AFRIAT'S THEOREM FOR GENERAL BUDGET SETS

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Abstract. Afriat (1967) proved the equivalence of a variant of the strong axiom of revealed preference and the existence of a solution to a set of linear inequalities. From this solution he constructed a utility function rationalizing the choices of a competitive consumer. We extend Afriat's theorem to a class of nonlinear, nonconvex budget sets. We thereby obtain testable implications of rational behavior for a wide class of economic environments, and a constructive method to derive individual preferences from observed choices. We also show that by increasing in a regular way the number of observed choices from our class of budget sets one can fully identify the underlying preference relation.

Keywords: GARP, rational choice, revealed preferences, SARP, WARP.

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Introduction

Richter (1966) developed an abstract version of revealed preference theory, allowing for 'budget sets' which are just non empty subsets of a given universal set of possible choices. He proved that (a suitably generalized version of) the Strong Axiom of Revealed Preference (SARP) is necessary and sufficient for the existence of a rationalizing preference. The proof is non constructive, using Zorn's Lemma.

Almost at the same time, Afriat (1967) developed the theory of revealed preference in a completely different direction. In the original context of the competitive consumer, he emphasized the operational aspects of the theory. He took as data a finite number of observations, each one consisting of the chosen bundle of goods and the prevailing prices, and proved that, if these data satisfy a variant of the SARP, a rationalizing utility can explicitly be constructed by elementary linear programming techniques.

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Afriat's method has subsequently been expanded and refined, notably by Diewert (1973), Varian (1982) and most recently Chung-Piaw and Vohra (2003) and Fostel, Scarf and Todd (2004). All these contributions deal with the case of linear, competitive budgets, and remain firmly in Afriat's constructive, finite observations setting.

Another important theoretical development, which can also be ascribed to the line of research initiated by Afriat, is the approximation theory of Mas Colell (1978). He asks whether by increasing in a regular way the number of observations one can fully identify the underlying preference of the consumer. Again for the case of linear competitive budgets, Mas Colell identifies conditions for a positive answer.

Samuelson's (1938) original ideas have thus been pursued in two quite different directions: Richter's very general, non constructive existence results, and, for the special case of competitive linear budgets, Afriat's constructive approach, supplemented by Mas Colell's analysis of uniqueness and approximation.

Our contribution in this paper is to identify a class of choice problems which is much more general than the competitive consumer's and still retains sufficient structure to allow us to recover the exact analog of Afriat's and Mas Colell's results. We develop each of these extensions in the next two sections of the paper.

Our choice space is the positive orthant of some Euclidean space, and we allow as admissible budget sets all subsets of the choice space that can be obtained as the comprehensive closure of a compact set. Every budget set that precludes infinite consumption and allows for free disposal fits in this class. In particular we do not impose any convexity assumption. Every budget set in our class admits a description by means of an increasing continuous function (Lemma 1), and this enables us to obtain the analog of the Afriat's inequalities (Proposition 1). As a possible application, we discuss the test of Nash behavior in strategic market games. We also show how the existence of a *concave* rationalization can possibly be tested in our framework (Proposition 2).

At the end of the first section we discuss the contributions of Matzkin (1991) and Chavas and Cox (1993), who, *under convexity assumptions*, provide constructive rationalizations of data generated by specific classes of non linear budget sets. Our budget sets encompass theirs and we show that their results can be recovered as immediate corollaries of ours. The gained generality is not only of a purely theoretical interest as illustrated by classical economic examples in e.g. Mas Colell, Whinston and Green (1995, chapter 2, section 2.D) or in the more recent literature on labor supply (see e.g. Blomquist and Newey (2002)).

In the second section, on identification and approximation, we derive the analog in our setting of Mas Colell's results (Propositions 3 and 4).

1. Existence of a Rationalization

Consider an individual choosing consumption bundles in R^L_+. A consumption experiment is a finite collection $(x_k, B_k)_{k=1,\dots n}$, where $x_k \in B_k$ and $B_k \subset R^L_+$. The interpretation is that x_k is the observed choice of the individual when she has access to the set of consumption bundles B_k. We consider sets of alternatives of the form $B_k = \{x \in R^L_+ \mid g_k(x) \leq 0\}$ with $g_k : R^L_+ \to R$ an increasing, continuous function and $g_k(x_k) = 0$, for all $k = 1, \dots n$.

