May lack of information be the real invisible hand? (Work in progress.)

Andrés Carvajal
Yale University and Royal Holloway, University of London

Herakles Polemarchakis
Brown University

January 20, 2005

Abstract

Pareto improving intervention may be impossible not because it does not exist, but because it is impossible for a planner to figure out one. In this paper, we show that finite sets of market data may not suffice for a planner to be able to design Pareto improving policies.

In an incomplete markets economy, after asset markets have closed, there typically exist Pareto-improving asset reallocations. The question that naturally arises is: how much does a planner need to know if he is to figure out one such asset reallocation? Existing identification results (Kubler et al., 2002, or Carvajal and Riascos, 2004) require knowledge of the equilibrium manifold that may be unrealistic. If the informational requirement for finding Pareto-improving policies is implausible, market performance is not questionable on efficiency grounds.

We show here that market information in a nonstationary economy does not provide enough information for the design of Pareto-improving policies, even when these policies are likely to exist. When the information available is a set of prices and endowments, we prove that there may exist multiple rationalizations in which the welfare effects of economic policy go in opposite directions.

1 Not everything will do

Consider a two-period economy with uncertainty. There are $I \in \mathbb{N}$ individuals, $S \in \mathbb{N}$ states of nature and $L \in \mathbb{N}$ commodities, with commodity 1 as numeraire. There are $A < S$ linearly independent numeraire assets. Asset payoffs in state s are $r_s \in \mathbb{R}^A$.

Let $\mathcal{U}$ be the class of all strongly concave, strictly monotone, C^2 functions $u : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$. For each individual i, state-contingent endowments are $e^i = \left(e^i_s \in \mathbb{R}^L_{++}\right)^S_{s=1}$ and state-contingent preferences are $\left(u^i_s \in \mathcal{U}\right)^S_{s=1}$. There is no date-zero consumption, and ex-ante preferences are $U^i : \left(\mathbb{R}^L_+\right)^S \longrightarrow \mathbb{R}; x = (x_s)^S_{s=1} \mapsto \sum^S_{s=1} u^i_s(x_s)$.

Let $\left(q, (p_s)^S_{s=1}, \left(z^i, (x^i_s)^S_{s=1}\right)^I_{i=1}\right) \in \mathbb{R}^A \times \mathcal{P}^S \times \left(\mathbb{R}^A \times \mathbb{R}^{LS}_+\right)^I$ be a financial markets equilibrium of this economy, where $\mathcal{P} = \left\{p \in \mathbb{R}^L_{++} \,|p_1 = 1\right\}$. Generically in the space of economies (Geanakoplos and Polemarchakis, 1986), there exists $\left(dz^i\right)^I_{i=1} \in \left(\mathbb{R}^A\right)^I$ such that $\sum^I_{i=1} dz^i = 0$

and $\left(\sum_{s=1}^S du_s^i\right)_{i=1}^I \gg 0$, where $\left(du_s^i\right)_{i=1}^I$ is the (spot) general equilibrium welfare effects resulting from revenue transfer $\left(r_s dz^i\right)_{i=1}^I$ at spot equilibrium $\left(p_s, \left(x_s^i\right)_{i=1}^I\right)$ in exchange economy $\left(u_s^i, e_s^i + r_s z^i \left(1, 0, ..., 0\right)^\top\right)_{i=1}^I$.

For each i, there exists $\left(\lambda_s^i\right)_{s=0}^S \in \mathbb{R}_{++}^{S+1}$ such that $\left(Du_s^i\left(x_s^i\right)\right)_{s=1}^S = \left(\lambda_s^i p_s\right)_{s=1}^S$ and $\sum_{s=1}^S \lambda_s^i r_s = \lambda_0^i q$, while $p_s x_s^i = p_s e_s^i + r_s z^i$ and $q z^i = 0$.

Define $b_s^i = -du_s^i / \lambda_s^i$. For each s, since $\sum_{i=1}^I dz^i = 0$, it follows that $\sum_{i=1}^I b_s^i = 0$ and, hence, by Donsimoni and Polemarchakis (1996), there exist an exchange economy $\left(f_s^i \in \mathcal{U}, \omega_s^i \gg r_s z^i \left(1, 0, ..., 0\right)\right)_{i=1}^I$ and $\left(\widetilde{p}_s, \left(\widetilde{x}_s^i\right)_{i=1}^I\right)$, such that $\left(\widetilde{p}_s, \left(\widetilde{x}_s^i\right)_{i=1}^I\right)$ is spot equilibrium for $\left(f_s^i, \omega_s^i\right)_{i=1}^I$, $Df_s^i\left(\widetilde{x}_s^i\right) = \widetilde{p}_s$ and, as a consequence of revenue transfer $\left(d\omega_{1,s}^i = r_s dz^i\right)_{i=1}^I$, $\left(df_s^i\right)_{i=1}^I = \left(b_s^i\right)_{i=1}^I$.

Define $\left(\left(\widetilde{u}_s^i, \widetilde{e}_s^i\right)_{s=1}^S\right)_{i=1}^I$ by $\widetilde{u}_s^i : \mathbb{R}_+^L \longrightarrow \mathbb{R}; x \mapsto \lambda_s^i f_s^i\left(x\right)$ and $\widetilde{e}_s^i = \omega_s^i - r_s z^i \left(1, 0, ..., 0\right)^\top \in \mathbb{R}_{++}^L$. By construction, $\widetilde{p}_s \widetilde{e}_s^i + r_s z^i = \widetilde{p}_s \omega_s^i$, and $D\widetilde{u}_s^i\left(\widetilde{x}_s^i\right) = \lambda_s^i \widetilde{p}_s$. Also, $\sum_s \lambda_s^i r_s = \lambda_0^i q$, $\sum_{i=1}^I \widetilde{x}_s^i = \sum_{i=1}^I \widetilde{e}_s^i$ and $\sum_{i=1}^I z^i = 0$, so $\left(q, \left(\widetilde{p}_s\right)_{s=1}^S, \left(z^i, \left(\widetilde{x}_s^i\right)_{s=1}^S\right)_{i=1}^I\right)$ is financial equilibrium for $\left(\left(\widetilde{u}_s^i, \widetilde{e}_s^i\right)_{s=1}^S\right)_{i=1}^I$.

On the other hand, for each s, $\sum_{s=1}^S d\widetilde{u}_s^i = \sum_{s=1}^S \lambda_s^i df_s^i = \sum_{s=1}^S \lambda_s^i b_s^i = -\sum_{s=1}^S \lambda_s^i du_s^i / \lambda_s^i = -\sum_{s=1}^S du_s^i$.

It follows that, in the basic setting of the previous introduction, observation from the asset markets does not suffice for the identification of Pareto-improving policies:

Remark 1 *Let*

$$\left(q, \left(p_s\right)_{s=1}^S, \left(z^i, \left(x_s^i\right)_{s=1}^S\right)_{i=1}^I\right)$$

be a financial markets equilibrium for economy $\left(\left(u_s^i, e_s^i\right)_{s=1}^S\right)_{i=1}^I$ *and let* $\left(dz^i\right)_{i=1}^I$ *be such that* $\sum_{i=1}^I dz^i = 0$*. Suppose that for each* $s \in \{1, ..., S\}$ *there is* $i \in \{1, ..., I\}$ *such that* $r_s dz^i \neq 0$ *and* $du_s^i \neq 0$*. There exists an economy* $\left(\left(\widetilde{u}_s^i, \widetilde{e}_s^i\right)_{s=1}^S\right)_{i=1}^I$ *such that:*

1. *There exists* $\left(\widetilde{p}_s, \left(\widetilde{x}_s^i\right)_{i=1}^I\right)_{s=1}^S \in \mathcal{P}^S \times \left(\mathbb{R}_+^L\right)^{IS}$*, such that* $\left(q, \left(\widetilde{p}_s\right)_{s=1}^S, \left(z^i, \left(\widetilde{x}_s^i\right)_{s=1}^S\right)_{i=1}^I\right)$ *is financial markets equilibrium for* $\left(\left(\widetilde{u}_s^i, \widetilde{e}_s^i\right)_{s=1}^S\right)_{i=1}^I$*.*

2. $\left(\sum_{s=1}^S d\widetilde{u}_s^i\right)_{i=1}^I = \left(-\sum_{s=1}^S du_s^i\right)_{i=1}^I$*, where* $\left(d\widetilde{u}_s^i\right)_{i=1}^I$ *is the (spot) general equilibrium welfare effects resulting from revenue transfer* $\left(r_s dz^i\right)_{i=1}^I$ *at spot equilibrium* $\left(\widetilde{p}_s, \left(\widetilde{x}_s^i\right)_{i=1}^I\right)_{s=1}^S$ *in exchange economy* $\left(\widetilde{u}_s^i, \widetilde{e}_s^i + r_s z^i\right)_{i=1}^I$*.*

That is, for a given asset reallocation, all the information available from the markets that have actually been open fails to distinguish the true economy from another in which the same reallocation has the opposite welfare effects. In particular, the information does not distinguish an economy in which the policy is Pareto-improving from one in which it is Pareto impairing.

