MONOTONICITY AND RATIONALIZABILITY IN A LARGE FIRST PRICE AUCTION

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ABSTRACT. This paper proves that the monotonicity of bidding strategies together with the rationality of bidders implies that the winning bid in a first price auction converges to the competitive equilibrium price as the number of bidders increases (Wilson (1977)). Instead of analyzing the symmetric Nash equilibrium, we examine rationalizable strategies (Bernheim (1984) and Pearce (1984)) among the set of monotonic bidding strategies to prove that any monotonic rationalizable bidding strategy must be within a small neighborhood of the "true" valuation of the object, conditioned on the signal received by the bidder. We obtain an information aggregation result similar to Wilson (1977), while dispensing with almost all symmetric assumptions and using a milder solution concept than the Nash equilibrium. In particular, if every bidder is ex ante identical, then any rationalizable bidding strategy must be within a small neighborhood of the symmetric Nash equilibrium. In a symmetric first price auction, the symmetry of outcomes is implied rather than assumed.
KEYWORDS. First price auction, Monotonic bidding strategy, Rationalizability, Information aggregation

1. INTRODUCTION

This paper investigates the information aggregation property of the first price auction that the delivery price converges to the highest valuation among bidders as the number of bidders increases. This result was first obtained by Wilson (1977), prompting a large literature on the information aggregation capability of auctions under various institutional and informational assumptions (e.g., Milgrom (1979), Satterthwaite and Williams (1989), Rustichini, Satterthwaite, and Williams (1994), and Pesendorfer and Swinkels (1997)). With a few remarkable exceptions, however, most existing models focused on the symmetric Nash equilibrium in first price auctions populated with ex ante identical bidders. While the symmetric Nash equilibrium and the ex ante identical bidders help us calculate an equilibrium and study its asymptotic properties, the same assumptions significantly restrict the scope of the result.

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Recently, Maskin and Riley (2000), Athey (2001), McAdams (2002), and Reny and Zamir (2002) have made significant progress in proving that a (pure strategy) Nash equilibrium exists in first price auctions populated with ex ante heterogeneous bidders. However, because the first price auction generally has many Nash equilibria, it is not clear whether the information aggregation result holds for *all* Nash equilibria. If we obtain the information aggregation result for a particular class of equilibria as in Wilson (1977) and Pesendorfer and Swinkels (1997), the validity of the result critically relies upon how sensible the selection criterion is.

We opt for a new tack in investigating the information aggregation property of first price auctions populated with ex ante heterogeneous agents. Our solution concept is built upon two mild, and rather natural, restrictions. First, instead of the entire bidding strategies, we let each bidder use a class of "well behaved" strategies that are strictly increasing Lipschitz continuous functions of his own signal. The important assumption is that this restriction on the bidding strategies is common knowledge among players. Because we allow a very large upper bound and a very small lower bound for the slope of the bidding function (i.e., Lipschitz constant), we admit virtually all increasing bidding strategies investigated in the literature.

Second, we use rationalizability (Bernheim (1984) and Pearce (1984)) over the set of restricted bidding strategies instead of Nash equilibrium as the solution concept. In our model, a bidder may not have perfect foresight about the other bidders' strategies. It is well known that the first price auction is not dominance solvable. The rationalizability, if invoked on the set of entire bidding strategies, has little power to narrow down the outcome (Dekel and Wolinsky (2001) and Battigalli and Siniscalchi (2002)). However, the combination of monotonicity imposed on the strategies and rationalizability sharply narrows down the outcome so that we can make a meaningful prediction from the model.[1]

This paper is closely related to Dekel and Wolinsky (2001) and Battigalli and Siniscalchi (2002) that investigate the first price auction through rationalizability. This paper admits a more general informational structure than Dekel and Wolinsky (2001) did by covering both the private value and the common value models, but also those in between as in Wilson (1977). Instead of discrete bidding strategies, we follow Wilson (1977) by assuming continuous distribution over private information and continuous bidding strategies. We impose monotonicity on the bidding strategies to obtain a result as sharp as in Dekel and Wolinsky (2001) under informational assumptions as general as in Battigalli and Siniscalchi (2002).

We demonstrate that for any small $\epsilon > 0$, if the number of bidders is sufficiently large, then any rationalizable monotonic bidding strategy must be located within ϵ neighborhood of the valuation of each bidder conditioned on the event that his bid is pivotal.[2] We eliminate all symmetric assumptions embedded in Wilson (1977). The asymptotic result holds as long as the asymptotic distribution over the ex ante "characteristic" of bidders, that is his utility function and the signal function, is non-atomic (Hildenbrand (1974)) or

[1] This approach of combining the restriction on the strategy space and rationalizability is first used by Cho (1994).

[2] In case of the private value model, the truthful valuation is his private value. If the auction has a common value component, we have to define the notion of truthful valuation properly.

can be obtained by replicating the bidders. In particular, if the agents are ex ante identical as in Wilson (1977), the asymptotic result prevails.

Because a rationalizable monotonic bidding strategy must be within a small neighborhood of a true valuation of the object, the outcome from a large first price auction must be close to the outcome from the competitive market. Moreover, all other losing bids must also be within a close neighborhood of the bidders' valuations of the object. In this sense, the outcome of the market and the behavior of the bidders converge to what we expect from an agent in a competitive market.

If the bidders are ex ante identical, then the symmetric equilibrium in Wilson (1977) is rationalizable, because the equilibrium strategy is a differentiable increasing function of the signal of each bidder. Although the first price auction is not dominance solvable, we essentially show that any sensible equilibrium of the first price auction should be close to the symmetric equilibrium. The symmetry is implied by the symmetry of the primitives of the model rather than arbitrarily imposed by the equilibrium selection rule.

The rest of the paper is organized as follows. In section 2, we rigorously describe the first price auction and the assumptions which we imposed on the informational structure and the strategy space. Section 3 characterizes the set of rationalizable strategies and obtains the asymptotic result. Section 4 concludes the paper.

2. MODEL

Let us consider a first price auction in which n bidders are competing to buy a single indivisible object whose value is v. Each bidder is identified as $i \in \{1, \dots, n\}$. The value is drawn from a probability distribution over $V = [\underline{v}, \overline{v}]$ according to $G(v)$. Conditioned on v, bidder type i receives a signal s_i generated according to a probability distribution function $F_i(s_i|v)$. Assume that for a given $v \in [\underline{v}, \overline{v}]$, s_i and s_j are mutually independent for $\forall i \neq j$. Let

$$[\underline{s}_i(v), \overline{s}_i(v)] = S_i(v)$$

be the support of $F_i(s_i|v)$. Let $S_i = \bigcup_{v \in V} S_i(v)$ be the set of all possible signals that bidder i can receive. Define $\overline{s}_i = \sup S_i$ and $\underline{s}_i = \inf S_i$.

Assumption 2.1. (1) *G and F_i are continuously differentiable over the interior of the support. In addition, for any v in the interior of the support of $G(v)$, $G'(v) > 0$ and $F_i'(s_i|v) > 0$ for $\forall i$, and $\forall s_i$. Let $f_i(s_i|v) = F_i'(s_i|v)$ be the probability density function of F_i. For $\forall i$, $F_i(s_i|v)$ is a weakly decreasing function of v for a fixed s_i and satisfies the monotonic likelihood ratio property. In particular, if $0 < F_i(s_i|v) < 1$, then for a small $h > 0$,*

$$\frac{F_i(s_i|v-h)}{F_i(s_i|v)} > 1.$$

(2) *$\overline{s}_i(v)$ and $\underline{s}_i(v)$ are strictly increasing.*

(3) *For $\forall \rho > 0$ and $\forall v$,*

$$\limsup_{n \to \infty} \max_{1 \le i \le n} F_i(\overline{s}_i(v) - \rho | v) < 1. \tag{2.1}$$

The first condition is standard in the literature. The second condition implies that if bidder i receives a higher signal, then his distribution over v conditioned on his signal s_i

must be increasing in a stochastic sense.[3] One can replace the last condition by a stronger condition used in Wilson (1977) which says

$$\liminf_{n\to\infty} \min_{1\le i\le n} F_i'(\overline{s}_i(v)|v) > 0$$

so that there is a positive probability mass around the neighborhood of $\overline{s}(v)$.

The last two conditions are restrictive because some distributions can be stochastically increasing while sharing the same support (Pesendorfer and Swinkels (1997)). Yet, we use these conditions in order to illuminate the key difference of our exercise from Wilson (1977).

Define

$$u_i : S_i \times V \to \mathbb{R}$$

as the utility function of bidder i that is strictly increasing with respect to both arguments.

