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On the strategic origin of Brownian motion in finance

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Abstract This paper is concerned with the strategic use of a private information on the stock market. A repeated auction model is used to analyze the evolution of the price system on a market with asymmetric information.

The model turns out to be a zero-sum repeated game with one-sided information, as introduced by Aumann and Maschler.

The stochastic evolution of the price system can be explicitly computed in the n times repeated case. As n grows to ∞, this process tends to a continuous time martingale related to a Brownian Motion.

This paper provides in this way an endogenous justification for the appearance of Brownian Motion in Finance theory.

1 Introduction

Since the pioneer work of Bachelier [2], finance theory often uses a Brownian Motion to model the evolution of the price system on the stock markets. In Black and Scholes model [4] for instance, the price p_t of the underlying asset follows the dynamic:

$$dp_t := p_t \cdot (a \cdot dB_t + r \cdot dt), \qquad (1)$$

where B is a Brownian motion, a, a volatility parameter and r, an interest rate.

The question we address in this paper is that of the origin of this Brownian motion.

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A first justification for the appearance of the Brownian motion is exogenous: both the utility functions of the agents and the productivity of the firms depend on exogenous variables that are subject to random changes over time. To generate a Brownian motion, these random changes must be infinitesimal and must happen on a continuous time basis: one can easily conceive that the sum of these infinitesimal changes will aggregate in a kind of Brownian motion as an intuitive consequence of the central limit theorem.

However the above explanation is quite unsatisfactory: for a lot of exogenous variables, the changes are typically of discrete nature as is for instance the discovery by a firm of a new production process. This kind of change is not infinitesimal and does not happen on a continuous time basis. Intuitively, one could think that such a shock will have a dramatic consequence on the market and will generate a discontinuity of the price process.

In this paper, we provide an endogenous justification for the appearance of the Brownian motion and we explain how the market will deaden the effect of these macro random changes to preserve the continuity of the price system.

We interpret this smoothing property of the market as a result of an asymmetry of information between the agents: some agents are informed about the shock while others are not. By the day after day actions he will take, the informed agent will partially reveal his information to the other agents. This phenomenon is crucial: the informed agents are loosing in this way their strategic advantage. As it will appear later the better way for these agents to take benefit of their information without revealing it too fast will be to slightly randomize each of their actions. Thus, for strategic reasons, the exogenous random shock will be followed in the day after day transactions by a sequence of small endogenous randomizations. The joint effect of all these randomizations will be the appearance of a Brownian Motion and the continuity of the price process will follow.

To emphasize this phenomena, we analyze a highly simplified model with only one initial exogenous random shock and we study its influence on the price process.

We consider the interactions between two market makers, player 1 and 2, that are trading two commodities N and R. Commodity N is used as numéraire and has a final value of 1. Commodity R (R for Risky asset) has a final value depending on the state k of nature $k \in \mathcal{K} := \{L, H\}$. The final value of commodity R is 0 in state L and 1 in state H. By final value of an asset, we mean the conditional expectation of its liquidation price at a fixed horizon T, given the state of the world.

The state of nature k is initially chosen at random once for all. The probability of H and L being respectively P and $1 - P$. Both players are aware of this probability. Player 1 is informed of the resulting state k while player 2 is not.

The transactions between the players, up to date T, take place during n consecutive rounds. At round q ($q = 1, \cdots, n$), player 1 and 2 propose simultaneously a price $p_{1,q}$ and $p_{2,q}$ for 1 unit of commodity R. The maximal bid wins and one unit of commodity R is transacted at this price. If both bids are equal, no transaction happens.

In other words, if $y_q = (y_q^R, y_q^N)$ denotes player 1's portfolio after round q, we have $y_q = y_{q-1} + t(p_{1,q}, p_{2,q})$, with

$$t(p_{1,q}, p_{2,q}) := \mathbb{1}_{p_{1,q} > p_{2,q}}(1, -p_{1,q}) + \mathbb{1}_{p_{1,q} < p_{2,q}}(-1, p_{2,q}).$$

The function $\mathbb{1}_{p_{1,q} > p_{2,q}}$ takes the value 1 if $p_{1,q} > p_{2,q}$ and 0 otherwise.

At each round the players are supposed to remind the previous bids including these of their opponent.

The final value of player 1's portfolio y_n is then $\mathbb{1}_{k=H} y_n^R + y_n^N$, and we consider that the players are risk neutral, so that the utility of the players is the expectation of the final value of their own portfolio. Let V denote the final value of player 1's initial portfolio: $V = E[\mathbb{1}_{k=H}\ y_0^R + y_0^N]$. Since V is a constant that does not depend on players' strategies, removing it from player 1's utility function will have no effect on his behavior. This turns out to be equivalent to suppose $y_0 = (0, 0)$ (negative portfolios are then allowed). Similarly, there is no loss of generality to take $(0, 0)$ for player 2's initial portfolio . With that convention player 2's final portfolio is just $-y_n$ and player 2's utility is just the opposite of player 1's.

We further suppose that both players are aware of the above description. The game thus described will be denoted $G_n(P)$. It is essentially a zero-sum repeated game with one-sided information as introduced by Aumann and Maschler [1], the only difference being that both players have at each stage a continuum of possible actions instead of a finite number in Aumann and Maschler's model.

The problem the informed player is confronted to in the above framework is essentially the following: to take benefit of his information when told that the state is H, he may wish to propose a higher price for commodity R than in state L. This is a fully revealing behavior: his opponent could deduce the actual state just by observing the proposed price. When playing in this way, player 1 is losing his strategic advantage for the next stages.

Another behavior player 1 may adopt is a completely non revealing one: to hide his information he may propose the same price in both states of the nature. This would be equivalent for player 1 to ignore the state and he could take no extra benefit from his information.

One can easily conceive that the optimal behavior for player 1 will be a compromise between the above described behaviors: player 1 will randomize his proposed prices with state dependent lotteries. Player 2 could then revise his probability distribution (*a posteriori* distribution) on the states of nature by applying Bayes rules. The revelation is in general not total: the *a posteriori* may be close to the *a priori* distribution.

The above discussion explain why randomized strategies are needed in this setting and we will see in the next section that all these randomizations will lead in the limit to the appearance of a Brownian motion

The model proposed is maybe not completely realistic, but it has to be taken as a first model to emphasize our main idea: *part of the Brow-*

nian motion originates in the mixed behavior of the agents. In Kyle [8], the trading between agent with asymmetric information is analyzed and a Brownian motion also appears in that setting, but as a consequence the behavior of a set of noisy traders that is supposed random in an exogenous way. Typically, the optimal behavior of the informed agent in Kyle's model is deterministic (pure strategy). In Kyle's paper the action of the informed agent is not directly observable by the market maker, (orders are anonymous, and only the aggregate order flow is observable). On many markets however, the action of the leading investors are publicly observable. This feature is better captured by our model.

The transaction mechanism analyzed here is maybe not completely realistic: both market makers are committed to set prices at each round and to make transactions accordingly. Typically, since we are concerned with a zero-sum game where player 2 has less strategic power than player 1, he would be better off if he could avoid trading with player 1. He is not allowed to do so in our model.

If a market maker is willing to avoid transaction on a particular market, he will put large spread bid-ask on that asset. That the bid ask spread are in practice quite constant in relative value seems to indicate that market-makers are in fact almost always participating. This makes maybe our assumption a bit more realistic.

The zero-sum property however highly simplify the analysis since it allows us to use the duality methods. We have some hope in fact to generalize the results of this paper to more realistic models as this introduced in Calcagno-Lovo [5]: two market makers, an insider and an uninformed are trading with liquidity traders (not together). At each period, they both propose bid and ask prices, the liquidity trader buy or sell a fixed amount at the better price.

2 The main results of the paper

Our first result is:

Theorem 1 *The game $G_n(P)$ has a value $V_n(P)$ and both players have optimal strategies. $V_n(P)$ is a concave function of $P \in [0,1]$.*

In Aumann and Maschler's paper the existence of a value and of the optimal strategies for the players was a straightforward consequence of finiteness of the action space. In this framework, this result has to be proved since the players have at each round a continuum of possible actions. We will prove this theorem by providing explicitly the optimal strategies. In turn, this will allow us to compute explicitly $V_n(P)$:

Definition 1 *Let f_n denote the probability density of the random variable $S_n := \sum_{q=1}^{n} U_q/\sqrt{n}$, where $U_1, \ldots, U_n$ are n independent random variables uniformly distributed on the interval $[-1,1]$.*

The above definition completely determines the function f_n: in particular, $f_1(x) := \frac{1}{2}1\!\!1_{[-1,1]}(x)$ and, for $n > 1$:

$$f_n(x) = \frac{\sqrt{n}}{2^n} \sum_{k=0}^{n} C_n^k (-1)^{(n-k)} \left((x\sqrt{n} - 2k + n)^+\right)^{n-1},$$

where $(a)^+ := \max\{0, a\}$ and $C_n^k := \frac{n!}{k!(n-k)!}$.

With this definition, we get the explicit value of $V_n(P)$:

Theorem 2 *For all $P \in [0,1]$:*

$$\psi_n(P) := \frac{V_n(P)}{\sqrt{n}} = \int_{x_P}^{\infty} s f_n(s) ds,$$

where x_P is such that $P = \int_{x_P}^{\infty} f_n(s) ds$.

For $n = 1$ for instance, the above result yields: $P = \int_{x_P}^{1} 1/2ds$, so $x_P = 1 - 2P$, and $V_1(P) = \int_{1-2P}^{1} s/2ds = P(1-P)$.

The main result of Aumann and Maschler paper is the convergence of the value $v_n(P)$ ($v_n := V_n/n$) of the n times repeated game with one sided information to $cav(u)(P)$: the concavification on $[0,1]$ of the function u, where $u(P)$ is the value of the 1-round game $G(P)$ where no player is informed. In our framework, $u(P)$ would be the value of the game where a lottery P selects $k \in \mathcal{K}$ without informing the players. Both players propose a price p_1 and p_2 and the resulting payoff for player 1 is $E[\langle(1\!\!1_{k=H}, 1), t(p_1, p_2)\rangle]$. Since k is here independent of p_1, p_2, this payoff is also equal to

$$E[\langle(P, 1), t(p_1, p_2)\rangle].$$

This zero sum game is symmetric and is value $u(P)$ is thus 0 (the optimal strategies are $p_1 = p_2 = P$). So, in our framework, we would have, $\forall P \in [0,1]$, $cav(u)(P) = 0$. Last theorem leads us to a much stronger result than the convergence of $V_n(P)/n$ to 0: we have the convergence of $\psi_n(P) := \frac{V_n(P)}{\sqrt{n}}$. Indeed, the central limit theorem indicates that S_n converges in law to $Z/\sqrt{3}$, where Z is a centered Gaussian random variable with variance 1. So, the density of Z is $f(z) := \exp(-z^2/2)/\sqrt{2\pi}$, and $f_n(x)$ converges to $\sqrt{3}f(\sqrt{3}x)$. This in turn will imply:

Theorem 3 *As n goes to ∞, $\sqrt{3}\psi_n(P)$ converges to $f(z_P) = \int_{z_P}^{\infty} s f(s) ds$, where z_P is such that $P = \int_{z_P}^{\infty} f(s) ds$.*

In the finite action case, games with this kind of asymptotic behavior for $v_n(P)$ were already analyzed in Mertens and Zamir [10], De Meyer [6,7]. However, the class of game where this phenomena was described is characterized by the property that the optimal strategy of the informed player in the average game $G(P)$ does not depend on P. This is not the case in our model.

Our next result deals with the asymptotic behavior of the price process: since we know the optimal strategies of both players, we may compute the distribution of the process of proposed prices $\{(p^n_{1,q}, p^n_{2,q})\}_{q=1,\ldots,n}$(the superior index n indicates that we consider a game of length n) and thus also the distribution of the transaction prices $\{p^n_q\}_{q=1,\ldots,n}$ where $p^n_q := \max(p^n_{1,q}, p^n_{2,q})$. This process p^n can be represented by a continuous time process π^n by setting $\pi^n_t := p^n_q$ if $t \in [(q-1)/n, q/n[$. With this convention, we have:

Theorem 4 *As n goes to ∞, the process π^n converges in law (in the sense of finite distributions) to the following process π:*

$$\pi_t := F\left(\frac{z_P + B_t}{\sqrt{1-t}}\right), \tag{2}$$

where B is a standard Brownian motion issued at 0 and $F(x) := \int_x^\infty f(s)ds$. This process is a $[0,1]$-valued continuous martingale starting at P at time 0. Furthermore π_1 belongs almost surely to $\{0,1\}$.

