Dynamic Systems of Social Interactions

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Abstract

We state conditions for existence and uniqueness of equilibria in dynamic microeconomic models with an infinity of locally and globally interacting agents. Agents face repeated discrete choice problems. Their utility depends on the actions of some designated neighbors and the average choice throughout the whole population. We show that the dynamics on the level of aggregate behavior can be described by a deterministic measure-valued integral equation. If some form of positive complementarities prevails we establish convergence results for aggregate activities. If the coupling between different agents is sufficiently weak, the dynamics is ergodic: macroscopic quantities converge to a unique limit independently of the initial conditions. We apply our convergence results to study a glass of population games with random matching.

Preliminary - Comments Wecome

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1 Introduction

A common observation in economics and the social sciences is the emergence of large differences in long run aggregate variables in the absence of corresponding differences in initial conditions. To accommodate this phenomenon, a model must generate an amplifier effect that transforms small changes in initial conditions into large changes in aggregate outcomes. Models of social interactions are capable of displaying amplifier effects. In these models an agent's behavior depends on the choices of other agents. In the presence of positive complementarities a change in initial conditions has a direct effect on the behavior of an agent and an indirect effect through the interaction with others that are of the same sign. If these complementarities are powerful enough, small differences in initial conditions may be amplified as time passes and hence different aggregate activities may emerge in the long run.

Much of the literature on social interactions assumes that interactions are either local or global. Agents interact locally when each agent interacts with only a small, stable set of peers in an otherwise large economy. Local interactions are designed to capture economic environments where markets do no exist to mediate all of the agents' choices. Agents interact global if they only care about the distribution of actions or preferences throughout the whole population. Global interactions naturally capture market interactions and uniform random matching. Pure local and pure global interaction models are well understood.[1] Horst & Scheinkman (2006a, 2006b) recently established existence and convergence results for equilibria in static economies with an infinity of locally and globally interacting agents. Combining local and global interactions allows for an integration of global externalities with neighborhood effects and to consider economies in which agents interact with peers and, at the same time, act as price takers in competitive markets. Local technological complementarities and counterparty relations between business partners, for instance, have been identified as important determinants of economic growth (Durlauf 1993) and channels for the spread of financial distress (Allen & Gale 2000), respectively, but firms also interact globally through market; rumors in financial markets spread through word-of-mouth communication (Kosfeld 2002) but they are also transmitted when agents meet at random (Banerjee 1993).

In this paper we establish the existence of equilibria in a class of dynamic discrete choice models with infinitely many interacting agents. Following the seminal approach by Blume (1993), we assume that agents face repeated binary choice problems. The agents change their actions at random points in time at a rate that depends on the current states of some designated neighbors and the average situation throughout the whole population. The randomness in the agents' behavior prevents choices from converging pathwise to some steady state so

[1] See, e.g., Kandori, Mailath & Rob (1993), Blume (1993), Benaim & Weibull (2003) and references therein.

an appropriate notion of equilibrium is an invariant distribution.[2] Due to the dependence of the transition dynamics on aggregate quantities the microscopic process of individual states standard results on existence and uniqueness of invariant distributions do not apply. To overcome this problem we apply the method of separating the local and global interaction introduced by Föllmer & Horst (2001). Such a separation is not necessary in models with a finite population as in Ioannides & Soetevent (2005). However when global interactions are present the analysis is most naturally done in the context of an infinity of agents where one can appeal to laws of large numbers. We prove indeed that despite the strong correlations between individual choices aggregate quantities follow a recursive deterministic dynamics. This implies that our model can be viewed as a time-inhomogeneous model of purely local interaction with global externalities. If some form of positive complementarities prevail and the flip rates strictly positive, we establish the existence of maximal and minimal equilibria. Under a weak interaction condition the equilibrium is unique and aggregate quantities settle down to a unique limit. Uniqueness breaks down if the interaction becomes too strong. Our results are applied to a class of coordination games with local interaction and random matching. We show that self-reinforcing "negative" global externalities may prevent the agents from coordinating on the Pareto optimal equilibrium. This distinguishes our results from those of, for instance, Blume (1993) and Ellison (2000).

Dynamic systems of social interactions is introduced in Section 2. Existence of equilibria is proved in Section 3 while Section 4 outlines an application to population games.

2 Dynamic Systems with Discrete Choice

In this section we define dynamic systems of social interactions. The agents are located on the d-dimensional lattice with integer entries $\mathbb{A}$ and choose actions from the binary set $C = \{-1, +1\}$. A *configuration* is a list of actions for each agent. The configuration space

$$E := \{\eta = (\eta^a)_{a\in\mathbb{A}} : \eta^a \in C\}$$

is equipped with the product topology and hence it is compact. The agents myopically switch their states from η^a to $-\eta^a$ at Poisson random times. The probabilities with the agent a switches her state in s short period of time given that all the other agents are configured at η^{-a} are governed by the *flip rates* $c(\eta, a)$. The impact of the current configuration on her transition dynamics is felt *locally* through the actions taken by the agents in her *neighborhood*

$$N(a) := \{b : |a - b| = 1\} \tag{1}$$

[2]The idea of using stationary distributions to describe run long behavior in economic models has gained a lot of popularity in the last decade; see, e.g. Foster & Young (1990), Kirman (1992), and Young (1993).

and *globally* through the distribution of choices throughout the entire population. To accommodate the global component we assume that the flip rates depend on the *empirical field* $R(\eta)$ associated with an action profile $\eta \in E$:

$$c(\eta, a) \equiv c(\eta, a, R(\eta)). \tag{2}$$

The empirical field $R(\eta)$, which is defined in (17) below, is a probability measure on the configuration space. It can be viewed as an object that carries all macroscopic information about η. The empirical distribution of choices, for instance, is given by the one-dimensional marginal distribution. The proportion of adjacent agents configured in the "plus" state is given by the integral of the map $f : E \to \mathbb{R}$ that takes the value one if $\eta^a = 1$ for all $|a| \leq 1$ and zero else with respect to $R(\eta)$.