Assumption 2.2. *u_i is continuously differentiable and*

$$\frac{\partial u_i(s_i,v)}{\partial s_i} \ge 0, \quad \frac{\partial u_i(s_i,v)}{\partial v} \ge 0 \ \textit{and} \ \ \frac{\partial u_i(s_i,v)}{\partial s_i} + \frac{\partial u_i(s_i,v)}{\partial v} > 0 \qquad \forall s_i, \forall v.$$

Important special cases are the private value model ($u_i(s_i,v) = s_i$) and the common value model ($u_i(s_i,v) = v$). Following Wilson (1977), we interpret s_i as the private value component, and v as the common value component. To simplify the analysis, we focus on the two most important cases: the pure private value model where

$$\frac{\partial u_i(s_i,v)}{\partial v} = 0 \ \text{ and } \ \frac{\partial u_i(s_i,v)}{\partial s_i} > 0 \qquad \forall s_i, \forall v$$

and another case where the payoff function is strictly increasing with respect to the common value component:

$$\frac{\partial u_i(s_i,v)}{\partial v} > 0 \qquad \forall s_i, \forall v,$$

which includes the pure common value model.

Since $\overline{s}_i(v)$ and $\underline{s}_i(v)$ are strictly increasing, we can define their inverse functions a_i and a_i' as $a_i(\overline{s}_i(v)) = v$ and $a_i'(\underline{s}_i(v)) = v$, respectively. By definition,

$$u_i(\overline{s}_i(v), v) = u_i(s, a_i(s)),$$

where $s = \overline{s}_i(v)$.

Because we admit ex ante heterogeneous bidders, we need to differentiate heterogeneity at the ex ante stage from the heterogeneity caused by a difference in private signals.

Definition 2.3. *By a characteristic of bidder i, we mean (u_i, F_i) which represents his utility function and the information function. Let us define a metric over the space of characteristics of bidders as*

$$d(i,j) = \|u_i - u_j\| + \|F_i - F_j\| \tag{2.2}$$

where $\|\cdot\|$ is the sup norm metric. If $d(i,j) = 0$, we say that i and j are ex ante identical.

[3] If $\overline{s}_i(v)$ or $\underline{s}_i(v)$ is weakly increasing, the main result of the paper holds after some modifications but the analysis requires considerably more complex notation.

Our model covers the first price auction populated with ex ante identical bidders as a special case. But we can also investigate the information aggregation problem in which some bidders are different in an ex ante sense as illustrated in the following example.

Example 2.4. *Suppose that n is an even number. Each bidder $i \in \{1, \ldots, n/2\}$ is trying to purchase the object for his own consumption and regards the auction as the private value model: $u_i(s_i, v) = s_i$ for $i = 1, \ldots, n/2$. The main objective of the remaining half of the bidders is to resell the item later. They view the auction as the common value model: $u_i(s_i, v) = v$ for $i = n/2 + 1, \ldots, n$.*[4]

We increase the number of bidders in order to model the increase in the competition in the auction. However, in order to capture the competitive pressure arising from an increased number of bidders, we need to restrict how we can "expand" the game.

Definition 2.5. *We say that i is not isolated asymptotically if for $\forall \mu > 0$, $\forall i$,*

$$\liminf_{n\to\infty} \frac{\#\{j | d(i,j) < \mu\}}{n} > 0, \tag{2.3}$$

where $\#\{\cdot\}$ is the number of elements in $\{\cdot\}$.

If i is isolated asymptotically, then i can be differentiated from other bidders and consequently, can still exercise a monopolistic power. Condition (2.3) forces each agent to be "surrounded" by many agents who are only slightly different. This condition is satisfied if the asymptotic distribution of the characteristics is non-atomic, but also if we expand the game by replicating the agents with the same characteristics as in Example 2.4. In particular, if every agent is ex ante identical as in Wilson (1977), then (2.3) holds.[5]

A bidding function or a strategy of bidder i is

$$p_i : S_i \to \mathbb{R}.$$

Let $p = (p_1, \ldots, p_n)$ be the profile of bidding strategies. By subscript "$-i$", we mean a profile that includes all but the i-th component. For example, by p_{-i}, we mean a strategy profile of all bidders except for i. Let $G(v|s_i)$ be the posterior distribution over v conditioned on s_i and $F_i(s_i)$ be the marginal distribution of s_i. Define for each realized bid p_i,

$$\Pi_i(p_i, s_i) = \int_v (u_i(s_i, v) - p_i)\, Q_{n,i}(p_i|v) dG(v|s_i)$$

as the interim (subjective) expected profit of bidder i conditioned on signal s_i and his bid p_i, where

$$Q_{n,i}(p_i|v) = \prod_{j\neq i} F_j\left(p_j^{-1}(p_i)|v\right)$$

is the probability that p_i is a winning bid. Note that p_{-i} represents bidder i's subjective conjecture about other players' strategies, which can be different from the actual strategies selected by the other players. Player i is pivotal if his bid p_i satisfies

$$p_i = \max_{j\neq i} p_j.$$

[4] I am grateful for Preston McAfee for the example.

[5] This condition renders each bidder informationally small in the sense of Gul and Postlewaite (1992).

Recall that $a_i(s_i)$ is the lowest v and $a'_i(s_i)$ is the highest v that are consistent with signal s_i:

$$\begin{aligned} a_i(s_i) &= \inf \{v | f_i(s_i|v) > 0\} \\ a'_i(s_i) &= \sup \{v | f_i(s_i|v) > 0\}. \end{aligned}$$

Definition 2.6. *We say that $u_i(s_i, a_i(s_i))$ is the truthful valuation of bidder i conditioned on s_i. We simply call $u_i(s_i, a_i(s_i))$ the truthful valuation whenever the meaning is clear from the context.*

Our notion of the truthful valuation represents an extremely pessimistic view of the bidders. Yet, this pessimism is due to the winner's curse in the first price auction. In a symmetric equilibrium, every bidder is using a strictly increasing bidding strategy. The bidder with the most optimistic estimate about the underlying value of the object becomes the winner. As the number of bidders increases, the winner should be the one with the highest signal conditioned on v. Because each bidder is well aware of the winner's curse, it is very much sensible to place the lowest possible value conditioned on the winning signal: $a_i(s_i)$. We shall show that the winner's curse is a direct implication of the monotonic bidding strategies, largely independent of the symmetry or the equilibrium hypothesis.

Definition 2.7. *A profile $p^*_n = (p^*_{n,1}, \ldots, p^*_{n,n})$ of bidding strategies is a Nash equilibrium if for $\forall i$, $\forall v$, $\forall s_i \in S_i(v)$,*

$$\Pi_i(p^*_{n,i}(s_i), s_i) \geq \Pi_i(p_i, s_i) \qquad \forall p_i \in \mathbb{R}_+.$$

In particular, if p^*_n is a Nash equilibrium and

$$p^*_{n,1}(s) = \cdots = p^*_{n,n}(s) \qquad \forall s,$$

then p^*_n is a symmetric Nash equilibrium.

Wilson (1977) analyzed the first price auction populated with ex ante identical bidders. Consider the following strategy:

$$p^*_{n,i}(s_i) = \int_{\underline{s}_i}^{\overline{s}_i} \overline{u}_n(t) dH_n(t|s)^{n-1} \tag{2.4}$$

where

$$H_n(t|s) = \exp\left(- \int_t^s \hat{\phi}_n(\tau) d\tau \right),$$

$$\hat{\phi}_n(s_i) = \int_{a_i(s_i)}^{a'_i(s_i)} \frac{f(s_i|v)}{F(s_i|v)} dG(v|s_i)$$

and

$$\overline{u}_n(t) = E\left[u_i(s_i, v) | s_i, p^*_{n,i} = \max_{j \neq i} p^*_{n,j} \right].$$

Notice that if every bidder is ex ante identical, then

$$p^*_{n,i}(s) = p^*_{n,j}(s) \qquad \forall i \neq j.$$

Theorem 2.8. *If every bidder is ex ante identical, then $p_n^* = (p_{n,1}^*, \ldots, p_{n,n}^*)$ is a symmetric Nash equilibrium and the winning bid of the symmetric Nash equilibrium converges to*

$$\max_{i \in \{1,\ldots,n\}} u_i(\overline{s}_i(v), v) \qquad \forall v$$

as $n \to \infty$.

Proof. See Wilson (1977).

The first price auction aggregates the diverse private information to achieve the efficient allocation as the number of bidders increases. In the limit, the good is allocated to the bidder with the highest valuation, even though s_i remains private information of each bidder. Yet, the theory says little about the losing bids, which could deviate significantly from $u_i(s_i, a_i(s_i))$. We shall derive a result close to Theorem 2.8 without relying on the equilibrium hypothesis or the symmetry of equilibrium strategies. Still, we can say a lot about the losing bid as well as the winning bid from the first price auction when the number of bidders is very large.

