INTERTEMPORAL PRICE DISCRIMINATION WITH STOCHASTIC VALUES

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ABSTRACT. We study the infinite-horizon pricing problem of a seller facing a buyer with single-unit demand, whose private valuation changes over time. This evolution is modeled as a stochastic shock to the buyer's valuation arriving at a random time that is unanticipated by both the buyer and the seller. The arrival of the shock is unobserved by the seller. We show that the seller's optimal contract with commitment consists of two prices: he will charge a low introductory price at the first instant, and a constant higher price thereafter. We also study a version of the model that allows for multiple shocks whose arrival times follow a Poisson process. It is assumed that the buyer can only make a purchase when she receives a shock. We derive the optimal contract with commitment and show that it consists of an increasing sequence of prices that converges in the limit to the highest buyer type. We show that, without commitment, the worst equilibrium for the seller is stationary, featuring a constant price over time. We characterize the set of equilibrium payoffs for the seller using this worst equilibrium as an optimal penal code.

Keywords: Intertemporal price discrimination, dynamic pricing, durable goods monopoly, impulse purchasing, stochastic values, dynamic mechanism design.

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1. INTRODUCTION

There is a large body of research in economic theory devoted to explaining the prevalence of intertemporal price discrimination in markets. A typical model for such a pricing problem consists of a seller facing a single buyer[1] who is interested in purchasing a "durable good." In our context, these goods are nonperishable goods that the buyer will only purchase once, for which resale or renting is not possible and that have the same cost across time for the seller. The buyer stays in the market until she purchases the good. The vast majority of the literature studies incentives for the seller to lower his price over time - a phenomenon which is common in practice. However, there are a variety of products for which the price increases over time. Amusement parks and ski resorts offer beginning of season discounts, stores offer back to school sales at the beginning of the academic year and many new products offer discounts when they are launched. In such markets, the seller often commits to the price increase by advertising the date of expiration of the introductory price and the regular price to be charged thereafter. If a buyer has rational expectations, then there is no reason to expect a buyer with a constant valuation to wait and buy the product at a higher price. The literature has explained this phenomenon by arguing that changing market conditions, such as the arrival of new buyers, can cause the seller to raise his price. By contrast, in this paper, we develop a model in which the buyer has a stochastic valuation. We show that the optimal commitment contract for a seller facing a buyer with a changing valuation features increasing prices.

A consumer's valuation can be influenced by a variety of different media - product reviews, advertising, word of mouth etc. In practice, the seller can observe neither when the buyer's valuation has been revised as a result of new information nor what the revised valuation is. No seller can plausibly hope to know when his customers have read or will read a product review or for that matter which review they read and as a result what their revised valuation will be. Other well known behavioral factors, such as impulse purchasing, can also affect a buyer's valuation.

The canonical intertemporal pricing model considers a buyer with a private valuation that does not change across time. In this model, the seller offers a sequence of prices and if the buyer chooses to purchase the good at any time the game is over. The fact that the buyer chooses not to purchase

[1] For ease of exposition, we refer to the seller using masculine pronouns and the buyer using feminine pronouns throughout this paper.

the good at a certain time reveals information about her value and provides incentives for the seller to lower his price in the future to serve the buyer, who has revealed that she has a low valuation. However, when prices fall over time it would induce some optimally behaving buyers with rational expectations to delay their purchases. When the monopolist cannot commit to a sequence of prices this intertemporal competition can be severe. The celebrated Coase conjecture (formally shown by Stokey (1981), Gul, Sonnenschein and Wilson (1986) for stationary strategies) states that, without commitment, it may not be possible for the monopolist to exercise any market power whatsoever. When the seller can make offers frequently, the competitive market outcome occurs despite the fact that the durable good is being supplied by a monopolist. There has been a large body of work examining market conditions where the Coase conjecture does and doesn't apply.

A complementary line of research studies optimal pricing when the seller has commitment power. In her seminal work, Stokey (1979) showed that when the monopolist can commit to a sequence of prices at the beginning of time, he chooses to offer the monopoly price at each instant of time. Hence, he makes sales only in the first instant, forgoing all future sales. This shows that the driving concern for the monopolist is to restrict the buyer's option value (from postponing her purchasing decision), to the extent that he makes no future sales, in order to dissuade the buyer from waiting. Since every equilibrium of the pricing game without commitment can be implemented by the seller when he has commitment power, the seller cannot expect to get more than the single period monopoly profit in equilibrium. As a result, the solution with commitment constitutes an upper bound for the revenue the seller can receive in equilibrium. In an influential paper, Ausubel and Deneckere (1989) show that a sufficiently patient seller can get arbitrarily close to the single period monopoly profit even without being able to commit to a sequence of prices.

The above results depend critically on the assumption that the buyer's valuation is constant across time. As a motivating example that highlights the tradeoffs, consider the anecdotal problem of a person deciding whether to go to watch a movie on the weekend of its release.[2] She has information about the movie (having watched a trailer for example) but knows that it is possible

[2]While a movie is not thought of as a standard durable good, Conlisk (1984) pointed out that it satisfies all the properties of such a good. And while we use this example solely to highlight the factors influencing a buyer's decision when she has a stochastic value, it should be pointed out that intertemporal price discrimination by movie theaters is practiced in many countries and in a few theaters across the US. The fact that this practice is not widespread in the US is because of economic and legal factors peculiar to the motion picture industry (see Orbach (2004)). In fact, Orbach and Einav (2007) conclude that if legal constraints were lifted, exhibitors could increase profits by price discrimination.

that after reading reviews, she might revise her opinion. Will such a person end up watching the movie if it cost her 10 dollars to do so while knowing that the price would be the same next week? What if the theater offered an opening weekend discount price of 5 dollars and she knew that in the subsequent week the price would go up? Would this override the fact that she had not read a review yet? Finally, how would the person decide if this lack of information was combined with a lack of opportunity? Some dates are more convenient than others depending on a person's work schedule, the availability of friends to go with, and so on. She might not know for sure when she will get the opportunity to watch the movie again. How does this influence her decision?

In this paper, we present a model where changes to a buyer's valuation arrive in the form of unanticipated stochastic shocks. Conditional on receiving a shock, the buyer draws a new independent valuation. In the first half of the paper, we focus on a model where there is a single shock to the buyer's valuation and both the arrival of the shock and the resulting valuation are not known ex-ante by either the buyer or the seller. The shock arrives via an exponential process and, conditional on receiving the shock, the buyer draws a new valuation independently. The seller does not observe when the buyer's valuation changes. This models the arrival of information that allows the buyer to reassess her value of the good. We derive the optimal pricing contract of the seller when he has commitment power.

In the second half of the paper, we extend the model to allow the buyer to receive multiple shocks to her valuation. These shocks are assumed to arrive from a Poisson process and their arrival is not observed by the seller. Once again, conditional on receiving a shock, the buyer draws an independent private value. We assume, in addition, that the buyer can only make a purchase when she receives a shock. An interpretation of these shocks is that they represent opportunities to make a purchase, or an impulse to buy. Impulse purchasing has been studied extensively by a large marketing literature starting with the seminal work of Clover (1950).[3] In fact, recently, there have been interesting studies of impulse purchasing in settings such as airports (Crawford and Melewar (2003)) and on the internet (Madhavaram and Laverie (2004)).[4] We derive the optimal contract with commitment and show that it features an increasing price function. When the seller cannot commit, we show that the worst equilibrium for the seller features a constant price

[3]Stern (1962), Kollat and Willet (1967), Rook (1987), Rook and Fisher (1995) are a few of the many other important papers.
[4]The latter being the medium of choice for impulse purchasing by the author of this paper.

over time. This is in sharp contrast with the rapidly decreasing prices suggested by the Coase conjecture.

1.1. *Overview and Discussion of Results*

As we mentioned above, conditional on receiving a shock, the buyer draws a new valuation independently from a distribution F. A natural interpretation of independent draws is that of noisy valuations. Assume that the seller knows that the buyer has an intrinsic value ψ for the object which is subject to fluctuations. These fluctuations can either be a result of information or the result of behavioral factors such as impulses. When the buyer receives a shock, her realized valuation is $\theta = \psi + \varepsilon$ where ε is drawn independently from a distribution F' and is not observed by the seller. It should be pointed out that the assumption of an independent noise parameter is standard in almost all regression models. The distribution F' on the noise parameter ε induces the distribution F on θ.

We show that when the buyer receives a single shock to her valuation, the optimal contract of the seller consists of two prices. The seller charges a price $\hat{p}$ in the first instant and then charges a higher price $\hat{q} > \hat{p}$ thereafter. Both these prices are strictly higher than the monopoly price. If a buyer does not purchase at the first instant, she will only buy the good in the future if she receives the shock and her resulting valuation is higher than $\hat{q}$. Since the buyer can receive the shock at any time, the seller has a non-zero probability of making a sale at all points of time. As we discussed above, Stokey (1979) shows that when the buyer's valuation is constant, the seller's optimal contract is equivalent to that of a static single period monopoly problem. By contrast, we show that the solution to the single shock model is essentially the solution to a two period model where the buyer has a new independent private valuation in each period.

There are considerable technical difficulties associated with deriving the optimal contract. Optimal behavior by the buyer involves solving a complex optimal stopping problem that depends on the prices set by the seller and the seller must take the strategic behavior of the buyer into account when designing the optimal contract. Moreover, the expected distribution of buyer types at any time also depends on the prices and on the time that the buyer received the shock - an event that is unobserved by the seller. We argue that it is difficult for the seller to solve this optimal control problem and define instead an appropriate "relaxed problem." In this relaxed problem, we assume that the seller can observe the shock and can condition his prices on this information (the conditional price functions are declared at the beginning of time). This removes one level of asymmetric information, namely, the time of arrival of the shock. The seller can always do weakly better in the relaxed problem as he can choose to ignore the extra information. We solve for the optimal contract by showing that the solution to the relaxed problem can be implemented with unconditional prices.

In the multiple shock extension, we show that the optimal commitment contract of the seller features a monotonically increasing price function that asymptotically converges to the highest buyer type in the limit. This means that, in the limit, the seller chooses to price out the buyer. Moreover, increasing prices imply that the continuation payoff of the buyer is decreasing over time and this makes it unattractive for the buyer to wait. Once again, this demonstrates that the seller's primary concern is reduce the buyer's outside option even at the expense of making fewer sales.

We show that there exists a unique stationary equilibrium (the seller charges the same price at all histories) when the seller cannot commit. We prove that this equilibrium provides the seller with the lowest payoff $\underline{R}^*$ amongst all perfect equilibria irrespective of the discount rate or the arrival rate of the shocks (of course, the stationary equilibrium itself depends on these parameters). Using this equilibrium as an optimal penal code, we argue that only prices which give the seller a continuation revenue weakly higher than $\underline{R}^*$ at all points of time can be supported in equilibrium. In particular, this implies that the optimal commitment contract cannot be supported as an equilibrium because the seller's revenue converges to zero as time goes to infinity.

Finally, we would like to point out that the analysis in this paper does not restrict attention to differentiable price paths - an assumption commonly made in such models. This assumption is primarily made to solve the buyer's optimal stopping problem - if the price path is differentiable, the buyer's best response can be summarized by a first order condition. In contrast, by working in the space of cutoff types, we can allow the seller to choose over the space of measurable price functions. In fact, given that the optimal contract in the single shock case is discontinuous, there are real benefits from relaxing this assumption.

1.2. *Related Literature*

This paper is related to two distinct strands of literature - the above mentioned literature on durable goods problems and the recent literature on dynamic mechanism design. The classical literature on durable goods makes two critical assumptions - that no new buyers enter the market and that the valuation of the buyer is constant across time. There are well known models that relax the former assumption. Conlisk, Gerstner and Sobel (1984) and Sobel (1991) allow the entry of an identical cohort of new buyers in each period. Like Stokey (1979), they show that the optimal strategy involves setting the static monopoly price in each period. Since new buyers enter, there are sales at each period but no buyer ever delays her purchase. Board (2008) introduces a model in which new heterogeneous consumers can enter in each period and he derives the optimal seller contract under commitment. By contrast, there are considerably fewer papers that relax the latter assumption of constant valuations. Conlisk (1984) and Biehl (2001) analyze a two period, two type model. Conlisk derives the optimal contract with and without commitment, whereas Biehl compares sales to leasing and shows that under certain parameter values sales may dominate leasing.

The single shock model in this paper most closely resembles the model of Fuchs and Skrzypacz (2009). In their model, there is a single exogenous event that arrives from an exponential process and this event terminates the game. Upon termination, the seller and the buyer receive payoffs given by an exogenous function that depends on the buyer's private valuation. They derive the stationary equilibrium and show that the revenue of the seller is driven down to his outside option as the time between successive offers goes to 0. Apart from the fact that we study the optimal contract when the seller has commitment power, the single shock model in this paper differs in two additional dimensions. Firstly, in this paper, upon arrival of the event, the continuation values of the buyer and the seller are determined endogenously by the prices set by the seller. Secondly, and most importantly, the seller does not observe the arrival of the event. This implies that the game can continue well after the arrival of the event.