2.1 Separating local and global interactions

The general form of the flip rates (2) is not convenient for proving the existence of equilibrium distributions for the *microscopic process* $\{\eta_t\}$ that describes the evolution of all the individual sates. If the flip rates depend on aggregate quantities in a non-trivial manner they are not continuous in the product topology so standard existence and uniqueness results for stationary distributions of Markov processes do not apply. To overcome this difficulty we define the flip rates as *continuous* mappings $c(\cdot, a, \cdot) : E \times \mathcal{M}_h \to \mathbb{R}_+$ on the extended state space $E \times \mathcal{M}_h$ where $\mathcal{M}_h$ denotes the class of all spatially homogeneous probability measures on E. The flip rates $c(\eta, a, \mu)$ describe the transition of choices in a benchmark model of purely local interactions when the agents' assessment about the overall distribution of choices is "frozen" to μ. In view of (1) they satisfy the Markov property

$$c(\eta, a, \mu) = c(\xi, a, \mu) \quad \text{if} \quad \eta^b = \xi^b \text{ for } b \in N(a). \tag{3}$$

In order to construct a model where at any point in time the instantaneous flip rate is of the form (3) with μ replaced by the prevailing empirical field we need to assume some form of spatial homogeneity. Otherwise there is no reason to assume that configurations have empirical averages. We therefore assume that all agents are identical ex-ante so that the flip rates are invariant with respect to simultaneous shifts of agents and configurations.

Definition 2.1 *A dynamic system of social interactions is defined by a family of translation invariant flip rates* $\{c(\eta, a, \mu)\}_{\eta \in E, a \in \mathbb{A}, \mu \in \mathcal{M}_h}$ *that satisfy (3).*

We denote the *generator* associated with the rates $c(\eta, a, \mu)$ by A_μ. It acts on the class $\mathcal{L}$ of all *local functions* that depend only on finitely many coordinates according to

$$A_\mu f(\eta) = \sum_{x \in \mathbb{A}} c(\eta, x, \mu)[f(\eta_x) - f(\eta)] \tag{4}$$

where the configuration η_x coincides with η at $x \neq u$ and $\eta_x^x = -\eta^x$. The operator A_μ generates a *Markov semi-group* (S_t^μ) and hence a Markov process on E. If the assessments about the distribution of choices follow a measure-valued process $\Psi = \{\Psi_t\}$, the infinitesimal generator at time t is given by A_{Ψ_t} and we denote by $\mathbb{P}^\Psi$ the distribution of the associated time-inhomogeneous Markov process on the canonical path space:

$$\mathbb{P}^\Psi[\eta_{t+h}^a \neq \eta_t^a | \eta_t] = c(\eta_t, a, \Psi_t)h + o(h) \quad \text{as} \quad h \downarrow 0. \tag{5}$$

Remark 2.2 *If the measure Ψ_t is ergodic, i.e., trivial on the σ-fields of all shift invariant events, an agent may either react to the expected behavior of an individual under Ψ_t or the average choice throughout the entire population. The two quantities are almost surely equal:*

$$\int \eta_t^0 \Psi_t(d\eta_t) = \lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}} \eta_t^a \qquad \Psi_t\text{-a.s.}$$

where $\mathbb{A}_n := [-n,n]^d \cap \mathbb{A} \uparrow \mathbb{A}$. We denote the class of all ergodic measures by $\mathcal{M}_0$.

2.2 The microscopic and the macroscopic process

The construction of the *microscopic* process $\{\eta_t\}$ where the assessment about aggregate behavior in (5) is replaced by the actual empirical field requires a continuity condition on the dependence of the flip rates on the agents' assessment about the situation at the global scale. To make this more precise, we denote by

$$\Delta_y(f) := \sup\{|f(\eta) - f(\xi)| : \eta^x = \xi^x \,\forall\, x \neq y\} \quad \text{and} \quad \mu(f) := \int f d\mu$$

the *oscillation at site* y of the function f on E and the integral of f with respect to μ, respectively, fix some constant $r > 0$ and recall that the metric

$$d_r(\mu,\nu) := \sup_{f\in\mathcal{L}} \frac{|\mu(f) - \nu(f)|}{\sum_y 2^{r|y|}\Delta_y(f)} \tag{6}$$

induces the weak topology on the class $\mathcal{M}_h$ of spatially homogeneous probability measures on E.

Assumption 2.3 *The flip rates depend in a Lipschitz continuous manner on μ, i.e.,*

$$|c(\cdot,\cdot,\mu) - c(\cdot,\cdot,\nu)| \leq L d_r(\mu,\nu).$$

Our continuity condition is sufficient for the dynamics of the microscopic process to be well defined. It is a mild assumption that is satisfied if, for instance, the rates depend on μ only through the expected behavior,

$$m(\mu) := \int \eta^0 \mu(d\eta), \tag{7}$$

of an individual under μ and if this dependence is Lipschitz continuous; see Example 3.5 below for details. A formal construction of the particle process is given in the appendix.

Theorem 2.4 *If the flip rates $c(\eta, x, \mu)$ satisfy Assumption 2.3, there exists a unique Markov process (η_t) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $E_0 \subseteq E$ such that*

$$\mathbb{P}\left[\eta^a_{t+h} \neq \eta^a_t | \eta_t = \eta\right] = c\left(\eta, a, R(\eta)\right) h + o(h) \quad \textit{as} \quad h \downarrow 0 \qquad (\eta \in E_0).$$

For any $\eta \in \mathbb{E}_0$ the empirical field $R(\eta)$ exists as an ergodic probability measure on E.

