's theorem yields eqn. (7). It also follows that the effort and belief (x, u) trajectories satisfy

$$c''(u)(1+e^x)\dot{u} = (\lambda + u)c'(u) - c(u) + w_t - v \quad (14)$$

$$\dot{x} = \lambda + u \quad (15)$$

with boundary conditions

$$x_0 = x^0 \quad (16)$$

$$ke^{-x_T} = \left(1 + e^{-x_T}\right)c'(u_T). \quad (17)$$

Differentiating (14) further, we obtain

$$\begin{aligned}\left(c''(u)(1+e^x)\right)^2 u_t'' &= \left((\lambda+u)c''(u)u_t' + w_t'\right)c''(u)(1+e^x) \\ &\quad - \left((\lambda+u)c'(u) + w_t - c(u) - v\right)\left(c'''(u)u_t'(1+e^x) + e^x(\lambda+u)c''(u)\right).\end{aligned}$$

So that when $u_t' = 0$ we obtain

$$c''(u)(1+e^x)u_t'' = w_t'.$$

This immediately implies the first conclusion.

In the linear case, mimicking the proof of Lemma 2.1, Pontryagin's principle applies, and yields the existence of an absolutely continuous function $\gamma : [0,T] \to \mathbb{R}$ such that

$$\gamma_t - \alpha\left(1 + e^{-x_t}\right) > (<) 0 \Rightarrow u_t = \bar{u} \;\; (u_t = 0).$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(w_t - \alpha u_t - v), \gamma_T = ke^{-x_T}.$$

Define ϕ by $\phi_t := \gamma_t - \alpha(1 + e^{-x_t})$. Note that $\phi_t > 0$ (resp. < 0) $\Rightarrow u_t = \bar{u}$ (resp. $= 0$). Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda + w_t - v), \phi_T = (k - \alpha)e^{-x_T} - \alpha.$$

(This is the formal derivation of eqn. (8).) Observe now that if w is monotone, so is $\alpha\lambda + w_t - v$, and hence $\dot{\phi}$ changes signs only once. Conclusion 1 follows for the linear case. If it is strictly monotone, ϕ is equal to zero at most at one date t, and so the optimal strategy is extremal, yielding the second conclusion of the lemma. □

C Proofs for Section 4

We shall start with Lemma 4.3 before turning to Theorem 4.2.

Proof of Lemma 4.3. We prove the following:

1. If there exists $t \in (0, T)$ such that $\phi_t > 0$, then there exists $t' \in [t, T]$ such that $u_s = \bar{u}$ for $s \in [t, t']$, $u_s = 0$ for $s \in (t', T]$.
2. If there exists $t \in (0, T)$ such that $\phi_t < 0$, then either $u_s = 0$ for all $s \in [t, T]$ or $u_s = 0$ for all $s \in [0, t]$,

which implies the desired decomposition. For the first part, note that either $u_s = \bar{u}$ for all $s > t$, or there exists t'' such that both $\phi_{t''} > 0$ (so in particular $u_{t''} = \bar{u}$) and $\dot{\phi}_{t''} < 0$. Because p_t decreases over time, and $u_s \leq u_{t''}$ for all $s > t''$, it follows that $w_s < w_{t''}$, and so $\dot{\phi}_s < \dot{\phi}_{t''} < 0$. Hence ϕ can cross 0 only once for values above t, establishing the result. For the second part, note that either $u_s = 0$ for all $s \geq t$, or there exists $t'' \geq t$ such that $\phi_{t''} < 0$ (so in particular $u_{t''} = 0$) and $\dot{\phi}_{t''} > 0$. Because p_t decreases over time, and $u_s \geq u_{t''}$ for all $s < t''$, it follows that $w_s \geq w_{t''}$, and so $\dot{\phi}_s > \dot{\phi}_{t''} > 0$. For all $s < t''$, $\phi_s < 0$ and $\dot{\phi}_s > 0$. Hence, $u_s = 0$ for all $s \in [0, t]$. □

Proof of Theorem 4.2. We study the linear and convex cases in turn.

Proof of Theorem 4.2 (Linear case). We start by establishing uniqueness.
Uniqueness: assume an equilibrium exists, and note that, given a final belief x_T, the pair of differential equations for ϕ and x (along with the transversality condition) admit a unique solution, pinning down, in particular, the effort exerted by, and the wage received by the agent. Therefore, if two (or more) equilibria existed for some values (x_0, T), it would have to be the case that each of them is associated with a different terminal belief x_T. However, we shall show that, for any x_0, the time it takes to reach a terminal belief x_T is a continuous, strictly increasing function $T(x_T)$; therefore, no two different terminal beliefs can be reached in the same time T.
We start with a very optimistic initial belief $x_0 < x_1$, as this allows for the richest paths (the other cases are subsets of these).
Clearly, we have $T(x_0) = 0$. As long as $x_0 < x^*$, we have a first range for x_T over which full effort is always exerted. For these terminal beliefs, we have $T(x_T) = (x_T - x_0) / (\lambda + \bar{u})$, increasing. If for all $x_T \leq x^*$ the following expression is strictly positive

$$(k - \alpha) e^{-x_T} - \alpha - \int_{x_0}^{x_T} e^{-x} \left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) \mathrm{d}x, \tag{18}$$

then we always have full effort, until $x_T = x^*$. If so, go to the section "Long Terminal Beliefs." Otherwise, go to the section "Short Terminal Beliefs."

Short Terminal Beliefs

For these beliefs, we have a full effort phase at the end. We assume $x_0 < x_1 < x^*$, as the other cases are subsets of those discussed here. Full effort is exerted at the end typically for short deadlines. If $x_T < x^*$ then the full effort region is given by $[x_2, x_T]$, where x_2 solves

$$(k-\alpha)\, e^{-x_T} - \alpha - \int_{x_2}^{x_T} e^{-x} \left(\frac{1}{1+e^x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) \mathrm{d}x = 0.$$

Therefore, we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_T} = \left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right)^{-1} \left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) e^{x_2 - x_T}.$$

The denominator is positive by construction ($\phi(x)$ hits zero going backwards).

1. Suppose $x_2 > x_1$. Then the time to get to x_T is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda+\bar{u}} + \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\lambda + u(x)} + \frac{x_1 - x_0}{\lambda}.$$

Using the formula for interior effort,

$$u(x) = (v - \alpha\lambda)(1+e^x) - \lambda,$$

we can write

$$\begin{aligned} T'(x_T) &= \frac{1}{\lambda+\bar{u}} + \frac{\mathrm{d}x_2}{\mathrm{d}x_T} \frac{\bar{u} - u(x_2)}{(\lambda+\bar{u})(\lambda+u(x_2))} \\ &\propto \lambda + u(x_2) + \frac{\mathrm{d}x_2}{\mathrm{d}x_T}(\bar{u} - u(x_2)) \\ &= (v-\alpha\lambda)(1+e^{x_2}) + (\bar{u} - u(x_2)) \frac{\mathrm{d}x_2}{\mathrm{d}x_T}. \end{aligned}$$

We want to show $T'(x_T) > 0$. Clearly, if $\mathrm{d}x_2/\mathrm{d}x_T > 0$, we are done. If not, then we have

$$\begin{aligned} T'(x_T) &> (v-\alpha\lambda)(1+e^{x_2}) + (\lambda + \bar{u} - (v-\alpha\lambda)(1+e^{x_2})) \frac{\mathrm{d}x_2}{\mathrm{d}x_T} e^{-(x_2 - x_T)} \\ &= (v-\alpha\lambda)(1+e^{x_2}) + (1+e^{x_2})(\lambda+u) \left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) \\ &\propto k - \alpha + \frac{1}{1+e^{x_T}} > 0. \end{aligned}$$

2. Now suppose $x_0 < x_2 < x_1$, and so no effort is exerted on $[x_0, x_2]$. Notice that if $x_2(x_T) \leq x_0$ then $T(x_T)$ is clearly increasing, in x_T (since we have full effort throughout). If $x_2(x_T) > x_0$, the time necessary to reach the terminal belief is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda+\bar{u}} + \frac{x_2 - x_0}{\lambda}.$$

Therefore,

$$\lambda(\lambda+\bar{u})\, T'(x_T) = \lambda + \bar{u} \frac{\mathrm{d}x_2}{\mathrm{d}x_T}.$$

It is immediate that if x_2 is increasing in x_T then $T'(\cdot) > 0$. If not, then we have

$$\begin{aligned} T'(x_T) &\propto \lambda + \bar{u}\frac{\mathrm{d}x_2}{\mathrm{d}x_T} > \lambda + \bar{u}\frac{\mathrm{d}x_2}{\mathrm{d}x_T}e^{-(x_2 - x_T)} \\ &\propto \lambda\left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) + \bar{u}\left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right). \end{aligned}$$

We also know $e^{x_2} < e^{x_1} = \lambda/(v-\alpha\lambda) - 1$, and thus

$$\begin{aligned} T'(x_T) &> \lambda\left(\frac{v-\alpha\lambda}{\lambda} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) + \bar{u}\left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) \\ &= \bar{u}\left(k-\alpha+\frac{1}{1+e^{x_T}}\right) > 0. \end{aligned}$$

Longer Terminal Beliefs

For $x_T > x^*$ we can have four possible patterns: never work (in which case the time to x_T is clearly increasing), zero-mixed-zero, zero-mixed-full-zero, or zero-full-zero. We now show that $T(x_T)$ is increasing under any of these patterns. In addition the times at which the equilibrium path switches between the various effort regions are continuous functions of x_T, so it suffices to establish $T'(x_T)$ in each of these cases separately.

Zero and Mixed Effort Phases

We again consider the time necessary to reach a given terminal belief x_T. We consider beliefs $x_T > x^*$, for which the agent does not work at the end. If there is no full effort phase, the agent works at a rate

$$u(x) = (v - \alpha\lambda)(1+e^x) - \lambda$$

until the switching belief x_3, then stops until x_T. The two thresholds are linked by the equation

$$(k-\alpha)e^{-x_T} - \alpha - \int_{x_3}^{x_T} e^{-x}\left(\frac{1}{1+e^x} + \alpha - \frac{v}{\lambda}\right)\mathrm{d}x = 0.$$

From the state equation, we know beliefs increase at rate $\lambda + u(x)$ in the first phase, and at rate λ afterwards. The time to x_T is therefore given by

$$T(x_T) = \int_{x_1}^{x_T} \frac{1}{\lambda+u(x)}\mathrm{d}x = \int_{x_1}^{x_3(x_T)} \frac{1}{(v-\alpha\lambda)(1+e^x)}\mathrm{d}x + \frac{x_T - x_3(x_T)}{\lambda}.$$

Consider the derivative of T with respect to x_T,

$$\lambda T'(x_T) = 1 + \left(\frac{\lambda}{\lambda + u(x_3)} - 1\right)\frac{\mathrm{d}x_3}{\mathrm{d}x_T},$$

where $\mathrm{d}x_3/\mathrm{d}x_T$ is given by

$$\frac{\mathrm{d}x_3}{\mathrm{d}x_T} = \left(\frac{1}{1+e^{x_3}} - \frac{v}{\lambda} + \alpha\right)^{-1}\left(k + \frac{1}{1+e^{x_T}} - \frac{v}{\lambda}\right)e^{x_3 - x_T}. \tag{19}$$

Now, we know $(1+e^{x_3})^{-1}+\alpha-v/\lambda<0$ for all $x>x_1$. Therefore, if $k\geq v/\lambda$ (or more generally if $(1+e^{x_3})^{-1}+k-v/\lambda>0$), then $\mathrm{d}x_3/\mathrm{d}x_T<0$, the whole expression is positive and we are done.

Conversely, suppose that $(1+e^{x_3})^{-1}+k-v/\lambda<0$. We then check whether $T'(x_T)$ can be negative. We obtain

$$\begin{aligned}\lambda T'(x_T) &= 1-e^{x_3-x_T}\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\Big/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right)\\ &> 1-\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\Big/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right).\end{aligned}$$

Now plug in the expression for $u(x_3)$, notice that the x_3 drops out, and obtain

$$\lambda T'(x_T)>\lambda\frac{k-\alpha}{v-\alpha\lambda}>0.$$

Full and Mixed Effort Phases

Now suppose the path involves mixing on $[x_1,x_2]$, full effort on $[x_2,x_3]$ and zero effort on $[x_3,x_T]$. The time it takes to reach x_T is then given by

$$\lambda T(x_T)=\int_{x_1}^{x_2(x_T)}\frac{\lambda}{(v-\alpha\lambda)(1+e^x)}\mathrm{d}x+\frac{\lambda}{\lambda+\bar{u}}(x_3(x_T)-x_2(x_T))+x_T-x_3(x_T).$$

Hence

$$\lambda T'(x_T)=1-\frac{\bar{u}}{\lambda+\bar{u}}\frac{\mathrm{d}x_3}{\mathrm{d}x_T}+\frac{\mathrm{d}x_2}{\mathrm{d}x_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right).$$

Notice that x_2 is the solution to

$$\int_{x_2}^{x_3}e^{-x}\left(\frac{1}{1+e^x}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)\mathrm{d}x=0. \tag{20}$$

We then have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_T}=-\frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}{e^{-x_2}\left(\frac{1}{1+e^{x_2}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}\frac{\mathrm{d}x_3}{\mathrm{d}x_T}, \tag{21}$$

and so

$$\lambda T'(x_T)=1-\frac{\mathrm{d}x_3}{\mathrm{d}x_T}\frac{\bar{u}}{\lambda+\bar{u}}+\frac{\mathrm{d}x_2}{\mathrm{d}x_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right).$$

Clearly, $k\geq v/\lambda$ implies $\mathrm{d}x_3/\mathrm{d}x_T<0$, $\mathrm{d}x_2/\mathrm{d}x_T>0$ and $T'(x_T)>0$.

Conversely, suppose $k<v/\lambda$. Plug in the explicit formula for $u(x_2)$ and for $\mathrm{d}x_3/\mathrm{d}x_T$ to obtain the following expression for $\lambda T'(x_T)$:

$$\frac{(e^{x_T}(v-k\lambda)-(k+1)\lambda+v)\left(-\lambda e^{x_2}\left(\bar{u}+e^{x_3}(\alpha\lambda-v)-v+\alpha\lambda+\lambda\right)-\bar{u}\left(e^{x_3}+1\right)e^{x_3}(v-\alpha\lambda)\right)}{e^{x_T}\left(\bar{u}+\lambda\right)\left(e^{x_T}+1\right)(v-\alpha\lambda)\left(e^{x_3}(v-\alpha\lambda)+v-(\alpha+1)\lambda\right)}+1.$$

To simplify, let $V=\lambda/(v-\alpha\lambda)-1, U=(\lambda+\bar{u})/(v-\alpha\lambda)-1$, $k=\alpha(K+1)$, and $X_i=e^{x_i}$ to get

$$1-\frac{(K(V+1)(X_3+1)\alpha+V-X_T)\left(U\left(VX_2+X_2+X_3^2+X_3\right)-X_3(V(X_2+X_3+1)+X_2)\right)}{(U+1)X_T(X_T+1)(V-X_3)}.$$

The constraints are: $0 < V < X_2 < U < X_3 < X_T$, $0 < K < X_T$, and $\alpha > 0$. Note that the conditions $v > \alpha\lambda$ and $\bar{u} > 0$ follow from $U > V > 0$. The condition $\alpha < k < \alpha(X_T + 1)$ is captured by $0 < K < X_T$. Finally, note that if $v > k\lambda$ (which is equivalent to $\alpha K < (1+V)^{-1}$) then this expression is positive, as it is linear in $A = K(1+V)\alpha$, and it is positive both for $A = 0, 1$.[23]

Only the Full Effort Phase

In this case, the incentives to exert effort hit zero when beliefs are at a level that does not allow mixing, or $x_2 < x_1$. The candidate equilibrium involves zero-full-zero effort. The time required is then given by

$$\begin{aligned} \lambda T(x_T) &= x_T - x_3 + (x_3 - x_2)\frac{\lambda}{\lambda + \bar{u}} + x_2 - x_0 \\ &= x_T - (x_3 - x_2)\frac{\bar{u}}{\lambda + \bar{u}} - x_0, \end{aligned}$$

where x_3 and x_2 solve the same equations as before. Therefore,

$$\lambda T'(x_T) = 1 - \frac{\bar{u}}{\lambda + \bar{u}}\left(\frac{\mathrm{d}x_3}{\mathrm{d}x_T} - \frac{\mathrm{d}x_2}{\mathrm{d}x_T}\right),$$

where the last two terms are given by equations (19) and (21) respectively.

