A Derivation of Expected Utility Maximization in the Context of a Game*

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Abstract

A decision maker faces a decision problem, or a game against nature. For each probability distribution over the state of the world (nature's strategies), she has a weak order over her acts (pure strategies). We formulate conditions on these weak orders guaranteeing that they can be jointly represented by expected utility maximization with respect to an almost-unique state-dependent utility, that is, a matrix assigning real numbers to act-state pairs. As opposed to a utility function that is derived in another context, the utility matrix derived in the game will incorporate all psychological or sociological determinants of well-being that result from the very fact that the outcomes are obtained in a given game.

1 Introduction

1.1 Motivation

Do players maximize expected utility when playing a game? The experimental outcomes involving ultimatum and dictator games might seem to suggest that they do not. (See Guth and Tietz (1990) and Roth (1992) for surveys.)

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For instance, a player who moves second in an ultimatum game, and rejects an offer of a positive amount of money, evidently does not maximize her monetary payoff. Similarly, a dictator in a dictator game, who chooses to leave some money to her dummy opponent, fails to maximize her payoff under conditions of certainty, let alone her expected payoff under conditions of uncertainty.

Some authors argue that these experimental results constitute a violation of game theoretic predictions. Indeed, if one insists that the utility function be defined over monetary payoffs alone, such a conclusion appears unavoidable. But many game theorists hold that the utility function need not be defined on monetary prizes alone. Indeed, an "outcome" should specify all the relevant features of the situation, including feelings of envy, guilt, preferences for fairness, and so forth. Moreover, recent developments in economic theory call for explicit modeling of such determinants of utility. (See, for instance, Frank (1989), Elster (1998), Rabin (1998), and Loewenstein (2000).) Further, if one adopts a purely behavioral approach, one has no choice but to incorporate into the utility function all psychological and sociological effects on well-being. The very fact that, say, a dictator prefers taking less money to taking more money implies that the utility of the former exceeds that of the latter. As long as players satisfy the axioms of von-Neumann and Morgenstern (vNM, 1944), they can be described as if they are maximizing the expected value of an appropriately chosen utility function. From this viewpoint, the experimental results of dictator and ultimatum games might challenge the implicit assumption that monetary payoff is the sole determinant of utility, but not the assumption of expected utility maximization itself.

We find that this argument is essentially correct: the debate aroused by dictator and ultimatum games is about determinants of the utility function, not about expected utility theory (EUT). Yet, we do not believe that vNM's axiomatic derivation of EUT is a very compelling argument in this context.

vNM's result assumes a preference relation over lotteries with given probabilities, and derives a utility function over outcomes, such that the maximization of its expectation represents preferences over lotteries. vNM then assumed that, when players evaluate mixed strategies in a game, they use the same utility function for the calculation of expected payoff, and attempt to maximize this expectation. Thus, the vNM derivation implicitly assumes that the utility function that one obtains in the context of a single person decision problem will apply to the context of a game.

This assumption seems implausible precisely in the context of games such as ultimatum and dictator, where utility is heavily dependent on interpersonal comparisons and interactions. For instance, if player two considers an outcome of 10% of the pie, she cannot ignore the fact that player one is about to pocket 9 times as much. Similarly, player one (the dictator) in a dictator game cannot be assumed to treat the outcome "I get $90" as equivalent to "I chose to take $90 and to leave $10 to my opponent." Preferences over fairness distinguish the former from the latter. Moreover, the very fact that the dictator has *chosen* a particular division of the money implies that she might experience guilt even if she has no preference for fairness *per se*. Finally, suppose that player two in an ultimatum game chooses to reject an offer not because it is unfair, but because she finds it insulting. That is, she does not envy player one, but she finds that he should be punished for his greed. In this case, she distinguishes between "I get $10, player one gets $90, and this was decided by Nature" and "I get $10, player one gets $90, and this was decided by Player one." Such distinctions are precisely about the difference between a single-player decision problem and a game. If one were to measure a player's utility over such outcomes in a laboratory, one would have to generate outcomes that simulate all the interactive effects of a game. That is, one would have to measure utility *in the context of the game itself.*

Similar issues arise when a single player is concerned. Consider, for instance, the effect of regret. It has long been argued that regret may color the way individuals evaluate outcomes. (See, for instance, Luce and Raiffa (1957), Loomes and Sugden (1982), and Gul (1991).) Thus, the utility function of a certain outcome, when measured in isolation, may not reflect the way this outcome is perceived in a game. "Getting $10" is not the same as "Getting $10 when I could have gotten $20." In order to measure the relevant utility of the latter, one would have to simulate the entire choice situation, that is, to measure utility in the context of the game.

In order to defend the expected utility paradigm in face of experimental evidence as well as of the theoretical considerations mentioned above, it does not suffice to show that it *can* explain the data with an appropriate definition of the utility function. One needs to show that this new definition also relies on sound axiomatic foundations. That is, one needs an axiomatic derivation of EUT that would parallel that of vNM, but will only use preferences in the game itself as data.

1.2 The present contribution

In this paper we axiomatize expected utility maximization in a two-person game. We assume that every player can rank his pure strategies, given *any* mixed strategy of the other player. That is, we assume that for every mixed strategy of the opponent, the player has a weak order over her pure strategies. Equivalently, one may consider a single person decision problem under uncertainty (a "game against nature"), where, for each vector of probabilities over the state of nature (representing the decision maker's beliefs), the decision maker has a weak order over the possible acts. The set of acts may be finite or infinite, and it is not assumed to have any algebraic, topological, or other structure.