Instead of all possible strategies, we consider a class of "well behaved" strategies in the following sense.

Assumption 2.9. *Fix $M > 0$. For $\forall i$, let Σ_i^M be the set of feasible bidding strategies for player i. If $p_i \in \Sigma_i^M$, then $p_i(s_i)$ is a continuous function of s_i satisfying*

$$\frac{1}{M} \leq \frac{p_i(s_i) - p_i(s_i')}{s_i - s_i'} \leq M \qquad \forall s_i \neq s_i' \tag{2.5}$$

and

$$p_i(\underline{s}_i) = u_i(\underline{s}_i, a_i(\underline{s}_i)). \tag{2.6}$$

Let $\Sigma^M = \prod_{i=1}^n \Sigma_i^M$. Σ^M is common knowledge among players.

The first condition (2.5) says that the bid should be a strictly increasing uniformly Lipschitz continuous function of the private signal. As $M \to \infty$, we can "approximate" any strictly increasing bidding functions that are used in the equilibrium models for the first price auctions. The second condition (2.6) says that if bidder i receives the worst possible signal, then his bid will be precisely his expected valuation of the good under the most pessimistic assumption $a_i(\underline{s}_i)$ about the valuation conditioned on the worst possible $\underline{s}_i$.[6] If bidder i receives the worst possible signal, then he concludes that it is not worthwhile placing any meaningful bid.

Let $\{p_i^k\}_{k=1}^\infty \subset \Sigma_i^M$ be a sequence of feasible bidding strategies of player i, and $p_i \in \Sigma_i^M$. If $p_i^k(s_i) \to p_i(s_i)$ pointwise for $\forall s_i \in S_i$, then we write $p_i^k \to p_i$.

Lemma 2.10. *For $\forall i$, Σ_i^M is compact.*

Proof. See Appendix A

[6] A similar condition can be found in Battigalli and Siniscalchi (2002).

Remark 2.11. *To simplify notation, we shall assume that the strategies are differentiable, and the interim expected payoff function $\Pi_i(p_i, s_i)$ is differentiable with respect to p_i. This differentiability can be dispensed without affecting any results.*

Instead of Nash equilibrium, we invoke rationalizability over Σ^M (Bernheim (1984) and Pearce (1984)). Given a profile p_{-i} of bidding strategies other than bidder i, $p_i \in \mathbb{R}$ is a best response to p_{-i} conditioned on s_i if

$$\Pi_i(p_i, s_i) \geq \Pi_i(\tilde{p}_i, s_i) \qquad \forall \tilde{p}_i.$$

We write

$$p_i = b_i(s_i, p_{-i})$$

as the best response of player i conditioned on s_i against p_{-i}. Fix $p_i \in \Sigma_i^M$. Let $\Sigma_i' \subset \Sigma_i^M$ for $\forall i$. If for $\forall s_i \in S_i$, there exists $p_{-i} \in \Sigma_{-i}'$ such that $p_i(s_i)$ is a best response to p_{-i}, then we write

$$p_i \in BR_i(\Sigma_{-i}' | \Sigma_i').$$

Let $\Sigma_i^{M,0} = \Sigma_i^M$. Suppose that $\Sigma_i^{M,0}, \ldots, \Sigma_i^{M,k-1}$ are defined. Let

$$\Sigma_i^{M,k} = BR_i\left(\Sigma_{-i}^{M,k-1} | \Sigma_i^{M,k-1}\right).$$

Definition 2.12. *A bidding strategy $p_i \in \Sigma_i^M$ is rationalizable if*

$$p_i \in \bigcap_{k=0}^{\infty} \Sigma_i^{M,k} \equiv R_{n,i}^M.$$

A profile $p = (p_1, \ldots, p_n)$ of bidding strategies is rationalizable if for each i, p_i is rationalizable.

The symmetric Nash equilibrium strategy $p_{n,i}^*$ calculated in (2.4) is a strictly increasing Lipschitz continuous function. Thus,

$$\exists M', \ \forall M \geq M', \ \forall n, \ \forall i, \quad p_{n,i}^* \in R_{n,i}^M \neq \emptyset \tag{2.7}$$

in the first price auction populated with ex ante identical bidders.

Remark 2.13. *In principle, we have to admit a belief that is a probability distribution over the bidding strategies of others. However, we opt for our more restrictive notion of rationalizability that admits only a belief over pure strategy profiles, because we have the same conclusion with a general notion of rationalizability at the cost of significantly more complicated notation.*

If $S_i = S_j$ for $\forall i \neq j$, then one can easily show that

$$\forall i, \forall p_i \in R_{n,i}^M, \forall s_i > \underline{s}_i \ \ \Pi_i(p_i(s_i), s_i) > 0.$$

That is, except for the lowest signal, player i can win the object with a positive probability against any rationalizable bidding strategies. However, as we admit ex ante heterogenous bidders, in particular $F_i(s_i|v)$ with different supports conditioned on v, it is possible that $\Pi_i(p_i(s_i), s_i) = 0$ for some rationalizable strategy p_i, as illustrated in the next example.

Example 2.14. *Suppose that n is an even number. Every player i has utility function $u(s_i, v) = s_i + v$. For $i \in \{1, \ldots, n/2\}$, $F_i(s_i|v)$ is the uniform distribution over $[v, 2v]$ for $v \in [0, 2]$, while for $i \in \{n/2+1, \ldots, n\}$, $F_i(s_i|v)$ is the uniform distribution over $[v+5, 2v+5]$.*

Conditioned on $s_1 = 3$, the support of the posterior distribution $G(v|s_i)$ is $[1, 2]$. For $v \in [1, 2]$, every possible signal for bidder 1 is smaller than any possible signal for $i \geq n/2 + 1$. Since no player $i \geq n/2 + 1$ places a bid smaller than 5, bidder 1 knows that his chance of winning the object is precisely 0 when he receives $s_1 = 3$. In this case, every bidding strategy in Σ_1^M is rationalizable.

Our analysis applies only for a player who has a positive probability of winning the object and therefore, generates a positive expected profit from the game. That is, conditioned on s_i, there exists $v \in [a_i(s_i), a_i'(s_i)]$ such that s_i is the highest signal in the population. This property is guaranteed if the support of the private signal of each player is identical. Indeed, in calculating a symmetric equilibrium in a model populated with ex ante identical bidders, we practically take it for granted that the players can generate a positive expected profit. To simplify the notation, we shall maintain the assumption that the expected profit of player i conditioned on s_i is positive, unless we specifically say otherwise.

Still, some bidding strategy in Σ_i^M appears to be more sensible than the others even if player i has no chance of winning the object. Because we are interested in the behavior of each player as well as the outcome of the game, it is important to see whether we can discriminate some strategies against the others. We shall return to discuss this issue at the end of Section 3.

3. Analysis

Define

$$\overline{p}_{n,i}(s_i) = \max\left\{p_i(s_i) | p_i \in R_{n,i}^M\right\}$$

and

$$\underline{p}_{n,i}(s_i) = \min\left\{p_i(s_i) | p_i \in R_{n,i}^M\right\}.$$

Lemma 3.1. *If* $p_i, \tilde{p}_i \in R_{n,i}^M$, *then* $\max(p_i, \tilde{p}_i), \min(p_i, \tilde{p}_i) \in R_{n,i}^M$.

Proof. See Appendix B.

By Lemma 2.10 and Lemma 3.1, $\overline{p}_{n,i}(s_i), \underline{p}_{n,i}(s_i) \in R_{n,i}^M$. Define $\overline{p}(s_i)$ and $\underline{p}(s_i)$ as the pointwise limits of $\{\overline{p}_{n,i}(s_i)\}_{n=1}^{\infty}$ and $\{\underline{p}_{n,i}(s_i)\}_{n=1}^{\infty}$.

It is fairly straightforward to characterize the lower bound of the set of rationalizable strategies when the number of bidders is large.

Proposition 3.2. $\forall \epsilon$, $\exists N$ *such that* $\forall n \geq N$, $\forall i$, $\forall s_i$,

$$\underline{p}_{n,i}(s_i) \geq u_i(s_i, a_i(s_i)) - \epsilon.$$

Proof. See Appendix C.

