Decision Making in Phantom Spaces

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Abstract

The phantom space, with its phantom probability measure, allows the framing of ambiguous probabilities of events, as well as their vague consequences. This paper introduces a decision-making model under uncertainty, established on a phantom generalization of the von Neumann-Morgenstern axiomatization. Beliefs in our model are subjective variations of objective probabilities, recorded in a framework comprising not only risk but also phantom effects. Uncertainty measures are carried over naturally into this setting such that many of the familiar attributes of objective probabilities are preserved. The degree of uncertainty, which is determined by the available information and subjective beliefs of the decision maker, is distinguished from the attitude toward uncertainty, which is drawn from her preferences. Decision making under ambiguity is a special case of our model in which probabilities are vague but outcomes of events are clearly forecasted. The Ellsberg paradox and an insurance dilemma are the main examples we present.

Keywords and Phrases: Phantom probability measure, Phantom spaces, Decision making, Subjective probability, Risk, Ambiguity, Uncertainty, Ellsberg paradox.
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Contents

1 Introduction

The fundamental assumption of the von Neumann-Morgenstern (vNM, 1944) and the Savage (1954) paradigms is that a decision maker (DM), whose preferences have an expected utility representation, knows, or acts as if she knows, the probabilities of all states. Critics of these models mainly attack their ability to model decision making in real life: they are not realistic and do not record the full picture of uncertainty. In the real world, uncertainty has no numerical form agreed upon by all DMs; for example, real investments, financial assets, and insurance have no common prospects for financial decision making.

These paradigms were first challenged by Ellsberg (1961), who suggested the following two-part experiment. Consider an urn with 90 colored balls, 30 of them red and the others either black or yellow. In each part of the experiment, before one ball is drawn at random from the urn, a DM is offered two alternative bets; winning the bet entitles her \$9. In the first part, the DM has to choose between the two alternative bets: the ball drawn is red (R) or it is black (B). Then in the second part, the DM has to choose between betting on: the ball drawn is red or yellow (RY) or alternatively the ball is black or yellow (BY). Behavioral experiments have demonstrated that individuals usually prefer (R) over (B) but (BY) over (RY). These preferences can be attributed to lack of information about likelihoods and to the DM's attitude toward vagueness.

Clearly, there is no classical probability measure supporting these preferences through the expected utility paradigm. Furthermore, since classical probability theory does not support the formulation of such bets, the violation of expected utility's underlying axioms is revealed in such an ambiguous context, i.e. vNM's Independence axiom and Savage's axiom P2 (the Sure Thing Principle). The ***phantom probability measure***, a generalization of the classical probability measure introduced by Izhakian and Izhakian(I&I, 2009), allows the formulation of ambiguous odds. For example, the likelihood of drawing a black ball in the Ellsberg example can be identified in terms of phantoms as $\mathcal{P}(B) = \wp \frac{2}{3}$, the likelihood of drawing a yellow ball as $\mathcal{P}(Y) = \frac{2}{3} - \wp \frac{2}{3}$, and the likelihood of drawing a red ball as $\mathcal{P}(R) = \frac{1}{3}$.

One may take the Ellsberg decision problem one step further and assume that the prizes are also vague, as in the following situation. Assume a farmer who has a bag of wheat seeds that she is saving for consumption in the coming year; the farmer has two alternatives: storing the seeds or sowing them. Choosing the latter, the expected crop depends on the amount of rain; the yield in a drought year may vary from none to a single bag while in a rainy year it can be two or three bags of seeds. Based on her experience, the farmer estimates the chances of a drought year as $30\% - 40\%$, and the chances of a rainy year as $60\% - 70\%$. In reality most decisions are in fact taken in circumstances of this type.

The goal of this paper is to present a theory that aims to model decision making in such vague situations. We consider a DM who faces three sources of ***uncertainty***:

(i) A-priori, not knowing the event to be realized (referred to as ***risk*** in the literature);

(ii) The odds of events are ambiguous (called ***ambiguity*** or ***Knightian uncertainty***);

(iii) A-priori, consequences of events are not perfectly clear.

Decision making under ambiguity is then a special case in which consequences of events are precisely known but their probabilities are not. Vague odds of events, or vague outcomes, are termed ***phantoms*** in our language, and are represented by phantom numbers. Our suggested model is smooth and applicable to dynamic models.

The main advantage of our model lies in its natural generalization of classical expected utility theory to support ambiguous probabilities and vague consequences. Its underlying axiomatization generalizes the vNM axioms to the phantom setting as well as the functional representation of preferences, such that for a world without phantoms our model collapses to the vNM expected utility model. Our model distinguishes preferences from beliefs; A DM is characterized by beliefs, attitude toward risk, attitude toward phantoms (ambiguity), and pessimism.

Unlike multiple-priors or non-additive models, beliefs in our model are captured by a single additive probability measure. The unique attributes of the phantom probability measure, particularly that it obeys the Bayes' rule, support the modeling of learning processes (Bayesian updates) and allow the formalization of compound ambiguous lotteries. Phantom probabilities also preserve the key feature of statistics, leading to a natural forming of uncertainty measures, thus allowing the measurement of risk, phantoms, and their aggregation to uncertainty. Since phantom probability extends classical probability, statistical instruments can be used for empirical tests.

Our model is especially applicable to financial decision making. For example, it can be used for modeling an investor who has to select an optimal portfolio while the distributions of future returns are ambiguous due to limited information or because of her personal doubts. A second example could be an insurance company that needs to price an insurance policy against earthquakes, an infrequent event with ambiguous odds and vague monetary consequences. Phantoms can be applied not only to probabilities but also to parameters governing assets' distributions and stochastic processes.

Classical probability theory supports probability measures that assign a fixed positive real value to each event. Aiming to formulate real-life occurrences mathematically, Izhakian and Izhakian (2009) introduced the ***theory of phantom probability***, established on a ring structure that generalizes the field of real numbers. The main innovation of phantom theory is a probability measure that permits the assignment of varying odds to an event. Yet, this model is Bayesian, resembling the classical model, such that most of the attributes of classical probability theory are preserved. Phantom theory naturally allows the construction of a decision-making model when consequences are vague and probabilities are ambiguous. Phantom space, in general, can be of any order N greater than 1, where a higher order means additional layers of uncertainty. Focusing on the applicative side of this theory, to make our exposition clearer, we give the explicit description for the case of decision making in a phantom space of order 1, which is suitable enough for the scope of this paper. Although we concentrate on the phantom space of order 1, all the results can be extended to phantom spaces of order higher than 1 with respect to both probabilities and consequences.

The vNM expected utility theory (1944) assumes that probabilities are objective, i.e. the odds of events are known and shared by all DMs. De Finetti (1930), Ramsey (1931) and Savage (1954) suggested an alternative approach in which personal beliefs are involved when an individual makes a decision; accordingly, subjective probabilities can be derived from observable choice behavior. [1] In the past, different authors have suggested models that combine the advantages of both objective and subjective probabilities. For example, Anscombe and Aumannn (1963) adjusted the vNM theory to subjective probabilities; the ***probabilistic sophistication*** model of Machina and Schmeidler (1992) adopts additive numerical probabilities into a subjective framework that preserve attributes of objective probabilities. Roughly speaking, since beliefs might violate basic assumptions of classical probability theory, subjective probabilities may not preserve all the properties of exogenous objective probabilities. Simplicity, engendered by the underlying structural phantom measure, is one of the main advantages of our model, in which beliefs are formulated as phantom probabilities. Probabilities in our model can be objective, subjective, or a subjective variation of objective probabilities guided by the available

[1] De Finetti (1930) and Ramsey (1931) considered maximization of expected value, rather than expected utility, conditional on a subjective prior.

information and subjective beliefs of the DM.

In economics it was Knight (1921) who introduced the distinction between risk (perfectly known probabilities) and uncertainty (non-unique assigned probabilities). Ellsberg (1961) presented examples, known as the ***Ellsberg paradox***, demonstrating the systematic violation of the independence axiom in ambiguous contexts. Since the mid-1980s utility theory research has been making a concerted effort to treat decision processes with ambiguity. The multiple-priors discipline, suggested by Gilboa and Schmeidler (1989), set aside the assumption that DMs have a unique prior, and presumes they have a set of priors. In their ***max-min expected utility with multiple priors*** (MEU) model, Gilboa and Schmeidler asserted that an ambiguity-averse DM evaluates her ex-ante welfare by computing the expected utility conditional on the worst prior.

The weighted MEU (α-MEU), extending the MEU by also considering best priors, was discussed by Ghirardato, Maccheroni, and Marinacci (2004). [2] Klibanoff, Marinacci and Mukerji (KMM, 2005) generalized the MEU model further to a smooth version, assuming non-reducted beliefs on sets of priors. Other studies that used the multiple-prior concept include: Epstein and Wang (1994), Epstein (1999), Casadesus-Masanell, Klibanoff and Ozdenoren (2000), Chen and Epstein (2002), Epstein and Miao (2003) and Epstein and Schneider (2003), to name a few. The second class of multiple-priors models assumes the uncertainty of true probability law in governing the realization of states; see, for example, Anderson, Hansen and Sargent (1999), Uppal and Wang (2003) and Maenhout (2004). This literature investigates the ***robustness*** of decision rules in modeling ***misspecification*** with respect to the underlying probability. [3]

In contrast to multiple-priors models, our model presumes states of nature with a unique probability measure, such that each of the DM's alternatives is encoded by a single prior. Our model distinguishes between beliefs and preferences, as does the KMM (2005) model, i.e. between ***phantom degree***, determined by available information and the DM's subjective belief, and ***phantom attitude***, drawn from her tastes. Decision making under ***ambiguity*** becomes a special case of our model in which probabilities are ambiguous while outcomes of events are clearly forecasted. The distinction between degree of ambiguity and attitude toward ambiguity is sustained in our model, which also distinguishes between ambiguity (phantom) attitude and pessimism. Each is identified and parameterized uniquely. Besides preserving smoothness, as does the KMM (2005), the phantom model specifies pessimism and optimism as subclasses of phantom aversion and phantom seeking.

Several past models have used a single probability measure to formalize decision making under uncertainty. The subjective non-additive probabilities of Gilboa (1987) and the Choquet expected utility (CEU) of Schmeidler (1989) state that belief can be represented by a single, but non-additive, prior. In the CEU model, uncertainty aversion results in a sub-additive prior. [4] Machina (2004) introduced the almost-objective uncertainty (AOU) model, which relies on the Poincaré (1912) insight that some subjective events appear to be more objective than others. In contrast to these models, whose evaluations are given by real numbers, ours consists of a single additive probability measure that encodes a range of probabilities as unique phantom numbers.

The MEU, CEU and AOU models assume absolutely predicted consequences for all possible events, which makes them quite limited, especially when one tries to model real-life situations in which estimation of outcomes is imprecise or even impossible earthquakes for instance. Furthermore, given that the outcome is unsure, the utility produced by the outcome is vague. In

[2]Further discussion of α-MEU can also be found in Segal and Spivak (1990), Loomes and Segal (1994), Ghirardato, Klibanoff and Marinacci (1998) and Siniscalchi (2006).

[3]Others studies use entropy models: Hansen, Sargent, and Tallarini (1999), Hansen and Sargent (2001), Hansen and Sargent (2003), Epstein and Schneider (2003), Hansen and Sargent (2007) and others.

[4]See also Gilboa and Schmeidler (1993) and Wakker (1989).

addition to risk and ambiguous probabilities, our model also comprises a third level of uncertainty: vagueness of consequences, which is new in the literature. This allows the existence of phantoms, without enforcing exposure to risk.

