Rationally discounting an uncertain future

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Assessing the relative value of a benefit (or harm) in the future vs. the present is a fundamental problem in economics[1]. The conventional approach is to discount the future exponentially. This can be motivated by the opportunity cost relative to an investment paying fixed interest[2]. In contrast, real interest rates vary randomly, so the resulting effective discounting function is no longer exponential, but is rather a function that gives more weight to the far future[3,4]. We extend this by investigating a random interest rate model and showing that the effective discounting function asymptotically decays as a power law. This demonstrates that the proper way to discount over long time horizons is very sensitive to fluctuations in interest rates. It suggests that for problems such as global warming, standard exponentially based present value calculations might underestimate the economic impact of future harm by several orders of magnitude. In a broader context our results suggest that the non-exponential discounting used by humans and many other animals[5,6] may be rational.

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When we address a problem such as global warming we are forced to compare the benefit or harm of an action today, such as an investment in an alternative energy technology, against its consequences in the future, such as environmental improvement[7]. To do an economic analysis we must compare the utility of consumption (which one can think of as having a monetary value) in the future to that in the present. Consider a consumption stream $x = (x_0, x_1, x_2, \ldots)$ for an agent who gets instantaneous utility $u(x_t)$ from consuming at time t. Contemplating at time s her future consumption, her utility $U_s(x)$ is usually assumed to be a sum of the form

$$U_s(x) = u(x_s) + \sum_{t=s+1}^{\infty} D_s(t-s)u(x_t). \tag{1}$$

$D_s(t-s)$ is the *discount function* at time s associated with consumption at a future time t, where $t > s$. It weights the relative importance of the future vs. the present.

A natural justification for discounting is the opportunity cost of foregoing an investment. A dollar at time s can be placed in the bank to collect interest at rate r, and if the interest rate is constant, it will generate $\exp(r(t-s))$ dollars at time t. A dollar at time t is therefore equivalent to $\exp(-r(t-s))$ dollars at time s. Letting $\tau = t - s$, this motivates the exponential discount function $D_s(\tau) = D(\tau) = \exp(-r\tau)$, independent of s.

Real people and animals, in contrast, do not use exponential discounting, but rather give more weight to events that are very immediate or very distant in time, and less weight at intermediate times[5,6,8]. This kind of attitude toward time is referred to as *hyperbolic discounting*, and is often written in the functional form

$$D_s(\tau) = D(\tau) = (1 + \alpha\tau)^{-\beta}, \tag{2}$$

where α and β are constants. This functional form is generally believed to fit empirical data better than an exponential. In the limit as $\tau \to \infty$, $D(\tau)$ is proportional to $\tau^{-\beta}$, i.e. it follows a power law. Our main contribution in this paper is to propose a rational explanation for this.

A dramatic example of hyperbolic discounting was provided by Thaler[8]. He asked a group of subjects how much money they would be willing to accept in the future in lieu of receiving \$15 immediately. The average responses were: One month later \$20, one year later \$50, and ten years later \$100. The exponential function fits Thaler's data poorly, as can be seen by writing the discount factors in the form $D(\tau) = K^{\tau}$. Assuming $u(x) = x$ and measuring time in months, $D(1) = 15/20 = .75^1$, $D(12) = 15/50 = .9^{12}$, and $D(120) = 15/100 = .98^{120}$. The value of K needed varies from 0.75 to 0.98, in contrast to the constant value predicted under exponential discounting. The hyperbolic functional form, in contrast, fits the Thaler data quite well, with $\beta \approx 0.5$.

An attractive property of hyperbolic discounting is that it is time stationary: $D_s(\tau) = D(\tau)$. However, economists have largely been skeptical because it violates time consistent rationality. For example[9], suppose Susy is contemplating how painful it will be to clean her room, and suppose that she uses hyperbolic discounting as in Eq. (2) with $\alpha = \beta = 1$. Since $1/D_0(1) = 2$, she feels that it is twice as bad to clean the room today as opposed to postponing it until tomorrow, but she views it as only $1.003 = D_0(365)/D_0(366)$ times as bad to clean it in $t = 365$ days vs. $t + 1 = 366$ days. The inconsistency comes because when she is asked one year later about cleaning her room on those same two days, she will give a different answer: Just as before, she will say it is twice as bad to clean her room immediately rather than do it the next day: $1/D_{365}(1) = 1/D_0(1) = 2$. The overwhelming empirical evidence that real people like Susy use hyperbolic discounting is thus often viewed as contradicting rationality.