Remark 1 Notice here that applying Ito's formula to (2) provides a diffusion equation for π_t:

$$d\pi_t = -\frac{f\left(\frac{z_P+B_t}{\sqrt{1-t}}\right)}{\sqrt{1-t}} \cdot dB_t = -\frac{f(F^{-1}(\pi_t))}{\sqrt{1-t}} \cdot dB_t.$$

This is not exactly Back and Scholes' dynamic as in (1), but it indicates however that π is a diffusion process. To get exactly Back and Scholes' dynamic, the model should be generalized allowing the final value of the risky asset to be log-normally distributed. As mentioned in the conclusion of this paper, the second chapter of H. Moussa's ph-D thesis is devoted to this kind of generalization.

To derive all these results, we make an intensive use of duality techniques: the Fenchel conjugate ψ^*_n of the concave function ψ_n is defined as $\psi^*_n(x) := \inf_{P\in[0,1]} xP - \psi_n(P)$. For a reference on Fenchel duality see for instance in [12]. A similar definition holds for V^*_n. These techniques introduced in [6,7] for the finite action case simplify the analysis of the uninformed player's behavior and theorem 2 takes a very simple form in the dual framework:

Theorem 5 *For all $x \in I\!R$: $\psi^*_n(x) = E[\phi(x - S_n)]$, where $\phi(x) := \min\{x, 0\}$, with $S_n := \sum_{q=1}^n U_q/\sqrt{n}$, and where $U_1, \ldots, U_n$ are independent random variables uniformly distributed on the interval $[-1,1]$.*

The dual function $V^*_n(x)$ can be interpreted as the value of the so called dual game $G^*_n(x)$ where player 1 is allowed to chose initially the state of the world k without informing player 2. The transactions between the players happen then during n successive rounds as in $G_n(P)$, but at the end of the game, player 1 has to pay a penalty x to player 2 if he chosen $k = H$.

This game could also be used to model default problems in finance: if commodity R is interpreted as a zero coupon bound, issued by player 1, for a principal of 1 unit of N at maturity date T. Player 1 decides whether he will default or not by selecting the state $k = L$ or H. Defaulting for player 1 will have dramatic consequences for him in the future, after date T. If we model these consequences by a penalty $Q_n \in I\!R$ he has to pay to player 2 if he defaults, i.e. if he chooses $k = L$, then the game played is essentially $G_n^*(-Q_n)$. The penalty Q_n depends on n since it must be proportional to the damage caused to player 2 by a default of player 1, which in turn depends on the number of shares of R in player 2's portfolio at date T and therefore also depends on the number n of transaction rounds.

At each transaction round q in $G_n^*(-Q_n)$, player 2 may compute the probability P_q^n of not defaulting by player 1, conditionally on player 1's previous actions. This martingale P^n may also be viewed as a continuous time process Π^n, by setting $\Pi_t^n := P_q^n$ if $t \in [(q-1)/n, q/n[$. We then have:

Theorem 6 *If* $\lim_{n\to\infty} Q_n/\sqrt{n} = \alpha$, *then, as* n *goes to* ∞, *the process* Π^n *converges in law (in the sense of finite distributions) to the following process* Π:

$$\Pi_t := F\left(\frac{-\alpha + B_t}{\sqrt{1-t}}\right),$$

where B *is a standard Brownian motion issued at 0 and* $F(x) := \int_x^\infty f(s)ds$.

The remaining sections are devoted to the proof of all these results.

Remark 2 Since the final value of the asset R is either 0 or 1, it seems quite natural that the players will always propose prices in the interval $I := [0, 1]$. For technical reasons, we will first analyze the game $H_n(P)$ where the players have by rule to play in that interval. At the very end of the paper (section 8), we will prove that all the results developed for $H_n(P)$ also apply to $G_n(P)$.

In the next section, after a definition of the strategy spaces, we analyze the recursive structure of the game $H_n(P)$. Using backward induction, this will allow us to define a strategy of player 1 in $H_n(P)$ that will guarantee him the payoff announced in theorem 2.

We then have to prove that player 2 may guarantee the same amount. As mentioned above, the best way to study player 2's behavior is to analyze the dual game $H_n^*(x)$. The recursive structure of that game is analyzed in section 4, and we provide by backward induction a strategy of player 2 that guarantees him the payoff $E[\phi(x - U_1 - \cdots - U_n)]$ in $H_n^*(x)$.

Section 5 is concerned with the duality relationships between H_n and H_n^*. It results from these relations that the strategies introduced in sections 3 and 4 are optimal and that both games $H_n(P)$ and $H_n^*(x)$ have a value.

The asymptotic of player 1's behavior is analyzed in section 6, that of player 2, in section 7.

3 The recursive structure of $H_n(P)$

3.1 The strategy spaces in $H_n(P)$.

Let h_q denote the sequence

$$h_q := (p_{1,1}, p_{2,1}, \ldots, p_{1,q}, p_{2,q})$$

of the proposed prices up to round q. When playing round q, player 1 has observed (k, h_{q-1}). A strategy to select $p_{1,q}$ is thus a probability distribution σ_q on I depending on (k, h_{q-1}). This leads us to the following definition:

Definition 2 *A strategy σ for player 1 in $H_n(P)$ is a sequence $(\sigma_1, \ldots, \sigma_n)$ where σ_q is a transition probability from $(\mathcal{K} \times I^{2(q-1)})$ to $(I, \mathcal{B}_I)$ (i.e. a mapping from $(\mathcal{K} \times I^{2(q-1)})$ to the set $\Delta(\mathcal{B}_I)$ of probabilities on the Borel σ-algebra $\mathcal{B}_I$ on I, such that $\forall A \in \mathcal{B}_I$: $\sigma_q(\cdot)[A]$ is measurable on $(\mathcal{K} \times I^{2(q-1)})$.)*

Similarly, a strategy τ for player 2 is a sequence $\tau = (\tau_1, \ldots, \tau_n)$ where τ_q is a transition probability from $I^{2(q-1)}$ to $(I, \mathcal{B}_I)$.

The initial probability P joint to a pair (σ, τ) of strategies induces inductively a probability distribution $\Pi(P, \sigma, \tau)$ on $(\mathcal{K} \times I^{2n})$. The payoff $g_n(P, \sigma, \tau)$ of player 1 corresponding to a pair of strategies (σ, τ) in $H_n(P)$ is then

$$g_n(P, \sigma, \tau) = E_{\Pi(P,\sigma,\tau)}[\langle (1\!\!1_{k=H}, 1), y_n \rangle].$$

The maximal payoff $V_{1,n}(P)$ player 1 can guarantee in $H_n(P)$ is

$$V_{1,n}(P) := \sup_{\sigma} \inf_{\tau} g_n(P, \sigma, \tau).$$

A strategy σ^* is optimal for player 1 if $V_{1,n}(P) = \inf_\tau g_n(P, \sigma^*, \tau)$.

Similarly, the best payoff for player 2 (i.e. the minimal payoff player 2 can guarantee not to pay more) is

$$V_{2,n}(P) := \inf_{\tau} \sup_{\sigma} g_n(P, \sigma, \tau),$$

and an optimal strategy τ^* for a player 2 is such that

$$V_{2,n}(P) = \sup_{\sigma} g_n(P, \sigma, \tau^*).$$

The game $H_n(P)$ is said to have a value if $V_{1,n}(P) = V_{2,n}(P)$.

Lemma 1 *For all $P \in [0,1]$: $0 \leq V_{1,n}(P) \leq V_{2,n}(P) \leq 2nP(1-P)$. Furthermore $V_{1,n}(P)$ and $V_{2,n}(P)$ are concave continuous functions of P.*

proof: Player 2 can force player 1's payoff to be less than $V_{2,n}(P)$. Clearly, player 1 will not be able to guarantee a higher payoff. This is the content of the middle inequality in the first statement. The constant strategy $p_{1,q} = P$, $\forall q$ is not revealing and is optimal in the average game $G(P)$. It guarantees at least $u(P) = 0$ to player 1. This is the left hand inequality.

If player 2 uses the constant strategy $p_{2,q} = P$, $\forall q$, his behavior is not influenced by the past moves of player 1. So, when replying to that strategy of player 2, player 1 has not to take care of revelation problems: he may just maximize his stage payoff at each stage. He can do this, up to $n\epsilon > 0$, by playing $P + \epsilon$ at each stage in case $k = H$ and $P - \epsilon$ in case $k = L$. The stage payoff is then:

$$E[1\!\!1_{k=H}(1 - P - \epsilon) + 1\!\!1_{k=L}P] = 2P(1 - P) - P\epsilon.$$

This leads to the right hand inequality.

We proceed now to the proof of the concavity of $V_{1,n}(P)$ and $V_{2,n}(P)$. The argument is classical and can be found in [1].

Let λ, P_0, P_1 belong to $[0, 1]$, let P_λ denote $\lambda P_1 + (1 - \lambda)P_0$. The lottery P_λ that selects k in $H_n(P_\lambda)$ may be viewed as a two stage lottery: At the first stage, a number $l \in \{0, 1\}$ is selected, the probability of $l = 1$ being λ. Then a lottery P_l is performed to select $k \in \mathcal{K}$.

Obviously, the knowledge of l gives no extra advantage to player 1 in $H_n(P_\lambda)$ since he knows k. So we may suppose that player 1 is indeed informed in $H_n(P_\lambda)$ about l. Let us then compare $H_n(P_\lambda)$ with the game H' where both players are informed of l.

Since Player 2 is more informed in H' than in $H_n(P_\lambda)$, the better payoff V_2' he will be able to guarantee in H' will be less than $V_{2,n}(P_\lambda)$. In the same way, player 1 will be able to guarantee a lesser payoff V_1' in H' than in $H_n(P_\lambda)$. The concavity of $V_{2,n}$ and $V_{1,n}$ follows, since H' is just equivalent to playing $H_n(P_0)$ with probability $1 - \lambda$, and $H_n(P_1)$ with probability λ, so, $V_2' = (1 - \lambda)V_{2,n}(P_0) + \lambda V_{2,n}(P_1)$ and $V_1' = (1 - \lambda)V_{1,n}(P_0) + \lambda V_{1,n}(P_1)$.

A bounded concave function on $[0, 1]$ is continuous except maybe at 0 or 1. The first statement implies the continuity of $V_{1,n}(P)$ and $V_{2,n}(P)$ at these extreme points, and the continuity of these functions is thus proved. ■

3.2 The recursive structure of $H_n(P)$

To display the recursive structure of the game, we will have to view $H_n(P)$ as a family of games indexed by $P \in [0, 1]$.

Definition 3 *An ϵ-optimal selection for player 1 is a function $\sigma(\cdot)$ that maps P on an ϵ-optimal strategy $\sigma(P)$ in $H_n(P)$. (a strategy σ of player 1 is said ϵ-optimal in $H_n(P)$ if $\inf_\tau g_n(P, \sigma, \tau) \geq V_{1,n}(P) - \epsilon$).*

When, $\forall q$, $\forall A \in \mathcal{B}_I$, $\sigma_q(P; k, h_{q-1})[A]$ is measurable with respect to $(P; k, h_{q-1})$, we say that the ϵ-optimal selection for player 1 is measurable.

We then have the following result:

Lemma 2 *For all $\epsilon > 0$, there exists a measurable ϵ-optimal selection for player 1.*

proof: Let us fix $\epsilon > 0$. Since $V_{1,n}$ is continuous on the interval $[0,1]$, it is also uniformly continuous. In particular, there exists a $\delta \in]0, \epsilon/6n]$ such that $\forall P_1, P_2 \in [0,1]$ with $|P_1 - P_2| \leq \delta$, we have $|V_{1,n}(P_1) - V_{1,n}(P_2)| \leq \epsilon/3$. Let the $A_1, \ldots, A_M$ be a partition of $[0,1]$ in intervals of length less than δ. Let $P_1, \ldots, P_M$ a sequence of points of $[0,1]$ with, $\forall m$: $P_m \in A_m$, and let σ_m denote an $\epsilon/3$-optimal strategy in $H_n(P_m)$.

We prove now that if $P \in A_m$, then σ_m is ϵ-optimal in $H_n(P)$: Indeed, let τ be a strategy of player 2. For $k \in \mathcal{K} := \{L, H\}$, let $g(k)$ denote $E_{\Pi(P,\sigma_m,\tau)}[\langle(1\!\!1_{k=H}, 1); y_n\rangle|k]$. Thus: $g_n(P, \sigma_m, \tau) = Pg(H) + (1-P)g(L)$. Since the conditional probability of $\Pi(P, \sigma_m, \tau)$ on (h_n) given k depends on (σ_m, τ), but not on P, we also have: $g_n(P_m, \sigma_m, \tau) = P_m g(H) + (1-P_m)g(L)$. So, $g_n(P, \sigma_n, \tau) = g_n(P_m, \sigma_m, \tau) + (P - P_m)(g(H) - g(L))$. Since the proposed prices are in the interval I, it is easy to prove that $|g(k)| \leq n$. Therefore, $(P - P_m)(g(H) - g(L)) \geq -\epsilon/3$, since $|P - P_m| \leq \delta \leq \epsilon/6n$. Furthermore, σ_m is $\epsilon/3$-optimal in $H_n(P_m)$. So, we have $g_n(P_m, \sigma_m, \tau) \geq V_{1,n}(P_m) - \epsilon/3 \geq V_{1,n}(P) - 2\epsilon/3$, since $|P - P_m| \leq \delta$. So, all together: $g_n(P, \sigma_n, \tau) \geq V_{1,n}(P) - \epsilon$. This inequality holds for all strategy τ: as announced σ_m is ϵ-optimal in $H_n(P)$.