It should be noticed that this analysis assumes that $\left(\left(e_s^i\right)_{s=1}^S\right)_{i=1}^I$ is unknown. This is used in the argument only for the purpose of satisfying nonnegativity constraints, and neither the fact that $\left(q, \left(z^i\right)_{i=1}^I\right)$ is financial equilibrium for the constructed economy nor the fact that $\left(d\widetilde{u}_s^i\right)_{i=1}^I = -\left(du_s^i\right)_{i=1}^I$ depend on the exact value of $\left(\left(\widetilde{e}_s^i\right)_{s=1}^S\right)_{i=1}^I$.

This basic result is subject to criticism if it is plausible to assume that a planner may have available: (i) information from the commodity markets, in a longer history of observed data; (ii) information of different equilibria of the economy.

2 Longer histories of data

Let Σ be a finite tree of events. Denote by Σ^T the set of terminal nodes. For every $\sigma \in \Sigma$, let $F(\sigma) \subseteq \Sigma$ be the set containing σ and all nodes that follow σ (whether immediately or not). For every $\sigma \in \Sigma$, let $f(\sigma) \subseteq \Sigma$ be the set containing all immediate successors of σ. Denote by Σ^{T-1} the set of pre-terminal nodes (that is, $\sigma \in \Sigma^{T-1} \Longleftrightarrow f(\sigma) \subseteq \Sigma^T$). Let σ_0 be the initial node and for every $\sigma \in \Sigma \backslash \{\sigma_0\}$, let $b(\sigma)$ denote its immediate predecessor

For each $\sigma \in \Sigma$, individual i has preferences $u_\sigma^i \in \mathcal{U}$ and endowments $e_\sigma^i \in \mathbb{R}_{++}^L$.

At each $\sigma \in \Sigma \backslash \Sigma^T$, there is a finite set A_σ of one-period numeraire assets. For each $\sigma \in \Sigma \backslash \Sigma^T$, $\alpha \in A_\sigma$ and $\sigma' \in f(\sigma)$, $r_{\sigma'}^\alpha \in \mathbb{R}$ is the return of asset α in state σ', and $r_{\sigma'}$ is $\left(r_{\sigma'}^\alpha\right)_{\alpha \in A_\sigma}$.

At each $\sigma \in \Sigma$, given prices of commodities and assets $\left(p_{\sigma'}\right)_{\sigma' \in F(\sigma)}$ and $\left(q_{\sigma'}\right)_{\sigma' \in F(\sigma) \backslash \Sigma^T}$ and a portfolio $z \in \mathbb{R}^{\# A_{b(\sigma)}}$ carried from $b(\sigma)$, individual i solves

$$\max_{\left(x_{\sigma'}\right)_{\sigma' \in F(\sigma)}, \left(z_{\sigma'}\right)_{\sigma' \in F(\sigma) \backslash \Sigma^T}} \sum_{\sigma' \in F(\sigma)} u_{\sigma'}^i\left(x_{\sigma'}\right)$$

$$st. \begin{cases} \left(\forall \sigma' \in F(\sigma) \backslash \Sigma^T\right) : p_{\sigma'} x_{\sigma'} + q_{\sigma'} z_{\sigma'} \leq p_{\sigma'} e_{\sigma'}^i + r_{\sigma'} z_{b(\sigma')} \\ \left(\forall \sigma' \in F(\sigma) \cap \Sigma^T\right) : p_{\sigma'} x_{\sigma'} \leq p_{\sigma'} e_{\sigma'}^i + r_{\sigma'} z_{b(\sigma')} \end{cases}$$

where $z_{b(\sigma)} = z$ and if $\sigma = \sigma_0$ then $z = 0$.

Necessary and sufficient conditions for $\left(x_{\sigma'}^i\right)_{\sigma' \in F(\sigma)}$ and $\left(z_{\sigma'}^i\right)_{\sigma' \in F(\sigma) \backslash \Sigma^T}$ to be a solution are that for some $\left(\lambda_{\sigma'}^i\right)_{\sigma' \in F(\sigma)} \in \mathbb{R}_{++}^{\# F(\sigma)}$ the following conditions be satisfied:

$$\begin{aligned} \left(\forall \sigma' \in F(\sigma)\right) &: \quad Du_{\sigma'}^i\left(x_{\sigma'}^i\right) = \lambda_{\sigma'}^i p_{\sigma'} \\ \left(\forall \sigma' \in F(\sigma) \backslash \Sigma^T\right) &: \quad \lambda_{\sigma'}^i q_{\sigma'} = \sum_{\sigma'' \in f(\sigma')} \lambda_{\sigma''}^i r_{\sigma''} \\ \left(\forall \sigma' \in F(\sigma) \backslash \Sigma^T\right) &: \quad p_{\sigma'} x_{\sigma'} + q_{\sigma'} z_{\sigma'} = p_{\sigma'} e_{\sigma'}^i + r_{\sigma'} z_{b(\sigma')} \\ \left(\forall \sigma' \in F(\sigma) \cap \Sigma^T\right) &: \quad p_{\sigma'} x_{\sigma'} = p_{\sigma'} e_{\sigma'}^i + r_{\sigma'} z_{b(\sigma')} \end{aligned}$$

Theorem 1 *Let* $\left(\left(q_\sigma\right)_{\sigma \in \Sigma \backslash \Sigma^T}, \left(p_\sigma\right)_{\sigma \in \Sigma}, \left(\left(z_\sigma^i\right)_{\sigma \in \Sigma \backslash \Sigma^T}, \left(x_\sigma^i\right)_{\sigma \in \Sigma}\right)_{i=1}^I\right)$ *be a financial markets equilibrium for* $\left(\left(u_\sigma^i, e_\sigma^i\right)_{\sigma \in \Sigma}\right)_{i=1}^I$. *There exist state-contingent preferences* $\left(\left(\widetilde{u}_\sigma^i \in \mathcal{U}\right)_{\sigma \in \Sigma}\right)_{i=1}^I$ *such that:*

1. *Observed*

$$\left(\left(q_\sigma\right)_{\sigma \in \Sigma \backslash \Sigma^T}, \left(p_\sigma\right)_{\sigma \in \Sigma}, \left(\left(z_\sigma^i\right)_{\sigma \in \Sigma \backslash \Sigma^T}, \left(x_\sigma^i\right)_{\sigma \in \Sigma}\right)_{i=1}^I\right)$$

is a financial markets equilibrium for $\left(\left(\widetilde{u}_\sigma^i, e_\sigma^i\right)_{\sigma \in \Sigma}\right)_{i=1}^I$.

2. For every $\sigma \in \Sigma^{T-1}$ and every $\left(dz^i\right)_{i=1}^I \in \left(\mathbb{R}^{\#A_\sigma}\right)^I$, such that $\sum_{i=1}^I dz^i = 0$, it is true that

$$\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} d\widetilde{u}^i_{\sigma'} = -\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} du^i_{\sigma'}$$

and if $I = 2$, then, moreover

$$\left(\sum_{\sigma' \in f(\sigma)} d\widetilde{u}^1_{\sigma'}, \sum_{\sigma' \in f(\sigma)} d\widetilde{u}^2_{\sigma'} \right) = \left(-\sum_{\sigma' \in f(\sigma)} du^2_{\sigma'}, -\sum_{\sigma' \in f(\sigma)} du^1_{\sigma'} \right)$$

where $\left(du^i_{\sigma'}\right)_{i=1}^I$ and $\left(d\widetilde{u}^i_{\sigma'}\right)_{i=1}^I$ are the (spot) general equilibrium welfare effects resulting from revenue transfer $\left(r_{\sigma'} dz^i\right)_{i=1}^I$ at spot equilibrium $\left(p_{\sigma'}, \left(x^i_{\sigma'}\right)_{i=1}^I\right)$ in exchange economies $\left(u^i_{\sigma'}, e^i_{\sigma'} + r_{\sigma'} z^i \left(1, 0, ..., 0\right)^\top\right)_{i=1}^I$ and $\left(\widetilde{u}^i_{\sigma'}, e^i_{\sigma'} + r_{\sigma'} z^i \left(1, 0, ..., 0\right)^\top\right)_{i=1}^I$, respectively.