If $k \geq v/\lambda$, then $\mathrm{d}x_3/\mathrm{d}x_T < 0$, $\mathrm{d}x_2/\mathrm{d}x_T > 0$, and we are done by the same argument as before.

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[23] This requires a little bit of work. Consider the case $A = 0$. The derivative w.r.t. U of the expression is

$$-\frac{(1+V)(X_3+1)(X_3+X_2)(X_T-V)}{(1+U)^2(X_3-V)X_T(1+X_t)} < 0,$$

so the expression is minimized by choosing U as high as possible given the constraints, i.e. $U = X_3$, in which case the expression simplifies to

$$\frac{X_3V + X_T(1+X_T-X_3)}{X_T(1+X_T)} > 0.$$

Consider now $A = 1$. Similarly, the derivative w.r.t. U does not depend on U itself, so the expression is minimized at one of the extreme values of U; if $U = X_3$, it is equal to

$$\frac{X_3(1+X_3+V) + X_T(X_T-X_3+1)}{X_T(1+X_T)} > 0;$$

if $U = X_2$, the resulting expression's derivative w.r.t. X_2 is independent of X_2, so we can again plug in one of the two extreme cases, $X_2 = X_3$ or $X_2 = V$; the values are then, respectively,

$$\frac{X_3(1+V+X_3) + X_T(X_T-X_3+1)}{X_T(1+X_T)} > 0$$

and

$$\frac{X_T(1+X_T+V) - V(1+V+X_3)}{X_T(1+X_T)} \geq \frac{X_3(X_3+1) - V(V+1)}{X_T(1+X_T)} \geq 0.$$

If $v > k\lambda$, then $\mathrm{d}x_3/\mathrm{d}x_T > 0$ and we proceed as follows. Substituting the expressions in (19) and (21), and using the same change of variable as before, we want to show that

$$1 - \frac{X_2 (X_2+1)(U-V)(U-X_3)(X_T(K(V+1)\alpha-1) + K(V+1)\alpha + V)}{(U+1)X_T (X_T+1)(U-X_2)(V-X_3)} > 0.$$

To establish this inequality, it is simpler to bound α. Setting the expression to zero, this is equivalent to requiring that

$$\alpha K < \frac{(U+1)X_T (U-X_2)(V-X_3)}{(V+1)X_2 (X_2+1)(U-V)(U-X_3)} - \frac{1}{X_T+1} + \frac{1}{V+1},$$

a sufficient condition for this is that $\alpha K < (1+V)^{-1}$, which is equivalent to $v > k\lambda$.

Existence: We have established that the time necessary to reach the terminal belief is a continuous and strictly increasing function. Therefore, the terminal belief reached in equilibrium is itself given by a strictly increasing function

$$x_T(T) : \mathbb{R}_+ \to [x_0, \infty).$$

Since there exists a unique path consistent with optimality for each terminal belief, given a deadline T we can establish existence by constructing the associated equilibrium outcome, and in particular, the equilibrium wage path. Existence and uniqueness of an optimal strategy for the worker, after any (on or off-path) history, follows then from Lemma 3.1.

Proof of Theorem 4.2 (Convex case). We proceed as in the linear case. Fix T. The two differential equations obeyed by the (x,u)-trajectory are

$$\begin{aligned} \dot{x} &= \lambda + u \\ \dot{u} &= \frac{(\lambda+u)\,c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{c''(u)(1+e^x)}. \end{aligned}$$

We have, using that $\mathrm{d}x = (\lambda+u)\,\mathrm{d}t$,

$$\frac{\mathrm{d}u}{\mathrm{d}x} =: f(u,x) = \frac{(\lambda+u)\,c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{(\lambda+u)\,c''(u)(1+e^x)}. \tag{22}$$

Recall also that the transversality curve is given by

$$(1+e^x)\,c'(u) = k,$$

and so

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'(u)\,e^x}{(1+e^x)\,c''(u)} < 0.$$

Note that the slope of $u(x)$ at the deadline T is at most first positive then negative. To see this, differentiate the numerator in (22) and impose (22) equal to zero. We obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_T}\frac{\mathrm{d}u(x_T)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x_T}\left[(\lambda+u(x_T))\,c'(u(x_T)) - c(u(x_T)) + \frac{\lambda+u(x_T)}{1+e^{x_T}}\right] \\ &= \left((\lambda+u(x_T))\,c''(u(x_T)) + \frac{1}{1+e^{x_T}}\right)\frac{\mathrm{d}u_T}{\mathrm{d}x_T} - \frac{(\lambda+u(x_T))\,e^{x_T}}{(1+e^{x_T})^2} < 0. \end{aligned}$$

Suppose now we had

$$u'(x) > \frac{\mathrm{d}u_T}{\mathrm{d}x_T}, \text{ at } x = x_T,$$

so that the trajectory does not cross the transversality line from above. Then we would be done. A path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach.

Denote the difference in the slopes of the effort and transversality lines by

$$\Delta(u,x) := \frac{(\lambda+u)\,c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{(\lambda+u)\,c''(u)\,(1+e^x)} + \frac{c'(u)\,e^x}{(1+e^x)\,c''(u)}.$$

Note that $\Delta = 0 \Rightarrow \Delta'(x_T) < 0$ so our trajectory crosses transversality (at most) first from below then from above.

More generally, the time required to reach terminal belief x_T is given by

$$T = \int_{x_0}^{x_T} \frac{1}{\lambda + u(x)}\mathrm{d}x = \int_{x_0}^{x_T} \left(\lambda + u(x_T) - \int_x^{x_T} u'(x)\,\mathrm{d}x\right)^{-1} \mathrm{d}x.$$

Differentiating with respect to x_T we obtain

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}x_T} &= \frac{1}{\lambda + u(x_T)} + \left(u'(x_T) - \frac{\mathrm{d}u_T}{\mathrm{d}x_T}\right) \int_{x_0}^{x_T} \frac{1}{(\lambda + u(x))^2}\mathrm{d}x \\ &= \frac{1}{\lambda + u(x_T)} + \Delta(u_T, x_T) \int_{x_0}^{x_T} \frac{1}{(\lambda + u(x))^2}\mathrm{d}x. \end{aligned}$$

Clearly, if the function $u(x)$ crosses the transversality line from below ($\Delta > 0$) then we are done: a path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach. Imposing transversality and simplifying we obtain that a necessary condition for $\Delta > 0$ for all x_T is

$$k \geq v/\lambda.$$

Because we do not wish to assume that, note that the function $f(u,x)$ in (22) has the following properties

$$\begin{aligned} f(u,x) &\leq 0 \Rightarrow f_u(u,x) > 0 \\ f(u,x) &\geq 0 \Rightarrow f_x(u,x) < 0. \end{aligned}$$

In words, a trajectory at $(x, u + \mathrm{d}u)$ comes down not as fast as a trajectory at (x,u) if (x,u) is such that $\dot{u} \leq 0$. Conversely, a trajectory at $(x - \mathrm{d}x, u)$ climbs faster than a trajectory at (x,u) if (x,u) is such that $\dot{u} \geq 0$.

Therefore, consider a trajectory $u(x)$ such that $\Delta(x_T) < 0$. As we increase x_T, the new trajectory lies everywhere above the original one. For a small increase in x_T, because the trajectory changes continuously, the two properties of $f(u,x)$ ensure that the vertical distance between the two trajectories is maximized at x_T.

We then have the condition

$$\frac{\mathrm{d}T}{\mathrm{d}x_T} > \frac{1}{\lambda + u(x_T)} + (x_T - x_0)\frac{\Delta(u_T, x_T)}{(\lambda + u(x_T))^2}.$$

Using transversality and rewriting $\Delta(u_T, x_T)$ we obtain the condition

$$c''(u(x_T))(\lambda + u(x_T)) \geq \frac{x_T - x_0}{1 + e^{x_T}} \left(\frac{c(u(x_T)) + v}{\lambda + u(x_T)} - k - \frac{1}{1 + e^{x_T}} \right). \tag{23}$$

Note that (23) clearly holds at $x_T = x_0$. Furthermore, if $c''(0) > 0$, (23) also holds in the limit for $x_T \to \infty$. Finally, since $c''(u)(\lambda + u)$ was assumed increasing, a sufficient condition for (23) to be satisfied is given by

$$c''(0) > \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right) h(x_0) \geq \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right) e^{-x_0},$$

which is the condition for uniqueness. Existence is established as in the linear case.

Single-peakedness: Single-peakedness of effort is almost immediate. Ssubstituting the equilibrium expression $w_t = (\lambda + u_t) / (1 + e^{x_t})$ in the boundary value problem (14). Differentiating u'_t further, we obtain

$$u'_t = 0 \Rightarrow c''(u)\left(1 + e^{-x}\right) u''_t = -(w_t)^2,$$

which implies that the function u is at most first increasing then decreasing.

We now argue that the wage is single-peaked. In terms of x, the wage is given by

$$\begin{aligned} w(x) &= \frac{\lambda + u(x)}{1 + e^x}, \text{ and so} \\ w'(x) &= \frac{u'(x)}{1 + e^x} - \frac{\lambda + u(x)}{(1 + e^x)^2} e^x, \end{aligned}$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x) e^x.$$

Note that $w' = 0$ only if $u'(x) > 0$. We know that

$$u'(x) = \frac{(\lambda + u) c'(u) - c(u) + \frac{\lambda + u}{1 + e^x} - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

Let $g(u) = v + c(u) - (\lambda + u) c'(u)$ and study $u''(x)$ when $w'(x) = 0$. We have

$$\begin{aligned} u''(x) &= \frac{c''(1 + e^x)(\lambda + u)(w'(x) + c''(\lambda + u) u'(x))}{(c''(u)(1 + e^x)(\lambda + u))^2} \\ &\quad - \frac{(w - g)((1 + e^x)(c'' + (\lambda + u) c''') u'(x) + e^x c''(\lambda + u))}{(c''(u)(1 + e^x)(\lambda + u))^2} \\ &= \frac{c''(\lambda + u) u'(x)}{c''(1 + e^x)(\lambda + u)} - \frac{u'(x)((1 + e^x)(c'' + (\lambda + u) c''') u'(x) + e^x c''(\lambda + u))}{c''(1 + e^x)(\lambda + u)} \\ &= -\frac{u'(x)((1 + e^x)(c'' + (\lambda + u) c''') u'(x) + e^x c''(\lambda + u) - c''(\lambda + u))}{c''(1 + e^x)(\lambda + u)} \\ &= -\frac{u'(x)((2c'' + (\lambda + u) c''') e^x (\lambda + u) - c''(\lambda + u))}{c''(1 + e^x)(\lambda + u)}. \end{aligned}$$

Now consider the second derivative of $w(x)$. We have

$$w''(x) = -\frac{e^x}{1 + e^x} w'(x) + \frac{1}{1 + e^x}\left(u''(x) - w(x) e^x - e^x w'(x)\right),$$

so that when $w'(x) = 0$ we have

$$w''(x) = \frac{u''(x) - u'(x)}{1 + e^x}.$$

We therefore consider the quantity

$$\begin{aligned} u''(x) - u'(x) &= -\frac{u'(x)\left(\left(2c'' + (\lambda + u)\, c'''\right) e^x - c'' + c''\left(1 + e^x\right)\right)}{c''\left(1 + e^x\right)} \\ &= -\frac{u'(x)\left(3c'' + (\lambda + u)\, c'''\right) e^x}{c''\left(1 + e^x\right)} < 0, \end{aligned}$$

if as we have assumed, $c'' + (\lambda + u)\, c''' > 0$. Therefore, we also have single-peaked (at most increasing then decreasing) wages. (More generally, if $c''' < 0$ but $3c'' + (\lambda + u)\, c'''$ is increasing in u then the wage can be increasing on at most one interval.) □

Proof of Lemma 4.4. An important distinction is whether a full effort region occurs right before the terminal belief $x_T = x^*$. This depends on the sign of

$$\phi'(x^* \mid \bar{u}) := \frac{1}{1 + e^{x^*}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} = \frac{\alpha}{k} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \lesseqgtr 0.$$

(1.) Fix a terminal belief $x_T = x_3 + \lambda (T - t)$ and consider the equation defining the no effort frontier, which is given in (47). The left hand side of (47) is decreasing in x_3, because fixing x_T the derivative is simply given by $\phi'(x_3 \mid u = 0)$, which is negative by construction. In addition, it is immediate to show that the left hand side of (47) is increasing in k and v and decreasing in α and λ, which establishes the result.

(2.) We analyze the cases of $x_T \leq x^*$ and $x_T > x^*$ separately.

Fix a terminal belief $x_T \leq x^*$ and consider the definition of the full effort frontier, which is obtained by setting $x_0 = x_2$ in equation (18). The left hand side of (18) is increasing in x_2, because fixing x_T the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the left hand side of (18) is increasing in k and v and decreasing in α, λ, and $\bar{u}$,which establishes the result.

Fix a terminal belief $x_T > x^*$ and consider the equation defining the full effort frontier, which in this case is given in (20) and depends on $x_3(x_T)$ as well. The left hand side of (20) is increasing in x_2, because fixing x_T and hence $x_3(x_T)$ the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the integrand in (18) is increasing in α, λ, and $\bar{u}$, and decreasing in v. Finally, the left hand side of (18) is decreasing in x_3 (the derivative is given by $\phi'(x_3 \mid u = \bar{u}) < 0$). Combining these facts with the comparative statics of x_3 from part (1.) establishes the result. □

D Proofs for Section 5

D.1 Proofs for Subsection 5.1

Proof of Theorem 5.1. The proof is divided in several steps.

D.1.1 Original and Relaxed Programs

Suppose the firm can commit to a wage path $w_t : [0,T] \to \mathbb{R}_+$, which includes firing the agent at T. The agent cannot commit to staying in the firm, but he *is* committed to staying in the sector (cannot quit earlier and take the loss k). By the competitive employers assumption, the firm cannot make positive profits on any interval $[t,T]$. If it did, then the agent would be poached by a competing firm (that can offer, for example, the same wage schedule plus a signing bonus).

Therefore, even if the firm wanted to frontload wages as much as possible, it would not be able to pay more than $\int_0^t \frac{\lambda+u_s}{1+e^{x_s}} \mathrm{d}s$ on any $[0,t]$ interval. If it did, it would make losses on $[0,t]$. If the firm obtained zero *ex ante* profits, it would make profits on $[t,T]$, which would lead to the agent's early departure.

More formally, we can write the firm's problem as

$$W_t = \max_w \int_t^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T}, \tag{24}$$

$$\text{s.t. } u = \arg\max \int_t^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T},$$

$$\forall \tau \geq t : \int_\tau^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T} \geq W_\tau, \tag{25}$$

$$0 = \int_t^T (e^{-x_t}(\lambda + u_t) - (1+e^{-x_t})w_t)\mathrm{d}t. \tag{26}$$

We now proceed in the following steps.

1. Conjecture a full-zero (or "FO") solution, i.e. a solution in which the agent first exerts maximum effort, then no effort.

2. Consider the relaxed problem

$$\begin{aligned} & \max_w \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T} \\ \text{s.t. } u \quad = \quad & \arg\max \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T} \\ 0 \quad \geq \quad & \int_\tau^T \left(1+e^{-x_t}\right)\left(\frac{\lambda+u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned} \tag{27}$$

3. Use the fact that our solution involves maximal frontloading of effort (given x_T) to conclude that any other solution must yield a higher x_T in order to beat it.