We define now $\sigma(P) := \sigma_m$ if $P \in A_m$. This is clearly a measurable ϵ-optimal selection for player 1. ∎

We are now ready to analyze the recursive structure of $H_n(P)$: after the first stage of $H_{n+1}(P)$, the remaining part of the game is essentially a game of length n. Such an observation leads, in Aumann and Maschler's paper, to a recursive formula of the value V_n of the n-stages game.

At this level of our analysis however we have no argument to prove the existence of V_n and we are only able to provide a lower bound for $V_{1,n+1}(P)$. This is the content of the forthcoming lemma 3.

Let us now consider a strategy σ of player 1 in $H_{n+1}(P)$. The first stage strategy σ_1 is a conditional probability on $p_{1,1}$ given k. Joint to P it induces a probability distribution $\pi(P, \sigma_1)$ on $(k, p_{1,1})$ such that $\pi(P, \sigma_1)[k = H] = P$.

The remaining part $(\sigma_2, \ldots, \sigma_{n+1})$ of player 1's strategy σ in $H_{n+1}(P)$ is in fact a family of strategies $\tilde{\sigma}$ in H_n depending on the first stage actions $(p_{1,1}, p_{2,1})$. In the same way, a strategy τ of player 2 in $H_{n+1}(P)$ can be viewed as a pair $(\tau_1, \tilde{\tau})$, where τ_1 is the first stage strategy, and $\tilde{\tau}$ is family of a strategies in H_n depending on $(p_{1,1}, p_{2,1})$.

Let $P(p_{1,1})$ denote $\pi(P, \sigma_1)[k = H|p_{1,1}]$. Since $p_{2,1}$ is independent of k, we also have

$$\Pi(P, \sigma, \tau)[k = H|p_{1,1}, p_{2,1}] = P(p_{1,1}).$$

Then, conditionally on $(p_{1,1}, p_{2,1})$, the distribution of

$$(k, p_{1,2}, p_{2,2}, \ldots, p_{1,n+1}, p_{2,n+1})$$

is $\Pi(P(p_{1,1}), \tilde{\sigma}(p_{1,1}, p_{2,1}), \tilde{\tau}(p_{1,1}, p_{2,1}))$. Therefore $g_{n+1}(P, \sigma, \tau) =$

$$g_1(P, \sigma_1, \tau_1) + E_{\Pi(P,\sigma_1,\tau_1)}[g_n(P(p_{1,1}), \tilde{\sigma}(p_{1,1}, p_{2,1}), \tilde{\tau}(p_{1,1}, p_{2,1})].$$

Let $\sigma()$ be an ϵ-optimal selection in H_n. If, for $\tilde{\sigma}(p_{1,1}, p_{2,1})$, player 1 chooses the strategy $\sigma(P(p_{1,1}))$, then

$$g_n(P(p_{1,1}), \tilde{\sigma}(p_{1,1}, p_{2,1}), \tilde{\tau}(p_{1,1}, p_{2,1})) \geq V_{1,n}(P(p_{1,1})) - \epsilon$$

and thus $g_{n+1}(P, \sigma, \tau) \geq g_1(P, \sigma_1, \tau_1) + E_{\pi(P,\sigma_1)}[V_{1,n}(P(p_{1,1})) - \epsilon]$.

The above described strategy of player 1 guarantees then, up to ϵ, a payoff of

$$\min_{p_{2,1}} g_1(P, \sigma_1, p_{2,1}) + E_{\pi(P,\sigma_1)}[V_{1,n}(P(p_{1,1}))],$$

and thus the following lemma is proved:

Lemma 3 *For all $P \in [0, 1]$:*

$$V_{1,n+1}(P) \geq \max_{\sigma_1} \min_{p_{2,1}} g_1(P, \sigma_1, p_{2,1}) + E_{\pi(P,\sigma_1)}[V_{1,n}(P(p_{1,1}))].$$

Notice that, as compared with Aumann-Maschler recursive formula, we only get here an inequality. The reversed inequality will be proved later in section 5, after analyzing the recursive structure of the dual game in section 4, which will provide upper bounds on $V_{2,n}$.

In the next subsection, using a new parameterization of player 1's strategy space, we will transform the inequality of lemma 3 in a more tractable one.

3.3 Another parameterization of player 1's strategy space.

The first stage strategy space of player 1 may be identified with the space of probability distributions π on $(k, p_{1,1})$ satisfying

$$\pi[k = H] = P. \tag{3}$$

In turn, such a probability π may be represented as a pair of functions (f, Q) $[0, 1] \to [0, 1]$ satisfying:

$$\begin{array}{l} a)\ f \text{ is increasing.} \\ b)\ \int_0^1 Q(u)du = P \\ c)\ \forall x, y \in [0, 1] : f(x) = f(y) \Rightarrow Q(x) = Q(y). \end{array} \tag{4}$$

Given such a pair (f, Q), player 1 generates the probability π as follows: he first selects a random number u uniformly distributed on $[0, 1]$, he plays then $p_{1,1} := f(u)$ and $k \in \mathcal{K}$ is then chosen at random with a lottery such that $Proba[k = H] = Q(u)$.

Any probability π satisfying (3) may be generated in this way. Indeed, if f is the left inverse of the distribution function F of the marginal of π on $p_{1,1}$, then $f(u)$ will have the same law as $p_{1,1}$. f is clearly increasing.

Next, let $R(p_{1,1})$ denote $R(p_{1,1}) := \pi[k = H|p_{1,1}]$, and let $Q(u)$ be defined as $Q(u) := R(f(u))$. This pair (f, Q) generates π, and Q satisfy clearly to (4)-c). Finally, (3) implies (4)-b).

So, we may now view player 1' first stage strategy space as the set of functions (f, Q) satisfying (4).

The question we answer now is how to retrieve the first stage strategy $\sigma_1 = (\sigma_1(H), \sigma_1(L))$ from its representation (f, Q). If $A \in \mathcal{B}_I$, $\sigma_1(H)[A]$ is just equal to

$$\pi[p_{1,1} \in A|k = H] = \frac{\pi[p_{1,1} \in A \cap k = H]}{\pi[k = H]} = \int_0^1 1\!\!1_{f(u)\in A} Q(u)du/P.$$

Therefore, if player 1 is told H, he picks a random number u in $[0, 1]$ according to a probability density $Q(u)/P$, and he plays $p_{1,1} = f(u)$. Similarly, if he is told L, he uses the probability density $(1-Q(u))/(1-P)$ to select the random number u and he then plays $p_{1,1} = f(u)$.

We next proceed to the transformation of the recursive formula of lemma 3: If player 1 plays the strategy σ_1 represented by (f, Q), he gets

$$g_1(P, \sigma_1, p_{2,1}) = \int_0^1 \left\{ 1\!\!1_{f(u)>p_{2,1}}(Q(u) - f(u)) + 1\!\!1_{f(u)<p_{2,1}}(p_{2,1} - Q(u)) \right\} du.$$

On the other hand, $P(p_{1,1}) = \pi[k = H|f(u)] = Q(u)$, and thus

$$E[V_{1,n}(P(p_{1,1}))] = \int_0^1 V_{1,n}(Q(u))du.$$

All together, we have proved the following recursive formula for $V_{1,n}$:

Theorem 7 $V_{1,n+1}(P) \geq \sup_{(f,Q)} \inf_{p_{2,1}} F((f, Q), p_{2,1})$ *with*

$$\begin{aligned} F((f, Q), p_{2,1}) := & \int_0^1 1\!\!1_{f(u)>p_{2,1}}(Q(u) - f(u))du \\ & + \int_0^1 1\!\!1_{f(u)<p_{2,1}}(p_{2,1} - Q(u))du \\ & + \int_0^1 V_{1,n}(Q(u))du, \end{aligned}$$

where (f, Q) *satisfy to (4).*

In the next section we seek for an optimal candidate (f^*, Q^*) to the left hand side of the recurrence formula.

3.4 An optimal candidate (f^*, Q^*).

We want to find an optimal strategy (f^*, Q^*) in the zero sum game where player 1 chooses a pair (f, Q), player 2 a probability distribution τ_1 on $p_{2,1}$ and player 1's payoff is $F((f, Q), p_{2,1})$. Let us suppose for a while that this game has a value v and both player have optimal strategies. The optimal strategy τ_1^* of player 2 must then be a best reply to player 1's optimal (f^*, Q^*).

As a consequence $F((f^*,Q^*),p_{2,1})$ must be equal to v at all point $p_{2,1}$ of the support S of τ_1^*. A strategy pair (f,Q) such that $F((f,Q),p_{2,1})$ assume the same value for all $p_{2,1}$ in a set A will be referred to as A-equalizing. So, the optimal pair (f^*,Q^*) should be S-equalizing.

At this level of our analysis, we have no information on S. We will conjecture that $f([0,1])$ is included in S (In fact we will prove later that $S=f([0,1])$).

Based on that conjecture, we will seek for the optimal (f^*,Q^*) among the pair (f,Q) that are $f([0,1])$-equalizing strategies. Since any point $p_{2,1}$ in $f([0,1])$ can be written as $p_{2,1}=f(\alpha)$ for some $\alpha\in[0,1]$, a $f([0,1])$-equalizing pair (f,Q) is such that $F((f,Q),f(\alpha))$ does not depend on $\alpha\in[0,1]$.

If f is strictly increasing at α, then

$$\begin{aligned}F((f,Q),f(\alpha)) = &\textstyle\int_\alpha^1 (Q(u)-f(u))du + \int_0^\alpha (f(\alpha)-Q(u))du\\ &+\textstyle\int_0^1 V_{1,n}(Q(u))du.\end{aligned}$$

At any point α where f is differentiable, we must then have:

$$0=\frac{d}{d\alpha}F((f,Q),f(\alpha)).$$

So,

$$0=2(f(\alpha)-Q(\alpha))+\alpha f'(\alpha). \tag{5}$$

Multiplying this by α, we get after integration:

$$\alpha^2 f(\alpha)=\int_0^\alpha 2sQ(s)ds. \tag{6}$$

So, for all function Q there exists an f given by formula (6), such that the pair (f,Q) is $f([0,1])$-equalizing. Let now $G(Q)$ denote the corresponding quantity $F((f,Q),f(\alpha))$ that does not depend on α. So:

$$\begin{aligned}G(Q) &= F((f,Q),f(1))\\ &= f(1)-\textstyle\int_0^1 Q(u)du+\int_0^1 V_{1,n}(Q(u))du\\ &= \textstyle\int_0^1 (2u-1)Q(u)du+\int_0^1 V_{1,n}(Q(u))du.\end{aligned}$$

We suspect that the pair (f,Q) will guarantee $G(Q)$ to player 1 in our game. It is then natural to conjecture that Q^* will maximize $G(Q)$ among the functions Q satisfying $\int_0^1 Q(u)du=P$.

This leads us to consider the Lagrange's functional

$$L(Q,\lambda):=G(Q)+\lambda(P-\int_0^1 Q(u)du),$$

and to find the pair (λ,Q) maximizing $L(Q,\lambda)$. Using variational methods, we find that, $\forall u\in[0,1]$:

$$\lambda+1-2u=V'_{1,n}(Q(u)). \tag{7}$$

Let us introduce here the Fenchel conjugate $V^*_{1,n}$ of $V_{1,n}$:

$$V^*_{1,n}(x) := \inf_{P\in[0,1]} xP - V_{1,n}(P).$$

Fenchel Lemma (see for instance in [12]) indicates that $P \in \partial V^*_{1,n}(x)$ iff $x \in \partial V_{1,n}(P)$ where ∂ stands for the upper gradient of a concave function:

$$\partial f(x) := \{\beta | \forall y : f(y) \leq f(x) + \beta(y - x)\}.$$

So, if we suppose that both $V_{1,n}$ and $V^*_{1,n}$ are differentiable, equation (7) reads

$$Q(u) = V^{*\prime}_{1,n}(\lambda + 1 - 2u). \tag{8}$$

Since a concave function on $I\!R$ is differentiable everywhere except maybe on a countable subset of $I\!R$, this equation defines Q^* up to a countable set of values of u. Since, in the following, Q^* always appears inside an integral, the definition of Q^* on this countable set is not essential. However, it is convenient to assume that Q^* is the unique left-continuous selection of the correspondence $u \to \partial V^*_{1,n}(\lambda + 1 - 2u)$. In particular Q^* is an increasing left-continuous function.