The paper is organized as follows. For completeness, Section 2 reviews the phantom space and its structural probability measure. Section 3 establishes the bases for decision making in the phantom framework. Section 4 introduces the generalized axiomatization of preferences. The main results are presented in Section 5, which studies phantom functional representations of preferences. Section 6 discusses the DM's attitudes toward risk. Section 7 explores attitudes toward phantoms and ambiguity. Section 8 investigates risk, phantoms, and degrees of uncertainty, and develops a measure for ranking phantom choices. Section 9 applies our model to the Ellsberg paradox and to insurance problems. Section 10 concludes.

2 Phantom Spaces

The phantom framework, introduced by Izhakian and Izhakian (2009), provides a phantom probability measure whose target is a phantom ring, an algebraic structure that allows the modeling of uncertain scenarios in which outcomes are vague and probabilities are ambiguous. The phantom approach generalizes the field $(\mathbb{R}, +, \cdot)$ of real numbers to a ring structure whose binary operations are induced by the familiar addition and multiplication of $\mathbb{R}$.

The ***phantom ring*** $(\mathbb{PH}, \oplus, \otimes)$, consists of ***phantom numbers*** of the form $z = a + \wp b$, where a and b are real numbers called, respectively, the ***real term*** and the ***phantom term*** of z; it is equipped with the operations

$$\begin{aligned}(a_1 + \wp b_1) \oplus (a_2 + \wp b_2) &:= (a_1 + a_2) + \wp\,(b_1 + b_2)\,, \\ (a_1 + \wp b_1) \otimes (a_2 + \wp b_2) &:= a_1 a_2 + \wp\,(a_1 b_2 + b_1 a_2 + b_1 b_2)\,,\end{aligned}$$

addition and multiplication, respectively, and with the unit element $1 := 1 + \wp 0$ and the zero element $0 := 0 + \wp 0$.

Formally, $(\mathbb{PH}, \oplus, \otimes)$ is a commutative ring, but not a field, since it has zero divisors, all of them having the form

$$z = 0 + \wp a \qquad \text{or} \qquad z = a - \wp a,$$

for some $a \in \mathbb{R}$. The set of zero divisors is denoted by $Z_{\text{div}}(\wp)$ and Z^0_{div} denotes the union $Z_{\text{div}} \cup \{0\}$. $\mathbb{PH}$ is assumed to be provided with a weak order $\precsim_{\text{wk}}$ which is compatible with the order of $\mathbb{R}$, in the sense that $z_1 \precsim_{\text{wk}} z_2 \iff a_1 \leq a_2$, for any $z_1 = a_1 + \wp 0$ and $z_2 = a_2 + \wp 0$.

The ***phantom conjugate*** $\overline{z}$ and the (real) ***reduction*** $\widehat{z}$, of $z = a + \wp b$ are defined as

$$\overline{z} := (a + b) - \wp b, \qquad\qquad \widehat{z} := a + b.$$

The ***projection*** of phantom number z, denoted by $[z]$, is given by $[z] = \frac{z + \overline{z}}{2}$. The reduction plays a main role in the realization property of phantoms. A function f, taking phantom arguments, which can be written in terms of two real functions, f_{re} the real component of f and $\widehat{f}$ the reduction of f, as

$$f\,(a + \wp b) = f_{\text{re}}\,(a) + \wp\,\left(\widehat{f}\,(\widehat{z}) - f_{\text{re}}\,(a)\right),$$

is said to have a ***realization form***. It turns out that many of the basic functions admit the realization property. The following is a partial list:

- $z_1 \otimes z_2 = a_1 a_2 + \wp\,(\widehat{z_1}\widehat{z_2} - a_1 a_2)$;
- $z_1 \oslash z_2 = \frac{a_1}{a_2} + \wp\,\left(\frac{\widehat{z_1}}{\widehat{z_2}} - \frac{a_1}{a_2}\right)$;

- $z^n = a^n + \wp\left(\widehat{z}^n - a^n\right)$;
- $e^z = e^a + \wp\left(e^{\widehat{z}} - e^a\right)$;

A phantom number $z = a + \wp b$ is said to be ***strictly positive***, denoted by $z \gg 0$, if $a > 0$ and $b > 0$, and it is said to be ***positive***, denoted by $z \ggeq 0$, if $a \geq 0$ and $b \geq 0$. When $a > 0$ and $\widehat{z} > 0$ we say that z is ***pseudo positive***, and when z is pseudo positive or 0 it is termed a ***pseudo nonnegative***.

Notation 2.1. *For the rest of this paper, assuming that the reader is familiar with the arithmetical nuances, we write $z_1 z_2$ for the product $z_1 \otimes z_2$, $\frac{z_1}{z_2}$ for the division $z_1 \oslash z_2$, and z^n for $z \otimes \cdots \otimes z$ repeated n times.*

Next, we review the probability measure associated with the phantom space. Let $\bar{\Lambda}$ be the set

$$\bar{\Lambda} = \left\{ z \in \mathbb{PH} \;\middle|\; z = a + \wp b,\ a \in [0,1],\ 0 \leq a + b \leq 1 \right\}.$$

A triple $(\Omega, \Sigma, \mathcal{P})$, where Σ is a σ-algebra of subsets of Ω, and $\mathcal{P} : \Sigma \to \bar{\Lambda}$ is a ***phantom probability measure*** satisfying the axioms below, is called a ***phantom probability space***. We write $\mathcal{P} = \mathcal{P}_{\mathrm{re}} + \wp \mathcal{P}_{\mathrm{ph}}$, where $\mathcal{P}_{\mathrm{re}}$ (the real component of $\mathcal{P}$) and $\mathcal{P}_{\mathrm{ph}}$ (the phantom component of $\mathcal{P}$) are real-valued functions. The value $\mathcal{P}(A)$ is called the ***phantom probability*** of the event $A \in \Sigma$.

Axiom 2.2 (***Phantom probability measure***). *The order $\leq$ is the standard order of $\mathbb{R}$.*

(i) Nonnegativity: $0 \leq \mathcal{P}_{\mathrm{re}}(A) \leq 1$ *for each* $A \in \Sigma$,

(ii) Normalization: $\mathcal{P}(\Omega) = 1$,

(iii) Additivity: $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B)$ *for any pair of disjoint events A and B in Σ,*

(iv) Phantomization: $0 \leq \mathcal{P}_{\mathrm{re}}(A) + \mathcal{P}_{\mathrm{ph}}(A) \leq 1$ *for each* $A \in \Sigma$.

The phantom probability measure generalizes the classical probability measure $P : \Sigma \to [0,1]$ to a phantom-valued function $\mathcal{P} : \Sigma \to \bar{\Lambda}$, whose real component is a real standard measure $\mathcal{P}_{\mathrm{re}} : \Sigma \to [0,1]$. To avoid nonzero divisors, we usually take the target of $\mathcal{P}$ to be the set

$$\Lambda = \left\{ \lambda \;\middle|\; \lambda \in \Lambda \text{ and } \lambda \notin Z_{\mathrm{div}} \right\},$$

which is called the ***phantom probability zone***. We also define the set $\Lambda^{\times} = \Lambda \setminus \{0,1\}$ whose elements are called ***phantom weights***. In order to make phantom probability theory appropriately defined, the following requirement is enforced on the weak order provided with $\mathbb{PH}$:

$$0 \precsim_{\mathrm{wk}} z \precsim_{\mathrm{wk}} 1, \qquad \text{for each } z \in \Lambda.$$

Example 2.3. *Consider two urns A and B. The likelihood of drawing a red (R) ball from urn A, which contains red and yellow balls, is between 40% and 60%. The likelihood of drawing a black (B) ball from urn B, which contains black and yellow balls, is between 50% and 80%. In phantom representation, the likelihood of drawing a red ball from urn A is $\mathcal{P}(R) = 0.4 + \wp 0.2$, and the likelihood of drawing a black ball from urn B is $\mathcal{P}(B) = 0.8 - \wp 0.3$.* [5] *If one ball is drawn from each urn the likelihood of having red and black balls is*

$$\begin{aligned} \mathcal{P}(R) \otimes \mathcal{P}(B) &= (0.4 + \wp 0.2) \otimes (0.8 - \wp 0.3) \\ &= 0.4 \times 0.8 + \wp\left(-0.3 \times 0.4 + 0.2 \times 0.8 - 0.3 \times 0.2\right) \\ &= 0.32 - \wp 0.02 \end{aligned}$$

[5] An alternative representation could be the conjugates $\mathcal{P}(R) = 0.6 - \wp 0.2$ and $\mathcal{P}(B) = 0.5 + \wp 0.3$, respectively.

Phantom numbers, and thus phantom probabilities, can be extended to higher orders which capture a larger number of uncertainty layers.

Example 2.4. *Consider an urn containing 90 colored balls, 30 of them red and the others either black or yellow, the number of yellow balls being greater than the number of black balls. In phantom representation the likelihood of drawing red* (R)*, yellow* (Y) *and black* (B) *are, respectively:* $\mathcal{P}(R) = \frac{1}{3}$, $\mathcal{P}(Y) = \frac{2}{3} - \wp_1 \frac{2}{3} + \wp_2 \frac{2}{3}$ *and* $\mathcal{P}(B) = \wp_1 \frac{2}{3} - \wp_2 \frac{2}{3}$.

Section A.1 in the Appendix explores some inequalities and functions in the phantom environment that will be useful in the sequel. Since this paper concentrates on applicative aspects of phantom theory, throughout we consider only a phantom ring of the order 1, $\mathbb{PH}$, but all the results presented can be extended to a phantom ring of the order N.

3 Preliminaries

This section builds the decision-making framework over the phantoms, establishing the object of choice, and the basics for modeling of preferences and their functional representations in the phantom environment.

3.1 Intuition

Assume a DM who faces three levels of ***uncertainty***:

(i) The event to be realized.

(ii) The estimated odds of events happening.

(iii) The outcome of the realized event.

For example, take a rare event such as earthquake. We do not know the likelihood of a disaster with any accuracy, nor do we know the expected monetary damage if the disaster occurs.

The philosophy beyond the phantom numbers is that the real term should be understood as an "expected" value, while the phantom term stands for a signed variation (either, positive or negative) assigned to that value. With this interpretation three tiers of uncertainty can be formulated naturally:

(i) Vague consequences, referred to as ***phantom effect***, and modeled by phantom numbers;

(ii) ***Risk***, as in classical expected utility theory, obtained by assigning probabilities to events;

(iii) ***Ambiguity***, formulated by a phantom probability measure, indicating ambiguous odds.

Practically, the phantom model extends ambiguity to cases when consequences of events are vague, even if their realization is given. Therefore, the term phantom is used to indicate both ambiguity and the phantom effect.

As in classical probability theory, the axioms of phantom probability determine the summation of probabilities to be 1; this implies that phantom terms of probabilities are summed to 0. The motivation for the latter emerges from the fact that phantom probabilities of an event and its complement will share the same phantom term but with opposite sign. Recalling that the real and the phantom terms of a probability have bounded values in Λ, a realized event might have a probability with a real term < 1, but the sum of the real term and the phantom term always lies in the real interval $[0, 1]$.

In the phantom representation, the real term of a phantom probability can be seen as an objective probability agreed upon by all DMs while the phantom term corresponds to some subjective doubts on whether to rely on known information or on personal beliefs. These topics are discussed later in Section 8.