The preceding theorem states that under Assumption 2.3 the microscopic processes induces the *macroscopic process* $\{R(\eta_t)\}$ of ergodic empirical fields. Ergodicity of the macroscopic process implies that the configurations have spatial averages at any point in time so a combination of local and global interactions makes sense. The next result shows that aggregates evolve through time as if they were deterministic. Moreover, empirical fields are a sufficient statistic: knowledge of the current empirical field is enough to predict aggregates at future times; such a recursion typically does not hold on the level of empirical averages. Our model can hence be viewed as a model of purely local interactions with a time-inhomogeneous externality generated by the global interaction. The proof is postponed to the appendix.

Theorem 2.5 *Under the assumption of Theorem 2.4 the macroscopic process $\{R(\eta_t)\}$ is given as the unique solution to the measured-valued integral equation*

$$R_t(f) = R(\eta_0) + \int_0^t R_u(A_{R_u} f)\, du \quad (f \in \mathcal{L}). \tag{8}$$

For a model of mean-field interactions where the agents only care about the distribution of choices in the population the integral equation in (8) reduces to an ordinary differential equation. In this case the microscopic process for an individual agent can be constructed by analogy to the so-called Mckean process or, more directly, using Kurtz's (1978) strong approximation result for Poisson processes by Browninan motion. The former method has recently been applied by Tanabe (2006) to prove propagation of chaos results of large economies. Both approaches are based on an approximation of infinite economies by large but finite ones and explicitly use the mean-field interaction. Neither approach carries over to local interactions; for mean-field models all three approaches are equivalent.

Remark 2.6 *Ordinary differential equations and not measure-valued equations commonly arise in many discrete-time models of adaptive learning and fictitious play. In both cases the state is a vector of accumulated empirical frequencies of strategies used in the previous rounds. The ODE arises in the limit when time tends to infinity so the one-step changes in* *time-averages become small. Our approach is qualitatively quite different. The macroscopic process describes the evolution of spatial rather than temporal averages. As such our approach is conceptually much closer to that of Benaim & Weibull (2003). They consider a sequence of discrete time mean-field models and study the evolution of population averages. The limiting ODE arises from passing to a continuous time limit.*

3 Equilibria of Monotone Systems

In a random economy with many agents where probabilistic choices prevent configurations from converging pathwise to some steady state, an appropriate notion of equilibrium is not a particular state, but rather a distribution of states which reflects the proportion of time the agents spend in the states. This calls for an existence result for invariant distributions for the microscopic process. When started with an invariant measure μ the distribution of individual choices does not change over time. Since any invariant measure can be written as a convex combination of ergodic ones we may with no loss of generality assume that μ is ergodic. In this case the empirical field satisfies $R(\eta_t) \equiv \mu$ so aggregate quantities are in a steady state while individual choices fluctuate in accordance with μ.

Definition 3.1 *A probability measure μ^* on E is an equilibrium of the system of social interactions $\{c(\eta, x, \mu)\}$ if μ^* is an ergodic stationary distribution for the associated Markov process. It is globally stable if the macroscopic process converges to μ^* independently of the initial conditions.*

We assume that an agent is more likely to flip to the opposite state in a short period of time if she generally disagrees with her environment than if it generally agrees with it. In a benchmark model of purely local interactions this translates into the following condition on the flip rates: if $\eta^a \leq \xi^a$ for all $a \in N(x)$, then

$$c(\eta, x, \mu) \leq c(\xi, x, \mu) \text{ if } \eta^x = \xi^x = -1 \quad \text{and} \quad c(\eta, x, \mu) \geq c(\xi, x, \mu) \text{ if } \eta^x = \xi^x = 1. \quad (9)$$

In order to state a corresponding condition when the agents also react to the population average, we write $\nu \geq \mu$ for two probability measures if the integral of monotone functions with respect to ν dominates the integrals of the same functions with respect to μ. The idea that an agents' propensity to switch to a different state increases in the number of agents the proportion of agents configured in that state is captured by the following definition.

Definition 3.2 *The system is monotone if $\nu \geq \mu$ and $\eta^a \leq \xi^a$ for all $a \in N(x)$ implies*

$$c(\eta, x, \mu) \leq c(\xi, x, \nu) \;\; \textit{if} \;\; \eta^x = \xi^x = -1 \;\; \textit{and} \;\; c(\eta, x, \mu) \geq c(\xi, x, \nu) \;\; \textit{if} \;\; \eta^x = \xi^x = 1. \quad (10)$$

Our monotonicity condition can be viewed as a version of the positive complementarity conditions commonly assumed in the theory of supermodular games,[3] suitably adapted to capture the random and discrete nature of the agents' choice dynamics. In supermodular games an agents' utility of increasing an action is increasing in the other players actions while in our model an agent's propensity to switch to a higher state is increasing in the neighbors states and overall average. If the flip rates were differentiable assumption (10) would translate into a positivity condition on the cross-partial derivatives of the rate function with respect to an agent's own state and, respectively, the neighbors' choices and the average.

The monotonicity condition allows us to prove that when started in an "all low" or "all high" configuration, aggregate quantities settle down in the long run.

Theorem 3.3 *Let $\{R_t^{\pm\mathbf{1}}\}$ be the macroscopic processes of a monotone system when $\eta_0^a \equiv \pm 1$.*

(i) For all $s \leq t$ the processes $(R_t^{-\mathbf{1}})$ and $(R_t^{+\mathbf{1}})$ satisfy the monotonicity condition

$$R_s^{-\mathbf{1}}(f) \leq R_t^{-\mathbf{1}}(f) \quad \textit{and} \quad R_s^{\mathbf{1}}(f) \geq R_t^{+\mathbf{1}}(f)$$

for every real-valued functions f on E for with $f(\eta) \leq f(\xi)$ if $\eta^a \leq \xi^a$ for all $a \in \mathbb{A}$.

(ii) The weak limits $\underline{\mu} := \lim_{t\to\infty} R_t^{-\mathbf{1}}$ and $\overline{\mu} := \lim_{t\to\infty} R_t^{+\mathbf{1}}$ exist.