There is a recent literature in dynamic mechanism design that analyzes repeated contracting between a principal and an agent with a stochastic valuation. A few influential papers that generalize the design of efficient mechanisms in static mechanism design to a dynamic environment are Athey and Segal (2007), Bergemann and Välimäki (2008). Battaglini (2005) derives the optimal mechanism for a buyer with two types whose valuation follows a Markov process. Pavan, Segal and Toikka (2009) study incentive compatibility and revenue maximization in a general dynamic model with a continuum of types. They also generalize the results of Baron and Besanko (1984) from two periods to an arbitrary finite time horizon. Their work differs from this paper in three major ways. Firstly, they assume that the agent desires to contract with the principal in every period. This assumption allows them to use an envelope theorem to eliminate the pricing rule from the incentive compatibility constraints as is standard in static mechanism design. By contrast, the buyer in this paper only wants to purchase the object once. Moreover, the analysis in this paper is in continuous time and there is no obvious analogue of their model in continuous time. Secondly, they only require the individual rationality condition to hold at the first period. The principal makes the agent post a bond at the beginning of time and if the agent decides to break the contract at any period before the end of the game, the bond is forfeit. At the end of the game, if the agent has not broken the contract then the bond is returned. In expectation, the contract is individually rational at the beginning of time but along certain paths the agent may end up losing money. By contrast, in a durable goods problem the lowest type that contracts at any time, gets rent equal to their continuation payoff which is determined endogenously. Lastly, in their model the agent's valuation changes at each period of time and the principal knows that her valuation is changing at each period. In this paper the seller does not observe when the buyer receives the shock. While this can, in principle, be modeled as a Markov process, the discrete time analogue of this process is disallowed in their model as the conditional distribution of types at any time t is not differentiable.

1.3. *Organization*

This paper has been organized into fairly self contained sections. Section 2 describes the single shock model. In Section 3.1, we set up the seller's problem and describe some of the inherent difficulties in solving this optimal control problem. In Section 3.2, we define the relaxed problem and derive its solution. Section 3.3 presents the optimal contract and discusses its properties. In Section 4, we set up the multiple shock extension and derive the optimal contract with commitment. In Section 5, we describe formally the game without commitment, derive a stationary equilibrium and show that this equilibrium is the worst equilibrium for the seller. Section 6 provides some concluding remarks. The proofs not included in the body of the paper are in the appendix.

2. THE SINGLE SHOCK MODEL

We consider a continuous time, infinite horizon model consisting of a single buyer and a single seller where time is indexed by $t \in [0, \infty)$. The seller wants to sell a single unit of a perfectly durable good, the cost of which is assumed to be constant over time and is normalized to 0. He is facing a buyer with single unit demand. The buyer stays in the market until she makes a purchase (if ever). The game ends when the buyer purchases the object. We assume that both the seller and the buyer discount the future exponentially with a common discount rate $r \in (0, \infty)$.

We normalize the buyer's valuation to lie in the set $[0, 1]$. Her valuation at time t is denoted by $\theta(t)$. The buyer draws an initial private valuation $\theta(0)$ from a cdf F at the beginning of time 0. There is an exogenous shock which arrives in the market from an exponential distribution with parameter λ. If the shock arrives at the beginning of time t, the buyer draws a new valuation $\theta(t)$ independently from the same distribution F and the valuation is constant thereafter . Stated formally, $\theta(t) \sim F$ and $\theta(t') = \theta(t)$ for all $t' > t$. The seller can observe neither the arrival of the shock nor the realized valuation. In general, if the buyer does not receive the shock at time t then $\theta(t) = \theta(t_-)$ where $\theta(t_-)$ is the limit from the left[5] or, in the other words, the value just before time t. If the buyer does not receive a shock at t, her type is referred to as persistent.

The seller offers a measurable price function $p(\cdot)$ at the beginning of time 0, where

$$p : [0, \infty) \to [0, 1].$$

If the buyer purchases the good at time t, she receives discounted payoff $e^{-rt}[\theta(t) - p(t)]$ and the seller gets $e^{-rt}p(t)$. Since we want to focus on the best commitment contract of the seller, we formally define only the optimal strategies of the buyer. Given any price function $p(\cdot)$ set by the seller at the start of the game, the optimal strategy requires the buyer to behave optimally at all points of time conditional on her information. The buyer's information is her initial value and if the shock has arrived, its date of arrival and the valuation realized from it. This saves on notation, as we do not need to define the buyer's payoff and seller's revenue from an arbitrary nonoptimal strategy of the buyer. We will describe formally the buyer's payoff from optimal behavior in Section 3.1.

[5]This limit is well defined. If the buyer has not received the shock then $\theta(t_-) = \theta(0)$. If the buyer received the shock at $t' < t$ then $\theta(t_-) = \theta(t')$.

The following assumptions about the distribution F are used in the paper:

A1 F is smooth[6] and has strictly positive density throughout the support.

A2 F satisfies the monotone hazard rate condition or $\frac{1-F(\cdot)}{f(\cdot)}$ is decreasing.

A2′ F belongs to the family of polynomial distributions or $F(\theta) = \theta^{\alpha}$ where $\alpha > 0$.

Assumption A1 is an innocuous assumption satisfied by all distributions used in practice. Smoothness is not required for our results although it makes the presentation easier. Assumption A2 is standard in the mechanism design literature and it implies increasing virtual values. The stronger assumption A2′ is the family of polynomial distributions. When $\alpha \geq 1$ then this implies that F has a nondecreasing density which in turn implies the monotone hazard rate condition A2. When $0 < \alpha < 1$ then F has a decreasing density which does not satisfy the monotone hazard rate condition.[7] Assumptions A1 and A2 are assumed to hold throughout the paper except when A2 is explicitly replaced by A2′. Finally, we define

$$\begin{aligned} p_M &\equiv \text{The profit maximizing static monopoly price at distribution } F \\ &\equiv \text{The solution to the equation: } p - \frac{1-F(p)}{f(p)} = 0. \end{aligned}$$

3. Optimal Contract with a Single Shock

In this section, we derive the optimal contract for the infinite horizon model with a single exogenous shock to the buyer's valuation when the seller has commitment. In this game, the seller sets a price function $p(\cdot)$ at the beginning of time and the buyer decides strategically when to buy depending on her current valuation and future expected draw. The seller's problem is to choose a price function which maximizes his revenue given that the buyer is strategic and has rational expectations of future prices.

This section is organized as follows. Subsection 3.1 describes the seller's problem and highlights some of the key difficulties associated with a durable goods problem with stochastic values. Subsection 3.2 describes the relaxed problem. This problem can be used to circumvent some of the

[6] There is some disagreement about the definition of smoothness. We use it to refer to an infinitely differentiable function.

[7] The first order condition from monopoly profit maximization has a unique solution even when $0 < \alpha < 1$. Moreover, this distribution satisfies the monotone hazard rate condition to the right of the monopoly price which is really all we require.

issues associated with the fact that the seller cannot observe when the buyer receives the shock. Finally, subsection 3.3 presents the result and discusses some comparative statics.

3.1. *The Seller's Problem*

As in the standard durable goods model, a strategic buyer with a stochastic valuation will utilize a cutoff strategy (a formal statement for the relaxed problem is Lemma 2) where the cutoffs depend on whether the buyer has received the shock or not. This implies that if it is optimal for a type $\theta(t)$ to buy the good at price $p(t)$, then purchasing is also optimal for all types $\theta > \theta(t)$. The intuition for this behavior is the following. For the buyer who has already received the shock, the existence of the cutoff type follows from the linearity of the buyer's payoff function (see for instance Stokey (1979)). For the buyer who is yet to receive the shock, the continuation payoff has two components - the expected payoff conditional on receiving the shock and the expected payoff conditional on not receiving the shock. The expected payoff to the buyer from receiving the shock is type independent. The type affects neither the arrival rate of the shock nor the valuation after a shock as draws are independent. The existence of the cutoff type now once again follows from the linearity of the buyer's payoff.

We denote the continuation payoffs of type θ at time t by $V^S(t,\theta), V^N(t,\theta)$ where

$$V^S(t,\theta) = \text{The continuation payoff of type } \theta \text{ at time } t \text{ when the shock has been realized,}$$
$$V^N(t,\theta) = \text{The continuation payoff of type } \theta \text{ t time } t \text{ when the shock has not been realized.}$$

Cutoff types at time t are denoted by by $c^S(t), c^N(t)$ where

$$\begin{aligned} c^S(t) &= \text{The lowest type willing to buy at time } t \text{ when the shock has been realized,} \\ &= \inf\{\theta : \theta - V^S(t,\theta) \geq p(t)\} \\ c^N(t) &= \text{The lowest type willing to buy at time } t \text{ when the shock has not been realized} \\ &= \inf\{\theta : \theta - V^N(t,\theta) \geq p(t)\}. \end{aligned}$$

At time t, if the valuation of the buyer $\theta(t)$ is higher than $c^S(t)$ and she has already received the shock then it is optimal for the buyer to purchase the good at price $p(t)$. Similarly, if she has not received the shock, then strategic behavior requires that she purchases at $p(t)$ if $\theta(t) \geq c^N(t)$.

These cutoffs are determined by optimal responses to the price function $p(\cdot)$. Notice also that these cutoff functions need not be continuous or decreasing, as the seller is not restricted to offering a continuous or monotonically decreasing price function. They are useful to simplify notation in the expressions that follow.

Stopping times are denoted by

$$\eta_t^S(\theta) = \inf\{t' : \theta \geq c^S(t') \text{ and } t' \geq t\},$$

$$\eta_t^N(\theta) = \inf\{t' : \theta \geq c^N(t') \text{ and } t' \geq t\}.$$

$\eta_t^S(\theta)$ denotes the earliest time after time t at which type θ will make a purchase conditional on having already received the shock. Since the cutoffs reflect optimal behavior, this time is the actual time at which θ will make a purchase. Similarly, $\eta_t^N(\theta)$ denotes the earliest time after time t that type θ will purchase should she not have received the shock up to that time. Stated differently, if type θ does not receive the shock till time $\eta_t^N(\theta)$, then it is optimal for her to buy the good at price $p(t)$. This takes into account the expected payoff of receiving the shock at some time $t' > \eta_t^N(\theta)$ in the future. Unlike $\eta_t^S(\theta)$, the buyer may not actually purchase the object at time $\eta_t^N(\theta)$ as she might receive the shock before then.

We can now use the cutoffs and stopping times to calculate the continuation payoffs of the buyer. By definition, the following relationship must hold between the prices, cutoffs and continuation payoffs:

$$p(t) = c^S(t) - V^S(t, c^S(t)) = c^N(t) - V^N(t, c^N(t)). \tag{1}$$

This says that the cutoff type must receive her continuation payoff and is hence indifferent between purchasing at t or waiting. $V^S(t,\theta)$ satisfies

$$V^S(t,\theta) = e^{-r(\eta_t^S(\theta)-t)}[\theta - p(\eta_t^S(\theta))]. \tag{2}$$

If the buyer has already received the shock, her continuation payoff is simply the discounted payoff of purchasing the object at optimal time $\eta_t^S(\theta)$. Similarly

$$V^N(t,\theta) = \lambda \int_t^{\eta_t^N(\theta)} e^{-(r+\lambda)(s-t)} \int_0^1 V^S(s,\phi) dF(\phi) ds + e^{-(r+\lambda)(\eta_t^N(\theta)-t)}[\theta - p(\eta_t^N(\theta))]$$

When the buyer has yet to receive the shock, her continuation payoff has two components. If the buyer does not receive the shock till time $\eta_t^N(\theta)$, she will make a purchase at that time because by definition $\theta \geq c^N(\eta_t^N(\theta))$. Hence, she makes the purchase without realizing the shock and receives discounted payoff $e^{-r(\eta_t^N(\theta)-t)}[\theta - p(\eta_t^N(\theta))]$. The probability of not receiving the shock till $\eta_t^N(\theta)$ is $e^{-\lambda(\eta_t^N(\theta)-t)}$. If the buyer receives a shock at time $t < t' < \eta_t^N(\theta)$, she gets continuation payoff $e^{-r(t'-t)} \int_0^1 V^S(t', \phi) dF(\phi)$. The probability density of receiving the shock at time t' is given by $\lambda e^{-\lambda(t'-t)}$. The first term is then simply the expectation of the event that the shock arrives before $\eta_t^N(\theta)$.

These expressions begin to highlight the difficulties associated with solving such a problem. The continuation payoff of the buyer is type dependent and also depends on whether the buyer has received the shock or not - something that the seller cannot observe. Moreover, the buyer is solving an optimal stopping problem in continuous time where the shock is unanticipated, the realization of the shock is random and the prices need not be monotone.[8] As equation (1) shows, the continuation payoffs of the cutoff types are critical in determining the revenue as they relate prices to cutoffs. The above expression shows that the flow of the continuation payoff $dV^N(t, c^N(t))/dt$ depends potentially on future prices. Thus we cannot express the evolution of $V^N(t, c^N(t))$ in terms of local time t variables. This is in contrast to evolution of the continuation payoff of the cutoff type in a standard durable goods problem with persistent values or for that matter to the evolution of continuation payoff $V^S(t, c^S(t))$. It is easy to express $dV^S(t, c^S(t))/dt$ in terms of $V^S(t, c^S(t))$ and $c^S(t)$ (this can be seen in equation (6)). This precludes using standard optimal control to maximize the revenue where the seller maximizes over the set of measurable functions $p(\cdot)$.