The number λ must be chosen such that

$$P = \int_0^1 Q(u)du = \int_0^1 V^{*\prime}_{1,n}(\lambda + 1 - 2u)du = K'(\lambda), \tag{9}$$

where K is the concave continuously differentiable function that maps $x \in I\!R$ to $K(x) := \int_0^1 V^*_{1,n}(x + 1 - 2u)\,du$. Since $V_{1,n}$ is bounded on $[0,1]$, it follows that $V^{*\prime}_{1,n}$ decreases on $I\!R$ from 1 to 0. As a consequence, such a number λ always exists.

So, up to now, we derived heuristic formulas (9), (8) and (6) to define a pair of functions (f, Q). We now have to prove that these functions fulfill the requirements of (4) for the pair (f, Q) to be a strategy: as mentioned above the function Q given by (8) is increasing and valued in $[0,1]$. The number λ has been chosen appropriately for relation (4)-b) to hold.

It follows then from (6) that

$$0 \leq f(\alpha) = \alpha^{-2}\int_0^\alpha 2sQ(s)ds \leq \alpha^{-2}\int_0^\alpha 2sQ(\alpha)ds = Q(\alpha) \leq 1.$$

We next prove that f is increasing: If $\beta > \alpha$, then, since Q is increasing and $f(\alpha) \leq Q(\alpha)$, we get

$$\begin{aligned}
f(\beta) &:= \beta^{-2}\textstyle\int_0^\beta 2uQ(u)du \\
&= \beta^{-2}\left(\textstyle\int_0^\alpha 2uQ(u)du + \int_\alpha^\beta 2uQ(u)du\right) \\
&\geq \beta^{-2}\left(\alpha^2 f(\alpha) + \textstyle\int_\alpha^\beta 2uQ(\alpha)du\right) \\
&\geq \beta^{-2}\left(\alpha^2 f(\alpha) + (\beta^2 - \alpha^2)f(\alpha)\right) \\
&= f(\alpha)
\end{aligned}$$

So (4)-a) holds and we next prove (4)-c): if $f(\alpha) = f(\beta)$ with $\alpha < \beta$, then the inequalities used in the previous reasoning are in fact equalities: for almost every point $u \in [\alpha, \beta]$: $Q(u) = Q(\alpha)$. Since Q is left-continuous, this implies in particular $Q(\beta) = Q(\alpha)$, which is just (4)-c). Let us observe here that the inequality $Q(\alpha) \geq f(\alpha)$ is an equality, so that, $\forall u \in [\alpha, \beta]$:

$$Q(u) = Q(\alpha) = f(\alpha) = f(u) \tag{10}$$

We may now compute the amount guaranteed to player 1 by this pair (f, Q):

case 1: If player 2 plays $p_{2,1}$ in the range $[f(0), f(1)]$, then $p_{2,1} = f(\alpha)$ for some $\alpha \in [0, 1]$. So, if we define $\alpha^- := \inf\{s | f(s) = f(\alpha)\}$ and $\alpha^+ := \sup\{s | f(s) = f(\alpha)\}$, we get

$$\begin{aligned} F((f,Q), p_{2,1}) = &\int_{\alpha^+}^1 (Q(u) - f(u))du + \int_0^{\alpha^-} (f(\alpha) - Q(u))du \\ &+ \int_0^1 V_{1,n}(Q(u))du. \end{aligned}$$

If $\alpha^+ = \alpha^-$, this leads to

$$\begin{aligned} F((f,Q), p_{2,1}) = &\int_{\alpha}^1 (Q(u) - f(u))du + \int_0^{\alpha} (f(\alpha) - Q(u))du \\ &+ \int_0^1 V_{1,n}(Q(u))du, \end{aligned} \tag{11}$$

and we are led to the same result in case $\alpha^- \neq \alpha^+$. Indeed, due to the definition of α^+ and α^-, f is constant on $[\alpha^-, \alpha^+]$, and by relation (10), we find $f(s) - Q(s) = 0$ for $s \in [\alpha^-, \alpha^+]$, and therefore $\int_{\alpha^+}^1 (Q(u) - f(u))du = \int_{\alpha}^1 (Q(u) - f(u))du$. Similarly, $\int_0^{\alpha^-} (f(\alpha) - Q(u))du = \int_0^{\alpha} (f(\alpha) - Q(u))du$, and thus, as announced, equation (11) is also true in case $\alpha^- \neq \alpha^+$.

The integral $\int_{\alpha}^1 f(u)du$ appearing in (11) can be computed with (6):

$$\begin{aligned} \int_{\alpha}^1 f(u)du &:= \int_{\alpha}^1 u^{-2} \int_0^u 2sQ(s)dsdu \\ &= \int_0^{\alpha} \int_{\alpha}^1 u^{-2} 2sQ(s)duds + \int_{\alpha}^1 \int_s^1 u^{-2} 2sQ(s)duds \\ &= \int_0^{\alpha} (\alpha^{-1} - 1)2sQ(s)ds + \int_{\alpha}^1 (s^{-1} - 1)2sQ(s)duds \end{aligned}$$

Similarly: $\int_0^{\alpha} f(\alpha)du = \alpha f(\alpha) = \alpha^{-1} \int_0^{\alpha} 2sQ(s)$. So, after rearranging the terms in (11), we find:

$$F((f,Q), p_{2,1}) = \int_0^1 (2u - 1)Q(u)du + \int_0^1 V_{1,n}(Q(u))du.$$

So, since by (8), $Q(u) \in \partial V_{1,n}^*(\lambda + 1 - 2u)$, we get:

$$\begin{aligned} F((f,Q), p_{2,1}) &= \lambda \int_0^1 Q(u)du + \int_0^1 \{(2u - 1 - \lambda)Q(u) + V_{1,n}(Q(u))\} du \\ &= \lambda P - \int_0^1 (\lambda + 1 - 2u)Q(u) - V_{1,n}(Q(u))du \\ &= \lambda P - \int_0^1 V_{1,n}^* (\lambda + 1 - 2u) du \\ &= \lambda K'(\lambda) - K(\lambda)) \\ &= K^*(K'(\lambda)) \\ &= K^*(P), \end{aligned}$$

where $K^*(q) := \inf_{x\in I\!R} xq - K(x)$.
case 2: If player 2 plays $p_{2,1} > f(1)$ against (f,Q), then

$$F((f,Q),p_{2,1}) = \int_0^1 (p_{2,1} - Q(u)) + V_{1,n}(Q(u))du \tag{12}$$

Next, due to relation (11) with $p_{2,1} = f(1)$, we have:

$$F((f,Q),f(1)) = \int_0^1 (f(1) - Q(u))du + \int_0^1 V_{1,n}(Q(u))du.$$

Thus:

$$\begin{aligned} K^*(P) &= F((f,Q),f(1)) \\ &= \textstyle\int_0^1 (f(1) - Q(u))du + \int_0^1 V_{1,n}(Q(u))du \\ &\leq \textstyle\int_0^1 (p_{2,1} - Q(u))du + \int_0^1 V_{1,n}(Q(u))du \\ &= F((f,Q),p_{2,1}). \end{aligned}$$

case 3: Finally, if player 2 plays $p_{2,1} < f(0)$, he gets

$$F((f,Q),p_{2,1}) = \int_0^1 (Q(u) - f(u)) + V_{1,n}(Q(u))du.$$

By playing $p_{2,1} = f(0)$, he would get, according to (11):

$$F((f,Q),f(0)) = \int_0^1 (Q(u) - f(u))du + \int_0^1 V_{1,n}(Q(u))du$$

Therefore $F((f,Q),p_{2,1}) = F((f,Q),f(0)) = K^*(P)$.

The next theorem follows then from the above discussion:

Theorem 8 *The strategy (f,Q) defined by equations (9), (8) and (6) guarantees the payoff $K^*(P)$ to player 1 in the recurrence formula of theorem 7. Therefore $V_{1,n+1}(P) \geq K^*(P)$. This inequality turns out to be equivalent to the dual one: $V^*_{1,n+1}(x) \leq K(x) := \int_0^1 V^*_{1,n}(x + 1 - 2u)\,du$*

In section 5, we will prove that the inequalities in the last theorem may be replaced by equalities, and the strategy (f,Q) will then be optimal.

4 The recursive structure of the dual game

Fenchel duality appeared quite naturally in the last section to express properly the recurrence formula. In this section, we analyze the dual game $H^*_n(x)$ which is the natural framework to analyze this Fenchel duality. This will allow us to analyze indirectly the recursive structure of $V_{2,n}(P)$.

4.1 The dual game $H_n^*(x)$.

In this game, player 1 chooses privately the state of nature k, then the game follows as H_n. The final payoff is the same as in H_n except that player 1 has to pay to player 2 an additional penalty x if he chose $k = H$. The strategies for player 2 are thus the same as in H_n. The lottery P that selects the state of nature in $H_n(P)$ is controlled by player 1 in $H_n^*(x)$.

So, a strategy for player 1 in $H_n^*(x)$ is just a pair (P, σ), where $P \in [0, 1]$, and σ is a strategy in H_n.

The final payoff of player 2 in $H_n^*(x)$ is then $g_n^*(x, (P, \sigma), \tau) := x \cdot P - g_n(P, \sigma, \tau)$. Player 2 maximizes this payoff.

The optimal amounts that player 1 and 2 can guaranty in $H_n^*(x)$ are respectively

$$W_{1,n}(x) := \inf_{P,\sigma} \sup_{\tau} g_n^*(x, (P, \sigma), \tau) \text{ and } W_{2,n}(x) := \sup_{\tau} \inf_{P,\sigma} g_n^*(x, (P, \sigma), \tau).$$

A strategy τ is said ϵ-optimal in $H_n^*(x)$ for player 2 if $\forall (P, \sigma)$:

$$g_n^*(x, (P, \sigma), \tau) \geq W_{2,n}(x) - \epsilon.$$

Lemma 4 *$W_{2,n}$ is a concave function of x. Furthermore:*

$$\forall x \in \mathbb{R} : \partial W_{2,n}(x) \subset [0, 1]. \tag{13}$$

In particular, $W_{2,n}$ is Lipschitzian.

proof: Let x_0, x_1 be to real numbers, let $\lambda \in [0, 1]$, and let x_λ denote $x_\lambda := (1 - \lambda)x_0 + \lambda x_1$. For $\epsilon > 0$ and $i \in \{0, 1\}$, let also τ^i be an ϵ-optimal strategy in $H_n^*(x_i)$. Suppose that player 2 uses the following strategy τ^λ in $H_n^*(x_\lambda)$: at the beginning of the game, he picks at random a number $i \in \{0, 1\}$, the probability of getting 0 or 1 being respectively $(1 - \lambda)$ and λ. He then plays the corresponding strategy τ^i.

If player 2 uses τ^λ against a strategy (P, σ) of player 1, then

$$\Pi(P, \sigma, \tau^\lambda) = (1 - \lambda)\Pi(P, \sigma, \tau^0) + \lambda \Pi(P, \sigma, \tau^1). \tag{14}$$

So: $g_n(P, \sigma, \tau^\lambda) = (1 - \lambda)g_n(P, \sigma, \tau^0) + \lambda g_n(P, \sigma, \tau^1)$ and thus

$$\begin{aligned} g_n^*(x_\lambda, (P, \sigma), \tau^\lambda) &= (1 - \lambda)g_n^*(x_0, (P, \sigma), \tau^0) + \lambda g_n^*(x_1, (P, \sigma), \tau^1) \\ &\geq (1 - \lambda)(W_{2,n}(x_0) - \epsilon) + \lambda(W_{2,n}(x_1) - \epsilon). \end{aligned}$$

This holds for all (P, σ) and thus τ^λ guarantees in $H_n^*(x_\lambda)$ the amount

$$(1 - \lambda)W_{2,n}(x_0) + \lambda W_{2,n}(x_1)$$

up to ϵ. As a consequence, $W_{2,n}(x_\lambda) \geq (1-\lambda)W_{2,n}(x_0) + \lambda W_{2,n}(x_1)$, as announced[1].

Since the stage payoff in H_n is less than 1, we get for all P, σ, τ:

$$g_n(P, \sigma, \tau) \leq n,$$

and thus $g_n^*(x, (P, \sigma), \tau) \geq xP - n$. Therefore, $W_{2,n} \geq h$, where $h(x) := x - n$ if $x < 0$ and $h(x) = -n$ if $x \geq 0$. Since both $W_{2,n}$ and h are concave, the last inequality entails that $\partial W_{2,n}(x) \subset \cup_y \partial h(y) = [0, 1]$.