The link between supermodular games and our dynamic discrete choice model suggests that the existence results for smallest and largest equilibria in supermodular games carries over to our framework. This is in fact true. In order to characterize the set of equilibrium distributions in terms of equilibrium distributions of the purely local systems we recall that a Markov processes is called ergodic if it has a unique invariant distribution and time averages converge to their expected values under the invariant measure.

Theorem 3.4 *If the Markov processes with generators A_μ are ergodic with invariant distribution ν_μ, then the following holds:*

(i) Every ergodic equilibrium distribution μ satisfies the fixed point condition $\mu = \nu_\mu$.

(ii) The weak limits $\underline{\mu}$ and $\overline{\mu}$ are equilibria. These equilibria are extremal in the sense that

$$\underline{\mu} = \inf\{\mu \;:\; \mu = \nu_\mu\} \in \mathcal{M}_0 \quad \textit{and} \quad \overline{\mu} = \sup\{\mu \;:\; \mu = \nu_\mu\} \in \mathcal{M}_0. \tag{11}$$

(iii) The system has a unique equilibrium if $\underline{\mu} = \overline{\mu}$, i.e., if the following fixed point condition has a unique solution:

$$\mu^* = \nu_{\mu^*}. \tag{12}$$

In this case the equilibrium is globally stable: the macroscopic process converges to μ^* independently of the initial distribution.

[3] See, e.g., Milgron & Roberts (1990), Topkis (1979) and Vives (1990) and references therein

Before proceeding with an example where all our assumption can be verified we recall a fundamental theorem of statistical mechanics. It states that when the flip rates $c(\eta, a, \mu)$ are strictly positive, they can be represented as exponentials of some potential function and the set of stationary distributions of the associated Markov process with the generator A_μ is given by a set of Gibbs measures; see Blume (1993) and references therein for details[4]. In particular the Markov process is ergodic with unique stationary distribution ν_μ if this set contains a single element. This is guaranteed if Dobrushin's uniqueness condition (Georgii 1988) holds, i.e., if the interaction between different agents is sufficiently weak. Under a slightly stronger condition the map $\mu \mapsto \nu_\mu$ is a contraction with respect to the metric d_r introduced in (6). In this case the fixed point condition (12) has a unique solution and as a result the macroscopic process of monotone systems is globally stable.

Example 3.5 *Let us assume that $\mathbb{A} = \mathbb{Z}^2$ and that $c(\eta, a, \mu)$ depends on μ only through the expected action $m(\mu)$ of an individual under μ. Suppose furthermore that the flip rates take the form of an Ising model of statistical mechanics as in Blume (1993) or Föllmer (1974):*

$$c(\eta, a, \mu) = \frac{1}{1 + \exp\left[2\beta\eta^a \left(h + J_1 m(\mu) + J_2 \sum_{b \in N(a)} \eta^b\right) .\right]} \tag{13}$$

As usual $\beta \geq 0$ and $J_2 \geq 0$ measures the strength of interaction and an agents desire for conformity, respectively ad $h + J_1 m(\mu)$ is an intrinsic value associated with an agent's choice. The case $J_1 = 0$ corresponds to Blume's (1993) local interaction model while $J_2 = 0$ yields a version of the mean-field model of Brock and Durlauf (2001). The function on the right hand side of equation (13) is Lipschitz continuous in $m(\mu)$. Since $m(\mu) = \int f d\mu$ with $f(\eta) = \eta^0$ and $\Delta_0(f) = 2$ while $\Delta_x(f) = 0$ for $x \neq 0$ we see that Assumption 2.3 holds because

$$|c(\eta, a, \mu) - c(\eta, a, \nu)| \leq L| \int f d\mu - \int f d\nu| \leq 2L d_r(\mu, \nu).$$

Furthermore, $\nu \geq \mu$ implies $m(\nu) \geq m(\mu)$ so the monotonicity condition holds, too. Hence the microscopic process is well defined and the macroscopic process generates the deterministic externality $\{m(R(\eta_t))\}$. When started in an "all high" or "all low" configuration its settles down in the long run. The Markov processes with generator A_μ has a unique stationary distribution ν_μ if β is sufficiently small or $|h| > J_1$. By Proposition 2.22 in Horst (2001)

$$d_r(\nu_{\mu_1}, \nu_{\mu_2}) < \theta d_r(\mu_1, \mu_2) \quad \textit{for some} \quad \theta < 1$$

if the Lipschitz constant L is less than one half, i.e., if $\beta \cdot J_1$ is sufficiently small. In this case the fixed point condition (12) has a unique solution.

[4]Notice that the model of Kosfeld (2005) is not covered by this method. His flip rates do not satisfy the strict positivity condition.

In our model maximal and minimal equilibria exists. However, unlike with many popular interaction and learning models we cannot use stochastic approximation algorithms to identify their basins of attractions. Such an approach require some form of limit taking - either by letting the number of agents tend to infinity (Benaim & Weibull 2003) or by letting the time between successive choices (Chen & White 1998) or the “noise” tend to zero. Foster & Young (1990), Kandori, Mailath & Rob (1993) and Young (1993), for instance, studied the long run dynamics in discrete time interaction games with uniform matching and identified the asymptotically stable configurations of actions when the agents play a perturbed best response strategy. It turns out for the special case of a 2×2 coordination game the agents eventually coordinate on the risk-dominant equilibrium. Ellison (2000) provides a unified framework for analyzing evolutionary games with small noise in discrete time that is flexible enough to allow for both local and global interactions. He shows that the selection of the risk dominant equilibrium “is robust to local vs. global interaction” (p.27). His analysis is, however, restricted to finite populations and lacks an explicit representation for the dynamics of aggregate play. We start right away with an infinite set of agents where we can appeal to a law of large numbers. In the following section we show by means of numerical simulations that in our framework the selection of equilibria is *not* robust to “local vs. global interaction”. This suggests that for models with an infinity of agents the distinctions between “local” and “global” is important.