4. Consider the even more relaxed problem, in which the objective is given by

$$\begin{aligned} & \max_w \; x_T \\ \text{s.t. } u \quad = \quad & \arg\max \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T} \\ 0 \quad \geq \quad & \int_\tau^T \left(1+e^{-x_t}\right)\left(\frac{\lambda+u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned}$$

5. Argue that full-zero solves this problem by:

   (a) Showing that gaps in effort provision should be achieved with zero wages and lump sums.

   (b) Ruling out terminal zero-full-zero ("OFO") phases (that is, a structure in which no effort is followed by maximum effort and then by no effort until T).

   (c) Ruling out overall zero-full-zero (that is, an OFO phase which does not necessarily extend to T), which always yields a lower x_T than a full-zero ("FO") phase (that is, a structure in which full effort is followed by zero effort; here we need to find a precise bound on $\bar{u}$).

   (d) Ruling out interior effort (that is, showing that $u_t \in \{0, \bar{u}\}$ a.e.).

6. Prove that for our candidate solution, (25) never binds (except at $t = 0$), and hence that we have found a solution to the original problem.

We then finally explain how the bound on $\bar{u}$ we have assumed ensures the validity of some of the steps above.

D.1.2 Conjectured Solution

In formulating a candidate solution, we will make repeated use of the following facts:

1. Under any payment function, the agent works strictly less than the social planner. Since the agent receives the entire surplus, it is always welfare-improving to induce more effort.

2. If the firm wants to hold the ex-ante profit level constant, and shift wages across time periods, it can do so by setting

$$\Delta w_1 = -\frac{1 + e^{-x_2}}{1 + e^{-x_1}} \Delta w_2.$$

   Then by construction,

$$\Delta w_1 + \Delta w_2 = -\left(e^{-x_1} \Delta w_1 + e^{-x_2} \Delta w_2\right).$$

   Therefore, by delaying payments (in a profit-neutral way, and without affecting effort), incentives at time t can be increased. Consider the function

$$\phi_t = \phi_T - \int_t^T e^{-x_s} \left(w_s - v + \alpha\lambda\right) \mathrm{d}s$$

   and two times t_1 and t_2. Indeed, if $\Delta w_2 > 0$, then $\Delta w_1 < 0$ and $\Delta w_1 + \Delta w_2 > 0$, which increases ϕ_1.

   The intuition for conjecturing this solution is based on the equation for the incentives to work. Remember the agent has incentives to work whenever

$$\alpha\left(1 + e^{-x_t}\right) \leq \int_t^T e^{-x_s} \left(v - w_s + \alpha u_s\right) \mathrm{d}s + k e^{-x_T}. \tag{28}$$

Therefore, the firm can actually induce higher incentives at time t by shifting wages to the future while holding the ex-ante profit level constant. This corresponds to reducing the integral $\int_t^T e^{-x_s} w_s \mathrm{d}s$. The reason behind this result is that expected profit levels include the probability $1/(1-p_t)$ of reaching each time t, while the agent's incentives depend on the *change* in this probability as a function of effort $p_t/(1-p_t)$. Therefore, suppose that at the optimal contract the agent shirks on a last interval $[t_3, T]$. It would then be beneficial (for incentives) to backload all wages from t_3 on, and pay them at the deadline (keeping the ex-ante profit constraint binding).

In the limit case $T \to \infty$, the planner's solution coincides with the agent's best response to a wage of $w_t \equiv 0$. But the firm could approach the planner's payoff by promising the agent a lump sum payment arbitrarily far in the future (and flow wages equal to marginal product thereafter). This would "count" almost as $w_t = 0$ in the agent's incentives, and induce the efficient effort level. The lump sum payment would then be essentially equal to $p_0/(1-p_0)$.

We now go over a conjectured solution.

For a short enough deadline, and high initial beliefs p_0, the agent will work throughout even if paid his marginal product. For these deadlines, the commitment and the non commitment solutions achieve the same payoff. However, for longer deadlines, the agent might shirk at the beginning if paid his marginal product.

Consider then following scheme: pay a wage of zero at the beginning, $w = v - \alpha\lambda$ in an intermediate phase, and a lump-sum L at the end. The agent exerts maximal effort throughout, therefore $x_T = x_0 + (\lambda + \bar{u})T$. The conditions that pin down this solution are given by indifference at T and by zero profits at $t = 0$.

$$\begin{aligned} \phi_T = (k - \alpha - L) e^{-x_T} - \alpha &= 0 \\ \int_0^{t_1} \left(1 + e^{-x_s}\right) \frac{\lambda + u}{1 + e^{x_s}} \mathrm{d}s + \int_{t_1}^T \left(1 + e^{-x_s}\right) \left(\frac{\lambda + \bar{u}}{1 + e^{x_s}} - v + \alpha\lambda\right) \mathrm{d}s - \left(1 + e^{-x_T}\right) L &= 0. \end{aligned}$$

As T increases, $t_1 \to 0$. Let T^* denote the longest deadline for which this solution induces full effort throughout. No other scheme can induce full effort: any further delay of payments from t to t' would induce the agent to shirk at t' (since he is now indifferent throughout). Anticipating payments would lead the agent to shirk at t (by property 2 above).

For $T > T^*$, we cannot obtain full effort throughout. The proposed solution is then the unique path characterized by full effort on $[0, t_1]$ with flat wages $w = v - \alpha\lambda$, zero wages and shirking on $[t_1, T]$, and a lump-sum severance pay L. Therefore, we have $x_T = x_t + \lambda(T - t) + \bar{u}(t_1 - t)$. The two further conditions that pin down this path are given by continuity of ϕ at t_1, and zero profits at $t = 0$.

$$\begin{aligned} \phi_{t_1} = (k - \alpha - L) e^{-x_T} - \alpha + \int_{t_1}^T e^{-x_s} (v - \alpha\lambda) \mathrm{d}s &= 0, \\ \int_0^{t_1} \left(1 + e^{-x_s}\right) \left(\frac{\lambda + \bar{u}}{1 + e^{x_s}} - v + \alpha\lambda\right) \mathrm{d}s + \int_{t_1}^T \left(1 + e^{-x_s}\right) \frac{\lambda}{1 + e^{x_s}} \mathrm{d}s - \left(1 + e^{-x_T}\right) L &= 0. \end{aligned}$$

It can be more useful to rewrite the three conditions in beliefs space. We then have

$$(k-\alpha-L)\,e^{-x_T}-\alpha+(v/\lambda-\alpha)\left(e^{-x_1}-e^{-x_T}\right) = 0 \quad (29)$$

$$e^{-x_0}-e^{-x_T}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0}-e^{-x_1}+x_1-x_0\right)-\left(1+e^{-x_T}\right)L = 0 \quad (30)$$

$$\frac{x_T-x_1}{\lambda}+\frac{x_1-x_0}{\lambda+\bar{u}}-T = 0 \quad (31)$$

for the three variables (x_1, x_T, L) as a function of x_0 and T. We can clearly solve the second one for L and the third for x_T and obtain one equation in one unknown for x_1.

Similarly, we can compute the agent's payoff under this scheme

$$\begin{aligned}
Y &= \int_0^{t_1}\left(1+e^{-x_s}\right)(v-\alpha\lambda-\alpha\bar{u}-v)\,\mathrm{d}s-\int_{t_1}^{T}\left(1+e^{-x_s}\right)v\mathrm{d}s+\left(1+e^{-x_T}\right)(L-k) \\
&= -\int_{x_0}^{x_1}\left(1+e^{-x}\right)\alpha\mathrm{d}x-\int_{x_1}^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x+\left(1+e^{-x_T}\right)(L-k) \\
&= -\int_{x_0}^{x_1}\left(\frac{v-\alpha\lambda}{\lambda+\bar{u}}+e^{-x}\left(\alpha+\frac{v-\alpha\lambda}{\lambda+\bar{u}}-1\right)\right)\mathrm{d}x-\int_{x_1}^{x_T}\left(\frac{v}{\lambda}+e^{-x}\frac{v-\lambda}{\lambda}\right)\mathrm{d}x-\left(1+e^{-x_T}\right)k \\
&= -\int_{x_0}^{x_1}\left(\frac{v-\alpha\lambda}{\lambda+\bar{u}}+e^{-x}\left(\frac{v+\bar{u}\alpha}{\bar{u}+\lambda}-1\right)\right)\mathrm{d}x-\int_{x_1}^{x_T}\left(\frac{v}{\lambda}+e^{-x}\frac{v-\lambda}{\lambda}\right)\mathrm{d}x-\left(1+e^{-x_T}\right)k
\end{aligned}$$

and express it as a function of the initial belief x:

$$J(x_0,T)=-\alpha\left(e^{-x}-e^{-x_2}+x_2-x\right)-\frac{v}{\lambda}\left(e^{-x_2}-e^{-x_T}+x_T-x_2\right)+\left(1+e^{-x_T}\right)(L-k),$$

where (L, x_2, x_T) are all functions of (x, T). For the initial contract, for $x < x_1$, write

$$Y(x)=-\alpha\left(e^{-x}-e^{-x_1}+x_1-x\right)-\frac{v}{\lambda}\left(e^{-x_1}-e^{-X_T}+X_T-x_1\right)+\left(1+e^{-X_T}\right)(L_0-k).$$

It can be more useful to write the continuation payoff as

$$Y=\int_t^{t_1}\left(1+e^{-x_s}\right)(v-\alpha\lambda-\alpha\bar{u}-v)\,\mathrm{d}s-\int_{t_1}^{T}\left(1+e^{-x_s}\right)v\mathrm{d}s+\left(1+e^{-x_T}\right)(L-k).$$

To construct the best competing offer under this scheme, we need to solve the system (38)-(31) again, and impose $J(x)=Y\left(x,T-\frac{x-x_0}{\lambda+\bar{u}}\right)$ when $x<x_1$ and $J(x)=Y\left(x,T-\frac{x_1-x_0}{\lambda+\bar{u}}-\frac{x-x_1}{\lambda}\right)$ when $x \geq x_1$. A necessary condition for this to make sense is

$$\frac{\lambda+\bar{u}}{1+e^{x}} \geq v-\alpha\lambda$$

for all x such that maximal effort is exerted, so that offering the agent the value $J(x)$ does not yield positive continuation profits to the firm.

D.1.3 Competing Contracts

We assume $\bar{u}$ is high enough, relative to T. In particular, let

$$\frac{\lambda+\bar{u}}{1+e^{x_t}} > v-\alpha\lambda$$

for all t. This means paying $v-\alpha\lambda$ during full effort contributes positively to the lump sum at the end. The assumption can be probably weakened to all t such that effort is exerted, or something similar.

Consider a full-zero competing contract offered when the agent's belief is x, and denote by $x_2(x)$ the new switching belief. We obtain

$$J(x)=\int_x^{x_2}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^x}-v-\alpha\bar{u}\right)\mathrm{d}x+\int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda}\left(\frac{\lambda}{1+e^x}-v\right)\mathrm{d}x-\left(1+e^{-x_T}\right)k,$$

with

$$\begin{aligned}x_T(x_2) &= x+\lambda\left(T-\frac{x-x_0}{\lambda+\bar{u}}\right)+\bar{u}\frac{x_2-x}{\lambda+\bar{u}}\\ &= x_0+\lambda T+\bar{u}\frac{x_2-x_0}{\lambda+\bar{u}}.\end{aligned}$$

We compare this to the continuation value under the original contract (here the timing of payments matters).

Case 1: the original contract induces full effort at x.

$$\begin{aligned}Y(x) &= \int_0^{t_1}\left(1+e^{-x_s}\right)(v-\alpha\lambda-\alpha\bar{u}-v)\,\mathrm{d}s-\int_{t_1}^{T}\left(1+e^{-x_s}\right)v\mathrm{d}s+\left(1+e^{-x_T}\right)(L-k)\\ &= -\int_x^{x_1}\left(1+e^{-x}\right)\alpha\mathrm{d}x-\int_{x_1}^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x+\left(1+e^{-x_T}\right)(L_0-k),\end{aligned}$$

where (x_1,x_T,L_0) are the switching and terminal beliefs, and the lump sum, under the original contract. We then evaluate the difference in continuation payoffs:

$$\frac{\mathrm{d}\left(Y(x)-J(x)\right)}{\mathrm{d}x}=\alpha\left(1+e^{-x}\right)+e^{-x}-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right)-\frac{\mathrm{d}J}{\mathrm{d}x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x}.$$

Note that because $\mathrm{d}J/\mathrm{d}x_2>0$ and because, by assumption, we have $(\lambda+\bar{u})/(1+e^x)>v-\alpha\lambda$, a sufficient condition for the difference to increase is $\mathrm{d}x_2/\mathrm{d}x<0$. Furthermore, notice that the social surplus under a competing contract evolves according to:

$$\begin{aligned}\frac{\mathrm{d}J}{\mathrm{d}x_2} &= -\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x_2}\right)+\frac{v}{\lambda}\left(1+e^{-x_2}\right)+\frac{\bar{u}}{\lambda+\bar{u}}\left(-\frac{v}{\lambda}\left(1+e^{-x_T}\right)+(1+k)\,e^{-x_T}\right)\\ &= \frac{\bar{u}}{\lambda+\bar{u}}\left(\left(1+k-\frac{v}{\lambda}\right)e^{-x_T}-\frac{v}{\lambda}+\left(\frac{v}{\lambda}-\alpha\right)\left(1+e^{-x_2}\right)\right).\end{aligned}\tag{32}$$

Using equation (39), and substituting into (38), we also have an expression characterizing the switching x_2 as a function of x and $x_T(x_2)$ only:

$$\begin{aligned}&(k-v/\lambda)\left(1+e^{-x_T}\right)+\left((v/\lambda-\alpha)\,e^{-x_2}-\alpha\right)\left(1+e^{x_T}\right)+e^{-x_T}\\ +\;&\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(-e^{-x_2}+x_2\right)-e^{-x}+\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x}-x\right)=0.\end{aligned}\tag{33}$$

Totally differentiating with respect to x yields

$$\frac{\mathrm{d}x_2}{\mathrm{d}x}=\frac{e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)}{\frac{\bar{u}}{\lambda+\bar{u}}e^{-x_T}\left(1+k-v/\lambda\right)+(v/\lambda-\alpha)\frac{\lambda}{\lambda+\bar{u}}e^{x_T}e^{-x_2}+\alpha\frac{\bar{u}}{\lambda+\bar{u}}e^{x_T}-\left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}+(v/\lambda-\alpha)\,e^{-x_2}}.\tag{34}$$

Therefore

$$\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x}=\left(e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)\right)\left(1-\frac{\mathrm{d}J}{\mathrm{d}x_{2}}\frac{\mathrm{d}x_{2}}{\mathrm{d}x}\right),$$

and computing the difference between (32) and the denominator of (34) we obtain

$$\frac{\left(e^{-x_{2}}e^{x_{T}}-1\right)\left(v-\alpha\lambda\right)+\alpha\bar{u}\left(1+e^{x_{T}}\right)}{\bar{u}+\lambda}>0. \tag{35}$$

Case 2: the original contract induces no effort at x. If there is no effort to be exerted under the competing contract, the comparison is immediate. Assuming there is still effort to be exerted under the competing contract, we have

$$\begin{aligned}Y\left(x\right) &= -\int_{x}^{x_{T}}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x+\left(1+e^{-x_{T}}\right)\left(L-k\right),\\ J\left(x\right) &= \int_{x}^{x_{2}}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}}-v-\alpha\bar{u}\right)\mathrm{d}x+\int_{x_{2}}^{x_{T}}\frac{1+e^{-x}}{\lambda}\left(\frac{\lambda}{1+e^{x}}-v\right)\mathrm{d}x-\left(1+e^{-x_{T}}\right)k,\end{aligned}$$

with

$$\begin{aligned}x_{T}\left(x,x_{2}\right) &= x+\lambda\left(T-\frac{x_{1}-x_{0}}{\lambda+\bar{u}}-\frac{x-x_{1}}{\lambda}\right)+\bar{u}\frac{x_{2}-x}{\lambda+\bar{u}}\\ &= \lambda T+\frac{\bar{u}x_{1}+\lambda x_{0}}{\lambda+\bar{u}}+\bar{u}\frac{x_{2}-x}{\lambda+\bar{u}}.\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x} &= \frac{v}{\lambda}\left(1+e^{-x}\right)+e^{-x}-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right)\\ &+ \frac{\bar{u}}{\lambda+\bar{u}}\left(e^{-x_{T}}\left(1+k\right)-\frac{v}{\lambda}\left(1+e^{-x_{T}}\right)\right)-\frac{\mathrm{d}J}{\mathrm{d}x_{2}}\frac{\mathrm{d}x_{2}}{\mathrm{d}x}.\end{aligned}$$