To explain the intuition of Proposition 3.2, let us suppress for a moment the restriction imposed on the bidding strategies, and assume that every bidder is ex ante identical. Consider the first derivative of the interim expected payoff:

$$\frac{\partial \Pi(p_i, s_i)}{\partial p_i} = \int_v \left[-1 + (u_i(s_i, v) - p_i) K_{n,i}(p_i, v)\right] Q_{n,i}(p_i, |v) dG(v|s_i) \tag{3.8}$$

where

$$K_{n,i}(p_i, v) = \sum_{j \neq i} \frac{f_j(s_j|v)}{p'_j(s_j)F_j(s_j|v)}$$

in which

$$s_j = p_j^{-1}(p_i(s_i)). \tag{3.9}$$

If $p_i(s_i)$ is a best response to p_{-i}, then the first order derivative must vanish whenever the objective function is differentiable. Note that $K_{n,i}(p_i(s_i), v)$ cannot be equal to 0 for every v if $p_i \in R^M_{n,i}$. Otherwise, bidder i can increase his payoff by lowering his bid slightly. Hence,

$$\begin{aligned} \frac{\partial \Pi(p_i(s_i), s_i)}{\partial p_i} &\propto \mathsf{E}\left[-\frac{1}{K_{n,i}(p_i(s_i), v)} + (u_i(s_i, v) - p_i(s_i)) \mid p_i(s_i) = \max_{j \neq i} p_j\right] \\ &\geq \mathsf{E}\left[-\frac{1}{K_{n,i}(p_i(s_i), v)} + (u_i(s_i, a(s_i)) - p_i(s_i)) \mid p_i(s_i) = \max_{j \neq i} p_j\right] \end{aligned} \tag{3.10}$$

since $u_i(s_i, a_i(s_i)) \leq u_i(s_i, v)$ for any v in the support of $G(v|s_i)$.

Note that the expectation is conditioned on the event that p_i is pivotal. If bidder i can win the object by placing $p_i < u_i(s_i, a_i(s_i))$, then he knows for sure that his surplus is strictly positive. Thus, he can afford to increase his bid slightly in order to improve his chance of winning the object. If

$$K_{n,i}(p_i, v) \to \infty, \tag{3.11}$$

then for any $p_i(s_i) < u_i(s_i, a(s_i))$, the right hand side of inequality (3.10) becomes positive for a sufficiently large n. Player i can indeed increase his payoff by slightly increasing his bid (subject to the constraint imposed on the strategy space).

However, we face a main complication in formalizing this simple intuition because (3.11) may not hold for some strategy profiles. If $p_j(s_j)$ is sufficiently small, say

$$p_j(s_j) = \frac{s_j - \underline{s}_j}{M} + u_j(\underline{s}_j, a(\underline{s}_j)),$$

it is possible that for $j \neq i$,

$$p_j(s_j) \leq p_j(\overline{s}_j(v)) < p_i(s_i).$$

In such a case, a slight increase of a bid may change the chance of winning the object only by a very small amount, and $K_{n,i}(p_i(s_i), v)$ may not increase indefinitely. Then, the intuition does not apply.

In response, we start from small values of s_i where (3.9) holds for any $p_i(s_i) \leq u_i(s_i, a(s_i))$. If every bidder is ex ante identical, then

$$\frac{s_j - \underline{s}_j}{M} + u_j(\underline{s}_j, a_j(\underline{s}_j)) \tag{3.12}$$

is the lower bound of all feasible bidding strategies. Then,

$$\frac{\overline{s}_j - \underline{s}_j}{M} + u_j(\underline{s}_j, a_j(\underline{s}_j))$$

is the highest possible price that can ever be realized under (3.12). Consider the set of private signals of bidder i

$$\left\{ s_i | u_i(s_i, a(s_i)) < \frac{\overline{s}_j - \underline{s}_j}{M} + u_j(\underline{s}_j, a_j(\underline{s}_j)) \right\}$$

which is a line segment, and let

$$\overline{s}_i^1 = \sup \left\{ s_i | u_i(s_i, a_i(s_i)) < \frac{\overline{s}_j - \underline{s}_j}{M} + u_j(\underline{s}_j, a_i(\underline{s}_j)) \right\}.$$

Then, for $\forall s_i < \overline{s}_i^1$, $\forall p_i < u_i(s_i, a_i(s_i))$ and $\forall j$, there exists s_j such that (3.9) holds. Thus, $K_{n,i}(p_i, v)$ increases indefinitely as $n \to \infty$ and therefore, no $p_i < u_i(s_i, a_i(s_i))$ will be realized by a rationalizable strategy of bidder i in the limit. We can invoke the same logic recursively to cover the whole set S_i of private signals of player i. Since we admit ex ante heterogeneous agents, however, the construction is a little bit more involved than what we just described.

We need to characterize the upper bound of $R_{n,i}^M$. In the case of the pure private value model in which

$$u_i(s_i, v) = s_i,$$

it is clear that

$$p_i(s_i) \leq s_i \qquad \forall s_i, \forall p_i \in R_{n,i}^M.$$

However, if u_i has a common value component, then the valuation of the object by player i is influenced by the event that $p_i = \max_{j \neq i} p_j$. Because his bid is influenced by his own evaluation of the object, the model has a feature of "self-reference," which significantly complicates the informational inference problem.

As a first step of characterizing the upper bound of $R_{n,i}^M$, we show that if every bidder j other than i uses $u_j(s_j, a_j(s_j)) - \epsilon$, then the best response of bidder i does not exceed $u_i(s_i, a_i(s_i))$. To simplify notation, define

$$w_i(s_i) = u_i(s_i, a_i(s_i)) \qquad \forall i, \forall s_i.$$

Recall that $b_i(s_i, p_{-i})$ is the best response of player i conditioned on s_i, when he believes that the other players use p_{-i}.

Lemma 3.3. $\forall \epsilon > 0$, $\exists N$ *such that* $\forall n \geq N$,

$$b_i(s_i, w_{-i} - \epsilon) \leq w_i(s_i)$$

where $w_{-i} - \epsilon$ *means that for* $j \neq i$, *player* j *uses* $u_j(s_j, a_j(s_j)) - \epsilon$.

Proof. See Appendix D.

The cost of making an aggressive bid in the first price auction is clear: the winner has to pay more. On the other hand, the aggressive bid increases the chance of winning the object. Because a rationalizable bid must generate a positive expected profit, it is necessary that if player i places a bid $p_i > w_i(s_i)$, then there exists $v'' \in (a_i(s_i), a_i'(s_i))$ such that $p_i = u_i(s_i, v'')$. Against $v > v''$, player i can generate a positive surplus. However, in order to win the object, it is necessary (if not sufficient) to beat the "neighbors" who bid according to a bidding strategy close to $w_i(s_i)$. Consequently, if there is a neighbor

j whose signal s_j is higher than s_i, then his bid is higher than $w_i(s_i)$. To rationalize $p_i > w_i(s_i)$, player i has to win when $v \geq v'' > a_i(s_i)$. However, conditioned on such v, it is more likely that his neighbors receive a signal higher than that conditioned on $a_i(s_i)$. Moreover, with a strictly positive probability, his neighbors receive a signal higher than s_i which induces a higher bid than $w_i(s_i)$. Hence, the probability that none of his neighbors receives a signal higher than s_i conditioned on $v \geq v''$ vanishes to 0, as the number of bidders increases. Thus, conditioned on $v \geq v''$, player i's chance of winning the object is 0 and his expected payoff cannot be positive.

Proposition 3.2 implies that for a fixed $\epsilon > 0$, any rationalizable strategy must be bounded from below by $u_i(s_i, a_i(s_i)) - \epsilon$. In fact, if every player j other than i is using a bidding strategy close to $u_j(s_j, a_j(s_j))$, then the best response of player i should be close to $u_i(s_i, a_i(s_i))$. Therefore, in the "limit" game populated with infinitely many bidders, $\{u_i(s_i, a_i(s_i))\}_{i=1}^{\infty}$ is a Nash equilibrium. Since $u_i(s_i, a_i(s_i))$ is a feasible bidding strategy, $u_i(s_i, a_i(s_i))$ is a rationalizable strategy of the "limit" game.

In the second price auction, one can prove that the best response of player i is increasing, if the bidding strategies of the other players "decrease." That is, if player i wins against more aggressive bids of the other players, he can infer that the underlying value is smaller thatn if the bidding strategies of the other players were less aggressive, which eventually discourages him from bidding aggressively.[7] Then, the most aggressive rationalizable bidding strategy is rationalized by the least aggressive rationalizable bidding strategy, and vice versa. In the first price auction, however, we cannot obtain the same kind of monotonicity of the best response function, which could simplify the analysis significantly, because the delivery price is the winning bid. Still, we can establish a "limited" version of the monotonic relationship when the number of bidders is large.

Remember that $w_i(s_i) = u_i(s_i, a_i(s_i))$. Proposition 3.2 says that for a large n, any rationalizable bid of player i conditioned on s_i must be larger than $w_i(s_i) - \epsilon$. The next lemma says that if every other bidder uses a bid lower than $w_i(s_i) - \epsilon$, then the best response should be larger than the largest rationalizable bid.