Pairs of acts and states (or combinations of pure strategies) can be thought of as defining outcomes. We do not assume that the player can compare any pair of lotteries defined over the possible outcomes. Rather, we assume that the decision maker can compare only lotteries derived from two rows in the matrix for the *same* probability vector over the columns. In particular, the data assumed in our results will not include a comparison of a certain outcome (i.e., a degenerate lottery) to a non-degenerate lottery.

We assume that the rankings over the acts (the player's pure strategies) satisfy two axioms that relate preferences given different beliefs (different mixed strategies of the opponent): first, we assume *convexity*: if act a is preferred to b given probability p, as well as given probability q, then the same preference will be observed for any convex combination of p and q. Second, we assume *continuity*: if a is strictly preferred to b given belief p, the same preference should hold in a neighborhood of p. Finally, we also need an axiom of *diversity*, requiring that any four pure acts can be ranked, in any given strict order, for at least one belief vector p.[1] These axioms imply that there is a utility matrix, such that, for every belief (opponent's mixed strategy) p, the decision maker (player) ranks her acts (pure strategies) according to their expected utility, computed for the relevant p.

The utility matrix we obtain treats every entry in the game as a different outcome. More precisely, the formal model does not have a separate notion of outcomes. Utility is defined over act-state pairs (pairs of pure strategies). In this sense, the utility function we derive is state-dependent. (See Dreze (1961) and Karni, Schmeidler, and Vind (1983).) This also implies that there is some freedom in the choice of the utility function: one may add a separate constant to each column in the matrix without changing the expected utility rankings. Indeed, our utility matrix is unique up to such shifts, and up to multiplication of the entire matrix by a positive number.

The diversity axiom implies that the matrix we obtain satisfies a certain condition, which we dub "diversification": no row in the matrix is dominated by an affine combination of (up to) three other rows in it. In particular, it does not allow domination relations between pure strategies. However, in the absence of the diversity assumption, the other axioms do not imply the

[1] See the following section for a more precise formulation of the axioms.

existence of the numerical representation we seek.

2 Results

The results presented in this section are reminiscent of the main results in Gilboa and Schmeidler (1997, 1999). All these results derive a representation of a family of weak orders by a matrix of real numbers, as follows. The objects to be ranked corresponds to rows in the matrix. A "context", which induces a weak order over these objects, is defined by a vector, attaching a real number to each column. Given such a context, the ranking corresponding to it is represented by the inner products of the context with each of the rows in the matrix. While our new results are similar in spirit to those in previous papers, some differences exist. In particular, the extension of our result to an infinite state space is new.

2.1 The result for finite spaces

Assume that a decision maker is facing a decision problem with a non-empty set of *acts* A and a finite, non-empty set of *states of the world* Ω. Such problems are often represented by a "decision matrix", or a "game against nature", attaching an outcome to each act-state pair (a, ω). We do not assume any knowledge about this set of outcomes or about the structure of the matrix, and hence suppress it completely. (Equivalently, one introduces a formal set of abstract outcomes that is simply the set of pairs $A \times \Omega$.) Let $\Delta = \Delta(\Omega)$ be the set of *probability distributions* on Ω. We assume that, for every probability vector $p \in \Delta$, the decision maker has a binary preference relation $\succsim_p$ over A. We now formulate axioms on $\{\succsim_p\}_{p \in \Delta}$:

A1 Order: For every $p \in \Delta$, $\succsim_p$ is complete and transitive on A.

A2 Combination: For every $p, q \in$ ¢ and every $a, b \in A$, if $a \succsim_p b$ ($a \succ_p b$) and $a \succsim_q b$, then $a \succsim_{\alpha p+(1-\alpha)q} b$ ($a \succ_{\alpha p+(1-\alpha)q} b$) for every $\alpha \in (0, 1)$.

A3 Archimedean Axiom: For every $a, b \in A$ and $p \in \Delta$, if $a \succ_p b$, then for every $q \in \Delta$ there exists $\alpha \in (0,1)$ such that, $a \succ_{\alpha p+(1-\alpha)q} b$.

A4 Diversity: For every list (a, b, c, d) of distinct elements of A there exists $p \in \Delta$ such that $a \succ_p b \succ_p c \succ_p d$. If $|A| < 4$, then for any strict ordering of the elements of A there exists $p \in \Delta$ such that $\succ_p$ is that ordering.

We need the following definition: a matrix of real numbers is called *diversified* if no row in it is dominated by an affine combination of three (or less) other rows in it. Formally:

Definition: A matrix $u : A \times Y \to \mathbb{R}$, where $|A| \geq 4$, is *diversified* if there are no distinct four elements $a, b, c, d \in A$ and $\lambda, \mu, \theta \in \mathbb{R}$ with $\lambda + \mu + \theta = 1$ such that $u(a, \cdot) \leq \lambda u(b, \cdot)) + \mu u(c, \cdot) + \theta u(d, \cdot)$. If $|A| < 4$, u is diversified if no row in u is dominated by an affine combination of the others.

Theorem 1 *: The following two statements are equivalent:*

(i) $\{\succsim_p\}_{p \in \Delta}$ *satisfy A1 - A4;*

(ii) There is a diversified matrix $u : A \times \Omega \to R$ *such that:*

$$(*) \qquad \begin{array}{c} \text{for every } p \in \Delta \text{ and every } a, b \in A, \\ a \succsim_p b \quad \text{iff} \quad \sum_{\omega \in \Omega} p(\omega) u(a, \omega) \geq \sum_{\omega \in \Omega} p(\omega) u(b, \omega) \end{array},$$

Furthermore, *in this case the matrix* u *is unique in the following sense:* u *and* w *both satisfy* $(*)$ *iff there are a scalar* $\lambda > 0$ *and a matrix* $v : A \times \Omega \to R$ *with identical rows (i.e., with constant columns) such that* $w = \lambda u + v$ *.*

Theorem 1 follows easily from its more general version, Theorem 2 below.