We now derive an expression for the time 0 expected distribution of types remaining in the market at time t in response to a price $p(\cdot)$. Naturally, this distribution depends on the time at which the shock arrived (if ever). Moreover, it depends both on the prices before time t and on the prices after, as these determine the continuation payoffs. This dependence on prices can be

[8]Zuckerman (1986) and Stadje (1991) study a job search model with random wage draws in continuous time where offers arrive stochastically. While they allow for multiple draws, they assume that the cost of job search is nondecreasing in time which corresponds to nondecreasing prices $p(\cdot)$ in our context. To the best of our knowledge, the problem with non-monotone costs has not been solved.

expressed succinctly using the cutoff types. We denote this distribution at time t by $\tilde{F}(t,\cdot)$.

$$\tilde{F}(t,\theta) = \text{The probability that the buyer will be in the market at } t$$
$$\text{and that she will have a valuation less than or equal to } \theta.$$

This implies that $\tilde{F}(t,1)$ is simply the probability that the buyer is still in the market at time t. Clearly, it is possible that $\tilde{F}(t,1) < 1$. While $\tilde{F}$ is not a probability measure, it is the relevant measure for the seller for profit maximization. Before we derive the expression for $\tilde{F}$ we need the following notation:

$$\underline{c}^N(s,s') = \inf\left\{c^N(t) : s \leq t < s'\right\},$$
$$\underline{c}^S(s,s') = \inf\left\{c^S(t) : s \leq t < s'\right\}.$$

These cutoffs are the minimum cutoffs in a given interval. Clearly, these are the relevant cutoffs that determine the distribution of types. $\tilde{F}$ can be written in terms of the cutoffs and has two components as shown below.

$$\tilde{F}(t,\theta) = e^{-\lambda t} \min\left\{F(\theta), F\left(\underline{c}^N(0,t)\right)\right\} + \int_0^t \lambda e^{-\lambda s} F\left(\underline{c}^N(0,s)\right) \min\left\{F(\theta), F\left(\underline{c}^S(s,t)\right)\right\} ds. \tag{3}$$

The first term is the event that the shock does not arrive till t, the probability of which is $e^{-\lambda t}$. The probability that the buyer is still in the market is just the probability that her type is lower than the lowest cutoff type till t. This is given by $F\left(\underline{c}^N(0,t)\right)$. Finally, the probability that the buyer's type is lower than θ is simply the minimum of $F(\theta)$ and $F\left(\underline{c}^N(0,t)\right)$. The second term deals with the event of the shock occurring before t. The density of the shock arriving at time s is $\lambda e^{-\lambda s}$. The probability that the buyer is still in the market at s is $F\left(\underline{c}^N(0,s)\right)$. Conditional on being in the market at s, the probability that the valuation she realizes as a result of the shock at s is low enough that she does not make a purchase between s and t is $F\left(\underline{c}^S(s,t)\right)$. The integral reflects that the fact that the shock can arrive at any time prior to and including t.

Any price function $p(\cdot)$ set by the seller induces a measure over time that gives the probability of making a sale . For any measurable subset $T \subset \mathbb{R}$, we define

$$\mu(T) = \text{The probability of making a sale at a time in the set } T.$$

For example $\mu((t,t'))$ is the probability that the seller makes a sale between times t and t'. Clearly, the probability of making a sale before t is given by $\mu([0,t)) = 1 - \tilde{F}(t,1)$. Since prices are not restricted to be continuous, it is possible that the seller chooses to discontinuously change the price at some point of time, thereby serving a positive mass of buyers at that instant. As a result, the measure μ might have atoms. We can decompose μ into two parts μ_L and μ_C where μ_L is absolutely continuous with respect to the Lebesgue measure and μ_C is absolutely continuous with respect to the counting measure. These measures have densities $\mu'_L(\cdot)$ and $\mu'_C(\cdot)$ respectively. We use $\mu'(t) = \mu'_L(t) + \mu'_C(t)$ to denote the sum of the densities. This allows us to define the revenue as a single integral.

The optimal revenue $\hat{R}$ of the seller can now be written as maximization problem in terms of the price function $p(\cdot)$ as follows

$$\hat{R} \equiv \max_{p(\cdot)} \int_0^\infty e^{-rt} p(t)\mu'(t)dt. \tag{4}$$

There is no obvious way to solve the above optimal control problem. The density $\mu'(t)$ depends on the prices via the cutoffs. As previously mentioned these cutoffs are the solution to a nonstandard optimal stopping problem. As we can see from equation (3) the density $\mu'(t)$ depends on the rate of change of the distribution $\tilde{F}$. The rate of change of the distribution depends potentially on future prices and as a result it is not possible to use standard optimal control and reduce (4) to a local optimization problem. A common strategy employed in such problems is to work in the cutoff space (see for example Board (2008)). This strategy involves rewriting the problem, expressing the seller's revenue function only in terms of the cutoff types thereby allowing the seller to maximize revenue by choosing these types. The optimal prices are then backed out from the optimal cutoffs.[9] In the above problem, this would involve eliminating the prices and allowing the seller to choose $c^S(\cdot)$ and $c^N(\cdot)$. Even if the price function could be eliminated in a straightforward way, in our setting, the seller cannot choose these cutoffs functions arbitrarily as they must be generated by a price path. Restricting attention to the space of cutoffs generated by prices introduces similar difficulties to working in the space of prices directly. We now define an appropriate relaxed problem that allows us to circumvent these issues.

[9]This strategy is analogous to the use of the envelope theorem in mechanism design to eliminate the pricing rule from the principal's objective function and expressing it solely in terms of the allocation function (see Myerson (1981)).

3.2. *The Relaxed Problem*

The difficulty for the seller in solving for the optimal contract lies partly in the fact that he could not observe when the buyer received the shock. In this section, we define a relaxed problem where the seller does observe the shock. Moreover, we allow the seller to condition the prices he offers on this information. Any contract in the original problem can be implemented in this problem because the seller can simply choose to ignore the information about the shock. Hence, the highest revenue in the relaxed problem must be weakly greater than in the original. Finally, we show that the optimal contract in the relaxed problem can be "implemented" in the original problem, which implies that it in turn is optimal.

In this relaxed problem, the seller offers conditional price functions $p(\cdot)$, $q_{\cdot}(\cdot)$ where

$$\begin{aligned} p(t) &= \text{Price offered at } t \text{ if buyer has not received the shock yet,} \\ q_t(t') &= \text{Price offered at } t' \text{ if buyer received shock at } t \leq t'. \end{aligned}$$

The seller sets these functions at the beginning of the game with the intention of maximizing revenue. The price at time 0 is $p(0)$. If the buyer has not received the shock until time t, the price she faces is $p(t)$. If the buyer receives the shock at time t, she faces price $q_t(t)$ at t and prices $q_t(t')$ at all times $t' > t$ in the future. We assume that $p(\cdot)$ is measurable with respect to the Lebesgue measure on $[0, \infty)$ and $q_{\cdot}(\cdot)$ is measurable with respect to the Lebesgue measure on $[0, \infty) \times [0, \infty)$. The buyer knows these price functions at the beginning of time and behaves optimally in response to these prices. The solution to this problem involves choosing price functions $p(\cdot)$, $q_{\cdot}(\cdot)$ that maximize revenue. We denote the maximum revenue that the seller can achieve in this problem by R. The following lemma is the formal statement of the fact that the seller can be weakly better in this problem than the original.

Lemma 1. *Let $\hat{R}$ be the revenue to the seller from the optimal contract of the original problem* (4) *and let R be the revenue from the optimal solution to the relaxed problem. Then $R \geq \hat{R}$.*

Proof. Let prices $\hat{p}(\cdot)$ correspond to the optimal contract of the original problem (4) that gives the seller revenue $\hat{R}$. Setting prices

$$p(t) = \hat{p}(t),$$

and

$$q_s(t) = \hat{p}(t) \qquad \text{for all } s \leq t.$$

in the relaxed problem gives the seller $\hat{R}$ in the relaxed problem. This is because setting these prices induces the same behavior for the buyer as in the original problem. But the revenue from the optimal solution to the relaxed problem must be weakly better than this particular choice of contract or $R \geq \hat{R}$. □

The price functions $p(\cdot), q_\cdot(\cdot)$ induce continuation payoffs for each type at each point of time conditional on the information regarding the arrival (or not) of the shock. The continuation payoff of a type θ who has not received the shock till time t is denoted by

$$V(t,\theta) = \text{Continuation payoff of type } \theta \text{ at time } t \text{ when the shock has not arrived.}$$

In other words, $V(t,\theta)$ is the payoff to a type θ from not purchasing the good at time t and behaving optimally in the future where she will face different prices depending on when the shock arrives. If the buyer receives the shock at time t, the continuation payoff of a type θ (which is drawn as a realization of the shock) at time $t' \geq t$ is denoted by

$$W_t(t',\theta) = \text{Continuation payoff of type } \theta \text{ at time } t' \text{ when the shock arrived at } t.$$

We define $X^N(t)$ as the set of types remaining in the market at time t conditional on the buyer not having received the shock till time t. Similarly, we define $X_t^S(t')$ to be the set of types remaining in the market at time t' conditional on the buyer having received the shock at t. As in the previous section, it is more convenient to work in the space of cutoff types. These types summarize optimal behavior by the buyer and are denoted as follows:

$$c(t) = \sup\{\theta : \theta \in X^N(t) \text{ and } \theta - V(t,\theta) \leq p(t)\},$$
$$b_t(t') = \sup\{\theta : \theta \in X_t^S(t') \text{ and } \theta - W_t(t',\theta) \leq q_t(t')\}.$$

Cutoff type $c(t)$ represents the highest persistent type left in the market who is unwilling to purchase the good at $p(t)$. Notice that this does not imply that type $c(t)$ is indifferent between purchasing at $p(t)$ or waiting. It is possible that all the remaining types in the market strictly prefer to wait if the price is too high. $b_t(t')$ is the analogous cutoff type at t' when the shock was observed at t. Hence, it follows that

$$c(t) \text{ is non-increasing in } t \text{ and } b_t(t') \text{ is non-increasing in } t' \text{ for all } t.$$

We say that the cutoff functions defined above describe optimal buyer behavior in response to the price functions $p(\cdot), q_\cdot(\cdot)$.

In this relaxed problem, it is straightforward to express the continuation payoffs of the buyer solely in terms of the cutoffs $c(\cdot)$ and $b_\cdot(\cdot)$. As we show below, this allows us to maximize over the cutoff space and then use the cutoffs to back out prices. The continuation payoff $V(t, c(t))$ of the cutoff type $c(t)$ is denoted by shortened notation $V(t)$:

$$V(t) = V(t, c(t)) = c(t) - (r+\lambda)\int_t^\infty e^{-(r+\lambda)(s-t)}c(s)ds + \lambda\int_t^\infty e^{-(r+\lambda)(s-t)}W(s)ds, \tag{5}$$

where

$$W(t) = \text{Expected continuation payoff if the buyer receives the shock at } t.$$

A derivation of the above equation is given in the appendix. The continuation payoff of the cutoff type has two components. The first consists of the payoff from waiting and not receiving the shock. The second part is the expected benefit from receiving the shock where the expectation is taken over time. We can now compute an expression for $W(t)$.

Clearly

$$W(t) = \int_0^1 W_t(t,\theta)dF(\theta).$$

Finally, we define

$$\eta_t(\theta) = \inf\{t' : \theta \geq b_t(t') \text{ and } t' \geq t\}$$

or in other words, $\eta_t(\theta)$ is the time at which type θ drawn at time t purchases the object. We can now write

$$W_t(t,\theta) = e^{-r(\eta_t(\theta)-t)}\theta - r\int_{\eta_t(\theta)}^\infty e^{-r(s-t)}b_t(s)ds. \tag{6}$$

We can also express the total surplus and the revenue for the seller in terms of η_t. The expected surplus $S(\cdot)$ and expected revenue $R(\cdot)$ conditional on receiving a shock at time t are given by

$$S(t) = \int_0^1 e^{-r(\eta_t(\theta)-t)}\theta dF(\theta)$$

and

$$R(t) = S(t) - W(t).$$

Before we continue, it is important to point out that it is without loss of generality to work in the cutoff space. This is stated in the following lemma.

Lemma 2. *Given any measurable price functions $p(\cdot)$, $q.(\cdot)$, there are unique non-decreasing measurable cutoff functions $c(\cdot)$, $b.(\cdot)$ that describe optimal buyer behavior in response to the price. Conversely, given any non-decreasing measurable cutoff functions $c(\cdot)$, $b.(\cdot)$, there are measurable price functions $p(\cdot)$, $q.(\cdot)$ such that the given cutoff functions describe optimal behavior in response to these price functions.*

The first part of the above lemma simply says that optimal behavior of the buyer has the cutoff property. This comes from the linearity of the buyer's payoff and from the fact that, conditional on receiving the shock, the buyer draws a value independently. The second part of the lemma says the converse. It says that we can always construct price functions to correspond to the given cutoffs and hence it is without loss of generality to focus on the cutoff space. The sequence of prices corresponding to given cutoff functions need not be unique, however, they will be outcome equivalent.

