Bubbles and Self-Enforcing Debt[*]

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Abstract

We characterize equilibria with endogenous debt constraints for a general equilibrium economy with limited commitment in which the only consequence of default is losing the ability to borrow in future periods. First, we show that equilibrium debt limits must satisfy a simple condition that allows agents to exactly roll over existing debt period by period. Second, we provide an equivalence result, whereby the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with unbacked public debt or rational bubbles. In contrast to the classic result by Bulow and Rogoff (AER, 1989), positive levels of debt are sustainable in our environment because the interest rate is sufficiently low to provide repayment incentives.

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1 Introduction

Consider a world with integrated capital markets in which countries can borrow and lend at the world interest rate. Suppose that the only punishment for a country that defaults on its debt is the denial of credit in all future periods. What is the maximum amount of borrowing that can be sustained in equilibrium? In this paper, we address this question in the context of a general equilibrium model with complete securities markets and endogenous debt constraints. Following Alvarez and Jermann (2000), debt limits are set at the largest possible levels such that repayment is always individually rational. We show that, under some conditions, positive levels of debt are sustainable in equilibrium and we characterize the joint behavior of intertemporal prices and debt limits. In the process, we identify a novel connection between the sustainability of debt by reputation and the sustainability of rational asset price bubbles.

The key observation for our sustainability result is that the incentives for debt repayment rely not only on the amount of credit to which agents have access, but also on the interest rate at which borrowing and lending takes place. After default, agents are excluded from borrowing, but are allowed to save. Lower interest rates make both borrowing more appealing and saving after default less appealing. We show that sustaining positive levels of debt requires "low interest rates," that is, intertemporal prices such that the present value of the agents' endowments is infinite.[1]

Our result stands in contrast to the classic result of Bulow and Rogoff (1989a, henceforth BR), who analyze the repayment incentives of a small open economy borrowing at a fixed world interest rate and show that no debt can be sustained in equilibrium if the only punishment for default is the denial of future credit.[2] The crucial difference is that BR take the world interest rate as given and assume that the net present value of the borrower's endowment, evaluated at the world interest rate, is finite. This rules out precisely the intertemporal prices that emerge in our general equilibrium setup.

The difference between BR and this paper can also be interpreted in terms of unilateral versus multilateral lack of commitment. BR assume that, after default, the borrowing country can save by entering a "cash-in-advance" contract with some external agent (some other country or some international bank), paying upfront in exchange for future state-contingent payments. BR implicitly assume that this agent has full commitment power for exogenous reasons, so he will never default

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[1] This terminology follows Alvarez and Jermann (2000).

[2] This result has sparked a vast literature that considers alternative ways to sustain positive levels of debt. A non-exhaustive list of contributions includes Eaton and Gersovitz (1981), Bulow and Rogoff (1989b), Cole and Kehoe (1995, 1998), Kletzer and Wright (2000), Kehoe and Perri (2002), Amador (2003), Gul and Pesendorfer (2004).

on the cash-in-advance contract.[3] In our model, instead, there is multilateral lack of commitment: no agent can commit to future repayments. Therefore, one country's ability to save after default, and hence its default incentives, rest on the other countries' repayment incentives.[4]

To illustrate our results, we first present a simple stationary example where, in equilibrium, positive borrowing arises and the interest rate is zero. More generally, debt is self-enforcing as long as the real interest rate is less than or equal to the growth rate of debt limits, which, in steady state, is equal to the growth rate of the aggregate endowment. As is well known, this is the same condition that makes bubbles sustainable in equilibrium (Tirole, 1985).

In the rest of the paper, we give a complete characterization of the conditions under which debt is sustainable. Our first general result (Theorem 1) states that debt limits and intertemporal prices are consistent with self-enforcement if and only if all agents are able to exactly re-finance outstanding obligations by issuing new claims. The result relies on arbitrage arguments that compare the feasible consumption sequences with and without a default (as in BR), as well as weak restrictions on preferences that guarantee the monotonicity of the agent's optimal asset holdings in initial wealth.

Our second result then establishes conditions for the existence of an equilibrium with self-enforcing debt and gives a characterization of sustainable equilibrium allocations by means of an equivalence result. Consider an alternative environment with no private debt, but where the government issues state-contingent debt that is not backed by any fiscal revenue, that is, the government must finance all existing claims by issuing new debt. If this unbacked public debt is valued in equilibrium, it is a rational bubble.[5] Theorem 2 shows that any equilibrium allocation of the economy with self-enforcing private debt can also be sustained as an equilibrium allocation of the economy with unbacked public debt, and vice versa. Therefore, conditions that ensure the existence of valued unbacked public debt, or, more generally, the existence of rational bubbles, also ensure the sustainability of positive levels of private debt in our economy. The possibility

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[3] The role of this assumption in BR was first pointed out by Eaton (1990) and Chari and Kehoe (1993a).

[4] Two papers that share with BR the assumptions of one-sided commitment and credit exclusion after default are Chari and Kehoe (1993b) and Krueger and Uhlig (2006), the first in the context of a model of government debt with distortive taxes and lack of government commitment, the second in the context of competitive insurance markets with one-sided commitment by insurers but not households. Krueger and Uhlig (2006) show that BR's assumption about credit exclusion emerges naturally from competition among insurers with one-sided commitment. Two papers which, instead, assume multilateral lack of commitment are Kletzer and Wright (2000) and Kehoe and Perri (2002). In both papers borrowing can arise in equilibrium, but the mechanism is quite different from ours: all transfers are publicly observable and risk-sharing and debt are sustained by a folk-theorem type argument.

[5] In a deterministic environment, unbacked public debt is equivalent to fiat money.

of rational bubbles in models with exogenous borrowing constraints *à la* Bewley (1980) has been recognized in Scheinkman and Weiss (1986), Kocherlakota (1992) and Santos and Woodford (1997). Our equivalence result shows that self-enforcing private debt can play the same role as a bubble and can take its place in facilitating intertemporal exchange.

Our model is closely related to models with limited commitment and endogenous borrowing constraints (Kehoe and Levine, 1993; Kocherlakota, 1996; Alvarez and Jermann, 2000). The central difference is our assumption about default consequences: while they assume that agents are completely excluded from financial markets after default, our exclusion only rules out future credit. From a methodological point of view, the major difference is that under our punishment, the equilibrium utility after default is endogenous and depends on equilibrium prices. This prevents us from starting from a social planner's problem with exogenous participation constraints and then decentralize its solution, as in Alvarez and Jermann (2000).

In Section 2, we describe our general model and define competitive equilibria with self-enforcing private debt and unbacked public debt. In Section 3, we illustrate our main results with a simple example. In Section 4, we study repayment incentives for individual agents, and characterize debt limits consistent with no default (Theorem 1). In Section 5, we study the resulting general equilibrium implications (Theorem 2). An online appendix contains additional results and extensions for the example in Section 3.

2   The Model

Uncertainty, preferences and endowments: Consider an infinite-horizon endowment economy with a single non-storable consumption good at each date $t \in \{0, 1, 2, ...\}$. For each t, there is a positive finite set S^t of date-t events s^t. Each s^t has a unique predecessor $\sigma\left(s^t\right) \in S^{t-1}$, and a positive, finite number of successors $s^{t+1} \in S^{t+1}$, for which $\sigma\left(s^{t+1}\right) = s^t$. There exists a unique initial date-0 event s^0. Event $s^{t+\tau}$ is said to follow event s^t (denoted $s^{t+\tau} \succ s^t$) if $\sigma^{(\tau)}\left(s^{t+\tau}\right) = s^t$. The set $\mathcal{S}\left(s^t\right) = \left\{s^{t+\tau} : s^{t+\tau} \succ s^t\right\} \cup \left\{s^t\right\}$ denotes the subtree of all events starting from s^t, and $\mathcal{S} \equiv \mathcal{S}\left(s^0\right)$ the complete event tree.

At date 0, nature draws a sequence $\left\{s^0, s^1, ...\right\}$, such that $s^{t-1} = \sigma\left(s^t\right)$ for all t. At date t, s^t is then publicly revealed. The unconditional probability of s^t is denoted by $\pi\left(s^t\right)$, where $\pi\left(s^t\right) > 0$ for all $s^t \in \mathcal{S}$. For $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, $\pi\left(s^{t+\tau}|s^t\right) = \pi\left(s^{t+\tau}\right)/\pi\left(s^t\right)$ denotes the conditional probability of $s^{t+\tau}$, given s^t.

There is a finite number J of consumer types, each represented by a unit measure of agents,

and indexed by j. Each consumer type is characterized by a sequence of endowments of the consumption good, $Y^j \equiv \left\{ y^j \left(s^t \right) \right\}_{s^t \in \mathcal{S}}$.[6] Preferences over consumption sequences $C \equiv \left\{ c \left(s^t \right) \right\}_{s^t \in \mathcal{S}}$ are represented by the lifetime expected utility functional

$$U\left(C \right) = \sum_{s^t \in \mathcal{S}} \beta^t \pi \left(s^t \right) u(c\left(s^t \right)) \tag{1}$$

where $\beta \in (0,1)$, and $u(\cdot)$ is strictly increasing, concave, bounded, twice differentiable, and satisfies standard Inada conditions.

Markets: At each date s^t, agents can issue and trade a complete set of contingent securities, which promise to pay one unit of period $t+1$ consumption, contingent on the realization of event $s^{t+1} \succ s^t$, in exchange for current consumption. If no agent ever defaults (as will be the case in equilibrium), securities issued by different agents are perfect substitutes and trade at a common price.

If agents had the ability to fully commit to their promises, they would be able to smooth all type-specific endowment fluctuations. In our model, however, agents cannot commit: at any date s^t, they can refuse to honor the securities they have issued and default. Any default becomes common knowledge and the defaulting agent loses the ability to issue claims in all future periods. Creditors can seize the financial assets he holds at the moment of default (i.e., his holdings of claims issued by other agents), but they are unable to seize any of his current or future endowments nor any of his future asset holdings. In sum, after a default, an agent loses the ability to issue debt, starts with a net financial position of 0, but he retains the ability to purchase assets.[7]

This form of punishment follows the assumptions of BR. It captures the idea that it is much easier for market participants to coordinate on not accepting the claims issued by a given borrower, than to enforce an outright ban from financial markets. As the future denial of credit eliminates the incentive to repay, a potential lender will assign zero value to the claims issued by a borrower who has defaulted in the past. Enforcing an outright ban from financial markets, on the other hand, requires that potential borrowers are dissuaded from accepting loans at market prices from agents who have defaulted in the past. The denial of future credit thus only requires the issuer of each security to be known, while a ban from financial markets requires that the identity of buyers

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[6]Throughout the paper, for any variable x, $x \left(s^t \right)$ denotes the realization of x at event s^t, X denotes the sequence $\left\{ x \left(s^t \right) \right\}_{s^t \in \mathcal{S}}$, and $X \left(s^t \right)$ denotes the subsequence $\left\{ x \left(s^{t+\tau} \right) \right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$.