4 Population Games with Random Matching

Blume (1993, 1995) was the first to explore the link between continuous-time interaction games in large population, dynamic discrete choice models and Gibbs distribution theory. In his model an agent $a \in \mathbb{A}$ configured η_t^a receives an instantaneous payoff $G(\eta_t^a, \eta_t^b)$ from each of her neighbors $b \in N(a)$. The agent switches his state at Poisson random times where the rate depends on the difference of the payoffs that the different states achieve. For coordination games with payoff functions

$$G = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \tag{14}$$

for some $x \in (0, 1)$ he shows that in the limit of a best response dynamics the agents eventually coordinate on the Pareto optimal equilibrium. Here player a is the row player and $\eta^a = -1$ corresponds to choosing the top row. Each agent $b \in N(a)$ is column player and $\eta^b = -1$ means that she choose the left column. In the sequel we introduce an additional “anonymous” component and show that the agents mail fail to coordinate on the Pareto dominant equilibrium, due to negative global externalities.

4.1 Introducing an anonymous component

Let us assume that the agents are located on the two-dimensional integer lattice and consider the 2×2 coordination game with payoff matrix (14). The agents receive a payoff from their nearest neighbors while, at the same time, being matched with four (for reasons of symmetry) other agents whose actions are unobservable. In order to have a unified framework within which to embed purely local and purely global interactions as special cases, the payoffs from the neighbors and random matches are weighted by a factor J_1 and J_2, respectively. The case $J_1 = 1$ and $J_2 = 0$ corresponds to a purely local interaction while $J_1 = 0$ and $J_2 = 1$ yields an interaction of mean-field type.

Given the average m_t at time t, the action η^a yields a payoff $J_1 G(\eta^a, \eta^b)$ from each neighbor and an expected payoff $J_2 \left\{\frac{1-m}{2} G(\eta^a, -1) + \frac{1+m}{2} G(\eta^a, +1)\right\}$ per random match. The payoff is thus equivalent to that of a local interaction game with payoff matrix

$$G_t = \begin{pmatrix} J_1 + J_2 \frac{1-m_t}{2} & J_2 \frac{1-m_t}{2} \\ x \cdot J_2 \frac{1+m_t}{2} & x \cdot (J_1 + J_2 \frac{1+m_t}{2}) \end{pmatrix}$$

at time t. Assuming the same log-linear strategy adjustment process as in Blume (1993) the dynamics of the choices can be described by a system of social interactions with flip rates

$$c(\eta, a, \mu) = \frac{1}{1 + \exp\left[2\beta\eta^a \left(a(m(\mu)) + \frac{b}{4}\sum_{b \in N(a)} \eta^b\right)\right]}. \tag{15}$$

Here $m(\mu)$ denotes the expected action of an individual agent under μ as defined in (7) and

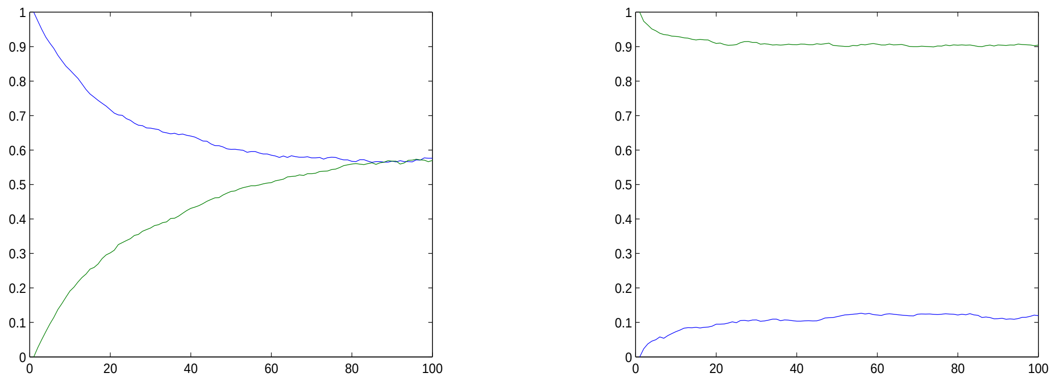
$$a(m(\mu)) = J_1(1-x) + J_2\left\{1 - m(\mu) - x(1 + m(\mu))\right\} \quad \text{and} \quad b = J_1(1+x). \tag{16}$$

In view Theorem 3.4 and Example 3.5 we the following result about the long run dynamics of aggregate behavior.

Proposition 4.1 *When started in an "all -1" or "all +1" configuration the macroscopic processes associated with the flip rates (15) settle down in the long run. They converge to the same limit if the interaction between players is sufficiently weak. In this case, the unique equilibrium is globally stable.*

Figure 1 displays the evolution of the proportions of agents configured in state "-1" when started in the "all high" and "all low" configuration, respectively. The simulations is based on an array of 75×75 agents with $J_1 = J_2 = 1$ and boundary condition "-1" when $p_0 = 1$ and "+1" if $p_0 = -1$. For $x = \frac{1}{2}$ and a sufficiently weak interaction ($\beta = 0.1$), the proportion settles down at approximately 57% independently of the initial condition as illustrated by Figure 1(a). If the interaction grows stronger ($\beta = 0.4$) ergodicity breaks down.

While the proportions still converge monotonically when started in an “all high” or “all low” configuration, they converge to different limits. If initially all agent are configured at +1, the long run proportion decreases to about 90% while it increases to about 12% when started in the “all low” configuration as shown in Figure 1(b).



(a) Dynamics for $x = \frac{1}{2}$ and $\beta = 0.1$.

(b) Dynamics for $x = \frac{4}{5}$ and $\beta = 0.4$.