Lemma 3.4. $\forall \epsilon > 0$, $\exists N$, $\forall n \geq N$, $\forall s_i$,

$$\overline{p}_{n,i}(s_i) \leq b_i(s_i, w_{-i} - \epsilon) + \epsilon.$$

Proof. See Appendix E.

We are ready to prove that the upper bound of $R^M_{n,i}$ is within a small neighborhood of $u_i(s_i, a_i(s_i))$ by combining Lemma 3.4, Proposition 3.2 and Lemma 3.3.

Theorem 3.5. *For* $\forall \epsilon > 0$, *there exists* N *such that* $\forall n \geq N$, $\forall i$, $\forall p_i \in R^M_{n,i}$, $\forall s_i$,

$$|p_i(s_i) - u_i(s_i, a_i(s_i))| \leq \epsilon.$$

Proof. Recall that $\underline{p}_{n,i}$ and $\overline{p}_{n,i}$ are the lower and the upper bounds of $R^M_{n,i}$. Fix $\epsilon > 0$. By Proposition 3.2, there exists N_1 such that for $\forall n \geq N_1$,

$$w_i(s_i) - \epsilon \leq \underline{p}_{n,i}(s_i).$$

[7] For a complete proof, see Cho (2002).

By the definition,

$$\underline{p}_{n,i}(s_i) \leq \overline{p}_{n,i}(s_i).$$

By Lemma 3.4, there exists N_2 such that for $\forall n \geq N_2$,

$$\overline{p}_{n,i}(s_i) \leq b_i(s_i, w_{-i} - \epsilon) + \epsilon.$$

By Lemma 3.3, there exists N_3 such that for $\forall n \geq N_3$,

$$b_i(s_i, w_{-i} - \epsilon) \leq w_i(s_i).$$

Let $N = \max(N_1, N_2, N_3)$. Then, for $\forall n \geq N$,

$$w_i(s_i) - \epsilon \leq \underline{p}_{n,i}(s_i) \leq \overline{p}_{n,i}(s_i) \leq w_i(s_i) + \epsilon.$$

Q.E.D.

From Lemma 3.3, we can infer that the set of rationalizable strategies collapses into a single bidding strategy $u_i(s_i, a_i(s_i))$ if the lower bound $\underline{p}_{n,i}(s_i)$ of $R^M_{n,i}$ converges to $u_i(s_i, a_i(s_i))$. Thus, by examining the rate at which $\underline{p}_{n,i}(s_i)$ converges to $u_i(s_i, a_i(s_i))$, we can estimate the rate of convergence.

Proposition 3.6. *For $\forall \epsilon > 0$, there exists N such that $\forall n \geq N$, $\forall p_i \in R^M_{n,i}$, $\forall s_i$,*

$$|p_i(s_i) - u_i(s_i, a_i(s_i))| < \epsilon.$$

Moreover, we can choose $\epsilon = O(1/n)$ where $\lim_{h \to 0} O(h) = 0$.

Proof. See Appendix F.

In particular, if every bidder is ex ante identical, there exists a symmetric Nash equilibrium, in which the equilibrium strategy of each player belongs to Σ^M_i for a sufficiently large $M > 0$. Since every Nash equilibrium is rationalizable, Theorem 3.5 can be viewed as offering a theoretical foundation for selecting a symmetric Nash equilibrium in the first price auction in which every bidder is ex ante identical as in Wilson (1977) and Milgrom (1979).

Corollary 3.7. *Suppose that every bidder is ex ante identical. For $\forall \epsilon > 0$, there exist M' and N such that for $\forall M \geq M'$, $\forall n \geq N$, $\forall p_i \in R^M_{n,i}$,*

$$|p^*_{n,i}(s_i) - p_i(s_i)| \leq \epsilon \qquad \forall s_i,\ \forall i$$

where $p^*_{n,i}$ is (2.4). In particular, we can choose $\epsilon = O(1/n)$.

We have maintained the assumption that

$$\Pi_i(p_i(s_i), s_i) > 0 \qquad \forall p_i \in R^M_{n,i}.$$

Therefore, our asymptotic result holds only for s_i, which can be the highest signal for some common value v. We claim that the same intuition can be extended for the case where

$$\Pi_i(p_i(s_i), s_i) = 0$$

for some $p_i \in R^M_{n,i}$ to identify

$$p_i(s_i) = u_i(s_i, a_i(s_i))$$

as the sensible strategy.

The characteristic of player i is (u_i, F_i). Recall that $S_i(v)$ is the support of $F_i(s_i|v)$ and $S_i = \bigcup S_i(v)$. Also, remember that $S = \bigcup_{i=1}^n S_i$. Consider a sequence $(u_i, F_{k,i})$ of characteristics of player i in which the support of $F_{k,i}(s_i|v)$ is S for $\forall v$, $\forall i$ and $\forall n$. Assume that $F_{k,i}(s_i|v) \to F_i(s_i|v)$ in distribution as $k \to \infty$. Given any bidding strategy p_i in the original game, we can find a bidding strategy $p_{k,i}$ which "embeds" p_i:

$$\forall s_i \in S_i, \;\; p_{k,i}(s_i) = p_i(s_i).$$

Let Γ be the original first price auction, and Γ_k be the auction in which each player i's characteristic is $(u_i, F_{k,i})$. As $k \to \infty$, we can regard Γ_k as being close to Γ. Moreover, for $\forall k \geq 1$, every player generates a positive expected profit from any rationalizable bidding strategies in Γ_k. Thus, any rationalizable bidding strategy in Γ_k must converge to

$$w_i(s_i) \equiv u_i(s_i, a_i(s_i)) \qquad \forall i, \forall s_i \in S.$$

Note that $w_i(s_i)$ is now defined over S which contains S_i. Therefore, when k is very large, player i's bidding strategy must be close to $w_i(s_i)$ over S_i. In this sense, we believe that if $\Pi_i(p_i(s_i), s_i) = 0$ for some s_i and $p_i \in R^M_{n,i}$, it is still reasonable to select $p_i(s_i) = u_i(s_i, a_i(s_i))$ as the sensible strategy.

4. Concluding Remarks

We believe that the restriction on the set of bidding strategies is mild. However, the informational structure can be subject to the same criticism that Pesendorfer and Swinkels (1997) made on the informational structure of Wilson (1977) and Milgrom (1979). As long as the portion of bidders who win the object vanishes as the number of bidders increases,[8] it seems necessary that the support of the signal distribution should be bounded, and the probability weight of the neighborhood of the upper bound of the support should remain positive. Because the good is essentially delivered to the bidder who has the highest signal, the first order statistics converges (weakly) to the degenerate distribution concentrated at the upper bound of the signal distribution. Thus, without a well behaved upper bound of the support of the signal distribution, the asymptotic statistical properties of the highest bid could be very complicated, and we might not derive the information aggregation result.

This paper focuses on the first price auction where the private signal is drawn from a real line. It is known that if the signal is drawn from higher dimensional space, the information aggregation may not lead to efficient allocation (Pesendorfer and Swinkels (2000)). It remains to be investigated whether or not the rationalizability in combination with intuitive restrictions on the strategy space can narrow down the outcome substantially for the models with richer signals.

One of the most important advantages of our approach over the equilibrium model is that we largely bypass the need to calculate the equilibrium strategy precisely. By simply showing that the rationalizable strategies should be contained within a narrow neighborhood of the truthful valuation of the object, we can obtain the information aggregation result under milder conditions than those used in equilibrium models.

[8] This class of auctions is called the concentrated auctions (Jackson and Kremer (2001)).

This observation prompts us to investigate the information aggregation problem in the double auction that admits the common value component. This is important because "pure" private value or common value auctions hardly exist. Rather, most major auctions exhibit features of both private and common values, such as PCS auctions. The information aggregation results of Satterthwaite and Williams (1989) and Rustichini, Satterthwaite, and Williams (1994) remain to be examined when there is a common value component in the object. The calculation of a symmetric equilibrium is significantly complicated by the presence of the common value component, which is the main reason why the extension has not been attempted. The continuing research will address this important question.

APPENDIX A. PROOF OF LEMMA 2.10

Choose any sequence $\{p_i^k\}_{k=1}^\infty \subset \Sigma_i^M$. Let $\tilde{S}_i = \{s_i^\ell\}_{\ell=1}^\infty$ be a countable dense subset of S_i. Since $\{p_i^k(s_i^\ell)\}_{k=1}^\infty$ is contained in a compact set for $\forall \ell \geq 1$, we can find a convergent subsequence $\{p^{k_\ell}(s_i^\ell)\}_{\ell=1}^\infty$ such that

$$p^{k_\ell}(s_i) \to p(s_i) \qquad \forall s_i \in \tilde{S}_i$$

for some p_i. Since

$$\frac{1}{M} \leq \frac{p^{k_\ell}(s_i) - p^{k_\ell}(s_i')}{s_i - s_i'} \leq M \qquad \forall s_i, s_i' \in \tilde{S}_i,$$

$$\frac{1}{M} \leq \frac{p(s_i) - p(s_i')}{s_i - s_i'} \leq M \qquad \forall s_i, s_i' \in \tilde{S}_i.$$

Thus, p_i is Lipschitz continuous over $\tilde{S}_i \subset S_i$. Let p_i^o be the continuous function that coincides with p_i over $\tilde{S}_i$. Clearly, $p_i^{k_\ell}(s_i) \to p_i^o(s_i)$ for $\forall s_i \in S_i$ and p_i^o is Lipschitz continuous. Hence, $p_i^o \in \Sigma_i^M$.