[7]The assumption that any positive holdings of other agents' claims are confiscated in case of default implies that agents can default only on their *net* financial position. This assumption is made only for analytic and expositional purposes, and, as we will show later, it can be relaxed without changing our results. Therefore, the only disciplining element that may prevent agents from defaulting is losing the privilege to borrow in future periods.

and sellers in all financial transactions are observable, so that agents can be punished for dealing with others who have defaulted in the past.

Let $q\left(s^t\right)$ denote the price of a s^t-contingent bond at the preceding event $\sigma\left(s^t\right)$. The date-0 price of consumption at s^t, $p\left(s^t\right)$, is defined recursively by $p\left(s^t\right) = q\left(s^t\right) \cdot p\left(\sigma\left(s^t\right)\right)$ for all $s^t \in \mathcal{S}$. Let $a^j\left(s^t\right)$ denote the agent's net financial position at s^t, that is, the amount of s^t-contingent securities he holds net of the amount of s^t-contingent securities he has issued. An agent chooses a profile of consumption and asset holdings $C^j \equiv \left\{c^j\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ and $A^j \equiv \left\{a^j\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ subject to the sequence of flow budget constraints

$$c^j\left(s^t\right) \leq y^j\left(s^t\right) + a^j\left(s^t\right) - \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) a^j\left(s^{t+1}\right) \text{ for each } s^t \in \mathcal{S}. \tag{2}$$

The amount of securities an agent issues is observable, and subject to a state-contingent upper bound $-\phi^j\left(s^{t+1}\right)$, which then determines a lower bound on his net financial position at s^{t+1}:

$$a^j\left(s^{t+1}\right) \geq \phi^j\left(s^{t+1}\right) \text{ for all } s^{t+1} \succ s^t, s^t \in \mathcal{S}. \tag{3}$$

Given the initial asset position $a^j\left(s^0\right)$, the optimal consumption and asset profile for an agent who never defaults maximizes (1), subject to the constraints (2) and (3).

Self-enforcing private debt: Since several arguments in the paper require the manipulation of budget sets, it is convenient to denote by $\mathcal{C}^j(a, \Phi^j\left(s^t\right); s^t)$ the set of feasible consumption profiles $C^j\left(s^t\right)$ for a type-j agent starting at event s^t with an asset position $a \in \mathbb{R}$, facing the sequence of debt limits $\Phi^j\left(s^t\right)$, that is, the set of profiles $C^j\left(s^t\right)$ which satisfy (2) and (3) at all events in $\mathcal{S}\left(s^t\right)$ for some asset holdings profile $A^j\left(s^t\right)$ with $a^j\left(s^t\right) = a$. If the agent chooses never to default, his lifetime expected utility is given by:

$$V^j(a, \Phi^j\left(s^t\right); s^t) \equiv \max_{C(s^t) \in \mathcal{C}^j(a, \Phi^j(s^t); s^t)} \sum_{s^{t+\tau} \in \mathcal{S}(s^t)} \beta^\tau \pi\left(s^{t+\tau}|s^t\right) u(c\left(s^{t+\tau}\right)). \tag{4}$$

The lifetime utility of a consumer who has defaulted in the past is $V_D^j\left(a; s^t\right) \equiv V^j(a, O\left(s^t\right); s^t)$, where $O\left(s^t\right)$ denotes the sequence of borrowing constraints equal to zero at every $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. Notice that $V^j(a, \Phi^j\left(s^t\right); s^t)$ is increasing in a. Therefore, if $\phi^j\left(s^t\right)$ is such that $V^j(\phi^j\left(s^t\right), \Phi^j\left(s^t\right); s^t) = V_D^j\left(0; s^t\right)$, then an agent is exactly indifferent between default and no default if $a = \phi^j\left(s^t\right)$, no default is strictly preferred if $a > \phi^j\left(s^t\right)$, and default is strictly preferred if $a < \phi^j\left(s^t\right)$. This leads to the following definition of debt limits which are "not too tight," following the terminology in Alvarez and Jermann (2000).

Definition 1 *The debt limits $\Phi^j \equiv \left\{ \phi^j \left(s^t \right) \right\}_{s^t \in \mathcal{S}}$ are not too tight if and only if*

$$V^j(\phi^j \left(s^t \right), \Phi^j \left(s^t \right); s^t) = V_D^j \left(0; s^t \right) \ \text{for all } s^t \in \mathcal{S}. \tag{5}$$

Debt limits that are not too tight allow for the maximum amount of credit that is compatible with repayment incentives at all histories. In principle, any set of debt limits for which $V^j(\phi^j \left(s^t \right), \Phi^j \left(s^t \right); s^t) \geq V_D^j \left(0; s^t \right)$ for all $s^t \in \mathcal{S}$ eliminates default incentives. However, if this was a strict inequality for some s^t, an agent facing a binding debt constraint at $\phi^j \left(s^t \right)$ would be willing to borrow at a rate slightly higher than the market interest rate and market participants would not be willing to refuse him credit. Our debt limits are thus set so that (i) no borrower has an incentive to default, and (ii) no lender has an incentive to extend credit beyond a borrower's debt limit.[8]

A competitive equilibrium with self-enforcing private debt is then defined as follows:

Definition 2 *For given $\left\{ a^j \left(s^0 \right) \right\}_{j=1,\dots,J}$ with $\sum_{j=1}^J a^j \left(s^0 \right) = 0$, a competitive equilibrium with self-enforcing private debt consists of consumption profiles, asset holdings and debt limits $\{C^j, A^j, \Phi^j\}_{j=1,\dots,J}$, and state-contingent bond prices Q, such that (i) for each j, $\{C^j, A^j\}$ maximize (1), subject to (2) and (3), for given $a^j \left(s^0 \right)$, (ii) the debt limits Φ^j are not too tight for each j, and (iii) markets clear: $\sum_{j=1}^J c^j \left(s^t \right) = \sum_{j=1}^J y^j \left(s^t \right)$ and $\sum_{j=1}^J a^j \left(s^t \right) = 0$ for all $s^t \in \mathcal{S}$.*

Our equilibrium definition follows exactly Alvarez and Jermann (2000), except for the default punishment, which only allows for the denial of future credit, instead of complete autarky. Conceptually, the debt limits are similar to prices in Walrasian markets, in that individuals optimize taking prices and debt limits as given, but both are endogenously determined by the market equilibrium to satisfy the market-clearing and self-enforcement conditions.

Unbacked public debt: For our equivalence result, we consider an alternative economy with unbacked public securities. As before, there are sequential markets with complete contingent securities. However, unlike before, agents can no longer issue these claims. Claims are only supplied by a government, which rolls over a fixed initial stock of claims $d \left(s^0 \right)$ period by period, by issuing new securities. The government must satisfy its budget constraint $d \left(s^t \right) \leq \sum_{s^{t+1} \succ s^t} q \left(s^{t+1} \right) d \left(s^{t+1} \right)$, for all $s^t \in \mathcal{S}$, that is, at s^t the amount of resources raised by issuing new claims for all $s^{t+1} \succ s^t$ must be sufficient to honor the previous period's commitments. This budget constraint captures

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[8]Moreover, since $V^j(\cdot, \Phi^j \left(s^t \right); s^t)$ is increasing in $\left\| \Phi^j \left(s^t \right) \right\|$, the maximum extension in later periods can only help increase the sustainability of credit in earlier periods. In this sense, our notion of debt limits not being too tight really expands the provision of credit to the maximum sustainable level at all horizons.

the notion that these securities are not backed by any tax revenues or other government income. We assume that the government's budget constraint is satisfied with equality each period:

$$d\left(s^{t}\right) = \sum_{s^{t+1} \succ s^{t}} q\left(s^{t+1}\right) d\left(s^{t+1}\right) \text{ for all } s^{t} \in \mathcal{S}. \tag{6}$$

Given initial asset holdings $a^{j}\left(s^{0}\right) \geq 0$, optimal consumption allocations and asset holdings maximize (1), subject to (2) and the non-negativity constraint $a^{j}\left(s^{t}\right) \geq 0$ for all $s^{t} \in \mathcal{S}$. A competitive equilibrium with unbacked public debt is then defined as follows:

Definition 3 *For given initial asset positions $a^{j}\left(s^{0}\right) \geq 0$ for all j, and debt supply $d\left(s^{0}\right) = \sum_{j=1}^{J} a^{j}\left(s^{0}\right)$, a competitive equilibrium with unbacked public debt consists of consumption and asset profiles $\{C^{j}, A^{j}\}_{j=1,...,J}$, a debt supply profile D, and bond prices Q, such that (i) for each j, $\{C^{j}, A^{j}\}$ are optimal given $a^{j}\left(s^{0}\right)$, (ii) D satisfies (6), and (iii) markets clear: $\sum_{j=1}^{J} c^{j}\left(s^{t}\right) = \sum_{j=1}^{J} y^{j}\left(s^{t}\right)$ and $\sum_{j=1}^{J} a^{j}\left(s^{t}\right) = d\left(s^{t}\right)$ for all $s^{t} \in \mathcal{S}$.*

3 An Example

In this section, we illustrate the main results of our paper by means of a simple example with two types of consumers. In each period, one type receives the high endowment $\bar{e}$ and the other the low endowment $\underline{e}$, with $\bar{e} + \underline{e} = 1$. The types switch endowment with probability α from one period to the next. Formally, uncertainty is captured by the Markov process s_{t}, with state space $S = \{s_{1}, s_{2}\}$ and symmetric transition probabilities $\Pr\left[s_{t+1} = s_{1} | s_{t} = s_{2}\right] = \Pr\left[s_{t+1} = s_{2} | s_{t} = s_{1}\right] = \alpha$. The event s^{t} corresponds here to the sequence $\{s_{0}, ..., s_{t}\}$ and the endowments $y^{j}\left(s^{t}\right)$ only depend on the current realization of s_{t}, with $y^{j}\left(s^{t}\right) = \bar{e}$, if $s_{t} = s_{j}$ and $y^{j}\left(s^{t}\right) = \underline{e}$ if $s_{t} \neq s_{j}$.

We construct a symmetric Markov equilibrium, in which consumption allocations, asset holdings, debt limits, and the prices of state-contingent bonds depend only on the current state s_{t}, consumption allocations and asset holdings are symmetric across types and states, and the debt limit is binding in the high-endowment state for each agent. To focus on a stationary equilibrium, assume that the economy begins in state $s_{0} = s_{1}$ and the initial asset positions are $a^{1}\left(s_{0}\right) = -\omega$ and $a^{2}\left(s_{0}\right) = \omega$, where $-\omega$ is the debt limit for both agents. Proposition 1 shows that equilibria with positive debt levels can exist.