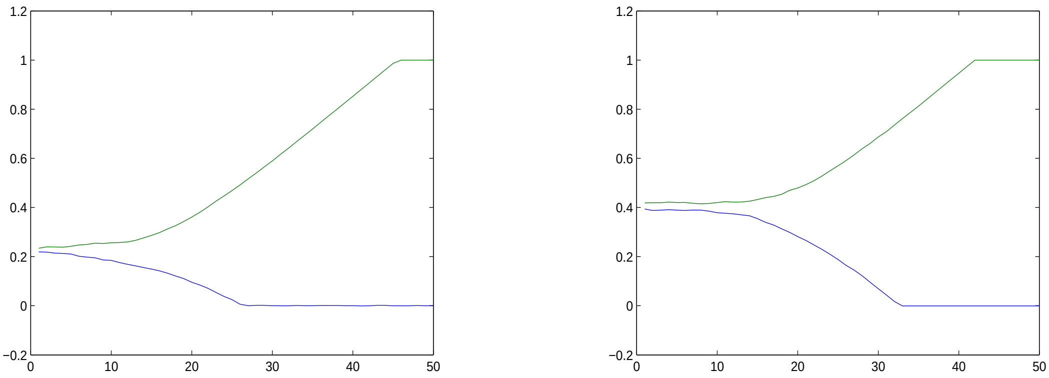
Figure 1: Proportion of agents in state “-1” for $p_0 = 0$ and $p_0 = 1$.

4.2 Coordination failure: a numerical analysis

In this section we illustrate that in the presence of global interactions the agents may coordinate on the Pareto inferior equilibrium, due to negative global externalities. This feature not shared by models with only local interactions where the agents always coordinate in the optimal equilibrium. In the benchmark model with frozen macroscopic component μ

$$\lim_{\beta\to\infty}\lim_{t\to\infty}\mathbb{P}[\eta_t^a = -1] = \begin{cases} 1 & \text{if } a(m(\mu)) > 0 \\ 0 & \text{if } a(m(\mu)) < 0 \end{cases}.$$

The fact that the run long behavior of agents in local interaction models depends only on the sign of $a(m(\mu))$ suggests that in a population game with global interactions the agents may eventually coordinate on the Pareto inferior equilibrium if a large enough proportion of agents is initially configured at “+1” so that $a(m_0)$ is negative. Agents configured at “-1” have a strong incentive to change their state as they cannot change the average situation in one step. This effect may become self-reinforcing: when more agents witch to +1 the external field decreases further thereby increasing an agents’ incentive to switch from -1 to $+1$. This way, the agents may eventually get trapped in the Pareto inferior equilibrium. We leave it for future research to substantiate these arguments in a mathematically rigorous and rely



(a) Dynamics for $x = \frac{1}{2}$; $p_0 = .22$ and $p_0 = 0.23$.

(b) Dynamics for $x = \frac{3}{4}$; $p_0 = .37$ and $p_0 = .40$.

Figure 2: Proportion of agents in state "-1"; $\beta = 2$; boundary condition "-1".

instead on a numerical analysis. Figure 2 displays the evolution of the proportion of agents configured in state "-1" for $\beta = 2$ when $x = \frac{1}{2}$ and $x = \frac{3}{4}$, respectively. The simulations show that a slight difference in the initial proportions p_0 of agents initially configured at "-1" may have a major effect on the long run dynamics. It turned out that for $x = \frac{1}{2}$ and $p_0 = 23\%$ the agents coordinate on the unique Pareto optimal equilibrium in the long run. However, if the initial proportion drops to about 22% the agents eventually settle on the Pareto inferior "all +1" equilibrium. Furthermore when x increases the "basin of attraction" of the inferior equilibrium grows as "negative" complementarities from global interactions become more powerful. At the same time we see from (16) that there are no negative external effects from global interactions if the impact of the macroscopic component is sufficiently weak. In this case the "all -1" equilibrium will emerge in the long run.

A Particle systems with macroscopic interaction

A *random field* is a probability measure on E. The space $\mathcal{M}$ of all random fields is compact with respect to the topology of weak convergence induced by the metric d_r in (6); see Horst (2002) for details. Since $\mathcal{L} \subset \mathcal{C}$ is dense a sequence (μ_t) converges weakly to $\mu \in \mathcal{M}$ if and only if integrals of local functions with respect to μ_t converge to the integrals with respect to μ. Ergodic random fields are concentrated on the set E_0 of all *ergodic configurations* whose associated *empirical field* defined as the weak limit

$$R(\eta) := \lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}_n} \delta_{\tau^a \eta}(\cdot) \tag{17}$$

exists along the increasing sequence $\mathbb{A}_n := [-n, n]^d \cap \mathbb{A} \uparrow \mathbb{A}$ and is ergodic. Here $\delta_\eta(\cdot)$ denotes the Dirac measure that puts all mass on η and τ^a is the shift operator: $\tau^a \eta = (\eta^{a+b})_{b\in\mathbb{A}}$.

A.1 Construction of the particle process

In the next section we outline how to construct the particle process with global interactions. Following Föllmer & Horst (2001) the idea is to redefine the Markov process (η_t) by a process with the empirical field replaced by the solution to some integral equation. To this end, let $\Psi = (\Psi_t)_{t\geq 0}$ be a sequence homogeneous random fields on E and consider a time-inhomogeneous particle process (ξ_t) whose infinitesimal generator at time t is

$$A_t f(\xi) \equiv A_{\Psi_t} f(\xi) := \sum_x c(\xi, x, \Psi_t)[f(\xi_x) - f(\xi)] \quad (f \in \mathcal{L}).$$

The corresponding semi-group is denoted by (S_t^Ψ). The particle process describes the evolution of the agents' states when the dynamics of their assessments about the empirical distribution of choices is given by Ψ. For any ergodic initial distribution R it follows from Theorem I.4.15 in Liggett (2005) that the macroscopic process $\{R^\Psi(\xi_t)\}$ exists almost surely as a sequence of ergodic empirical fields and satisfies the integral equation

$$R_t^\Psi(f) = R(f) + \int_0^t R_u^\Psi(A_{\Psi_u} f)du \quad \text{and} \quad R_t \in \mathcal{M}_0. \tag{18}$$

The goal is then to prove that the "externality" Ψ can be chosen to coincide with the sequence of empirical fields generated by the microscopic process of individual states, i.e., that there exists a unique solution to the integral equation

$$R_t(f) = R(f) + \int_0^t R_u\left(A_{R_u} f\right) du \quad \text{for all} \quad f \in \mathcal{L}. \tag{19}$$

To this end, one first applies Lemma 1.6.2 in Ethier & Kurtz (1986) and Theorem 1.3.9 in Liggett (2005) to prove that the distributions of states depend continuously on the environment. In a second step one can then prove the existence of a unique solution to the integral equation (19). Here one uses standard techniques from ODEs with Lipschitz coefficients.