Proof. For each i, fix $\left(\lambda^i_\sigma\right)_{\sigma \in \Sigma} \in \mathbb{R}^{\#\Sigma}_{++}$ such that

$$\begin{aligned}
(\forall \sigma \in \Sigma) &: \quad Du^i_\sigma \left(x^i_\sigma\right) = \lambda^i_\sigma p_\sigma \\
\left(\forall \sigma \in \Sigma \backslash \Sigma^T\right) &: \quad \lambda^i_\sigma q_\sigma = \sum_{\sigma' \in f(\sigma)} \lambda^i_{\sigma'} r_{\sigma'} \\
\left(\forall \sigma \in \Sigma \backslash \Sigma^T\right) &: \quad p_\sigma x_\sigma + q_\sigma z_\sigma = p_\sigma e^i_\sigma + r_\sigma z_{b(\sigma)} \\
\left(\forall \sigma \in \Sigma \cap \Sigma^T\right) &: \quad p_\sigma x_\sigma = p_\sigma e^i_\sigma + r_\sigma z_{b(\sigma)}
\end{aligned}$$

Define preferences as

$$\widetilde{u}^i_\sigma (x) = \frac{\sum_{j \in \{1,...,I\} \backslash \{i\}} \lambda^j_\sigma}{\lambda^i_\sigma} u^i_\sigma (x)$$

To see that $\left((q_\sigma)_{\sigma \in \Sigma \backslash \Sigma^T}, (p_\sigma)_{\sigma \in \Sigma}, \left(\left(z^i_\sigma\right)_{\sigma \in \Sigma \backslash \Sigma^T}, \left(x^i_\sigma\right)_{\sigma \in \Sigma} \right)_{i=1}^I \right)$ is a financial markets equilibrium for $\left(\left(\widetilde{u}^i_\sigma, e^i_\sigma\right)_{\sigma \in \Sigma} \right)_{i=1}^I$, it suffices to let $\left(\widetilde{\lambda}^i_\sigma\right)_{\sigma \in \Sigma} = \left(\sum_{j \in \{1,...,I\} \backslash \{i\}} \lambda^j_\sigma\right)_{\sigma \in \Sigma} \in \mathbb{R}^{\#\Sigma}_{++}$ and observe that $(\forall \sigma \in \Sigma) : D\widetilde{u}^i_\sigma \left(x^i_\sigma\right) = \widetilde{\lambda}^i_\sigma p_\sigma$ and $\left(\forall \sigma \in \Sigma \backslash \Sigma^T\right)$:

$$\sum_{\sigma' \in f(\sigma)} \widetilde{\lambda}^i_{\sigma'} r_{\sigma'} = \sum_{\sigma' \in f(\sigma)} \sum_{j \in \{1,...,I\} \backslash \{i\}} \lambda^j_{\sigma'} r_{\sigma'} = \sum_{j \in \{1,...,I\} \backslash \{i\}} \sum_{\sigma' \in f(\sigma)} \lambda^j_{\sigma'} r_{\sigma'} = \sum_{j \in \{1,...,I\} \backslash \{i\}} \lambda^j_\sigma q_\sigma = \widetilde{\lambda}^i_\sigma q_\sigma$$

Now, fix $\sigma \in \Sigma^{T-1}$ and $\left(dz^i\right)_{i=1}^I \in \left(\mathbb{R}^{\#A_\sigma}\right)^I$, with $\sum_{i=1}^I dz^i = 0$.

For each $\sigma' \in f(\sigma)$, since dp_σ is the same in exchange economies $\left(u^i_{\sigma'}, e^i_{\sigma'} + r_{\sigma'} z^i \left(1, 0, ..., 0\right)^\top\right)_{i=1}^I$ and $\left(\widetilde{u}^i_{\sigma'}, e^i_{\sigma'} + r_{\sigma'} z^i \left(1, 0, ..., 0\right)^\top\right)_{i=1}^I$, it follows that $d\widetilde{u}^i_{\sigma'} = \frac{\sum_{j \in \{1,...,I\} \backslash \{i\}} \lambda^j_{\sigma'}}{\lambda^i_{\sigma'}} du^i_{\sigma'}$, which means

$$d\widetilde{u}^i_{\sigma'} = \left(\frac{\sum_{j=1}^I \lambda^j_{\sigma'} - \lambda^i_{\sigma'}}{\lambda^i_{\sigma'}} \right) du^i_{\sigma'} = \left(\frac{\Lambda_{\sigma'}}{\lambda^i_{\sigma'}} - 1 \right) du^i_{\sigma'}$$

and, hence,

$$\sum_{i=1}^{I} d\widetilde{u}_{\sigma'}^{i} = \Lambda_{\sigma'} \sum_{i=1}^{I} \frac{du_{\sigma'}^{i}}{\lambda_{\sigma'}^{i}} - \sum_{i=1}^{I} du_{\sigma'}^{i} = -\sum_{i=1}^{I} du_{\sigma'}^{i}$$

(because $\sum_{i=1}^{I} dz^{i} = 0$). It then follows that

$$\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} d\widetilde{u}_{\sigma'}^{i} = \sum_{\sigma' \in f(\sigma)} \sum_{i=1}^{I} d\widetilde{u}_{\sigma'}^{i} = -\sum_{\sigma' \in f(\sigma)} \sum_{i=1}^{I} du_{\sigma'}^{i} = -\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} du_{\sigma'}^{i}$$

In particular, if $I = 2$, then, $\widetilde{u}_{\sigma'}^{i} = \frac{\lambda_{\sigma'}^{\neg i}}{\lambda_{\sigma'}^{i}} u_{\sigma'}^{i}$, where $\neg i \in \{1,2\} \setminus \{i\}$, and, by Roy's identity,

$$\begin{aligned} du_{\sigma'}^{i} &= \lambda_{\sigma'}^{i} \left(dp_{\sigma'} \left(e_{\sigma'}^{i} - x_{\sigma'}^{i} \right) + r_{\sigma'} dz^{i} \left(1, 0, ..., 0\right)^{\top} \right) \\ &= \lambda_{\sigma'}^{i} \left(dp_{\sigma'} \left(x_{\sigma'}^{\neg i} - e_{\sigma'}^{\neg i} \right) + r_{\sigma'} \left(-dz^{\neg i} \right) \left(1, 0, ..., 0\right)^{\top} \right) \\ &= -\frac{\lambda_{\sigma'}^{i} du_{\sigma'}^{\neg i}}{\lambda_{\sigma'}^{\neg i}} \end{aligned}$$

so $\sum_{\sigma'' \in f(\sigma')} d\widetilde{u}_{\sigma''}^{i} = \sum_{\sigma'' \in f(\sigma')} \frac{\lambda_{\sigma''}^{\neg i}}{\lambda_{\sigma''}^{i}} du_{\sigma''}^{i} = -\sum_{\sigma'' \in f(\sigma')} d\widetilde{u}_{\sigma''}^{\neg i}$. ■

The theorem implies that even the observation of all equilibrium information at given (observed) endowments does not allow a planner to discern between economies in which, for any policy, the aggregate of individual welfare effects go in opposite direction. In particular, it does not distinguish between an economy in which a policy is Pareto-improving and another in which at least one consumer loses. When there are only two consumers, the implication is stronger: the information fails to distinguish an economy in which a policy is Pareto-improving and one in which it is Pareto impairing.

Corollary 1 *Suppose that for some $u^{i} \in \mathcal{U}$ it is true that for every $\sigma \in \Sigma$, there exists $\pi_{\sigma}^{i} \in \mathbb{R}_{++}$ such that $u_{\sigma}^{i} = \pi_{\sigma}^{i} u^{i}$. Let $\left((q_{\sigma})_{\sigma \in \Sigma \setminus \Sigma^{T}}, (p_{\sigma})_{\sigma \in \Sigma}, \left((z_{\sigma}^{i})_{\sigma \in \Sigma \setminus \Sigma^{T}}, (x_{\sigma}^{i})_{\sigma \in \Sigma} \right)_{i=1}^{I} \right)$ be a financial markets equilibrium for $\left(u^{i}, (\pi_{\sigma}^{i}, e_{\sigma}^{i})_{\sigma \in \Sigma} \right)_{i=1}^{I}$. There exist preferences $\left(\widetilde{u}^{i} \in \mathcal{U} \right)_{i=1}^{I}$ and $\left(\left(\widetilde{\pi}_{\sigma}^{i} \right)_{\sigma \in \Sigma} \right)_{i=1}^{I} \in \left(\mathbb{R}_{++}^{\#\Sigma} \right)^{I}$ such that:*

1. *Observed*

$$\left((q_{\sigma})_{\sigma \in \Sigma \setminus \Sigma^{T}}, (p_{\sigma})_{\sigma \in \Sigma}, \left((z_{\sigma}^{i})_{\sigma \in \Sigma \setminus \Sigma^{T}}, (x_{\sigma}^{i})_{\sigma \in \Sigma} \right)_{i=1}^{I} \right)$$

   is a financial markets equilibrium for $\left(\widetilde{u}^{i}, \left(\widetilde{\pi}_{\sigma}^{i}, e_{\sigma}^{i} \right)_{\sigma \in \Sigma} \right)_{i=1}^{I}$.