Proposition 1 *Let $\bar{c}$ be defined by $1 - \beta\left(1 - \alpha\right) = \beta\alpha u'\left(1 - \bar{c}\right) / u'\left(\bar{c}\right)$. If $\bar{c} < \bar{e}$, there exists a stationary equilibrium with self-enforcing private debt in which:*

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(*i*) *State-contingent bond prices are* $q\left(s^{t+1}\right) = q_c = 1 - \beta\left(1 - \alpha\right)$ *if* $s_{t+1} \neq s_t$, *and* $q\left(s^{t+1}\right) = q_{nc} = \beta\left(1 - \alpha\right)$ *if* $s_{t+1} = s_t$;[9]

(*ii*) *Consumption allocations are* $c^j\left(s^t\right) = \bar{c}$ *if* $s_t = s_j$ *and* $c^j\left(s^t\right) = \underline{c}$ *if* $s_t \neq s_j$, *where* $\underline{c} = 1 - \bar{c}$;

(*iii*) *Asset holdings are* $a^j\left(s^t\right) = -\omega$ *if* $s_t = s_j$ *and* $a^j\left(s^t\right) = \omega$ *if* $s_t \neq s_j$, *where* $\omega = \left(\bar{e} - \bar{c}\right) / \left(2 q_c\right)$;

(*iv*) *Debt limits are* $\phi^j\left(s^t\right) = -\omega$ *for all* $s^t \in \mathcal{S}$.

Proof. Given the conjectured equilibrium prices, the proposed allocations satisfy the consumer's first-order conditions $q_c u'\left(\bar{c}\right) = \beta \alpha u'\left(\underline{c}\right)$, $q_{nc} = \beta\left(1 - \alpha\right)$, and $q_c u'\left(\underline{c}\right) \geq \beta \alpha u'\left(\bar{c}\right)$. In addition, budget constraints and market-clearing conditions are satisfied by construction, given our definition of ω. We therefore only need to check that debt limits are not too tight. Theorem 1 below shows that a sequence of debt limits Φ satisfies (5) if and only if $\phi\left(s^t\right) = \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right)$ for all $s^t \in \mathcal{S}$. With constant debt limit $\phi^j\left(s^t\right) = -\omega$, this reduces to $1 = q_c + q_{nc}$, which the equilibrium prices satisfy by construction. ∎

Notice that $\bar{c}$ is well defined and unique, given Inada conditions and strict concavity. Moreover, $\bar{c}$ is defined independently of $\bar{e}$, so the condition $\bar{c} < \bar{e}$ is satisfied for an open set of parameters, under any smooth parametrization of $u\left(.\right)$.

Proposition 1 illustrates our first general result: positive levels of debt are sustainable in equilibrium if interest rates are sufficiently low. In particular, the no-default condition requires that $q_c + q_{nc} = 1$, tying the risk free interest rate to zero. To explain where this result comes from and why it relates the incentives for repayment to bond prices, let us suppose the prices q_c and $q_{nc} = \beta\left(1 - \alpha\right)$ are stationary, and restrict agents to stationary consumption allocations of c_h in high-endowment periods and c_l in low endowment periods.[10] For a consumer in the high-endowment state, the expected lifetime utility associated to $\left(c_h, c_l\right)$ is

$$v\left(c_h, c_l\right) = \frac{1}{1 - \beta + 2\beta\alpha} \left(\left(1 - \beta\left(1 - \alpha\right)\right) u\left(c_h\right) + \beta \alpha u\left(c_l\right)\right).$$

A consumer who chooses constant asset positions of $-\omega$ in high-endowment states and $a \geq -\omega$ in low-endowment states, faces the budget constraints $c_h = \bar{e} - \omega + q_{nc}\omega - q_c a$ and $c_l = \underline{e} + a - q_{nc}a + q_c\omega$. Substituting for a gives the intertemporal budget constraint

$$\left(1 - q_{nc}\right) c_h + q_c c_l = \left(1 - q_{nc}\right) \bar{e} + q_c \underline{e} + \left[q_c^2 - \left(1 - q_{nc}\right)^2\right] \omega. \tag{7}$$
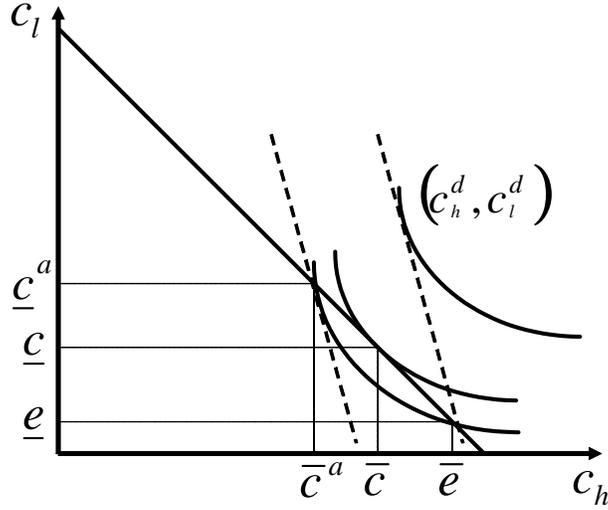

Figure 1: Feasible stationary allocations and default incentives

If the consumer never defaults, his optimal allocation (c_h, c_l) maximizes $v(c_h, c_l)$ subject to (7). After defaulting in a high-endowment period, the agent maximizes the same objective function subject to the same constraint (7), but with $\omega = 0$. A default thus represents a parallel shift of the intertemporal budget constraint and is optimal as long as the shift is positive, which is the case whenever $1 - q_{nc} > q_c$. On the other hand, not defaulting is strictly preferred when $1 - q_{nc} < q_c$, and the agent is exactly indifferent when $q_c = 1 - q_{nc}$. This illustrates how repayment incentives depend on the interest rate: the higher the interest rate, the less appealing is the opportunity to borrow and the more appealing the option to lend after default.

Figure 1 illustrates this argument graphically and shows how the equilibrium allocation is determined. The figure depicts the space of stationary allocations (c_h, c_l), along with the indifference curves for the function $v(c_h, c_l)$. The solid straight line going through $(\bar{e}, \underline{e})$, with slope -1, depicts the aggregate resource constraint $c_h + c_l = \bar{e} + \underline{e} = 1$. Our equilibrium allocation $(\bar{c}, \underline{c})$ corresponds to the allocation that maximizes $v(c_h, c_l)$, subject to the resource constraint.

Consider now an allocation along the resource constraint to the upper-left of $(\bar{c}, \underline{c})$, for example $(\bar{c}^a, \underline{c}^a)$. The intertemporal budget constraint that supports this allocation as an equilibrium (the dashed line through $(\bar{c}^a, \underline{c}^a)$) must be tangent to the indifference curve going through $(\bar{c}^a, \underline{c}^a)$; its slope in turn equals the ratio of state prices $(1 - q_{nc})/q_c$. Since by construction the slope of the indifference curve at $(\bar{c}, \underline{c})$ is -1, the state prices supporting $(\bar{c}^a, \underline{c}^a)$ satisfy $(1 - q_{nc})/q_c > 1$. However, our previous argument implies that the no-default condition is violated. In the figure, the

consumer can default in the high-endowment state, trade along the post-default budget constraint (the dashed line through $(\bar{e}, \underline{e})$), and reach the consumption bundle (c_h^d, c_l^d) which gives strictly higher utility than $(\bar{c}^a, \underline{c}^a)$.

The same argument applies to all allocations to the upper-left of $(\bar{c}, \underline{c})$, but notice that as the allocation moves closer to $(\bar{c}, \underline{c})$, the two intertemporal budget constraints become flatter, and the gap between them, and hence the benefit from defaulting, becomes smaller. At the allocation $(\bar{c}, \underline{c})$, at which the supporting state prices satisfy $(1 - q_n)/q_c = 1$, the two intertemporal budget constraints exactly coincide with each other and with the aggregate resource constraint, implying that agents are indifferent between defaulting and not defaulting. This corresponds to our equilibrium allocation with self-enforcing private debt.

If we move to the lower right of $(\bar{c}, \underline{c})$, the allocations between $(\bar{c}, \underline{c})$ and $(\bar{e}, \underline{e})$ along the feasibility constraint require supporting state prices that satisfy $(1 - q_{nc})/q_c < 1$. At these prices, the intertemporal budget constraint with no default is to the right of the one with default, and hence agents would strictly prefer no default to default. Finally, the default and no-default budget constraints converge again at $(\bar{e}, \underline{e})$, which corresponds to the autarkic equilibrium with zero borrowing and prices equal to $q_c^{aut} = \beta \alpha u'(\underline{e})/u'(\bar{e})$ and $q_{nc}^{aut} = \beta(1 - \alpha)$. Moreover, all allocations to the right of $(\bar{e}, \underline{e})$ or to the left of $(\bar{c}^a, \underline{c}^a)$ can be immediately ruled out since they are dominated by the autarky allocation, which is always within the agents' budget sets.

Our example allows for a simple comparison between our results and those obtained when default is punished by complete exclusion from financial markets. In particular, the environment in our example is the same as the one in Krueger and Perri (2006), with the exception that they consider autarky as the consequence of default. We have chosen the point $(\bar{c}^a, \underline{c}^a)$ in Figure 1 so that it also represents the stationary equilibrium allocation under the autarky punishment. To see this, notice that $(\bar{c}^a, \underline{c}^a)$ is on the same indifference curve as the autarky allocation $(\bar{e}, \underline{e})$.[11] If agents are not allowed to save after default, when they make their default decision they only need to compare their expected utility at $(\bar{c}^a, \underline{c}^a)$ and $(\bar{e}, \underline{e})$. The interest rate plays no role in this comparison.

As the figure shows, in our equilibrium there is strictly less risk sharing than in the economy with the autarky punishment, that is, $\underline{c} < \underline{c}^a$. This is not surprising, given that our punishment is weaker. Somewhat more surprisingly and contrary to what one would expect from BR's argument, the conditions for the existence of equilibria with positive debt are identical in the two environments: in both cases, a non-autarkic equilibrium exists only if $\bar{e} > \bar{c}$. The figure shows why this condition is

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[11]Notice that in the figure $(\bar{c}^a, \underline{c}^a)$ lies to the lower right of the 45 degree line. If that was not the case, the stationary equilibrium under autarky punishment would feature perfect risk sharing.

necessary for sustaining positive levels of risk sharing: if $\bar{e} \leq \bar{c}$, any departure from autarky towards more risk-sharing would strictly lower the expected utility of a high-endowment consumer and would not be incentive compatible under either punishment. The condition $\bar{e} > \bar{c}$ is equivalent to the condition $1/(q_c^{aut} + q_{nc}^{aut}) < 1$. In the terminology of Alvarez and Jermann (2000) a non-autarkic equilibrium exists if the autarky allocation displays "low implied interest rates," that is, interest rates such that the present value of the aggregate endowment is infinite.[12] However, the two forms of punishment have different implications for interest rates and debt levels *in equilibrium*: Under the autarky punishment, if positive levels of debt are sustainable, the equilibrium allocation displays "high implied interest rates," that is, interest rates such that the present value of the aggregate endowment is finite.[13] Under the no-borrowing punishment, instead, the equilibrium allocation displays low implied interest rates. The difference comes from the fact that, in our environment, utility after default is endogenous and depends on market interest rates. As long as interest rates are high, the BR result applies and the incentives for repayment disappear.