We do not know of a set of axioms that are necessary and sufficient for a representation as in $(*)$ by a matrix u that need not be diversified. We do know that dropping A4 will not do. (The counter-examples in Gilboa and Schmeidler (1997, 1999) can be easily adapted to our case.) It will be clear from the proof that weaker versions of A4 suffice for a representation as in $(*)$. Ashkenazi and Lehrer (2001) also offer a condition that is weaker than A4, and that also suffices for such a representation. The diversity axiom is stated here in its simplest and most elegant form, rather than in its mathematically weakest form.

2.2 The general result

Assume that a decision maker is facing a decision problem with a non-empty set of acts A and a measurable space of states of the world (Ω, Σ), where Σ is a σ-algebra of subsets of Ω. Further, assume that Σ includes all singletons. Let $\mathsf{B}(\Omega, \Sigma)$ be the space of bounded Σ-measurable real-valued functions on Ω. Recall that $\mathsf{ba}(\Omega, \Sigma)$, the space of finitely additive bounded measures on Σ, is the dual of $\mathsf{B}(\Omega, \Sigma)$. Let P denote the subset of $\mathsf{ba}(\Omega, \Sigma)$ consisting of finitely additive probability measures on Σ. Assume that, for every probability measure $p \in \mathsf{P}$, the decision maker has a binary preference relation $\succsim_p$ over A. The axioms on $\{\succsim_p\}_{p \in \mathsf{P}}$ now take the following form:[2]

A1 Order: For every $p \in \mathsf{P}$, $\succsim_p$ is complete and transitive on A.

A2 Combination: For every $p, q \in \mathsf{P}$ and every $a, b \in A$, if $a \succsim_p b$ ($a \succ_p b$) and $a \succsim_q b$, then $a \succsim_{\alpha p + (1-\alpha) q} b$ ($a \succ_{\alpha p + (1-\alpha) q} b$) for every $\alpha \in (0, 1)$.

A3* Continuity: For every $a, b \in A$ the set $\{p \in \mathsf{P} \mid a \succ_p b\}$ is open in the relative weak* topology.

A4 Diversity: For every list (a, b, c, d) of distinct elements of A there exists $p \in \mathsf{P}$ such that $a \succ_p b \succ_p c \succ_p d$. If $|A| < 4$, then for any strict ordering of the elements of A there exists $p \in \mathsf{P}$ such that $\succ_p$is that ordering.

We can now state

Theorem 2 *: The following two statements are equivalent :*

[2] Axioms 1,2, and 4 are literally identical to those of the finite case (apart from the space of probability distributions over Ω). They are also denoted as above.

(i) $\{\succsim_p\}_{p\in\mathcal{P}}$ *satisfy A1,A2,A3*,A4;*

(ii) For every $a \in A$ *there exists a* $u(a,\cdot) \in B(\Omega,\Sigma)$ *such that:*

$$(**)\qquad \begin{array}{c} \text{for every } p \in \mathcal{P} \text{ and every } a, b \in A, \\ a \succsim_p b \quad \text{iff} \quad \int_\Omega u(a,\cdot)dp \geq \int_\Omega u(b,\cdot)dp\ , \end{array}$$

and the matrix $u(\cdot,\cdot)$ *is diversified.*

Furthermore, *in this case the functions* $\{u(a,\cdot)\}_{a\in A}$ *are unique in the following sense:* $\{u(a,\cdot)\}_{a\in A}$ *and* $\{w(a,\cdot)\}_{a\in A}$ *both satisfy* $(**)$ *iff there are a scalar* $\lambda > 0$ *and a function* $v \in B(\Omega,\Sigma)$ *such that* $w(a,\cdot) = \lambda u(a,\cdot) + v$ *for all* $a \in A$.

The proof of this theorem is given in an appendix.

3 Discussion

Fishburn (1976) and Fishburn and Roberts (1978) provide derivations of expected utility maximization in the context of a game. In these papers, a player is assumed to have preferences over lotteries that are generated by her own mixed strategies and by mixed strategies of the opponents. These results do not suffice for our purposes for two reasons. First, they assume that all lotteries, obtained by independent mixed strategies, can be compared. But each player in a game can only choose her own strategies. Thus, to make such preferences observable one would, again, have to resort to experimental settings that are external to the game. Second, a player's preference over her own mixed strategies has been criticized as shaky data. It is not clear when a player's actual choices (of pure strategies) reflect preferences over mixed strategies. Moreover, it has been argued that players never actually play mixed strategies. (See Rubinstein (2000).)

Our results are limited in other ways, however. First, as mentioned above, our result can only produce "diversified" utility matrices. Second, our result for a 2-person game assumes that a player's preferences are given for all mixed strategies of her opponent. In an n-player game, this would include all correlated strategies of the $(n-1)$ opponents, and not only those obtained by independent mixing. Third, because our result does not assume preferences over a player's own mixed strategies, it cannot represent such preferences. Thus, it does not imply that the utility matrix we obtain for a given player can be used to evaluate a mixed strategy of that player, should she consider actual randomization.