2. *For every $\sigma \in \Sigma^{T-1}$ and every $\left(dz^{i} \right)_{i=1}^{I} \in \left(\mathbb{R}^{\# A_{\sigma}} \right)^{I}$, such that $\sum_{i=1}^{I} dz^{i} = 0$, it is true that*

$$\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} \widetilde{\pi}_{\sigma'}^{i} d\widetilde{u}_{\sigma'}^{i} = -\sum_{i=1}^{I} \sum_{\sigma' \in f(\sigma)} \pi_{\sigma'}^{i} du_{\sigma'}^{i}$$

where $\left(du^i_{\sigma'}\right)_{i=1}^I$ *and* $\left(d\widetilde{u}^i_{\sigma'}\right)_{i=1}^I$ *are the (spot) general equilibrium welfare effects resulting from revenue transfer* $\left(r_{\sigma'}dz^i\right)_{i=1}^I$ *at spot equilibrium* $\left(p_{\sigma'}, \left(x^i_{\sigma'}\right)_{i=1}^I\right)$ *in exchange economies* $\left(u^i, e^i_{\sigma'} + r_{\sigma'}z^i\right)_{i=1}^I$ *and* $\left(\widetilde{u}^i, e^i_{\sigma'} + r_{\sigma'}z^i\right)_{i=1}^I$*, respectively.*

Proof. Define $\left(\lambda^i_\sigma\right)_{\sigma\in\Sigma} \in \mathbb{R}^{\#\Sigma}_{++}$ as in the previous proof. It suffices to let $\widetilde{u}^i = u^i$ and $\widetilde{\pi}^i_\sigma = \frac{\sum_{j\in\{1,\dots,I\}\setminus\{i\}} \lambda^j_\sigma}{\lambda^i_\sigma}\pi^i_\sigma$, for every $\sigma \in \Sigma$. ■

3 Multiple observations and rationalizability

Literature on the empirical content of theories distinguishes two problems: rationalizability (existence) and identification (uniqueness) of fundamentals consistent with observed data and theory. Typically, theoretical work concentrates on only one of the two problems and, in particular, literature on identification takes rationalizability for granted (e.g. Kubler et al (2002), Carvajal and Riascos (2004)). So far we have done the same: we have assumed the existence of a profile of preferences that explains observed data and have concentrated on finding a second profile which (i) is consistent with the data as well, and (ii) would give welfare effects that are opposite to the ones given by the original profile, for any policy. This is obviously a nonidentification result (i), which is stronger and of particular interest, because of (ii), yet it still assumes rationalizability.

At the same time, the analysis so far has assumed that all the endogenous variables are observed for a given value of endowments. The existing literature on testabiliy and identification has considered a different kind of data set: series of pairs of prices and endowments, only. We now study the rationalizability and identification problems simultaneously: we now consider the question of whether a data set of prices of endowments is rationalizable (is a subset of the equilibrium manifold for some preferences) but, still, fails to identify Pareto-improving policies.

3.1 Main theorem

Back in the original two-period setting of section 1, there are $L \in \mathbb{N}$ commodities, $S \in \mathbb{N}$ states of nature and $I \in \mathbb{N}$ individuals. State-contingent preferences are $\left(\left(u^i_s \in \mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$ and ex-ante preferences are $U^i = \sum_{s=1}^S u^i_s$. Let $T \in \mathbb{N}$. A data set is a sequence $\left(q^t, \left(p^t_s\right)_{s=1}^S, \left(\left(e^{it}_s\right)_{s=1}^S\right)_{i=1}^I\right)_{t=1}^T \in \left(\mathbb{R}^A \times \mathcal{P}^S \times \left(\left(\mathbb{R}^L_{++}\right)^S\right)^I\right)^T$.

Theorem 2 *There exists a finite, noncontradictory system (R) of polynomial inequalities in*

$$\left(\mathbb{R}^A \times \mathcal{P}^S \times \left(\left(\mathbb{R}^L_{++}\right)^S\right)^I\right)^T$$

such that if a data set satisfies (R) then there exist preferences $\left(\left(u^i_s \in \mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$ *and* $\left(\left(\widetilde{u}^i_s \in \mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$ *such that*

1. *For every* $t \in \{1, ..., T\}$*, observed*

$$\left(q^t, \left(p^t_s\right)_{s=1}^S\right)$$

are financial markets equilibrium prices for economy $\left(\left(u^i_s, e^{it}_s\right)_{s=1}^S\right)_{i=1}^I$.

2. For every $t \in \{1, ..., T\}$, observed

$$\left(q^t, (p_s^t)_{s=1}^S\right)$$

are financial markets equilibrium prices for economy $\left(\left(\widetilde{u}_s^i, e_s^{it}\right)_{s=1}^S\right)_{i=1}^I$.

3. For every $\left(dz^i\right)_{i=1}^I \in \left(\mathbb{R}^{\# A_\sigma}\right)^I$*, such that* $\sum_{i=1}^I dz^i = 0$*, it is true that*

$$\sum_{i=1}^I \sum_{s=1}^S d\widetilde{u}_s^{iT} = -\sum_{i=1}^I \sum_{s=1}^S du_s^{iT}$$

and if $I = 2$, then, moreover

$$\left(\sum_{s=1}^S d\widetilde{u}_s^1, \sum_{s=1}^S d\widetilde{u}_s^2\right) = \left(-\sum_{s=1}^S du_s^2, -\sum_{s=1}^S du_s^1\right)$$

Suppose that a planner has available a finite data set of prices and profiles of endowments. He may ask whether the data set can be explained by the Walrasian model and, if so, whether it can be used to identify a Pareto-improving policy. The theorem says that it may well happen that the answer to the second part of the question is negative, although the answer to the first part is affirmative.

The next two subsections prove the theorem. We first argue the existence of system (R) and then show that it need not a contradiction.

3.2 Definition of system (R)

Suppose that there exist

$$\left(\left(\left(u_s^{it}, \mu_s^{it}, x_s^{it}, \lambda_s^{it}\right)_{s=1}^S, \lambda_0^{it}, k^{it}, z^{it}\right)_{i=1}^I\right)_{t=1}^T \in \left(\left(\left(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++}^L \times \mathbb{R}_{++}\right)^S \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^A\right)^I\right)^T$$

such that $\left(q^t, (p_s^t)_{s=1}^S, \left(\left(u_s^{it}, \mu_s^{it}, x_s^{it}, e_s^{it}, \lambda_s^{it}\right)_{s=1}^S, \lambda_0^{it}, k^{it}, z^{it}\right)_{i=1}^I\right)_{t=1}^T$ satisfies the following system:

i. $(\forall i)\,(\forall s)\,(\forall t, t' \in \{1, ..., T\} : t \neq t') : x_s^{it} \neq x_s^{it'}$

ii. $(\forall i)\,(\forall s)\,(\forall t, t' \in \{1, ..., T\} : t \neq t') : u_s^{it'} < u_s^{it} + \lambda_s^{it} p_s^t \left(x_s^{it'} - x_s^{it}\right)$

iii. $(\forall i)\,(\forall t \in \{1, ..., T\})\,(\forall a) : \lambda_0^{it} q_a^t = \sum_{s=1}^S \lambda_s^{it} r_s^a$

iv. $(\forall i)\,(\forall s)\,(\forall t \in \{1, ..., T\}) : p_s^t \left(x_s^{it} - e_s^{it}\right) = r_s z^{it}$

v. $(\forall i)\,(\forall t \in \{1, ..., T\}) : q^t z^{it} = 0$

vi. $(\forall s)\,(\forall t \in \{1, ..., T\}) : \sum_{i=1}^I x_s^{it} = \sum_{i=1}^I e_s^{it}$

vii. $(\forall t \in \{1, ..., T\}) : \sum_{i=1}^I z^{it} = 0$

viii. $(\forall i)\,(\forall s)\,(\forall t \in \{1, ..., T-1\}) : k^{it} \lambda_s^{it} p_s^t \left(x_s^{it} - x_s^{iT}\right) < \mu_s^{it}$

ix. $(\forall i)(\forall s)(\forall t \in \{1,...,T-1\}) : \mu_s^{it} < k^T \sum_{j\in\{1,...,I\}\setminus\{i\}} \lambda_s^{jT} p_s^T \left(x_s^{it} - x_s^{iT}\right)$

x. $(\forall i)(\forall s)(\forall t, t' \in \{1,...,T-1\} : t \neq t') : \mu_s^{t'} < \mu_s^t + k^t \lambda_s^{it} p_s^t \left(x_s^{it'} - x_s^{it}\right)$