More formally, we call the utility *time consistent* if for all s

$$U_s(x) = u(x_s) + D_s(1)U_{s+1}(x). \tag{3}$$

This says that the utility U_s from contemplating consumption stream x at time s is equal to the utility of consuming at time s plus the utility of contemplating that same stream at time $s+1$, discounted by one time period. Time consistency requires the expected utility at one time to be consistent with the expected utility at another time. Writing the one period discount factors in the form $D_s(1) = e^{-r_s}$, where r_s is the *discount rate* at time s, in the Supplementary Material we show that time consistency is equivalent to $D_s(\tau + 1) = D_s(\tau)D_{s+\tau}(1)$ or

$$D_s(\tau) = D_s(1) \dots D_{s+\tau-1}(1) = e^{-r_s} \dots e^{-r_{s+\tau-1}}. \tag{4}$$

The discount rate r_t can be thought of as an interest rate, or it can be thought of as a psychological state representing the attitude of an agent on day t about receiving utility on day $t+1$. For example, in the Thaler experiment an agent's patience might vary from day to day. Alternatively, an agent might update her probability that Thaler will flee to Brazil that night, or that she herself might die that night, in which case r_t can be interpreted as a hazard rate.

Now consider a duller Susy who has the same hyperbolic utility as Susy starting from time 0, but has time consistent utility. What does her sequence of one-period discount rates need to be? Today $D_0(1) = 1/2$, i.e. $r_0 = 0.7$, so she is very impatient. A year ahead, though, $D_{365}(1) = 1/1.003$, i.e. $r_{365} = 0.003$, so she is very patient. Thus, unlike original Susy, duller Susy must get steadily more patient, with her one-period discount rates trending toward zero. She certainly is not time stationary.

We take it as axiomatic that people do not systematically grow more patient with time; for one thing death hazard rates on average grow with age, indicating that a rational person should grow less patient. In a world of certainty, the argument of the previous paragraph shows that growing impatience and time consistency rule out hyperbolic discounting.

In contrast, we shall now show that if Susy's one-period discount rate is stochastic, she can be hyperbolic and time consistent even if her impatience is increasing on average.. When Susy's psychological state r_s follows a stochastic Markov process, her utility $U_{s,r_s}(x)$ and her discounting $D_{s,r_s}(\tau)$ both become state dependent. Time consistency now takes the form

$$U_{s,r_s}(x) = u(x_s) + D_s(1) \sum_{r_{s+1}} P(r_{s+1}|r_s)U_{s+1,r_{s+1}}(x), \tag{5}$$

where $P(r_{s+1}|r_s)$ is the probability of interest rate r_{s+1} given interest rate r_s. Define a feasible path as any sequence of possible future discount rates $\vec{r} = \{r_{s+1}, r_{s+2}, \dots, r_{s+\tau-1}\}$. Assuming that P gives equal probability for each of the $N(\tau)$ feasible paths $\vec{r}$, define the *certainty equivalent discount function* $\bar{D}_{s,r_s}(\tau)$ as the average over all possible paths, i.e.

$$\bar{D}_{s,r_s}(\tau) = \frac{1}{N(\tau)} \sum_{\vec{r}} e^{-r_s} e^{-r_{s+1}} \dots e^{-r_{s+\tau-1}}. \tag{6}$$

In the supplementary material we show that time consistency implies

$$U_{s,r_s}(x) = u(x_s) + \sum_{\tau=1}^{\infty} \bar{D}_{s,r_s}(\tau)u(x_{s+\tau}). \tag{7}$$

This is the same as Eq. (1), but with the important difference that the discount rate is now the certainty equivalent rate, which depends on the psychological state r_s.

Following a long tradition in finance[10], Weitzman observed that serially correlated uncertainty in interest rates yields less discounting in the long run than when interest rates are certain at the mean level[3]. To take the simplest example, assume that in the future there are two possible feasible paths, each of which has constant interest rate, either r or r', with $r < r'$. Then the certainty equivalent discount function is $\bar{D}_{0,r_0}(\tau) = e^{-r_0}(e^{-r(\tau-1)} + e^{-r'(\tau-1)})/2$. For sufficiently large times $\bar{D}(\tau) \sim \exp(-r\tau)$, i.e. the smaller interest rate dominates.

The *geometric random walk* is the most standard stochastic model of interest rates. At each timestep the current interest rate is either multiplied by a factor e^v, yielding $r_{t+1} = r_t e^v$, or divided by the same factor, $r_{t+1} = r_t e^{-v}$. The two choices have equal probability. If the initial interest rate r_0 is positive, r_t is always positive. The geometric mean is constant and the arithmetic mean is an increasing function of time.

Thus if her discount rates follow a geometric random walk, stochastic Susy can have time consistent utility even though her average discount rate is increasing. As we now show, she will display hyperbolic discounting with $\beta = 1/2$ from every starting point, and yet at the same time display growing impatience on average. In a stochastic world hyperbolic discounting is compatible with rationality (time consistency) and exponential discounting is not.