3.2 Elementary definitions

We use a Savage-style formalism for decision making under uncertainty, adjusted to the phantom probability space. For a nonempty ***set of states*** Ω, each of whose elements $\omega \in \Omega$ is called a ***state of nature***, we define the σ-algebra Σ of subsets of Ω; an element $\mathcal{E} \in \Sigma$ is called an ***event***. We often abuse the notation and write $\omega \in \Sigma$, i.e. events are replaced by states. Recall that the phantom probability measure $\mathcal{P} : \Sigma \to \bar{\Lambda}$ assigns to each event $\mathcal{E} \in \Sigma$ a phantom number in $\bar{\Lambda}$. The phantom probability measure is additive, with the real probability measure $\mathcal{P}_{\text{re}} : \Sigma \to [0, 1]$ as its private case. The DM's belief is represented by a phantom probability measure. We denote the set $\{\mathcal{P} \mid \mathcal{P} : \Sigma \to \bar{\Lambda}\}$ of all phantom probability measures as $\mathfrak{P}_{\Sigma}$.

The set of ***consequences*** $\mathcal{X}$ (i.e. payoffs, prizes, or outcomes) is a nonempty subset of $\mathbb{PH}$ endowed with a σ-algebra $\mathfrak{D}$ of subsets. Since $\mathbb{PH}$ is a separable topological space, whose metric is induced by $|\cdot|$, each subset $\mathcal{X} \subset \mathbb{PH}$ is also a separable metric. A consequence $x \in \mathcal{X}$ is a ***real consequences*** if it has no phantom term; we denote the subset of real consequences by $\mathcal{X}_{\text{re}} \subseteq \mathcal{X}$.

A ***phantom act*** f, or (Savage) ***act*** for short, is a measurable function $f : \Omega \to \mathcal{X}$, i.e., $f^{-1}(D) \in \Sigma$ for all $D \in \mathfrak{D}$; Ω is the ***domain*** of f. We may also let a measurable function $f : \Sigma \to \mathcal{X}$ be an act, assuming all states of each event take the same value; in such a case the domain of f is Σ. An act is a ***real act***, denoted f_{re}, if $f : \Omega \to \mathcal{X}_{\text{re}}$. Consider an order $\succsim$ on $\mathcal{X}$. An act is said to be bounded (with respect to $\succsim$) if there exist finite $x, y \in \mathcal{X}$ such that $x \succsim f \succsim y$. The ***set of acts*** $\mathcal{F}_{\Sigma} = \left\{ f \;\middle|\; f : \Sigma \to \mathcal{X} \right\}$ of Σ is a set of bounded acts, where $\mathcal{F}_{\Sigma;\text{re}} \subseteq \mathcal{F}_{\Sigma}$ stands for the subset of real acts.

An act f is a ***constant act***, denoted f^{c}, if all the states of Ω are mapped to a fixed consequence x_0. (Abusing the notation we write $f : \Omega \to x_0$ for $f : \Omega \to \{x_0\}$.) In the usual way, we write $\mathcal{P}(x)$ for $\sum_j \mathcal{P}(\mathcal{E}_j)$, where j runs over all the events for which $f : \mathcal{E}_j \mapsto x$. Thus, in this view, the probability measure plays no role in constant acts.

An object of choice is a map $v : \Sigma \to \mathcal{X} \times \bar{\Lambda}$, sending each event to a pair composed of a consequence and a probability; such a map (whose domain is Σ) is termed a ***vision*** and $\mathcal{V}_{\Sigma} = \left\{ v \;\middle|\; v : \Sigma \to \mathcal{X} \times \bar{\Lambda} \right\}$ is called the ***set of visions*** of Σ. The ***support*** of a vision consists of all the events which have nonzero probability. Usually, to make the exposition clearer, when considering a set of visions $\mathcal{V}_{\Sigma}$ we identify the supports of the visions with its domain.

We assume that visions are compound maps of bounded acts and phantom probability measures, and thus write $v = (f, \mathcal{P})$. Sometimes we also write f_v and $\mathcal{P}_v$, respectively, for the act and probability measures of v. Then, the set of visions can be rewritten as $\mathcal{V}_{\Sigma} = \left\{ (f, \mathcal{P}) \;\middle|\; f \in \mathcal{F}_{\Sigma},\ \mathcal{P} \in \mathfrak{P}_{\Sigma} \right\}$. Thus, when a DM chooses a vision she gets a double assignment of events to consequences and to ambiguous probabilities, which can take phantom values. Visions are generalization of acts; the joining of a varied probability measure to a fixed act paves the way to wider range of applications, such as comparing phantom degree across DMs.

The ***conjugate*** $\overline{v}$ of a vision $v = (f, \mathcal{P})$ is defined by the conjugate act $\overline{f}$ and the conjugate probability measure $\overline{\mathcal{P}}$, i.e. $\overline{v} = \left(\overline{f}, \overline{\mathcal{P}}\right)$. A ***constant vision*** $v^{\text{c}} = (f^{\text{c}}, \mathcal{P})$ is a vision whose act is constant. Identifying each constant act $f^{\text{c}} : \Omega \to x_0$ with its target x_0, we abuse the notation and write $v^{\text{c}} = (x_0, \mathcal{P})$; then $\mathcal{V}^{\text{c}}_{\Sigma} = \{(x, \mathcal{P}) \mid x \in \mathcal{X}, \mathcal{P} \in \mathfrak{P}_{\Sigma}\}$. A vision composed of a real act and a real probability measure is a ***real vision*** and is denoted v_{re}; the subset of these visions is denoted as $\mathcal{V}_{\Sigma;\text{re}} \subseteq \mathcal{V}_{\Sigma}$.

The **expectation** and the **variance** of a vision $v = (f, \mathcal{P}) \in \mathcal{V}_{\Sigma}$ are respectively

$$\mathrm{E}(v) = \sum_{\mathcal{E} \in \Sigma} f(\mathcal{E})\, \mathcal{P}(\mathcal{E}) \quad \text{and} \quad \mathrm{Var}(v) = \sum_{\mathcal{E} \in \Sigma} \left(f(\mathcal{E}) - \mathrm{E}(v)\right)^2 \mathcal{P}(\mathcal{E}).$$

For the remainder of this paper we assume that Σ is a Borel σ-algebra of the separable metric space Ω and $\mathfrak{A}$ is a Borel σ-algebra of $\bar{\Lambda}$. Thus $\mathcal{X} \times \bar{\Lambda}$ endowed with the product Borel σ-algebra $\mathfrak{D} \otimes \mathfrak{A}$. In this paper we consider a single-period decision problem. In real life a decision problem is sometimes spanned over multiple periods; these problems are related to strategic games in which the DM a-priori chooses a strategy.

Example 3.1. *A football league consist of 10 games. For a \$2 ticket, one is offered the following bet: if she guesses the results of the all* 10 *games correctly she wins* \$200*, but the prize might be reduced to* \$100 *depending on the results of other participants. If she guesses the results of* 9 *games the reward varies between* \$50 *and* \$100*, otherwise she wins nothing. She estimates her chances to guess* 10 *games between* 10% *and* 30% *and to guess* 9 *games between* 30% *and* 20%*. The gain of this decision problem is framed in terms of vision as:*

$$v : \begin{cases} 10 & \textit{guesses} & \longmapsto & (198 - \wp 100, & 0.1 + \wp 0.2), \\ 9 & \textit{guesses} & \longmapsto & (48 + \wp 50, & 0.3 - \wp 0.1), \\ 0 - 8 & \textit{guesses} & \longmapsto & (-2 + \wp 0, & 0.5 - \wp 0.1). \end{cases}$$

Assume $v_1, \ldots, v_n$ are visions in $\mathcal{V}_\Sigma$, i.e. all have the same domain, and let $\alpha_1, \ldots, \alpha_n$ be elements in Λ that sum up to 1. The ***compound vision*** $\tilde{v}$ of $v_1, \ldots, v_n$, written $\oplus_i \alpha_i v_i$, is given by the map

$$\tilde{v} : \mathcal{E} \mapsto \left(\sum \alpha_i f_i (\mathcal{E}), \sum \alpha_i \mathcal{P}_i (\mathcal{E}) \right);$$

for example, when $n = 2$, $\tilde{v} = \alpha v_1 \oplus (1 - \alpha) v_2$ for $\alpha \in \Lambda$. Since the $\alpha_1, \ldots, \alpha_n$ can be realized as a probability measure on $\{1, \ldots, n\}$, the ***compound vision*** $\tilde{v}$ can be understood as the event-wise phantom expectation (or mean) of the vision $v_1, \ldots, v_n$.

Remark 3.2. *Any vision* $v = (f, \mathcal{P}) \in \mathcal{V}_\Sigma$ *can be written as a compound vision* $\oplus_i \alpha_i v_i$ *of constant visions* $v_1, \ldots, v_n \in \mathcal{V}_\Sigma$*. Indeed, for each event* $\mathcal{E}_i \in \Sigma$*, define the constant vision whose action is* $f_i : \mathcal{E} \mapsto f(\mathcal{E}_i)$ *for any* $\mathcal{E} \in \Sigma$*; the* α_i *are then provided as* $\mathcal{P}(\mathcal{E}_i)$*.*

Having defined the notion of visions as the elementary objects of choice, we turn to formulating lotteries, situations in which more than one DM is involved, each of whom has a vision in the same $\mathcal{V}_\Sigma$. A ***lottery*** $\ell \in \mathcal{V}_\Sigma$ is a vision whose domain Σ is countable. A lottery which is a real vision is called a ***roulette lottery***. A lottery is a ***horse lottery*** if its act is a real act. A lottery is a ***constant lottery***, denoted l^{c}, if its act is constant. A ***set of lotteries*** $\mathcal{L}_{\Sigma;f}$ on $\mathcal{V}_\Sigma$ is a set consisting of lotteries sharing the same act f; we denote the set of constant lotteries as $\mathcal{L}^{\mathrm{c}}_{\Sigma;f}$.

3.3 Preferences

A preference represents the subjective attitude of a DM toward uncertainty, which can be either uncertainty averse, uncertainty seeking, or indifferent to uncertainty. Attitude toward uncertainty is composed of two components: attitude toward risk and attitude toward phantoms. The former (i.e. aversion, loving or neutrality to risk) is framed here as in classical expected utility theory (cf. Section 6).

The attitude toward phantoms stands for a DM's tastes for situation in which phantom values (representing not-perfectly-known values) are involved, either in outcomes or in their odds. As before, we identify three tastes: phantom aversion, phantom seeking, and phantom neutrality. When the phantom attitude applies to vague probabilities, it is called an attitude toward ambiguity (cf. Section 7). In our model attitude toward phantoms and attitude toward ambiguity coincide, since they share the same representation.

Formally, the DM's attitudes are recorded by a weak preference $\succsim$, which is a complete (weak) order with extra axioms on the set of visions $\mathcal{V}_\Sigma$; i.e. $v \succsim w$ means that the vision v is ***preferred over or equivalent to*** the vision w. Thus, one have $\neg(v \succsim w) \equiv (v \prec w)$, and then the strict preference $\succ$ and indifference $\backsim$ are derived in the usual way. In the standard way, $\backsim$ induces a partition of $\mathcal{V}_\Sigma$ into equivalence classes, denoted as $\mathcal{V}_\Sigma/\backsim$. Consider a preference $\succsim$ on $\mathcal{V}_\Sigma$; we need the following assumption:

Assumption 3.3. *For any vision $v \in \mathcal{V}_\Sigma$ there is a constant vision $w^{\mathrm{c}} \in \mathcal{V}_\Sigma$ such that $v \backsim w^{\mathrm{c}}$. For any $x \in \mathcal{X}$ there is a constant vision $w^{\mathrm{c}} \in \mathcal{V}_\Sigma$ whose act f^{c} sends each $\mathcal{E} \in \Sigma$ to x.*

Then, having this assumption, $\mathcal{V}_\Sigma/\backsim$ is isomorphic to a subset of $\mathcal{X} \times \mathfrak{P}_\Sigma$. Note that a preference $\succsim$ on the set of visions $\mathcal{V}_\Sigma$ induces a relation on the set of consequences $\mathcal{X}$, denoted also by $\succsim$. For example, fixing a measure $\mathcal{P}$, the relation between $x, y \in \mathcal{X}$ is determined by that of the constant visions $(x, \mathcal{P})$ and $(y, \mathcal{P})$. Note that each member of $\mathcal{V}_\Sigma/\backsim$ contains at most one real-constant vision, since otherwise one would obtain an incomputability with the order of the reals (cf. Assumption A.6). Since $\succsim$ induces an order on the set of consequences $\mathcal{X}$, consequence-monotonicity and convexity of relations can be defined.