For each i and s, conditions **i**, **ii** and **iv** imply, by Theorem 2 in Matzkin and Richter (1991) that there exists $u_s^i \in \mathcal{U}$ such that for each $t \in \{1,...,T\}$,

$$x_s^{it} \in \arg\max_x u_s^i(x) \text{ s.t. } p_s^t x \le p_s^t e_s^{it} + r_s z^{it}$$

and $Du_s^i\left(x_s^{it}\right) = \lambda_s^{it} p_s^t$. It then follows from **iii** and **v** that

$$\left(\left(x_s^{it}\right)_{s=1}^S, z^{it}\right) \in \arg \max_{\left((x_s)_{s=1}^S, z\right)} \sum_{s=1}^S u_s^i(x_s) \text{ s.t. } \left\{ \begin{array}{c} (\forall s \in \{1,...,S\})\, p_s^t x_s \le p_s^t e_s^{it} + r_s z \\ q^t z = 0 \end{array} \right.$$

and, hence, from **vi** and **vii**, it follows that, for every $t \in \{1,...,T\}$, $\left(q^t, \left(p_s^t\right)_{s=1}^S, \left(\left(x_s^{it}\right)_{s=1}^S, z^{it}\right)_{i=1}^I\right)$ is a financial markets equilibrium for economy $\left(\left(u_s^i, e_s^{it}\right)_{s=1}^S\right)_{i=1}^I$, which proves part 1 of the theorem.

Since for every i, s and $t \in \{1,...,T\}$, $Du_s^i\left(x_s^{it}\right) = \lambda_s^{it} p_s^t$, parts 2 and 3 follow from lemma 2, given **viii**, **ix** and **x**.

Since system **i-x** is a finite system of polynomial inequalities, the definition of system (R) follows from the Tarski-Seidenberg quantifier elimination theorem (see Mishra, 1993).

3.3 System (R) is noncontradictory

Some cases in which the system has a solution are given next. It is straightforward that conditions **i** to **vii** of the system defined in the previous section are necessary and sufficient for a data set to be rationalized by some preferences $\left(\left(u_s^i \in \mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$, so we now concentrate on conditions **viii**, **ix** and **x**, and construct preferences $\left(\left(\widetilde{u}_s^i \in \mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$. For simplicity, we do this for given i, taking as given the rest of the system. The following cases point out that all the remaining variables in the system, namely $\left(\left(k^t\right)_{t=1}^T, \left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1}\right)$ for individual i, are useful and also that there may be instances in which conditions are very easy to check.

3.3.1 Perturbing the gradients elsewhere

For every t and every s, let $\Lambda_s^{it} = \sum_{j\in\{1,...,I\}\setminus\{i\}} \lambda_s^{jt}/\lambda_s^{it}$ and define

$$\begin{aligned} N_s^{it} &= \left\{t' \in \{1,...,T-1\} : Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right) < 0\right\} \\ P_s^{it} &= \left\{t' \in \{1,...,T-1\} : Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right) > 0\right\} \end{aligned}$$

For every $t \in \{1, ..., T-1\}$, suppose that $\forall s$, $N_s^{it} \cup P_s^{it} = \{1, ..., T-1\}$ and

$$\begin{aligned} L_s^{it} &= \max_{t' \in P_s^{it}} \frac{\Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right) - \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)}{Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right)} \\ &< \min_{t' \in N_s^{it}} \frac{\Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right) - \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)}{Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right)} \\ &= U_s^{it} \end{aligned}$$

while

$$\max_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) < 0} \Lambda_{s'}^{it} < \min_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) > 0} \Lambda_{s'}^{it}$$

and

$$\left(L_s^{it}, U_s^{it}\right) \cap \left(\max_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) < 0} \Lambda_{s'}^{it}, \min_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) > 0} \Lambda_{s'}^{it}\right) \neq \varnothing$$

In this case, for each $t \in \{1, ..., T-1\}$ let

$$k^t \in \left(L_s^{it}, U_s^{it}\right) \cap \left(\max_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) < 0} \Lambda_{s'}^{it}, \min_{s' \in \{1,...,S\}: Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) > 0} \Lambda_{s'}^{it}\right)$$

and let $k^T = 1$ and $\left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1} = \left(\left(\Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)\right)_{s=1}^S\right)_{t=1}^{T-1}$.

Since

$$\begin{aligned} (\forall s)\left(\forall t \in \{1, ..., T-1\}\right) &: u_s^i\left(x_s^{it}\right) < u_s^i\left(x_s^{iT}\right) + Du_s^i\left(x_s^{iT}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) \\ (\forall s)\left(\forall t \in \{1, ..., T-1\}\right) &: u_s^i\left(x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) + Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{iT} - x_s^{it}\right) \\ (\forall s)\left(\forall t, t' \in \{1, ..., T-1\} : t \neq t'\right) &: u_s^i\left(x_s^{it'}\right) < u_s^i\left(x_s^{it}\right) + Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right) \end{aligned}$$

we have that

$$\begin{aligned} (\forall s)\left(\forall t \in \{1, ..., T-1\}\right) &: u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) < Du_s^i\left(x_s^{iT}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) \\ (\forall s)\left(\forall t \in \{1, ..., T-1\}\right) &: Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) \\ (\forall s)\left(\forall t, t' \in \{1, ..., T-1\} : t \neq t'\right) &: u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) + Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right) \end{aligned}$$

Fix s and $t \in \{1, ..., T-1\}$. Since $\Lambda_s^{it} > 0$,

$$(\forall s)\left(\forall t \in \{1, ..., T-1\}\right) : \mu_s^t < k^T \Lambda_{s'}^{it} Du_s^i\left(x_s^{iT}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$$

while

$$(\forall s)\left(\forall t \in \{1, ..., T-1\}\right) : \mu_s^t > \Lambda_{s'}^{it} Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$$

so, if $Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) < 0$, since $k^t > \Lambda_s^{it}$ then $\mu_s^t > \Lambda_{s'}^{it} Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) > k^t Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$, whereas if $Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) > 0$, then, since $k^t < \Lambda_s^{it}$, it follows that $\mu_s^t > \Lambda_{s'}^{it} Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) > k^t Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$. If, alternatively, $Du_{s'}^i\left(x_{s'}^{it}\right) \cdot \left(x_{s'}^{iT} - x_{s'}^{it}\right) = 0$, then $\mu_s^t > 0 = k^t Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$.

Now, fix s and $t,t' \in \{1,...,T-1\}$ with $t \neq t'$. If $Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it'} - x_s^{it}\right) > 0$, since

$$k^t > \frac{\Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right) - \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)}{Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)}$$

it follows that

$$\Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right) < \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right) + k^t Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)$$

which means that $\mu_s^{t'} < \mu_s^t + k^t Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)$.

If, on the other hand, $Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right) > 0$, then

$$k^t < \frac{\Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right) - \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)}{Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)}$$

which implies that

$$k^t Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right) + \Lambda_s^{it}\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right) > \Lambda_s^{it'}\left(u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right)\right)$$

or, equivalently, $\mu_s^{t'} < \mu_s^t + k^t Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)$.

Finally, notice that $Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right) = 0$ does not occur, by assumption.

3.3.2 Perturbing the gradients at T only

Suppose that $\exists s, s'$ such that $P_s^{iT} \neq \varnothing$ and $N_{s'}^{iT} \neq \varnothing$

$$\max_{s:P_s^{iT}\neq\varnothing}\max_{t\in P_s^{iT}} \frac{u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)}{Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right)} < \min_{s:N_s^{iT}\neq\varnothing}\min_{t\in N_s^{iT}} \frac{u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)}{Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right)}$$

In this case, for each $t \in \{1,...,T-1\}$ let $k^t = 1$, let

$$k^T \in \left(\max\left\{0, \max_{s:P_s^{iT}\neq\varnothing}\max_{t\in P_s^{iT}} \frac{u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)}{Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right)}\right\}, \min_{s:N_s^{iT}\neq\varnothing}\min_{t\in N_s^{iT}} \frac{u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)}{Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right)}\right)$$

and $\left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1} = \left(\left(u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right)\right)_{s=1}^S\right)_{t=1}^{T-1}$.