Furthermore, notice that $\mathrm{d}J/\mathrm{d}x_2$ is still given by (32). Therefore we can write

$$\begin{aligned}\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x} &= \left(\frac{v}{\lambda}-\alpha\right)\left(1+e^{-x}\right)+e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)\\ &+ \frac{\bar{u}}{\lambda+\bar{u}}\left(e^{-x_{T}}\left(1+k\right)-\frac{v}{\lambda}\left(1+e^{-x_{T}}\right)\right)-\frac{\mathrm{d}J}{\mathrm{d}x_{2}}\frac{\mathrm{d}x_{2}}{\mathrm{d}x},\end{aligned}$$

and with the notation $J'\left(x\right):=\left[\mathrm{d}J\left(x_{2}\left(x\right)\right)/\mathrm{d}x_{2}\right]_{x_{2}=x}$, we obtain

$$\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x}=e^{-x}+J'\left(x\right)-J'\left(x_{2}\right)\frac{\mathrm{d}x_{2}}{\mathrm{d}x}.$$

Since $\mathrm{d}J/\mathrm{d}x_2$ is decreasing in x_2 it will be sufficient to show that

$$e^{-x}+J'\left(x_{2}\right)\left(1-\frac{\mathrm{d}x_{2}}{\mathrm{d}x}\right)>0.$$

Assume $\mathrm{d}x_2/\mathrm{d}x>1$ otherwise the result is immediate. Now consider equation (33) for $x_2\left(x\right)$, but keep in mind now x_T is given by (36). Differentiating with respect to x yields

$$\frac{\mathrm{d}x_{2}}{\mathrm{d}x}=\frac{e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)-\frac{\bar{u}}{\lambda+\bar{u}}\left(-e^{-x_{T}}\left(1+k-v/\lambda\right)+e^{x_{T}}\left(\left(v/\lambda-\alpha\right)e^{-x_{2}}-\alpha\right)\right)}{e^{-x_{2}}\left(v/\lambda-\alpha\right)\left(1+e^{x_{T}}\right)-\left(1+e^{-x_{2}}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}-\frac{\bar{u}}{\lambda+\bar{u}}\left(-e^{-x_{T}}\left(1+k-v/\lambda\right)+e^{x_{T}}\left(\left(v/\lambda-\alpha\right)e^{-x_{2}}-\alpha\right)\right)}$$

Notice that the denominator (and the derivative $J'(x_2)$) are unchanged from the previous case. Therefore, if $\mathrm{d}x_2/\mathrm{d}x > 1$, we can use the result in (35) to conclude

$$\begin{aligned}
& e^{-x} + \frac{\mathrm{d}J}{\mathrm{d}x_2}\left(1 - \frac{\mathrm{d}x_2}{\mathrm{d}x}\right) \\
> \; & e^{-x} - \left(e^{-x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right) - \left(e^{-x_2}\left(v/\lambda - \alpha\right)\left(1+e^{x_T}\right) - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\right) \\
= \; & \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right) + e^{-x_2}\left(v/\lambda-\alpha\right)\left(1+e^{x_T}\right) - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}} \\
> \; & \frac{v-\alpha\lambda}{\lambda+\bar{u}}e^{-x} + e^{-x_2}\left(\frac{v-\alpha\lambda}{\lambda}\left(1+e^{x_T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) > 0.
\end{aligned}$$

Case 3: the competing contract induces full effort throughout. Consider the scenario in which the original contract induces full effort at x and the competing contract full effort throughout the remaining time. We obtain

$$\begin{aligned}
Y(x) &= -\int_x^{x_1}\left(1+e^{-x}\right)\alpha \mathrm{d}x - \int_{x_1}^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)\left(L_0 - k\right) \\
J(x) &= \int_x^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)k,
\end{aligned}$$

with (for the competing contract)

$$x_T = x_0 + (\lambda+\bar{u})\,T.$$

Therefore

$$Y'(x) - J'(x) = \alpha\left(1+e^{-x}\right) + e^{-x} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right) > 0$$

as before. If conversely the original contract induces zero effort for the remaining time, we have

$$\begin{aligned}
Y(x) &= -\int_x^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)(L-k), \\
J(x) &= \int_x^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)k,
\end{aligned}$$

and

$$x_T = x_0 + (\lambda+\bar{u})\,T + \frac{\bar{u}}{\lambda}\left(x_1 - x\right).$$

Also remember it must be the case that

$$\phi_{x_T} := (k-\alpha)\,e^{-x_T} - \alpha > 0.$$

Therefore,

$$\begin{aligned}
Y'(x) - J'(x) &= \left(1+e^{-x}\right)\frac{v}{\lambda} + \frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right) + \frac{\bar{u}}{\lambda}\left(e^{-x_T} - (v+\alpha\bar{u})\frac{1+e^{-x_T}}{\lambda+\bar{u}} + ke^{-x_T}\right) \\
&> \left(1+e^{-x}\right)\frac{v}{\lambda} + \frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right) + \frac{\bar{u}}{\lambda}\left(e^{-x_T} - (v-\alpha\lambda)\frac{1+e^{-x_T}}{\lambda+\bar{u}}\right) \\
&= e^{-x} + \frac{e^{-x} - e^{-x_T}}{\lambda+\bar{u}}\frac{\bar{u}}{\lambda}(v-\alpha\lambda) + \frac{\bar{u}}{\lambda}e^{-x_T} > 0.
\end{aligned}$$

This ends this step of the proof.

D.1.4 Ruling out final OFO

Consider a final O-F-O phase. We must have non-positive profits at the beginning of the first O phase (let x_0 denote the belief at the beginning of O). In addition, if O-F-O is not the initial phase, we have $\phi_{x_0} = 0$. Conversely, since O-F-O is final, at the end of the second O phase, we have ϕ_T as given by transversality. We now hold both ϕ_0 and π_0 constant, and we consider shrinking the initial O phase by varying the belief x_1 at the end of the first O. Without loss, we consider O phases achieved through lump sums at the end.

Let M and L denote the lump sums, let x_2 denote the second switching point, and write the equations for all endogenous variables as follows:

$$\begin{aligned}
\phi_T - Le^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_2} - e^{-x_T}\right) &= 0 \\
-Me^{-x_1} + (v/\lambda - \alpha)\left(e^{-x_0} - e^{-x_1}\right) &= \phi_0 = 0 \\
e^{-x_0} - e^{-x_T} - M\left(1 + e^{-x_1}\right) - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_2} + x_2 - x_1\right) - L\left(1 + e^{-x_T}\right) &= \pi_0 \\
x_0 + \frac{x_2 - x_1}{\lambda + \bar{u}}\bar{u} + \lambda T &= x_T \\
(k - \alpha)\, e^{-x_T} - \alpha &= \phi_T
\end{aligned}$$

We then wish to show that x_T is decreasing in x_1, or equivalently that

$$\mathrm{d}x_2/\mathrm{d}x_1 < 1.$$

Therefore, solve for L and M and substitute into the first equation, letting $b := x_2 - x_1$. We obtain

$$\begin{aligned}
0 &= \phi_T + (v/\lambda - \alpha)\left(e^{-x_1 - b} - e^{-x_0 - \frac{b}{\lambda + \bar{u}}\bar{u} - \lambda T}\right) \\
&\quad - \frac{e^{-x_0} - \pi_0 - (v/\lambda - \alpha)\left(e^{-x_0} - e^{-x_1}\right)\left(1 + e^{x_1}\right) - e^{-x_0 - \frac{b}{\lambda + \bar{u}}\bar{u} - \lambda T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_1 - b} + b\right)}{1 + e^{x_0 + \frac{b}{\lambda + \bar{u}}\bar{u} + \lambda T}}.
\end{aligned}$$

Multiplying by $\left(1 + e^{x_0 + \frac{b}{\lambda + \bar{u}}\bar{u} + \lambda T}\right) e^{x_1}$ and collecting the terms in e^{x_1}, we obtain

$$\begin{aligned}
0 &= e^{2x_1}(v/\lambda - \alpha)\, e^{-x_0} + (v/\lambda - \alpha)\left(e^{x_0 - \frac{\lambda}{\lambda + \bar{u}}b + \lambda T} + \bar{u}\frac{e^{-b} - 1}{\lambda + \bar{u}}\right) \\
&\quad + e^{x_1}\left(\begin{array}{c} (k - v/\lambda + 1)\left(1 + e^{-x_0 - \frac{b}{\lambda + \bar{u}}\bar{u} - \lambda T}\right) - \alpha e^{x_0 + \frac{b}{\lambda + \bar{u}}\bar{u} + \lambda T} \\ - \left((1 + \alpha - v/\lambda)\, e^{-x_0} - \pi_0 + 1 + v/\lambda - \frac{v - \alpha\lambda}{\lambda + \bar{u}}b\right) \end{array}\right). \qquad (37)
\end{aligned}$$

For $\bar{u}$ large enough (see Section D.1.8), this expression is decreasing in b. Furthermore, it is quadratic in e^{x_1} with a positive coefficient on e^{2x_1}. Therefore

$$\frac{\mathrm{d}b}{\mathrm{d}x_1} = -\frac{\mathrm{d}\,[37]\,/\mathrm{d}x_1}{\mathrm{d}\,[37]\,/\mathrm{d}b}$$

has the same sign as $\mathrm{d}\left[37\right]/\mathrm{d}x_1$, computed at the relevant root. Let

$$\begin{aligned}
A &= (v/\lambda-\alpha)\,e^{-x_0},\\
B &= (k-v/\lambda+1)\left(1+e^{-x_0-\frac{b}{\lambda+\bar{u}}\bar{u}-\lambda T}\right)-\alpha e^{x_0+\frac{b}{\lambda+\bar{u}}\bar{u}+\lambda T}\\
&- \left((1+\alpha-v/\lambda)\,e^{-x_0}-\pi_0+1+v/\lambda-\frac{v-\alpha\lambda}{\lambda+\bar{u}}b\right),\\
C &= (v/\lambda-\alpha)\left(e^{-b}e^{x_0+\frac{b}{\lambda+\bar{u}}\bar{u}+\lambda T}+\bar{u}\frac{e^{-b}-1}{\lambda+\bar{u}}\right),
\end{aligned}$$

and consider the two roots

$$e^{x_1}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

Because $A>0$, if the relevant solution to (37) is the left root, then $\mathrm{d}\left[37\right]/\mathrm{d}x_1<0$ (which is the desired result here).

Now consider profits at the beginning of the full effort phase.

$$\begin{aligned}
\pi_1 &= e^{-x_1}-e^{-x_T}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_1}-e^{-x_1-b}+b\right)-L\left(1+e^{-x_T}\right)\\
&= e^{-x_1}-\left(e^{-x_0}-M\left(1+e^{-x_1}\right)\right)\\
&= e^{-x_1}-e^{-x_0}+(1+e^{x_1})\,(v/\lambda-\alpha)\left(e^{-x_0}-e^{-x_1}\right)\\
&= \left(e^{-x_0}-e^{-x_1}\right)\frac{v-\lambda+ve^{x_1}-\alpha\lambda-\alpha\lambda e^{x_1}}{\lambda}\\
&\propto (e^{x_1}+1)\,(v-\alpha\lambda)-\lambda.
\end{aligned}$$

Note that profits are increasing in x_1 (actually, their sign). Impose the solution $e^{x_1}=\frac{-B+\sqrt{B^2-4AC}}{2A}$, and find

$$\begin{aligned}
\pi_1 &= \left(\frac{-B+\sqrt{B^2-4AC}}{2A}+1\right)(v-\alpha\lambda)-\lambda\\
&> \left(-\frac{B}{2A}+1\right)(v-\alpha\lambda)-\lambda\\
&\propto 2\,(v/\lambda-\alpha-1)\,e^{-x_0}-(k-v/\lambda+1)\left(1+e^{-x_0-\frac{b}{\lambda+\bar{u}}\bar{u}-\lambda T}\right)\\
&\quad+\alpha e^{x_0+\frac{b}{\lambda+\bar{u}}\bar{u}+\lambda T}+(1+\alpha-v/\lambda)\,e^{-x_0}-\pi_0+1+v/\lambda-\frac{v-\alpha\lambda}{\lambda+\bar{u}}b.
\end{aligned}$$

Since $b\geq 0$ we have

$$\pi_1>2e^{-x_0}\,(v/\lambda-\alpha-1)+\alpha e^{x_0}+(1+\alpha-v/\lambda)\,e^{-x_0}+1+v/\lambda-(k-v/\lambda+1)\left(1+e^{-x_0}\right)-\frac{v-\alpha\lambda}{\lambda+\bar{u}}b,$$

and for $\bar{u}$ large enough we can ignore the last term and obtain

$$\pi_1>(e^{x_0}+1)\,(v-k\lambda)+ve^{x_0}-\lambda+\alpha\lambda e^{2x_0}.$$

We can now find conditions for this expression to be positive (such as v or x_0 large enough), which leaves the lower root as the only possibility for having $\pi_1<0$. But maybe we can find weaker conditions too?

Whatever the condition, if we select the lower root, then $\mathrm{d}\left[37\right]/\mathrm{d}x_1<0$ and $\mathrm{d}x_T/\mathrm{d}x_1<0$.

D.1.5 Ruling out initial OFO

Suppose the optimal contract induced a single O-F-O phase. Then we would have $\pi_0 = 0$ and $\phi_0 \leq 0$. In addition, it would be optimal for the firm to keep profits as low as possible at the beginning of the F phase, so we also have $\pi_1 = 0$. We now shrink the initial O phase while holding profits constant and equal to zero at the beginning of the F phase. At the end of the second O phase, we again have the transversality condition.

We have the following equations for the endogenous variables x_2, x_T and L:

$$\begin{aligned}
\phi_T - Le^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_2} - e^{-x_T}\right) &= \phi_0 = 0 \\
e^{-x_1} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_2} + x_2 - x_1\right) - L\left(1 + e^{-x_T}\right) &= \pi_1 = 0 \\
x_0 + \frac{x_2 - x_1}{\lambda + \bar{u}}\bar{u} + \lambda T &= x_T \\
(k - \alpha) e^{-x_T} - \alpha &= \phi_T.
\end{aligned}$$

We then wish to show that x_T is decreasing in x_1, or

$$\mathrm{d}x_2/\mathrm{d}x_1 < 1.$$

Therefore, substitute the second and third into the first equation, and let $b := x_2 - x_1$. Collecting the terms with e^{-x_1}, we obtain

$$\begin{aligned}
0 &= \phi_T - \frac{e^{-x_1} - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_1} - e^{-x_1 - b} + b\right)}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} \\
&+ (v/\lambda - \alpha)\left(e^{-x_1 - b} - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\
&= e^{-x_1}\left((v/\lambda - \alpha) e^{-b} - \frac{1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1 - e^{-b}\right)}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\
&+ \phi_T + \left(\frac{e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} - (v/\lambda - \alpha)\right) e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T},
\end{aligned}$$

then solving for $e^{-x_1}(b)$ we get (just plug-in to verify)

$$\begin{aligned}
e^{-x_1} &= \frac{-\phi_T\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) + (v/\lambda - \alpha - 1) e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + v/\lambda - \alpha - \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{e^{-b}\left((v/\lambda - \alpha)\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \left(1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)} \\
&= \frac{\alpha e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} + (v/\lambda - k - 1) e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + v/\lambda - k + \alpha - \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{e^{-b}\left((v/\lambda - \alpha)\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \left(1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)}.
\end{aligned}$$

For $\bar{u}$ high enough, the expression is increasing in b. Therefore, $x_1'(b) < 0$ and again we have

$$x_2'(x_1) = 1 + b'(x_1) = 1 + \frac{1}{x_1'(b(x_1))} < 1,$$

which is the desired result.