The last two restrictions do not pose major difficulties if one views a mixed strategy of a player merely as the beliefs of other players regarding her (pure strategy) choice. (See Aumann and Brandenburger (1995).) Adopting this interpretation, a player's belief about her opponents' joint strategy need not reflect independent mixing on part of the other players. Rather, each player may believe that the other players actions depend on some correlation device, and that they are independent only given the outcome of this device. This implies that the set of possible beliefs contains all correlated strategies of the opponents. More concretely, if there is an experiment scenario in which a player can be made to believe that the other players play according to a joint distribution p, and another scenario that induces belief q, one can easily generate a scenario in which the player would be led to have beliefs $\alpha p + (1-\alpha)q$: the player should be told that the first scenario was played out with probability α, and the second – with the complementary probability. Thus, according to this interpretation of mixed strategies, a player need not have preferences over her own mixed strategies, but she should be able to entertain any beliefs over the strategies of her opponents.

The assumption that a player can rank pure strategies given any belief over the opponents' strategies introduces another limitation. If a player is matched with another player, whom she knows, there might be beliefs p that the player would not entertain regarding her opponent. For instance, if Mary is happily married to John, and the two are matched to play an ultimatum game, Mary might be convinced that John, as player 1, will never make an ungenerous offer. In this case, we will never know what she would do if she did believe that John is ungenerous. In particular, we will not be able to tell whether Mary is nice to John because he is generous in his dealings with her, or because she will like him even if he treats her badly. But if players are anonymously matched with other players, and each player is given statistical data regarding past plays of the game by other players from the same population, it is reasonable that any probability vector p might be induced as a player's beliefs.

Appendix: Proof

Proof of Theorem 2:

Theorem 2 is reminiscent of the main results in Gilboa and Schmeidler (1997, 1999). Although the spaces discussed are different, some steps in the proof are practically identical. For the sake of completeness, we provide here a complete proof.

We present the proof for the case $|A| \geq 4$. The proofs for the cases $|A| = 2$ and $|A| = 3$ will be described as by-products along the way. For $u \in \mathsf{B}(\Omega, \Sigma)$ and $p \in \mathsf{P}$, let $u \cdot p$ denote $\int_\Omega u dp$.

The following notation will be convenient for stating the first lemma. For every $a, b \in A$ let

$$Y^{ab} \equiv \{p \in \mathsf{P} \mid a \succ_p b\} \text{ and}$$
$$W^{ab} \equiv \{p \in \mathsf{P} \mid a \succsim_p b\}.$$

Observe that by definition and A1: $Y^{ab} \subset W^{ab}$, $W^{ab} \cap Y^{ba} = \emptyset$, and $W^{ab} \cup Y^{ba} = \mathsf{P}$. The first main step in the proof of the theorem is:

Lemma 1 *For every distinct $a, b \in A$ there exists $u^{ab} \in \mathsf{B}(\Omega, \Sigma)$ such that,*

(i) $W^{ab} = \{p \in \mathsf{P} \mid u^{ab} \cdot p \geq 0\}$;
(ii) $Y^{ab} = \{p \in \mathsf{P} \mid u^{ab} \cdot p > 0\}$;
(iii) $W^{ba} = \{p \in \mathsf{P} \mid u^{ab} \cdot p \leq 0\}$;
(iv) $Y^{ba} = \{p \in \mathsf{P} \mid u^{ab} \cdot p < 0\}$;
(v) Neither $u^{ab} \leq 0$ nor $u^{ab} \geq 0$;
(vi) $-u^{ab} = u^{ba}$.

Moreover, the function u^{ab} satisfying (i)-(iv), is unique up to multiplication by a positive number.

The lemma states that we can associate with every pair of distinct acts $a, b \in A$ a separating hyperplane defined by $u^{ab} \cdot p = 0$ ($p \in \mathsf{P}$), such that $a \succsim_p b$ iff p is on a given side of the plane (i.e., iff $u^{ab} \cdot p \geq 0$). Observe that if there are only two acts, Lemma 1 completes the proof of sufficiency: for instance, one may set $u^a = u^{ab}$ and $u^b = 0$. It then follows that $a \succsim_p b$ iff $u^{ab} \cdot p \geq 0$, i.e., iff $u^a \cdot p \geq u^b \cdot p$. More generally, we will show in the following lemmata that one can find a function u^a for every act a, such that, for every $a, b \in A$, u^{ab} is a positive multiple of $(u^a - u^b)$.

For a subset B of P let $int(B)$ denote the set of interior points of B (relative to P).

Proof of Lemma 1:

The continuity axiom implies that the sets Y^{ba} are open (in the relative topology). This, in turn, implies that the sets W^{ab} are closed and therefore compact in the weak* topology. This allows the use of a (weak) separating hyperplane theorem between two disjoint and convex sets, one of which is compact: the convex hull of W^{ab} and the origin (i.e., $\{\alpha p \mid \alpha \in [0,1], p \in W^{ab}\}$) on the one hand, and $\{\alpha p \mid \alpha \in (0,1],\ p \in Y^{ba}\}$ on the other. That is, we obtain a non-zero function $u^{ab} \in \mathsf{B}(\Omega, \Sigma)$ such that $u^{ab} \cdot p \geq 0$ for all $p \in W^{ab}$ and $u^{ab} \cdot p \leq 0$ for all $p \in Y^{ba}$. Further, we argue that u^{ab} does not vanish on $W^{ab} \cup Y^{ba} = \mathsf{ba}^1_+(\Omega, \Sigma)$. If it did, then it would also vanish on $\mathsf{ba}_+(\Omega, \Sigma)$, and therefore also on $\mathsf{ba}_-(\Omega, \Sigma)$. But in this case it would vanish on all of $\mathsf{ba}(\Omega, \Sigma)$, in view of Jordan's decomposition theorem, in contradiction to the fact that u^{ab} is non-zero.