Theorem A.1 *Suppose that the flip rates $c(x, \eta, \mu)$ satisfy Assumption 2.3.*

(i) There exists a constant L_T such that, for any two environments Ψ^1 and Ψ^2, we have

$$d_r\left(S_t^{\Psi^1}, S_t^{\Psi^2}\right) \leq L_T \int_0^t d_r(\Psi_u^1, \Psi_u^2)\, du \quad (t \geq T). \tag{20}$$

(ii) The integral equation (19) has a unique ergodic solution (Ψ_t^R) for any $R \in \mathcal{M}_0$.

We are now ready to define the particle process with global interaction.

Definition A.2 *Let R be an ergodic initial distribution of states. A particle system with macroscopic interaction with initial distribution R is given by the unique Markov process with local interactions associated with the rates $\{c(\eta, a, \Psi_t^R)\}_{\eta \in E, a \in \mathbb{A}}$ and initial distribution R.*

A.2 Attractive Particle Systems with Macroscopic Interactions

In this section we prove a convergence result for attractive spin systems. A spin system is called *attractive*, if the flip rates $c(\eta, x, \mu)$ satisfy (9). The general theory of attractive spin systems is well developed (Liggett 2005, Chapter III). To state some of the convergence results let $\mathbf{M}$ be the class of all monotone functions f on E and write $\mu_1 \leq \mu_2$ for two probability measures μ_1 and μ_2, if $\mu_1(f) \leq \mu_2(f)$ for all $f \in \mathbf{M}$. The semi-group S_t^μ associated with the flip rates $c(\eta, x, \mu)$ maps the class $\mathbf{M}$ into itself, and for any initial distribution ν

$$\delta_{-\mathbf{1}} S_t^\mu \leq \nu S_t^\mu \leq \delta_{\mathbf{1}} S_t^\mu \tag{21}$$

where $\delta_{-\mathbf{1}}$ and $\delta_{\mathbf{1}}$ denote the Dirac measures putting all mass on the configurations $\eta^a \equiv -1$ and $\eta^a \equiv +1$, respectively. Moreover, the weak limits $\underline{\nu}^\mu = \lim_{t\to\infty} \delta_{-\mathbf{1}} S_t^\mu$ and $\overline{\nu}^\mu = \lim_{t\to\infty} \delta_{\mathbf{1}} S_t^\mu$ exist. Hence, the spin system is ergodic if an only if $\underline{\nu}^\mu = \overline{\nu}^\mu$ (Liggett 2005, Theorem III.2.3). The semi-groups (S_t^μ) and (S_t^ν) also satisfy a monotonicity condition with respect to the "frozen" assessments about averages (Liggett (2005), Corollary III.1.7):

$$\mu_1 S_t^\mu \leq \mu_2 S_t^\nu \quad \text{for} \quad \mu \leq \nu \quad \text{if} \quad \mu_1 \leq \mu_2.$$

Proposition A.3 *Let $R^{-\mathbf{1}}, R^{\mathbf{1}}$, and R^η be the macroscopic processes of a monotone system with macroscopic interactions and initial distributions $\delta_{-\mathbf{1}}, \delta_{\mathbf{1}}$ and δ_η, respectively.*

(i) For any $t \geq 0$ we have $R_t^{-\mathbf{1}} \leq R_t^\eta \leq R_t^{\mathbf{1}}$.

(ii) The processes $(R_t^{-\mathbf{1}})$ and $(R_t^{\mathbf{1}})$ are monotone, i.e., for $s \leq t$,

$$R_s^{-\mathbf{1}} \leq R_t^{-\mathbf{1}} \quad \textit{and} \quad R_s^{\mathbf{1}} \geq R_t^{\mathbf{1}}.$$

(iii) The weak limits $\underline{\mu} := \lim_{t\to\infty} R_t^{-\mathbf{1}}$ and $\overline{\mu} := \lim_{t\to\infty} R_t^{\mathbf{1}}$ exist. In particular,

$$\lim_{t\to\infty} R_t^\eta = \mu \quad \textit{if} \quad \underline{\mu} = \overline{\mu} = \mu.$$

PROOF: Let us fix a monotone function f and put $k_s^n := \frac{i}{n}$ for $\frac{i}{n} \leq s < \frac{i+1}{n}$. For any $\eta \in \widetilde{E}$ we define processes $(R_t^{n,\eta})$ of ergodic random fields on E by

$$R_t^{n,\eta}(f) := R(\eta)(f) + \int_0^t R_{k_s^n}^{n,\eta}(A_{R_{k_s^n}^{n,\eta}} f) ds.$$