We use $x(t)$ to denote the probability that the buyer is still in the market at time t if she has not received the shock till t.[10] The flow of $x(t)$ is the mass of buyers being served at any instant. $c(t)$ is non-increasing and hence

$$x(t) = F(c(t)).$$

Since $x(\cdot)$ may be discontinuous, we can once again decompose it into two parts - one absolutely continuous with respect to the Lebesgue measure and the other absolutely continuous with respect to the counting measure. We use $-x'(t)$ to denote sum of the densities corresponding to these measures at time t (it is negative because the probability that the buyer remains in the market is decreasing). We are now in a position to write the seller's revenue R from the relaxed problem in terms of $c(\cdot), V(\cdot), R(\cdot)$ and $W(\cdot)$.

$$R \equiv \left\{ \int_0^\infty \lambda e^{-(r+\lambda)t} R(t) F(c(t)) dt - \int_0^\infty e^{-(r+\lambda)t} [c(t) - V(t)] x'(t) dt \right\}.$$

[10]This $x(t)$ is simply the measure of the set $X^N(t)$ with respect to probability measure F.

The seller maximizes R by choosing cutoff functions $c(\cdot)$ and $b_{\cdot}(\cdot)$. This implies that the seller's optimal contract involves a continuum of functions, an inherently nonstandard problem. We show below that the choice space of the seller can be reduced to two functions. We first plug in $c(t) - V(t)$ from (5) to get

$$(7) \qquad R \equiv \int_0^\infty \lambda e^{-(r+\lambda)t} R(t) F(c(t)) dt \\ - \int_0^\infty e^{-(r+\lambda)t} \left[\int_t^\infty (r+\lambda) e^{-(r+\lambda)(s-t)} c(s) ds + \lambda \int_t^\infty e^{-(r+\lambda)(s-t)} W(s) ds \right] x'(t) dt.$$

Using integration by parts on (7) (the details are in the appendix) and simplifying, we get

$$(8) \qquad R \equiv \int_0^\infty \lambda e^{-(r+\lambda)t} \left\{ S(t) F(c(t)) - W(t) \right\} dt + (r+\lambda) \int_0^\infty e^{-(r+\lambda)t} c(t) [1 - F(c(t))] dt.$$

We first observe that after the buyer receives a shock at t, we are in a setting similar to that of the standard durable goods setting of Stokey (1979). The key difference is that the prices $q_t(t')$ affect the continuation payoffs at earlier points of time. The following lemma shows that the intuition from a standard durable goods problem can be extended here. In the optimal solution of the relaxed problem, the seller charges $q_t(t') = q_t(t)$ for all $t' \geq t$, that is, a constant price conditional on the shock arriving. This means that if the seller observes that the buyer received a valuation shock at t, then the seller only makes sales at t and forgoes sales to lower types in the future. This is summarized and formally shown in the following lemma.

Lemma 3. *In the optimal contract for the relaxed problem, the seller sets cutoffs $b_t(t') = b_t(t)$ or equivalently prices $q_t(t') = q_t(t)$ for all $t' \geq t$ and for all t.*

This result implies that we can reduce the dimension of the seller's problem. The seller now simply needs to choose cutoffs $c(t)$ and cutoffs $b_t(t)$ as it is not optimal for him to make sales to any persistent types at a time $t' > t$ conditional on having observed a shock at t. Notice that this also implies that in the optimal solution $q_t(t) = b_t(t)$. For notational convenience, we drop the subscript on b and denote $b_t(t)$ as simply $b(t)$. We can now state the seller's maximization problem

(9)

$$\max_{c(\cdot), b(\cdot)} \left\{ \int_0^\infty \lambda e^{-(r+\lambda)t} \int_{b(t)}^1 \left[F(c(t))\theta - \frac{1-F(\theta)}{f(\theta)} \right] dF(\theta) dt + (r+\lambda) \int_0^\infty e^{-(r+\lambda)t} c(t) [1 - F(c(t))] dt \right\},$$

where $c(\cdot)$ is a non-increasing measurable function and $b(\cdot)$ is measurable.

This is a well defined calculus of variations problem. We ignore the monotonicity restriction on $c(\cdot)$ and show instead that the solution of the unconstrained problem features a non-increasing $c(\cdot)$. The following lemma characterizes the solution.

Lemma 4. *We can say the following about the calculus of variations problem* (9)*:*

i *It has a solution.*

ii *Any solution satisfies the system of two Euler equations in two variables*

$$c(t) - \frac{1 - F(c(t))}{f(c(t))} = \frac{\lambda}{r + \lambda} \int_{b(t)}^{1} \theta dF(\theta),$$

$$F(c(t))b(t) - \frac{1 - F(b(t))}{f(b(t))} = 0.$$

iii *Moreover, every solution lies in the open interval* $(p_M, 1)$.

This shows that the cutoff types $c(t), b(t)$ satisfy the same two equations for all t. Stokey (1979) argued that when values are completely persistent, the first order condition at each point in time is the same and is independent of the discount rate. In that case, the monotone hazard rate assumption guarantees that the necessary first order condition has a unique solution and is thus also sufficient. In turn this implies that the seller charges the static monopoly price at each point of time. By contrast, in the relaxed problem with stochastic values, the Euler equations depend on the arrival rate and the discount rate. Moreover, it is no longer immediate that this system of two equations in two variables has a solution or that the solution is unique. Lemma 4 demonstrates that this system has a solution and Lemma 5 shows that when F satisfies A2′ these equations have a unique solution.

Since there are only two cutoffs in the relaxed problem, what we have effectively shown is that the solution to the relaxed problem consists of two prices such that

$$p(t) = p \qquad \text{for all } t \geq 0$$

$$q_t(t') = q \qquad \text{for all } 0 \leq t \leq t'.$$

The seller chooses to make some sales in the first instant. If the buyer does not purchase in the first instant, she waits until she receives the shock at some time t. If her value $\theta(t)$ is greater than the price q she makes a purchase else she never buys the good. This is because the seller never chooses to drop the price and serve any of the persistent types. The following lemma states that this contract features increasing prices.

Lemma 5. *When F satisfies A2′, the system of equations in Lemma 4 have a unique solution. Moreover, the solution features an increasing price or $p < q$.*

It should be pointed out that while assumption A2′ is required to show this result analytically, both the uniqueness of the solution of the equations in Lemma 4 and the property of increasing prices seem to be fairly robust in simulations with a variety of different distributions.

3.3. *The Optimal Contract*

We can derive the optimal contract by using the solution to the relaxed problem. We have argued that the seller is always weakly better in the relaxed problem as he has added information that he can choose to ignore. Consider the following contract using the optimal prices from the relaxed problem. The seller charges the price p at the first instant and q thereafter. With these prices, types that do not purchase at the first instant will not do so until they receive a new shock as the price has gone up. When they do receive a shock they will only make a purchase if their type θ is greater than q. This behavior is identical to the behavior of buyer in the relaxed problem. Hence, this price function yields the same revenue for the seller as that in optimal solution to the relaxed problem. This yields following theorem under the assumption that F satisfies A2′.

Theorem 1. *The optimal contract consists of two prices - an introductory price $\hat{p}$ at time 0 and price $\hat{q} > \hat{p}$ at all times $t > 0$. These prices induce cutoff types $\hat{c}, \hat{b}$ where these are the unique solutions to following two equations*

$$\hat{c} - \frac{1 - F(\hat{c})}{f(\hat{c})} = \frac{\lambda}{r + \lambda} \int_{\hat{b}}^{1} \theta dF(\theta) \tag{10}$$

$$F(\hat{c})\hat{b} - \frac{1 - F(\hat{b})}{f(\hat{b})} = 0 \tag{11}$$

Prices are given by

$$\hat{p} = \frac{1 - F(\hat{c})}{f(\hat{c})} + \frac{\lambda}{r + \lambda} \hat{b}[1 - F(\hat{b})] \tag{12}$$

$$\hat{q} = \hat{b} \tag{13}$$

That the optimal contract consists of two increasing prices is intuitive. The increasing price implies that if the buyer does not have a high enough valuation in the first instant to make a purchase, she will not do so in the future unless she receives new information about the product that makes her revise her valuation upwards. From the seller's perspective, the standard intuition for the durable goods monopoly problem applies. It is too costly to serve a persistent type. The gain the seller gets from serving a type who chose to not purchase at time 0 and who has not received a shock is offset by the loss due to the additional rent he has to give the types who make a purchase in the first instant. This is because the seller can only serve persistent types by dropping the price and this increases the continuation payoff at time 0.

Once the seller infers that it is not optimal for him to make a sale to any type who did not purchase at time 0 and who has not received a shock, the stationarity of the price after time 0 follows from the arrival process of the shock. The seller only wants to serve the buyer when she receives a shock and her valuation as a result is sufficiently high. By an identical argument to that given above, if the buyer receives a shock at time t and her type $\theta(t)$ is too low to be served at time t, the seller has no incentive to serve that type in the future. But since the shock arrives from an exponential process, the expected duration of arrival of the shock is the same at every point of time conditional on not having received it. Hence, in essence, the problem after period 0 is 'stationary'.

Stokey showed that the standard durable goods problem essentially boiled down to a static single period monopoly problem. When the buyer can receive a shock, Theorem 1 shows that the seller's problem is effectively a two period problem. In this problem, the buyer has a private value in the first period and draws a new independent private value in the second period from the same distribution. The discount factor of this two period problem depends on the ratio of the arrival rate of the shock to the discount rate. The solution to such a two period problem with discount rate $\lambda/(r+\lambda)$ is precisely characterized by first order conditions (10) and (11).

While it is intuitive for the optimal contract to feature increasing prices, it should be pointed out that the result depends on the assumptions we made about the distribution F and is not true in general for all distributions. The following is a well known example of a two period two type model from Conlisk (1984) in which, for appropriate parameter values, it is optimal for the monopolist to drop the price from the first to the second period. Given the equivalence stated above between our model and the two period model, this example can be modified in the obvious way to our infinite time horizon. We denote the discount rate in the example below by $0 < \beta < 1$ that is simply equal to $\lambda/(r+\lambda)$ in our setting.

Example 1 (Conlisk (1984))**.** Consider a two period model where types can only take two values: $\theta \in \{\theta_L, \theta_H\}$. Moreover, in this model, the buyer draws an independent private value from the distribution in both periods. Let the probability function f be given by

$$f(\theta) = \begin{cases} \alpha & \text{if } \theta = \theta_H \\ 1-\alpha & \text{if } \theta = \theta_L \end{cases}$$

When

$$1 - \frac{\beta(1-\alpha)^2}{(1-\beta\alpha)} < \alpha\frac{\theta_H}{\theta_L} < 1 - \alpha(1-\alpha),$$

it is optimal for the seller to charge a decreasing price sequence $\hat{p} > \hat{q}$ given by

$$\hat{p} = \beta\alpha\theta_L + (1-\beta\alpha)\theta_H$$
$$\hat{q} = \theta_L$$

The seller chooses to serve the high type at the first period and both types in the second period. The high type does not want to delay her purchase both due to discounting and due to the possibility that she might draw a low valuation in period two. It is easy to approximate the discrete distribution in this example by a smooth distribution peaked about θ_H and θ_L and not affect the results.

Finally, we derive a simple comparative static result. When λ increases and r decreases, this raises the effective discount rate of the seller. As the seller loses less in discounting, he would rather make a sale in the future as opposed to at time 0. This is because types served at time 0 must receive rent at least equal to their continuation payoff. After time 0, the seller offers continuation payoff 0 to indifferent type (this type is only present after the buyer has received the shock). This is summarized in the following proposition.

Proposition 1. *The cutoff types move in the following way as the arrival rate of the shock and the discount rate change:*

(i) *When r increases, $\hat{c}$ decreases and $\hat{b}$ increases.*
(ii) *When λ increases, $\hat{c}$ increases and $\hat{b}$ decreases.*

The above proposition implies that the price at time $t > 0$ decreases as the effective discount rate $\lambda/(r+\lambda)$ goes up. The price at time 0 is effected by two different factors - the cutoff type and the continuation payoff. The cutoff type increases, however, the continuation payoff of the buyer increases as well. This implies that the impact of an increase of the effective discount rate on the price at time 0 is ambiguous.

4. Multiple Shocks

In this section, we examine a model with multiple shocks. The buyer can now get repeated shocks that arrive from a Poisson process with parameter λ. We assume that the buyer can only purchase the object when she receives a shock (in particular, this implies that she may not want to purchase at time 0). Specifically, if the buyer receives a shock to her valuation at time t and the next shock at time $t' > t$, the buyer cannot make a purchase at any time s ($t < s < t'$) in between. As we pointed out in the introduction, this model captures the ubiquitous phenomenon of impulse purchasing.

In spite of allowing multiple shocks, the restriction of purchasing conditional on receiving the shock somewhat simplifies the seller's problem. This is because the buyer's continuation payoff is no longer type dependent, moreover, the seller expects that at any point in the future she is either facing a buyer unwilling to make a purchase or a buyer with distribution F. This simplification allows us to study both the case where the seller has commitment and where she cannot commit. In the subsection below, we derive the optimal contract with commitment and then in the next section, we discuss the set of equilibria when the buyer cannot commit.