Under the geometric random walk the one-period interest rate r_t at any given time t is a Markov process, so the certainty equivalent discount function $\bar{D}_{s,r}(\tau) = \bar{D}_r(\tau)$ is time stationary. Each r_t is determined randomly by the number of increases minus the number of decreases along the path leading up to r_t. One can therefore visualize the possible states (t, r) as the nodes in a recombining binary tree in which each interior node at a given level comes from a pair of nodes at the previous level, as shown in Figure 1. (Note the tree is turned on its side). Within the tree interest rates are constant horizontally and increase vertically at an exponential rate.

The certainty equivalent discount $\bar{D}_r(\tau)$ can be computed numerically using equation 6. An example is shown in Figure 2. The parameters are chosen so that a single time step of the simulation corresponds to a year, with an initial interest rate of $r_0 = 4\%$ and v chosen so that $e^v = 1.5$, corresponding to an annual volatility of 50%. For roughly the first eighty years the average discount rate stays fairly close to the exponential, but afterward the two diverge substantially, with the geometric random walk giving a much larger weight to the future. A comparison using more realistic parameters is given in Table 1. For large times the difference is dramatic.

In Figure 2 we plotted the result in double logarithmic scale to highlight that for large times the average discounting function approaches a power law, corresponding to a straight line on double logarithmic scale. In this

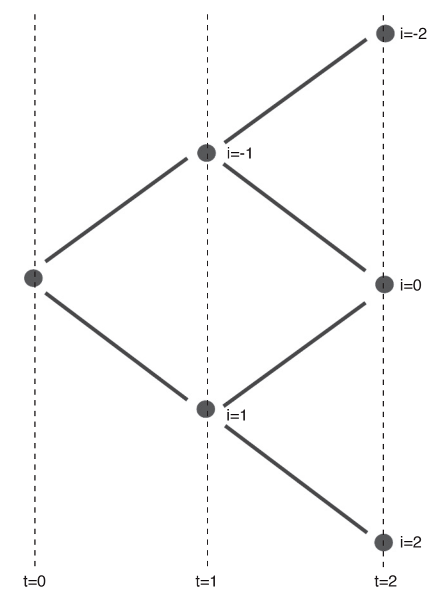

Figure 1: A schematic representation of the possible future interest rates under the geometric random walk. Each level of the tree corresponds to the time $t = 0, \ldots, \tau$, which increases to the right, while vertically the interest rates increase exponentially.

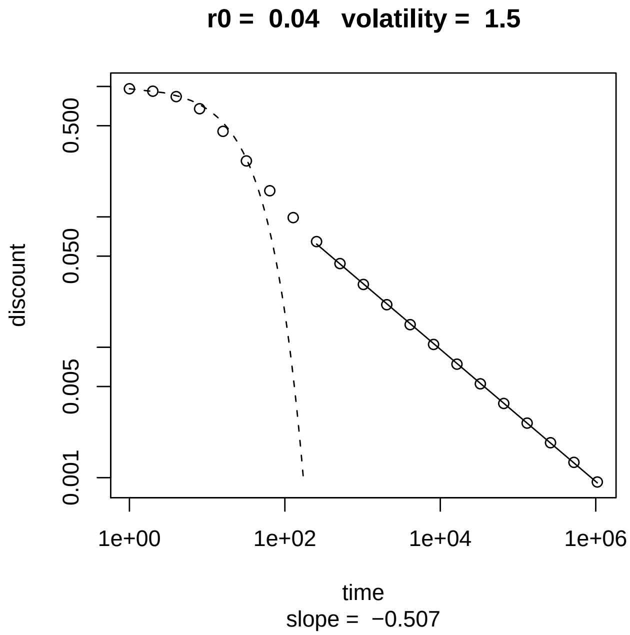

Figure 2: Certainty equivalent discount function vs. time for the geometric random walk, plotted in double logarithmic scale. The parameters correspond to an annual interest rate of 4% and an annual volatility of 50%. The dashed line corresponds to exponential discounting, and the solid line is a least squares fit to the indicated part of the tail.

year	GRW	exponential
20	0.462	0.456
60	0.12	0.095
100	0.51	0.20
500	0.008	2×10^{-9}
1000	0.005	4×10^{-18}

Table 1: Comparison of effective discounting functions $D_{r_0}(\tau)$ on different time horizons τ. The first column is the time in years; the second column is the certainty equivalent discounting function for the geometric random walk with an initial interest rate of 4% and a volatility of 15%; the third column is for exponential discounting with the same rate.

case the exponent is -0.507. We have performed many simulations with different parameters, and providing we simulate the discounting out to a sufficiently large horizon, we always observe power law tails with exponents near $-1/2$.