1.1. Budget sets.

Budget sets. The only restriction that we impose on our class of admissible budget sets is that they can be described as $B = \{x \in R^L_+ \mid g(x) \leq 0\}$ with $g : R^L_+ \to R$ an increasing, continuous function.

As an illustration, we provide a specific construction of the function g for a rather large class of budgets sets whose main characteristics is to prevent infinite consumption and to allow for free disposal.

Let $B \subset R^L_+$, be compact, monotone[1] and such that it contains at least one strictly positive vector: $\exists x \in B$, $x \gg 0$. The upper boundary of B is $b(B) = \{x \in B \mid y \gg x \Rightarrow y \notin B\}$. To obtain a natural representation of each budget set by means of a continuous function, we make the additional technical assumption that any line from the origin intersects the boundary in at most one point:

$$\text{Assumption H} : x \in b(B) \implies \forall k \in [0, 1), \quad kx \in B \setminus b(B)$$

For any $x \in R^L_+$, let $\gamma_B(x) = \inf\{\lambda > 0 \mid x \in \lambda B\}$. When B is a convex set, the function γ_B is known in convex analysis as the gauge of B (see e.g. Rockafellar (1972)). The following Lemma can then be proved.

Lemma 1. *Let $B = B \subset R^L_+$ be compact, monotone, such that $\exists x \in B$, $x \gg 0$ and assumption H is satisfied. Then:*

1. *$\gamma_B : R^L_+ \to R$ is homogeneous of degree one: for any $k > 0$, and $x \in R^L_+$, $\gamma_B(kx) = k\gamma_B(x)$*
2. *$\gamma_B : R^L_+ \to R$ is a continuous function*
3. *$\gamma_B : R^L_+ \to R$ is increasing: for any $x, y \in R^L_+$, $y \gg x$ implies $\gamma_B(y) > \gamma_B(x)$*
4. *$B = \{x \in R^L_+ \mid \gamma_B(x) \leq 1\}$, and*
5. *$b(B) = \{x \in R^L_+ \mid \gamma_B(x) = 1\}$*

[1] $\forall\, y \in R^L_+$, if $\exists\, x \in B$, $y \leq x$, then $y \in B$.

Letting $g(x) = \gamma_B(x) - 1$, we can write $B = \{x \in R^L_+ \mid g(x) \leq 0\}$ so that our class of admissible budget sets contains all the sets satisfying the assumptions of Lemma 1[2].

1.2. **Afriat's inequalities.** Fix a consumption experiment $(x_k, B_k)_{k=1,\dots n}$. We say that the function $v : X \to R$ rationalizes the experiment if, for all k, $g_k(x) \leq 0$ implies $v(x) \leq v(x_k)$. We say that x_k is revealed preferred to x_j, $x_k R x_j$, if $g_k(x_j) \leq g_k(x_k) = 0$. Let H be the transitive closure of the relation R. The standard competitive case corresponds to $g_k(x_j) = p_k(x_j - x_k)$, where p_k is the price vector. The following Generalized Axiom of Revealed Preference (GARP) is a variation of the SARP introduced by Varian (1982), in the linear case, to deal with the possibility of indifference.

Definition (GARP): the experiment $(x_k, B_k)_{k=1,\dots n}$ satisfies GARP if, for any k, j, $x_k H x_j$ implies $g_j(x_k) \geq 0$.

It is convenient to express GARP as a condition on the elements of a square matrix. To each consumption experiment $(x_k, B_k)_{k=1,\dots n}$, we associate an $(n \times n)$ matrix A with elements $a_{kj} = g_k(x_j)$.

Definition (Cyclical Consistency): a square matrix A of dimension n is cyclically consistent if $a_{jj} = 0$ for every $j \in \{1, \dots n\}$, and for every chain $\{k, j, l, \dots m\} \subset \{1, \dots n\}$, $a_{kj} \leq 0, a_{jl} \leq 0, \dots a_{mk} \leq 0$ implies that all terms are zero.