Definition 3.4. *We say that $\succsim$ satisfies **consequence-monotonicity** if*

$$a_1 \geq a_2 \ \wedge \ b_1 \geq b_2 \Longrightarrow x_1 \succsim x_2,$$

*for any consequences $x_1 = a_1 + \wp b_1$ and $x_2 = a_2 + \wp b_2$ in $\mathcal{X}$. Similarly, $\succ$ is **strictly consequence-monotonic** if $a_1 > a_2 \ \wedge \ b_1 > b_2 \Longrightarrow x_1 \succ x_2$.*

Monotonicity gives the meaning of "goods" in the sense that more is better. Throughout this paper we assume that all the preferences satisfy consequence-monotonicity.

Definition 3.5. *Let $x, y \in \mathcal{X}$ be two consequences: the binary relation $\succsim$ satisfies **convexity** if $x \succsim y$ and $\lambda \in \Lambda$ then $\lambda x + (1-\lambda)\, y \succsim y$, and satisfies **strict convexity** if $x \succ y$ and $\lambda \in \Lambda$ then $\lambda x + (1-\lambda)\, y \succ y$.*

4 Axiomatization

This section axiomatically settles preferences of DMs over the phantoms and provides the basis for proving that phantom preferences have a functional representation, i.e. utility representation. Our axioms generalize the vNM axiomatization [6] to the phantom framework, where in the case of roulette lotteries our axiomatization collapses to the well-known vNM model. Recalling that a preference is a complete weak order, we still present the full axioms upon a binary relation.

Axiom 4.1. *A binary relation $\succsim$ on $\mathcal{V}_\Sigma$ is a* **preference** *if it satisfies:*

I. ***Completeness:*** *Either $v \succsim w$ or $w \succsim v$, for all $v, w \in \mathcal{V}_\Sigma$.*

II. ***Transitivity:*** *If $v \succsim w$ and $w \succsim u$ then $v \succsim u$, for all $v, w, u \in \mathcal{V}_\Sigma$.*

III. ***Archimedean:*** *If $v \succ w \succ u$ then there are $\alpha, \beta \in \Lambda^\times$ for which $\alpha v + (1-\alpha)\, u \succ w$ and $w \succ \beta v + (1-\beta)\, u$.*

IV. ***Independence:*** *$v \succsim w$ iff $\alpha v + (1-\alpha) u \succsim \alpha w + (1-\alpha) u$, for each $u \in \mathcal{V}_\Sigma$ and any $\alpha \in \Lambda^\times$.*

[6] Our axiomatization is similar in approach to that of Jensen (1967), not exactly to the original von Neumann-Morgenstern (1944) axiomatization. Initially, vNM did not include an independence axiom. This was added in later works: Herstein and Milnor (1953) Fishburn (1970) and others.

These axioms refer to both visions and compound visions, where phantom weights are taken from $\Lambda^{\times}$ (replacing the open real interval $(0,1)$ in vNM) which does not contain zero divisors. Condition I and II are those of the weak order axiom. The independence condition says that when one observes preferences over compound visions, she can ignore identical summands. The Archimedean condition simulates a continuity property on preferences, meaning that for any triplet of visions $v \succ w \succ u$, there is always a vision "in between" v to w, and w to u. Moreover, it also imposes a convex structure on the set of phantom visions $\mathcal{V}_\Sigma$.

Corollary 4.2. *Suppose $\succ$ is a preference. For any visions $v \succ u$ in $\mathcal{V}_\Sigma$ there are neighborhoods, in the phantom topology, $\mathcal{M}(v)$ and $\mathcal{M}(u)$, such that $v' \succ u'$ for all $v' \in \mathcal{M}(v)$ and $u' \in \mathcal{M}(u)$.*

Proof. By the Archimedean Axiom there exists a vision $w \in \mathcal{V}_\Sigma$ such that $v \succ w \succ u$. So, we can take neighborhoods $\mathcal{M}(v) = \{v' \in \mathcal{V}_\Sigma \mid v' \succ w\}$ and $\mathcal{M}(u) = \{u' \in \mathcal{V}_\Sigma \mid w \succ u'\}$, clearly nonempty. The proof is then completed by transitivity. □

Section A.2 in the Appendix studies additional properties of preferences $\succsim$, namely, Intermediate (Lemma A.10), Mixture monotonicity (Lemma A.11) and Unique solvability (Lemma A.12).

5 Representation Theorem

This section proves that a preference, as defined in Axiom 4.1, has a functional representation. To prove this we progress as follows: we open by defining our framework which includes functional representations and some supporting statements (Section 5.1); these results are then applied to phantom-valued functions which represent preferences numerically (Section 5.2); finally, we identify utility functions and phantom-choice functions and use them to prove the existence of complete functional representation (Section 5.3).

5.1 Framework and main assumptions

Aiming to characterize preferences, we introduce a functional structure of DM's attitude toward uncertainty; later we will show that this structure provides a suitable representation for preferences. By representation we mean a sequence of real numbers whose order stand for preferences over visions; for this purpose we need:

Definition 5.1. *Consider a DM and a set of visions $\mathcal{V}_\Sigma$. We have the following functions:*

*(i) The **utility function** $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ determines the DM's attitude toward risk;*

*(ii) The **derived utility** $\mathrm{U} : \mathcal{V}_\Sigma \to \mathbb{PH}$;*

*(iii) The **(phantom) value function** $\psi : \mathbb{PH} \to \mathbb{R}$ represents the DM's attitude toward phantoms;*

*(iv) The **(phantom) choice function** $\mathrm{C} : \mathcal{V}_\Sigma \to \mathbb{R}$, given by $\mathrm{C} = \psi \circ \mathrm{U}$, which stands for the DM's attitude toward uncertainty.*

Note that a choice function encodes both risk and phantom attitudes. A DM has two sources of phantom values, consequences and expected consequences, the latter being derived from probabilities which take phantom values (standing for ambiguous probabilities).

Assumption 5.2. ***(Uniformity)*** *A DM retains the same preferences over phantom consequences and over expected consequences and thus the same preferences over phantom utility and over expected phantom utility; therefore, the value function ψ stands for both preferences.*

The derived utility $\mathrm{U}:\mathcal{V}_\Sigma \to \mathbb{PH}$ can be either expected utility, state dependent, or probabilistic sophisticated; this paper focuses on state-independent expected-utility-style representation.

Assumption 5.3. *The derived utility* $\mathrm{U}:\mathcal{V}_\Sigma \to \mathbb{PH}$ *is state-independent and additive-separable; that is, it takes the expected utility form:*

$$\mathrm{U}(v) = \sum_{\mathcal{E}\in\Sigma} \mathrm{u}(f(\mathcal{E}))\,\mathcal{P}(\mathcal{E}) = \mathrm{E}_{\mathcal{P}}(\mathrm{u}\circ f), \qquad \forall v=(f,\mathcal{P})\in\mathcal{V}_\Sigma.$$

When a DM evaluates her implicit welfare for a given vision, she first calculates the expected utility, which is a phantom value, and then appraises it in terms of real numbers. Clearly, the first step becomes meaningless in the case of a constant vision.

Definition 5.4. *Consider a value function* $\psi:\mathbb{PH}\to\mathbb{R}$ *and utility function* $\mathrm{u}:\mathcal{X}\to\mathbb{PH}$*; the choice-function* $\mathrm{C}:\mathcal{V}_\Sigma\to\mathbb{R}$ *is defined as:*

$$\mathrm{C}(v) = \psi(\mathrm{U}(v)) = \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}(f(\mathcal{E}))\,\mathcal{P}(\mathcal{E})\right) = \psi(\mathrm{E}_{\mathcal{P}}(\mathrm{u}\circ f)).$$

The main results of this paper consider visions having a finite number of events that take nonzero probabilities.

Assumption 5.5. *The set of visions* $\mathcal{V}_\Sigma$ *has a finite domain, i.e. the cardinality of* Σ *is finite. The utility function* $\mathrm{u}:\mathcal{X}\to\mathbb{PH}$ *and the value function* $\psi:\mathbb{PH}\to\mathbb{R}$ *are assumed to be continuous.*

5.2 Phantom value representation

A DM is asked to rank a set of optional phantom consequences according to her preference $\succsim$. She does this via her phantom-value function, which aims to be an order preserving function. In this way, phantom numbers are mapped to real numbers, and are then translated to the standard order over the reals. So the goal is to extract the value function out of the DM's preference, and to show that it is order preserving. However, we need some preparatory work before we can mach this goal.

Proposition 5.6. *A preference* $\succsim$ *on* $\mathcal{V}_\Sigma$*, cf. Axiom 4.1, induces a complete transitive order on the set of consequences* $\mathcal{X}$*; this relation is continuous and is compatible with the order of* $\mathbb{R}$ *when* $\succsim$ *is.*

Proof. By Assumption 3.3, we can restrict the set of visions $\mathcal{V}_\Sigma$ to the constant visions. Since we identify constant visions with their constant act, i.e. $(x_1,\mathcal{P}) \backsim (x_1,\mathcal{P}')$, we have $(x_1,\mathcal{P}) \succsim (x_2,\mathcal{P}) \Rightarrow x_1 \succsim x_2$. Accordingly, the properties of $\succsim$ are preserved over x as well. □

The order $\succsim$ on $\mathcal{X}$ induces a partition to equivalence classes, where each class has at least one phantom consequence and at most one pure-real consequence (since otherwise it would contradict the computability of $\succsim$ with the order of the reals). Every class determines a set of elements for which the DM is indifferent.

The following theorem proves that an order $\succsim$ on $\mathcal{X}$ has a functional representation, i.e. an order-preserving function, which is unique up to a strictly increasing transformation. Since preferences and the utility functions u representing them are monotonic increasing, we abuse the notation and refer to $x\in\mathcal{X}$ also as its resultant utility $\mathrm{u}(x)$.

Theorem 5.7. *Let $\succsim$ be a preference on $\mathcal{X}$. Then there exists a continuous real order-preserving function (phantom-valued representation) $\psi : \mathbb{PH} \to \mathbb{R}$, such that*

$$x_1 \succsim x_2 \iff \psi(x_1) \geq \psi(x_2), \qquad \forall x_1, x_2 \in \mathcal{X}.$$

This function ψ is unique up to strictly increasing transformations.

Proof. The complete proof, which is based on Debreu's (1954) Theorem I, is presented in Section A.3 of the Appendix. □

It is clear that any order-preserving function must also be monotonic increasing.