APPENDIX B. PROOF OF LEMMA 3.1

Since $p_i, \tilde{p}_i \in R_{n,i}^M$, for each s_i, $\exists p_{-i}, \exists \tilde{p}_{-i} \in R_{n,-i}^M$ such that $p_i(s_i)$ is a best response to p_{-i} and $\tilde{p}_i(s_i)$ is to $\tilde{p}_{-i}$. If $p_i(s_i) \geq \tilde{p}_i(s_i)$, then $\max(p_i(s), \tilde{p}_i(s_i))$ is a best response to p_{-i} and if $p_i(s_i) < \tilde{p}_i(s_i)$, then $\max(p_i(s), \tilde{p}_i(s_i))$ is a best response to $\tilde{p}_{-i}$. The other case follows from the same logic.

APPENDIX C. PROOF OF PROPOSITION 3.2

Fix $\epsilon > 0$. Choose a large $M > 0$ such that

$$\frac{1}{M} \leq \frac{du_i(s, a_i(s))}{ds} \leq M \qquad \forall i. \tag{C.13}$$

Since $u_i(s_i, a_i(s_i))$ is a strictly increasing function of s_i, we can choose $\delta > 0$ so that

$$u_i(\overline{s}_i - \delta, a_i(\overline{s}_i - \delta)) = u_i(\overline{s}_i, a_i(\overline{s}_i)) - \epsilon,$$

which is well defined, because $u_i(s_i, a_i(s_i))$ is strictly increasing with respect to s_i. Also, choose $\mu \in (0, \epsilon]$ so that

$$\frac{\delta}{M} - 2\mu > 0. \tag{C.14}$$

Let $\overline{s}_i^0 = \underline{s}_i$. Suppose that $\overline{s}_i^1, \ldots, \overline{s}_i^{k-1}$ are defined. Define

$$\overline{p}^k = \frac{(\overline{s}_i - \delta) - \overline{s}_i^{k-1}}{M} + u_i(\overline{s}_i^{k-1}, a_i(\overline{s}_i^{k-1}))$$

and define $\overline{s}_i^k$ implicitly by

$$u_i(\overline{s}_i^k, a(\overline{s}_i^k)) = \overline{p}^k - \mu.$$

Since $u_i(s_i, a_i(s_i))$ is strictly increasing in s_i, $\overline{s}_i^k$ is well defined. By choosing $\delta, \mu > 0$ sufficiently small, we can ensure

$$\overline{s}_i^{k-1} < \overline{s}_i^k \qquad \forall k.$$

By (C.13), this process must stop in a finite number of steps, say k^*, as soon as

$$u_i(\overline{s}_i^{k^*}, a_i(\overline{s}_i^{k^*})) \geq u_i(\overline{s}_i, a_i(\overline{s}_i)) - \epsilon + \mu.$$

Given $\underline{s}_i = \overline{s}_i^0, \ldots, \overline{s}_i^{k^*}$, we can partition $[\underline{s}_i, \overline{s}_i]$ into

$$[\overline{s}_i^0, \overline{s}_i^1), \ldots, [\overline{s}_i^{k^*-1}, \overline{s}_i^{k^*}), [\overline{s}_i^{k^*}, \overline{s}_i].$$

Note that k^* can be chosen according to M and u_i, independently of ϵ and μ. We shall show recursively that over each interval of the private signals, no rationalizable bid can be too far below $u_i(s_i, a_i(s_i))$.

Fix $s_i \in [\overline{s}_i^0, \overline{s}_i^1)$ and $p_i \leq u_i(s_i, a_i(s_i)) - \mu$. Define

$$I_{\frac{\mu}{2}} = \left\{ j \mid d(i,j) < \frac{\mu}{2} \right\}.$$

By the construction of $\overline{s}_i^1$, for any bidding strategy in R_j^M where $j \in I_{\frac{\mu}{2}}$,

$$F_j\left(p_{n,j}^{-1}(p_i) | v\right) < 1 \qquad \forall j \neq i.$$

Since F_j is continuously differentiable over a compact set, there exist $\overline{f} > \underline{f} > 0$ such that

$$0 < \underline{f} < F_i'(s_i | v) < \overline{f} \qquad \forall s_i, \forall v \in V.$$

Since $p_j'(s_j) \in [1/M, M]$ for $\forall s_j$,

$$0 < Q_{n,i}(p_i|v) \frac{\underline{f}}{M} \left[\# I_{\frac{\mu}{2}}\right] \leq \frac{dQ_{n,i}(p_i|v)}{dp_i} = Q_{n,i}(p_i|v) \sum_{j \neq i} \frac{f_j(s_j|v)}{p_j'(s_j) F_j(s_j|v)} = Q_{n,i}(p_i|v) K_{n,i}(p,v), \tag{C.15}$$

where s_j satisfies $p_j(s_j) = p_i$ and $\# I_{\frac{\mu}{2}}$ means the number of elements in $I_{\frac{\mu}{2}}$. Since $p_j' \leq M$, $\liminf_{n \to \infty} F_j(s_j|v) > 0$. Recall

$$K_{n,i}(p_i, v) = \sum_{j \neq i} \frac{f_j(s_j|v)}{p_j'(s_j) F_j(s_j)}$$

where $p_j(s_j) = p_i$. Consider

$$\begin{aligned} \frac{d\Pi_i(p_i, s_i)}{dp_i} &= \int_v \left[-Q_{n,i}(p_i|v) + (u_i(s_i,v) - p_i) \frac{dQ_{n,i}(p_i|v)}{dp_i} \right] dG(v|s_i) \\ &= \int_{a(s_i)}^{a'(s_i)} \left[-\frac{1}{K_{n,i}(p_i,v)} + (u_i(s_i,v) - p_i) \right] \frac{dQ_{n,i}(p_i|v)}{dp_i} dG(v|s_i). \end{aligned} \tag{C.16}$$

For $\forall v \in [a(s), a'(s)]$, $u_i(s_i, v) \geq u_i(s_i, a(s_i))$. We also know from (C.15) that

$$K_{n,i}(p_i, v) \geq \frac{\underline{f}}{M} \left[\# I_{\frac{\mu}{2}}\right].$$

Thus,

$$\frac{d\Pi_i(p_i, s_i)}{dp_i} \geq \int_{a(s)}^{a'(s)} \left[-\frac{M}{\overline{f}\left[\# I_{\frac{\mu}{2}}\right]} + \mu \right] \frac{dQ_{n,i}(p_i|v)}{dp_i} dG(v|s_i)$$

because $p_i \leq u_i(s_i, a(s_i)) - \mu$. By (2.3), there exists N_0 such that $\forall n \geq N_0$,

$$\# I_{\frac{\mu}{2}} \geq \frac{M}{\mu \underline{f}}.$$

Then, for $\forall n \geq N_0$,

$$\frac{d\Pi_i(p_i, s_i)}{dp_i} > 0$$

which implies that bidder i can increase his expected payoff by increasing his bid slightly from p_i. Since the lower bound is independent of the choice of $p_{n,-i}$, p_i is never a best response conditioned on s_i.