The condition $\bar{e} > \bar{c}$ also ensures that this economy admits a stationary equilibrium with valued unbacked public debt, with consumption allocations and bond prices that are identical to the ones in the equilibrium with self-enforcing private debt.

Proposition 2 *If $\bar{e} > \bar{c}$, there exists an equilibrium with unbacked public debt, in which consumption allocations and prices are the same as in Proposition 1, asset holdings are $a^j(s^t) = 0$ if $s_t = \mathsf{s}_j$ and $a^j(s^t) = 2\omega$ if $s_t \neq \mathsf{s}_j$, and the government's supply of debt is $d(s^t) = 2\omega$ for all $s^t \in \mathcal{S}$.*

Proof. Optimality of the proposed consumption allocations and asset holdings, as well as market-clearing in goods and asset markets follows by construction. Moreover, since $d(s^t)$ is constant for all $s^t \in \mathcal{S}$ and $q_c + q_{nc} = 1$, the government's roll-over condition is also satisfied. ∎

Proposition 2 illustrates our second general result: equilibrium allocations in an economy with self-enforcing private debt are equivalent to equilibrium allocations in an economy with unbacked public debt. This is proved in full generality in Theorem 2 and allows us to use equilibrium characterizations that apply in known environments with unbacked public debt to establish the existence and characterization of equilibria with positive levels of self-enforcing private debt.

In an online appendix, we extend the analysis of this example in several dimensions. First, we augment the example to include aggregate endowment growth, showing that the equilibrium

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[12]When $1/(q_c^{aut} + q_{nc}^{aut}) \geq 1$, autarky is the unique stationary equilibrium in both regimes. Proposition 4.8 in Alvarez and Jermann (2000) proves this result in general, under the autarky punishment.

[13]See Proposition 4.10 in Alvarez and Jermann (2000) for a general statement of this result.

interest rate must equal the economy's growth rate, which must equal the growth rate of the aggregate debt supply and the individual debt limits in steady-state. We also show the existence of non-stationary equilibria in which there is a self-fulfilling collapse in the real value of debt (analogous to hyperinflations in economies with fiat money). Finally, we show how transitional dynamics depend on type-specific debt limits and initial asset holdings.

4   Characterizing Repayment Incentives

In this section, we characterize the repayment incentives of an individual borrower. The main result in this section (Theorem 1) is that debt limits are not too tight if and only if they allow for "exact roll-over," i.e., if and only if at each history, the agent is able to exactly repay his maximum outstanding debt $-\phi\left(s^t\right)$ by issuing new debt, up to the limit $-\phi\left(s^{t+1}\right)$, for each $s^{t+1} \succ s^t$. Since we are exclusively concerned with the single-agent problem, we simplify notation throughout this section by dropping the superscript j.

Theorem 1 *The debt limits Φ are not too tight if and only if they allow for exact roll-over:*

$$\phi\left(s^t\right) = \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right) \ \text{for all } s^t \in \mathcal{S}. \tag{8}$$

Moreover, if the debt limits Φ are not too tight, $\mathcal{C}(a, \Phi\left(s^t\right); s^t) = \mathcal{C}(a - \phi\left(s^t\right), O\left(s^t\right); s^t)$ for all $s^t \in \mathcal{S}$.

This theorem also shows that the budget set of an agent facing debt limits Φ which are not too tight is identical to that of an agent facing zero debt limits who starts with a higher initial asset position. Hence, optimal consumption and asset profiles are the same for the two agents. This simplifies the equilibrium characterization, since, rather than computing the fixed point between the consumer's optimization problem and the self-enforcement condition (8), we only need to compute optimal consumption allocations for agents with zero debt limits, and these are identical to the optimal consumption allocations without default. Equilibrium debt limits are then constructed so as to satisfy (8) and market clearing in the asset market.

Condition (8) states that debt can only be sustained, if, instead of repaying, the borrower is able to infinitely roll over outstanding debt. This leads to a simple comparison with BR's no lending result, which follows almost immediately from Theorem 1.

12

Proposition 3 *(Bulow and Rogoff) Suppose prices and endowments are such that*

$$w\left(s^t\right) \equiv \sum_{s^{t+\tau} \in \mathcal{S}(s^t)} y\left(s^{t+\tau}\right) p\left(s^{t+\tau}\right) / p\left(s^t\right) < \infty \text{ for all } s^t.$$

Suppose the debt limits Φ *are not too tight and satisfy* $\phi\left(s^t\right) \geq -w\left(s^t\right)$ *for all* s^t, *then* $\phi\left(s^t\right) = 0$ *for all* s^t.

Proof. If $w\left(s^t\right) < \infty$, it must be the case that $\lim_{T \to \infty} \sum_{s^{t+T} \succ s^t} p\left(s^{t+T}\right) w\left(s^{t+T}\right) / p\left(s^t\right) = 0$. Now, using the exact roll-over condition and the condition $\phi\left(s^t\right) \geq -w\left(s^t\right)$, we have

$$\phi\left(s^t\right) = \sum_{s^{t+T} \succ s^t} \frac{p\left(s^{t+T}\right)}{p\left(s^t\right)} \phi\left(s^{t+T}\right) \geq - \sum_{s^{t+T} \succ s^t} \frac{p\left(s^{t+T}\right)}{p\left(s^t\right)} w\left(s^{t+T}\right)$$

for all T, which implies

$$\phi\left(s^t\right) \geq - \lim_{T \to \infty} \sum_{s^{t+T} \succ s^t} \frac{p\left(s^{t+T}\right)}{p\left(s^t\right)} w\left(s^{t+T}\right) = 0.$$

∎

BR show that positive levels of self-enforcing debt are ruled out if two conditions hold: (i) endowments are finite-valued at the prevailing state prices, i.e., interest rates are high, and (ii) the agent's debt is bounded by the present value of his future endowments, the "natural debt limit." Condition (i) is imposed as an exogenous restriction on state prices. Condition (ii) on the other hand is a standard restriction which is usually imposed to rule out Ponzi games.[14] With high interest rates, if debt limits are not zero and are not too tight, they are eventually inconsistent with the natural debt limits. With low interest rates, however, natural debt limits are infinite and impose no restriction on debt levels. While BR only focus on the first scenario, our characterization in Theorem 1 is sufficiently general to encompass both.[15]

To sustain positive debt levels, we must abandon condition (i) or (ii). From a partial equilibrium point of view, relaxing either one can lead to self-enforcing debt. However, a general equilibrium argument shows that the interesting case arises when we dispose of condition (i). If we relax condition (ii), but maintain high interest rates, Theorem 1 implies that, if there are positive levels

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[14]In the working-paper version, Bulow and Rogoff (1988, p. 5) hint at the idea that relaxing this condition may lead to positive debt, when they remark that this assumption rules out "Ponzi-type reputational contracts."

[15]Our formulation of the BR result is slightly different from the original version of the no-lending result, which stated that any asset sequence that gives an agent no incentive to ever default must always remain non-negative. The formal equivalence between the two statements follows from the additional (almost immediate) observation that any sustainable asset position can be bounded below by a sequence of debt limits that are not too tight.

of debt, the aggregate stock of debt, and thus the savings of some lender, will eventually exceed the value of aggregate endowments. This clearly cannot happen in general equilibrium. On the other hand, it is possible to construct economies where, in general equilibrium, condition (*i*) fails to hold, as shown in the example in Section 3 and, more generally, in Section 5 below.

Self-enforcement and exact roll-over: Before we proceed to the proof of Theorem 1, we illustrate the relation between self-enforcement and exact roll-over through a series of figures. For this, we assume that endowment fluctuations are deterministic, and agents trade a single uncontingent bond.[16] The agents' budget constraint can then be rewritten as $c_t = y_t + (p_t a_t - p_{t+1} a_{t+1})/p_t$. For a given sequence of prices $\{p_t\}$, we can thus compare the consumption profiles resulting from different asset plans simply by comparing the period-by-period changes in the present value of asset holdings, $p_t a_t - p_{t+1} a_{t+1}$. In the following figures, we plot the time paths $\{p_t a_t\}$ of the present values of asset profiles with and without defaults to evaluate repayment incentives.
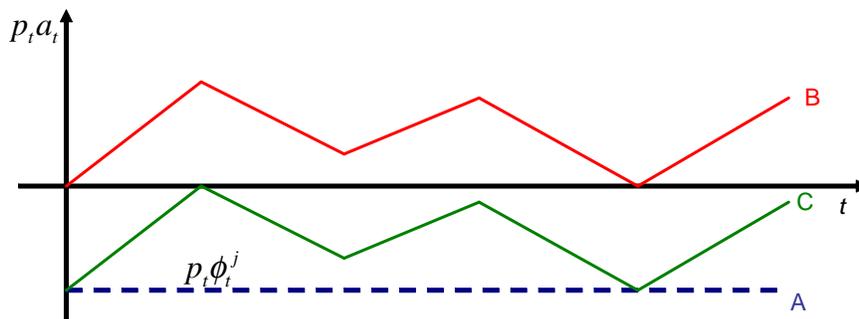
Figure 2: Debt limits satisfying exact roll-over

Figure 2 considers repayment incentives when debt limits allow for exact roll-over. In a deterministic environment, this requires that $p_t \phi_t$ is constant over time; such debt limits are represented by the dotted line A. Line B represents an arbitrary asset profile that is feasible for an agent who defaults at some date t. Line C represents a parallel downwards shift of the asset profile B, to an initial asset position of ϕ_t. Notice that C generates the same consumption sequence as B. Moreover, since profile B remains non-negative, C always remains above A, and is therefore feasible for an agent who starts with an asset position of ϕ_t and does not default. Hence, this agent must be weakly better off not defaulting at date t. On the other hand, for any asset profile C that is feasible without default starting from an asset position of ϕ_t, there is some asset profile B that is feasible starting from a default at date t, and gives the agent the same consumption profile as C. If debt

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[16]We thus replace the dependence on s^t by a time subscript to simplify notation.

14

limits allow for exact roll-over, the agent must therefore be exactly indifferent between defaulting on an asset position of ϕ_t, and not defaulting, i.e. the self-enforcement condition is satisfied with equality.
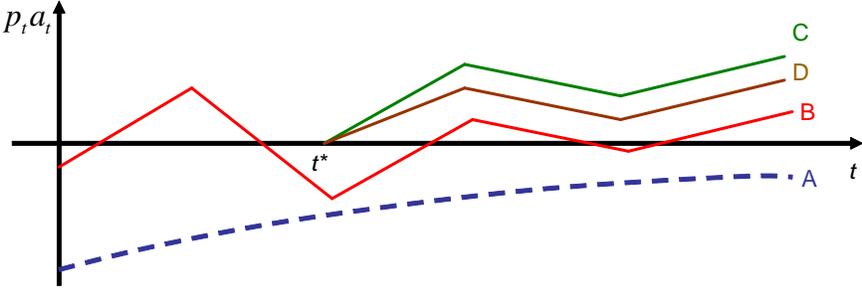
Figure 3: Shrinking debt limits

Along similar lines, we can illustrate how repayment incentives are violated when the present value of debt limits is shrinking over time. Figure 3 plots the case of BR, in which the natural debt limits are finite and act as a lower bound on the agent's asset profile. In this case, their present value falls over time (line A). Then, given any asset profile B that is consistent with these debt limits and admits positive debt at some date t, there exists a date $t^* \geq t$ at which the present value of debt reaches a maximum. At that point, the agent can default and replicate the same consumption profile as B, just using positive asset holdings (line C), and even improve upon the no-default profile by strictly increasing consumption at date t^* (line D).