4.1. *Commitment*

Once again, any price function will induce a function of cutoff types. This is because the probability of drawing a shock is type independent and conditional on a shock, values are drawn independently. The buyer only makes a purchase if her type is above the cutoff type at the given time. The continuation payoff of the buyer is no longer type dependent. Each type will not purchase the object in the future unless she gets the impulse to do so and conditional on receiving the impulse, the new valuation is drawn independently. We denote the continuation payoff of the buyer at time t by $Y(t)$ and it is given by

$$Y(t) = \lambda \int_t^\infty e^{-r(s-t)} \int_{c(s)}^1 [\theta - c(s)] dF(\theta) \tag{14}$$

This expression has the following interpretation. Since the shocks arrive from a Poisson distribution, the probability that the buyer receives a shock during a small period of time Δt is simply $\lambda \Delta t$. Conditional on receiving the shock, she will buy only if her type θ is above the cutoff type $c(t)$. The rent she will get is $\theta - p(t)$ which is simply $\theta - c(t) + Y(t)$. This intuition is captured in the discrete time approximation using small time increments Δt below

$$\begin{aligned} Y(t) &= e^{-r\Delta t} \lambda \Delta t \int_{c(t)}^1 [\theta - p(t)] + e^{-r\Delta t} F(c(t)) Y(t + \Delta t) \\ &\approx \lambda \Delta t \int_{c(t)}^1 [\theta - c(t)] + e^{-r\Delta t} Y(t + \Delta t) \end{aligned}$$

Plugging in the recursive expression for $Y(t + \Delta t)$ and so on, we can obtain a discrete time summation analogous to the integral in (14). We denote the probability that the buyer is still in the market at time t by $w(t)$. This is given by

$$w(t) = e^{-\lambda \int_0^t [1 - F(c(s))] ds}.$$

We observe that $w(t)$ is differentiable and hence it follows the flow equation

$$w'(t) = -\lambda [1 - F(c(t))] w(t).$$

In the above flow equation, λ is the instantaneous probability of receiving the shock and the probability the buyer leaves the market after receiving the shock is $1 - F(c(t))$. In addition, the buyer is only present at time t with probability $w(t)$.

The seller's revenue equation can be expressed in terms of this flow equation and is given by

$$\tilde{R} \equiv -\int_0^\infty e^{-rt} [c(t) - Y(t)] w'(t) dt.$$

Plugging in the expression for $Y(t)$ from (14), integrating by parts and simplifying (details in the appendix), we get

$$\tilde{R} \equiv \int_0^\infty e^{-rt} \left\{ \lambda \int_{c(t)}^1 \left(\theta w(t) - \frac{1-F(\theta)}{f(\theta)} \right) dF(\theta) \right\} dt \tag{15}$$

Setting a cutoff $c(t)$ at time t has three effects. It sets the profit of a sale made at time t, it affects the continuation payoffs to the buyer at all times before t and it determines the probability of making a sale after time t through $w(t)$. The surplus from making a sale at time t depends on the probability $w(t)$ that the buyer is still in the market. The rents, however, are not just paid to the buyer at time t but to all types at all times in the past via their continuation payoffs. Whether or not the buyer actually ends up waiting till time t to make a purchase, she always has the option of doing so and this option value provides her a payoff in expectation. This explains why the surplus term θ is weighted down by $w(t)$ but not the rent term $[1 - F(\theta)]/f(\theta)$.

The seller maximizes $\tilde{R}$ by choosing a cutoff function $c(t)$. We observe that the principle of optimality applies to the above problem and hence we can rewrite it as a dynamic programming problem where the state variable is $w(t)$. It is interesting that the seller's commitment problem can be written as a dynamic programming problem that is normally the method of choice in problems without commitment. Assuming that $\tilde{R}$ is differentiable, the Hamilton-Jacobi-Bellman equation is

$$r\tilde{R}(w) = \max_c \left\{ \lambda \int_c^1 \left(\theta w - \frac{1-F(\theta)}{f(\theta)} \right) dF(\theta) - \lambda w[1 - F(c)]\tilde{R}'(w) \right\} \tag{16}$$

The Hamilton-Jacobi-Bellman equation is a sufficient condition for optimality because the above problem is bounded and has discounting. Solving this Hamilton-Jacobi-Bellman equation, we can derive the following properties of the optimal contract.

Theorem 2. *The optimal contract satisfies the following properties*

(i) *$c(t)$ is smooth.*
(ii) *$c(t)$ is monotonically increasing and as $t \to \infty$, $c(t) \to 1$.*
(iii) *The price function $p(t)$ that generates $c(t)$ is smooth, monotonically increasing and $p(t) \to 1$.*

In the proof, we also derive an expression for the cutoffs. $c(t)$ is given by

$$w(t)c(t) - \frac{1-F(c(t))}{f(c(t))} = \lambda \int_t^\infty e^{-r(s-t)} w(s) \int_{c(s)}^1 \theta dF(\theta) ds. \tag{17}$$

Since F is smooth and the right side of the above equation is differentiable, we can conclude that $c(t)$ is differentiable (as $w(\cdot)$ is differentiable). But if $c(\cdot)$ is differentiable, then the right side of the above equation is twice differentiable which implies that $c(\cdot)$ is twice differentiable. Repeating this argument, we can conclude that $c(\cdot)$ is smooth.[11] The term on the right hand side is the expected surplus from time t onwards. This is clearly bounded below by 0 and above by 1. As time progresses, the probability $w(t)$ that the buyer is still in the market gets smaller. Hence the benefits from making a sale get smaller relative to the loss from the rent this sale will provide at all times in the past. In the limit, we show that this tradeoff is severe and as a result the seller chooses not to make any sales. This limit behavior can then be used to establish the monotonicity of the cutoff function. We show that any inflection point of $c(\cdot)$ must be a local maximum. This implies that if an inflection point were to exist, the cutoff would decrease to its right and the only way in which the limit result can hold is if the cutoff increases at some point in the future which would result in a local minimum. This gives the desired contradiction.

This result once again corroborates the intuition of the single shock model. When the seller has commitment, her primary concern is to reduce the continuation payoff of the buyer. When the buyer can receive multiple shocks the seller needs to "defend" against each shock. In the single shock model, the seller raised the price once to counter the effect of the single shock. In this case, the seller needs to keep increasing prices and price out the buyer in the limit to ensure that her incentives to wait are reduced to a minimum while still allowing for the possibility of making sales in the future.

The best commitment solution is an upper bound for the revenue that the seller can achieve in equilibrium of the dynamic game without commitment. This is because the seller can always commit to charging the same prices as the price function on the equilibrium path. In the next section, we study the set of equilibria of the dynamic game without commitment. We derive the worst equilibrium for the seller and show that a variety of different price paths can be supported as equilibria using the worst equilibrium as an optimal penal code. Finally, we point out that the seller is strictly better off with commitment power.

[11]It is also clear from the above expression that F need not be smooth for the result. Instead, the degree of differentiability of $c(\cdot)$ will depend on the degree of differentiability of F.

5. Multiple Shocks Without Commitment

When the monopolist does not have commitment power, we are in the setting of a dynamic game. In this game, the seller's strategy at time t is a price $p(t)$ whereas the buyer's strategy is a binary decision to purchase or not. We restrict attention to measurable histories. Histories at time t are

$$\mathscr{H}_S^t \equiv \{p(s) : 0 \leq s < t \, , \, p(\cdot) \text{ measurable } \},$$

$$\mathscr{H}_B^t \equiv \{(\phi(s), p(s)) : 0 \leq s \leq t \, , \, p(\cdot) \text{ measurable}\}, ^{12}$$

where

$$\phi(s) = \begin{cases} \theta(s) & \text{if the buyer receives the shock at time } s, \\ -1 & \text{if the buyer does not receive the shock at time } s. \end{cases}$$

$\mathscr{H}_B^t$ is the set of histories of the buyer at time t. It consists of all the prices offered by the seller prior to time t, the price offered at the current time and the history of shocks and valuations drawn including the current valuation. The fact that the buyer's history contains the price at time t, reflects the fact that the seller offers the price first and then the buyer makes her purchase decision. Hence, she can condition her strategy on the price offered by the seller at the current time. The seller cannot observe the buyer's valuation so his history $\mathscr{H}_S^t$ consists only of the prices he has offered prior to time t. Clearly, the fact that the game has not ended by t implies that the buyer has chosen to not purchase the object until t. The value $\phi(t) = -1$ corresponds to the buyer not receiving a shock at time t. Setting this negative valuation is purely for notational convenience as, in equilibrium, the buyer will never find it optimal to make a purchase with a negative valuation. This ensures that, in equilibrium, purchases are only made during shocks.

The set of all possible histories are given by

$$\mathscr{H}_S \equiv \cup_{t \in [0,\infty)} \mathscr{H}_S^t \qquad \text{and} \qquad \mathscr{H}_B \equiv \cup_{t \in [0,\infty)} \mathscr{H}_B^t .$$

The seller sets a price conditional on the previous prices he has offered and the buyer makes a purchase decision conditional on the previous prices, valuations and on the current price and

[12] $\phi(\cdot)$ is measurable by definition as it is simply the realized independent draws from the Poisson process.

value. We define $\mathscr{F}_S^t$ to be the filtration generated by the price path or

$$\mathscr{F}_S^t = \sigma(\{p(s) : s < t\}).$$

The seller's strategy

$$\sigma_S : \mathscr{H}_S \to [0,1]$$

is an $\mathscr{F}_S$ adapted process. Similarly, we define the filtration $\mathscr{F}_B^t$ generated by the values and prices as

$$\mathscr{F}_B^t = \sigma(\{(\phi(s), p(s)) : s \leq t\}).$$

The buyer's strategy

$$\sigma_B : \mathscr{H}_B \to \{0,1\}$$

is an $\mathscr{F}_B$ predictable process. The seller's strategy is simply to set a price between 0 and 1 at each history. The buyer makes a purchasing decision where 1 denotes buying the object and 0 denotes not buying and waiting instead.

We should point out that restricting attention to pure strategies is without loss of generality. Since the seller posts his price before the buyer makes her decision, in equilibrium, he has no incentive to mix as the buyer will only condition her strategy on the realized price. Due to linearity of the buyer's payoff, there will be a only single type for the buyer (the cutoff type) that will be indifferent between buying and waiting. This type is a set of measure 0 and hence her decision does not affect the outcome.

For any histories $h_S^t \in \mathscr{H}_S$ for the seller, $h_B^t \in \mathscr{H}_B$ for the buyer, there is a continuation game (which is the infinite horizon dynamic game that begins at time t) following history h_S^t or h_B^t. While we are not in the setting of a repeated game, the above dynamic game has a recursive structure that allows us to define continuation strategies in the standard way. For any strategy profile σ_S, σ_B we denote

$$\sigma_S|_{h_S^t}(h_S^{t'}) = \sigma_S(h_S^t h_S^{t'}), \qquad \forall h_S^{t'} \in \mathscr{H}_S,$$
$$\sigma_B|_{h_B^t}(h_B^{t'}) = \sigma_B(h_B^t h_B^{t'}), \qquad \forall h_B^{t'} \in \mathscr{H}_B,$$

as the behavior implied by strategies σ_S, σ_B in the continuation game following h_S^t, h_B^t, where $h_S^t h_S^{t'}, h_B^t h_B^{t'}$ denote the concatenation of histories. For notational convenience, we define the notion of *compatible histories*.

Definition 5.1 (Compatible Histories). Consider histories h_S^t, h_B^t and a strategy σ_s. These histories are said to be *compatible* with respect to strategy σ_S if the prices charged by the seller in history h_S^t at all points of time $t' < t$ due to strategy σ_s, are the same as the prices in history h_B^t . Moreover, the price at time t at h_B^t is $\sigma_S(h_S^t)$.

A compatible history is simply a tuple of histories that can be jointly reached by a strategy σ_S. Consider any history h_S^t for the seller and a compatible history h_B^t. Let $p(t) = \sigma_S(h_S^t)$ and let $\phi(t)$ be the value of the buyer at time t. The payoffs u_S, u_B corresponding to strategies σ_S, σ_B at time t are given by

$$u_S|_{h_S^t}(\sigma_S, \sigma_B) = \begin{cases} p(t) & \text{if } \sigma_B(h_B^t) = 1 \\ 0 & \text{if } \sigma_B(h_B^t) = 0 \end{cases}$$

$$u_B|_{h_B^t}(\sigma_S, \sigma_B) = \begin{cases} \phi(t) - p(t) & \text{if } \sigma_B(h_B^t) = 1 \\ 0 & \text{if } \sigma_B(h_B^t) = 0 \end{cases}$$

Since there are shocks that affect the buyer's valuation throughout time, there is a probability measure over histories corresponding to each pair of strategies. We refer to this measure by $\nu(\sigma_B, \sigma_S)$. The probability measure of future histories conditional on having reached history h_S^t for the seller is denoted by $\nu|_{h_S^t}(\sigma_B, \sigma_S)$ and by $\nu|_{h_B^t}(\sigma_S, \sigma_B)$ at h_B^t. Expected revenue and payoff at any pair of compatible histories can now be defined informally in terms of ν and u as

$$U_S|_{h_S^t}(\sigma_S|_{h_S^t}, \sigma_B|_{h_B^t}) = \text{Expected continuation revenue generated by } u_S \text{ and } \nu|_{h_S^t}(\sigma_B, \sigma_S),$$

$$U_B|_{h_B^t}(\sigma_S|_{h_S^t}, \sigma_B|_{h_B^t}) = \text{Expected continuation payoff generated by } u_B \text{ and } \nu|_{h_B^t}(\sigma_B, \sigma_S).$$

It should be pointed out that in continuous time, it is well known that even well defined strategies need not lead to outcomes (see for example, Simon and Stinchcombe (1989)). However, as can be seen above, outcomes correspond to strategies trivially in our setting. Perfect equilibrium can now be defined naturally.