We can apply the same logic to the next interval $[\overline{s}_i^1, \overline{s}_i^2)$, since any rationalizable strategy p_i must satisfy

$$p_i(\overline{s}_i^1) \geq u_i(\overline{s}_i^1, a(\overline{s}_i^1)) - \mu$$

for $\forall n \geq N_0$. Hence, there exists $N_k \geq N_{k-1}$ such that $\forall k \in \{1, \ldots, k^*\}$, $\forall i$, $\forall p_i \in R^M_{n,i}$, $\forall s_i \in [\overline{s}_i^{k-1}, \overline{s}_i^k]$, $\forall n \geq N_k$,

$$p_i(s_i) \geq u_i(s_i, a(s_i)) - \mu \geq u_i(s_i, a(s_i)) - \epsilon.$$

The only possible exception is the last interval $[\overline{s}_i^{k^*}, \overline{s}_i]$. By the construction of $\overline{s}_i^{k^*}$,

$$0 < u_i(\overline{s}_i, a(\overline{s}_i)) - \left[\frac{1}{M}(\overline{s}_i - \overline{s}_i^{k^*}) + u_i(\overline{s}_i^{k^*}, a_i(\overline{s}_i^{k^*})) - \mu\right] < \epsilon.$$

Recall that $\forall n \geq N_{k^*-1}$, any rationalizable strategy p_i must satisfy

$$p_i(\overline{s}_i^{k^*-1}) \geq u_i(\overline{s}_i^{k^*-1}, a_i(\overline{s}^{k^*-1})) - \mu.$$

Since the slope of any feasible strategy must be bounded from below by $1/M$, $\forall p_i \in R^M_{n,i}$, $\forall s_i \in [\overline{s}_i^{k^*}, \overline{s}_i]$,

$$p_i(s_i) \geq u_i(s_i, a_i(s_i)) - \epsilon.$$

Since

$$[\underline{s}, \overline{s}] = \left(\bigcup_{k=0}^{k^*-1} [\overline{s}_i^k, \overline{s}_i^{k+1})\right) \cup [\overline{s}_i^{k^*}, \overline{s}_i],$$

and $\mu \leq \epsilon$, we have that for $\forall n \geq N = \max(N_0, \ldots, N_{k^*-1})$, $\forall i$, $\forall p_i \in R^M_{n,i}$, $\forall s_i \in [\underline{s}, \overline{s}]$, $\forall n \geq N$,

$$p_i(s_i) \geq u_i(s_i, a_i(s_i)) - \epsilon$$

as desired.

Appendix D. Proof of Lemma 3.3

The proof has three parts. We first examine the pure private model. The pure common value model is examined after the general case in which the utility function $u_i(s_i, v)$ is strictly increasing with respect to the private signal s_i and the common value v.

Pure Private Value Model: $\partial u_i(s_i, v)/\partial v = 0$

If $u_i(s_i, v) = s_i$ as in the private value model, it is straightforward that any $p_i > u_i(s_i, v)$ is never a best response. Thus, $\overline{p}_i(s_i) \leq u_i(s_i, v)$. The conclusion follows from combining this observation with Proposition 3.2.

Increasing in both arguments: $\partial u_i(s_i, v)/\partial s_i > 0$ and $\partial u_i(s_i, v)/\partial v > 0$.

Suppose that there exist a rationalizable bidding strategy and a signal s_i such that

$$p_i \equiv p_i(s_i) > u_i(s_i, a_i(s_i)).$$

Since p_i is generating a positive expected payoff, $\exists v'' \in [a_i(s_i), a_i'(s_i)]$ such that

$$u_i(s_i, v'') - p_i = 0.$$

Note that $v'' > a_i(s_i)$ and $s_i \in [\underline{s}_i(v''), \overline{s}_i(v'')]$. Thus, $\overline{s}_i(v'') > s_i$. Since $u_i(s_i, v)$ is strictly increasing in s_i,

$$u_i(\overline{s}_i(v''), v'') - p_i > 0. \tag{D.17}$$

Define

$$\epsilon = u_i(\overline{s}_i(v''), v'') - p_i > 0.$$

Since u_i is continuous and increasing in both arguments, $\exists \rho > 0$ and $\exists \kappa > 0$ such that

$$p_i = u_i(\overline{s}_i(v'') - \rho, a_i(\overline{s}_i(v'') - \rho)) - \frac{\epsilon}{\kappa}. \tag{D.18}$$

By Proposition 3.2, we can choose N' such that $\forall n \geq N'$, $\forall i$, $\forall s_i$, $\forall p_i \in R^M_{n,i}$,

$$p_i(s_i) \geq u_i(s_i, a_i(s_i)) - \frac{\epsilon}{2\kappa}.$$

Let $\epsilon' = \frac{\epsilon}{2\kappa}$.

Let us consider the interim expected payoff of player i conditioned on s_i, which is generated from bidding p_i:

$$\begin{aligned}\Pi_i(p_i, s_i) &= \int_v (u_i(s_i, v) - p_i) Q_{n,i}(p_i, v) dG(v|s_i) \\ &= \int_{a_i(s_i)}^{v''} (u_i(s_i, v) - p_i) Q_{n,i}(p_i, v) dG(v|s_i) + \int_{v''}^{a_i'(s_i)} (u_i(s_i, v) - p_i) Q_{n,i}(p_i, v) dG(v|s_i).\end{aligned}$$

By the definition of v'', the first term is strictly negative, as long as $Q_{n,i}(p_i, v) > 0$. We shall prove that the second term, which is positive, has to vanish as $n \to \infty$, but also does so "sufficiently" more quickly than the first term, in case the first term also vanishes to 0.

If $Q_{n,i}(p_i, v) = 0$ for $\forall v \in [a_i(s_i), a_i'(s_i)]$, then player i's chance of winning against any rationalizable strategies is 0. Assume that $Q_{n,i}(p_i, v) > 0$ for some $v \in [a_i(s_i), a_i'(s_i)]$. Recall that

$$Q_{n,i}(p_i, v) = \prod_{j \neq i} F_j(p_j^{-1}(p_i)|v)$$

where $p_j(s_j) = u_j(s_j, a_j(s_j)) - \epsilon$ for $j \neq i$.

We need to consider several cases. First, suppose that $\exists h \in (a_i(s_i), v'')$ such that $Q_{n,i}(p_i|v'' - h) = 0$. Since $Q_{n,i}(p_i|v)$ is weakly decreasing in v, $Q_{n,i}(p_i|v) = 0$ for $\forall v \geq v'' - h$, in particular for $\forall v \geq v''$. Then, $\Pi_i(p_i, s_i) < 0$.

Second, suppose that $\exists h \in (a_i(s_i), v'')$ such that $Q_{n,i}(p_i|v'' - h) = 1$. Then, $\forall v \leq v'' - h$, $Q_{n,i}(p_i|v) = 1$. Since i is not asymptotically isolated, we can choose $\mu > 0$ sufficiently small so that if $d(i, j) \leq \mu$, then

$$p_j(s_j) \geq u_j(\overline{s}(v'') - \rho, a(\overline{s}(v'') - \rho)) - \frac{\epsilon}{4\kappa}$$

and

$$\left| u_j(\overline{s}_i(v'') - \rho, a_i(\overline{s}_i(v'') - \rho)) - u_i(\overline{s}_i(v'') - \rho, a_i(\overline{s}_i(v'') - \rho)) \right| < \frac{\epsilon}{4\kappa}.$$

Hence, if $d(i, j) < \mu$, then $\forall s_j \in [\overline{s}_i(v'') - \rho, \overline{s}_i(v'')]$,

$$p_{n,j}(s_j) > p_i.$$

Hence, for $\forall v \in [v'', a_i'(s_i)]$,

$$\begin{aligned}Q_{n,i}(p_i|v) &\leq \Pr\left(\max_{d(i,j) \leq \mu} s_j \leq [\overline{s}(v'') - \rho] \mid v\right) \leq \Pr\left(\max_{d(i,j) \leq \mu} s_j \leq [\overline{s}(v'') - \rho] \mid v''\right) \\ &= \prod_{d(i,j) \leq \mu} F_j(\overline{s}(v'') - \rho \mid v'') \leq \prod_{d(i,j) \leq \mu} \max_{d(i,j) \leq \mu} F_j(\overline{s}(v'') - \rho \mid v'').\end{aligned}$$

Condition (2.1) implies that for $\forall n \geq 1$, and $\forall i \in \{1, \ldots, n\}$,

$$\max_i F_i(\overline{s}(v'') - \rho | v'') \equiv \overline{F} < 1,$$

and therefore,

$$Q_{n,i}(p_i|v) \leq \prod_{j \in I_{\frac{\epsilon}{4}}} \overline{F}$$

which converges to 0 at a geometric rate. Since $Q_{n,i}(p_i|v) \to 0$ for $\forall v \geq v''$, $\Pi_i(p_i, s_i) < 0$.

Finally, suppose that $\forall h \in (a_i(s_i), v'')$,

$$0 < Q_{n,i}(p_i|v'' - h) < 1.$$

Thus, there exists j^* so that $F_{j^*}(p_{j^*}^{-1}(p_i)|v'') < 1$. Since $F_{j^*}(s_{j^*}|v'')$ is strictly decreasing with respect to v for a fixed s_{j^*} in its support, for any sufficiently small $\epsilon > 0$, $\exists h > 0$ such that

$$\frac{F_j(s_j|v'' - h)}{F_j(s_j|v'')} > 1 + \epsilon.$$

Choose $\mu > 0$ sufficiently small so that if $d(j, j^*) < \mu$, then

$$\frac{F_j(s_j|v'' - h)}{F_j(s_j|v'')} > 1 + \frac{\epsilon}{2}.$$

Since j^* is not asymptotically isolated, $I(\mu) = \{j|d(j,j^*) < \mu\}$ increases at the rate of n. Thus,

$$\frac{Q_{n,i}(p_i|v'' - h)}{Q_{n,i}(p_i|v')} \geq \left(1 + \frac{\epsilon}{2}\right)^{\# I(\mu)} \to \infty$$

as $n \to \infty$. By applying the same argument for all $v \in [a_i(s_i), a_i'(s_i)]$, we conclude that $Q_{n,i}(p_i|v) \to 0$ at $v > v''$ geometrically more quickly than $Q_{n,i}(p_i|v)$ at $v < v''$. Thus, for any large n, $\Pi_i(p_i, s_i) < 0$.