Likewise, if the present value of debt limits is expanding over time (Figure 4), for every profile B that is feasible after a default at some date t, there exists a profile C that is feasible without default
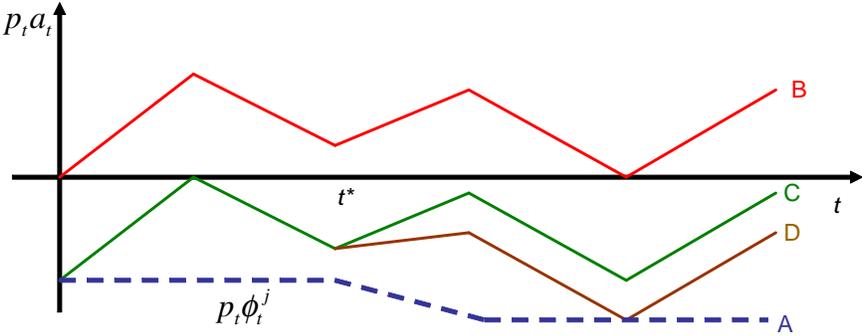
Figure 4: Expanding debt limits

starting from asset position ϕ_t and implements the same consumption. Moreover, profile D remains feasible without default starting from asset position ϕ_t, but delivers strictly higher consumption at date t^*, where the debt limits are expanding. Hence, agents strictly prefer not to default, when the present value of debt limits is expanding over time.

Proof of Theorem 1: Proposition 4 establishes the sufficiency part of Theorem 1 by generalizing the graphical argument of Figure 2.

Proposition 4 *Suppose that the debt limits Φ allow for exact roll-over. Then*

$$V\left(a, \Phi\left(s^t\right); s^t\right) = V_D\left(a - \phi\left(s^t\right); s^t\right) \text{ for all } s^t \in \mathcal{S}, \text{ and any } a \geq \phi\left(s^t\right). \tag{9}$$

Proof. Starting from an arbitrary s^t, consider asset profiles $\left\{a\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ and $\left\{\hat{a}\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ that satisfy $\hat{a}\left(s^{t+\tau}\right) = a\left(s^{t+\tau}\right) - \phi\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, with $a\left(s^t\right) = a \geq \phi\left(s^t\right)$ and $\hat{a}\left(s^t\right) = a - \phi\left(s^t\right)$. By construction, $\left\{a\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ is feasible under no default, if and only if $\left\{\hat{a}\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ is feasible after defaulting at s^t. Moreover, the exact roll-over condition (8) implies $a\left(s^{t+\tau}\right) - \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) a\left(s^{t+\tau+1}\right) = \hat{a}\left(s^{t+\tau}\right) - \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \hat{a}\left(s^{t+\tau+1}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, and therefore, starting from $a\left(s^t\right) = a$, asset plan $\left\{a\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ implements the same consumption allocation $\left\{c\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ as asset plan $\left\{\hat{a}\left(s^{t+\tau}\right)\right\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ starting from $\hat{a}\left(s^t\right) = a - \phi^j\left(s^t\right)$. But then, $\mathcal{C}\left(a, \Phi\left(s^t\right); s^t\right) = \mathcal{C}\left(a - \phi\left(s^t\right), O\left(s^t\right); s^t\right)$, and $V\left(a, \Phi\left(s^t\right); s^t\right) = V_D\left(a - \phi\left(s^t\right); s^t\right)$. ∎

The self-enforcement condition (5) follows from setting $a = \phi\left(s^t\right)$ in (9). Condition (9) further implies that an agent who defaults on his maximum gross amount of debt $-\phi\left(s^t\right)$, but keeps his own asset holdings $a - \phi\left(s^t\right)$ after a default is always exactly indifferent between defaulting and not defaulting. The assumption that agents start with a net financial position of zero after default can therefore be relaxed without weakening repayment incentives.

Proposition 5 establishes the necessity part of Theorem 1 by showing that debt limits are not too tight only if they allow for exact roll-over. This was already suggested by the graphical arguments in Figures 2-4. However, this graphical intuition is incomplete, since it only applies to sequences of debt limits whose present values are monotone increasing or decreasing. The argument for arbitrary non-monotone sequences of debt limits turns out to be considerably more involved.

Proposition 5 *If the debt limits Φ are not too tight they allow for exact roll-over.*

The proof of this proposition is in the Appendix, here we sketch the key steps. Consider a sequence of debt limits Φ which are not too tight. Starting from some arbitrary event s^t, we first

16

construct a sequence of auxiliary debt limits $\tilde{\Phi}\left(s^t\right)$, as follows:

$$\tilde{\phi}\left(s^{t+\tau}\right) = \begin{cases} \phi\left(s^{t+\tau}\right) & \text{if } a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \\ \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), \tilde{\phi}\left(s^{t+\tau+1}\right)\right\} & \text{otherwise} \end{cases},$$
(10)

where $A^*\left(s^t\right)$ denotes the optimal no-default asset profile of an agent starting from s^t with asset position $a^*\left(s^t\right) = \phi\left(s^t\right)$. The first step of the proof (and the major technical hurdle) consists in showing that this sequence is well-defined and finite-valued. This is complicated by the fact that present discounted values need not be well-defined in our environment since we cannot rely on an assumption of high interest rates. The characterization in turn makes use of the time-separability, concavity and boundedness of $u\left(\cdot\right)$.

Next, using arbitrage arguments, we show that (i) $\tilde{\phi}\left(s^{t+\tau}\right) \leq \phi\left(s^{t+\tau}\right)$, and $\tilde{\phi}\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$ whenever $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$, and (ii) $\tilde{\phi}\left(s^{t+\tau}\right) \geq \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right)\tilde{\phi}\left(s^{t+\tau+1}\right)$, for every $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. The first property states that $\tilde{\Phi}$ is a lower bound of Φ, and is equal to Φ whenever the actual debt limit is binding. The second property states that $\tilde{\Phi}$ satisfies (ER) with a weak inequality, so that under $\tilde{\Phi}$, at any event, the maximum outstanding debt obligations are weakly less than the funds that can be raised by exhausting debt limits on the continuation events.

Property (i), together with the concavity of $u\left(\cdot\right)$ then implies that the value of the no-default problem starting from asset holdings of $a\left(s^t\right) = \phi\left(s^t\right)$ at s^t is the same under the original debt limits, $\Phi\left(s^t\right)$, as under the auxiliary debt limits, $\tilde{\Phi}\left(s^t\right)$ (since the latter only relaxes non-binding debt limits). Since the original debt limits are not too tight and $\phi\left(s^t\right) \geq \tilde{\phi}\left(s^t\right)$, we thus obtain $V_D\left(0; s^t\right) = V\left(\phi\left(s^t\right); \Phi\left(s^t\right), s^t\right) = V(\phi\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t) \geq V(\tilde{\phi}\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t)$. On the other hand, using the same arbitrage argument as Proposition 4, property (ii) implies that $\mathcal{C}(0, O\left(s^t\right); s^t) \subseteq \mathcal{C}(\tilde{\phi}\left(s^t\right), \tilde{\Phi}\left(s^t\right), s^t)$ and $V(\tilde{\phi}\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t) \geq V_D\left(0; s^t\right)$, with strict inequality if $\tilde{\phi}(s^{t+\tau}) > \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$ for some $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. Therefore, both inequalities must hold with equality, which requires $\phi\left(s^t\right) = \tilde{\phi}\left(s^t\right)$, and that $\tilde{\Phi}\left(s^t\right)$ satisfies the exact roll-over condition as an equality, for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$.

To complete the proof, we show that $\tilde{\phi}\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$, for all $s^{t+1} \succ s^t$. Repeating the same steps as above, we construct additional sequences of auxiliary debt limits $\hat{\Phi}\left(s^{t+1}\right)$, together with optimal asset holdings $\hat{A}\left(s^{t+1}\right)$, for each $s^{t+1} \succ s^t$. Clearly, $\hat{\Phi}\left(s^{t+1}\right)$ satisfies (ER), and $\phi\left(s^{t+1}\right) = \hat{\phi}\left(s^{t+1}\right)$. Moreover, due to concavity and additive separability of U, optimal asset profiles are monotone in initial asset holdings, so that $\hat{a}\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}\right)$, whenever $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) = \tilde{\phi}\left(s^{t+\tau}\right)$. Together with (10), this implies that $\tilde{\phi}\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^{t+1}\right)$, and hence $\tilde{\phi}\left(s^{t+1}\right) = \hat{\phi}\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$, which completes our proof.

17

Remark: Proposition 5 is the only result where we use the assumptions of additive time-separability, concavity and boundedness of $u(\cdot)$. All other results rely purely on arbitrage arguments and therefore require only strict monotonicity. The boundedness assumption is a strong restriction, but it is required only for a partial equilibrium characterization. If one restricts attention to debt limits Φ such that optimal consumption allocations are bounded above by aggregate endowments (a condition that must hold in general equilibrium), Proposition 5 holds under the following weaker restriction.

Assumption 1 *For all C, such that $U(C) \geq \min_j U\left(Y^j\right)$ and $c\left(s^t\right) \in \left[0, \sum_{j=1}^{J} y^j\left(s^t\right)\right]$ for all $s^t \in \mathcal{S}$, $\sum_{s^t \in \mathcal{S}} \beta^t \pi\left(s^t\right) c\left(s^t\right) u'(c\left(s^t\right)) < \infty$.*

This regularity condition bounds the rate at which individual and aggregate endowments can grow or decline, relative to the discount factor β. When relative risk aversion is bounded, this assumption holds whenever $U\left(\sum_{j=1}^{J} Y^j\right)$ and $\min_j U\left(Y^j\right)$ are both finite.

5   General Equilibrium Characterization

We now turn to the question whether there exist equilibria with positive levels of self-enforcing debt, and how they can be characterized. Theorem 2 shows that a given consumption allocation and price vector constitute a competitive equilibrium with self-enforcing private debt, if and only if the same allocation and prices are an equilibrium of the corresponding economy with unbacked public debt. For the latter economy, there are known existence and characterization results (e.g. Santos and Woodford 1997), which then extend immediately to the economy with self-enforcing private debt.