An experiment satisfies GARP if and only if the associated matrix A is cyclically consistent. Suppose A is cyclically consistent, and let $x_k H x_m$. This means that there are indices $\{j, l, \dots h\}$ such that $a_{kj} \leq 0, a_{jl} \leq 0, \dots a_{hm} \leq 0$. If $g_m(x_k) < 0$, $\{k, j, l, \dots h, m\}$ would be a chain satisfying the premise in the definition of cyclical consistency. But then $a_{mk} = g_m(x_k) = 0$ would lead to a contradiction. Thus we must have $g_m(x_k) \geq 0$, i.e. GARP holds. In the other direction, let the experiment satisfy GARP, contruct the associated matrix A and take a chain $\{k, j, l, \dots h, m\}$ satisfying the premise of the definition of cyclical consistency. For any two adjacent elements in the chain, say (j, l), by going through the chain we have $x_l H x_j$. Applying GARP, it must be that $g_j(x_l) = a_{jl} \geq 0$, so that $a_{jl} = 0$. This is true for any couple of adjacent elements in the chain, i.e. Cyclical Consistency holds.

Lemma 2. *If a square matrix A of dimension n is cyclically consistent, there exist numbers $(v_k, \lambda_k)_{k=1,\dots n}$, $\lambda_k > 0$, such that, for all $k, j =$*

[2]Our admissible class includes also budget sets that do not satisfy the assumptions of Lemma 1. For example, if a set B satisfies all the assumptions of Lemma 1 except assumption H, we may use $g(x) = \gamma_{B(e)}(x + e) - 1$, where $e = (1, 1, \dots, 1)$ and $B(e)$ is the monotone hull of $B + e$ in R^L_+ (the smallest monotone set in R^L_+ containing $B + e$). See also Bonnisseau - Crettez (2007) for alternative ways to describe a compact monotone set by means of a continuous, increasing function.

$1, \dots n$,

$$v_j \leq v_k + \lambda_k a_{kj}$$

Proof: See Fostel, Scarf and Todd (2004), sections 2 and 3. □

We are now able to state our generalization of Afriat's Theorem.

Proposition 1. *Let* $B_k = \{x \in R^L_+ \mid g_k(x) \leq 0\}$ *with* $g_k : R^L_+ \to R$ *an increasing, continuous function and* $g_k(x_k) = 0$*, for* $k = 1, \dots n$*. The following conditions are equivalent:*

(1) *there exists a locally non satiated, continuous utility function* v *rationalizing the experiment* $(x_k, B_k)_{k=1,\dots n}$

(2) *the experiment* $(x_k, B_k)_{k=1,\dots n}$ *satisfies GARP*

(3) *there exist numbers* $(v_k, \lambda_k)_{k=1,\dots n}$*,* $\lambda_k > 0$*, such that, for all* $k, j = 1, \dots n$*,*

$$v_j \leq v_k + \lambda_k g_k(x_j)$$

Proof:

(1) → (2): Let $x_k H x_j$: there exist indices $(g, \dots m)$ such that $x_k R x_g R \dots R x_m R x_j$. We want to show that $g_j(x_k) \geq 0$. Using the definition of R, $g_k(x_g) \leq 0$, ..., $g_m(x_j) \leq 0$. If v rationalizes the experiment, we must have $v(x_k) \geq v(x_g) \geq \dots v(x_m) \geq v(x_j)$, implying $v(x_k) \geq v(x_j)$. If $g_j(x_k) < 0$, by the local non satiation of v and the continuity of g_j we could find $x \in X$ such that $g_j(x) < 0$ and $v(x) > v(x_k) \geq v(x_j)$, contradicting the fact that v rationalizes the experiment.

(2) → (3): construct the matrix A associated with the experiment. By (2), A is cyclically consistent. Then use Lemma 2.