Remark 5.8. *When $\succsim$ is (strictly) consequence-monotonic, cf. Definition 3.4, the order-preserving function (i.e. value function) ψ must be (strictly) monotonic as well.*

Remark 5.9. *Any function $\psi : \mathbb{PH} \to \mathbb{R}$ induces a weak order $\succsim_{\mathrm{wk}}$ on $\mathbb{PH}$, denoted as $\geq_\psi$, which is defined by $z_1 \geq_\psi z_2$ iff $\psi(z_1) \geq \psi(z_2)$; equivalence and strictly greater are denoted, respectively, by $=_\psi$ and $>_\psi$. Thus, in view of Theorem 5.7, a given weak order $\succsim_{\mathrm{wk}}$ on $\mathbb{PH}$ is identified with the induced order $\geq_\psi$.*

For example, the α-function $[\ ]_\alpha : a + \wp b \ \mapsto \ a + \frac{b}{\alpha}$, with a real parameter $\alpha \in \mathbb{R}^\times$, or the absolute value $|\ | : \mathbb{PH} \to \mathbb{R}_+$ (cf. I&I, Section 1.15, Equation 1.18), determine an order on $\mathbb{PH}$. Furthermore, ψ determines a partition of $\mathbb{PH}$, and thus of $\mathcal{X} \subset \mathbb{PH}$, into equivalence classes

$$[x]_\psi = \left\{ z \in \mathbb{PH} \;\middle|\; z =_\psi x \right\}.$$

*When $\mathcal{X}$ is continuous, these equivalence classes are called **indifference curves**; the family of indifference curves constitutes an **indifference map**. When ψ is an order-preserving function, each class $[x]_\psi$ contains elements that are equivalent to real consequences $y \in \mathbb{R}$.*

Figure 1 below gives a diagrammatic representation of two curves of the indifference map. A detailed analysis of the phantom-value function, its characteristics and its economic meaning is presented in Section 7.

5.3 Utility representation

Having a functional representation for subjective preferences over consequences, we now extend it to preferences over visions. aiming to provide the utility function u and the choice function C. As before, a representation means an order-preserving function, but this time is $\mathcal{V}_\Sigma \to \mathbb{R}$, i.e. $v \succsim w$ iff $\mathrm{C}(v) \geq \mathrm{C}(w)$ for any $v, w \in \mathcal{V}_\Sigma$:

Theorem 5.10. ***(Representation Theorem)*** *A binary relation $\succsim$ on $\mathcal{V}_\Sigma$ is a preference (cf. Axiom 4.1) iff there is a choice function $\mathrm{C} : \mathcal{V}_\Sigma \to \mathbb{R}$, such that for every $v, w \in \mathcal{V}_\Sigma$:*

$$v \succsim w \iff \mathrm{C}(v) \geq \mathrm{C}(w).$$

Proof. Our proof follows the steps used in Mas-Colell, Whinston, and Green (1995) and others, in proving classical expected utility theory. The complete proof is presented in Section A.3 of the Appendix. □

Theorem 5.10 provides the connection between a DM's preference $\succsim$ to her (phantom) choice function C, which we recall is composed of a utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ and a value function $\psi : \mathbb{PH} \to \mathbb{R}$; the latter is unique up to a strictly positive transformation, cf. Theorem 5.7. Next, we study more properties of C via these functions.

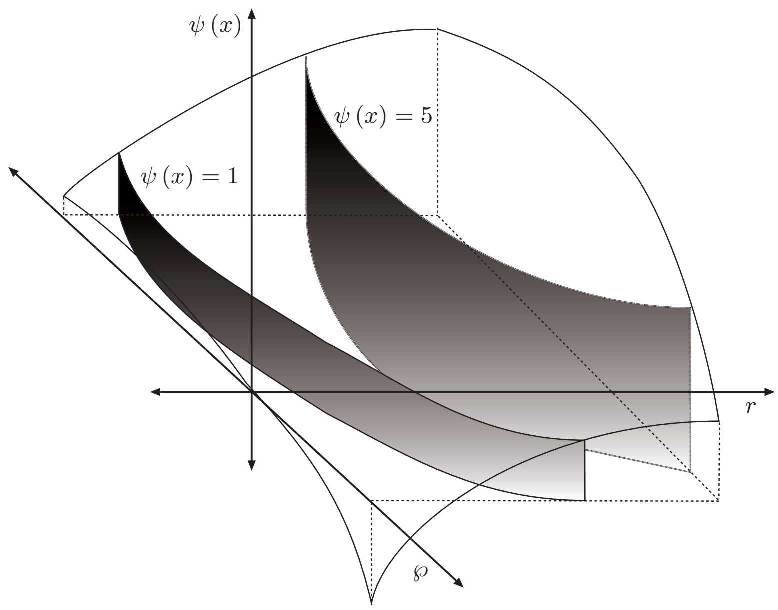


Figure 1: **Indifference map**

Proposition 5.11. *A choice function* C *is invariant under strictly increasing transformation.*

Proof. Since ψ is invariant to strictly increasing transformations and $\mathrm{C} = \psi \circ \mathrm{U}$, so is C. ☐

As in classical expected utility theory, our representation is ordinal, so that cardinal values have no additional meaning, and thus it allows the ranking of visions. Recall that the vNM model is a private case of our model in which all values are reals, i.e. probabilities and consequences. Because of the structure of our choice function $\mathrm{C} = \psi \circ \mathrm{U}$ as a composition of a value function ψ and a derived utility U, any modification of ψ or U, should be accompanied by modification of the other. Next we turn to additional characteristics: the utility function u and the choice function C, including their reciprocal relations.

Proposition 5.12. *If* $\mathrm{C} = \psi \circ \mathrm{U}$ *represents the preference* $\succsim$*, then* $\succsim$ *is monotonic (strictly monotonic) iff* ψ *and* u *are monotonic (strictly monotonic).*

Proof. Straightforward, since C is a composition of two order-preserving functions ψ and U. ☐

Proposition 5.13. *If a preference* $\succsim$ *is represented by concave (strictly concave) functions* ψ *and* u *then* $\succsim$ *is convex (strictly convex).*

Proof. The proof is derived directly from the concavity of functions (cf. Definition A.8) and the convexity of preferences (cf.Definition 3.5) together with the representation theorems (Theorems 5.7 and 5.10). ☐

Our ground structure for the above discussion has been the phantom ring $\mathbb{PH}_{(1)}$ of the order 1. However, one can also apply this exposition for consequences or probabilities that take phantom values of the order n; in such cases a DM is exposed to $2n + 1$ "levels" of uncertainty. For example, when $n = 2$, the DM might have a different attitude to each of the three parts involved , two phantoms and one real. The first phantom term is understood as a variation of the real estimation (given by the real term) while the second phantom term stands for a distortion of this variation.

6 Risk Attitude

The risk attitude of a DM is explained through her utility function u, which classically is a real-valued function $\mathrm{u} : \mathbb{R} \to \mathbb{R}$ that stands for welfare produced by real monetary outcomes; in our model this utility function is generalized by a phantom-valued function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$. Our next goal is to incorporate risk attitude into our phantom model, letting it carry the classical meaning and preserving its key features. In the standard way the DM's attitude toward risk is specified with respect to expectations:

- Risk aversion: if she prefers the expectation of the vision to the vision itself;
- Risk neutrality: if she is indifferent between a vision and its expectation.
- Risk loving: if she prefers the vision itself to its expectation;

In expected utility theory, as well as in our model, these risk attitudes are described, respectively, by the property of u being a concave function, linear function, or convex function. Note that $\mathrm{E}(v) \in \mathbb{PH}$ can be identified with a constant vision and thus it is comparable with any vision $w \in \mathcal{V}_\Sigma$.

Proposition 6.1. *For a preference $\succsim$ on $\mathcal{V}_\Sigma$, satisfying Theorem 5.10, the risk attitude is one of the three types:*

(a) ***Risk aversion:*** *(i)* $\mathrm{E}(v) \succsim v$ $\iff$ *(ii)* u *is concave* $\iff$ *(iii)* $\mathrm{u}(\mathrm{E}_{\mathcal{P}}(f)) \geq_\psi \mathrm{E}_{\mathcal{P}}(\mathrm{u}(f))$,

(b) ***Risk neutrality:*** *(i)* $\mathrm{E}(v) \backsim v$ $\iff$ *(ii)* u *is linear* $\iff$ *(iii)* $\mathrm{u}(\mathrm{E}_{\mathcal{P}}(f)) =_\psi \mathrm{E}_{\mathcal{P}}(\mathrm{u}(f))$,

(c) ***Risk seeking:*** *(i)* $\mathrm{E}(v) \precsim v$ $\iff$ *(ii)* u *is convex* $\iff$ *(iii)* $\mathrm{u}(\mathrm{E}_{\mathcal{P}}(f)) \leq_\psi \mathrm{E}_{\mathcal{P}}(\mathrm{u}(f))$,

for any vision $v = (f, \mathcal{P}) \in \mathcal{V}_\Sigma$.

Proof. The proof is presented in Section A.3 of the Appendix. □

The phantom utility function $\mathrm{U} : \mathbb{PH} \to \mathbb{PH}$ preserves other properties of the classical case, in particular features of risk aversion representation; these are demonstrated for the well-known functions (complying to the Arrow-Pratt measure [7]): the constant relative risk aversion (CRRA) and the constant absolute risk aversion (CARA).

Proposition 6.2. *The (phantom) utility function*

$$\mathrm{u}(x) = -\frac{e^{-\gamma x}}{\gamma}, \qquad \gamma \in \mathbb{R} \tag{1}$$

*implies **constant absolute risk aversion**.*

Proof. Compute the Arrow-Pratt measure $A(x) = -\frac{\mathrm{u}''(x)}{\mathrm{u}'(x)}$, which in the sense of phantom derivatives is $A(x) = \gamma$, cf. I&I (2009, Section 1.7) . □

Corollary 6.3. *The constant absolute risk aversion utility function takes the form*

$$\mathrm{u}(x = a + \wp b) = -\frac{e^{-\gamma a}}{\gamma} - \wp \left(\frac{e^{-\gamma(a+b)}}{\gamma} - \frac{e^{-\gamma b}}{\gamma} \right).$$

Proof. Apply Proposition 1.18 in I&I (2009) to $\mathrm{u}(x) = -\frac{e^{-\gamma x}}{\gamma}$ (see Equation (1)). □

[7]See Arrow (1963), Pratt (1964), and Arrow (1965)

This corollary shows that the phantom component of u can be viewed as a possible marginal utility, derived from a potential realization of the phantom term of consequences. Later in Section 7, we investigate how the real and the phantom parts of utility functions are aggregated by preferences.

Proposition 6.4. *The (phantom) utility function*

$$\mathrm{u}(x) = \frac{x^{1-\gamma}}{1-\gamma}, \qquad \gamma \in \mathbb{Q}, \tag{2}$$

models ***constant relative risk aversion****.*

Proof. Use the phantom derivative to obtain $A(x) = -\frac{\mathrm{u}''(x)}{\mathrm{u}'(x)}x = \gamma$, cf. I&I (2009, Section 1.7). ☐

In comparison to the real CRRA utility function, which allows the coefficient of risk aversion γ to be any real number, in the phantom case this coefficient is restricted to be a rational number; this provides the following forms for Equation (2):

Corollary 6.5. *The constant relative risk aversion utility function can be written as*

$$\mathrm{u}(x = a + \wp\, b) = \frac{a^{1-\gamma}}{1-\gamma} + \wp \left(\frac{(a+b)^{1-\gamma}}{1-\gamma} - \frac{b^{1-\gamma}}{1-\gamma} \right).$$

Proof. Suppose $\gamma = \frac{m}{n}$. Then Equation (2) can be rewritten as $\mathrm{u}(x) = \frac{n}{n-m}\sqrt[n]{x^{m-n}}$ and the required function is obtained by the realization property for exponents and roots of phantoms, cf. I&I (2009, Section 1.4). ☐

As obtained for the CARA, the separation of u in CRRA to its real and phantom components indicates a real utility plus a possible marginal utility. In some sense, u can be referred to as carrying two commodities, a real utility and a marginal utility, where the choice function aggregates them together. The latter results show that the measures of risk attitude used in classical expected utility theory are also applicable to our framework, while here one also distinguishes between attitude toward risk and attitude toward phantoms.