Hence we approximate the macroscopic process by a sequence of macroscopic processes associated with spin systems with local interactions. In fact, on the time interval $[\frac{i}{n}, \frac{i+1}{n})$ the systems behaves as if it were purely local. In view of (9) and (10) and the monotonicity properties of locally interacting spin systems theses processes satisfy

$$R_s^{n,-\mathbf{1}} \leq R_t^{n,-\mathbf{1}} \leq R_t^{n,\eta} \leq R_t^{n,\mathbf{1}} \leq R_s^{n,\mathbf{1}}$$

for all $n \in \mathbb{N}$ and $s \leq t$. One can show that all the processes $(R_t^{n,\eta})$ $(n \in \mathbb{N})$ are uniformly Lipschitz continuous on compact time intervals. It follows from the theorem of Ascoli and Arzela that they have accumulation points. Continuity of the flip rates shows that each such accumulation point satisfies the integral equation

$$R_t^{\eta}(f) = R(\eta)(f) + \int_0^t R_s^{\eta}(A_{R_s^{\eta}} f)\, ds.$$

Since this equation has a unique solution, we see that $\lim_{n\to\infty} R_t^{n,\eta} = R_t^{\nu}$ for all $t \geq 0$ and the convergence is uniform over compact time intervals. This proves (i) and (ii). In order to establish (iii), let μ_1 and μ_2 be weak accumulation points of the process $(R_t^{-\mathbf{1}})$:

$$\lim_{k\to\infty} R_{t_k}^{-\mathbf{1}} = \mu_1 \quad \text{and} \quad \lim_{k\to\infty} R_{l_k}^{-\mathbf{1}} = \mu_2.$$

We may with no loss of generality assume that $\ldots t_k \leq l_k \leq t_{k+1} \leq l_{k+1} \leq \ldots$. Since the map $t \mapsto R_t^{-\mathbf{1}}(f)$ is monotone for any monotone function f we see that

$$\ldots R_{t_k}^{-\mathbf{1}}(f) \leq R_{l_k}^{-\mathbf{1}}(f) \leq R_{t_{k+1}}^{-\mathbf{1}}(f) \leq R_{l_{k+1}}^{-\mathbf{1}}(f) \leq \ldots$$

This shows that $\mu_1(f) = \mu_2(f)$ for all monotone functions f, and hence $\mu_1 = \mu_2$. Similar arguments show that the weak limit $\lim_{t\to\infty} R_t^{\mathbf{1}}$ exists. □

To state our ergodicity result for attractive particle systems let (R_t) be the unique weak solution to the integral equation (19) with initial condition $R_0 = R$. Since

$$R_{t+s}(f) = R_t(f) + \int_0^s R_{t+u}(A_{R_{t+u}} f)\, du \quad (f \in \mathcal{L})$$

weak convergence of the process (R_t) to a random field R^* yields

$$\lim_{t\to\infty} \int_0^s R_{t+u}\left(A_{R_{t+u}} f\right)\, du = 0$$

for any $f \in \mathcal{L}$, and all $s \in \mathbb{R}_+$. Since the flip rates $c(\cdot,\cdot,\mu)$ satisfy Assumption 2.3 we obtain

$$\lim_{t\to\infty} \sup_u \|A_{R_{t+u}} f - A_{R^*} f\|_\infty = 0.$$

Thus, weak convergence of the macroscopic process yields

$$\lim_{t\to\infty} R_{t+u}\left(A_{R_{t+u}} f\right) = R^*(A_{R^*} f)$$

because the functions $A_{R_{t+u}} f$ are uniformly bounded. Hence dominated convergence yields

$$\begin{aligned}\lim_{t\to\infty}\int_0^s R_{t+u}\left(A_{R_{t+u}}f\right)\,du &= \int_0^s \lim_{t\to\infty} R_{t+u}\left(A_{R_{t+u}}f\right)\,du\\ &= \int_0^s R^*\left(A_{R^*}f\right)\,du.\end{aligned}$$

As a result, the limiting random field satisfies the fixed point condition

$$\mu\left(A_\mu f\right) = 0 \quad \text{for all} \quad f \in \mathcal{L}. \tag{22}$$

On the other hand, if μ is an ergodic equilibrium distribution, then the macroscopic process (R_t) is almost surely constant and equal to μ. From the representation (19) of (R_t), we thus deduce that μ satisfies (22). To characterize the class of all limiting distribution and to prove Theorems 3.4 and 3.4 and hence remains to establish the following result.

Theorem A.4 *Suppose that the flip rates satisfy Assumption 2.3. If for any homogeneous random field μ on E the semi-group (S_t^μ) is ergodic with unique stationary distribution ν_μ, then an attractive particle system with global interaction satisfies*

$$\underline{\mu} = \inf\{\mu \;:\; \mu = \nu_\mu\} \in \mathcal{M}_0 \quad \text{and} \quad \overline{\mu} = \sup\{\mu \;:\; \mu = \nu_\mu\} \in \mathcal{M}_0 \tag{23}$$

In particular, the system is ergodic if there exists a unique μ^* such that

$$\mu^* = \nu_{\mu^*}. \tag{24}$$

Proof: The semi-group (S_t^μ) is ergodic with unique stationary distribution ν_μ. By Proposition I.2.13 in Liggett (2005) ν_μ is the only measure that satisfies

$$\nu_\mu(A_\mu f) = 0 \quad \text{for all} \quad f \in \mathcal{L}.$$

Thus, (22) shows that when the process (R_t) converges to some random field μ^*, the limit must satisfy (24). If the fixed point condition has a unique solution we see that

$$\underline{\mu} = \overline{\mu} = \mu^*$$

and (R_t) converges to μ^*, due to Proposition A.3. Moreover, uniqueness of stationary distributions for (S_t^μ) implies $\nu_\mu \in \mathcal{M}_0$ (Liggett (2005), Chapter 2). Thus, for any μ that satisfies the fixed point condition (24), we that the distribution of the Markov process (ξ_t^*) does not change through time if the starting point is chosen according to μ. Hence, if T denotes the semi-group of our microscopic process, then $\mu T(t) = \mu$. Now, the monotonicity conditions on the flip rates yields (23) because

$$\delta_{-\mathbf{1}} T(t) \le \mu T(t) \le \delta_{\mathbf{1}} T(t) \quad \text{for all} \quad t \ge 0.$$

□

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