(3) → (1): Let $v(x) = \min_k\{v_k + \lambda_k g_k(x)\}$. The function v is increasing and continuous on R^L_+. To show that it rationalizes the data, notice first that, for all j, $v(x_j) = \min_k\{v_k + \lambda_k g_k(x_j)\} = v_j$, using the Afriat's inequalities in (3) and the fact that $g_j(x_j) = 0$. Then, if we consider x such that $g_j(x) \leq 0$ we have $v(x) \leq v_j + \lambda_j g_j(x) \leq v_j = v(x_j)$. □

Let us compare our result to the original Afriat's (1967) theorem for the case of linear budget sets[3]. Varian (1982, p.970) offers a heuristic argument to illustrate the Afriat's inequalities. His argument refers to the implication (1) → (3) and can be paraphrased as follows. Suppose that the budget sets can be described by differentiable functions g_k and that the experiment is rationalized by an increasing and differentiable

[3]We thank an anonymous referee for suggesting this discussion.

function v. Then each observation x_k solves the constrained maximization problem $\max v(x)$ over $\{x \mid g_k(x) \leq 0\}$ and there exists a sequence of multipliers $\lambda_k > 0$, $k = 1, \ldots, n$ such that, for all k, the following equations hold:

$$\nabla v(x_k) - \lambda_k \nabla g_k(x_k) = 0,$$

$$g_k(x_k) = 0,$$

where ∇ indicates the gradient.

Now, *if* the function v were concave, for all j, k we would have:

$$v(x_j) \leq v(x_k) + \nabla v(x_k)(x_j - x_k)$$

which can be rewritten, using the first equation above, as:

$$v(x_j) \leq v(x_k) + \lambda_k \nabla g_k(x_k)(x_j - x_k).$$

Moreover, *if* the functions g_k were convex, for all j, k:

$$\nabla g_k(x_k)(x_j - x_k) \leq g_k(x_j) - g_k(x_k).$$

Using the second equation above, $g_k(x_k) = 0$, one could then conclude:

$$v(x_j) \leq v(x_k) + \lambda_k \nabla g_k(x_k)(x_j - x_k) \leq v(x_k) + \lambda_k g_k(x_j).$$

Setting $v_k = v(x_k)$ and $v_j = v(x_j)$ these are exactly the Afriat's inequality appearing in (3).

What our result shows is that this heuristics, useful as it is for the linear case, does not go to the heart of the matter. Indeed, as it is clear from the discussion preceding Lemma 2, independently from any convexity hypothesis, if an experiment satisfies GARP the existence of numbers $(v_k, \lambda_k)_{k=1,\ldots n}$ satisfying the inequalities in (3) follows for *any* matrix A having the property that $a_{kj} \leq 0$ if x_k is revealed preferred to x_j.

Of course, the existence of these numbers does not provide, in general, any hint on how to define a rationalizing utility function on the whole consumption set (i.e. on how to prove the implication (3) $\rightarrow$ (1)). Afriat's clever idea was to notice that, in the competitive setting, $a_{kj} = p_k(x_j - x_k)$ so that starting form the inequalities in (3) one could extend the utility function from the finite observations to the entire consumption set by letting $v(x) = \min_k\{v_k + \lambda_k p_k(x - x_k)\}$. We carry this approach one step further and show that, in the more general setting that we consider, one can choose $a_{kj} = g_k(x_j)$, and still obtain, in the last step of the proof, an explicit utility function.

1.3. **Market Games.** Nonlinear budgets are the hallmark of imperfect competition. In this subsection we consider the application of Proposition 1 to strategic market games, a rather general class of models of imperfect competition. Consider I individuals trading L commodities. A market mechanism consists of a set A^i of strategies (bids, offers, etc.) for every agent and an outcome function $x : \times_i A^i \rightarrow R_+^{LI}$ that maps strategy profiles into allocations of commodities. Fixing the strategies of the others, each player generates a set of consumption bundles as she varies her strategy. The individual problem can thus be expressed as the maximization of the player's preferences over a 'budget set':

$$B^i(a^{-i}) = \{x^i \in R_+^L \mid \exists a^i \in A^i s.t. x^i \leq x^i(a^i, a^{-i})\}$$

Typically, the strategy chosen by a player has some influence on the 'terms of trade', and we should not expect the frontier of the budget set to be linear. Strategic market games are thus a natural setting for the application of our generalization of Afriat's theory. For example, for the broad class of market mechanisms axiomatized by Dubey and Sahi (2003), individual 'budget sets' are exactly of the form covered by our Lemma 1. An experiment now consists of a collection of observations of individual choices and of the rules of the game, from which the budget sets of each individual can be reconstructed. As an immediate corollary of Proposition 1 we then obtain a set of testable restrictions which are necessary and sufficient to interpret the observed choices as Nash equilibrium outcomes.