Definition 5.2. A perfect equilibrium is a pair of strategies σ_B, σ_S such that for all σ_B', σ_S':

(i) $U_S|_{h_S^t}(\sigma_S|_{h_S^t}, \sigma_B|_{h_B^t}) \geq U_S|_{h_S^t}(\sigma_S'|_{h_S^t}, \sigma_B|_{h_B'^t})$ at all $h_S^t \in \mathscr{H}_S$ and for each pair of histories $h_B^t, h_B'^t$ compatible with h_S^t and with common buyer valuations such that the price at time t in $h_B^t, h_B'^t$ is $\sigma_S(h_S^t), \sigma_S'(h_S^t)$ respectively.

(ii) $U_B|_{h_B^t}(\sigma_S|_{h_S^t}, \sigma_B|_{h_B^t}) \geq U_B|_{h_B^t}(\sigma_S|_{h_S^t}, \sigma_B'|_{h_B^t})$ at all $h_B^t \in \mathscr{H}_B$ where h_S^t is compatible with h_B^t.

An important aspect of equilibrium in this game is that the seller cannot punish the buyer for deviating as he cannot observe her type and hence has no way of knowing that she is not following the equilibrium strategy. We can also define a stationary equilibrium in this environment.

Definition 5.3. A stationary equilibrium is a perfect equilibrium where

(1) The seller offers the same price p^* at all histories.

(2) There is history independent function $y : [0,1] \times [0,1] \to \{0,1\}$ that describes the buyer's purchasing decision at any history. At any time t, the buyer will purchase the good when her current value is $\phi(t)$ and the current price is $p(t)$ if and only if $y(\phi(t), p(t)) = 1$.

Thus in a stationary equilibrium, the seller offers the price p^* irrespective of the prices he has offered previously. In equilibrium, the buyer best responds and hence follows a cutoff strategy. The stationarity of the buyer's strategy implies that there is a constant time independent continuation payoff Y^*, such that at any time t, if the seller offers a price $p(t)$, the buyer will purchase if and only if

$$\phi(t) \geq p(t) + Y^*.$$

A price p^* at every point of time induces a constant cutoff $c^* = p^* + Y^*$. We can express the rent Y^* which is the continuation payoff to the buyer from a constant price function p^* in terms of the cutoff c^* as

$$Y^* = \frac{\lambda}{r} \int_{c^*}^{1} (\theta - c^*) dF(\theta).$$

For such an equilibrium to exist, it must be optimal for the seller to charge the same price p^* at any history when the buyer's behavior is stationary. Moreover, in equilibrium, Y^* should be the continuation payoff from this constant p^*. The seller's maximization problem can then be written in terms of the cutoffs at any history $h_S^t \in \mathscr{H}_S$ as follows

$$R_{h_S^t}^* \equiv \max_{\{c(s): s \geq t\}} - \int_t^\infty \left\{ e^{-r(s-t)} [c(s) - Y^*] w'(s) \right\} ds,$$

where

$$w'(s) = -\lambda[1 - F(c(s))]w(s).$$

We observe that this problem is identical at all histories, so we can drop the subscript on $R^*_{h^t_S}$. Once again this calculus of variations problem can be expressed as the following dynamic programming problem:

$$rR^*(w) = \max_c \left\{ \lambda[c - Y^*][1 - F(c)]w - \lambda w[1 - F(c)] \frac{dR^*(w)}{dw} \right\}.$$

In the appendix, we solve the above dynamic programming problem and show that there is a unique stationary equilibrium. The solution involves proving first that the best response of the seller to a stationary strategy by the buyer is a stationary strategy and then showing that Y^* is sequentially rational. The following proposition summarizes this.

Proposition 2. *A stationary equilibrium exists and is unique. The stationary cutoff contract c^*, given by the unique solution to the following equation, is the unique stationary equilibrium.*

$$c - \frac{1 - F(c)}{f(c)} = \frac{\lambda}{r + \lambda[1 - F(c)]} \int_c^1 \theta dF(\theta) \tag{18}$$

The equation in the above proposition is easy to interpret. On the left side is the marginal loss in revenue by slightly increasing the cutoff type served. On the right side is the marginal gain from having an additional mass of buyers in the market in the continuation game. This additional mass is the result of serving fewer types at the current time. Unlike the contract with commitment, the seller is not concerned about how the price today effects the continuation payoff of the buyer at all previous times.

An interesting comparison is when we compare the stationary equilibrium to the optimal stationary contract with commitment. It is easy to show that the optimal stationary contract with commitment $\bar{c}$ is the solution to the following equation

$$c - \left(\frac{r + \lambda[1 - F(c)]}{r} \right) \left(\frac{1 - F(c)}{f(c)} \right) = \frac{\lambda}{r + \lambda[1 - F(c)]} \int_c^1 \theta dF(\theta) \tag{19}$$

By definition, the stationary contract with commitment $\bar{c}$ offers a higher revenue to the seller than c^*. Moreover, the term on the left hand side now indicates that the seller's choice of cutoff is also determined by the rent it provides in expectation to the buyer through her continuation payoff. By comparing the equations (19) and (18) we can conclude that $\bar{c} > c^*$. In the commitment solution, the seller can commit to offering the buyer a lower outside option which results in higher revenue even though it implies that fewer sales are being made at each point of time. Moreover, it is possible to show that the total surplus[13] is lower under $\bar{c}$ than c^*. In other words, not only is the seller better off, the buyer is strictly worse off when the seller has commitment power.

Finally, we show that the stationary equilibrium is in fact the worst equilibrium for the seller. This result is similar to what Mason and Välimäki (2008) find in a dynamic moral hazard model. They show that when a principal hires an agent to complete a project, the worst equilibrium for the principal features a constant wage to the agent over time.

Proposition 3. *Let $\underline{R}^*$ be the lowest revenue that the seller can achieve in a perfect equilibrium. Then the revenue from the stationary equilibrium $R^* = \underline{R}^*$.*

Proof. To make the intuition transparent, we prove this result by considering the discrete time game with small time increments Δt and this can be generalized in the obvious way to continuous time (in particular, taking the limit $\Delta t \to 0$ yields the stationary equilibrium in continuous time). The stationarity of our dynamic game implies that we can use the well known argument of Abreu (1988). This states that every equilibrium can be supported by optimal penal equilibrium $\underline{R}^*$. Specifically, any deviation by the seller results in reverting to seller worst equilibrium $\underline{R}^*$. We denote the expected ex-ante payoff to the buyer in the seller worst equilibrium by $\underline{Y}$. Let the price offered at the first instant of equilibrium $\underline{R}^*$ be $\underline{p}$. Then for any price deviation $p \neq \underline{p}$ that induces a cutoff c it must be the case that

$$\underline{R}^* \geq \lambda \Delta t (c - \underline{Y})[1 - F(c)] + e^{-r\Delta t}(1 - \lambda \Delta t[1 - F(c)])\underline{R}^*,$$

as equilibrium deviations must be punished. By continuity in c we get

$$\underline{R}^* \geq \max_c \{\lambda \Delta t (c - \underline{Y})[1 - F(c)] + e^{-r\Delta t}(1 - \lambda \Delta t[1 - F(c)])\underline{R}^*\}.$$

The right side corresponds to the maximum over all prices that the seller can offer at the current time. But this implies that the right side corresponds to an equilibrium payoff and therefore by minimality of $\underline{R}^*$ we get

$$\underline{R}^* = \max_c \{\lambda \Delta t (c - \underline{Y})[1 - F(c)] + e^{-r\Delta t}(1 - \lambda \Delta t[1 - F(c)])\underline{R}^*\}. \tag{20}$$

[13]Note that a uniformly higher cutoff does not necessarily imply higher surplus as values are stochastic.

The above expression immediately tells us that any worst equilibrium must offer the same ex-ante expected payoff to the buyer. Let us assume that there are two equilibria which gave the seller the same payoff $\underline{R}^*$ and give different ex-ante expected payoffs $\underline{Y} \neq \underline{Y}'$ to the buyer. Since both equilibria can act as optimal penal codes, this would imply that the above equation (20) must hold for both $\underline{Y}$ and $\underline{Y}'$ which is not possible. Hence, the worst equilibrium must offer a unique ex-ante expected payoff $\underline{Y}$ to the buyer. But then if $\underline{c}$ is the argmax of equation (20), it must be the case that

$$\underline{Y} = \frac{e^{-r\Delta t}\lambda\Delta t}{1 - e^{-r\Delta t}} \int_{\underline{c}}^{1} [\theta - \underline{c}] dF(\theta).$$

But now equation (20) is simply the problem of the seller in the stationary equilibrium and this completes the proof. □

This result allows for a simple characterization of the set of perfect equilibria. Any price function $p(\cdot)$ can be supported as an equilibrium of the dynamic game as long as the expected continuation revenue to the seller does not fall below R^* at any time t. Moreover, since at each point of this dynamic game, the seller charges the price first and then the buyer decides, any deviation by the seller can be immediately punished by the buyer. Therefore this set of equilibria can be supported irrespective of discount rate r. In particular, this implies that the best stationary commitment contract can be supported as a perfect equilibrium. Finally, the continuation vale of the seller in the optimal commitment contract goes to 0 as $t \to \infty$. It is immediate then that the optimal commitment contract cannot be supported as an equilibrium of the dynamic game.

6. Concluding Remarks

6.1. *Robustness*

The analysis in the paper was done in continuous time primarily because it makes the intuition transparent and it reduces the algebra to a minimum. The majority of the results in the paper are robust to analysis in discrete time. Theorem 1 is restated in the obvious way where the introductory price is charged in the first period as opposed to the first instant. The limit result for the cutoffs in the multiple shock case (Theorem 2) also holds. However, the monotonicity of prices no longer necessarily holds in discrete time. The proof of this result relied crucially on calculus through the fact that the instantaneous probability of receiving a shock is zero as shocks are drawn from an exponential process. In the equivalent discrete time model, shocks will be drawn from a Bernoulli process and hence between periods, there is a strictly positive probability that the buyer receives the shock. Finally, the results without commitment generalize as well. The details of the analysis in discrete time can be found in a supplemental appendix to the paper.

We assumed that conditional on receiving a shock, the seller always draws a new valuation from the same distribution F. This assumption has different implications for the single and multiple shock case. In the single shock case, we are free to assume that the shock comes from some other distribution G as long as the solution to the relaxed problem features an increasing price. This can be determined in a straightforward way by analyzing the solution to the system of two equations in Theorem 1 rewritten in terms of G. These equations are rewritten below

$$\hat{c} - \frac{1 - F(\hat{c})}{f(\hat{c})} = \frac{\lambda}{r + \lambda} \int_{\hat{b}}^{1} \theta dG(\theta),$$

$$F(\hat{c})\hat{b} - \frac{1 - G(\hat{b})}{g(\hat{b})} = 0.$$

When the solution to the above system of cutoffs results in an increasing price, it will coincide with the optimal contract. The intuition is identical. If the buyer does not purchase the good at the first instant, she will not buy the good unless she receives the shock and has a suitably high valuation. If the solution to the relaxed problem has a decreasing price due to the distribution G, then the buyer can find it optimal to purchase the good even without receiving the shock which implies that the solution to the relaxed problem gives strictly higher revenue.

The multiple shock model can allow draws from different distributions. At time t, we can assume that the buyer draws a valuation from distribution $G(t)$ (which needs to be appropriately "well behaved" across time) conditional on receiving the shock. The analysis will follow and we can get appropriately redefined versions of the necessary conditions for the cutoffs. While, the limit result Theorem 2 will still follow, it is no longer necessary that the prices are monotone. Naturally, we will also lose the stationarity in the equilibrium for the game without commitment.

Finally, we assume in this paper that conditional on receiving a shock, the new valuation is drawn independently. This assumption is hard to relax. Allowing correlation, conditional on a shock, significantly complicates the analysis. The primary difficulty is that optimal behavior may no longer be a cutoff strategy. A higher type may find it optimal to wait when a lower type finds it optimal to purchase. As a result, we can no longer to work in the cutoff space and optimal buyer behavior must be derived by solving complicated optimal stopping problems.

6.2. *Multiple Shocks with Persistence*

Another interesting model is one where a buyer's valuation is subject to multiple shocks but where she can purchase the good between the arrival of shocks. This models the arrival of repeated information that causes the buyer to adjust her valuation. This model is simply the multiple shock model in this paper where the buyer is not restricted to purchasing conditional on a shock. This problem is far more complicated than the single shock case in this paper as the buyer can receive new private information repeatedly and the seller cannot observe when this information arrives. Surprisingly, we conjecture that the solution to this problem will be identical to the solution of the multiple shock model in this paper. This is explained below.

Theorem 2 shows that the optimal contract for the multiple shock model features increasing prices. This implies that even if the buyer were allowed to purchase between shocks it will not be optimal for her to do so in response to these prices. As argued earlier, the only incentive for the seller to ever lower the price is to serve the lower buyer types. This incentive is most stark when the buyer has a completely persistent valuation. However, we know that when the buyer has a completely persistent value, the seller never chooses to lower his price. Hence, there is no reason to expect him to do so in this stochastic environment where there is a positive probability of making a sale even by increasing the price. The proof of this conjecture is left for future research.