7 Decision Making

One of the key features of our decision-making model is the distinction of four arguments which play a role in making a decision: risk and attitude towards risk (inherited from classical expected utility theory) recorded here by the utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$, and phantoms and attitude towards phantoms which are new arguments recorded by $\psi : \mathbb{PH} \to \mathbb{R}$, cf. Definition 5.1. The first two have been already studied in the previous section, showing that classical notions of risk attitude are also feasible in our model. Next we study phantom attitude – a characteristic of a DM's personal approach to phantoms that is assumed to be consistent on the two sources of phantoms, vague consequences and ambiguous probabilities, cf. Remark 5.2. Later, in Section 8.1, we introduce the notion of phantom degree, which estimates the quantity of phantom embedded in the system.

Remark 7.1. *In the study of phantom attitude we shall focus on the value function ψ, which will be extracted from a given preference $\succsim$ on a set of visions $\mathcal{V}_\Sigma$. This extraction is performed by taking the choice function $\mathrm{C} : \mathcal{V}_\Sigma \to \mathbb{R}$ (representing $\succsim$) restricted to a subdomain of visions*

$v = (f, \mathcal{P}_0) \in \mathcal{V}_\Sigma|_{\mathcal{P}_0}$ having a common fixed probability measure $\mathcal{P}_0 \in \mathfrak{P}_\Sigma$. Accordingly, each vision in $\mathcal{V}_\Sigma|_{\mathcal{P}_0}$ is identified with its act, which in turn is described by its consequences. So in this view we formulate this part in terms of consequences, abusing the notation and writing $x_1 \succsim x_2$, for consequences and $x_1 \oplus x_2$ for a compound of consequences. Moreover, since $\mathrm{C} = \psi \circ \mathrm{U}$, and the translation between $\mathcal{V}_\Sigma|_{\mathcal{P}_0}$ and $\mathbb{PH}$ is given by expected utility $\mathrm{E}(\mathrm{u} \circ f)$, we work directly with the latter. Thus consequences $x \in \mathcal{X}$ can have the interpretation of expected utility.

Throughout this section we assume that DMs follow the axiomatization of Section 4 and the representation theorems in Section 5. Since preferences, and their functional representation, are monotonically increasing, we sometimes abuse the notation and identify consequences $x \in \mathcal{X}$ with expected utilities produced by some vision.

7.1 Attitude towards phantoms (ambiguity)

Preferences of DMs on visions are represented by the functions $\psi \circ \mathrm{U}$, where the attitude toward phantoms is characterized by the properties of ψ. As before, we specify three attitudes: aversion, loving, and neutrality. The phantom averse DM dislikes circumstances in which she has to choose when the outcomes are vague (recorded by phantom consequences) or their odds of occurrence are unclear (recorded by phantom probabilities) or both. Conversely, a phantom lover seeks these circumstances, while a phantom neutral is indifferent to them.

A phantom averse DM would prefer to make decisions when all the arguments are unmistakably known; accordingly she will try to minimize phantom effects and therefore prefer real lotteries over phantom visions. A phantom lover behaves in the opposite way. Attitude toward ambiguity (situations with clear outcomes) is a special case of phantom attitude relating to a setting whose consequences take only real values.

As in classical theory, subjective beliefs of DMs are recoded by probability measures $\mathcal{P} \in \mathfrak{P}$. It is commonly assumed that a DM's attitude toward risk is independent of her subjective beliefs [8] ; we adopt this approach for phantom attitude too:

Assumption 7.2. *Attitudes toward risk and phantoms, derived from a DM's preferences, are independent of her beliefs.*

In formal language this means that the preference $\succsim$ on $\mathcal{V}_\Sigma|_{\mathcal{P}_0}$ is unchanged for all possible restrictions to $\mathcal{P}_0 \in \mathfrak{P}$. Recall that a DM holds the same preference on consequences and expected consequences, induced by her preference on visions, cf. Assumption 5.2 and Remark 7.1. For our purpose, consequences can be considered as expected utilities in terms of phantoms. Thus, to study phantom attitude we may refer only to consequences.

There are several ways to define phantom attitude, each based on a different notion. We first define these notions and show that they coincide, then we provide an overall characterization of phantom attitude.

Suppose $x \in \mathcal{X}$ is a consequence and let $x_0 \in \mathcal{X}$ be a real consequence, such that $x \backsim x_o$. We say that x_0 is the **real equivalent (REQ)** of $x \in \mathcal{X}$ and denote it as x_{req}; therefore, $\psi(x_{\mathrm{req}}) = \psi(x)$ for any order-preserving function ψ. Since in this context consequences are expected phantom utilities, the real equivalent of vision is a real lottery. Note that real equivalents may take any value in $\mathbb{R}$. Recall that $[x] = \frac{x+\overline{x}}{2}$ stands for the projection of x to $\mathbb{R}$, always satisfying $[x] = [\overline{x}]$, and can be understood as a real estimation of x.

Remark 7.3. *Consider a weak order $\succsim$ on $\mathbb{PH}$, then $\frac{1}{2}x \oplus \frac{1}{2}\overline{x} \backsim \frac{1}{2}x_{\mathrm{req}} \oplus \frac{1}{2}\overline{x}_{\mathrm{req}}$ for any $x \in \mathbb{PH}$. Indeed, the equivalency is by the definition of REQ and by the transitivity of $\succsim$.*

[8] Similar assumptions on ambiguity attitude appear in KMM (2005).

Property 7.4. *Let $x \in \mathcal{X}$ be a phantom consequence. We say that the DM is:*

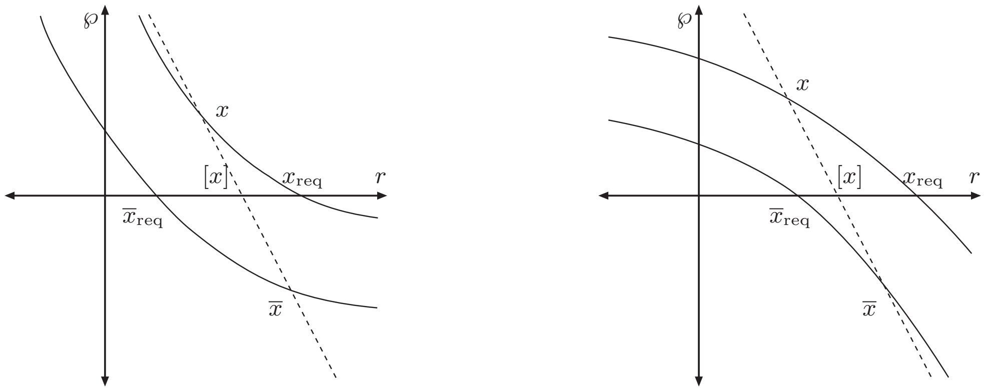
(i) ***Phantom averse*** *if* $[x] \succsim \frac{1}{2}x \oplus \frac{1}{2}\overline{x}$ $\left(\backsim \frac{1}{2}x_{\text{req}} \oplus \frac{1}{2}\overline{x}_{\text{req}}\right)$,

(ii) ***Phantom neutral*** *if* $[x] \backsim \frac{1}{2}x \oplus \frac{1}{2}\overline{x}$ $\left(\backsim \frac{1}{2}x_{\text{req}} \oplus \frac{1}{2}\overline{x}_{\text{req}}\right)$,

(iii) ***Phantom loving*** *if* $[x] \precsim \frac{1}{2}x \oplus \frac{1}{2}\overline{x}$ $\left(\backsim \frac{1}{2}x_{\text{req}} \oplus \frac{1}{2}\overline{x}_{\text{req}}\right)$.

The motivation for this definition is whether or not the DM sees any potential for a benefit (in real terms) greater than the real estimation of the consequence; if she does, she will prefer the phantom outcome over its real estimation, otherwise she will prefer its real estimation.

Figure 2 illustrates phantom aversion and phantom loving. The solid lines are drawn for equivalent consequences of x, also containing their REQ. The dashed line between a consequence and its conjugate passes through their common projection. Note that in this figure $x_{\text{req}} \backsim x \succ \overline{x} \backsim \overline{x}_{\text{req}}$, so $x_{\text{req}} \neq \overline{x}_{\text{req}}$. Fig. 2 (a) demonstrates the case of phantom aversion in which $\frac{1}{2}x \oplus \frac{1}{2}\overline{x} \prec [x]$, which in real terms is $\frac{1}{2}\left(x_{\text{req}} + \overline{x}_{\text{req}}\right) < [x]$. Panel (b) shows the indifference curve of a phantom lover, where $\frac{1}{2}x \oplus \frac{1}{2}\overline{x} \succ [x]$, for $x \succ \overline{x}$, that is $\frac{1}{2}\left(x_{\text{req}} + \overline{x}_{\text{req}}\right) > [x]$.



(a) Phantom aversion **(b) Phantom seeking**

Figure 2: **Phantom attitude**

Another way to define phantom attitude is via the convexity of the value function ψ:

Property 7.5. *For any phantom consequences $x \in \mathcal{X}$ a DM exhibits, cf. Definition A.8,*

(i) Phantom aversion if $\psi([x]) \geq \frac{\psi(x)+\psi(\overline{x})}{2}$, *i.e. ψ is quasi-concave,*

(ii) Phantom neutrality if $\psi([x]) = \frac{\psi(x)+\psi(\overline{x})}{2}$, *i.e. ψ is linear,*

(iii) Phantom loving if $\psi([x]) \leq \frac{\psi(x)+\psi(\overline{x})}{2}$, *i.e. ψ is quasi-convex.*

This notion of phantom attitude is stated using the value function, in comparison to Property 7.4, which was given in terms of preferences.

Proposition 7.6. *Having the conditions of Theorem 5.10 satisfied, properties 7.4 and 7.5 are equivalent.*

Proof. We prove only case (i), the proof of the other two cases is derived from the same considerations. Assuming ψ is concave, then by Lemma A.9 $\psi([x]) \geq \frac{\psi(x)+\psi(\overline{x})}{2}$, which by Theorem 5.10 yields $[x] \succsim \frac{1}{2}x \oplus \frac{1}{2}\overline{x}$. Conversely, suppose that $\succsim$ exhibits phantom aversion then $[x] \succsim \frac{1}{2}x \oplus \frac{1}{2}\overline{x}$, which by Theorem 5.10 implies $\psi([x]) \geq \frac{\psi(x)+\psi(\overline{x})}{2}$, i.e. ψ is concave. □

Example 7.7. *Take the value function* $\psi(x = a + \wp b) = 1 + a - \kappa e^{-\kappa b}$*, with* $\kappa \in \mathbb{R}$*. When* $\kappa > 0$ *then the DM exhibits phantom aversion, while if* $\kappa < 0$ *she exhibits phantom loving. Note that for* $\kappa = 0$ *this function collapses to* $\psi(x = a + \wp b) = a$ *and thus* $\psi(x) = x$ *for any* $x \in \mathbb{R}$*.*

7.2 Other attributes

Although risk attitude and phantom attitude have been separated, they preserve the fundamental attributes of classical expected utility. We say that a representation, assumed to have a continuous function, is **smooth** if its indifference curves have no kinks. Therefore, when ψ viewed as a real function $\mathbb{R}^2 \to \mathbb{R}$ is differentiable, then the functional representation is also smooth; for example

$$\psi(x = a + \wp b) = e^a e^{\frac{1}{2}b}$$

provides a smooth representation which is the standard exponential for real arguments.