Since, for all x, x', $(W_{2,n}(x') - W_{2,n}(x))/(x' - x)$ belongs to $\partial W_{2,n}(y)$ for some point $y \in [x, x']$, we conclude that

$$|W_{2,n}(x') - W_{2,n}(x)| \leq |x' - x|,$$

and $W_{2,n}$ is as announced Lipschitzian with constant 1. ■

4.2 The recursive structure of $H_n^*(x)$.

Definition 4 *An ϵ-optimal selection for player 2 is a mapping $\tau(.)$ that maps $x \in I\!\!R$ to an ϵ-optimal strategy $\tau(x)$ in $H_n^*(x)$. If, $\forall q$, $\forall A \in \mathcal{B}_I$: $\tau_q(x; h_{q-1})[A]$ is furthermore measurable with respect to $(x; h_{q-1})$ the selection is said measurable.*

Lemma 5 *For all $\epsilon > 0$, there exists a measurable ϵ-optimal selection for player 2.*

proof: Since $W_{2,n}$ is uniformly continuous, as was $V_{1,n}$, the proof of this result is very similar to the proof of lemma 2, and will not be detailed here. ■

A strategy τ in $H_{n+1}^*(x)$ can be viewed as a pair $\tau_1, \tilde{\tau}$ where τ_1 is the first stage strategy, and $\tilde{\tau}$ is a strategy in H_n^* depending on $p_{1,1}, p_{2,1}$.

Similarly for a strategy (P, σ) of player 1: the pair (P, σ_1) induces a probability distribution π_{P,σ_1} on $(k, p_{1,1})$ which can be viewed as a pair $(s, \tilde{P})$, where s is the marginal distribution of $p_{1,1}$ under π_{P,σ_1}, and $\tilde{P}(p_{1,1})$ is $\pi_{P,\sigma_1}[k = H|p_{1,1}]$. Let $\tilde{\sigma}$ denote the remaining part $(\sigma_2, \ldots, \sigma_{n+1})$ of σ. $\tilde{\sigma}$ is in fact a strategy in H_n depending on $(p_{1,1}, p_{2,1})$. Therefore $(\tilde{P}, \tilde{\sigma})$ is a strategy in H_n^* depending on $(p_{1,1}, p_{2,1})$ (with $\tilde{P}$ depending only on $p_{1,1}$).

[1] The problem with the above argument is that, stricto senso, τ^λ is not a strategy of player 2 as defined in the beginning of section 3. Indeed, τ_q^λ depends not only on h_{q-1}, but also on the result $i \in \{0, 1\}$ of the initial lottery player 2 performed. This kind of strategy is referred to as a "general" strategy in chapter 2 of [9] and theorem 1.6 in that chapter claims the equivalence between general strategies and behavior strategies, since the game under concern is of perfect recall. Therefore, there exists a behavior strategy $\hat{\tau}$ that will induce the same distribution on plays as τ^λ, whatever be the strategy (P, σ) of player 1. Relation (14) written with $\hat{\tau}$ instead of τ^λ will thus also hold.

With these notations, we get:

$$g_{n+1}^*(x,(P,\sigma),\tau) = E_s[P(p_{1,1})x] - E_s[g_1(P(p_{1,1}),p_{1,1},\tau_1)] \\ -E_{s,\tau_1}[g_n(P(p_{1,1}),\tilde{\sigma}(p_{1,1},p_{2,1}),\tilde{\tau}(p_{1,1},p_{2,1}))].$$

Let us suppose now that player 2's strategy is such that $\tilde{\tau}$ is independent of $p_{2,1}$, that is $\tilde{\tau}$ is just a function of $p_{1,1}$. If player 1 is replying against such a strategy, he will have to minimize for all $p_{1,1},p_{2,1}$ the quantity $-g_n\left(P(p_{1,1}),\tilde{\sigma}(p_{1,1},p_{2,1}),\tilde{\tau}(p_{1,1})\right)$. He may clearly do so with a strategy $\tilde{\sigma}$ that does not depend on $p_{2,1}$. So, if both $\tilde{\tau}$ and $\tilde{\sigma}$ are functions of $p_{1,1}$ we get:

$$g_{n+1}^*(x,(P,\sigma),\tau) = E_s[P(p_{1,1})x] - E_s[g_1(P(p_{1,1}),p_{1,1},\tau_1)] \\ -E_s[g_n(P(p_{1,1}),\tilde{\sigma}(p_{1,1}),\tilde{\tau}(p_{1,1}))].$$

Now, $g_1(P(p_{1,1}),p_{1,1},\tau_1) = P(p_{1,1})r(p_{1,1},\tau_1) + g_1(L,p_{1,1},\tau_1)$, where

$$r(p_{1,1},\tau_1) := g_1(H,p_{1,1},\tau_1) - g_1(L,p_{1,1},\tau_1). \tag{15}$$

Thus,

$$\begin{aligned} g_{n+1}^*(x,(P,\sigma),\tau) =& E_s[P(p_{1,1})(x - r(p_{1,1},\tau_1)) - g_n(P(p_{1,1}),\tilde{\sigma}(p_{1,1}),\tilde{\tau}(p_{1,1})) \\ & -g_1(L,p_{1,1},\tau_1)] \\ =& E_s[g_n^*(x - r(p_{1,1},\tau_1),(P(p_{1,1}),\tilde{\sigma}(p_{1,1})),\tilde{\tau}(p_{1,1})) \\ & -g_1(L,p_{1,1},\tau_1)]. \end{aligned}$$

Therefore, if $\tau^*(x)$ is an ϵ-optimal selection for player 2 in H_n^*, and player 2 plays $\tilde{\tau}(p_{1,1}) := \tau^*(x - r(p_{1,1},\tau_1))$, he gets at least, up to ϵ:

$$E_s[W_{2,n}(x - r(p_{1,1},\tau_1)) - g_1(L,p_{1,1},\tau_1)].$$

Since this quantity is linear in s, the minimum over s is just a minimum over $p_{1,1}$, and we get thus the lemma:

Lemma 6

$$W_{2,n+1}(x) \geq \sup_{\tau_1} \inf_{p_{1,1}} W_{2,n}(x - r(p_{1,1},\tau_1)) - g_1(L,p_{1,1},\tau_1),$$

where r is defined in (15).

4.3 Another parameterization of player 2's strategy space.

The first stage strategy τ_1 of player 2 is a probability distribution for $p_{2,1} \in [0,1]$. To pick $p_{2,1}$ at random, player 2 may proceed as follows: given a increasing function $h : [0,1] \to [0,1]$, he selects a random number u uniformly distributed on $[0,1]$ and he plays $p_{2,1} = h(u)$. Any distribution τ_1 can be generated in this way.

If h generates τ_1, then

$$g_1(L, p_{1,1}, \tau_1) = \int_0^1 1\!\!1_{h(u)<p_{1,1}}(-p_{1,1}) + 1\!\!1_{h(u)>p_{1,1}} h(u)du$$

and $r(p_{1,1}, \tau_1) = \int_0^1 1\!\!1_{h(u)<p_{1,1}} - 1\!\!1_{h(u)>p_{1,1}} du$. All together, we have proved the following recursive formula for $W_{2,n}$:

Theorem 9 $W_{2,n+1}(x) \geq \sup_h \inf_{p_{1,1}} G(p_{1,1}, h)$, *where* $h : [0,1] \to [0,1]$ *is an increasing function and where*

$$\begin{aligned} G(p_{1,1}, h) := W_{2,n} &\left(x - \textstyle\int_0^1 1\!\!1_{h(u)<p_{1,1}} - 1\!\!1_{h(u)>p_{1,1}} du\right) \\ &- \textstyle\int_0^1 1\!\!1_{h(u)<p_{1,1}}(-p_{1,1}) + 1\!\!1_{h(u)>p_{1,1}} h(u)du. \end{aligned}$$

4.4 An optimal strategy for player 2

To find an optimal h in this formula, we will proceed as in section 3.4 and conjecture that h should be $h[0,1]$-equalizing: for all α in $[0,1]$, $G(h(\alpha), h)$ is independent of α. If h is continuous and strictly increasing at α, then:

$$G(h(\alpha), h) = W_{2,n}\left(x - 2\alpha + 1)\right) - \left(\int_\alpha^1 h(u)du - \alpha h(\alpha)\right). \tag{16}$$

So we are looking for an h such that:

$$W_{2,n}\left(x - 2\alpha + 1\right) - \left(\int_\alpha^1 h(u)du - \alpha h(\alpha)\right) = C, \tag{17}$$

where C is a constant. If we set $H(\alpha) := -\int_\alpha^1 h(u)du$, this equation reads:

$$W_{2,n}\left(x - 2\alpha + 1\right) + \frac{d}{d\alpha}(\alpha H(\alpha)) = C.$$

Since $H(1) = 0$, we get by integrating this between α and 1:

$$\alpha H(\alpha) = C(\alpha - 1) + \int_\alpha^1 W_{2,n}(x - 2u + 1))du. \tag{18}$$

Since h is bounded, so is H and the right hand side of the last equation evaluated at $\alpha = 0$ must be 0. Therefore:

$$C = \int_0^1 W_{2,n}\left(x - 2u - 1\right) du. \tag{19}$$

Equation (17) multiplied by α reads:

$$\alpha^2 h(\alpha) = C\alpha - \alpha W_{2,n}\,(x - 2\alpha + 1) - \alpha H(\alpha).$$

Replacing $\alpha H(\alpha)$ with its value given in (18), we get:

$$\alpha^2 h(\alpha) = C - \int_\alpha^1 W_{2,n}(x - 2u + 1)du - \alpha W_{2,n}\,(x - 2\alpha + 1)\,.$$

Replacing C by its value given in (19), we obtain:

$$\begin{aligned}\alpha^2 h(\alpha) &= \textstyle\int_0^\alpha W_{2,n}\,(x - 2u + 1) - W_{2,n}\,(x - 2\alpha + 1)\,du \\ &= \textstyle\int_0^\alpha \left(\int_u^\alpha W'_{2,n}\,(x - 2s + 1)\,2ds\right) du \\ &= \textstyle\int_0^\alpha \left(\int_0^s W'_{2,n}\,(x - 2s + 1)\,2du\right) ds \\ &= \textstyle\int_0^\alpha W'_{2,n}\,(x - 2s + 1)\,2sds.\end{aligned}$$

So finally:

$$h(\alpha) = \alpha^{-2} \int_0^\alpha 2 \cdot s \cdot W'_{2,n}\,(x - 2s + 1)\,ds. \tag{20}$$

Due to the concavity of $W_{2,n}$ and relation (13), $W'_{2,n}$ is decreasing and $[0,1]$-valued. It results then from the last formula that h is $[0,1]$-valued continuous and increasing. Indeed, if we set $\tilde{Q}(s) := W'_{2,n}\,(x - 2s + 1)$, then $h(\alpha)$ is the barycenter of the values $\tilde{Q}(s)$ when s is randomly distributed on $[0,\alpha]$ with density $2s\alpha^{-2}$. Since $\tilde{Q}$ is increasing, we infer therefore that $h(\alpha) \le \tilde{Q}(\alpha)$. The equality holds here only if $\tilde{Q}$ is constant on $[0,\alpha]$. Differentiating (20), we get $\alpha^2 h'(\alpha) = 2\alpha(\tilde{Q}(\alpha) - h(\alpha))$. Thus h is increasing as announced and is therefore a strategy for player 2 in the sense of the last subsection. Furthermore, $h'(\alpha) = 0$ can only happens in case $h(s) = \tilde{Q}(s) = \tilde{Q(0)}$, for all s in $[0,\alpha]$.

Lemma 7 *The strategy h guarantees C in the recurrence formula of theorem 9.*

proof: We just have to prove that $\forall p_{1,1} \in [0,1] : G(p_{1,1}, h) \ge C$. Several cases are to be considered.
case 1: If $p_{1,1} < h(0)$, then

$$G(p_{1,1}, h) = W_{2,n}(x + 1) - \int_0^1 h(u)du.$$

Since h has been designed to fulfill formula (17) for all $\alpha \in [0,1]$, we get by identification with the left hand side of this equation for $\alpha = 0$: $G(p_{1,1}, h) = C$.
case 2: If $p_{1,1} = h(0)$, then

$$G(p_{1,1}, h) = W_{2,n}(x + 1 - \alpha^+) - \int_{\alpha^+}^1 h(u)du,$$

where $\alpha^+ := \inf\{s : h(s) > h(\alpha)\}$.

case 2-1: If $\alpha^+ = 0$, then again by identification with formula (17), we find $G(p_{1,1}, h) = C$.
case 2-2: If $\alpha^+ > 0$, then h is constant on the interval $[0, \alpha^+]$, and, as observed above, $h(s) = \tilde{Q}(s) = \tilde{Q}(0)$, for all s in $[0, \alpha^+]$. Due to the concavity of $W_{2,n}$, we get:

$$\begin{aligned} W_{2,n}(x + (1 - \alpha^+)) &\geq W_{2,n}(x + 1) - W'_{2,n}(x + (1 - \alpha^+))\alpha^+ \\ &= W_{2,n}(x + 1) - \tilde{Q}(\alpha^+/2)\alpha^+ \\ &= W_{2,n}(x + 1) - h(\alpha^+)\alpha^+ \\ &= W_{2,n}(x + 1) - \textstyle\int_0^{\alpha^+} h(u)du. \end{aligned}$$

We find therefore with (17):

$$G(p_{1,1}, h) \geq W_{2,n}(x + 1) - \int_0^1 h(u)du = C.$$

case 3: If $p_{1,1} \in]h(0), h(1)[$, then $p_{1,1} = h(\alpha)$ for some α in $]\alpha^+, 1[$. For this α, h is strictly increasing (on the contrary, we would have h constant on $[0, \alpha]$ and thus $\alpha^+ \geq \alpha$). Therefore formula (16) is valid and we get $G(p_{1,1}, h) = C$ with formula (17).
case 4: The case $p_{1,1} = h(1)$ has to be divided in two sub cases:
case 4-1: If $\alpha^+ < 1$ then h is strictly increasing at 1. So formula (16) also applies to this case: $G(p_{1,1}, h) = C$.
case 4-2: If $\alpha^+ = 1$, then

$$G(p_{1,1}, h) = W_{2,n}(x).$$

But in this case h is constant on $[0, 1]$, thus formula (17) yields:

$$C = W_{2,n}\left(x - 2\alpha + 1\right) - (1 - 2\alpha)h(\alpha).$$

With $\alpha = 1/2$, we get thus $C = W_{2,n}(x) = G(p_{1,1}, h)$.
case 5: The last case to consider is: $p_{1,1} > h(1)$. Then,

$$G(p_{1,1}, h) = W_{2,n}(x - 1) + p_{1,1} > W_{2,n}(x - 1) + h(1).$$

The right hand side of this inequality is just equal to C as it follows from (17), and thus $G(p_{1,1}, h) > C$. ■

The next result follows then from Theorem 9, Lemma 7 and equation (19).