Theorem 2 *An allocation $\{C^j\}_{j=1,\ldots,J}$ and prices Q are sustainable as a competitive equilibrium with self-enforcing private debt, if and only if $\{C^j\}_{j=1,\ldots,J}$ and Q are also sustainable as a competitive equilibrium with unbacked public debt.*

Proof. *Step 1:* If $\{C^j, A^j, \Phi^j, Q\}$ is a CE with self-enforcing private debt, for initial asset positions $\{a^j\left(s^0\right)\}_{j=1,\ldots,J}$, then $\{C^j, A^j - \Phi^j\}$ is optimal given initial asset holdings of $a^j\left(s^0\right) - \phi^j\left(s^0\right)$, state prices Q, and zero debt limits, $-\sum_{j=1}^{J} \Phi^j$ satisfies (6), and $\sum_{j=1}^{J}(A^j - \Phi^j) = -\sum_{j=1}^{J} \Phi^j$, so that market clearing is satisfied. $\{C^j, A^j - \Phi^j, -\sum_{j=1}^{J} \Phi^j, Q\}$ therefore constitutes a CE with unbacked public debt, for initial asset holdings $\{a^j\left(s^0\right) - \phi^j\left(s^0\right)\}_{j=1,\ldots,J}$.

Step 2: If $\{C^j, \hat{A}^j, D, Q\}_{j=1,\ldots,J}$ constitutes a CE with unbacked public debt, starting from initial asset positions $\{\hat{a}^j\left(s^0\right)\}_{j=1,\ldots,J}$, we can construct debt limits $\tilde{\Phi}^j = -\hat{a}^j\left(s^0\right)/d\left(s^0\right) \cdot D$,

and asset holdings $\tilde{A}^j = \hat{A}^j + \tilde{\Phi}^j$. By construction, debt limits $\tilde{\Phi}^j$ are not too tight, and the allocations $\{C^j, \tilde{A}^j\}$ are optimal, given initial asset holdings of $\tilde{a}^j(s^0) = 0$ and debt limits holdings $\tilde{\Phi}^j$. Moreover, since $\sum_{j=1}^J \tilde{A}^j = \sum_{j=1}^J (\hat{A}^j + \tilde{\Phi}^j) = \sum_{j=1}^J \hat{A}^j - D = 0$, these allocations clear markets in the private debt economy. Therefore $\{C^j, \tilde{a}^j, \tilde{\Phi}^j, p\}_{j=1,\ldots,J}$ is a CE with self-enforcing private debt, for initial asset positions of zero. ∎

The proof of Theorem 2 makes repeated use of the exact roll-over property. First, with exact roll-over, the feasible, and hence the optimal, consumption allocations in the default and no-default problems exactly coincide, for suitably chosen initial values of asset holdings. Second, the exact roll-over property implies a mapping from debt limits Φ^j which are not too tight to a sequence of public debt supply that satisfies the government roll-over constraint (6), and vice versa. Given this mapping from debt limits to public debt circulation and the equivalence of optimal consumption allocations, market-clearing conditions in the two economies turn out to be exactly equivalent.

Theorem 2 formally establishes the connection between sustaining repayment incentives for private debt and sustaining rational bubbles. In the private debt economy, the agents' commitment and enforcement power is so limited that any contract that, at some date, requires a positive transfer of resources in net present value is not sustainable. Likewise, in the economy with unbacked public debt, the government does not have the power to use taxation to guarantee its debt holders a positive net transfer of resources. In both cases, the result is that the only sustainable allocations roll over existing debt forever. The equivalence arises because neither side can credibly commit to future transfers, either via contract enforcement or via taxation.

6   Concluding Remarks

In this paper, we have studied a general equilibrium economy with self-enforcing private debt, in which borrowers, after default, are excluded from future credit, but retain the ability to save at market interest rates. For a partial equilibrium version of this model, in which a small open economy borrows internationally at fixed, positive interest rates, BR show that debt cannot be sustainable by reputational mechanisms only: eventually, the country always has an incentive to default. The BR result can be interpreted as follows: if there are some agents who are able to commit to intertemporal transfers at "high interest rates," then the remaining agents, who are unable to commit, will accumulate and decumulate the securities issued by the committed agents, but will never become net borrowers. Krueger and Uhlig (2006) provide the analytical foundations for this interpretation.

In contrast, we show that positive levels of debt can be sustained when no party has commitment power. The key to our result is that interest rates adjust downwards to provide repayment incentives to all the potential borrowing parties. As a result, "low interest rates" emerge in equilibrium.

These results helps to clarify the BR result in two directions. From a formal point of view, they highlight the role played by the interest rate in the BR argument. From a more substantive point of view, they clarify the role of multilateral vs. unilateral lack of commitment and the role of credit exclusion vs. stronger forms of punishment for the sustainability of debt in general equilibrium. This analysis helps to bridge the gap between the negative result of BR and the positive results obtained with stronger forms of punishment, in particular those in Kehoe and Levine (1993) and Alvarez and Jermann (2000).[17]

From an empirical point of view, the central implication of our model is that equilibrium borrowing requires "low interest rates." In principle, this prediction could be used to test the hypothesis that international debt is sustained by our reputational mechanism. In practice, this is tantamount to testing for rational bubbles in international debt.[18] One standard approach to such a test would be to compare the rate of return on international debt with the growth rate of the aggregate stock of debt in circulation. An alternative approach focuses on net flows, taking a sample of countries who borrow on the international capital markets and evaluating whether their debtor position is associated to a net outflow or to a net inflow of resources. This is analogous to the approach in Abel at al. (1989), who focus on the net flows between the corporate sector and consumers in a closed economy. Empirical work in either direction will face two significant challenges. First, as emphasized in Gourinchas and Rey (2005), net financial flows between countries are the outcome of gross flows which include instruments with very different rates of return and risk profiles (debt, equity, direct investment and so on). Second, especially when looking at emerging economies, default episodes do take place regularly in international financial markets.[19] In this

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[17]See the discussion in Section 3. Our analysis also relates to the model of private international capital flows of Jeske (2006) and Wright (2006). In their model, the ability of agents to participate in domestic capital markets after defaulting on external debt has the same effects as the ability to save in our model. See the discussion in Wright (2006) for formal details.

[18]See Ventura (2004) and Caballero and Krishnamurthy (2006) for related work on bubbles in international capital flows. Notice that rational bubbles are often ruled out on theoretical grounds if there is a tradable asset that pays an infinite stream of dividends. This argument does not necessarily apply in a world with sovereign borrowers and multilateral lack of commitment. Real assets are typically tied to a physical location where the dividend is generated. The country controlling that location may choose to expropriate foreign nationals and bar them from accessing the asset's dividends in the event of a default.

[19]After disentangling the various gross positions and estimating rates of returns on the various instruments, Gour-

paper, we allow for fully state-contingent securities, so we do not treat explicitly gross holdings of different assets and do not take a stand on how to interpret actual defaults within the context of our model. To map the model's predictions to observed data on financial flows that include defaults, it would probably be necessary to extend the analysis by modelling explicitly gross positions in various financial instruments and to formulate an explicit interpretation of default episodes.

Finally, our paper also has implications for the literature on inside and outside money. In particular, in our setup unbacked public debt and self-enforcing private debt are analogous, respectively, to outside (fiat) money and inside money. The existing monetary literature discusses the circulation of fiat money and inside money largely in separation from each other. The circulation of fiat money requires that an intrinsically useless asset (a rational bubble) is traded at a positive price. The circulation of inside money instead relies on having the proper reputational mechanisms in place to guarantee that outstanding claims are honored. Although on the surface these seem to be conceptually distinct problems, our analysis shows that they are closely related.[20]

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inchas and Rey (2005) find that the US after 1988 has become a net debtor, and, in spite of that, has been typically receiving a net positive inflow of resources from the rest of the world. This finding is consistent with our story. Klingen, Weder, and Zettelmeyer (2004) look at ex post realized returns for a broad sample of emerging markets over the last 30 years and argue that returns were very close to the returns on 10-year US treasuries — a level which could very well be consistent with our story. Eichengreen and Portes (1989) and Lindert and Morton (1989) find similar results based on historical evidence.

[20]See Cavalcanti, Erosa and Temzelides (1999) and Berentsen, Camera and Waller (2007) for matching models of private debt circulation under limited commitment. In Cavalcanti et al. (1999), a fixed subset of agents is allowed to issue notes, which are sustained by the loss of a non-competitive note-issuing rent if outstanding notes are not redeemed on demand. Berentsen et al. (2007) study a matching model with money and competitive supply of bank credit, and show among other things that with lack of commitment, such credit is sustainable only if the inflation rate is non-negative.

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7   Appendix: Proof of Proposition 5

Suppose that $V\left(\phi\left(s^t\right), \Phi\left(s^t\right); s^t\right) = V_D\left(0; s^t\right)$ for all $s^t \in \mathcal{S}$, and that $\phi\left(s^t\right) < 0$ for some s^t (otherwise, the proposition holds trivially). The budget set $\mathcal{C}\left(a, \Phi\left(s^t\right); s^t\right)$ is non-empty, and hence the problem is well-defined, only if $a \geq -y\left(s^t\right) + \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right)$, from which it follows that $\Phi\left(s^t\right)$ must satisfy $\phi\left(s^t\right) \geq -y\left(s^t\right) + \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right)$. Standard properties of the consumer problem (4) imply that $V\left(a, \Phi\left(s^t\right); s^t\right)$ is strictly increasing in a, and the sequences $C^*\left(s^t\right)$ and $A^*\left(s^t\right)$ of optimal consumption and asset holdings, starting from an initial asset position $a^*\left(s^t\right) = \phi\left(s^t\right)$ at history s^t, are non-decreasing in $\phi\left(s^t\right)$.