D.1.6 An aside

Note that we could have tried another way

$$\begin{aligned}
\int_{x_2}^{x_T}\left(e^{-x}-\frac{v-\alpha\lambda}{\lambda}e^{-x}\left(1+e^{x_T}\right)\right)\mathrm{d}x+\int_{x_1}^{x_2}\left(e^{-x}-\frac{v-\alpha\lambda}{\lambda+u\left(x\right)}\left(1+e^{-x}\right)\right)\mathrm{d}x-\phi_T\left(1+e^{x_T}\right) &= \pi_0\\
x_0+\int_{x_1}^{x_2}\frac{u\left(x\right)}{\lambda+u\left(x\right)}\mathrm{d}x+\lambda T &= x_T.
\end{aligned}$$

Now differentiate with respect to x_1.

$$\begin{aligned}
\frac{\mathrm{d}x_2}{\mathrm{d}x_1} &= -\frac{\mathrm{d}\pi/\mathrm{d}x_1}{\mathrm{d}\pi/\mathrm{d}x_2}\\
&= -\frac{-\mathrm{d}\pi/\mathrm{d}x_T\frac{u}{\lambda+u}-\left(e^{-x_1}-\frac{v-\alpha\lambda}{\lambda+u}\left(1+e^{-x_1}\right)\right)}{\mathrm{d}\pi/\mathrm{d}x_T\frac{u}{\lambda+u}+\left(v/\lambda-\alpha\right)e^{-x_2}\left(1+e^{x_T}\right)-\frac{v-\alpha\lambda}{\lambda+u}\left(1+e^{-x_2}\right)};
\end{aligned}$$

now check the denominator

$$\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}x_T} &= e^{-x_T}-e^{x_T}\left(v/\lambda-\alpha\right)\left(e^{-x_1}-e^{-x_T}\right)-\left(1+e^{x_T}\right)e^{-x_T}\left(v/\lambda-\alpha\right)\\
&= e^{-x_T}-e^{x_T}\left(\left(v/\lambda-\alpha\right)\left(e^{-x_1}-e^{-x_T}\right)+\phi_T\right)-\left(1+e^{x_T}\right)e^{-x_T}\left(v/\lambda-\alpha\right)\\
&= e^{-x_T}-\phi_T e^{x_T}-\left(e^{-x_1+x_T}+e^{-x_T}\right)\frac{v-\alpha\lambda}{\lambda}-\frac{\mathrm{d}}{\mathrm{d}x_T}\left[\phi_T\left(1+e^{x_T}\right)\right].
\end{aligned}$$

Notice that $-\frac{\mathrm{d}}{\mathrm{d}x_T}\left[\phi_T\left(1+e^{x_T}\right)\right]>0$ if either ϕ_T is a non-positive constant or if it is given by the transversality condition $\phi_T=\left(k-\alpha\right)e^{-x_T}-\alpha$. We use this fact in the following section.

D.1.7 Mixing

Consider a F-O phase that generates π_0 and continuation ϕ_T. We have the following equations:

$$\begin{aligned}
\phi_T-Le^{-x_T}+\left(v/\lambda-\alpha\right)\left(e^{-x_2}-e^{-x_T}\right) &= 0\\
e^{-x_0}-e^{-x_T}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0}-e^{-x_2}+x_2-x_0\right)-L\left(1+e^{-x_T}\right) &= \pi_0\\
x_0+\frac{x_2-x_0}{\lambda+\bar{u}}\bar{u}+\lambda T &= x_T.
\end{aligned}$$

We ask whether we can improve the final x_T by mixing and generating the same revenue. We would then have

$$\begin{aligned}
\phi_T-Me^{-x_T}+\left(v/\lambda-\alpha\right)\left(e^{-x_1}-e^{-x_T}\right) &= 0\\
e^{-x_0}-e^{-x_T}-\int_{x_0}^{x_1}\frac{v-\alpha\lambda}{\lambda+u\left(x\right)}\left(1+e^{-x}\right)\mathrm{d}x-M\left(1+e^{-x_T}\right) &= \pi_0\\
x_0+\int_{x_0}^{x_1}\frac{u\left(x\right)}{\lambda+u\left(x\right)}\mathrm{d}x+\lambda T &= x_T.
\end{aligned}$$

or (let $\phi_T = 0$)

$$\int_{x_1}^{x_T} \left(e^{-x} - \frac{v-\alpha\lambda}{\lambda} e^{-x}\left(1+e^{x_T}\right)\right) \mathrm{d}x + \int_{x_0}^{x_1} \left(e^{-x} - \frac{v-\alpha\lambda}{\lambda+u(x)}\left(1+e^{-x}\right)\right) \mathrm{d}x = \pi_0$$
$$x_0 + \int_{x_0}^{x_1} \frac{u(x)}{\lambda+u(x)} \mathrm{d}x + \lambda T = x_T.$$

Now consider increasing u at x. We obtain

$$\frac{\mathrm{d}x_T}{\mathrm{d}u} = \frac{\lambda}{\left(\lambda+u(x)\right)^2} + \frac{u_1}{\lambda+u_1}\frac{\mathrm{d}x_1}{\mathrm{d}u},$$

with

$$\frac{\mathrm{d}x_1}{\mathrm{d}u} = -\frac{\mathrm{d}\pi/\mathrm{d}u}{\mathrm{d}\pi/\mathrm{d}x_1} = -\frac{\mathrm{d}\pi/\mathrm{d}x_T \frac{\lambda}{(\lambda+u(x))^2} + \frac{v-\alpha\lambda}{(\lambda+u(x))^2}\left(1+e^{-x}\right)}{\mathrm{d}\pi/\mathrm{d}x_T \frac{u_1}{\lambda+u_1} + (v/\lambda-\alpha)\, e^{-x_1}\left(1+e^{x_T}\right) - \frac{v-\alpha\lambda}{\lambda+u_1}\left(1+e^{-x_1}\right)},$$

therefore (we know from before that the denominator is positive),

$$\begin{aligned}
\frac{\mathrm{d}x_T}{\mathrm{d}u} &= \frac{\lambda}{\left(\lambda+u(x)\right)^2} - \frac{\mathrm{d}\pi/\mathrm{d}x_T \frac{\lambda}{(\lambda+u(x))^2} + \frac{v-\alpha\lambda}{(\lambda+u(x))^2}\left(1+e^{-x}\right)}{\mathrm{d}\pi/\mathrm{d}x_T \frac{u_1}{\lambda+u_1} + (v/\lambda-\alpha)\, e^{-x_1}\left(1+e^{x_T}\right) - \frac{v-\alpha\lambda}{\lambda+u_1}\left(1+e^{-x_1}\right)} \frac{u_1}{\lambda+u_1} \\
&\propto \frac{\lambda}{\left(\lambda+u(x)\right)^2}\left((v/\lambda-\alpha)\, e^{-x_1}\left(1+e^{x_T}\right) - \frac{v-\alpha\lambda}{\lambda+u_1}\left(1+e^{-x_1}\right)\right) - \frac{v-\alpha\lambda}{\left(\lambda+u(x)\right)^2}\left(1+e^{-x}\right)\frac{u_1}{\lambda+u_1} \\
&\propto \left(\frac{u_1}{\lambda+u_1} e^{-x_1} + e^{x_T-x_1} - \frac{\lambda}{\lambda+u_1}\right) - \left(1+e^{-x}\right)\frac{u_1}{\lambda+u_1},
\end{aligned}$$

so this is increasing in x, meaning it is optimal to ask the agent for zero (or maximal) effort. But if we ask for zero effort, we do so at the beginning of the phase, which we know is suboptimal. □

D.1.8 Bound on $\bar{u}$

We derive a lower bound on $\bar{u}$ that ensures

$$\frac{\lambda+\bar{u}}{1+e^x} \geq v - \alpha\lambda$$

over all beliefs x for which the agent exerts maximal effort. This clearly requires finding an upper bound on the range of such beliefs. Under the conjectured strategy, the switching belief and the lump sum payment are determined by the two equations

$$(k-\alpha-L)\, e^{-x_T} - \alpha + (v/\lambda-\alpha)\left(e^{-x_1} - e^{-x_T}\right) = 0 \tag{38}$$

$$e^{-x_0} - e^{-x_T} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right) - \left(1+e^{-x_T}\right) L = 0. \tag{39}$$

Now solve (39) for L, and substitute into (38). We obtain

$$\left(k + \frac{1}{1+e^{x_T}} - v/\lambda\right) e^{-x_T} - \frac{e^{-x_0} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right)}{1+e^{x_T}} - \alpha + (v/\lambda-\alpha)\, e^{-x_1}. \tag{40}$$

Notice that as we let $x_T \to \infty$, x_1 must approach

$$\bar{x}_1 := \ln\left(v/\alpha\lambda - 1\right).$$

Furthermore, we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_T} = \frac{\left(\left(v/\lambda - \alpha\right)e^{-x_1} - \alpha\right)e^{x_T} - \left(k + 1 - v/\lambda\right)e^{-x_T}}{\left(v/\lambda - \alpha\right)\left(e^{x_T - x_1} - \frac{\lambda}{\lambda + \bar{u}} + \frac{\bar{u}}{\lambda(\bar{u}+\lambda)}e^{-x_1}\right)},$$

whose numerator is clearly positive.

Assume first $v/\lambda \leq 1 + k$. Notice that $x_1 > \bar{x}_1$ implies $\mathrm{d}x_1/\mathrm{d}x_T < 0$. Therefore, if $x_1\left(x_T\right)$ ever exceeds $\bar{x}_1$, it does so for all lower x_T compatible with work-shirk solutions. Now consider the equation characterizing the lowest such x_T:

$$\int_{x_0}^{x_T}\left(1 + e^{-x}\right)\left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right)\mathrm{d}x - \left(1 + e^{-x_T}\right)\left(k - \alpha\left(1 + e^{x_T}\right)\right) = 0.$$

As $x_0 \to -\infty$ the integrand is positive and grows without bound. This implies the solution to this equation x_T must also diverge to $-\infty$. But since we must have $x_1 \leq x_T$, this contradicts x_1 lying above $\bar{x}_1$ for some x_T. It follows that $\bar{x}_1$ is a tight upper bound on x_1 and that a lower bound on $\bar{u}$ is given by

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1\right)v - \lambda.$$

Assume now $v/\lambda \geq 1 + k$. Consider equation (40). Notice that the numerator of the second term is

$$e^{-x_0} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right) > \int_{x_0}^{x_1}\left(1 + e^{-x}\right)\left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right)\mathrm{d}x > 0.$$

Therefore the sum of the first two terms is negative, and hence the last two terms must be positive, which implies $x_1 < \bar{x}_1$ in this case as well. Notice that $x_1 \leq \bar{x}_1$ implies $\mathrm{d}x_1/\mathrm{d}x_T > 0$.

D.2 Proofs for Subsection 5.2

Proof of Lemma 5.2. Suppose that the equilibrium effort is zero on some open set Ω. Consider the sets $\Omega_{t'} = \{(x, s) : s \in (t', T]\}$ such that the trajectory starting at (x, s) intersects Ω. Suppose that u is not identically zero on Ω_0 and let $\tau = \inf\{t' : \ u = 0 \ \text{ on } \Omega_{t'}\}$. That is, for all $t' < \tau$, there exists $(x, s) \in \Omega_{t'}$ such that $u\left(x, s\right) > 0$. Suppose first that we take $(x, \tau) \in \Omega_\tau$. According to the definition of τ and Ω_τ, there exists $(x_k, k) \in \Omega$ such that the trajectory starting at (x, τ) intersects Ω at (x_k, k) and along the path the effort is zero. We can write the payoff

$$V\left(x, \tau\right) = \int_x^{x_k}\frac{1 + e^{-s}}{1 + e^{-x}}\left(\frac{\lambda}{1 + e^s} - v\right)\frac{1}{\lambda}\mathrm{d}s + \frac{1 + e^{-x_k}}{1 + e^{-x}}V\left(x_k, k\right),$$

or, rearranging,

$$\left(1 + e^{-x}\right)V\left(x, \tau\right) = -\left(e^{-x_k} - e^{-x}\right)\left(1 - \frac{v}{\lambda}\right) - \frac{v}{\lambda}\left(x_k - x\right) + \left(1 + e^{-x_k}\right)V\left(x_k, k\right),$$

where $V(x_k, k)$ is differentiable. The Hamilton-Jacobi-Bellman equation (a function of (x,τ)) can be derived from

$$\begin{aligned} V(x,\tau) &= \frac{\lambda+\hat{u}}{1+e^x}\mathrm{d}t - v\mathrm{d}t \\ &+ \max_u \left[-\alpha u \mathrm{d}t + \left(1 - \frac{\lambda+u}{1+e^x}\mathrm{d}t + o(\mathrm{d}t)\right)(V(x,\tau) + V_x(x,\tau)(\lambda+u)\mathrm{d}t + V_t(x,\tau)\mathrm{d}t + o(\mathrm{d}t))\right], \end{aligned}$$

which gives, taking limits,

$$0 = \frac{\lambda+\hat{u}}{1+e^x} - v + \max_{u\in[0,\bar{u}]}\left[-\alpha u - \frac{\lambda+u}{1+e^x}V(x,\tau) + V_x(x,\tau)(\lambda+u) + V_t(x,\tau)\right].$$

Therefore, if $u(x,\tau) > 0$,

$$-\frac{V(x,\tau)}{1+e^x} - \alpha + V_x(x,\tau) \geq 0, \text{ or } \left(1+e^{-x}\right)V_x(x,\tau) - e^{-x}V(x,\tau) \geq \alpha\left(1+e^{-x}\right),$$

or finally,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) \geq 0.$$

Notice, however, by direct computation, that, because low effort is exerted from (x,τ) to (x_k,k), for all points (x_s,s) on this trajectory, $s\in(\tau,k)$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x_s}\right)V(x_s,s)\right] - \alpha\left(1+e^{-x_s}\right) = -e^{-x_s}\left(1+\alpha-\frac{v}{\lambda}\right) + \frac{v}{\lambda} - \alpha \leq 0,$$

so that, because $x < x_s$, and $1+\alpha-v/\lambda > 0$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) < 0,$$

a contradiction to $u(x,\tau) > 0$.

If instead $u(x,\tau) = 0$ for all $(x,\tau)\in\Omega_\tau$, then there exists $(x',t')\to(x,\tau)\in\Omega_\tau$, $u(x',t') > 0$. Because u is upper semi-continuous, for every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of (x,τ) such that $u < \varepsilon$ on $\mathcal{N}$. Hence

$$\lim_{(x',t')\to(x,\tau)}\frac{\partial}{\partial x}\left[\left(1+e^{x'}\right)V(x',t')\right] - \alpha\left(1+e^{x'}\right) = \frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) < 0,$$

a contradiction. □

Proof of Theorem 5.3. We start with (i). That is, we show that $u(x,t) = \bar{u}$ for $x < \underline{x}_t$ in all equilibria. We first define $\underline{x}$ as the solution to the differential equation

$$(\lambda(1+\alpha) - v + (\lambda+\bar{u})\alpha e^{\underline{x}(t)} + \bar{u} - ((1+k)(\lambda+\bar{u}) - (v+\alpha\bar{u}))e^{-(\lambda+\bar{u})(T-t)})\left(\frac{\underline{x}'(t)}{\lambda+\bar{u}} - 1\right) = -\bar{u}, \tag{41}$$

subject to $\underline{x}(T) = x^*$. This defines a strictly increasing function of slope larger than $\lambda + \bar{u}$, for all $t \in (T - t^*, T]$, with $\lim_{t \uparrow t^*} \underline{x}(T-t) = -\infty$.[24] Given some equilibrium, and an initial value (x_t, t), let $u(\tau; x_\tau)$ denote the value at time $\tau \geq t$ along the equilibrium trajectory. For all t, let

$$\tilde{x}(t) := \sup\{x_t : \forall \tau \geq t : u(\tau; x_t) = \bar{u} \text{ in all equilibria}\},$$

with $\tilde{x}(t) = -\infty$ if no such x_t exists. By definition the function $\tilde{x}$ is increasing (in fact, for all $\tau \geq t$, $\tilde{x}(\tau) \geq \tilde{x}(t) + (\lambda + \bar{u})(\tau - t)$), and so it is a.e. differentiable (set $\tilde{x}'(t) = +\infty$ if x jumps at t). Whenever finite, let $s(t) = \tilde{x}'(t) / (\tilde{x}'(t) - \lambda) > 0$. Note that, from the transversality condition, $\tilde{x}(T) = x^*$. In an abuse of notation, we also write $\tilde{x}$ for the set function $t \to [\lim_{t' \uparrow t} \tilde{x}(t'), \lim_{t' \downarrow t} \tilde{x}(t')]$.