Consequences and probabilities $z \in \mathbb{PH}$ have dual values, given by their conjugates $\overline{z} \in \mathbb{PH}$. The phenomenon of the framing effect and its effects on choices is well-known in the literature; this phenomenon is also reflected over the phantoms in the representation through numbers or their phantom conjugates. Therefore, as in Tversky and Kahneman (1986), this effect can be eliminated:

Assumption 7.8. ***(Non-framing effect)*** *A DM is indifferent between a phantom consequence* $x \in \mathcal{X}$ *and its conjugate* $\overline{x} \in \mathcal{X}$*, i.e.* $x \backsim \overline{x}$*.*

Accordingly, noting that $\frac{\psi(x)+\psi(\overline{x})}{2} = \psi(x) = \psi(\overline{x})$, Property 7.5 is rephrased as:

Property 7.9. *Under Proposition 7.6 and Assumption 7.8 a DM is considered:*

(i) Phantom averse if $\psi([x]) \geq \psi(x) = \psi(\overline{x})$,

(ii) Phantom neutral if $\psi([x]) = \psi(x) = \psi(\overline{x})$,

(iii) Phantom loving if $\psi([x]) \leq \psi(x) = \psi(\overline{x})$,

for any $x \in \mathcal{X}$*.*

There are cases in which DMs are not affected by framing:

Example 7.10. *Suppose* $\psi : \mathbb{PH} \to \mathbb{R}$ *is a piecewise linear function given by*

$$\psi(x = a + \wp b) = \begin{cases} a + \kappa b, & b \geq 0 \\ a + (1-\kappa)b, & b \leq 0 \end{cases}, \qquad 0 < \kappa < 1 \text{ is in } \mathbb{R}.$$

Then $\psi(a + \wp b) = \psi(x) = \psi(\overline{x}) = \psi((a+b) - \wp b)$ *for any* $x \in \mathcal{X}$*. When* $\kappa < \frac{1}{2}$*, the function is quasi-convex (phantom loving), while* ψ *is quasi-concave (phantom aversion) for* $\kappa > \frac{1}{2}$*.*

7.3 Absolute and relative phantom (ambiguity) aversion

Our next goal of comparing phantom attitude across different DMs is performed with respect to specific anchors which in this case are constant visions identified with constant consequences.

Definition 7.11. *The* ***real certain equivalent (RCE)*** $v_{\text{rce}} \in \mathcal{V}_\Sigma$ *of a vision* $v \in \mathcal{V}_\Sigma$*, written* $\text{rce}(v)$*, is a real constant vision in* $\mathcal{V}_\Sigma$ *equivalent to* v*, i.e.* $v_{\text{rce}} \backsim v$*.*

Accordingly, $\psi(\mathrm{U}(v_{\mathrm{rce}})) = \psi(\mathrm{U}(v))$, and the RCE is unique by the strict monotonicity of the derived utility U and the value function ψ. Note that when v is constant, its RCE and its REQ are identical. Assume henceforth that the DM's risk attitude is represented by a concave increasing utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ with Assumption 5.2 and 7.2 being satisfied. That is, preference over consequences and over expected consequences are identical and probability assessment is separated from phantom attitude.

In order to rank phantom aversion one should first extract the phantom preference out of the DM's entire preferences, as given for visions. Since each vision is compounded of an act and a probability measure, when comparing preferences across different DMs we recall they are assumed to have identical beliefs. Therefore, the effects of different probability assessments are eliminated. [9] The extraction of phantom attitude also requires the separation of risk attitude, so we assume that comparable DMs have the same risk aversion. [10] Then we remain only with the effect of the phantom attitude, which brings us to defining a ranking rule for phantom aversion. For this purpose we give three definitions, based on different notions, and prove later that they coincide.

Definition 7.12. *Consider the preferences $\succsim_A$ and $\succsim_B$, represented respectively by the concave value functions ψ_A and ψ_B, of two DMs who share the utility function* u. *We say that A is* ***at least as phantom averse*** *as B if*

(i) for any vision $v \in \mathcal{V}_\Sigma$ and a real vision $\ell \in \mathcal{V}_{\Sigma;\mathrm{re}}$ (i.e. a lottery $\ell \in \mathcal{L}$), $v \succsim_A \ell \Longrightarrow v \succsim_B \ell$;

(ii) $\mathrm{rce}_A(v) \leq \mathrm{rce}_B(v)$ for any vision $v \in \mathcal{V}_\Sigma$;

(iii) ψ_A is a concave transformation of ψ_B, that is $\psi_A = g \circ \psi_B$, with $g : \mathbb{R} \to \mathbb{R}$ an increasing concave function.

The principle of notion (i) is that if the DM A is more phantom averse, whenever she prefers a vision over a real vision then DM B, who is less phantom averse, will follow the same choice. In fact one can weaken this notion to relations not necessarily obeying the axiomatization of Section 4. Note that notions (i) and (ii) assume no conditions on DMs' risk preferences, while (iii) assumes preferences which satisfy the axiomatization of Section 4.

Theorem 7.13. *The three notions of Definition 7.12 are equivalent.*

Proof. The proof is presented in Section A.3 of the Appendix. □

Next we focus on defining the coefficients of phantom aversion. Consider a continuous-differentiable function $f : \mathbb{PH} \to \mathbb{R}$, viewed as a function $f : \mathbb{R}^2 \to \mathbb{R}$. We denote by $\partial_\wp f$ the derivative of $f(s + \wp t)$ with respect to t; similarly $\partial_{\wp\wp} f$ denotes the second derivative. The derivatives $\partial_r f$ and $\partial_{rr} f$, taken with respect to s, are defined in the same way.

Definition 7.14. *The* ***coefficient of absolute phantom aversion*** *of a DM is defined as*

$$\vartheta = -\frac{\partial_{\wp\wp}\psi}{\partial_\wp \psi},$$

assuming that ψ, which represents the DM's preferences $\succsim$, is twice differentiable. [11]

[9] Such circumstances can be understood as symmetric information, i.e. all the DMs have the same information, leading to identical probability assessments that stand for the same beliefs.

[10] Similar assumptions were made in other models of decision making under ambiguity; for example, Ghirardato, Maccheroni, and Marinacci (2004), Epstein (1999), Ghirardato and Marinacci (2002), and KMM (2005).

[11] Neilson (1993) and KMM (2005) also used an Arrow-Pratt type index to measure ambiguity aversion.

We would like to show that this indicator of phantom aversion is correlated with the of phantom aversion as defined earlier.

Corollary 7.15. *Suppose that the conditions of Theorem 7.13 are satisfied and assume that the DMs A and B share the same utility function* u*. Then A is at least as phantom averse as B iff* $\vartheta_A(x) \geq \vartheta_B(x)$ *for any* $x \in \mathcal{X}$*, where the functions* ψ_A *and* ψ_B*, representing preferences* $\succsim_A$ *and* $\succsim_B$*, are assumed to be strictly increasing and twice differentiable.*

Proof. The proof is presented in Section A.3 of the Appendix. □

In fact, the structure of the coefficient of phantom aversion ϑ, i.e. the second derivative $\partial_{\wp\wp}\psi$ of the value function ψ normalized by its first derivative $\partial_{\wp}\psi$ (both taken with respect to the phantom component), provides the concavity of the value function ψ. Similarly to Arrow-Pratt analysis, we can define the ***coefficient of relative phantom aversion*** as

$$\vartheta(x) = -\frac{\partial_{\wp\wp}\psi(x)}{\partial_{\wp}\psi(x)}\mathrm{ph}(x).$$

Using a similar analysis, one can identify properties of absolute phantom aversion or relative phantom aversion; for example decreasing or increasing.

7.4 Pessimism and optimism

Optimism or pessimism is a matter of personal perception. Whenever an individual has to make a decision she has a subjective taste toward phantoms (phantom attitude) as well as beliefs (phantom probability measure) which might be crystalized from available objective information and her subjective doubts. Beside these arguments we would like also to identify the DM's personal perception of those beliefs. That is, observing an increase of the phantom term of a given expectation, one could see it as a promising occurrence (since it might have a better benefit), while others might see it as potentially disappointing. The former is cataloged as optimism, while the latter is pessimism. This section suggests an evaluation method, for pessimism, apathy or optimism, via preferences over visions.

Property 7.16. *A DM is said to be*

(i) ***Pessimistic*** *if* $[x] \succ x \backsim x_{\mathrm{req}}$ *(i.e.* $[x] > x_{\mathrm{req}}$*),*

(ii) ***Apathetic*** *if* $[x] \backsim x \backsim x_{\mathrm{req}}$ *(i.e.* $[x] = x_{\mathrm{req}}$*),*

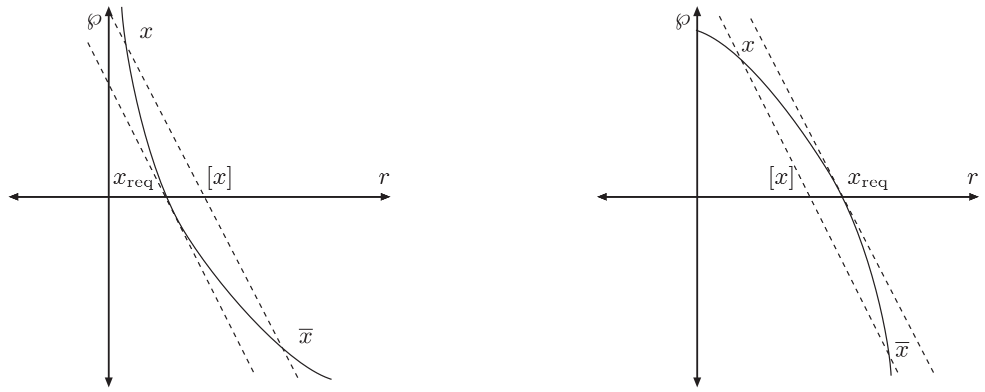
(iii) ***Optimistic*** *if* $[x] \prec x \backsim x_{\mathrm{req}}$ *(i.e.* $[x] < x_{\mathrm{req}}$*),*

for any consequence $x \in \mathcal{X}$*.*

Consider a phantom consequence, which may be any phantom value $x \in \mathcal{X}$. A pessimistic DM prefers the real projection of consequence over the consequence itself, the optimistic DM selects the opposite, while an apathetic DM is always indifferent between these alternatives.

Figure 3 illustrates indifference curves of pessimism and optimism. The dashed line connecting consequence x and its projection $\overline{x}$ addresses their projection on the real axis. Since, in this figure, x and $\overline{x}$ lie on the same indifference curve, $x \backsim \overline{x}$, they have identical real equivalence, $x_{\mathrm{req}} = \overline{x}_{\mathrm{req}} \in \mathbb{R}$, which means that the DM follows the non-framing effect (Assumption 7.8). Panel (a) demonstrates pessimism, for which one sees that $[x] \succ x \backsim \overline{x}$ and $[x] > x_{\mathrm{req}}$. Panel (b) shows the indifferent curve of a optimism, for which $x \backsim \overline{x} \succ [x]$ and $[x] < x_{\mathrm{req}}$.