We argue that for some $p \in Y^{ba}$, $u^{ab} \cdot p < 0$. If not, $u^{ab} \cdot p = 0$ for all $p \in Y^{ba}$. Since u^{ab} does not vanish on $W^{ab} \cup Y^{ba}$, there has to exist a $q \in W^{ab}$ with $u^{ab} \cdot q > 0$. But then for all $\varepsilon > 0$, $u^{ab} \cdot (\varepsilon q + (1-\varepsilon)p) > 0$, while $\varepsilon q + (1-\varepsilon)p \in Y^{ba}$ for small enough ε by the continuity axiom. Next, we argue that for all $q \in Y^{ab}$ we have $u^{ab} \cdot q > 0$. Indeed, if $u^{ab} \cdot q = 0$ for $q \in Y^{ab}$, $u^{ab} \cdot (\varepsilon p + (1-\varepsilon)q) < 0$ for all $\varepsilon > 0$. By a similar argument, $u^{ab} \cdot p < 0$ for all $p \in Y^{ba}$.

Thus $Y^{ba} \subset \{p \mid u^{ab} \cdot p < 0\}$. Since we also have $W^{ab} \subset \{p \mid u^{ab} \cdot p \geq 0\}$, $Y^{ba} \supset \{p \mid u^{ab} \cdot p < 0\}$. That is, $Y^{ba} = \{p \mid u^{ab} \cdot p < 0\}$ and $W^{ab} = \{p \mid u^{ab} \cdot p \geq 0\}$. We have also shown that $Y^{ab} \subset \{p \mid u^{ab} \cdot p > 0\}$. To show the converse inclusion, assume that $u^{ab} \cdot p > 0$ but $a \approx_p b$. Choose $q \in Y^{ba}$. By the combination axiom, $\alpha p + (1-\alpha)q \in Y^{ba}$ for all $\alpha \in (0,1)$. But for α close enough to 1 we have $u^{ab} \cdot (\alpha p + (1-\alpha)q) > 0$, a contradiction. Hence $Y^{ab} = \{p \,|\, u^{ab} \cdot p > 0\}$ and $W^{ba} = \{p \,|\, u^{ab} \cdot p \leq 0\}$.

Observe that u^{ab} can be neither non-positive nor non-negative due to the diversity axiom (applied to the pair a, b).

We now turn to prove uniqueness. Assume that u^{ab}, $v^{ab} \in \mathsf{B}(\Omega, \Sigma)$ both satisfy conditions (i)-(v) of Lemma 1. Consider a two-person zero-sum game with a payoff matrix $(u^{ab}, -v^{ab})$. Specifically, (i) the set of pure strategies of player 1 (the row player) is Ω; (ii) the set of pure strategies of player 2 (the column player) is $\{L, R\}$; and (iii) if player 1 chooses $\omega \in \Omega$, and player 2 chooses L, the payoff to player 1 will be $u^{ab}(\omega)$, whereas if player 2 chooses R, the payoff to player 1 will be $-v^{ab}(\omega)$. Since both u^{ab}, v^{ab} satisfy conditions (i)-(iv), there is no $p \in \mathsf{P}$ for which $u^{ab} \cdot p > 0$, $-v^{ab} \cdot p > 0$. Hence the maximin in this game is non-positive. Therefore, so is the minimax. It follows that there exists a mixed strategy of player 2 that guarantees a non-positive payoff against any pure strategy of player 1. In other words, there are $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ such that $\alpha u^{ab}(\omega) \leq \beta v^{ab}(\omega)$ for all $\omega \in \Omega$. Moreover, by condition (v) $\alpha, \beta > 0$. Hence for $\gamma = \beta/\alpha > 0$, $u^{ab} \leq \gamma v^{ab}$. Applying the same argument to the game $(-u^{ab}, v^{ab})$, we find that there exists $\delta > 0$ such that $u^{ab} \geq \delta v^{ab}$. Therefore, $\gamma v^{ab} \geq u^{ab} \geq \delta v^{ab}$ for $\gamma, \delta > 0$. In view of part (v), there exists $\omega \in \Omega$ with $v^{ab}(\omega) > 0$, implying $\gamma \geq \delta$. By the same token there exists $\omega' \in \Omega$ with $v^{ab}(\omega') < 0$, implying $\gamma \leq \delta$. Hence $\gamma = \delta$ and $u^{ab} = \gamma v^{ab}$.

Finally, we prove part (vi). Observe that both u^{ab} and $-u^{ba}$ satisfy (i)-(iv) (stated for the ordered pair (a, b)). By the uniqueness result, $-u^{ab} = \alpha u^{ba}$ for some positive number α. At this stage we redefine the functions $\{u^{ab}\}_{a,b \in A}$ from the separation result as follows: for every unordered pair $\{a, b\} \subset A$ one of the two ordered pairs, say (b, a), is arbitrarily chosen and then u^{ab} is rescaled such that $u^{ab} = -u^{ba}$. (If A is of an uncountable power, the axiom of choice has to be used.) ¤

Lemma 2 *For every three distinct acts, $f, g, h \in A$, and the corresponding vectors u^{fg}, u^{gh}, u^{fh} from Lemma 1, there are unique $\alpha, \beta > 0$ such that:*

$$\alpha u^{fg} + \beta u^{gh} = u^{fh} \ .$$

The key argument in the proof of Lemma 2 is that, if u^{fh} is not a linear combination of u^{fg} and u^{gh}, one may find $p \in \mathsf{P}$ for which $\succ_p$ is cyclical.