PURE COMMON VALUE MODEL: $\partial u_i(s_i, v)/\partial s_i = 0$

If $\partial u_i(s_i, v)/\partial s_i = 0$, then (D.17) does not hold. Define v'' as before, and ρ, κ as in (D.18). Choose small $\lambda > 0$ such that

$$p_i = u_i(\overline{s}_i(v'' + \lambda) - \rho, a_i(\overline{s}_i(v'' + \lambda) - \rho)) - \frac{\epsilon}{\kappa}.$$

We can repeat the same analysis as in the previous part after replacing v'' by $v'' + \lambda$. By the definition of v'',

$$\frac{\partial}{\partial \lambda} \int_{a_i(s_i)}^{v''+\lambda} (u_i(s_i, v) - p_i) Q_{n,i}(p_i, v) dG(v|s_i) = 0$$

and

$$\frac{\partial}{\partial \lambda} \int_{v''+\lambda}^{a_i'(s_i)} (u_i(s_i, v) - p_i) Q_{n,i}(p_i, v) dG(v|s_i) = 0.$$

Thus, if $\Pi_i(p_i, s_i) < 0$, then the interim expected utility remains negative when we replace v'' by $v'' + \lambda$ as long as $\lambda > 0$ is sufficiently small.

APPENDIX E. PROOF OF LEMMA 3.4

By Lemma 3.3, $\exists N_1$ such that $\forall n \geq N_1$, $b_i(s_i, w_{-i} - \epsilon) \leq w_i(s_i)$. Choose an arbitrarily small $\mu > 0$. Then, we claim that $\forall s_i \in (\underline{s}_i + \mu, \overline{s}_i - \mu)$,

$$\# \{j \mid w_j(s_j) - \epsilon = b_i(s_i, w_{-i} - \epsilon)\} \to \infty.$$

To see this, note that if $\{j \mid w_j(s_j) - \epsilon = b_i(s_i, w_{-i} - \epsilon)\} = \emptyset$, then player i can increase his payoff by slightly lowering his bid from $b_i(s_i, w_{-i} - \epsilon)$, which contradicts the hypothesis that $b_i(s_i, w_{-i} - \epsilon)$ is a best response. Thus, there exists j^* such that

$$w_i(\underline{s}_i + \mu) \leq w_{j^*}(s_{j*}) - \epsilon = b_i(s_i, w_{-i} - \epsilon) \leq w_i(s_i) \leq w_i(\overline{s}_i - \mu).$$

Since $w_i(s_i)$ is a strictly increasing Lipschitz continuous function, we can choose $\mu' > 0$ sufficiently small so that if $d(j, j^*) < \mu'$, then

$$w_i(\underline{s}_i + \mu/2) \leq w_j(s_{j^*}) - \epsilon \leq w_i(\overline{s}_i - \mu/2)$$

and therefore, $\exists s_j$ such that

$$w_j(s_j) - \epsilon = b_i(s_i, w_{-i} - \epsilon).$$

Since j^* is not asymptotically isolated, $\#\{j \mid d(j, j^*)\} \to \infty$. Therefore,

$$\# \{j \mid w_j(s_j) - \epsilon = b_i(s_i, w_{-i} - \epsilon)\} \to \infty \tag{E.19}$$

as desired. Because the right hand side is negative, while the left hand side becomes positive for $n \geq N_2$. for any $\delta > 0$, we can choose N_2 such that $\forall n \geq N_2$,

$$\delta + E\left[-\frac{1}{K_{n,i}(b_i(s_i, w_{-i} - \epsilon), v)} \mid b_i(s_i, w_{-i} - \epsilon) = \max_{j \neq i} w_j(s_j) - \epsilon\right]$$
$$\geq E\left[-\frac{1}{K_{n,i}(\overline{p}_{n,i}(s_i, v))} \mid \overline{p}_{n,i}(s_i) = \max_{j \neq i} p_j\right]$$

where p_{-i} is the profile of bidding strategies against which $\overline{p}_{n,i}(s_i)$ is a best response.

Consider the first order condition,

$$E\left(\frac{1}{K_{n,i}(p_i, v)} + u_i(s_i, v) \mid p_i = \max_{j \neq i} p_j\right) - p_i = 0$$

if

$$p_i = b_i(s_i, p_{-i}).$$

To simplify notation, let us write the first term as Ψ_i and the first condition as

$$\Psi_i(u_i, p_i, p_{-i}) - p_i = 0$$

if $p_i = b_i(s_i, p_{-i})$.

Since the second order condition holds at the local maximum,

$$\frac{\partial \Psi_i(u_i, p_i, p_{-i})}{\partial p_i} - 1 < 0 \tag{E.20}$$

around the neighborhood of $p_i = b_i(s_i, p_{-i})$. Recall that the bidding strategies are strictly increasing, and that the monotonic likelihood ratio property holds. Thus, if $u_i \geq \tilde{u}_i$ and $p_j \leq \tilde{p}_j$ for $\forall j \neq i$, then

$$b_i(s_i, p_{-i}) = \Psi(u_i, b_i(s_i, p_{-i}), p_{-i}) \geq \Psi(\tilde{u}_i, \tilde{b}_i(s_i, \tilde{p}_{-i}), \tilde{p}_{-i}) = \tilde{b}_i(s_i, \tilde{p}_{-i})$$

where $\tilde{b}_i(s_i, \tilde{p}_{-i})$ is a best response to $\tilde{p}_{-i})$ conditioned on s_i, when the utility function is $\tilde{u}_i$. Notice that this condition holds for all $n \geq 1$, and also in the limit.

It is most convenient to consider the first order condition in the limit. Abusing the notation, let $b_i(s_i, w_{-i} - \epsilon)$ be a best response to $w_{-i} - \epsilon$ in the "limit" game populated with infinitely many players, and $\overline{p}_i(s_i)$ be the upper bound of the rationalizable bids in the limit. By (E.19),

$$\lim_{n\to\infty} E\left[-\frac{1}{K_{n,i}(b_i(s_i, w_{-i} - \epsilon), v)} \mid b_i(s_i, w_{-i} - \epsilon) = \max_{j\neq i} w_j(s_j) - \epsilon\right] = 0.$$

Thus,

$$b_i(s_i, w_{-i} - \epsilon) = \mathsf{E}\left[u_i(s_i, v) \mid b_i(s_i, w_{-i} - \epsilon) = \max_{j\neq i} w_j(s_j) - \epsilon\right].$$

On the other hand, we do *not* know whether

$$\lim_{n\to\infty} E\left[-\frac{1}{K_{n,i}(\overline{p}_{n,i}(s_i, v))} \mid \overline{p}_{n,i}(s_i) = \max_{j\neq i} p_j\right] = 0.$$

In the limit,

$$\overline{p}_i(s_i) = \mathsf{E}\left[-\frac{1}{\lim_{n\to\infty} K_{n,i}(\overline{p}_i(s_i), v)} + u_i(s_i, v) \mid \overline{p}_i(s_i) = \max_{j\neq i} p_j\right].$$

Since p_j is rationalizable,

$$p_j(s_j) \geq w_j(s_j) - \epsilon \qquad \forall s_j, \forall j \neq i.$$

Clearly,

$$-\frac{1}{\lim_{n\to\infty} K_{n,i}(\overline{p}_i(s_i), v)} \leq 0.$$

Thus,

$$\overline{p}_i(s_i) \leq b_i(s_i, w_{-i} - \epsilon)$$

in the limit, as desired.

Appendix F. Proof of Proposition 3.6

We know

$$|p_i(s_i) - u_i(s_i, a_i(s_i))| \leq \mathsf{E}\left(\frac{1}{K_{n,i}(\underline{p}_{n,i}(s_i), v)} \mid \underline{p}_{n,i}(s_i) = \max_{j\neq i} p_{n,j}\right) \qquad \forall p_i \in R^M_{n,i}.$$

Since the right hand side of the inequality converges to 0 at the rate of n, we conclude that the set of rationalizable strategies converges to the truthful valuation at the rate of $1/n$.

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