As before, since

$$\begin{aligned}
(\forall s)\left(\forall t \in \{1,...,T-1\}\right) &: u_s^i\left(x_s^{it}\right) < u_s^i\left(x_s^{iT}\right) + Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right) \\
(\forall s)\left(\forall t \in \{1,...,T-1\}\right) &: u_s^i\left(x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) + Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{iT} - x_s^{it}\right) \\
(\forall s)\left(\forall t,t' \in \{1,...,T-1\} : t \neq t'\right) &: u_s^i\left(x_s^{it'}\right) < u_s^i\left(x_s^{it}\right) + Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)
\end{aligned}$$

we have that

$$\begin{aligned}
(\forall s)\left(\forall t \in \{1,...,T-1\}\right) &: u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) < Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it} - x_s^{iT}\right) \\
(\forall s)\left(\forall t \in \{1,...,T-1\}\right) &: Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it} - x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) \\
(\forall s)\left(\forall t,t' \in \{1,...,T-1\} : t \neq t'\right) &: u_s^i\left(x_s^{it'}\right) - u_s^i\left(x_s^{iT}\right) < u_s^i\left(x_s^{it}\right) - u_s^i\left(x_s^{iT}\right) + Du_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'} - x_s^{it}\right)
\end{aligned}$$

so

$$(\forall s)\left(\forall t\in\{1,...,T-1\}\right) \quad : \quad k^{t}Du_{s}^{i}\left(x_{s}^{it}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)<\mu_{s}^{t}$$
$$(\forall s)\left(\forall t,t'\in\{1,...,T-1\}:t\neq t'\right) \quad : \quad \mu_{s}^{t'}<\mu_{s}^{t}+k^{t}Du_{s}^{i}\left(x_{s}^{it}\right)\cdot\left(x_{s}^{it'}-x_{s}^{it}\right)$$

Now, fix s and $t\in\{1,...,T-1\}$. If $Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)>0$, then

$$k^{T}>\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}$$

so

$$k^{T}Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)>u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)$$

which means that $\mu_{s}^{t}<k^{T}Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)$.

If, on the other hand, $Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)<0$, then

$$k^{T}<\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}$$

so

$$k^{T}Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)>u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)$$

which is the same as $\mu_{s}^{t}<k^{T}Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)$.

Finally, if $Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)=0$, then $u_{s}^{i}\left(x_{s}^{it}\right)<u_{s}^{i}\left(x_{s}^{iT}\right)$, so

$$\mu_{s}^{t}=u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)<0=k^{T}Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)$$

It should be noticed that, by construction,

$$\min_{s:N_{s}^{iT}\neq\varnothing}\min_{t\in N_{s}^{iT}}\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}>0$$

so, for the assumption here to hold it would suffice that

$$Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)>0\Longrightarrow u_{s}^{i}\left(x_{s}^{it}\right)\leq u_{s}^{i}\left(x_{s}^{iT}\right)$$

in which case

$$\max\left\{0,\max_{s:P_{s}^{iT}\neq\varnothing}\max_{t\in P_{s}^{iT}}\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}\right\}=0$$

Alternatively, the condition implies that when $Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)>0$, then $u_{s}^{i}\left(x_{s}^{it}\right)$ should not be much larger than $u_{s}^{i}\left(x_{s}^{iT}\right)$, which makes explicit the cardinal character of the assumption.

It should also be noticed that if $\forall s$, $P_{s}^{iT}=\varnothing$ but $\exists s$ such that $N_{s}^{iT}\neq\varnothing$, then it suffices to define

$$k^{T}\in\left(0,\min_{s:N_{s}^{iT}\neq\varnothing}\min_{t\in N_{s}^{iT}}\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}\right)$$

whereas if $\forall s$, $N_{s}^{iT}=\varnothing$ but $\exists s$ such that $P_{s}^{iT}\neq\varnothing$, then it suffices to define

$$k^{T}\in\left(\max\left\{0,\max_{s:P_{s}^{iT}\neq\varnothing}\max_{t\in P_{s}^{iT}}\frac{u_{s}^{i}\left(x_{s}^{it}\right)-u_{s}^{i}\left(x_{s}^{iT}\right)}{Du_{s}^{i}\left(x_{s}^{iT}\right)\cdot\left(x_{s}^{it}-x_{s}^{iT}\right)}\right\},\infty\right)$$

3.3.3 Conditions on prices and quantities only

The condition

$$Du_s^i\left(x_s^{iT}\right)\cdot\left(x_s^{it}-x_s^{iT}\right)>0\Longrightarrow u_s^i\left(x_s^{it}\right)\leq u_s^i\left(x_s^{iT}\right)$$

is indeed awkward from a revealed preference perspective, but it allows us to illustrate that there are sufficient conditions on the prices and demands only: suppose that for every $t\in\{1,...,T-1\}$ and every s, $p_s^T\cdot\left(x_s^{it}-x_s^{iT}\right)\leq 0$, which is to say that the T^{th} observation is revealed preferred to all other observation in all the states of the world.

Similarly, the condition that for every $t\in\{1,...,T-1\}$ and every s, $p_s^T\cdot\left(x_s^{it}-x_s^{iT}\right)\geq 0$ suffices.

4 Concluding remarks

If an economy is not stationary, observation of all market information, on and off the realized path does not suffice to identify Pareto-improving policies. There exists a profile of preferences which would have yielded the exact same equilibrium information, but for which any reallocation policy would have opposite effects: if it is Pareto-improving in the real economy, there is at least one individual who is made worse off in the other economy, which cannot be ruled out. The intuition is that one equilibrium is not enough to pin down the vector of marginal utilities of income at the terminal nodes for each individual. One can scramble, or add, these vectors across individuals and still respect observed market behavior. Obviously, the fact that one cannot rule out these different profiles of marginal utilities is relevant when they are not collinear, which is the condition under which Pareto-improving reallocations exist in the first place. The result is then most meaningful: the same feature of the equilibrium that explains the existence of Pareto-improving policies implies that it is impossible to identify one such policy.

When the data set is a series of prices and endowments, the problem is more restrictive, and the nonidentification results are weakened. This is so, because the scramble or addition of marginal utilities of income are performed via noninfinitesimal perturbations of the utility functions at the points where policy is to be attempted. When multiple equilibria have to be rationalized, these perturbations must be local. But then, since we have additively separable preferences, guaranteeing concavity weakens the result. In this paper, we guarantee concavity via Afriat inequalities. It follows that there do exist conditions on the data under which one can ensure the existence of multiple rationalizations yielding different welfare effects for any policy.

In both settings, when identification fails, market performance is less questionable on grounds of its inefficiency. Granted, there may be Pareto-improving policies, but market information does not suffice for their design. Whether a mechanism can be designed in order to elicit information about the consumers' preferences is an open question. Also, this paper does not address the case of a stationary economy, which is interesting.

Appendix: lemmata

Lemma 1 *Fix* $u\in\mathcal{U}$, $\Lambda\in\mathbb{R}_{++}$ *and* $\left(x^t\right)_{t=1}^T\in\mathbb{R}_+^{LT}$ *such that* $t\neq t'\Longrightarrow x^t\neq x^{t'}$, *and let* $\left(\left(k^t\right)_{t=1}^T,\left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1}\right)\in\mathbb{R}_{++}^T\times\mathbb{R}^{S(T-1)}$ *solve the system*

$$(\forall t\in\{1,...,T-1\}):k^tDu\left(x^t\right)\cdot\left(x^t-x^T\right)<\mu^t<k^T\Lambda Du\left(x^T\right)\cdot\left(x^t-x^T\right)$$

$$(\forall t,t'\in\{1,...,T-1\}:t\neq t'):\mu^{t'}<\mu^t+k^tDu\left(x^t\right)\cdot\left(x^{t'}-x^t\right)$$

There exists $\widetilde{u} \in \mathcal{U}$ *such that*

1. *For every* $t \in \{1, ..., T-1\}$, $D\widetilde{u}\left(x_s^t\right) = \frac{k^t}{k^T} Du\left(x^t\right)$.
2. *There exists* $\epsilon > 0$ *such that* $\forall x \in B_\epsilon\left(x_s^{iT}\right)$, $\widetilde{u}\left(x\right) = \Lambda u\left(x\right)$.