1.4. **Testing concavity.** Consider now consumption experiments in which the function g_k describing B_k is not only increasing and continuous, but also quasi - convex and differentiable at x_k, for all $k = 1, \ldots n$. In this case, the gradient $\nabla g_k(x_k)$ identifies the unique supporting hyperplane of B_k at x_k.

For each (x_k, B_k), let $C_k = \{x \in R_+^L \mid \nabla g_k(x_k)(x - x_k) \leq 0\}$. Our definitions of R, H and GARP are easily adapted to the 'linearized' experiment $(x_k, C_k)_{k=1,\ldots n}$.

Proposition 2. *Let* $B_k = \{x \in R_+^L \mid g_k(x) \leq 0\}$ *with* $g_k(x_k) = 0$ *where* $g_k : R_+^L \rightarrow R$ *is increasing, continuous, quasi - convex and differentiable at* x_k*, for* $k = 1, \ldots n$*. The following conditions are equivalent:*

1') *there exists a locally non satiated, continuous and* concave *utility function* v *rationalizing the experiment* $(x_k, B_k)_{k=1,\ldots n}$

2') *the 'linearized' experiment* $(x_k, C_k)_{k=1,\ldots n}$ *associated with* $(x_k, B_k)_{k=1,\ldots n}$ *satisfies GARP*

3') *there exist positive numbers* $(v_k, \lambda_k)_{k=1,\ldots n}$, $\lambda_k > 0$, *such that, for all* $k, j = 1, \ldots n$,

$$v_j \leq v_k + \lambda_k \nabla g_k(x_k)(x_j - x_k)$$

The proof follows closely that of Proposition 1, and we omit it.

If the experiment satisfies the premise of Proposition 2, we may consider two sets of testable conditions: those in $2'$), and the 'non-linearized' ones, 2) of Proposition 1. If the experiment satisfies 2) but not $2'$) a rationalization is possible, but preferences cannot be represented by a concave utility function.

1.5. **Previous results.** Matzkin (1991) explicitly deals with nonlinear choice sets. She proves that the existence of a strictly concave rationalization is equivalent to the strong axiom of revealed preference when every choice (x, B) is either co-convex (i.e., B as in Lemma 1 and $B^c \cap R^L_+$ convex) or supportable (i.e., B as in Lemma 1, convex and supported by a unique hyperplane at x). Our Proposition 1 does not require any additional assumption besides those in Lemma 1. On the other hand, the utility function that we construct from Afriat's inequalities need not be concave.

For the convex case, the equivalence of the first two conditions of Proposition 2 above is similar to Matzkin's Theorem 2, but she does not derive the analog of Afriat's inequalities. In our setup, the inequalities follow from the representation of budget sets by means of the g functions, and this leads to a much easier construction of the concave utility function rationalizing the data[4].

To complete the comparison, in the co-convex case our construction immediately yields a concave rationalization of the data. Indeed, if B is co-convex, the function $\gamma_B : R^L_+ \to R$ is concave[5]. Then, if we let $g(x) = \gamma_B(x) - 1$ for all $x \in R^L_+$, the rationalization obtained in (3) of Proposition 1 is concave as a minimum of concave functions. This yields a short and straightforward proof of Matzkin's Theorem 1.

Independently of Matzkin (1991), Chavas and Cox (1993) consider a consumer facing nonlinear budget constraints. They do not formulate any axiom of revealed preference but derive inequalities as in (3) of Proposition 1 directly from the consumer's optimization problem

[4]In the convex case, Assumption H is redundant: when the set B is convex, the property stated in Assumption H follows from the other properties of our class of budget sets.