We first argue that the incentives to exert high effort decrease in x (when varying the value x of an initial condition (x, t) for a trajectory along which effort is exerted throughout). Indeed, recall that high effort is exerted iff

$$\frac{\partial}{\partial x}\left(V(x,t)\left(1 + e^{-x}\right)\right) \geq \alpha\left(1 + e^{-x}\right). \tag{42}$$

The value $V^H(x,t)$ obtained from always exerting (and being paid for) high effort is given by

$$\begin{aligned}(1+e^{-x})V^H(x,t) &= \int_t^T (1 + e^{-x_s})\left[\frac{\lambda + \bar{u}}{1 + e^{x_s}} - v - \alpha\bar{u}\right] \mathrm{d}s - k(1 + e^{-x_T}) \\ &= \left(e^{-x} - e^{-x_T}\right)\left(1 - \frac{v + \alpha\bar{u}}{\lambda + \bar{u}}\right) - (T - t)(v + \alpha\bar{u}) - k(1 + e^{-x_T}) \end{aligned} \tag{43}$$

where $x_T = x + (\lambda + \bar{u})(T - t)$. Therefore, using (42), high effort is exerted if and only if

$$k - \left(1 + k - \frac{v + \alpha\bar{u}}{\lambda + \bar{u}}\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right) \geq \alpha\left(1 + e^{x}\right).$$

Note that the left-hand side is independent of x, while the right-hand side is increasing in x. Therefore, if high effort is exerted throughout from (x,t) onward, then it is also from (x', t) for all $x' < x$.

Fix an equilibrium and a state (x_0, t_0) such that $x_0 + (\lambda + \bar{u})(T - t_0) < x^*$. Note that the equilibrium trajectory must eventually intersect some state $(\tilde{x}_t, t)$. We start again from the formula for the payoff

$$\begin{aligned}\left(1 + e^{-x_0}\right)V(x_0, t_0) &= \int_{t_0}^t \left[e^{-x_s}(\lambda + u(x_s, s)) - \left(1 + e^{-x_s}\right)(v + \alpha u(x_s, s))\right] \mathrm{d}s \\ &\quad + \left(1 + e^{-\tilde{x}_t}\right)V^H(\tilde{x}_t, t).\end{aligned}$$

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[24]The differential equation for $\underline{x}$ can be implicitly solved, which yields

$$\begin{aligned}\ln\frac{k-\alpha}{\alpha} &= \left(\underline{x}_t + (\lambda + \bar{u})(T - t\ )\right) + \frac{\bar{u}}{\lambda(1+\alpha) + \bar{u} - v}\ln(k - \alpha)\bar{u}(\lambda + \bar{u}) \\ &\quad - \frac{\bar{u}}{\lambda(1+\alpha) + \bar{u} - v}\ln\left(\begin{array}{c} e^{(\lambda+\bar{u})(T-t\ )}(\lambda(1+\alpha) + \bar{u} - v)(\lambda(1+\alpha) - v + \alpha(\lambda+\bar{u})e^{\underline{x}_t}) \\ -(\lambda(1+\alpha) - v)(\lambda(1+\alpha) + \bar{u} - v + (k-\alpha)(\lambda + \bar{u})) \end{array}\right).\end{aligned}$$

Let $W(\tilde{x}_t) = V^H(\tilde{x}_t, t)$ (since $\tilde{x}$ is strictly increasing, it is well-defined). Differentiating with respect to x_0, and taking limits as $(x_0, t_0) \to (\tilde{x}_t, t)$, we obtain

$$\begin{aligned} & \lim_{(x_0,t_0)\to(\tilde{x}_t,t)} \frac{\partial\left(1+e^{-x_0}\right)V\left(x_0,t_0\right)}{\partial x_0} \\ = \quad & \left[e^{-\tilde{x}_t}\lambda - \left(1+e^{-\tilde{x}_t}\right)v\right]\frac{s\left(\tilde{x}_t\right)-1}{\lambda} + s\left(\tilde{x}_t\right)\left[W'\left(\tilde{x}_t\right)\left(1+e^{-\tilde{x}_t}\right) - W\left(\tilde{x}_t\right)e^{-\tilde{x}_t}\right]. \end{aligned}$$

If less than maximal effort can be sustained arbitrarily close to, but before the state $(\tilde{x}_t, t)$ is reached, it must be that this expression is no more than $\alpha\left(1+e^{-\tilde{x}_t}\right)$ in some equilibrium, by (42). Rearranging, this means that

$$\left(1 - W(x) + (1+e^x)\left(W'(x) - \frac{v}{\lambda}\right)\right)s(x) + \left(\frac{v}{\lambda} - \alpha\right)e^x \le 1 + \alpha - \frac{v}{\lambda},$$

for $x = \tilde{x}_t$. Given the explicit formula for W (see (43)), and since $s(\tilde{x}_t) = \tilde{x}'_t / (\tilde{x}'_t - \lambda)$, we can rearrange this to obtain an inequality for $\tilde{x}_t$. The derivative $\tilde{x}'_t$ is smallest, and thus the solution $\tilde{x}_t$ is highest, when this inequality binds for all t. The resulting ordinary differential equation is precisely (41).

Next, we turn to (ii). That is, we show that $u(x,t) = 0$ for $x > \bar{x}_t$ in all equilibria. We define $\bar{x}$ by

$$\bar{x}_t = \ln\left[k - \alpha + \left(\frac{v + \bar{u}\alpha}{\lambda + \bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right)\right] - \ln\alpha, \tag{44}$$

which is well-defined as long as $k - \alpha + \left(\frac{v+\bar{u}\alpha}{\lambda+\bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right) > 0$. This defines a minimum time $T - t^*$ mentioned above, which coincides with the asymptote of $\underline{x}$ (as can be seen from (41)). It is immediate to check that $\bar{x}$ is continuous and strictly increasing on $[T - t^*, T]$, with $\lim_{t\uparrow t^*} \bar{x}_{T-t} = -\infty$, $x_T = x^*$, and for all $t \in (T - t^*, T)$, $\bar{x}'_t > \lambda + \bar{u}$.

Let us define $W(x,t) = (1+e^{-x})V(x,t)$, and re-derive the HJB equation. The payoff can be written as

$$W(x,t) = \left[(\lambda + u(x,t))e^{-x} - \left(1+e^{-x}\right)(v + \alpha u)\right]\mathrm{d}t + W(x + \mathrm{d}x, t + \mathrm{d}t),$$

which gives

$$0 = (\lambda + u(x,t))e^{-x} - v\left(1+e^{-x}\right) + W_t(x,t) + \lambda W_x(x,t) + \max_{u\in[0,\bar{u}]}\left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right)u.$$

As we already know (see (42)), effort is maximum or minimum depending on $W_x(x,t) \lesseqgtr \alpha(1+e^{-x})$. Let us rewrite the previous equation as

$$\begin{aligned} & v - \alpha\lambda - W_t(x,t) \\ = \quad & ((1+\alpha)\lambda - v + u(x,t))e^{-x} + \lambda\left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right) + \left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right)^+\bar{u}. \end{aligned}$$

Given W_x, W_t is maximized when effort $u(x,t)$ is minimized: the lower $u(x,t)$, the higher $W_t(x,t)$, and hence the lower $W(x, t - \mathrm{d}t) = W(x,t) - W_t(x,t)\,\mathrm{d}t$. Note also that, along any equilibrium trajectory, no effort is never strictly optimal (by (iv)). Hence, $W_x(x,t) \ge \alpha(1+e^{-x})$, and therefore, again $u(x,t)$ (or $W(x, t-\mathrm{d}t)$) is minimized when $W_x(x,t) = \alpha(1+e^{-x})$: to minimize $u(x,t)$, while preserving incentives to exert effort, it is best to be indifferent whenever possible.

Hence, integrating over the equilibrium trajectory starting at (x, t),

$$\begin{aligned} & (v-\alpha\lambda)(T-t)+k\left(1+e^{-x_T}\right)+W(x,t) \\ = & \int_t^T u(x_s,s)\,e^{-x_s}\mathrm{d}s+\int_t^T\left[((1+\alpha)\lambda-v)\,e^{-x_s}+(\lambda+\bar{u})\left(W_x(x_s,s)-\alpha\left(1+e^{-x_s}\right)\right)^+\right]\mathrm{d}s. \end{aligned}$$

We shall construct an equilibrium in which $W_x(x_s,s)=\alpha(1+e^{-x_s})$ for all $x>\underline{x}_t$. Hence, this equilibrium minimizes

$$\int_t^T u(x_s,s)\,e^{-x_s}\mathrm{d}s,$$

along the trajectory, and since this is true from any point of the trajectory onward, it also minimizes $u(x_s,s)$, $s\in[t,T]$; the resulting $u(x,t)$ will be shown to be increasing in x, and equal to $\bar{u}$ at $\bar{x}_t$.

Let us construct this interior effort equilibrium. Integrating (42) over any domain with non-empty interior, we obtain that

$$(1+e^x)V(x,t)=e^x(\alpha x+c(t))-\alpha, \tag{45}$$

for some function $c(t)$. Because the trajectories starting at (x,t) must cross $\underline{x}$ (whose slope is larger than $\lambda+\bar{u}$), value matching must hold at the boundary, which means that

$$(1+e^{\underline{x}_t})V^H(\underline{x}_t,t)=e^{\underline{x}_t}(\alpha\underline{x}_t+c(t))-\alpha,$$

which gives $c(t)$ (for $t\geq T-t^*$). From (45), we then back out $V(x,t)$. The HJB equation then reduces to

$$v-\alpha\lambda=\frac{\lambda+u(x,t)}{1+e^x}+V_t(x,t),$$

which can now be solved for $u(x,t)$. That is, the effort is given by

$$\begin{aligned} \lambda+u(x,t) &= (1+e^x)(v-\alpha\lambda)-\frac{\partial}{\partial t}\left[(1+e^x)V(x,t)\right] \\ &= (1+e^x)(v-\alpha\lambda)-e^xc'(t). \end{aligned}$$

It follows from simple algebra (c' is detailed below) that $u(x,t)$ is increasing in x. Therefore, the upper end $\bar{x}_t$ cannot exceed the solution to

$$\lambda+\bar{u}=\left(1+e^{\bar{x}}\right)(v-\alpha\lambda)-e^{\bar{x}}c'(t),$$

and so we can solve for

$$e^{\bar{x}}=\frac{\lambda(1+\alpha)-v+\bar{u}}{v-\alpha\lambda-c'(t)},$$

Note that, from totally differentiating the equation that defines $c(t)$,

$$\begin{aligned} c'(t) &= \underline{x}'(t)\,e^{-\underline{x}(t)}\left[(W'(\underline{x}(t))-\alpha)\left(e^{\underline{x}(t)}+1\right)-W(\underline{x}(t))\right] \\ &= v-\alpha\lambda+e^{-\underline{x}(t)}(v-(1+\alpha)\lambda), \end{aligned}$$

where we recall that $\underline{x}$ is the lower boundary below which effort must be maximal, and $W(\underline{x}) = V^H(\underline{x}_t, t)$. This gives

$$e^{\bar{x}} = e^{\underline{x}} \frac{\lambda(1+\alpha) - v + \bar{u}}{\lambda(1+\alpha) - v}, \text{ or } e^{\underline{x}} = \frac{\lambda(1+\alpha) - v}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}.$$

Because (41) is a differential equation characterizing $\underline{x}$, we may substitute for $\bar{x}$ from the last equation to obtain a differential equation characterizing $\bar{x}$, namely

$$\begin{aligned} & \frac{\bar{u}}{1 - \frac{\bar{x}'(t)}{\lambda+\bar{u}}} + ((1+k)(\lambda+\bar{u}) - (v+\alpha\bar{u}))\, e^{-(\lambda+\bar{u})(T-t)} \\ = \quad & \lambda(1+\alpha) + \bar{u} - v + \frac{\alpha(\lambda+\bar{u})(\lambda(1+\alpha)-v)}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}, \end{aligned}$$

with boundary condition $\bar{x}(T) = x^*$. It is simplest to plug in the formula given by (44) and verify that it is indeed the solution of this differential equation.

Finally, we prove (iii). The same procedure applies to both, so let us consider $\bar{\sigma}$, the strategy that exerts high effort as long as $x < \bar{x}_t$, (and no effort above). We shall do so by "verification." Given our closed-form expression for $V^H(x,t)$ (see (43)), we immediately verify that it satisfies the (42) constraint for all $x \leq \bar{x}_t$ (remarkably, $\bar{x}_t$ is *precisely* the boundary at which the constraint binds; it is strictly satisfied at $\underline{x}_t$, when considering $\underline{\sigma}$). Because this function $V^H(x,t)$ is differentiable in the set $\{(x,t) : x < \bar{x}_t\}$, and satisfies the HJB equation, as well as the boundary condition $V^H(x,T) = 0$, it is a solution to the optimal control problem in this region (remember that the set $\{(x,t) : x < \bar{x}_t\}$ cannot be left under any feasible strategy, so that no further boundary condition needs to be verified). We can now consider the optimal control problem with exit region $\Omega := \{(x,t) : x = \bar{x}_t\} \cup \{(x,t) : t = T\}$ and salvage value $V^H(\bar{x}_t, t)$ or 0, depending on the exit point. Again, the strategy of exerting no effort satisfies the HJB equation, gives a differentiable value on $\mathbb{R} \times [0,T] \setminus \Omega$, and satisfies the boundary conditions. Therefore, it is a solution to the optimal control problem. ☐

Proof of Lemma 5.4 The results can be obtained directly by differentiating expression (44) for the frontier $\bar{x}(t)$. ☐

Proof of Lemma 5.5 (1.) The equation defining the full effort frontier in the unobservable case $x_2(t)$ is given by

$$(k-\alpha)\, e^{-x_2 - (\lambda+u)(T-t)} - \alpha - \int_{x_2}^{x_2+(\lambda+u)(T-t)} e^{-x} \left(\frac{1}{1+e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) \mathrm{d}x. \tag{46}$$

Plug the expression for $\bar{x}(t)$ given by (44) into (46) and notice that (46) cannot be equal to zero unless $\bar{x}(t) = x^*$ and $t = T$, or $\bar{x}(t) \to -\infty$. Therefore, the two frontiers cannot cross before the deadline T, but they have the same vertical asymptote.

Now suppose that $\phi'(x^* \mid \bar{u}) > 0$ so that the frontier $x_2(t)$ goes through (T, x^*). Consider the slopes of $x_2(t)$ and $\bar{x}(t)$ evaluated at (T, x^*). We obtain

$$[\bar{x}'(t) - x_2'(t)]_{t=T} \propto (\bar{u} + \lambda)(k - \alpha) > 0,$$

so the unobservable frontier lies above the observable one for all t.

Next, suppose $\phi'(x^* \mid \bar{u}) < 0$, so there is no mixing at x^* and the frontier $x_2(t)$ does not go through (T, x^*). In this case, we still know the two cannot cross, and we also know a point on $x_2(t)$ is the pre-image of (T, x^*) under full effort. Since we also know the slope $\bar{x}'(t) > \lambda + \bar{u}$, we again conclude that the unobservable frontier $x_2(t)$ lies above $\bar{x}(t)$.