Proof. Define $\left(u^t\right)_{t=1}^{T-1} = \left(\Lambda u\left(x^T\right) + \mu^t / k^T\right)_{t=1}^{T-1}$. By construction,

$$\begin{aligned}
\left(\forall t \in \{1, ..., T-1\}\right) &: u^t < \Lambda u\left(x^T\right) + \Lambda Du\left(x^T\right) \cdot \left(x^t - x^T\right) \\
\left(\forall t \in \{1, ..., T-1\}\right) &: \Lambda u\left(x^T\right) < u^t + \frac{k^t}{k^T} Du\left(x^t\right) \cdot \left(x^T - x^t\right) \\
\left(\forall t, t' \in \{1, ..., T-1\} : t \neq t'\right) &: u^{t'} < u^t + \frac{k^t}{k^T} Du\left(x^t\right) \cdot \left(x^{t'} - x^t\right)
\end{aligned}$$

As in Matzkin and Richter (1991), define $h : \mathbb{R}^L \longrightarrow \mathbb{R}$, as $h\left(x\right) = \sqrt{\left\|x\right\|^2 + 1} - 1$. h is C^2 and strongly convex and satisfies $h\left(x\right) = 0 \iff x = 0$, $h\left(x\right) > 0 \iff x \neq 0$ and $\left(\forall l \in \{1, ..., L\}\right)$: $\frac{\partial h}{\partial x_l}\left(\cdot\right) \in [0, 1)$.

Since $T < \infty$, there exists $\gamma \in \mathbb{R}_{++}$ such that

$$\begin{aligned}
\left(\forall t, t' \in \{1, ...T-1\}, t \neq t'\right) : u^{t'} &< u^t + \frac{k^t}{k^T} Du\left(x^t\right) \cdot \left(x^{t'} - x^t\right) - \gamma h\left(x^{t'} - x^t\right) \\
\Lambda u\left(x^T\right) &< u^t + \frac{k^t}{k^T} Du\left(x^t\right) \cdot \left(x^T - x^t\right) - \gamma h\left(x^{t'} - x^t\right)
\end{aligned}$$

Now, for each $t \in \{1, ..., T-1\}$, define $\phi_t : \mathbb{R}^L \longrightarrow \mathbb{R}$ by

$$\phi_t\left(x\right) = u^t + \frac{k^t}{k^T} Du\left(x^t\right) \cdot \left(x - x^t\right) - \gamma h\left(x - x^t\right)$$

while $\phi_T = \Lambda u$. Notice that $\forall t \in \{1, ..., T\}$, ϕ_t is strictly concave, whereas, for each $t \in t \in \{1, ..., T-1\}$,

$$\left(\forall l \in \{1, ..., L\}\right) : \frac{\partial \phi_t}{\partial x_l}\left(\cdot\right) = \frac{k^t}{k^T} \frac{\partial u}{\partial x_l}\left(x^t\right) - \gamma \frac{\partial h}{\partial x_l}\left(x - x^t\right) > \frac{k^t}{k^T} \frac{\partial u}{\partial x_l}\left(x^t\right) - \gamma$$

so (since $L < \infty$ and $T < \infty$), γ may be taken small enough so as to make ϕ_t strictly monotone.

Define $\widetilde{u} : \mathbb{R}_+^L \longrightarrow \mathbb{R}$ by $\widetilde{u}\left(x\right) = \min_{t \in \{1, ..., T\}} \{\phi_t\left(x\right)\}$, a continuous, strongly concave, strictly monotone, differentiable almost everywhere (in Lebesgue measure) function. By continuity, there exists $\epsilon \in \mathbb{R}_{++}$ such that $x \in B_\epsilon\left(x^t\right) \implies u\left(x\right) = \phi_t\left(x\right)$, which also implies that for every $t \in \{1, ..., T-1\}$, $D\widetilde{u}\left(x^t\right) = \frac{k^t}{k^T} Du\left(x^t\right)$. Differentiability of $\widetilde{u}$ everywhere can be obtained as in Chiappori and Rochet (1987). ■

Lemma 2 *Let* $\left(q^t, \left(p_s^t\right)_{s=1}^S, \left(\left(x_s^{it}, e_s^{it}\right)_{s=1}^S, z^{it}\right)_{i=1}^I\right)_{t=1}^T$ *be such that, for every* $t \in \{1, ..., T\}$, $\left(q^t, \left(p_s^t\right)_{s=1}^S, \left(\left(x_s^{it}\right)_{s=1}^S, z\right.\right.$ *is a financial markets equilibrium for economy* $\left(\left(u_s^i, e_s^{it}\right)_{s=1}^S\right)_{i=1}^I$ *and for every* $t \neq t'$, *every* i *and every* s, $x_s^{it} \neq x_s^{it'}$. *Suppose that for every* $i \in \{1, ..., I\}$ *there exists a solution* $\left(\left(k^t\right)_{t=1}^T, \left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1}\right)$ *to the system of inequalities*

$$\left(\forall s\right)\left(\forall t \in \{1, ..., T-1\}\right) : k^t Du_s^i\left(x_s^{it}\right) \cdot \left(x_s^{it} - x_s^{iT}\right) < \mu_s^t < k^T \frac{\sum_{j \in \{1, ..., I\} \setminus \{i\}} \lambda_s^{jT}}{\lambda_s^{iT}} Du_s^i\left(x_s^{iT}\right) \cdot \left(x_s^{it} - x_s^{iT}\right)$$

$$(\forall s)\,(\forall t,t'\in\{1,...,T-1\}:t\neq t'):\mu_s^{t'}<\mu_s^t+k^tDu_s^i\left(x_s^{it}\right)\cdot\left(x_s^{it'}-x_s^{it}\right)$$

where $\left(\lambda_s^{jT}\right)_{s=0}^S$ *is such* $Du_s^j\left(x_s^{jT}\right)=\lambda_s^{jT}p_s^T$ *and* $\sum_s\lambda_s^{jT}r_s=\lambda_0^{jT}q^T$. *Then, there exist preferences* $\left(\left(\widetilde{u}_s^i\in\mathcal{U}\right)_{s=1}^S\right)_{i=1}^I$ *such that,*

1. *For every* $t\in\{1,...,T\}$, *observed* $\left(q^t,(p_s^t)_{s=1}^S,\left(\left(x_s^{it}\right)_{s=1}^S,z^{it}\right)_{i=1}^I\right)$ *is a financial markets equilibrium for economy* $\left(\left(\widetilde{u}_s^i,e_s^{it}\right)_{s=1}^S\right)_{i=1}^I$.
2. *For every* $\left(dz^i\right)_{i=1}^I\in\left(\mathbb{R}^{\#A_\sigma}\right)^I$, *such that* $\sum_{i=1}^I dz^i=0$, *it is true that* $\sum_{i=1}^I\sum_{s=1}^S d\widetilde{u}_s^{iT}=-\sum_{i=1}^I\sum_{s=1}^S du_s^{iT}$, *where* $\left(du_s^{iT}\right)_{i=1}^I$ *and* $\left(d\widetilde{u}_s^{iT}\right)_{i=1}^I$ *are the (spot) general equilibrium welfare effects resulting from revenue transfer* $\left(r_sdz^i\right)_{i=1}^I$ *at spot equilibrium* $\left(p_s^T,\left(x_s^{iT}\right)_{i=1}^I\right)_{s=1}^S$ *in exchange economies* $\left(u_s^i,e_s^{iT}+r_sz^{iT}\right)_{i=1}^I$ *and* $\left(\widetilde{u}_s^i,e_s^{iT}+r_sz^{iT}\right)_{i=1}^I$, *respectively.*

Proof. For each i, from a solution $\left(\left(k^t\right)_{t=1}^T,\left(\left(\mu_s^t\right)_{s=1}^S\right)_{t=1}^{T-1}\right)$ to i's system of inequalities, it follows from lemma 1 that for each s there exists $\widetilde{u}_s^i\in\mathcal{U}$ and such that:

1. $\forall t\in\{1,...,T-1\}$, $D\widetilde{u}_s^i\left(x_s^{it}\right)=\frac{k^t}{k^T}Du_s^i\left(x_s^{it}\right)$
2. $\exists\epsilon>0$ such that $\forall x\in B_\epsilon\left(x_s^{iT}\right)$, $\widetilde{u}_s^i(x)=\frac{\sum_{j\in\{1,...,I\}\setminus\{i\}}\lambda_s^{jT}}{\lambda_s^{iT}}u_s^i(x)$