If there are only three acts $f, g, h \in A$, Lemma 2 allows us to complete the proof as follows: choose a function $u^{fh} \in \mathsf{B}(\Omega, \Sigma)$ that separates between f and h. Then choose the multiples of u^{fg} and of u^{gh} defined by the lemma. Proceed to define $u^f = u^{fh}$, $u^g = \beta u^{gh}$, and $u^h = 0$. By construction, $(u^f - u^h)$ is (equal and therefore) proportional to u^{fh}, hence $f \succsim_p h$ iff $u^f \cdot p \geq u^h \cdot p$. Also, $(u^g - u^h)$ is proportional to u^{gh} and it follows that $g \succsim_p h$ iff $u^g \cdot p \geq u^h \cdot p$. The point is, however, that, by Lemma 2, we obtain the same result for the last pair: $(u^f - u^g) = (u^{fh} - \beta u^{gh}) = \alpha u^{fg}$ and $f \succsim_p g$ iff $u^f \cdot p \geq u^g \cdot p$ follows.

Proof of Lemma 2:

First note that for every three distinct acts, $f, g, h \in A$, if u^{fg} and u^{gh} are colinear, then for all p either $f \succ_p g \Leftrightarrow g \succ_p h$ or $f \succ_p g \Leftrightarrow h \succ_p g$. Both implications contradict diversity. Therefore any two functions in $\{u^{fg}, u^{gh}, u^{fh}\}$ are linearly independent. This immediately implies the uniqueness claim of the lemma. Next we introduce

Claim 1 *For every distinct $f, g, h \in A$, and every $\lambda, \mu \in \mathsf{R}$, if $\lambda u^{fg} + \mu u^{gh} \leq 0$, then $\lambda = \mu = 0$.*

Proof: Observe that Lemma 1(v) implies that if one of the numbers λ, and μ is zero, so is the other. Next, suppose, per absurdum, that $\lambda\mu \neq 0$,and consider $\lambda u^{fg} \leq \mu u^{hg}$. If, say, $\lambda, \mu > 0$, then $u^{fg} \cdot p \geq 0$ necessitates $u^{hg} \cdot p \geq 0$. Hence there is no p for which $f \succ_p g \succ_p h$, in contradiction to the diversity axiom. Similarly, $\lambda > 0 > \mu$ precludes $f \succ_p h \succ_p g$; $\mu > 0 > \lambda$ precludes

$g \succ_p f \succ_p h$; and $\lambda, \mu < 0$ implies that for no $p \in \Delta$ is it the case that $h \succ_p g \succ_p f$. Hence the diversity axioms holds only if $\lambda = \mu = 0$. ¤

We now turn to the main part of the proof. Consider a 2-person 0-sum game in which player 1 chooses a strategy $\omega \in \Omega$, player 2 chooses a strategy $u \in \{u^{fg}, u^{gh}, u^{hf}\}$, resulting in a payoff $u(\omega)$ for player 1. If the value of this game is positive, then there is an $p \in \mathsf{P}$ such that $u^{fg} \cdot p > 0$, $u^{gh} \cdot p > 0$, and $u^{hf} \cdot p > 0$. This, in turn, implies that $f \succ_p g$, $g \succ_p h$, and $h \succ_p f$ – a contradiction.

Therefore the value of the game is zero or negative. In this case there are $\lambda, \mu, \zeta \geq 0$, such that $\lambda u^{fg} + \mu u^{gh} + \zeta u^{hf} \leq 0$ and $\lambda + \mu + \zeta = 1$. The claim above implies that if one of the numbers λ, μ and ζ is zero, so are the other two. Thus $\lambda, \mu, \zeta > 0$. We therefore conclude that there are $\alpha = \lambda/\zeta > 0$ and $\beta = \mu/\zeta > 0$ such that

$$(*) \qquad \alpha u^{fg} + \beta u^{gh} \leq u^{fh}$$

Applying the same reasoning to the triple h, g, and f, we conclude that there are $\gamma, \delta > 0$ such that

$$(**) \qquad \gamma u^{hg} + \delta u^{gf} \leq u^{hf}.$$

Summation yields

$$(***) \qquad (\alpha - \delta)u^{fg} + (\beta - \gamma)u^{gh} \leq 0.$$

Claim 1 applied to inequality $(***)$ implies $\alpha = \delta$ and $\beta = \gamma$. Hence inequality $(**)$ may be rewritten as $\alpha u^{fg} + \beta u^{gh} \leq u^{fh}$, which together with $(*)$ yields the desired representation. ¤

Lemma 2 shows that, if there are more than three acts, the ranking of every triple of acts can be represented as in the theorem. The question that remains is whether these separate representations (for different triples) can be “patched” together in a consistent way.

Lemma 3 *There are functions* $\{u^{ab}\}_{a,b\in A,a\neq b} \subset \mathsf{B}(\Omega,\Sigma)$*, as in Lemma 1, and for any three distinct acts,* $f,g,h \in A$*, the Jacobi identity* $u^{fg}+u^{gh} = u^{fh}$ *holds.*

Proof: The proof is by induction, which is transfinite if A is uncountably infinite. The main idea of the proof is the following. Assume that one has rescaled the functions u^{ab} for all acts a,b in some subset of acts $X \subset A$, and one now wishes to add another act to this subset, $d \notin X$. Choose $a \in X$ and consider the functions u^{ad}, u^{bd} for $a,b \in X$. By Lemma 2, there are unique positive coefficients α,β such that $u^{ab} = \alpha u^{ad} + \beta u^{db}$. One would like to show that the coefficient α does not depend on the choice of $b \in X$. Indeed, if it did, one would find that there are $a,b,c \in X$ such that the vectors u^{ad}, u^{bd}, u^{cd} are linearly dependent, and this contradicts the diversity axiom.