Now, let $\mathcal{N}\left(s^t\right)$ denote the subtree of events starting from s^t, for which the debt limits are

24

non-binding: Starting from $\mathcal{N}_0\left(s^t\right) \equiv \left\{s^t\right\}$, define

$$
\begin{aligned}
\mathcal{N}_\tau\left(s^t\right) &= \left\{s^{t+\tau} \succ s^t : a^*\left(s^{t+\tau}\right) > \phi\left(s^{t+\tau}\right) \text{ and } \sigma\left(s^{t+\tau}\right) \in \mathcal{N}_{\tau-1}\left(s^t\right)\right\}, \\
\mathcal{B}_\tau\left(s^t\right) &= \left\{s^{t+\tau} \succ s^t : a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \text{ and } \sigma\left(s^{t+\tau}\right) \in \mathcal{N}_{\tau-1}\left(s^t\right)\right\}, \\
\mathcal{N}\left(s^t\right) &= \bigcup_{\tau=0}^{\infty} \mathcal{N}_\tau\left(s^t\right), \ \mathcal{B}\left(s^t\right) = \bigcup_{\tau=0}^{\infty} \mathcal{B}_\tau\left(s^t\right), \\
\mathcal{N}\left(s^{t+\tau}; s^t\right) &= \mathcal{N}\left(s^t\right) \cap \mathcal{S}\left(s^{t+\tau}\right) \text{ and } \mathcal{B}\left(s^{t+\tau}; s^t\right) = \mathcal{B}\left(s^t\right) \cap \mathcal{S}\left(s^{t+\tau}\right), \text{ for all } s^{t+\tau} \in \mathcal{N}\left(s^t\right).
\end{aligned}
$$

$\mathcal{N}_\tau\left(s^t\right)$ denotes the set of histories $s^{t+\tau}$ along which the debt limit was never binding between event s^t and $s^{t+\tau}$, and $\mathcal{N}\left(s^t\right)$ the union of all such sets, including s^t. $\mathcal{B}_\tau\left(s^t\right)$ denotes the set of histories $s^{t+\tau}$ at which the debt limit is binding for the first time after s^t, and $\mathcal{B}\left(s^t\right)$ the union of all such sets. $\mathcal{N}\left(s^{t+\tau}; s^t\right)$ defines the 'subtree' of $\mathcal{N}\left(s^t\right)$, which starts at $s^{t+\tau}$, and $\mathcal{B}\left(s^{t+\tau}; s^t\right)$ denotes the set of events at which the debt limit is binding for the first time after $s^{t+\tau}$. Finally, we define

$$
\hat{\phi}\left(s^{t+\tau}; s^t\right) = \sum_{s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} \phi\left(s^{t+\tau+k}\right) \text{ and } w\left(s^{t+\tau}; s^t\right) \equiv \sum_{s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} y\left(s^{t+\tau+k}\right)
$$

as the present value of the first binding debt limits and of the endowments over $\mathcal{N}\left(s^t\right)$.

The proof of Proposition 5 then proceeds in five steps, which are stated as separate lemmas. Lemma 1 establishes that under mild regularity conditions, $w\left(s^{t+\tau}; s^t\right)$ and $\hat{\phi}\left(s^{t+\tau}; s^t\right)$ are both finite-valued. Lemma 2 establishes the existence of a well-defined, finite-valued sequence of auxiliary debt limits $\tilde{\Phi}\left(s^t\right)$, which satisfies the following recursive characterization:

$$
\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \begin{cases} \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)\right\} & \text{if } a^*\left(s^{t+\tau}\right) > \phi\left(s^{t+\tau}\right) \\ \phi\left(s^{t+\tau}\right) & \text{if } a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \end{cases}
$$

(11)

for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. Lemma 3 establishes that $\tilde{\Phi}\left(s^t\right)$ bounds $\Phi\left(s^t\right)$ from below, and satisfies the (ER) condition as a weak inequality. Lemma 4 establishes that $\tilde{\Phi}\left(s^t\right)$ satisfies (ER) as an equality, and that $\tilde{\phi}\left(s^t; s^t\right) = \phi\left(s^t\right)$. Finally, Lemma 5 shows that the $\tilde{\phi}\left(s^{t+1}; s^t\right) = \phi\left(s^{t+1}\right)$ for all $s^{t+1} \succ s^t$.

Lemma 1 *Suppose that either that $u\left(\cdot\right)$ is bounded, or that $c^*\left(s^{t+\tau}\right) \leq \sum_{j=1}^J y^j\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, and that Assumption 1 holds. Then $w\left(s^{t+\tau}; s^t\right) < \infty$ and $\phi\left(s^{t+\tau}\right) + w\left(s^{t+\tau}; s^t\right) > \hat{\phi}\left(s^{t+\tau}; s^t\right) > -\infty$.*

Proof. Summing the agent's budget constraint over $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we get

$$\sum_{s^{t+\tau+k} \in \mathcal{N}(s^{t+\tau}; s^t)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} c^*\left(s^{t+\tau+k}\right) = a^*\left(s^{t+\tau}\right) + w\left(s^{t+\tau}; s^t\right) - \hat{\phi}\left(s^{t+\tau}; s^t\right)$$

$$- \lim_{K \to \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^{t+\tau}; s^t)} \frac{p\left(s^{t+\tau+K}\right)}{p\left(s^{t+\tau}\right)} a^*\left(s^{t+\tau+K}\right)$$

Now, using the agent's first-order condition for $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we have

$$\sum_{s^{t+\tau+k} \in \mathcal{N}(s^{t+\tau}; s^t)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} c^*\left(s^{t+\tau+k}\right) = \sum_{s^{t+\tau+k} \in \mathcal{N}(s^{t+\tau}; s^t)} \beta^k \frac{\pi\left(s^{t+\tau+k}\right)}{\pi\left(s^{t+\tau}\right)} \frac{u'\left(c^*\left(s^{t+\tau+k}\right)\right)}{u'\left(c^*\left(s^{t+\tau}\right)\right)} c^*\left(s^{t+\tau+k}\right) < \infty,$$

either because $u'\left(c\right) c \leq u\left(c\right) - u\left(0\right) \leq \bar{U}$, if $u\left(\cdot\right)$ is bounded, or because of Assumption 1, if $c^*\left(s^{t+\tau}\right) \leq \sum_{j=1}^{J} y^j\left(s^{t+\tau}\right)$ holds for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. In addition, the agent's first-order conditions over $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, along with the transversality condition imply

$$\lim_{K \to \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^{t+\tau}; s^t)} \frac{p\left(s^{t+\tau+K}\right)}{p\left(s^{t+\tau}\right)} a^*\left(s^{t+\tau+K}\right) = 0.$$

But then, it follows immediately that $w\left(s^{t+\tau}; s^t\right) < \infty$ and $\hat{\phi}\left(s^{t+\tau}; s^t\right) > -\infty$.

Finally, if $a\left(s^t\right) = -y\left(s^t\right) + \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right)$, the only feasible allocation without default yields $c\left(s^t\right) = 0$, and $a\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$ for all $s^{t+1} \succ s^t$, which yields a lifetime expected utility of $u\left(0\right) + \beta \sum_{s^{t+1} \succ s^t} \pi\left(s^{t+1}|s^t\right) V\left(\phi\left(s^{t+1}\right), \Phi\left(s^{t+1}\right), s^{t+1}\right)$. If instead the agent defaults and sets $d\left(s^{t+1}\right) = 0$ for all $s^{t+1} \succ s^t$, his lifetime expected utility is $u\left(y\left(s^t\right)\right) + \beta \sum_{s^{t+1} \succ s^t} \pi\left(s^{t+1}|s^t\right) V_D\left(0, s^{t+1}\right)$, which must be strictly preferred to no default if $\Phi\left(s^t\right)$ satisfies (SE). But then, $\phi\left(s^t\right) > -y\left(s^t\right) + \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right)$, for all $s^t \in \mathcal{S}$, and summing this inequality over $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we find $w\left(s^{t+\tau}; s^t\right) + \phi\left(s^{t+\tau}\right) - \hat{\phi}\left(s^{t+\tau}; s^t\right) > 0$. $\blacksquare$

Lemma 2 establishes the existence of a finite-valued sequence of 'auxiliary' debt limits $\tilde{\Phi}\left(s^t\right)$.

Lemma 2 *There exists a finite-valued sequence $\tilde{\Phi}\left(s^t\right)$ which satisfies (11), for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, and for which $\lim_{K \to \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^{t+\tau}; s^t)} p\left(s^{t+\tau+K}\right) \tilde{\phi}\left(s^{t+\tau+K}; s^t\right) = 0$.*

Proof. Step 1 establishes the existence of such a solution for $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, together with the limit property. Step 2 extends the construction to $\mathcal{S}\left(s^t\right)$.

Step 1: Let $\Phi^{(0)}$ be defined by $\phi^{(0)}\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}; s^t\right) - Y\left(s^{t+\tau}; s^t\right)$, and define $\Phi^{(K)}\left(s^t\right)$ recursively by $\Phi^{(K)} = \mathcal{T}\Phi^{(K-1)}$, where the operator $\mathcal{T}$ on sequences $B \in \mathbb{R}^{\mathcal{N}(s^t)}$ is defined by

$$\left(\mathcal{T}B\right)\left(s^{t+\tau}\right) = \sum_{s^{t+\tau+1} \in \mathcal{N}(s^{t+\tau}; s^t)} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), b\left(s^{t+\tau+1}\right)\right\} + \sum_{s^{t+\tau+1} \in \mathcal{B}(s^{t+\tau}; s^t)} q\left(s^{t+\tau+1}\right) \phi\left(s^{t+\tau+1}\right).$$

26

Since $\phi\left(s^{t+\tau}\right) \geq \hat{\phi}\left(s^{t+\tau}; s^t\right) - w\left(s^{t+\tau}; s^t\right) = \phi^{(0)}\left(s^{t+\tau}\right)$, we have $\phi^{(1)}\left(s^{t+\tau}\right) \geq \phi^{(0)}\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, and therefore $\Phi^{(1)} \geq \mathcal{T}\Phi^{(0)}$. But $\mathcal{T}$ is a monotone operator, and therefore, $\left\{\Phi^{(K)}\right\}$ is a non-decreasing sequence. Moreover, since $\phi^{(K)}\left(s^{t+\tau}\right) \leq 0$, for all K and $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, $\left\{\Phi^{(K)}\right\}$ must converge to a finite limit $\tilde{\Phi}\left(s^t\right)$ which satisfies (11). The limit property then follows immediately from $0 \geq \tilde{\phi}\left(s^{t+\tau+K}; s^t\right) \geq \phi^{(0)}\left(s^{t+\tau+K}\right) = \hat{\phi}\left(s^{t+\tau+K}; s^t\right) - w\left(s^{t+\tau+K}; s^t\right)$, and the fact that $\lim_{K\to\infty} \sum_{s^{t+\tau+K}\in\mathcal{N}(s^{t+\tau};s^t)} p\left(s^{t+\tau+K}\right)\left[\hat{\phi}\left(s^{t+\tau+K}; s^t\right) - w\left(s^{t+\tau+K}; s^t\right)\right] = 0$.

Step 2: Define $\mathcal{B}^{(1)}\left(s^t\right) = \mathcal{B}\left(s^t\right)$ and let

$$\mathcal{B}^{(k)}\left(s^t\right) = \bigcup_{\tau=k}^{\infty}\left\{s^{t+\tau} \succ s^t : a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \text{ and } \sigma\left(s^{t+\tau}\right) \in \mathcal{N}\left(s^{t+\tau'}\right) \text{ for some } s^{t+\tau'} \in \mathcal{B}^{(k-1)}\left(s^t\right)\right\}.$$

denote the subset of histories in $\mathcal{S}\left(s^t\right)$, at which the debt limit is binding for the k-th time after s^t. Since $A^*\left(s^t\right)$ solves the consumer problem starting from s^t with asset position $\phi\left(s^t\right)$, and $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$, $\left\{a^*\left(s^{t+\tau'}\right)\right\}_{s^{t+\tau'}\in\mathcal{S}(s^{t+\tau})}$ solves the consumer problem starting from any $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$ with asset position $\phi\left(s^{t+\tau}\right)$. Since s^t was chosen arbitrarily, we can replicate the same arguments as above for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$ to construct a solution $\tilde{\Phi}\left(s^t\right)$ to (11) for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. $\blacksquare$

Lemma 3 establishes that (i) $\tilde{\Phi}\left(s^t\right)$ bounds $\Phi\left(s^t\right)$ from below, and (ii) $\tilde{\Phi}\left(s^t\right)$ satisfies the exact roll-over property as a weak inequality. We prove these properties for $\mathcal{N}\left(s^t\right)$ and $\mathcal{B}\left(s^t\right)$; they can immediately be extended to $\mathcal{S}\left(s^t\right)$ using the construction of the previous proof.