[5]In the co-convex case, Assumption H is important; for example, the construction suggested in footnote 2 does not deliver a concave representation when $B^c \cap R^L_+$ is convex.

$\max v(x)$ over $\{x \mid g(x) \leq 0\}$, along the lines of our discussion following Proposition 1. More precisely, they assume the convexity of the following set:

$$K = \{(z_0, z_1) \in R^2 \mid \exists x \in R^L_+ \ v(x) \geq z_0, \ g(x) \leq z_1\}.$$

As the following example shows, this assumption is a strong one. It may fail even if v is linear and the set $\{x \mid g(x) \leq 0\}$ is convex (in the example, the function g is quasi-convex, but not convex).

Example: Let $L = 2$, $v(x) = x_1 + x_2$, $g(x) = 5 - \frac{17}{x_1+x_2}$. Both $(5, 2)$ and $(15, 4)$ are in K but $(10, 3)$ is not.

Our proof of Proposition 1 makes clear that the implication (1) $\rightarrow$ (3) does not require convexity assumptions of any sort[6].

For an example of the empirical relevance of budget sets which are neither convex nor co-convex we may refer, as mentioned in the Introduction, to the literature on the estimation of labour supply in which the wage is assumed to be an S-shaped function of hours worked (see e.g. Barzel (1973) Moffit (1984) and Blomquist and Newey (2002)).

2. Uniqueness and Approximation

In the theory of revealed preferences, besides the question of the existence of a rationalization, it is interesting to investigate the issue of uniqueness: can we fully identify the preferences of an individual by observing his behavior? The question has to be made precise. We cannot hope to identify preferences over a non finite choice set from observation of finitely many choices. Also, if we allow the individual to be indifferent among elements of a given set of alternatives, we must be able to observe *all* of his preferred choices at that set of alternatives, not just one. The spirit of the exercise is thus quite different from the 'finite observation' methodology we followed until now. The question has been investigated by Mas Colell ((1977), (1978)) in the case of the classical competitive consumer, facing linear budgets. In this section, we take the strictly positive orthant $X = R^L_{++}$ as our consumption set.

Let $\mathbb{B}$ be the set of *all* non-empty compact and monotone[7] $B \subset R^L_{++}$. Let $h : \mathbb{B} \rightarrow X$ be the individual choice correspondence, with $h(B) \subset B$, $h(B) \neq \emptyset$ for all $B \in \mathbb{B}$.

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[6]Chavas and Cox allow for budget sets of the form $\{x \mid g_1(x) \leq 0, \ldots, g_m(x) \leq 0\}$. This setting is not more general than the one we consider in Proposition 1. Indeed, given that we do not impose any convexity assumptions, we can always rewrite their budget set as $\{x \mid g(x) \leq 0\}$ for $g = \max[g_1, \ldots, g_m]$.

[7]$\forall \ y \in R^L_{++}$, if $\exists \ x \in B$, $y \leq x$, then $y \in B$.

For a given individual choice correspondence h, we say that x is revealed preferred to y, and we write $xR(h)y$, iff there exists $B \in \mathbb{B}$ such that $x, y \in B$ and $x \in h(B)$. We may also define the relation $P(h)$ as $xP(h)y$ iff there exists $B \in \mathbb{B}$ such that $x, y \in B$, $x \in h(B)$, $y \notin h(B)$. The Weak Axiom of Revealed Preference (WARP) can then be stated as follows

Definition (WARP): the individual choice correspondence satisfies WARP if $[xR(h)y] \implies [\neg yP(h)x]$.

We also introduce an additional condition which is natural, given our restriction to monotonic preferences

Definition (Monotonic choice): the individual choice correspondence is monotonic if, for all $B \in \mathbb{B}$, $[x \in h(B), y > x] \implies [y \notin B]$.