Claim 2 *Let* $X \subset A$*,* $|X| \geq 3$*,* $d \in A\backslash X$*. Suppose that there are functions* $\{u^{ab}\}_{a,b\in A,a\neq b} \subset \mathsf{B}(\Omega,\Sigma)$*, as in Lemma 1, and for any three distinct acts,* $f,g,h \in X$*,* $u^{fg}+u^{gh} = u^{fh}$ *holds. Then there are functions* $\{u^{ab}\}_{a,b\in X\cup\{d\},a\neq b} \subset \mathsf{B}(\Omega,\Sigma)$*, as in Lemma 1, and for any three distinct acts,* $f,g,h \in X \cup \{d\}$*,* $u^{fg} + u^{gh} = u^{fh}$ *holds.*

Proof: Choose distinct $a,b,c \in X$. Let u^{ad},u^{bd}, and u^{cd} be the functions provided by Lemma 1 when applied to the pairs (a,d), (b,d), and (c,d), respectively. Consider the triple $\{a,b,d\}$. By Lemma 2 there are unique coefficients $\lambda(\{a,d\},b), \lambda(\{b,d\},a) > 0$ such that

$$\text{(I) } u^{ab} = \lambda(\{a,d\},b)u^{ad} + \lambda(\{b,d\},a)u^{db}$$

Applying the same reasoning to the triple $\{a,c,d\}$, we find that there are unique coefficients $\lambda(\{a,d\},c), \lambda(\{c,d\},a) > 0$ such that

$$u^{ac} = \lambda(\{a,d\},c)u^{ad} + \lambda(\{c,d\},a)u^{dc}.$$

or

$$\text{(II) } u^{ca} = \lambda(\{a,d\},c)u^{da} + \lambda(\{c,d\},a)u^{cd}.$$

We wish to show that $\lambda(\{a,d\},b) = \lambda(\{a,d\},c)$. To see this, we consider also the triple $\{b,c,d\}$ and conclude that there are unique coefficients $\lambda(\{b,d\},c), \lambda(\{c,d\},b) > 0$ such that

$$\text{(III) } u^{bc} = \lambda(\{b,d\},c)u^{bd} + \lambda(\{c,d\},b)u^{dc}.$$

Since $a,b,c \in X$, we have

$$u^{ab} + u^{bc} + u^{ca} = 0$$

and it follows that the summation of the right-hand sides of (I), (II), and (III) also vanishes:

$$[\lambda(\{a,d\},b) - \lambda(\{a,d\},c)]u^{ad} + [\lambda(\{b,d\},c) - \lambda(\{b,d\},a)]u^{bd} + \\ [\lambda(\{c,d\},a) - \lambda(\{c,d\},b)]u^{cd} = 0.$$

If some of the coefficients above are not zero, the vectors $\{u^{ad}, u^{bd}, u^{cd}\}$ are linearly independent, and this contradicts the diversity axiom. For instance, if u^{ad} is a non-negative linear combination of u^{bd} and u^{cd}, for no p will it be the case that $b \succ_p c \succ_p d \succ_p a$.

We therefore obtain $\lambda(\{a,d\},b) = \lambda(\{a,d\},c)$ for every $b,c \in A \backslash \{a\}$. Hence for every $a \in X$ there exists a unique $\lambda(\{a,d\}) > 0$ such that, for every distinct $a,b \in X$, $u^{ab} = \lambda(\{a,d\})u^{ad} + \lambda(\{b,d\})u^{db}$. Defining $u^{ad} = \lambda(\{a,d\})u^{ad}$ completes the proof of the claim. ¤

The lemma is proved by an inductive application of the claim. In case A is not countable, the induction is transfinite. ¤

Note that Lemma 3, unlike Lemma 2, guarantees the possibility to rescale *simultaneously* all the u^{ab}-s from Lemma 1 such that the Jacobi identity will hold on A.

We now complete the proof that (i) implies (ii). Choose an arbitrary act, say, e in A. Define $u^e = 0$, and for any other act, a, define $u^a = u^{ae}$, where the u^{ae}-s are from Lemma 3.

Given $p \in \mathsf{P}$ and $a, b \in A$ we have:

$$a \succsim_p b \Leftrightarrow u^{ab} \cdot p \geq 0 \Leftrightarrow (u^{ae} + u^{eb}) \cdot p \geq 0 \Leftrightarrow$$
$$(u^{ae} - u^{be}) \cdot p \geq 0 \Leftrightarrow u^a \cdot p - u^b \cdot p \geq 0 \Leftrightarrow u^a \cdot p \geq u^b \cdot p$$

The first implication follows from Lemma 1(i), the second from the Jacobi identity of Lemma 3, the third from Lemma 1(vi), and the fourth from the definition of the u^a-s. Defining $u(a, \cdot) = u^a(\cdot)$, $(**)$ of the theorem has been proved.