Corollary 1

$$W_{2,n+1}(x) \geq \int_0^1 W_{2,n}\left(x - 2u + 1\right) du$$

5 Duality relations

Lemma 8 *For all $x \in I\!R$:*

$$V_{1,n}^*(x) = W_{1,n}(x) \geq V_{2,n}^*(x) \geq W_{2,n}(x). \tag{21}$$

proof: According to the definition of the Fenchel conjugate, we have:

$$\begin{aligned} V_{1,n}^*(x) &:= \inf_{P\in[0,1]} xP - V_{1,n}(P) \\ &= \inf_{P\in[0,1]}\{xP - \sup_\sigma \inf_\tau g_n(P,\sigma,\tau)\} \\ &= \inf_{(P,\sigma)} \sup_\tau g_n^*(x,(P,\sigma),\tau) \\ &= W_{1,n}(x). \end{aligned}$$

Since $V_{1,n} \leq V_{2,n}$ (see lemma 1), we get: $V_{1,n}^* \geq V_{2,n}^*$.
Next:

$$\begin{aligned} V_{2,n}^*(x) &= \inf_{P\in[0,1]} xP - \inf_\tau \sup_\sigma g_n(P,\sigma,\tau) \\ &= \inf_{P\in[0,1]} \sup_\tau \inf_\sigma g_n^*(x,(P,\sigma),\tau) \\ &\geq \sup_\tau \inf_{P\in[0,1]} \inf_\sigma g_n^*(x,(P,\sigma),\tau) \\ &= W_{2,n}(x), \end{aligned}$$

and (21) is proved. ∎

We are now ready to prove

Theorem 10 *For all $x \in I\!R$:*

$$V_{1,n}^*(x) = W_{1,n}(x) = V_{2,n}^*(x) = W_{2,n}(x). \tag{22}$$

The games $H_n(P)$ and $H_n^*(x)$ have therefore a value respectively denoted $V_n(P)$ and $W_n(x)$.

These values are Fenchel conjugates of each others:

$$W_n(x) = \inf_{P\in[0,1]} Px - V_n(P) \tag{23}$$

$$V_n(P) = \inf_{x\in I\!R} Px - W_n(x) \tag{24}$$

Furthermore:

$$W_{n+1}(x) = \int_0^1 W_n\left(x - 2u + 1\right) du \tag{25}$$

proof: This result is proved inductively. The game $H_0(P)$ where both players have no action to select, and the payoff is 0 has obviously a value $V_0(P) = 0$. In the game $H_0^*(x)$, player 1 chooses a state of nature k, player 2 has no action to select, and player 2's payoff is $x1\!\!1_{k=H}$. The value of this game is $W_0(x) = \phi(x) := \min\{x, 0\}$. Equations (23) and (24) hold then for $n = 0$, and

$$V_0^*(x) = V_{1,0}^*(x) = W_{1,0}(x) = V_{2,0}^*(x) = W_{2,0}(x) = W_0(x).$$

Let us suppose now that (22), (23), and (24) are proved for n. With theorem 8, we thus have

$$V^*_{1,n+1}(x) \leq \int_0^1 V^*_{1,n}\left(x-2u+1\right)du = \int_0^1 W_n\left(x-2u+1\right)du.$$

Corollary 1 claims that:

$$W_{2,n+1}(x) \geq \int_0^1 W_n\left(x-2u+1\right)du.$$

It follows then from the previous Lemma that

$$\begin{aligned} V^*_{1,n+1}(x) &= W_{1,n+1}(x) = V^*_{2,n+1}(x) \\ &= W_{2,n+1}(x) = \textstyle\int_0^1 W_n\left(x-2u+1)\right)du. \end{aligned} \tag{26}$$

So, (22) is proved for $n+1$, and since $W_{1,n+1}(x) = W_{2,n+1}(x)$, the game $H^*_{n+1}(x)$ has a value $W_{n+1}(x) = W_{1,n+1}(x) = W_{2,n+1}(x)$. Equation (25) is thus proved, and we also conclude from (26) that $V_{1,n+1}(P) = V_{2,n+1}(P) = W^*_{n+1}(P)$, so that the game $H_{n+1}(P)$ has a value $V_{n+1}(P) = W^*_{n+1}(P)$, which is exactly relation (24), and (23) follows from (26). ■

A recursive use of formula (25) indicates that:

Corollary 2 *Let $U_1, \cdots, U_n$ be a system of n independent random variables, uniformly distributed on $[-1,1]$, then:*

$$W_n(x) = E[\phi(x - \sum_{i=1}^n U_i)], \tag{27}$$

where $\phi(x) := \min\{x, 0\}$.

proof: The formula holds for $n = 0$, since $W_0(x) = \phi(x)$. If (27) holds for n, then we get with (25):

$$W_{n+1}(x) = \int_0^1 E\left[\phi\left(x - 2u + 1 - \sum_{i=1}^n U_i\right)\right]du.$$

Setting $U_{n+1} := 2u - 1$, we get then:

$$W_{n+1}(x) = E\left[\phi\left(x - U_{n+1} - \sum_{i=1}^n U_i\right)\right] = E\left[\phi\left(x - \sum_{i=1}^{n+1} U_i\right)\right],$$

and (27) holds thus also for $n+1$. ■

Corollary 3 *For $n \geq 1$, let f_n denote the probability density of $S_n := \frac{\sum_{i=1}^n U_i}{\sqrt{n}}$, then*

$$\psi_n(P) := \frac{V_n(P)}{\sqrt{n}} = \int_{x_P}^{\infty} s f_n(s) ds,$$

where x_P is such that: $P = \int_{x_P}^{\infty} f_n(s)ds$. Furthermore, $\psi'_n(P) = x_P$.

proof: We first observe that $\psi_n^*(x) = \frac{1}{\sqrt{n}} V_n^*(\sqrt{n}x) = \frac{1}{\sqrt{n}} W_n(\sqrt{n}x)$, so, with (27), we get:

$$\psi_n^*(x) = \frac{1}{\sqrt{n}} E[\phi(\sqrt{n}x - \sum_{i=1}^{n} U_i)] = E[\phi(x - S_n)], \tag{28}$$

since ϕ is 1-positively-homogeneous. This indicates that

$$\psi_n^*(x) = \int_{-\infty}^{\infty} \phi(x-s) f_n(s) ds = \int_x^{\infty} (x-s) f_n(s) ds.$$

The derivative of the last equation yields:

$$\psi_n^{*\prime}(x) = \int_x^{\infty} f_n(s) ds = Proba[S_n > x].$$

Next, if $P = \psi_n^{*\prime}(x_P)$, i.e. $P = \int_{x_P}^{\infty} f_n(s) ds$, then Fenchel lemma indicates that

$$\begin{aligned} \psi_n(P) &= x_P P - \psi_n^*(x_P) \\ &= x_P \int_{x_P}^{\infty} f_n(s) ds - \int_{x_P}^{\infty} (x_P - s) f_n(s) ds \\ &= \int_{x_P}^{\infty} s f_n(s) ds, \end{aligned}$$

and $\psi_n'(P) = x_P$. ■

Corollary 4 *Let $\psi^*(x)$ denote $\psi^*(x) := E[\phi(x + Z/\sqrt{3})]$, where Z is a centered gaussian random variable with variance 1. Then ψ_n^* and $\psi_n^{*\prime}$ converge uniformly to ψ^* and $\psi^{*\prime}$ as n grows to ∞. As a consequence: $\psi_n(P)$ converges uniformly to $\psi(P) = e^{-z_P^2/2}/\sqrt{6\pi}$, where z_P is such that $P = \int_{z_P}^{\infty} e^{-s^2/2}/\sqrt{2\pi} ds$.*

proof: Since $E[U_i] = 0$ and $Var[U_i] = 1/3$, the central limit theorem indicates that S_n converges in law to $Z/\sqrt{3}$, and implies thus the pointwise convergence of ψ_n^* to ψ^*. Since ϕ is Lipsitzian and U_i has a bounded support, a generalization of the central limit theorem due to Bhattachrya (see theorem 15.4 in [3]) implies the uniform convergence. Since $\psi_n^{*\prime}(x) = Proba[S_n > x]$, it results from Berry- Esseen theorem (see theorem 12.4 in [3]) that $\psi_n^{*\prime}$ converges uniformly to $Proba[Z/\sqrt{3} > x] = \psi^{*\prime}(x)$. Since Fenchel transform is an isometry in the uniform norm, the uniform convergence of ψ_n^* implies that of ψ_n to ψ.

Let next f denote the density of $Z/\sqrt{3}$, then $\psi^*(x) = \int_{-\infty}^{\infty} \phi(x-s) f(s) ds$. We find therefore that $\psi(P) = \int_{x_P}^{\infty} s f(s) ds$, where x_P is such that $P = \int_{x_P}^{\infty} f(s) ds$. So, if $g(s) := e^{-s^2/2}/\sqrt{2\pi}$, we have $f(s) = \sqrt{3} g(\sqrt{3}s)$, and thus

$$\psi(P) = \int_{x_P}^{\infty} s\sqrt{3} g(\sqrt{3}s) ds = \frac{1}{\sqrt{3}} \int_{\sqrt{3}x_P}^{\infty} s g(s) ds = \frac{1}{\sqrt{3}} g(\sqrt{3}x_P) = \frac{1}{\sqrt{3}} g(z_P),$$

where $z_P := \sqrt{3} x_P$ is such that

$$P = \int_{x_P}^{\infty} f(s) ds = \int_{\sqrt{3}x_P}^{\infty} g(s) ds = \int_{z_P}^{\infty} g(s) ds.$$

■

6 The optimal strategies of player 1 in $H_n(P)$.

6.1 Existence of a measurable optimal selection.

It results from the proof of theorem 10 that the inequalities in theorem 8 and 9 are in fact identities. The strategies defined in section 3.4 and 4.4 are thus optimal in the recurrence formula. We will now analyze the optimal behavior of player 1 in the whole game $H_n(P)$.

In the formalism of section 3, we analyzed the game $H_n(P)$ as if player 1 had to select jointly the state of nature k and his first stage price $p_{1,1}$. He is just restricted in the lotteries he may use by the condition that $Proba[k = H] = P$. The optimal joint lottery of player 1 consists of selecting a number u_1 uniformly distributed on $[0, 1]$, and then playing $p_{1,1} = f(u_1)$, and selecting a state k with a u_1-dependent lottery such that $Proba[k = H|u_1] = Q(u_1)$. The optimal $f(u_1)$ and $Q(u_1)$ are given by formulas (9), (8) and (6).

To induce the same joint probability on $(p_{1,1}, k)$ in the original game, where nature chooses the state, player 1 has to select the number u_1 with a state dependent lottery L_k and play $p_{1,1} = f(u_1)$. Since:

$$\begin{aligned} Proba[u_1 \le \alpha | k = H] &= Proba[u_1 \le \alpha \cap k = H]/Proba[k = H] \\ &= \int_0^\alpha Q(u_1) du_1/P, \end{aligned}$$

the probability density $L_H(u_1)$ of the random number u_1 when player 1 is told $k = H$ is just $L_H(u_1) = Q(u_1)/P$. A similar computation indicates that $L_L(u_1) = (1 - Q(u_1))/(1 - P)$. In particular, we find that the overall probability density of u_1 is $P \cdot L_H + (1 - P) \cdot L_L = 1$, so that the marginal distribution of u_1 is uniform on $[0, 1]$, as required.