Lemma 3 (i) *For all* $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, $\phi\left(s^{t+\tau}\right) \geq \tilde{\phi}\left(s^{t+\tau}; s^t\right) = \hat{\phi}\left(s^{t+\tau}; s^t\right)$ *and* $\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \sum_{s+\tau+1\succ s+\tau} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)$.

(ii) *For all* $s^{t+\tau} \in \mathcal{B}\left(s^t\right)$, $\tilde{\phi}\left(s^{t+\tau}; s^t\right) \geq \sum_{s+\tau+1\succ s+\tau} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)$.

Proof. Part (i): Suppose to the contrary that $\phi\left(s^{t+\tau}\right) < \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for some $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, and let $\left\{\tilde{a}\left(s^{t+\tau+k}\right)\right\}_{s^{t+\tau+k}\in\mathcal{S}(s^t)\setminus\{s^t\}}$ denote the optimal asset profile without default starting from a position of $\phi\left(s^{t+\tau}\right)$ at $s^{t+\tau}$. Consider an agent who defaults and sets

$$\hat{a}\left(s^{t+\tau+k}\right) = \begin{cases} \tilde{a}\left(s^{t+\tau+k}\right) - \min\left\{\phi\left(s^{t+\tau+k}\right), \tilde{\phi}\left(s^{t+\tau+k}; s^t\right)\right\} & \text{for all } s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right) \\ 0 & \text{for all } s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right) \end{cases},$$

which is feasible after a default. From the monotonicity of optimal asset holdings, $\tilde{a}\left(s^{t+\tau+k}\right) \leq a^*\left(s^{t+\tau+k}\right)$, and $\tilde{a}\left(s^{t+\tau+k}\right) = a^*\left(s^{t+\tau+k}\right) = \phi\left(s^{t+\tau+k}\right)$, whenever $s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)$. Moreover, because of (SE), $V\left(\phi\left(s^{t+\tau+k}\right), \Phi\left(s^{t+\tau+k}\right), s^{t+\tau+k}\right) = V_D\left(0, s^{t+\tau+k}\right)$, so that the default profile provides the same lifetime utility going forward from any $s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)$ as the non-default profile. For $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, the difference in consumption between default and no default

27

$\left\{\Delta c\left(s^{t+\tau+k}\right)\right\}$ is

$$
\begin{aligned}
\Delta c\left(s^{t+\tau+k}\right) &= -\min\left\{\phi\left(s^{t+\tau+k}\right),\tilde{\phi}\left(s^{t+\tau+k};s^t\right)\right\}+ \\
&\qquad \sum_{s^{t+\tau+k+1}\succ s^{t+\tau+k}} q\left(s^{t+\tau+k+1}\right)\min\left\{\phi\left(s^{t+\tau+k+1}\right),\tilde{\phi}\left(s^{t+\tau+k+1};s^t\right)\right\} \\
&= -\min\left\{\phi\left(s^{t+\tau+k}\right)-\tilde{\phi}\left(s^{t+\tau+k};s^t\right),0\right\}\geq 0,
\end{aligned}
$$

where the inequality is strict at $s^{t+\tau}$. Therefore, consumption is weakly higher after a default for all $s^{t+\tau+k}\in\mathcal{N}\left(s^{t+\tau};s^t\right)$, and strictly higher at $s^{t+\tau}$, implying that default must be optimal. Now, using $\phi\left(s^{t+\tau+1}\right)\geq\tilde{\phi}\left(s^{t+\tau+1};s^t\right)$ for all $s^{t+\tau+1}\in\mathcal{N}\left(s^{t+\tau};s^t\right)$ and $\phi\left(s^{t+\tau+1}\right)=\tilde{\phi}\left(s^{t+\tau};s^{t+\tau}\right)$ for all $s^{t+\tau+1}\in\mathcal{B}\left(s^{t+\tau};s^t\right)$ in (11) then implies (ER) for all $s^{t+\tau}\in\mathcal{N}\left(s^t\right)$. Solving (11) forward and using the limit property from lemma 2, we find $\tilde{\phi}\left(s^{t+\tau};s^t\right)=\hat{\phi}\left(s^{t+\tau};s^t\right)$.

Part (ii): Applying the same arbitrage argument as in part (i) at $s^{t+\tau}\in\mathcal{B}\left(s^t\right)$, we have $\phi\left(s^{t+\tau}\right)\geq\tilde{\phi}\left(s^{t+\tau};s^{t+\tau}\right)$, and $\tilde{\phi}\left(s^{t+\tau};s^{t+\tau}\right)=\sum_{s+\tau+1\succ s+\tau}q\left(s^{t+\tau+1}\right)\tilde{\phi}\left(s^{t+\tau+1};s^{t+\tau}\right)$. Moreover, by construction, $\tilde{\phi}\left(s^{t+\tau};s^t\right)=\phi\left(s^{t+\tau}\right)$ and $\tilde{\phi}\left(s^{t+\tau+1};s^t\right)=\tilde{\phi}\left(s^{t+\tau+1};s^{t+\tau}\right)$, for all $s^{t+\tau+1}\succ s^{t+\tau}$, from which the result follows immediately. ∎

Lemma 4 uses these two properties to show that $\phi\left(s^t\right)=\tilde{\phi}\left(s^t;s^t\right)$, and that $\tilde{\Phi}\left(s^t\right)$ satisfies (ER) with equality, for all $s^{t+\tau}\in\mathcal{S}\left(s^t\right)$.

Lemma 4 For all $s^t\in\mathcal{S}$, $\phi\left(s^t\right)=\tilde{\phi}\left(s^t;s^t\right)$, and $\tilde{\Phi}\left(s^t\right)$ satisfies (ER) with equality.

Proof. Consider the consumer problem with borrowing constraints equal to $\tilde{\Phi}(s^t)$. Since the objective is strictly concave, $\phi(s^{t+\tau})\geq\tilde{\phi}(s^{t+\tau};s^t)$ for all $s^{t+\tau}\in\mathcal{S}\left(s^t\right)$, and $\phi(s^{t+\tau})>\tilde{\phi}(s^{t+\tau};s^t)$ only if $a^*(s^{t+\tau})>\phi(s^{t+\tau})$, $\tilde{\Phi}(s^t)$ relaxes only non-binding constraints, and hence $A^*(s^t)$ is also optimal for the relaxed problem with borrowing constraints $\tilde{\Phi}(s^t)$, implying $V(\phi(s^t),\tilde{\Phi}(s^t),s^t)=V(\phi(s^t),\Phi(s^t),s^t)$. Using the self-enforcement hypothesis and the monotonicity of $V(a,\tilde{\Phi}(s^t),s^t)$ in a and $\phi(s^t)\geq\tilde{\phi}(s^t;s^t)$, this implies $V_D\left(0,s^t\right)=V(\phi(s^t),\Phi(s^t),s^t)=V(\phi(s^t),\tilde{\Phi}(s^t),s^t)\geq V(\tilde{\phi}(s^t;s^t),\tilde{\Phi}(s^t),s^t)$. On the other hand, since $\tilde{\Phi}(s^t)$ satisfies the exact roll-over property as a weak inequality, the same argument as proposition 4 implies that $V(\tilde{\phi}(s^t;s^t),\tilde{\Phi}(s^t),s^t)\geq V_D\left(0,s^t\right)$, where the inequality is strict whenever $\tilde{\phi}(s^{t+\tau};s^t)>\sum_{s^{t+\tau+1}\succ s^{t+\tau}}q(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1};s^t)$ for some $s^{t+\tau}\in\mathcal{S}(s^t)$. Together these inequalities can hold only as equalities, which requires that $\phi(s^t)=\tilde{\phi}(s^t;s^t)$, and $\tilde{\phi}(s^{t+\tau};s^t)=\sum_{s^{t+\tau+1}\succ s^{t+\tau}}q(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1};s^t)$ for all $s^{t+\tau}\in\mathcal{S}(s^t)$. ∎

To complete the proof that $\Phi\left(s^t\right)$ satisfies (ER), we thus need to show that $\phi\left(s^{t+1}\right)=\tilde{\phi}\left(s^{t+1};s^t\right)$ for all $s^{t+1}\in\mathcal{S}\left(s^t\right)$. Whenever $s^{t+1}\in\mathcal{B}_1\left(s^t\right)$, i.e. whenever the debt limit is binding at s^{t+1}, this is true by construction. Our final lemma shows that this is also true whenever the debt limit is not binding.

Lemma 5 *For all $s^{t+1} \in \mathcal{N}_1\left(s^t\right)$, $\phi\left(s^{t+1}\right) = \tilde{\phi}\left(s^{t+1}; s^t\right)$.*

Proof. Since s^t was chosen arbitrarily, applying all the preceding arguments to $s^{t+1} \in \mathcal{N}_1\left(s^t\right)$ implies that $\phi\left(s^{t+1}\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$, so it suffices to show that $\tilde{\phi}\left(s^{t+1}; s^t\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$. Now, from the monotonicity of $\left\{a^*\left(s^{t+\tau}\right)\right\}$ w.r.t. the initial asset holdings, it follows that $\mathcal{N}\left(s^{t+1}\right) \subseteq \mathcal{N}\left(s^{t+1}; s^t\right)$ and $\mathcal{B}\left(s^{t+1}\right) \subseteq \mathcal{B}\left(s^{t+1}; s^t\right) \cup \mathcal{N}\left(s^{t+1}; s^t\right)$, i.e. debt limits must be binding for an agent starting from s^{t+1} with assets $\phi\left(s^{t+1}\right)$, whenever they are binding for an agent starting from s^{t+1} with assets $a^*\left(s^{t+1}\right) > \phi\left(s^{t+1}\right)$. But then, by the definition of (11), $\tilde{\phi}\left(s^{t+\tau}; s^{t+1}\right) = \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for all $s^{t+\tau} \in \mathcal{B}\left(s^{t+1}; s^t\right)$, and both $\tilde{\Phi}\left(s^t\right)$ and $\tilde{\Phi}\left(s^{t+1}\right)$ satisfy (ER) for all $s^{t+\tau} \in \mathcal{N}\left(s^{t+1}; s^t\right)$, from which it follows immediately that $\tilde{\phi}\left(s^{t+\tau}; s^{t+1}\right) = \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for all $s^{t+\tau} \in \mathcal{N}\left(s^{t+1}; s^t\right)$, and hence $\tilde{\phi}\left(s^{t+1}; s^t\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$. ∎