Fix $t\in\{1,...,T-1\}$. Letting $\left(\widetilde{\lambda}_s^{it}\right)_{s=0}^S=\left(\frac{k^t}{k^T}\lambda_s^{it}\right)_{s=0}^S$, we have that, given 1

$$\begin{aligned}D\widetilde{u}_s^i\left(x_s^{it}\right)&=\frac{k^t}{k^T}Du_s^i\left(x_s^{it}\right)=\frac{k^t}{k^T}\lambda_s^{it}p_s^t=\widetilde{\lambda}_s^{it}p_s^t\\ \sum_{s=1}^S\widetilde{\lambda}_s^{it}r_s&=\sum_{s=1}^S\frac{k^t}{k^T}\lambda_s^{it}r_s=\frac{k^t}{k^T}\sum_{s=1}^S\lambda_s^{it}r_s=\frac{k^t}{k^T}\lambda_0^{it}q^t=\widetilde{\lambda}_0^{it}q^t\end{aligned}$$

from where $\left(q^t,(p_s^t)_{s=1}^S,\left(\left(x_s^{it}\right)_{s=1}^S,z^{it}\right)_{i=1}^I\right)$ is an equilibrium for economy $\left(\left(\widetilde{u}_s^i,e_s^{it}\right)_{s=1}^S\right)_{i=1}^I$.

To see that $\left(q^T,(p_s^T)_{s=1}^S,\left(\left(x_s^{iT}\right)_{s=1}^S,z^{iT}\right)_{i=1}^I\right)$ is an equilibrium for $\left(\left(\widetilde{u}_s^i,e_s^{iT}\right)_{i=1}^I\right)_{s=1}^S$, let $\left(\widetilde{\lambda}_s^{iT}\right)_{s=0}^S=\left(\sum_{j\in\{1,...,I\}\setminus\{i\}}\lambda_s^{jT}\right)_{s=0}^S\in\mathbb{R}_{++}^{S+1}$ and observe that $D\widetilde{u}_s^i\left(x_s^{iT}\right)=\widetilde{\lambda}_s^{iT}p_s^T$ and

$$\sum_{s=1}^S\widetilde{\lambda}_s^{iT}r_s=\sum_{s=1}^S\sum_{j\in\{1,...,I\}\setminus\{i\}}\lambda_s^{jT}r_s=\sum_{j\in\{1,...,I\}\setminus\{i\}}\sum_{s=1}^S\lambda_s^{jT}r_s=\sum_{j\in\{1,...,I\}\setminus\{i\}}\lambda_0^{jT}q^T=\widetilde{\lambda}_0^{iT}q^T$$

Now, since dp_s^T is the same in exchange economies $\left(u_s^i,e_s^{iT}+r_sz^{iT}\right)_{i=1}^I$ and $\left(\widetilde{u}_s^i,e_s^{iT}+r_sz^{iT}\right)_{i=1}^I$, it follows that $d\widetilde{u}_s^{iT}=\frac{\sum_{j\in\{1,...,I\}\setminus\{i\}}\lambda_s^{jT}}{\lambda_s^{iT}}du_s^{iT}$, which means

$$d\widetilde{u}_s^{iT}=\left(\frac{\sum_{j=1}^I\lambda_s^{jT}-\lambda_s^{iT}}{\lambda_s^{iT}}\right)du_s^{iT}=\left(\frac{\Lambda_s^T}{\lambda_s^{iT}}-1\right)du_s^{iT}$$

and, hence,

$$\sum_{j=1}^{I} d\widetilde{u}_s^{iT} = \Lambda_s^T \sum_{j=1}^{I} \frac{du_s^{iT}}{\lambda_s^{iT}} - \sum_{j=1}^{I} du_s^{iT} = -\sum_{j=1}^{I} du_s^{iT}$$

(because $\sum_{i=1}^{I} dz^i = 0$). It then follows that

$$\sum_{i=1}^{I}\sum_{s=1}^{S} d\widetilde{u}_s^{iT} = \sum_{s=1}^{S}\sum_{i=1}^{I} d\widetilde{u}_s^{iT} = -\sum_{s=1}^{S}\sum_{i=1}^{I} du_s^{iT} = -\sum_{i=1}^{I}\sum_{s=1}^{S} du_s^{iT}$$

■

Incidentally, the lemma implies that even the observation of all equilibrium information at observed a finite set of endowments may not allow a planner to discern between economies in which, at given endowments, for any policy, the aggregates of individual welfare effects go in opposite direction. In particular, it may not distinguish between an economy in which a policy is Pareto-improving and another in which at least one individual is made worse off.

The assumption of the previous lemma is not a necessary condition. First, it is sufficient for the profile of preferences that we construct, but there may be other profiles with the same implication. Secondly, for the case in which a policy is Pareto-improving in the original preferences, we only need the condition to be satisfied for the one individual who ends up with a welfare loss in the constructed preferences (at observation T) but not for other individuals; of course, we need to assume it for all individuals because we do not know ex-ante who the individual who loses will be.

Although the system resembles Brown-Matzkin, it must be noticed that it is different and that it does not amount to assuming the rationalization of the observed equilibria. The differences are: 1) this system is individual; 2) it does not involve prices; 3) it does not involve asset markets at all; 4) there are no market-clearing conditions. The only reason why the system has to be there is for strong concavity. It is crucial that in the system of inequalities $(k^t)_{t=1}^T$ is independent of s.

Corollary 2 *Let* $\left(q^t, (p_s^t)_{s=1}^S, \left((x_s^{it}, e_s^{it})_{s=1}^S, z^{it}\right)_{i=1}^I\right)_{t=1}^T$ *be such that, for every* $t \in \{1, ..., T\}$, $\left(q^t, (p_s^t)_{s=1}^S, \left((x_s^{it})_{s=1}^S, z^{it}\right)_{i=1}^I\right)$ *is a financial markets equilibrium for economy* $\left((u_s^i, e_s^{it})_{s=1}^S\right)_{i=1}^I$ *and for every* $t \neq t'$, *every* i *and every* s, $x_s^{it} \neq x_s^{it'}$. *There exists a finite system of polynomial inequalities on* $\left(\lambda_s^{iT}, \sum_{j\in\{1,...,I\}\setminus\{i\}} \lambda_s^{jT}, (p_s^t, x_s^{it})_{t=1}^T\right)_{s=1}^S$ *such that if each* i *satisfies those inequalities, then, there exist preferences* $\left((\widetilde{u}_s^i \in \mathcal{U})_{i=1}^I\right)_{s=1}^S$ *such that:*

1. *For every* $t \in \{1, ..., T\}$, *observed* $\left(q^t, (p_s^t)_{s=1}^S, \left((x_s^{it})_{s=1}^S, z^{it}\right)_{i=1}^I\right)$ *is a financial markets equilibrium for economy* $\left((\widetilde{u}_s^i, e_s^{it})_{s=1}^S\right)_{i=1}^I$.
2. *For every* $(dz^i)_{i=1}^I \in \left(\mathbb{R}^{\#A_\sigma}\right)^I$, *such that* $\sum_{i=1}^I dz^i = 0$, *it is true that* $\sum_{i=1}^I \sum_{s=1}^S d\widetilde{u}_s^{iT} = -\sum_{i=1}^I \sum_{s=1}^S du_s^{iT}$.

Proof. This is straightforward from the theorem, by the Tarski-Seidenberg theorem (see theorem 8.6.6 in Mishra, 1993). ■

References

Brown, D. and R. Matzkin, 1996, Testable restrictions on the equilibrium manifold, Econometrica 64, 1249-1262.

Carvajal, A. and A. Riascos, 2004, Global identification from the equilibrium manifold under random preferences, Royal Holloway Discussion Paper dpe04/30.

Donsimoni, M.-P. and H. Polemarchakis, 1994, Redistribution and welfare. Journal of Mathematical Economics, 23, 235-242.

Geanakoplos, J. and H. Polemarchakis, 1986, Existence, regularity and constrained suboptimality of competitive allocations when assets structure is incomplete, Essays in honor of K.J. Arrow, W. Heller, R. Starr and D. Starret (Eds), Vol 3, 65-95, Cambridge.

Kubler, F., P.-A. Chiappori, I. Ekeland and H. Polemarchakis, 2002. The identification of Preferences from Equilibrium Prices under Uncertainty. Journal of Economic Theory, 102, 403-420.

Matzkin, R. and M. Richter, 1991, Testing strictly concave rationality, Journal of Economic Theory 58, 287-303.

Mishra, B., 1993, Algorithmic Algebra (Springer-Verlag, New York).