Since $Q(u_1)$ is an increasing function, high values of u_1 are more likely than small values under the probability L_H. On the contrary, small values are more likely than high values under L_L. Since f is also increasing, player 1 has a probabilistic tendency to play higher prices in state H than in state L. However, player 2 may not completely infer the state k by observing $p_{1,1}$: using Bayes rules, the a posteriori probability of $k = H$ given $p_{1,1}$ is just $Q(u_1)$. ($Q(u_1)$ is $p_{1,1}$-measurable, since $f(u_1) = f(u_1') \Rightarrow Q(u_1) = Q(u_1')$.)

Let us now come back on the optimal f and Q in $H_n(P)$ as described by formulas (9), (8) and (6) written for H_n instead of H_{n+1}. The number λ in formula (9) will be denoted here λ_0^n. (n to indicate the length of the game, and 0 to indicate the number of stages already played). It is such that $K'(\lambda_0^n) = P$, where $K(x) := \int_0^1 V_{1,n-1}^*(x+1-2u_1)du_1$. Now, it results from theorem 10 that $V_{1,n-1}^* = W_{n-1}$, and so, $\forall x$, $K(x) = W_n(x)$. Since K, as a convolution is continuously differentiable, we also have, $\forall x$, $K'(x) = W_n'(x)$. So, we must have $P = W_n'(\lambda_0^n)$. With Fenchel lemma, this can be achieved by setting $\lambda_0^n = V_n'(P)$. From formulae (8) and (6), we have

$$Q(u_1) = W_{n-1}'(\lambda_0^n + 1 - 2u_1) \tag{29}$$

and

$$f(u_1) = \int_0^{u_1} 2sW'_{n-1}(\lambda_0^n + 1 - 2s_1)ds/u_1^2. \tag{30}$$

Since V'_n is continuous and $W'_{n-1}(x)$ is measurable, we conclude that, for all Borelian set A, $Proba[p_{1,1} \in A|k] = \int_{f^{-1}(A)} L_k(u_1)du_1$ is a measurable function of (P, k). As a consequence, with $n = 1$, we infer that player 1 has an optimal measurable selection in H_1.

Using backward induction we prove now that

Theorem 11 *$\forall n$, player 1 has a measurable optimal selection in H_n.*

proof: Indeed, if $\sigma(P)$ is a measurable selection in H_{n-1}, then the strategy consisting of playing $p_{1,1}$ as described above, and then playing $\sigma(Q(u_1))$ in the remaining stages of $H_n(P)$ is an optimal strategy as it results from the proof of theorem 7. Since Q is itself a measurable function of P, this strategy depends on a measurable way of P: it is a measurable optimal selection in H_n. ■

6.2 The a posteriori martingale.

Let us now analyze the evolution of player 2's a posteriori probability on H: As mentioned above, the a posteriori P_1 of player 2 after the first stage is $P_1 = W'_{n-1}(\lambda_0^n - U_1)$, where $U_1 := 2u_1 - 1$. Player 1 plays then at the second stage an optimal strategy in $H_{n-1}(P_1)$: he picks a number λ_1^n such that $P_1 = W'_{n-1}(\lambda_1^n)$. This is realized in particular for $\lambda_1^n = \lambda_0^n - U_1$. He then picks a random number u_2 with a density $L_H^2(u_2) = Q^2(u_2)/P_1$ if he observed $k = H$ and a density $L_L^2(u_2) = (1 - Q^2(u_2))/(1 - P_1)$ if $k = L$, where $Q^2(u_2) = W'_{n-2}(\lambda_0^n - U_1 + 1 - 2u_2)$. So, the a posteriori P_2 after the second stage is $P_2 = W'_{n-2}(\lambda_0^n - U_1 - U_2)$, where $U_2 := 2u_2 - 1$. Furthermore, the distribution of u_2 conditionally on $p_{1,1}$ is uniform on $[0, 1]$, since $P_1 L_H^2 + (1 - P_1)L_L^2 = 1$.

Applying inductively this argument, we find that the a posteriori P_m of player 2 after stage m is:

$$P_m = W'_{n-m}(\lambda_0^n - U_1 - \cdots - U_m), \tag{31}$$

where $U_1, \ldots, U_n$ appears to player 2 as a system of independent uniform random variables on $[-1, 1]$. Clearly, $P_0, \ldots, P_n$ is a martingale.

If $P_m = 0$ or 1, player 2 is completely informed about the state of nature. We prove now that full revelation to player 2 may happen before the end of $H_n(P)$, but the probability of that event tends to 0 as n grows to ∞.

With formula (27), we have $W'_n(x) = E[\phi'(x - \sum_{i=1}^n U_i)] = P[\sum_{i=1}^n U_i > x]$. Therefore, $W'_n(x) = 0$ is equivalent to $x \geq n$, and $W'_n(x) = 1$ is equivalent to $x \leq -n$. So full revelation can only happens at stage m if $|\lambda_0^n - U_1 - \cdots - U_m| \geq n - m$. This happens with probability 1 at stage n, but it can also happens before.

However, asymptotically, as n grows up, full revelation does not happens before the very end of the game: more precisely, if $t < 1$, the probability of full revelation before stage tn is

$$proba\left[\frac{|\lambda_0^n - U_1 - \cdots - U_{[tn]}|}{[tn]} \geq \frac{n}{[tn]} - 1\right] \leq proba\left[\frac{|\lambda_0^n - U_1 - \cdots - U_{[tn]}|}{[tn]} \geq \frac{1-t}{t}\right] \quad (32)$$

where $[tn]$ is the integer part of tn. Due to the law of large numbers, since $E[U_i] = 0$, $\frac{U_1 + \cdots + U_{[tn]}}{[tn]}$ tends to 0 in probability as n grows to ∞. On the other hand, λ_0^n also depends on n: it must be such that $P = W_n'(\lambda_0^n) = Proba[\frac{U_1 + \cdots + U_n}{n} < -\frac{\lambda_0^n}{n}]$. Since $\frac{U_1 + \cdots + U_n}{n}$ converge in probability to 0, this implies that if $0 < P < 1$ then λ_0^n/n also converges to 0. Therefore, the right hand side in (32) tends to 0 with n, and thus, as announced, the probability of full revelation before stage tn tends to 0 as n increases.

Let us now analyze the asymptotic of the a posteriori martingale. Let us represent the a posteriori martingale $P_0, \ldots, P_n$ in $H_n(P)$ by the continuous time process P^n: for $t \in [0, 1]$:

$$P_t^n := P_{[tn]}.$$

Let also ξ^n denotes the process $\xi_t^n := \frac{\sum_{i=1}^{[tn]} U_i}{\sqrt{n}}$. Then, equation (31) yields:

$$P_t^n = W'_{n-[nt]}(\lambda_0^n - \sqrt{n}\xi_t^n) = \psi^{*\prime}_{n-[nt]}(\frac{\lambda_0^n/\sqrt{n} - \xi_t^n}{\sqrt{1 - [tn]/n}}).$$

Since $\psi^{*\prime}(x)$ is a strictly increasing function, there is a unique root x_P to the equation $P = \psi^{*\prime}(x)$. On the other hand, λ_0^n must be such that $P = W_n'(\lambda_0^n) = \psi_n^{*\prime}(\lambda_0^n/\sqrt{n})$. Therefore, $\lambda_0^n/\sqrt{n}$ must converge to x_P, since $\psi_n^{*\prime}$ converges uniformly to $\psi^{*\prime}$.

Since, as it follows from the central limit theorem, ξ^n converges in finite distributions to $B/\sqrt{3}$, where B is a standard Brownian motion, we conclude that:

Theorem 12 *The process P^n converges in finite distributions to π, where*

$$\pi_t = \psi^{*\prime}\left(\frac{x_P - B_t/\sqrt{3}}{\sqrt{1-t}}\right) = F\left(\frac{z_P - B_t}{\sqrt{1-t}}\right),$$

where $F(x) = \int_{-\infty}^{x} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds$, and z_p is such that $F[z_P] = P$.

Let us now analyze the asymptotic of the price process $p_{1,1}, \ldots, p_{1,n}$ proposed by player 1 in $H_n(P)$. We have seen above that

$$Q(s) = W_{n-1}(\lambda_0^n + 1 - 2s) = \psi^{*\prime}_{n-1}\left(\frac{\lambda_0^n + 1 - 2s}{\sqrt{n-1}}\right).$$

Since $\psi^{*\prime}_{n-1}$ converges uniformly to $\psi^{*\prime}$, and since ψ^* has a bounded second order derivative, we conclude that $Q(1) - Q(0) \leq C/\sqrt{n}$, where C is a constant that does not depend on P. Next, since $Q(0) \leq f(s) \leq Q(s) \leq Q(1)$, we infer that $|f(s) - P_1| \leq C/\sqrt{n}$: the price $p_{1,1}$ proposed by player 1 at the first stage of $H_n(P)$ is very close to P_1 as far as n is large enough. Applying this reasoning at each stage of $H_n(P)$, we get:

Lemma 9 *There exists a constant C such that for all n, m, with $m < n$:*

$$|p_{1,m} - P_m| \leq C/\sqrt{n-m}.$$

We observe therefore that the price process in $H_n(P)$ which is not a martingale, will have the same asymptotic as the a posteriori martingale. Asymptotically, the price process becomes a martingale.

The next theorem indicates that player 1 has also an optimal strategy in $H_n^*(x)$:

Theorem 13 *A strategy σ is optimal in $H_n(P)$ if and only if (P,σ) is optimal in $H_n^*(x)$, with $x := V_n'(P)$.*

proof: If σ^* is optimal in $H_n(P)$ then, for all τ: $V_n(P) \leq g_n(P,\sigma^*,\tau)$ and thus $V_n^*(x) = xP - V_n(P) \geq xP - g_n(P,\sigma^*,\tau) = g_n^*(x,(P,\sigma^*),\tau)$. The strategy (P,σ^*) is thus optimal in $H_n^*(x)$.

On the other hand, if (P,σ^*) is optimal in $H_n^*(x)$ then $g_n^*(x,(P,\sigma^*),\tau) = xP - g_n(P,\sigma^*,\tau) \leq V_n^*(x)$ and thus $g_n(P,\sigma^*,\tau) \geq xP - V_n^*(x) = V_n(P)$. σ^* is thus optimal in $H_n(P)$. ∎

7 The optimal behavior of player 2 in $H_n^*(x)$.

Since formula (20) provides an optimal strategy in the recurrence formula of theorem 9, we may use backward induction to build up an optimal strategy for player 2 in $H_n^*(x)$: In $H_1(x)$, the optimal strategy for player 2 will be to select a random number u' uniformly distributed in $[0,1]$ and to propose the price $p_{2,1} := h(u')$, with

$$h(u) := u^{-2} \int_0^u 2sW_0'(x+1-2s)ds.$$

So, as a function of x, the probability $proba[p_{1,1} \in A]$ is measurable for all Borelian set A: Player 2 as an optimal selection in H_1^*.

If player 2 has a measurable optimal selection τ in H_{n-1}^*, he also has one in H_n^*: It consists in playing $p_{2,1} := h_1^n(u_1')$, with

$$h_1^n(u) := u^{-2} \int_0^u 2sW_{n-1}'(x+1-2s)ds \tag{33}$$

and then playing

$$\tau(\ x + \int_0^1 1\!\!1_{h_1^n(u)>p_{1,1}} - 1\!\!1_{h_1^n(u)<p_{1,1}} du) \tag{34}$$

in the remaining stages of the game. The measurability in x of this strategy makes no problem, and we then conclude by induction:

Lemma 10 *For all n, player 2 has an optimal selection in H_n^*.*

The next theorem indicates how player 2 will play in $H_n(P)$:

Theorem 14 *A strategy τ of player 2 is optimal in $H_n(P)$ if and only if it is optimal in $H_n^*(x)$, with $x := V_n'(P)$. In particular, player 2 has an optimal strategy in $H_n(P)$.*

proof: If τ^* is optimal in $H_n(P)$, then $\forall P'$:

$$\begin{aligned} V_n(P') &\leq \sup_\sigma g_n(P', \sigma, \tau^*) \\ &= \sup_\sigma P' g_n(H, \sigma^H, \tau^*) + (1 - P') g_n(L, \sigma^L, \tau^*) \\ &= P' \sup_{\sigma^H} g_n(H, \sigma^H, \tau^*) + (1 - P') \sup_{\sigma^L} g_n(L, \sigma^L, \tau^*), \end{aligned}$$

with an equality if $P' = P$. Therefore, as a function of P', $\sup_\sigma g_n(P', \sigma, \tau^*)$ is just an affine function that is tangent to V_n at $P' = P$. As a consequence, with $x := V_n'(P)$:

$$\sup_\sigma g_n(P', \sigma, \tau^*) = V_n(P) + x(P' - P) = -V_n^*(x) + xP'.$$

Hence, for all P': $\inf_\sigma g_n^*(x, (P', \sigma), \tau^*) = P'x - \sup_\sigma g_n(P', \sigma, \tau^*) = V_n^*(x)$. τ^* guarantees thus $V_n^*(x)$ to player 2 in $H_n^*(x)$ and is thus optimal.