Finally, consider the equation defining the no effort frontier $x_3(t)$,

$$(k-\alpha)\, e^{-x_3-\lambda(T-t)} - \alpha - \int_{x_3}^{x_3+\lambda(T-t)} e^{-x}\left(\frac{1}{1+e^{x}} - \frac{v-\alpha\lambda}{\lambda}\right) \mathrm{d}x = 0. \tag{47}$$

Totally differentiating with respect to t shows that $x_3'(t) < \lambda$ (might be negative). Therefore, the no effort region does not intersect the full effort region defined by $\bar{x}(t)$ in the observable case.

(2.) To compare the effort regions in the unobservable case and the full effort region in the social optimum, consider the planner's frontier $x_P(t)$, which is given by

$$x_P(t) = \ln\Big((1+k-v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\Big) - \ln \alpha.$$

The slope of the planner's frontier is given by

$$x_P'(t) = \lambda \frac{(1+k-v/\lambda)e^{-\lambda(T-t)}}{(1+k-v/\lambda)e^{-\lambda(T-t)} + v/\lambda - \alpha} \in [0, \lambda].$$

In the equilibrium with unobservable effort, all effort ceases above the frontier $x_3(t)$ defined in (47) above, which has the following slope

$$x_3'(t) = \lambda \frac{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1} + k - v/\lambda\right) e^{-\lambda(T-t)}}{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1} + k - v/\lambda\right) e^{-\lambda(T-t)} + v/\lambda - \alpha - (1+e^{x_3})^{-1}}.$$

We also know $x_3(T) = x^*$ and $x_P(T) = \ln((1+k-\alpha)/\alpha) > x^*$. Now suppose towards a contradiction that the two frontiers crossed at a point (t, x). Plugging in the expression for $x_P(t)$ in both slopes, we obtain

$$x_3'(t) = \left(1 + \frac{v/\lambda - \alpha - s(t)}{(1+k-v/\lambda+(1-s(t)))e^{-\lambda(T-t)}}\right)^{-1} > \left(1 + \frac{v/\lambda - \alpha}{(1+k-v/\lambda)e^{-\lambda(T-t)}}\right)^{-1} = x_P'(t),$$

with

$$s(t) = 1/\left(1 + e^{x_P(t)}\right) \in [0,1],$$

meaning the unobservable frontier would have to cross from below, a contradiction. □

D.3 Proofs for Subsection 5.3

This subsection starts by proving Lemma 5.6 in several steps.

If the agent is indifferent between continuing and stopping, then the flow expected payoff must be zero. His overall equilibrium payoff is given by

$$\int_0^T \left(e^{-x_t}\left((\lambda+u_t)-v\right)-\left(1+e^{-x_t}\right)c\left(u_t\right)\right)\mathrm{d}t-ke^{-x_T},$$

[Recall that, to analyze deadlines, we must take into account that v is premultiplied by the belief p_t in the original problem, see footnote 6.] and when he stops, we also know

$$ke^{-x_T}=\left(1+e^{-x_T}\right)c'\left(u_T\right).$$

Therefore, the terminal belief x_T must satisfy the following equation

$$(1+k)\frac{\lambda+u_T}{1+e^{x_T}}-\frac{v}{1+e^{x_T}}-c\left(u_T\right)=0,$$

where u_T is given above. Now, remember the boundary-value problem

$$c''\left(u\right)\left(1+e^{x}\right)u'=\left(\lambda+u\right)c'\left(u\right)-c\left(u\right)+\frac{\lambda+u_t}{1+e^{x_t}}-v.$$

We ask how effort behaves at the quitting belief. We have

$$\begin{aligned}
u'_T &\propto (\lambda+u)\,c'(u)+\frac{\lambda+u_t}{1+e^{x_t}}-c(u)-v\\
&= (\lambda+u)\,c'(u)-k\frac{\lambda+u_T}{1+e^{x_T}}+\frac{v}{1+e^{x_T}}-v\\
&= -\frac{e^{x_T}}{1+e^{x_T}}v<0.
\end{aligned}$$

This means wages and effort are decreasing at the deadline. Finally, compare the slope of the trajectory with the transversality curve at the stopping point.

Lemma D.1 *The equilibrium effort trajectory hits transversality from above if and only if* $v\geq k$.

Indeed, we have

$$\begin{aligned}
&-\frac{\frac{e^{x_T}}{1+e^{x_T}}v}{c''(u)\left(1+e^{x}\right)}+\frac{c'(u)\,e^{x}}{\left(1+e^{x}\right)c''(u)}\\
\propto\;&-\frac{1}{1+e^{x_T}}v+c'=-v+k.
\end{aligned}$$

Therefore $\Delta\left(x_T^*\right)<0$ iff $v>k$.

Finally, note that the planner quits when

$$(1+k)\frac{\lambda+u_T}{1+e^{x_T}}-\frac{v}{1+e^{x_T}}-c\left(u_T\right)=0. \tag{48}$$

He exerts effort given by

$$(1+k)\,e^{-x_T}=\left(1+e^{-x_T}\right)c'\left(u_T\right),$$

which means at the planner's quitting belief, the agent would be working less, making the left-hand side of (48) negative, and hence his solution must involve a lower belief x_T.

The planner's effort slope at the deadline is given by

$$u'_T \propto (\lambda + u)\, c'(u) - c(u) - v.$$

Therefore,

$$\begin{aligned} u'_T &= \frac{(\lambda+u)}{1+e^{x}}(1+k) - c(u) - v \\ &= -\frac{e^{x_T}}{1+e^{x_T}} v < 0. \end{aligned}$$

So it is efficient to have decreasing effort at the deadline. Of course, the planner's effort is decreasing throughout.

D.3.1 Convex Cost, with Commitment

Suppose that the agent commits to a deadline. With commitment, the first-order condition is different, as the agent takes into account the effect of x_T on all previous effort levels. Therefore we have

$$\begin{aligned} V(x_T) &= \int_0^T \left(e^{-x_t}((\lambda+u_t)-v) - \left(1+e^{-x_t}\right)c(u_t)\right)\mathrm{d}t - ke^{-x_T} \\ &= \int_{x_0}^{x_T}\left(e^{-x}\left(1-\frac{v}{\lambda+u(x)}\right) - \frac{1+e^{-x}}{\lambda+u(x)}c(u(x))\right)\mathrm{d}x - ke^{-x_T} \\ &= -\int_{x_0}^{x_T}\frac{e^{-x}v+(1+e^{-x})\,c(u(x))}{\lambda+u(x)}\mathrm{d}x - (1+k)\,e^{-x_T}, \end{aligned}$$

and so

$$\begin{aligned} V'(x_T) &= e^{-x_T}\left(1+k-\frac{v}{\lambda+u(x_T)}\right) - \frac{1+e^{-x_T}}{\lambda+u(x_T)}c(u(x_T)) \\ &\quad -\int_{x_0}^{x_T}\frac{d}{dx_T}\frac{e^{-x}v+(1+e^{-x})\,c(u(x))}{\lambda+u(x)}\mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} u(x) &= u(x_T) - \int_x^{x_T} u'(z)\,\mathrm{d}z, \\ \mathrm{d}u/\mathrm{d}x_T &= \mathrm{d}u_T/\mathrm{d}x_T - u'(x_T) = -\Delta(x_T), \end{aligned}$$

and therefore

$$\begin{aligned} V'(x_T) &= e^{-x_T}\left(1+k-\frac{v+c(u(x_T))}{\lambda+u(x_T)}\right) - \frac{c(u(x_T))}{\lambda+u(x_T)} \\ &\quad -\Delta(x_T)\int_{x_0}^{x_T}\frac{e^{-x}v+(1+e^{-x})\left(c(u(x))-(\lambda+u(x))\,c'(u(x))\right)}{(\lambda+u(x))^2}\mathrm{d}x. \end{aligned}$$

Let us study the sign of the numerator, *i.e.*

$$\left(1+e^{-x}\right)\left(v+c\left(u\left(x\right)\right)-\left(\lambda+u\left(x\right)\right)c'\left(u\left(x\right)\right)\right)-v.$$

Its derivative with respect to x is

$$\begin{aligned} & -\left(1+e^{-x}\right)\left(\lambda+u\left(x\right)\right)u'\left(x\right)c''\left(u\left(x\right)\right)-e^{-x}\left(c\left(u\left(x\right)\right)-\left(\lambda+u\left(x\right)\right)c'\left(u\left(x\right)\right)\right)-ve^{-x} \\ = \; & e^{-x}\left(-\left(1+e^{x}\right)\left(\lambda+u\left(x\right)\right)c''\left(u\left(x\right)\right)u'\left(x\right)+\left(\lambda+u\left(x\right)\right)c'\left(u\left(x\right)\right)-c\left(u\left(x\right)\right)-v\right). \end{aligned}$$

Note that the slope of our trajectory is given by

$$\left(\lambda+u\right)c''\left(u\right)\left(1+e^{x}\right)u'\left(x\right)=\left(\lambda+u\right)c'\left(u\right)-c\left(u\right)+\frac{\lambda+u}{1+e^{x}}-v,$$

Thus the derivative of the numerator is given by

$$-\frac{\lambda+u\left(x\right)}{1+e^{x}}e^{-x}<0.$$

Its value at x_T is given by (actually, proportional to)

$$c\left(u\left(x_T\right)\right)-\left(\lambda+u\left(x_T\right)\right)c'\left(u\left(x_T\right)\right)+\frac{v}{1+e^{x_T}}. \tag{49}$$

Now let us focus on the value of the numerator at x_T as we change x_T. We obtain

$$\begin{aligned} & -\left(\lambda+u\left(x_T\right)\right)c''\left(u\left(x_T\right)\right)\frac{\mathrm{d}u_T}{\mathrm{d}x_T}-\frac{ve^{x_T}}{\left(1+e^{x_T}\right)^2} \\ = \; & \frac{c'\left(u\right)e^{x}}{\left(1+e^{x}\right)c''\left(u\right)}\left(\lambda+u\left(x_T\right)\right)c''\left(u\left(x_T\right)\right)-\frac{ve^{x_T}}{\left(1+e^{x_T}\right)^2} \\ = \; & c'\left(u\right)\left(\lambda+u\left(x_T\right)\right)-\frac{v}{1+e^{x_T}} \\ = \; & k\left(\lambda+u\left(x_T\right)\right)-v \\ \leq \; & k\left(\lambda+u_T\left(x_0\right)\right)-v. \end{aligned}$$

We then impose the following condition

$$k\left(\lambda+u_T\left(x_0\right)\right)-v\leq 0 \tag{50}$$

which corresponds to the condition on the cost function

$$\begin{aligned} v \quad & \geq \quad k\left(\lambda+\zeta\left(\frac{k}{1+e^{x_0}}\right)\right), \\ \text{where } \zeta\left(v\right) \quad & := \quad \left(c'\right)^{-1}\left(u\right). \end{aligned}$$

Therefore, under this sufficient condition, the derivative of (49) is negative, and hence this expression is positive for all x_T (notice that it goes to zero from above as $x_T\to\infty$).

Lemma D.2 *Under condition (50), the optimal deadline with commitment is longer than the one without commitment if and only if $v>k$.*

This is the intuitive result that says if the trajectory obtained by moving past the non commitment deadline lies above the previous one then keep going.

D.3.2 The Role of Observable Effort

We now adapt our results to the linear model, in order to assess the role of observable effort. As before, given an equilibrium deadline T, we fix the off-equilibrium beliefs to specify $\hat{u}_t = \bar{u}$ if $x_t < x^*$, and $\hat{u}_t = 0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $x = x_T$ were the terminal belief.

In the linear model, the agent's payoff can be written as a function of the terminal belief as

$$V(x_T) = \int_{x_0}^{x_T} \frac{1+e^{-x}}{\lambda + u(x)} \left(\frac{\lambda + u(x)}{1+e^x} - \alpha u(x) - \frac{v}{1+e^x} \right) \mathrm{d}x - ke^{-x_T},$$

and its derivative is given by

$$V'(x_T) = \left(1 + k - \frac{v}{\lambda + u(x_T)}\right) e^{-x_T} - \left(1 + e^{-x_T}\right) \alpha \frac{u(x_T)}{\lambda + u(x_T)}.$$

Consider the unobservable case first.

If the agent is exerting full effort at x_T, his payoff is increasing if and only if $x_T \leq \hat{x}$, which is defined as the unique solution to equation (51) below.

$$(\lambda + \bar{u})(1+k) - v - \alpha\bar{u}\left(1 + e^{x_T}\right) = 0. \tag{51}$$

Note that $\hat{x} \leq x^*$ if and only if

$$v \geq \lambda(1+k) + \bar{u},$$

and that $\hat{x} < -\infty$ if and only if

$$v < \lambda(1+k) + \bar{u}(1+k-\alpha).$$

If the agent does not work at x_T, we obtain

$$V'(x_T) = \left(1 + k - \frac{v}{\lambda}\right) e^{-x_T},$$

which means that, while not working, the agent will quit immediately or never, depending on the value of $v - \lambda(1+k)$.

To summarize, we have the following characterization in terms of the final beliefs x_T.

Lemma D.3 *If effort is unobservable, the optimal deadline in the absence of commitment is given by*

$$x_{T^*} = \begin{cases} \max\{x_0, \hat{x}\} & \text{if} \quad v > \lambda(1+k) + \bar{u}, \\ \max\{x_0, x^*\} & \text{if} \quad v \in [\lambda(1+k), \lambda(1+k) + \bar{u}], \\ \infty & \text{if} \quad v \leq \lambda(1+k). \end{cases}$$

Proof of Lemma D.3 Suppose the agent quits before x^*. Then he must quit while exerting maximal effort. This can only occur at $x_T = \hat{x} < x^*$. If $\hat{x} \geq x^*$ then the agent can quit at x^*. This requires $V'(x^*) < 0$, where the payoff is computed assuming the market expects zero effort (for $x > x^*$) and the agent does not work going forward. Therefore, if $v > \lambda(1+k)$ the agent quits immediately at x^*. For $v \leq \lambda(1+k)$, he never does. □

We now turn to the case of observable effort.

Lemma D.4 *If effort is observable, the optimal deadline in the absence of commitment is given by*

$$x_{T^*} = \begin{cases} \tilde{x} \leq \hat{x} & \text{if} \quad v > \lambda(1+k) + \bar{u}, \\ \in [x_0, x^*] & \text{if} \quad v \in [\lambda(1+k), \lambda(1+k) + \bar{u}], \\ \infty & \text{if} \quad v \leq \lambda(1+k). \end{cases}$$

Furthermore, $V(\hat{x}) < V(x_0)$ *for* v *sufficiently close to* $\lambda(1+k) + \bar{u}$.

Proof of Lemma D.4 If $v < \lambda(1+k)$ the market never pays a wage corresponding to zero effort, and the worker chooses the optimal deadline as in the unobservable case, *i.e.* he never quits.