6.3. *Conclusion*

In this paper, we developed a model of a durable goods monopolist facing a buyer with a stochastic valuation. We examined the case where the buyer receives information about the product in the form of a single random shock and showed that, with commitment, the seller charges a low introductory price at time 0 and a higher price thereafter. This shows that the primary concern for the monopolist is to reduce the continuation payoff of the buyer even at the expense of making fewer sales. We demonstrated that this problem is equivalent to a two period problem in the same way that the standard durable goods problem with persistent values is equivalent to a static monopoly problem.

We extended the model to consider the case of multiple shocks arriving from a Poisson process. These shocks represented purchasing impulses for the buyer. We showed that the intuition from the single shock case generalizes, as the optimal commitment contract for the seller is an increasing price function that converges to the highest type in the limit. In this case, the seller constantly increases prices over time to reduce the continuation payoff of the buyer who may receive a shock at any point of time irrespective of what has happened in the past. We showed that without commitment, in worst equilibrium $\underline{R}^*$ for the seller, she charges a constant price. We argued that any price function can be supported as an equilibrium by threat of deviating to the aforementioned stationary equilibrium as long as the continuation revenue for the seller never falls below $\underline{R}^*$ at any history. Since the prices in the commitment contract converge to the highest type, the continuation revenue of the seller converges to zero and as a result the optimal commitment contract cannot be supported as an equilibrium without commitment.

APPENDIX: PROOFS FROM THE TEXT

Proof of Lemma 2. The fact that prices yield cutoff types is a standard result from the literature (see for example Lemma 1 in Board (2008). Hence, we focus on showing that it is possible to construct prices that correspond to a given cutoff functions $c(\cdot)$ and $b_t(\cdot)$. We set

$$q_t(t') = b_t(t') - W_t(t', b_t(t'))$$

where

$$W_t(t', b_t(t')) = b_t(t') - r\int_{t'}^{\infty} e^{-r(s-t')} b_t(s)ds.$$

Similarly

$$p(t) = c_t - V(t)$$

where V is given by equation (5). By definition these constructed prices correspond to the cutoffs.

□

Derivation of Equation (8)**.**

$$\begin{aligned}
&\int_0^{\infty} \lambda e^{-(r+\lambda)t} R(t)F(c(t))dt \\
&\quad - \int_0^{\infty} e^{-(r+\lambda)t}\left[\int_t^{\infty}(r+\lambda)e^{-(r+\lambda)(s-t)}c(s)ds + \lambda\int_t^{\infty} e^{-(r+\lambda)(s-t)}W(s)ds\right]x'(t)dt \\
&= \int_0^{\infty} \lambda e^{-(r+\lambda)t} R(t)F(c(t))dt - \int_0^{\infty}\left[\int_t^{\infty}(r+\lambda)e^{-(r+\lambda)s}c(s)ds + \lambda\int_t^{\infty} e^{-(r+\lambda)s}W(s)ds\right]x'(t)dt \\
&= \int_0^{\infty} \lambda e^{-(r+\lambda)t} R(t)F(c(t))dt - \int x'(s)ds\left[\int_t^{\infty}(r+\lambda)e^{-(r+\lambda)s}c(s)ds + \lambda\int_t^{\infty} e^{-(r+\lambda)s}W(s)ds\right]\Bigg|_0^{\infty} \\
&\quad - \int_0^{\infty}\int x'(s)ds\left[(r+\lambda)e^{-(r+\lambda)t}c(t)dt + \lambda e^{-(r+\lambda)t}W(t)dt\right]dt \\
&= \int_0^{\infty} \lambda e^{-(r+\lambda)t}\left\{R(t)F(c(t)) - [1-F(c(t))]W(t)\right\}dt + (r+\lambda)\int_0^{\infty} e^{-(r+\lambda)t}c(t)[1-F(c(t))]dt
\end{aligned}$$

Proof of Lemma 3. The seller maximizes (8) by choosing the cutoffs. As stated the objective function is additively separable. Hence, his optimization problem is equivalent to maximizing with respect to $b_t(\cdot)$ for fixed $c(t)$ and then maximizing over $c(t)$. The first term from (8) is the following

$$\int_0^{\infty} \lambda e^{-(r+\lambda)t}[S(t)F(c(t)) - W(t)]dt$$

The maximum for this term is achieved by maximizing $S(t)F(c(t)) - W(t)$ pointwise for each time t. Observe now that the term $S(t)F(c(t)) - W(t)$ is simply the standard durable goods problem of Stokey (1979) where types are drawn from $[0,1]$ and the valuation corresponding to a type θ is simply $F(c(t))\theta$. She shows that in the optimal solution to $S(t)F(c(t)) - W(t)$ the seller charges a constant price at each point of time. Hence $q_t(t) = q_t(t')$ for all $t' > t$. □

Proof of Lemma 4. We need to show that calculus of variations problem (9) has a solution where $c(\cdot)$ is non-increasing. We will ignore the constraint of non-increasing $c(\cdot)$ and show that the unconstrained problem satisfies this constraint. We observe that the flow revenue does not contain a derivative of the controls. Hence the revenue in the unconstrained problem can be maximized by the pointwise maximization of

$$\lambda \int_{b(t)}^{1} \left[F(c(t))\theta - \frac{1-F(\theta)}{f(\theta)} \right] dF(\theta) + (r+\lambda)c(t)[1-F(c(t))] \tag{21}$$

at each point of time t by choosing $c(t)$ and $b(t)$. Since these cutoffs are chosen from compact set $[0,1]$ the pointwise maximum must exist. Since it is never optimal to choose either 0 or 1, this implies that the pointwise maximum of (21) must satisfy the necessary first order conditions. Differentiating with respect to $c(t)$ and $b(t)$ we get the Euler equations of Lemma 4:

$$c(t) - \frac{1-F(c(t))}{f(c(t))} = \frac{\lambda}{r+\lambda} \int_{b(t)}^{1} \theta dF(\theta),$$

$$F(c(t))b(t) - \frac{1-F(b(t))}{f(b(t))} = 0.$$

It is immediate from the above equations that $c(t), b(t) \in (p_M, 1)$ for any $c(t), b(t)$ that satisfy the above equation. Let c and b be any maximizers of (21). We can set $c(t) = c$ and $b(t) = b$ for all t and these controls would be maximizers of the calculus of variations problem (9).

These equations must have a solution as they are necessary for optimality. We now show this explicitly while showing that the implicit functions $c(b)$ from each of the two first order conditions are strictly decreasing and that they cross an odd number of times between p_M and 1. Implicitly differentiating the first Euler equation we get

$$\frac{d}{dc}\left[c - \frac{1-F(c)}{f(c)}\right]\frac{dc}{db} = -\beta b f(b)$$
$$\implies \frac{dc}{db} = -\frac{\beta b f(b)}{\frac{d}{dc}\left[c - \frac{1-F(c)}{f(c)}\right]}$$

Since F satisfies the monotone hazard rate condition the denominator is positive and hence $\frac{dc}{db} < 0$. Moreover, for this equation $c(p_M) > 0$ and $c(1) = p_M$. Implicitly differentiating the second equation we get

$$f(c)b\frac{dc}{db} = -\frac{\partial}{\partial b}\left[F(c)b - \frac{1-F(b)}{f(b)}\right]$$
$$\implies \frac{dc}{db} = -\frac{1}{bf(c)}\frac{\partial}{\partial b}\left[F(c)b - \frac{1-F(b)}{f(b)}\right]$$

Again from the monotone hazard rate condition between p_M and 1, the right side is negative and hence $\frac{dc}{db} < 0$. Also, for this equation $c(p_M) = 1$ and $c(1) = 0$. Hence, The graphs of both these implicit equations must cross an odd number of times between p_M and 1.

□

Proof of Lemma 5. We begin by showing that the system of equations has a unique solution when $F \sim \theta^\alpha$. The second equation can be written as

$$c^\alpha b - \frac{1-b^\alpha}{\alpha b^{\alpha-1}} = 0$$
$$\implies b = [1+\alpha c]^{-\frac{1}{\alpha}}$$

Plugging this into the first equation we get

$$c^\alpha(1+\alpha) + \frac{\delta\alpha}{1+\alpha}[1+\alpha c]^{-\frac{\alpha+1}{\alpha}} - \frac{c^{1-\alpha}}{\alpha} = \frac{\delta\alpha}{1+\alpha}$$

For $\alpha > 0$ the left side is monotone when $c \geq p_M$ so there must be a unique solution. We now show that the prices are increasing. From the first order condition with respect to c we know that

$$p = \frac{1-F(c)}{f(c)} + \beta b[1-F(b)]$$

and

$$q = b.$$

Since $p < c$, if $c \leq b$ we are done. Hence, we consider the case of $c > b$. Now

$$\begin{aligned} p - q &= \frac{1-F(c)}{f(c)} + \beta b[1-F(b)] - b \\ &< \frac{1-F(c)}{f(c)} + b[1-F(b)] - b = \frac{1-F(c)}{f(c)} - bF(b) \end{aligned}$$

We now show that $\frac{1-F(c)}{f(c)} < bF(b)$ which in turn implies $p < q$. Using the first order condition

$$\begin{aligned} &\frac{1-F(c)}{f(c)} < bF(b) \\ \Longleftrightarrow &\frac{[1-F(c)]f(b)}{[1-F(b)]f(c)} < \frac{F(b)}{F(c)} \\ \Longleftrightarrow &\frac{F(c)[1-F(c)]}{f(c)} < \frac{F(b)[1-F(b)]}{f(b)} \end{aligned}$$

We finally show that $\frac{F(\cdot)[1-F(\cdot)]}{f(\cdot)}$ is decreasing for values greater than the monopoly price when $F \sim \theta^{\alpha}$. Since, $c > b > p_M$, this is sufficient to prove that $p < q$. Evaluating at arbitrary $\theta > p_M$, we get

$$\frac{F(\theta)[1-F(\theta)]}{f(\theta)} = \frac{1}{\alpha}\theta[1-\theta^{\alpha}].$$

But the right term is simply $1/\alpha$ times the monopoly profit by setting the price θ. But since this choice of F satisfies the monotone hazard rate (to the right of p_M when $\alpha < 1$), we know that the monopoly profit is decreasing when the price is greater than the monopoly price. Hence $p < q$. $\square$

***Proof of Proposition 1**.* Consider the implicit equation $\hat{c}(\hat{b})$ defined by the first equation. As $\lambda/(r+\lambda)$ increases this curve essentially shifts upwards. For each $\hat{b}$ the value of $\hat{c}(\hat{b})$ is strictly bigger. Since, both implicit equations are downward sloping and since the fist implicit equation starts below it must be the case that it intersects the second implicit equation earlier. This provides the relevant comparative static. The proof is summarized in the following figure $\square$

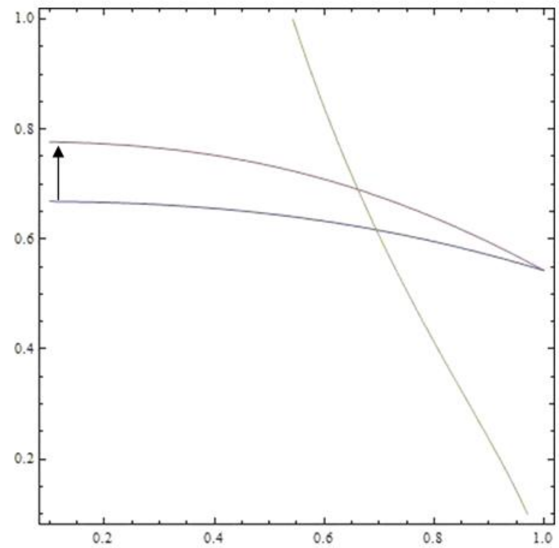

FIGURE 1

Derivation of Equation (15)

$$
\begin{aligned}
& -\int_0^\infty e^{-rt}[c(t)-Y(t)]w'(t)dt \\
=& -\int_0^\infty [e^{-rt}c(t)-\lambda\int_t^\infty e^{-rs}\int_{c(s)}^1[\theta-c(s)]dF(\theta)ds]w'(t)dt \\
=&\lambda\int_0^\infty e^{-rt}c(t)[1-F(c(t))]w(t)dt+\lambda w(t)\int_t^\infty e^{-rs}\int_{c(t)}^1[\theta-c(t)]dF(\theta)ds\Big|_0^\infty \\
& +\lambda\int_0^\infty w(t)e^{-rt}\int_{c(t)}^1[\theta-c(t)]dF(\theta)dt \\
=&\lambda\int_0^\infty e^{-rt}c(t)[1-F(c(t))]w(t)dt+\lambda\int_0^\infty e^{-rt}[w(t)-1]\int_{c(t)}^1[\theta-c(t)]dF(\theta)dt \\
=&\lambda\int_0^\infty e^{-rt}w(t)\int_{c(t)}^1\left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)dF(\theta)dt+\lambda\int_0^\infty e^{-rt}[w(t)-1]\int_{c(t)}^1\left(\frac{1-F(\theta)}{f(\theta)}\right)dF(\theta)dt \\
=&\lambda\int_0^\infty e^{-rt}\int_{c(t)}^1\left(w(t)\theta-\frac{1-F(\theta)}{f(\theta)}\right)dF(\theta)dt
\end{aligned}
$$

Proof of Theorem 2. We assume that the function $\tilde{R}(\cdot)$ is twice differentiable and later verify that this is true. Taking a derivative of the HJB with respect to w using the envelope theorem we get

$$0=\lambda\int_c^{\overline{\theta}}\theta dF(\theta)-\lambda[1-F(c)]\tilde{R}'(w)-\lambda w[1-F(c)]\tilde{R}''(w)-r\tilde{R}'(w).$$

Setting

$$\psi(t) = \tilde{R}'(w(t)) \qquad \Longrightarrow \psi'(t) = \tilde{R}''(w(t))w'(t) = -\lambda w(t)[1-F(c)]\tilde{R}''(w(t))$$

we get

$$\psi'(t) = (r + \lambda[1 - F(c(t))])\psi(t) - \lambda \int_{c(t)}^{\overline{\theta}} \theta dF(\theta) \tag{22}$$

Moreover

$$c(t) \in \operatorname*{argmax}_{c} \left\{ \lambda \int_{c}^{\overline{\theta}} \left(\theta w(t) - \frac{1-F(\theta)}{f(\theta)} \right) dF(\theta) - \lambda w(t)[1-F(c)]\psi(t) \right\}.$$

The first order condition yields

$$\lambda \left(c(t)w(t) - \frac{1-F(c(t))}{f(c(t))} \right) f(c(t)) = \lambda w(t) f(c(t)) \psi(t)$$

$$\Longrightarrow \psi(t) = c(t) - \frac{1}{w(t)} \frac{1-F(c(t))}{f(c(t))}. \tag{23}$$

We first observe from the above equation that $\psi(t) \in [0,1]$ for all t. We now prove the following property of the cutoffs.