Conversely, if τ^* is optimal in $H_n^*(x)$ then for all σ: $g_n^*(x, (P, \sigma)) \geq V_n^*(x)$ and thus $g_n(P, \sigma, \tau) \leq V_n^*(x) - xP = V_n(P)$: τ^* is also optimal in $H_n(P)$.

We conclude thus that player 2 has an optimal strategy in $H_n(P)$, since he has an optimal strategy in $H_n^*(x)$, as proved in lemma 10. ∎

We are now able to analyze the optimal behavior of player 2 in $H_n(P)$: setting $x := V_n'(P)$, player 2 picks a uniformly distributed random number u_1' and plays $p_{2,1} := h_1^n(u_1')$. Comparing this with the strategy of player 1 described in the last section, we observe that $x = \lambda_0^n$, and thus the function f (see (30)) used by player 1 to chose $p_{1,1}$ as $f(u_1)$ coincide with h_1^n (see (33)). Therefore, at the equilibrium, the random variables $p_{1,1}$ and $p_{2,1}$ are independent and have the same marginal distribution: An external observer could not deduce which player is informed by just observing the first proposed prices.

Furthermore, since $f = h_1^n$, we conclude that $Q(0) \leq h_1^n(u_1') \leq Q(1)$, and thus $|p_{2,1} - P_1| \leq C/\sqrt{n}$, with the same constant C as in lemma 9.

At the second stage of the game, player 2 computes

$$X_1 = x + \int_0^1 1\!\!1_{h_1^n(u) > p_{1,1}} - 1\!\!1_{h_1^n(u) < p_{1,1}} du = x + 1 - 2(h_1^n)^{-1}(p_{1,1}),$$

where $(h_1^n)^{-1}$ denotes the inverse function of h_1^n (we suppose here h_n^1 strictly increasing.) He then plays optimally in $H_{n-1}^*(X_1)$, so he picks a uniformly distributed random number u_2' and plays $p_{2,2} := h_2^n(u_2')$, with

$$h_2^n(u) := u^{-2} \int_0^u 2s W_{n-2}'(X_1 + 1 - 2s) ds.$$

If player 1 played optimally: $p_{1,1} = f(u_1)$, we get then $X_1 = \lambda_0^n + 1 - 2u_1$, since $f = h_1^n$. Therefore, $P_1 = Q(u_1) = W'_{n-1}(X_1)$, as it follows from (29), and due to Fenchel Lemma: $X_1 = V'_{n-1}(P_1)$. Therefore, the strategy used by player 2 from the second stage on is optimal in $H_n(P_1)$. A recursive use of the above arguments yields then:

Theorem 15 *If both players use their optimal strategies in $H_n(P)$, then*
1) Conditionally to the past proposed prices, $p_{1,q}$ and $p_{2,q}$ are independent and have the same distribution.
2) For q in $\{1, \dots, n-1\}$: $|p_{2,q} - P_q| \leq C/\sqrt{n-q}$, with the constant C of lemma 9.

8 Equivalence between $G_n(P)$ and $H_n(P)$.

In the previous sections, we analyzed the game $H_n(P)$, where both players are committed to propose prices in the interval $[0,1]$. We now prove that the results obtained in that framework, also apply to the original game $G_n(P)$, where the players are allowed to propose prices outside this interval.

We will first prove that player 1 may guarantee the same amount in $H_n(P)$ and in $G_n(P)$.

The optimal strategy σ^* of player 1 in $H_n(P)$ was defined inductively in section 6. This is a "reduced" strategy for player 1, i.e. a strategy that does not care on the past actions of player 2. Indeed, at stage q, player 1 computes the a posteriori P_{q-1}, and plays $p_{1,q}$ as if he was playing the first stage in $H_{n-q+1}(P_{q-1})$. In turn, P_{q-1} does not depend on the past actions of player 2.

The best reply against such a "reduced" strategy will be to minimize at each stage the stage payoff, since the price $p_{2,q}$ proposed by player 2 at stage q will not affect the joint distribution of $(k, p_{1,q+1}, \dots, p_{1,n})$.

Since both the price proposed by player 1 and the final value of the risky asset belong to $[0,1]$ with probability 1 (σ^* is a strategy in $H_n(P)$), player 2 has no advantage to play outside this interval: he is always better off proposing the price $\tilde{p}_{2,q} := \max(0, \min(1, p_{2,q}))$ instead of $p_{2,q}$.

So, the best reply to σ^* in $G_n(P)$ is a $[0,1]$-valued strategy, and is thus a strategy in $H_n(P)$. As a consequence:

Theorem 16 *The optimal strategy σ^* of player 1 in $H_n(P)$ defined in section 6 guarantees him $V_n(P)$ in $G_n(P)$.*

Let us next consider the optimal strategy τ^* of player 2 in $H_n^*(x)$ as defined in the last section. τ^* is in fact also a strategy in $G_n^*(x)$, since the quantity $X_1(p_{1,1}) := x + \int_0^1 1\!\!1_{h_1^n(u) > p_{1,1}} - 1\!\!1_{h_1^n(u) < p_{1,1}} du$ involved in (34) also makes sense for $p_{1,1}$ outside of $[0,1]$.

Since the pure strategy of player 2 $p_{2,q} = 0$ for all $q = 1, \dots, n-1$ guarantees him a payoff $\min_{P \in [0,1]} Px - (n-1)P$ in $H_{n-1}^*(x)$, it is optimal as far as $x \geq n-1$, and we may therefore consider that the optimal selection τ in H_{n-1}^* involved in (34) is such that $\tau(x) = \tau(n-1)$ for all $x > n-1$.

Similarly, the strategy $P_{2,q} = 1$ for $q = 1 \ldots, n-1$ is optimal in $H^*_{n-1}(x)$ as far as $x \leq 1-n$. We may also suppose that $\tau(x) = \tau(1-n)$ for $x < 1-n$.

Let now (P, σ) be a strategy of player 1 in $G^*_n(x)$, and let us consider the following strategy $(P, \tilde{\sigma})$: at the first stage player 1 picks a virtual price $p^v_{1,1}$ as if he was playing (P, σ), and he propose the price $\tilde{p}_{1,1} := \max(0, \min(1, p^v_{1,1}))$. At the next stages player 1 plays the strategy (P, σ) as if his first proposed price was $p^v_{1,1}$ instead of $\tilde{p}_{1,1}$.

As regarding to the first stage payoff, $(P, \tilde{\sigma})$ is a better reply to τ^* than (P, σ), since both the price $p_{2,1}$ and the liquidation value of the risky asset belong to $[0, 1]$.

On the other hand, we argue now that

Lemma 11

$$\tau(X_1(\tilde{p}_{1,1})) = \tau(X_1(p^v_{1,1})). \tag{35}$$

proof: Indeed, let us define α^+ and α^- as $\alpha^- := \inf\{u | h^n_1(u) > 0\}$ and $\alpha^+ := \sup\{u | h^n_1(u) < 1\}$.

If $p^v_{1,1} < 0$, then $X_1(p^v_{1,1}) = x + 1$ and $X_1(\tilde{p}_{1,1}) = x + 1 - \alpha^-$. So, either $\alpha^- = 0$, implying $X_1(p^v_{1,1}) = X_1(\tilde{p}_{1,1})$ and thus (35), or $\alpha^- > 0$. In this case, $h^n_1(u) = 0$ for $u \leq \alpha^-$ and thus $W'_{n-1}(x + 1 - 2\alpha^-) = 0$ as it follows from (33). With (27), this last equality implies:

$$n - 1 \leq x + 1 - 2\alpha^- < x + 1 - \alpha^- = X_1(\tilde{p}_{1,1}) < X_1(p^v_{1,1}).$$

So, since $\tau(x)$ is constant for $x \geq n-1$, (35) holds in this case.

If $0 \leq p^v_{1,1} \leq 1$, then $p^v_{1,1} = \tilde{p}_{1,1}$, and (35) follows.

Finally, if $p^v_{1,1} > 1$, then $X_1(p^v_{1,1}) = x - 1$, and $X_1(\tilde{p}_{1,1}) = x - \alpha^+$. So, either $\alpha^+ = 1$, implying obviously (35), or $\alpha^+ < 1$. In this case $h^n_1(1) = 1$ and thus $W'_{n-1}(x + 1 - 2s) = 1$ for $s \in]0, 1[$, as it follows from (33) ($\forall y : W'_{n-1}(y) \leq 1$). From (27), we infer then that:

$$1 - n \geq x + 1 > X_1(\tilde{p}_{1,1}) > X_1(p^v_{1,1}),$$

and thus (35) follows, since $\tau(x)$ is constant for $x \leq 1 - n$. ■

Relation (35) indicates that both player will play in the remaining part of the game as if they were playing $((P, \sigma), \tau^*)$, and the first stage price was $p^v_{1,1}$. We conclude then that $(P, \tilde{\sigma})$ is a better reply to τ^* than (P, σ).

So, to reply to τ^*, player 1 has no advantage to play a price $p_{1,1}$ outside of $[0, 1]$. An inductive use of this argument shows that the best reply to τ^* is a $[0, 1]$-valued strategy, and thus:

Theorem 17 *The optimal strategy τ^* of player 2 in $H^*_n(x)$ guarantees him $W_n(x)$ in $G^*_n(x)$.*

By a duality argument similar to lemma 8, it is now easy to prove:

Theorem 18 *The value of $G_n(P)$ is $V_n(P)$, the value of $G^*_n(x)$ is $W_n(x)$ and the strategies σ^* and τ^* of the last two theorems are optimal in these games.*

This conclude then the proof of theorem 1. Theorem 2 is then equivalent to corollary 3 and theorem 3 was proved in corollary 4.

Theorem 4 follows then from Theorem 12 joint to lemma 9 and theorem 15. Theorem 5 is just equivalent to corollary 2 (see equation (28)).

Finally, it remains for us to prove theorem 6. The optimal strategy (P, σ) of player 1 in $G_n^*(-Q_n)$ must be such that σ is optimal in $G_n(P(n))$, with $P(n) := W_n'(-Q_n)$, as it result from theorem 13. The process of not defaulting probabilities P_q^n involved in the definition of Π_t^n before theorem 6, may then be written as

$$P_q^n = W_{n-q}'(\lambda_0^n - U_1 - \cdots - U_q),$$

as it follows from formula (31), but here λ_0^n must be such that $W_n'(\lambda_0^n) = P(n) = W_n'(-Q_n)$. So, we may assume in particular $\lambda_0^n = -Q_n$, and since $Q_n/\sqrt{n}$ converges by hypothesis to α as n increases, the proof of theorem 6 follows word by word that of theorem 12.

9 Conclusion

All this paper is an attempt to emphasize that, in finance, the Browian motion appears partially for strategic reasons.

Aside from the Brownian motion related to the asymptotic of the price process in our model, there is another one related to the evolution of player 1's portfolio that was not mentioned before.

Indeed, as indicates theorem 15, the proposed prices $p_{1,q}$ and $p_{2,q}$ in $G_n(P)$ are independent and have the same distribution conditionally to the past proposed prices. If the distribution function of $p_{1,q}$ is continuous, which is essentially the case when the a posteriori P_q is far from 0 or 1, this implies in particular that $p_{1,q} > p_{2,q}$ with probability 1/2 conditionally to the the previous proposed prices. Therefore $\{y_q^R\}_{q=1,\cdots,n}$ is a random walk before full revelation. Since asymptotically, full revelation only happens in the very end of the game, we conceive easily that the portfolio process y^R scaled by $n^{-1/2}$ will converge in law to another Brownian motion.

To conclude this paper, we just mention some possible generalizations of the model:

1) Instead of considering that the final liquidation value of the risky asset is a bivalued (0 or 1) random variable, one can consider that it has a general distribution. This leads to analyze a zero-sum repeated game with a continuum of states of nature. The second chapter of H. Moussa's ph-D thesis is devoted to this problem. In the particular case of a Log-Normal distribution, the asymptotic dynamic of the price process coincides with Black and Scholes' dynamic (1).

2) The model can also be generalized by introducing various risky assets, and in particular one of these could be a derivative taking the others as underlying. This problem is currently the subject of a working paper by De Meyer and Gossner. One conclusion of that paper is that the derivative

market is also revealing, therefore, under asymmetrical information, the introduction of a derivative will modify the stochastic evolution of the price process of the underlying assets.

3) The crux point in our model for the appearance of a Brownian motion seems to be the complete symmetry between the players (aside from the informational asymmetry.) It would be interesting to analyze the same model with a more general symmetric transaction mechanism between the market makers.

4) This paper was concerned with the interaction between an insider and a non informed player. It is maybe more realistic to consider a situation where both players receive some private message on the liquidation value of the risky asset. This will lead to a zero sum game with incomplete information on both sides. Since duality appeared quite naturally in the one sided information case, this analysis could lead to a generalization of the duality technique to the case of incomplete information on both sides.

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