In the limit as $\tau \to \infty$ it is possible to prove that for all $r_0 > 0$, and all volatilities $v > 0$, $\bar{D}_{r_0}(\tau)$ satisfies hyperbolic discounting with $\beta = 1/2$. The simplest case is when the volatility v is so large that it can be considered infinite. In this case the tree can be divided into two regions: In the region below the median the interest rate is $r = r_0 e^{-\infty} = 0$, and in the region above the median it is $r = r_0 e^{\infty} = \infty$. The interest rate paths can be divided into three groups: (1) Paths that remain entirely below the median, which experience no discounting beyond the initial e^{-r_0}. (2) Paths that at any point go above the median; these experience infinite rates and thus contribute nothing. (3) Paths that hit the median exactly k times but never cross above it. This is thus a classic barrier crossing problem[11]. As shown in the supplementary material, by making repeated use of the reflection principle it is possible to place accurate bounds on $\bar{D}_{r_0}(t)$ and show that in the large time limit it goes as $t^{-1/2}$.

For volatilities that are not infinite the analysis becomes more complicated but the behavior remains essentially the same. The difference is that the dividing line between the two regions is no longer sharp because there is a band down the center where the interest rate can no longer be considered to be either zero or infinity. Nonetheless, it is still possible to show that the asymptotic scaling goes as $t^{-1/2}$ up to logarithmic corrections.

To demonstrate that the generalized random walk might generally apply outside the realm of interest rate modeling, we fit the experimental results obtained by Thaler[8] mentioned earlier. The result is shown in Figure 3. This also illustrates that the geometric random walk has the interesting property that, depending on the parameters, the certainty equivalent discount rate $\bar{r}(\tau) \equiv \log \bar{D}_{r_0}(\tau)/\tau$ can first increase and then decrease as a function of time[10].

How good a model is the geometric random walk for real interest rates? During the last 200 years the real interest rate for United States long bonds has varied from roughly 2% to roughly 8%. Newell and Pizer compared the geometric random walk model to several other

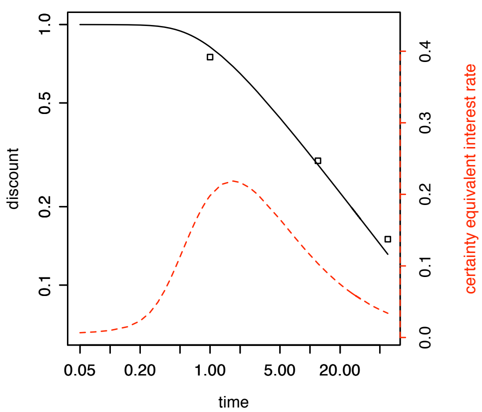

Figure 3: A comparison of Thaler's data on discounting (squares) to a fit of the discount function using the geometric random walk (solid line). The dashed plot shows the certainty equivalent discount rate $\bar{r}(\tau)$, which first increases and then decreases. Time is measured in months, with parameters $r_0 = 0.005$ and $v = 2.2$; the simulation is done by dividing each month into 100 subintervals.

stochastic interest rate models, including those with mean reversion, and found that it provided the best fit[4]. The constant interest rate model, in contrast, is obviously a much poorer approximation. The current economic crisis has reminded us that it is not uncommon for real interest rates to go to zero or even become negative.

The difference between the geometric random walk model and exponential discounting becomes stark when one considers really long horizons. Suppose we compare the cumulative weight that the two models give to the future beyond a given time τ by taking the integral of $D_{r_0}(t)$ from τ to ∞. For $\tau = 100$ years and an annual interest rate of 4%, under exponential discounting the far future only gets a weight of 2%. In contrast, when the same calculation is made for the geometric random walk model for interest rates, the contribution for the far future is always infinite, due to the fact that the integral to infinity of the function $t^{-1/2}$ is infinite.

We do not assert that the geometric random walk is necessarily the best model for interest rates. Allowing for strong mean reversion would destroy the hyperbolic discounting. What this analysis makes clear, however, is that the long term behavior of valuations depends extremely sensitively on the interest rate model. The fact that the present value of actions that affect the far future can shift from a few percent to infinity when we move from a constant interest rate to a geometric random walk calls seriously into question many well regarded analyses of the economic consequences of global warming. For example, Nordhaus[12] has used exponential discounting at 6% to evaluate long term climate effects, while the Stern report has used a discounting rate of $1/2\%$. Our results indicate that the correct way to discount is not even exponential. But if one has to choose between these two exponentials, our work, like Weitzman's[3], suggests the smaller discount rate is both more accurate and more rational.

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Acknowledgements We would like to thank Barclays Bank, Bill Miller, and National Science Foundation grant 0624351 for support, and Tanya Elliot for useful comments.

Competing Interests The authors declare that they have no competing financial interests.

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