If $v \in [\lambda(1+k), \lambda(1+k) + \bar{u}]$ and the agent were paid a wage corresponding to full effort, he would never stop as long as $x_T < x^*$. However, he would stop immediately if he were expected to quit at $x_T > x^*$ and were paid a wage corresponding to zero effort. We therefore construct a mixed strategy equilibrium in which the agent randomizes at each point in time between the following strategies: (a) exerting full effort and quitting, and (b) exerting zero effort and staying in the game. Denote by μ the instantaneous probability of quitting. The equations characterizing this equilibrium are given by agent indifference and zero profits, or

$$\begin{aligned} (w_t - p_t v - \alpha\bar{u})\,\mathrm{d}t - (1 - p_t(\lambda + \bar{u})\,\mathrm{d}t)\,k &= (w_t - p_t v)\,\mathrm{d}t + (1 - p_t\lambda\mathrm{d}t)\,V_{t+\mathrm{d}t} \\ \text{with } V_{t+\mathrm{d}t} &= (w_{t+\mathrm{d}t} - p_{t+\mathrm{d}t}v - \alpha\bar{u})\,\mathrm{d}t - (1 - p_{t+\mathrm{d}t}(\lambda + \bar{u})\,\mathrm{d}t)\,k, \\ \text{and } w_t &= p_t(\lambda + \mu_t\bar{u}). \end{aligned}$$

Deleting terms of order higher than dt (notice that terms of order 1 cancel), we obtain

$$w_t = p_t(v - \lambda k)$$

and hence

$$\mu \equiv \frac{v - \lambda(1+k)}{\bar{u}}.$$

Therefore, as v approaches $\lambda(1+k)$, μ vanishes. In particular, when $v = \lambda(1+k)$ the agent is indifferent between stopping immediately and never quitting, and our equilibrium places a mass point at $T = 0$. Conversely, as v approaches $\lambda(1+k) + \bar{u}$, μ goes to one, and the agent quits immediately. Finally, we need to verify that the agent's incentives to exert full and zero effort are satisfied. For all $x_T < x^*$ the transversality condition implies the agent exerts full effort at the deadline. Because the agent quits at rate μ, his strategy assigns positive probability to stopping at x^*. (Notice that its support cannot exceed x^*, as the agent would not exert effort at the quitting time then.) It follows that the agent is indifferent between stopping and at x^* he is indifferent between effort levels. We know from the analysis with a fixed deadline and observable effort that, when expected not to work, the agent would not exert effort. In this case, he is expected to exert a constant amount of work ($\mu\bar{u}$), independent of t and x. As the agent cannot alter is wage (beyond what he can do for a fixed deadline) he has strict incentives to shirk in this case too.

Finally, if $v \geq \lambda(1+k) + \bar{u}$, we have $\hat{x} < x^*$. The agent's payoff is increasing in the deadline as long as he is exerting full effort, receiving the maximum wage, and $x_T \leq \hat{x}$. Conversely, it is decreasing in x_T if the agent is receiving the minimum wage. We now construct a backward induction equilibrium. In the continuation game starting at x, for x close enough to $\hat{x}$, the agent is expected to quit at $\hat{x}$ and to exert effort throughout. For lower values of x, he is expected to quit at $\hat{x}$ and shirk initially. Denote by $\{x_j\}_{j=1,...J}$ a sequence of critical values. Let x_1 denote the belief x that leaves the agent indifferent between quitting and continuing until $\hat{x}$. This belief is well defined because as we decrease x the agent's payoff from continuing is first decreasing then increasing without bound. The agent is expected to quit at x_1. Therefore, he is expected to exert maximal effort for lower values of x, close enough to x_1. For even lower x he will shirk, then work, and then quit at x_1. Let x_2 denote the belief at which he is indifferent between quitting and continuing until x_1. Recursively define x_{j+1}. We can repeat this construction. Clearly, the value of x^0 determines the equilibrium terminal belief, and the resulting effort pattern. □

An immediate consequence of Lemmas D.3 and D.4 is that the total amount of effort exerted in equilibrium is weakly higher in the unobservable case. Thus, the comparison result carries over to the case of endogenous termination of the relationship. In the unobservable case, the effort patterns can then be traced back to x_0. In particular, when the agent quits at $\hat{x}$, the equilibrium phases are interior-full (because then $v > \lambda(1+\alpha)$); when he quits at x^* the phases are interior-full or always interior effort; and when he never quits, the equilibrium can have all four phases.

What about the social planner, in the non-commitment case? She follows exactly the same behavior, except she has a lower threshold $\hat{v}^P := \lambda(1+k) + \alpha\bar{u}$, above which the planner chooses an interior stopping point with full effort at the end. This follows from the fact that the planner can work at full speed for a larger set of parameters. Note that, when quitting is inefficient, the agent takes the efficient quitting decision (he never does).

D.3.3 Deadlines with commitment but competition

The agent's payoff may be written as (up to constants)

$$V(x_T) = -(1+k)e^{-x_T} - \int_{x_0}^{x_1} g(\bar{u}, x)\,dx - \int_{x_1}^{x_T} g(0,x)\,dx,$$

where

$$g(u,x) := e^{-x}\frac{v}{\lambda+u} + \left(1+e^{-x}\right)\frac{\alpha u}{\lambda+u}.$$

Its derivative is given by

$$\begin{aligned} V'(x_T) &= (1+k-v/\lambda)e^{-x_T} + \frac{((v/\lambda-\alpha)e^{-x_1}-\alpha)e^{x_T} - (1+k-v/\lambda)e^{-x_T}}{(v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \frac{\bar{u}(v-\alpha\lambda)}{\lambda(\bar{u}+\lambda)}e^{-x_1}}\left(g(0,x_1) - g(\bar{u},x_1)\right) \\ &= (1+k-v/\lambda)e^{-x_T} + \frac{((v/\lambda-\alpha)e^{-x_1}-\alpha)e^{x_T} - (1+k-v/\lambda)e^{-x_T}}{(v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \frac{\bar{u}(v-\alpha\lambda)}{\lambda(\bar{u}+\lambda)}e^{-x_1}}\left(e^{-x_1}\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) - \frac{(1+e^{-x_1})\alpha\bar{u}}{\lambda+\bar{u}}\right). \end{aligned}$$

Note that

$$g_u(u,x) \propto \alpha\lambda(1+e^x) - v.$$

Because we know from our bound that

$$e^{x_1} < e^{\bar{x}_1} = v/\alpha\lambda - 1.$$

We also know

$$\alpha\lambda\left(1+e^{x}\right) - v < 0.$$

Assume now $v/\lambda \leq 1+k$. Then we can write $V'(x_T)$ as

$$\begin{aligned}
V'(x_T) &\propto (1+k-v/\lambda)\, e^{-x_T}\left((v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \frac{\bar{u}\,(v-\alpha\lambda)}{\lambda\,(\bar{u}+\lambda)}e^{-x_1}\right) \\
&\quad - (1+k-v/\lambda)\, e^{-x_T}\left(e^{-x_1}\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) - \left(1+e^{-x_1}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}}\right) \\
&\quad + \left((v/\lambda-\alpha)\, e^{-x_1} - \alpha\right) e^{x_T}\left(e^{-x_1}\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) - \left(1+e^{-x_1}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}}\right) \\
&= (1+k-v/\lambda)\, e^{-x_T}\left((v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \bar{u}\frac{\alpha}{\bar{u}+\lambda}\right) \\
&\quad + e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1} - \alpha\right)^2,
\end{aligned}$$

which is positive as all terms in this expression are positive, and so the optimal deadline is infinite.

Conversely, if $v/\lambda \geq 1+k$ we know $\mathrm{d}x_1/\mathrm{d}x_T > 0$ and so

$$\begin{aligned}
V'(x_T) &= (1+k-v/\lambda)\, e^{-x_T}\left((v/\lambda-\alpha)\, e^{x_T-x_1} - \frac{v}{\lambda+\bar{u}} + \alpha\right) + e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1} - \alpha\right)^2 \\
&= (1+k-v/\lambda)\, e^{-x_1}(v/\lambda-\alpha) + \left(\alpha - \frac{v}{\lambda+\bar{u}}\right)(1+k-v/\lambda)\, e^{-x_T} \\
&\quad + e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1} - \alpha\right)^2.
\end{aligned}$$

This expression is negative for x_T large enough (because x_1 converges to $\bar{x}_1$ at rate x_T and hence the last term vanishes). Therefore, if $v/\lambda > 1+k$, the optimal deadline is finite. Finally, plugging in $x_1 = x_T$ we obtain

$$\begin{aligned}
V'(x_T) &= (1+k-v/\lambda)\, v\frac{\bar{u}}{\lambda\,(\bar{u}+\lambda)}e^{-x_T} + e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1} - \alpha\right)^2 \\
&\propto (1+k-v/\lambda)\frac{v}{\lambda} + \left(v/\lambda - \alpha\left(1+e^{x}\right)\right)^2 \\
&= \left(1+k-2\alpha\left(1+e^{x}\right)\right)\frac{v}{\lambda} + \alpha^2\left(1+e^{x}\right)^2. \qquad (52)
\end{aligned}$$

This is clearly positive for x_0 low enough (because we know in that case the lowest x_T yielding work-shirk is arbitrarily low). Now consider the equation determining the lowest x_T. We know this expression is increasing in x_0 (because the integrand is positive), and it is decreasing in x_T (all the terms go in the same direction). Therefore, we know $\mathrm{d}x_T/\mathrm{d}x_0 > 0$. We therefore look for the highest x_0 that allows for work-shirk, which is given by

$$x_0 = x^* = \ln\frac{k-\alpha}{\alpha}.$$

We then have condition (52), which is positive as $x \to -\infty$ and as $x = x^*$ (just plug in). Furthermore, it is quadratic in $\alpha(1+e^x)$ and decreasing at x^*. Therefore, it is everywhere positive.

Now consider the derivative of the social planner's payoff when exerting maximal effort throughout. We have

$$V'(x_T) = ((1+k)(\lambda+\bar{u}) - v)\, e^{-x_T} - \left(1+e^{-x_T}\right)\alpha\bar{u}.$$

Therefore, we have

$$V'(x^*) \propto (1+k)\lambda + \bar{u} - v.$$

We conclude with the following result.

Lemma D.5 *The socially optimal deadline is finite if and only if $v/\lambda \geq 1+k$.*

Furthermore, if $v/\lambda \leq 1+k+\bar{u}/\lambda$ the optimal deadline induces full then zero effort.

For values of v exceeding the upper bound, the optimal deadline is finite and may induce either full or full, then zero effort.

Planner's optimal deadline Contrast this with the planner's optimal deadline under full commitment. We again have

$$V'(x_T) = (1+k-v/\lambda)\, e^{-x_T} + \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(g(0,x_1) - g(\bar{u},x_1)\right),$$

with

$$(1+k-\alpha)\, e^{-x_T} - \alpha + (v/\lambda - \alpha)\left(e^{-x_1} - e^{-x_T}\right) = 0, \tag{53}$$

and so

$$\begin{aligned}
e^{-x_1} &= e^{-x_T}\left(1 - \frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}, \\
\frac{\mathrm{d}x_1}{\mathrm{d}x_T} &= \frac{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)}{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}}.
\end{aligned}$$

Therefore we can rewrite V' as

$$\begin{aligned}
V'(x_T) &= (1+k-v/\lambda)\, e^{-x_T} + \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}\right)\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) \\
&\quad - \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(1 + e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}} \\
&= (1+k-v/\lambda)\, e^{-x_T} - \frac{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)}{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}}\frac{\bar{u}}{\lambda}\frac{e^{-x_T}}{\bar{u}+\lambda}(\lambda - v + k\lambda) \\
&\propto e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)\frac{\lambda}{\bar{u}+\lambda} + \frac{\alpha}{v/\lambda-\alpha} \\
&\propto (1+k-v/\lambda)\left(\alpha - e^{-x_T}(1+k-v/\lambda)\frac{\lambda}{\bar{u}+\lambda}\right).
\end{aligned}$$

Notice that the second term is positive, because

$$\alpha - e^{-x_T}\left(1+k-v/\lambda\right)\frac{\lambda}{\bar{u}+\lambda} \geq \alpha\left(1-\frac{1+k-v/\lambda}{1+k-\alpha}\frac{\lambda}{\bar{u}+\lambda}\right) > 0$$

which follows from plugging $x_1 = x_T$ into (53). Therefore, $v \leq \lambda\left(1+k\right)$ is necessary and sufficient for the planner's problem to be increasing in the deadline, and we have the following characterization.

Lemma D.6 *If $v \in \left[\lambda\left(1+k\right), \lambda\left(1+k\right)+\alpha\bar{u}\right]$, the planner's optimal deadline is $x_P^* = \ln\left(\frac{1+k}{\alpha}-1\right)$.*
If $v > \lambda\left(1+k\right)+\alpha\bar{u}$, the optimal deadline is $\hat{x} < x_P^*$.
If $v < \lambda\left(1+k\right)$, the optimal deadline is infinite.

To summarize, when it is inefficient to stop the relationship, the socially optimal deadline is infinite, as is the planner's. However, when under full commitment (contractable output), the planner finds it optimal to work at full speed and stop, whereas the socially optimal deadline typically includes shirking (speculating, because the payment of a lump sum depresses incentives so much at the end that a bit of shirking is nevertheless beneficial –without shirking the highest attainable x_T is very low).

D.3.4 Finishing Lines

Proof of Lemma 5.7 Let $\hat{x}$ denote the stopping belief, fixed exogenously for now. The payoff to be maximized is

$$\begin{aligned} & \int_x^{\hat{x}} \frac{\left(\lambda+u\left(x\right)\right)e^{-x}-\left(1+e^{-x}\right)\left(v+\alpha u\right)}{\lambda+u}\mathrm{d}x - k\left(1+e^{-\hat{x}}\right)+v\int_0^{\hat{x}}\frac{\mathrm{d}x}{\lambda+u} \\ = & \int_x^{\hat{x}} \frac{\left(\lambda-v+u\left(x\right)\right)e^{-x}-\left(1+e^{-x}\right)\alpha u}{\lambda+u}\mathrm{d}x - k\left(1+e^{-\hat{x}}\right), \end{aligned}$$

where $u\left(x\right)$ is the expected effort given state x and u is the control variable (equal to $u\left(x\right)$ at x in equilibrium). [The last term on the first line corresponds to the term vT discussed in footnote 5.] Transversality requires that $u = u\left(\hat{x}\right)$ maximizes

$$\frac{\left(\lambda-v+u\left(\hat{x}\right)\right)e^{-\hat{x}}-\left(1+e^{-\hat{x}}\right)\alpha u}{\lambda+u},$$

whose derivative w.r.t. u is proportional to

$$v-u(\hat{x})-(1+\alpha)\lambda-\alpha\lambda e^{\hat{x}}.$$

Hence,

$$u\left(\hat{x}\right) = \begin{cases} \bar{u} \text{ if } e^{\hat{x}} < \frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \\ u \in \left(0,\bar{u}\right) \text{ if } e^{\hat{x}} = \frac{v-u-(1+\alpha)\lambda}{\alpha\lambda}, \\ 0 \text{ if } e^{\hat{x}} > \frac{v-(1+\alpha)\lambda}{\alpha\lambda}. \end{cases}$$

The intuition is straightforward: if $\hat{x}$ is high enough, there is no reason to work: the wage might be low, but then again the outside option v is not hurting, as it is unlikely to be collected.

More generally, the objective is maximized pointwise by setting:

$$u(x) = \begin{cases} \bar{u} \text{ for } e^x > \frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \\ v-(1+\alpha)\lambda-\alpha\lambda e^x \text{ if } e^x \in \left[\frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \frac{v-(1+\alpha)\lambda}{\alpha\lambda}\right], \\ 0 \text{ if } e^x > \frac{v-(1+\alpha)\lambda}{\alpha\lambda}, \end{cases}$$

for all relevant values of x (i.e., values such that $x < \hat{x}$). Note now that the derivative of the objective with respect to the finishing line is simply

$$\frac{(\lambda - v + u(\hat{x}))\, e^{-\hat{x}} - \left(1 + e^{-\hat{x}}\right) \alpha u(\hat{x})}{\lambda + u} + k e^{-\hat{x}},$$

which, given the formula for $u(\hat{x})$ is non-increasing in $\hat{x}$. When $1+k > v/\lambda$, then this derivative is positive for all values of $\hat{x}$: the optimal finishing line is infinite in that case. When $1+k \in [(v-\bar{u})/\lambda, v/\lambda]$, the optimal finishing line solves

$$e^{\hat{x}} = \frac{k-\alpha}{\alpha},$$

i.e. $\hat{x} = x^*$, and effort is interior at the finishing line. Finally, if $1+k < (v-\bar{u})/\lambda$, the optimal finishing line solves

$$1 + e^{\hat{x}} = \frac{(1+k)(\lambda+\bar{u}) - v}{\alpha\bar{u}},$$

with maximum effort throughout. Note that this finishing line coincides with the (belief at the) optimal deadline in the absence of commitment under non-observability (See Lemma D.3). □