It remains to be shown that the functions defined above form a diversified matrix. First, we quote the following result from Gilboa and Schmeidler (1999, revised version – 2001):

Proposition 3 *Let Y be a set. Assume first $|A| \geq 4$. A matrix $u : X \times Y \to R$ is diversified iff for every list (a, b, c, d) of distinct elements of A, the convex hull of differences of the row-vectors $(u(a, \cdot) - u(b, \cdot))$, $(u(b, \cdot) - u(c, \cdot))$, and $(u(c, \cdot) - u(d, \cdot))$ does not intersect R^Y_-. Similar equivalence holds for the case $|A| < 4$.*

Thus, we need to show that the functions $\{u^a\}_{a \in A}$ defined above are such that $conv(\{u^a - u^b, u^b - u^c, u^c - u^d\}) \cap \mathsf{R}^\Omega_- = \emptyset$. Indeed, in Lemma 1(v) we have shown that $u^a - u^b \notin \mathsf{R}^\Omega_-$. To see this one only uses the diversity axiom for the pair $\{a, b\}$. Lemma 2 has shown, among other things, that a non-zero linear combination of $u^a - u^b$ and $u^b - u^c$ cannot be in R^Ω_-, using the diversity axiom for triples. Linear independence of all three vectors was established in Lemma 3. However, the full implication of the diversity condition will be clarified by the following lemma. Being a complete characterization, we will also use it in proving the converse implication, namely, that part (ii) of the theorem implies part (i). The proof of the lemma below depends on Lemma

1. It therefore holds under the assumptions that for any distinct $a, b \in A$ there is an p such that $a \succ_p b$.

Lemma 4 *For every list (a, b, c, d) of distinct elements of A, there exists $p \in \mathsf{P}$ such that*

$$a \succ_p b \succ_p c \succ_p d \quad \text{iff} \quad conv(\{u^{ab}, u^{bc}, u^{cd}\}) \cap \mathsf{R}^{\Omega}_{-} = \emptyset \ .$$

Proof: There exists $p \in \mathsf{P}$ such that $a \succ_p b \succ_p c \succ_p d$ iff there exists $p \in \mathsf{P}$ such that $u^{ab} \cdot p, u^{bc} \cdot p, u^{cd} \cdot p > 0$.

Consider a 2-person 0-sum game in which player 1 chooses a strategy $\omega \in \Omega$, player 2 chooses a strategy $u \in \{u^{ab}, u^{bc}, u^{cd}\}$, resulting in a payoff $u(\omega)$ for player 1. The argument above implies that there exists $p \in \mathsf{P}$ such that $a \succ_p b \succ_p c \succ_p d$ iff the maximin in this game is positive. This is equivalent to the minimax being positive, which means that for every mixed strategy of player 2 there exists $\omega \in \Omega$ that guarantees player 1 a positive payoff. In other words, there exists $p \in \mathsf{P}$ such that $a \succ_p b \succ_p c \succ_p d$ iff for every convex combination of $\{u^{ab}, u^{bc}, u^{cd}\}$ at least one entry is positive, i.e., $conv(\{u^{ab}, u^{bc}, u^{cd}\}) \cap \mathsf{R}^{\Omega}_{-} = \emptyset$. ¤

This completes the proof that (i) implies (ii). ¤

Part 2: (ii) implies (i)

It is straightforward to verify that if $\{\succsim_p\}_{i \in \mathsf{P}}$ are representable by $\{u(a, \cdot)\}_{a \in A} \subset \mathsf{B}(\Omega, \Sigma)$ as in $(*)$, they have to satisfy Axioms 1-3. To show that Axiom 4 holds, we quote Lemma 4 and Proposition 3 of the previous part. ¤

Part 3: Uniqueness

It is obvious that if $w(a, \cdot) = \alpha u(a, \cdot) + v$ for some scalar $\alpha > 0$, a function $v \in \mathsf{B}(\Omega, \Sigma)$, and all $a \in A$, then part (ii) of the theorem holds with the functions $w(a, \cdot)$ replacing $u(a, \cdot)$.

Suppose that $\{u(a, \cdot)\}_{a \in A}$ and $\{w(a, \cdot)\}_{a \in A}$ both satisfy $(**)$, and we wish to show that there are a scalar $\alpha > 0$ and a function $v \in \mathsf{R}^{\Omega}$ such that for all $a \in A$, $w(a, \cdot) = \alpha u(a, \cdot) + v$. Denote $u^a = u(a, \cdot)$ and $w^a = w(a, \cdot)$

$(u^a, w^a \in \mathsf{B}(\Omega, \Sigma).)$ Recall that, for $a \neq b$, $u^a \neq \lambda u^b$ and $w^a \neq \lambda w^b$ for all $0 \neq \lambda \in \mathsf{R}$ by A4.

Choose $a \neq e$ ($a, e \in A$, e satisfies $u^e = 0$). From the uniqueness part of Lemma 1 there exists a unique $\alpha > 0$ such that $(w^a - w^e) = \alpha(u^a - u^e) = \alpha u^a$. Define $v = w^e$.

We now wish to show that, for any $b \in A$, $w^b = \alpha u^b + v$. It holds for $b = e$ and $b = a$, hence assume that $a \neq b \neq e$. Again, from the uniqueness part of Lemma 1 there are unique $\gamma, \delta > 0$ such that

$$(w^b - w^a) = \gamma(u^b - u^a)$$
$$(w^e - w^b) = \delta(u^e - u^b) \ .$$

Summing up these two with $(w^a - w^e) = \alpha(u^a - u^e)$, we get

$$0 = \alpha(u^a - u^e) + \gamma(u^b - u^a) + \delta(u^e - u^b) = \alpha u^a + \gamma(u^b - u^a) - \delta u^b.$$

Thus

$$(\alpha - \gamma)u^a + (\gamma - \delta)u^b = 0 \ .$$

Since $u^a \neq u^e = 0$, $u^b \neq u^e = 0$, and $u^a \neq \lambda u^b$ if $0 \neq \lambda \in \mathsf{R}$, we get $\alpha = \gamma = \delta$. Plugging $\alpha = \gamma$ into $(w^b - w^a) = \gamma(u^b - u^a)$ proves that $w^b = \alpha u^b + v$. ¤

This completes the proof of the Theorem. ¤ ¤

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