A preference relation $\succcurlyeq$ is a reflexive, complete, transitive binary relation on X. $\succcurlyeq$ is monotonic if $x \gg y$ implies $x \succ y$, i.e. $x \succcurlyeq y$ and $\neg y \succcurlyeq x$. $\succcurlyeq$ is upper semicontinuous if $\forall x \in X$ the set $\succcurlyeq (x) = \{y \in X \mid y \succcurlyeq x\}$ is closed in X. We say that a preference relation $\succcurlyeq$ generates the individual choice correspondence h on $\mathbb{B}$ if, for all $B \in \mathbb{B}$, $h(B) = \{x \in B \mid [y \in B] \implies x \succcurlyeq y\}$. This is stronger than simply requiring that $\succcurlyeq$ rationalizes h, which corresponds to the inclusion $\subset$.

2.1. **Uniqueness.** For the case of the competitive consumer, the issue of uniqueness has been settled by Theorem $2'$ of Mas-Colell (1977). In our setting, given the large class of admissible budget sets, both the statement and the proof are simpler. Indeed, under the assumption of monotonic preferences, we can always simulate a choice between any pair of alternatives $y, z \in X$ by proposing to the individual the budget set

$$B_{y,z} = \{x \in X \mid x \leq y\} \cup \{x \in X \mid x \leq z\}.$$

Thanks to this observation, the proof of the next Proposition follows the one given by Arrow (1959) for the case in which the class of admissible budget sets contains all finite sets of up to three alternatives, and we omit it[8].

Proposition 3. *If the individual choice correspondence h is monotonic and satisfies WARP, $R(h)$ is the unique preference relation which generates it.*

2.2. **Approximation.** It is interesting to investigate to what extent one can reconcile the approach above, which assumes that the whole choice correspondence is known, with the one of the previous section,

[8]The same idea has been used by Denicolò and Mariotti (2000) in their study of the Nash solution and social welfare ordering for nonconvex bargaining problems.

in which only finitely many budgets are included in each experiment. This is the question of approximation, first raised, for the case of the competitive consumer, by Mas-Colell (1978). Following his approach, we consider a sequence of finite experiments which becomes richer and richer at every step, and which 'tends' to the whole of $\mathbb{B}$. Let $\mathcal{K}(X)$ be the set of all non empty compact subsets of X. Endowed with the Hausdorff metric, $\mathcal{K}(X)$ is a separable metric space (See e.g. Aliprantis and Border (2000) chapter 3 for definitions and results, especially 3.76 and 3.77, p. 115). $\mathbb{B} \subset \mathcal{K}(X)$ inherits these properties. Let C_n be a collection of n elements of $\mathbb{B}$, and consider an increasing sequence of collection of sets $C_1 \subset C_2 \subset \ldots C_{n-1} \subset C_n \ldots$ such that their union is dense in $\mathbb{B}$: $\overline{\cup_n C_n} = \mathbb{B}$. For each collection C_n, let $\mathcal{R}_n$ be the set of upper semicontinuous, monotonic preference relations which generate h on C_n. We obtain a decreasing sequence $\ldots, \mathcal{R}_n \supset \mathcal{R}_{n+1}, \ldots$. The following Proposition shows that the limit of this sequence is well defined and coincides with the unique preference relation that generates h.

Proposition 4. *If the individual choice correspondence $h : \mathbb{B} \to X$ has closed values, is monotonic and upper hemi-continuous, and satisfies WARP, then $\cap_n \mathcal{R}_n = \{R(h)\}$.*

Proof:

We first show that $R(h) \in \cap_n \mathcal{R}_n$.

As in Proposition 3, $R(h)$ generates h on $\mathbb{B}$.
$R(h)$ is monotonic. Let $z \gg y$. We have to show that $zR(h)y$ and $\neg yR(h)z$. Take $B_z = \{x \in X \mid x \leq z\}$. Clearly, $y, z \in B_z$. If $z \notin h(B_z)$, $\exists x \leq z \; x \in h(B_z)$, contradicting the monotonicity of h; thus $z \in h(B_z)$ and $zR(h)y$. If we also had $yR(h)z$, we would have that for all B containing both z and y, if $z \in h(B)$ then $y \in h(B)$. Again, for $B = B_z$ this would contradict the monotonicity of h.
$R(h)$ is upper semicontinuous. We have to show that, $\forall x \in X$, the set $R(h)(x) = \{y \in X \mid yR(h)x\}$ is closed. Take a sequence $(y_n)_{n\geq 1}$ converging to y, such that $\forall n \; y_n R(h)x$. That is, $\forall n \; \exists B_n \in \mathbb{B}$ such that $y_n, x \in B_n$ and $y_n \in h(B_n)$. If we consider $B_{y_n,x}$, by monotonicity of h and WARP $y_n \in h(B_{y_n,x})$, $\forall n$. The sequence of sets $(B_{y_n,x})_{n\geq 1}$ converges in the Hausdorff metric to $B_{y,x}$, and, by u.h.c. of h, $y \in h(B_{y,x})$.