($\star$) For any $c < 1$ and for any t, there exists a time $t' > t$ such that $c(t') > c$.

We consider two cases. First consider $w(t) \to 0$. Since, $\psi(t) = \tilde{R}'(w) > 0$, it must be the case that for the first order condition (23) to hold at all t, $c(t)$ gets close to 1 as t gets large and this implies that property ($\star$) holds. If $c(t)$ does not get close to 1 as $t \to \infty$ then the rent term $[1 - F(c(t))]/[w(t)f(c(t))]$ would explode as $t \to \infty$ and this would make the right side of the above equation (23) negative which is not possible. Let us now consider the case $w(t) \nrightarrow 0$. This implies that $\int_0^\infty [1 - F(c(s))]ds < \infty$ which in turn implies that property $(*)$ holds. This is because if we could choose a $c < 1$ and a t such that $c(t') \leq c$ for all $t' > t$ then $\int_t^\infty [1 - F(c(s))]ds$ would diverge which is a contradiction.

Solving the above differential equation (22) by integrating from 0 to t, we get

$$\psi(t) = \frac{e^{rt}}{w(t)} \left[\psi(0) - \lambda \int_0^t e^{-rs} w(s) \int_{c(s)}^{1} \theta dF(\theta) ds \right].$$

As $t \to \infty$ we observe that $\frac{e^{rt}}{w(t)} \to \infty$. But since $\psi(t)$ is bounded, it must be the case that the term inside the brackets goes to 0. But that implies

$$\psi(0) = \lambda \int_0^\infty e^{-rs} w(s) \int_{c(s)}^1 \theta dF(\theta) ds,$$

which in turn yields

$$\psi(t) = \frac{\lambda}{w(t)} \int_t^\infty e^{-r(s-t)} w(s) \int_{c(s)}^1 \theta dF(\theta) ds. \tag{24}$$

From (23) we can express the above equation in terms of $c(t)$ to get

$$c(t) - \frac{1}{w(t)} \frac{1 - F(c(t))}{f(c(t))} = \frac{\lambda}{w(t)} \int_t^\infty e^{-r(s-t)} w(s) \int_{c(s)}^1 \theta dF(\theta) ds.$$

Clearly, the right side is differentiable as $w(\cdot)$ is differentiable. But since F is smooth this implies that $c(\cdot)$ is differentiable. But if $c(\cdot)$ is differentiable, this implies that the right side is twice differentiable which in turn would imply that $c(\cdot)$ is twice differentiable as F is smooth. We can keep differentiating the right side to conclude that $c(\cdot)$ is smooth. Finally, we know

$$\tilde{R}(w(t)) = \int_t^\infty \lambda e^{-r(s-t)} \int_{c(s)}^1 \left[w(s)\theta - \frac{1 - F(\theta)}{f(\theta)} \right] dF(\theta) ds.$$

Since $c(\cdot)$ is smooth, this implies $\tilde{R}$ is smooth. Moreover, we show below that $c(t)$ monotonically converges to 1 and from equation (24), we can conclude that $\psi(t) \to 0$.

We now show that $c(t)$ monotonically converges to 1. Differentiating (24) and equating it to $\psi'(t)$ from equation (22), we get

$$\left[1 - \frac{1}{w(t)} \frac{\partial}{\partial c} \left(\frac{1 - F(c(t))}{f(c(t))} \right) \right] c'(t) = \frac{\partial}{\partial t} \left(\frac{1}{w(t)} \right) \frac{1 - F(c(t))}{f(c(t))} + (r + \lambda[1 - F(c(t))])\psi(t) - \lambda \int_{c(t)}^{\overline{\theta}} \theta dF(\theta). \tag{25}$$

By the monotone hazard rate assumption the term in the square brackets on the left side is positive. Simplifying the right side term

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{w(t)}\right)\frac{1-F(c(t))}{f(c(t))}+(r+\lambda[1-F(c(t))])\psi(t)-\lambda\int_{c(t)}^{\overline{\theta}}\theta dF(\theta)\\
=&\frac{\lambda(1-F(c(t))}{w(t)}\frac{1-F(c(t))}{f(c(t))}+(r+\lambda[1-F(c(t))])\psi(t)-\lambda\int_{c(t)}^{\overline{\theta}}\theta dF(\theta)\\
=&r\psi(t)+\lambda[1-F(c(t))]c(t)-\lambda\int_{c(t)}^{\overline{\theta}}\theta dF(\theta)\\
=&r\psi(t)-\lambda\int_{c(t)}^{\overline{\theta}}[\theta-c(t)]dF(\theta)
\end{aligned}$$

We need to show $c'(t)>0$ which implies cutoffs are increasing. Consider any inflection point, that is, a time t^* such that $c'(t^*)=0$. We will now show that this inflection point must be a local maximum. Differentiating the left side of (25) at t^* we get.

$$\begin{aligned}
\frac{d}{dt}\left\{\left[1-\frac{1}{w(t^*)}\frac{\partial}{\partial c}\left(\frac{1-F(c(t^*))}{f(c(t^*))}\right)\right]c'(t^*)\right\}=&c'(t^*)\frac{d}{dt}\left\{\left[1-\frac{1}{w(t^*)}\frac{\partial}{\partial c}\left(\frac{1-F(c(t^*))}{f(c(t^*))}\right)\right]\right\}\\
&+\left[1-\frac{1}{w(t^*)}\frac{\partial}{\partial c}\left(\frac{1-F(c(t^*))}{f(c(t^*))}\right)\right]c''(t^*)\\
=&\left[1-\frac{1}{w(t^*)}\frac{\partial}{\partial c}\left(\frac{1-F(c(t^*))}{f(c(t^*))}\right)\right]c''(t^*)
\end{aligned}$$

Differentiating the right side with respect to time, we get

$$r\psi'(t^*)-\frac{\partial}{\partial c}\left[\lambda\int_{c(t^*)}^{\overline{\theta}}[\theta-c(t^*)]dF(\theta)\right]c'(t^*)=r\psi'(t^*)=-\frac{\partial}{\partial t}\left(\frac{1}{w(t^*)}\right)\frac{1-F(c(t^*))}{f(c(t^*))}<0.$$

The above follows from the fact that $c(t)$ is never equal to 1 at any time t and that $c'(t^*)=0$. This implies $c''(t^*)<0$ and therefore every inflection point t^* is a local maximum, that is, $c(t^*)$ must weakly decrease to the right of t^*. But then there cannot exist an inflection point as the choice of cutoff $c(t^*)$ and the time t^* would violate Property ($\star$). Moreover, Property ($\star$) also precludes a cutoff sequence that is always strictly decreasing. Hence, $c(\cdot)$ is strictly increasing.

Prices are given by

$$p(t)=c(t)-Y(t)=c(t)-\lambda\int_t^{\infty}e^{-r(s-t)}\int_{c(s)}^{1}[\theta-c(s)]dF(\theta).$$

Taking the derivative $Y(t)$, we get

$$\begin{aligned} Y'(t) &= -\lambda \int_{c(t)}^{1} [\theta - c(t)] dF(\theta) + r\lambda \int_{t}^{\infty} \left\{ e^{-r(s-t)} \int_{c(s)}^{1} [\theta - c(s)] dF(\theta) \right\} dt \\ &= -r\lambda \left[\int_{t}^{\infty} \left\{ e^{-r(s-t)} \int_{c(t)}^{1} [\theta - c(t)] dF(\theta) \right\} dt - \int_{t}^{\infty} \left\{ e^{-r(s-t)} \int_{c(s)}^{1} [\theta - c(s)] dF(\theta) \right\} dt \right] \end{aligned}$$

Since $c(\cdot)$ is increasing, when $s > t$

$$\int_{c(t)}^{1} [\theta - c(t)] dF(\theta) > \int_{c(s)}^{1} [\theta - c(s)] dF(\theta)$$

This implies $Y'(t) < 0$ and hence $p'(t) = c'(t) - Y'(t) > 0$ which completes the proof. $\square$

Proof of Proposition 2. We conjecture first the solution consists of constant cutoffs c^*, that is, the best response of the seller to a stationary strategy by the buyer is a constant cutoff function. Having assumed this, we can derive an expression for $\tilde{R}^*(y)$ given by

$$R^*(y) = \int_0^\infty e^{-rt} [c^* - Y^*] w'(t) dt$$

where $w(0) = y$. Using the expression $w(t) = w(0) e^{-\lambda[1 - F(c^*)]t}$ we solve

$$\begin{aligned} R^*(y) &= -\lambda [c^* - Y^*][1 - F(c^*)] \int_0^\infty e^{-rt} w(t) dt \\ &= \lambda y \frac{[c^* - Y^*][1 - F(c^*)]}{r + \lambda[1 - F(c^*)]} \end{aligned}$$

It is immediate that if the right hand side of the HJB is maximized at c^* then the above value function $R^*(\cdot)$ satisfies the HJB. Since the HJB equation is sufficient for an optimal solution, this would in turn imply that c^* constitutes a stationary equilibrium. We now find a value for c^* for which the right hand side of the HJB is maximized at c^*. This c^* must satisfy

$$\begin{aligned} &c^* \in \underset{c}{\operatorname{argmax}} \left\{ \lambda [c - Y^*][1 - F(c)] w - \lambda w [1 - F(c)] \frac{dR^*(w)}{dw} \right\} \\ \equiv &c^* \in \underset{c}{\operatorname{argmax}} \left\{ \lambda [c - Y^*][1 - F(c)] w - \lambda^2 w [1 - F(c)] \frac{[c^* - Y^*][1 - F(c^*)]}{r + \lambda[1 - F(c^*)]} \right\} \end{aligned}$$

Taking a first order condition with respect to c yields

$$\frac{1 - F(c)}{f(c)} - [c - Y^*] + \lambda \frac{[c^* - Y^*][1 - F(c^*)]}{r + \lambda[1 - F(c^*)]} = 0.$$

$c = c^*$ must satisfy the above expression. Plugging it in and using the expression for Y^* gives us the required equation

$$\frac{1-F(c^*)}{f(c^*)} - [c^* - Y^*] + \frac{[c^* - Y^*]\lambda[1-F(c^*)]}{r+\lambda[1-F(c^*)]} = 0$$
$$\implies \frac{r}{r+\lambda[1-F(c^*)]}c^* - \frac{1-F(c^*)}{f(c^*)} = \frac{r}{r+\lambda[1-F(c^*)]}Y^* = \frac{\lambda}{r+\lambda[1-F(c^*)]}\left\{\int_{c^*}^1 \theta dF(\theta) - c^*[1-F(c^*)]\right\}$$
$$\implies c^* - \frac{1-F(c^*)}{f(c^*)} = \frac{\lambda}{r+\lambda[1-F(c^*)]}\int_{c^*}^1 \theta dF(\theta)$$

If there exists a stationary equilibrium c^*, it must satisfy the above equation. We now show that this equation has a unique solution which in turn implies that the equation is also sufficient for equilibrium. Hence, the unique solution to the above equation would constitute the unique stationary equilibrium

We can rearrange the first order condition to get

$$c(r+\lambda[1-F(c)]) - \lambda\int_c^1 \theta dF(\theta) = (r+\lambda[1-F(c)])\left(\frac{1-F(c)}{f(c)}\right)$$

By evaluating both sides of the above equation at 0 and 1, we can show that the line given by the equation on the right side must cross the line given by the equation on the left at least once. This implies the existence of a solution. The right side is clearly decreasing because of the monotone hazard rate assumption. We will show that the left side is increasing and hence the equation has at most a single solution. Differentiating the left side we get

$$(r+\lambda[1-F(c)]) - \lambda c f(c) + \lambda c f(c) = (r+\lambda[1-F(c)]) > 0$$

Hence the left hand side is increasing and this completes the proof. □

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