It remains to show that there is no other element in $\cap_n \mathcal{R}_n$.

Suppose there exists $\succcurlyeq \in \cap_n \mathcal{R}_n$, $\succcurlyeq \neq R(h)$. Then we can find $x, y \in X$ such that $x \succcurlyeq y$ and $x \in R(h)(y)^c$. By u.s.c. of $R(h)$, $R(h)(y)^c$ is an open set. By monotonicity of $\succcurlyeq$, we can take $x, y \in X$ such that $x \succ y$ and $x \in R(h)(y)^c$.

Using again the u.s.c. and monotonicity of both $\succcurlyeq$ and $R(h)$ we can actually claim more. There exists $\eta > 0$ such that, if we define

$$x_\alpha = x + (1-\alpha)\eta\mathbf{1}$$

$$y_\beta = y + (1-\beta)\eta\mathbf{1}$$

then, for all $\alpha \in [0,1]$ and all $\beta \in [0,1]$,

$$y_\beta \notin \succcurlyeq (x_\alpha)$$

$$x_\alpha \notin R(h)(y_\beta).$$

Fix now $\alpha = \beta = \frac{1}{2}$. To simplify notation, let us denote the 'corner' budget $B_{x_{\frac{1}{2}},y_{\frac{1}{2}}}$ simply by $\hat{B}$. For any $\epsilon > 0$, consider the open set around $\hat{B}$ defined by

$$\mathcal{O}^\epsilon = \{F \in \mathbb{B} \mid \mathcal{H}(F,\hat{B}) < \epsilon\}$$

where $\mathcal{H}$ is the Hausdorff distance. We claim that, for any $\epsilon < \frac{\eta}{3}$, if $F \in \mathcal{O}^\epsilon$, then $x \in F$ and $y \in F$. Indeed, if e.g. x did not belong to F, then, by comprehensiveness of F, none of the points $y \geq x$ would be in F. But the closest point z to $x_{\frac{1}{2}}$ for which it is not the case that $z \geq x$ is at distance at least $\frac{\eta}{2}$ from $x_{\frac{1}{2}}$. Clearly, $\epsilon < \frac{\eta}{3} < \frac{\eta}{2}$ and the argument above contradicts the fact that $F \in \mathcal{O}^\epsilon$.

By a similar argument, if $F \in \mathcal{O}^\epsilon$, again for $\epsilon < \frac{\eta}{3}$, then $F \subset B_{x_0,y_0}$. Let us show that $h(F) \subset B_{y_0}$. Take $b \in h(F)$. Since $y \in F$ and $R(h)$ generates h, it must be that $bR(h)y$. On the other hand, $b \in F \subset B_{x_0,y_0} = B_{x_0} \cup B_{y_0}$. Assume $b \in B_{x_0}$; then, by monotonicity of $R(h)$, $x_0R(h)b$. But this and $bR(h)y$ imply $x_0R(h)y$, which is false by construction. Hence $b \in B_{y_0}$.

Using now the fact that $\cup_n C_n$ is dense in $\mathbb{B}$, there exist n and $F \in C_n$ such that $F \in \mathcal{O}^\epsilon$. Let $b \in h(F)$. Because $x \in F$, and $\succcurlyeq$ generates h on C_n, this implies $b \succcurlyeq x$. From $h(F) \subset B_{y_0}$, $y_0 \geq b$, and monotonicity and u.s.c. of $\succcurlyeq$, $y_0 \succcurlyeq b$. We therefore obtain $y_0 \succcurlyeq x$, a contradiction. □

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