Property 7.16 implies an additional characteristic of a DM; that is, in case of $x = a + \wp b$ with either positive or negative phantom component b, a pessimist DM would be willing to exchange a phantom consequence $a + \wp b$ for a real consequence that is less than $a + b/2$. This



(a) Pessimism **(b) Optimism**

Figure 3: **Phantom aversion**

view motivates the usage of phantoms to characterize pessimism or optimism. Understating the phantom component as a possible distortion of the real component, a pessimist would point to the worst case, reflected by minimizing positive phantom components; an optimist considers maximization.

Remark 7.17. ***Pessimism (Optimism)*** *is exhibited when a DM acts as if positive (negative) phantom components will probably not be realized but negative (positive) phantom components probably will.*

Having this insight we can identify levels of optimism:

Definition 7.18. *Assuming ψ, the function representing $\succsim$, is differentiable, the ratio*

$$\theta = \frac{\partial_r \psi}{\partial_\wp \psi}$$

is referred to as the ***coefficient of pessimism****.*

The coefficient of pessimism (which also stands for optimism) is the marginal rate of substitution of the real component of consequence and its phantom component. In other words, θ implicitly describes the shape of the indifferent curves of the value function ψ. When a DM sees no influence of the phantom component (i.e. $\partial_\wp \psi = 0$) the coefficient of pessimism θ is ∞. On the other hand, when she sees that the real component plays no role (i.e. $\partial_r \psi = 0$) then her pessimism level is zero, i.e. $\theta = 0$. Having the notion of θ, an alternative characterization is obtained:

Property 7.19. *The DM's perception can be classified as following ($x = a + \wp b$ is a consequence):*

(i) Pessimism: $\theta(x) > 2$ for any $b > 0$, and $\theta(x) < 2$ for any $b < 0$;

(ii) Apathy: $\theta(x) = 2$ for any x;

(iii) Optimism: $\theta(x) < 2$ for any $b > 0$, and $\theta(x) > 2$ for any $b < 0$.

Geometrically, the projection $[x] = a + \frac{b}{2}$ is the projection of $x = a + \wp b$ on the real axis in the direction of its conjugate value $\overline{x} = (a + b) - \wp b$. So the slope of the line connecting x to its projection is always -2. The economic meaning is that for a phantom consequence x with positive b, for giving up b a pessimist would ask for real compensation that is less than $b/2$; while when $b < 0$ she would be willing to give up real value that is greater than $b/2$. The reason for this is her perception about the potential realization of phantom components.

Proposition 7.20. *Under the conditions of Theorem 5.10, properties 7.16 and 7.19 are equivalent.*

Proof. The proof is presented in Section A.3 of the Appendix. □

We close the section by tying optimism and pessimism to phantom attitudes.

Proposition 7.21. *The DM's perception implies attitudes towards phantoms as follows:*

(i) The pessimistic DM is phantom averse,

(ii) The apathetic DM is phantom neutral,

(iii) The optimistic DM is phantom loving.

Proof. Property 7.16 is a special case of Property 7.4. □

This proposition indicates that a pessimistic (optimistic) DM does not exhibit phantom loving (aversion). For an illustration see Example 7.10, in which the coefficient κ is $< \frac{1}{2}$, so that the DM is pessimist. When $\kappa > \frac{1}{2}$ she is optimist, while $\kappa = \frac{1}{2}$ indicates apathy.

8 Phantom (Ambiguity) Versus Risk

The analysis in the previous two sections focused on characterizing the attitudes of a DM, and thus considered visions which have a common probability measure. This section addresses the question of comparing uncertainty degrees, consisting of phantom degrees which are derived from vagueness of consequences and their probabilities. Thus visions need not share the same probability measure but still have the same domain Σ. In our model, uncertainty degree captures both risk and phantoms.

Throughout this section visions are assumed to have the same domain Σ, which is identified by the state space Ω, assumed to have a countable number of element; thus visions are written $v \in \mathcal{V}_\Omega$. In this setting, acts are just (discrete) phantom random variables.

8.1 On priors

Priors are usually matters of personal belief, usually supported by preliminary knowledge and can not be entirely separated from the available information. Probabilities can be classified into three types: objective, subjective, and quasi-subjective. ***Objective probabilities*** rely totally on available information, either public or private, and they do not involve any personal beliefs. Therefore, DMs who hold the same information will share identical probabilities.

When a-priori there is no information about the odds of events, for example about events that have never happened, we say that the estimated probabilities are based only on personal beliefs and call them ***subjective probabilities***. These beliefs might rely on past experience or on educated analysis, but not on solid information like statistics on past occurrences. Subjective or objective probabilities can be forecasted clearly or ambiguously, where the latter being represented by phantom values.

In reality predicted probabilities are usually a mixture of information and beliefs, understood as doubts that a DM might have on a given objective probability, which we call ***quasi-subjective probabilities*** or parameter ***misspecification***. Our decision making model does not make the distinction between sources of vagueness: subjective reasons (i.e. doubts or subjective beliefs) or objective reasons (i.e. unclear information).

8.2 Uncertainty measures

To rank visions we need more information about a DM's preferences; however, this information can be replaced partially by conditions on the visions' distribution, leading to measuring of visions. The measures described below differentiate between two sources of uncertainty (risk and phantom) embedded in vision, and yield a consolidated measure of the uncertainty latent in visions. We first define uncertainty and introduce a measure of uncertainty in phantom spaces, then show that it can be decomposed into a phantom measure and a risk measure.

Consider visions $v, w \in \mathcal{V}_\Omega$. Considering their acts f_v and g_w as (phantom) random variables, we say that f_v and g_w are ***equal in distribution***, denoted by $f_v =_d g_w$, if they have the same distribution functions, i.e. $\mathcal{P}_v(f_v \precsim_{\text{wk}} x) = \mathcal{P}_w(g_w \precsim_{\text{wk}} x)$ for each $x \in \mathcal{X}$; this means that the left-hand side is a random variable which assumes the same values with the same probabilities as the random variable defined by the right-hand side. Note that $\precsim_{\text{wk}}$ not need be the DM's preferences $\precsim$.

Using the approach of Rothschild and Stiglitz (1970) and LeRroy and Werner (2001), we define an uncertainty ranking rule for phantom visions according to their acts.

Definition 8.1. *A vision $v \in \mathcal{V}_\Omega$ is **more uncertain** than $w \in \mathcal{V}_\Omega$ if there exists a vision $u \in \mathcal{V}_\Omega$, such that*

$$f_v - \mathrm{E}(v) =_d f_w - \mathrm{E}(w) + f_u, \qquad \textit{with} \quad \mathrm{E}(u) = 0,$$

where f_u and f_w are independent. [12] *If f_u is not identically zero, then v is strictly more uncertain than w.*

This definition does not assume that visions have an identical expectation or similar probability distributions. It is easily seen that if v is more uncertain than w, then the vision v' whose act is $f_{v'} = f_v - \mathrm{E}(v)$ is more uncertain than the vision w' having the act $f_{w'} = f_w - \mathrm{E}(w)$.

Remark 8.2. *Consider visions $v = (f, \mathcal{P})$ and $u = (g, \mathcal{Q})$ with $\mathrm{E}(u) = 0$, Definition 8.1 implies that the vision with act $f+g$ is more uncertain than v. One can also show that any non-constant vision is strictly more uncertain than its expectation (viewed as a constant vision).*

A DM who is risk averse or phantom averse is referred to as exhibiting uncertainty aversion. The next theorem aims to tie the DMs' preferences to the uncertainty measure, guided by the motivation that every uncertainty-averse DM prefers a less uncertain vision to a more uncertain one, assuming both have the same expectation.

Theorem 8.3. *Suppose v and w are visions in $\mathcal{V}_\Omega$ having identical expectation. Then v is (strictly) more uncertain than w iff every (strictly) uncertainty-averse DM (strictly) prefers w to v.*

Proof. The proof is presented in Section A.3 of the Appendix. □

Our next goal is to show that, like in classical expected utility theory, for visions whose acts are symmetrically distributed the vision with higher variance is more uncertain. To do this we need the following property of uncertainty:

[12] In phantom probability, as in classical probability theory, if two random variables are independent they are also uncorrelated and mean independent.

Proposition 8.4. *Suppose v and u are visions whose acts f_v and f_u are independent and $\mathrm{E}(u) = 0$. Let w_i, $i = 1, 2$, be the visions having the acts $f_v + \lambda_i f_u$, where $\lambda_1, \lambda_1 \in \mathbb{PH}$. Then w_1 is strictly more uncertain than w_2 for every pseudo positive $\lambda_1 \gg \lambda_2$.*

Proof. The proof is presented in Section A.3 of the Appendix. □

Corollary 8.5. *The vision $w' = (\lambda f_w, \mathcal{P})$ is strictly more uncertain than $w = (f_w, \mathcal{P})$ for any $\lambda \gg 1$.*

Proof. Immediate, by Proposition 8.4, applied to vision v with constant act 0 and vision u whose act is given by $w - \mathrm{E}(w)$. □

In order to define measure of uncertainty we need to impose some constrains on consequences and their distributions. [13]

Definition 8.6. *An act f with mass function $\mathcal{P}_f = \mathcal{P}_{f,\mathrm{re}} + \wp \mathcal{P}_{f,\mathrm{ph}}$ (cf. I&I (2009, Section 3.1)) is said to be a **symmetric act** if*

(i) The real part $\mathcal{P}_{f,\mathrm{re}}$ is symmetric around the expectation,

(ii) The phantom part $\mathcal{P}_{f,\mathrm{ph}}$ is anti-symmetric around the expectation.

*A vision, whose phantom component of the act is symmetric around the expectation and whose mass function is symmetric, is called **symmetric**.*

This definition of symmetry generalizes the classical definition of symmetric distribution (not only discrete); for example the phantom normal distribution is symmetric.

Theorem 8.7. *Suppose v and w are symmetric visions with identical expectation and variances σ_v^2 and σ_w^2, respectively. Then v is strictly more uncertain than w iff $\sigma_v^2 \gg \sigma_w^2$.*

Proof. The proof is presented in Section A.3 of the Appendix. □

Theorem 8.7 refers only to the first and the second moments of acts (i.e. phantom random variables); however moments of higher orders (which are defined similarly to the classical ones) can be used to extend our measure of uncertainty. Having the phantom variance $\sigma_v^2 \in \mathbb{PH}$ as a phantom measure for uncertainty, we next decompose it into measures of phantom degree and risk degree.

Definition 8.8. *The **risk measure** ρ of a symmetric vision v with variance σ_v^2 is defined as*

$$\rho(v) := \mathrm{re}\left(\sigma_v^2\right) + \frac{1}{2}\mathrm{ph}\left(\sigma_v^2\right),$$

*the **(relative) phantom measure** ξ is given by*

$$\xi(v) := \frac{1}{2}\left|\,\mathrm{ph}\left(\sigma_v^2\right)\right|,$$

*and the **uncertainty measure** is*

$$\kappa(v) := \sqrt{\rho(v)^2 + \xi(v)^2} = \left|\,\sigma_v^2\,\right|.$$

[13] The equivalent restriction in a continuous environment would be a phantom random variable with phantom normal distribution.

By the properties of conjugation and the fact that $\overline{\rho(v)^2} = \rho(\overline{v})^2$, it is easy to check that these measures are invariant under conjugation, i.e. $\rho(v) = \rho(\overline{v})$, $\xi(v) = \xi(\overline{v})$, and $\kappa(v) = \kappa(\overline{v})$. Note that increasing the real term of variance results in a higher level of risk, while increasing the absolute value of its phantom term might result in a lower level of risk, but a higher level of phantoms. Note that the phantom measure can be 0 even though the system includes phantom arguments; this happens when the phantom effect is perfectly canceled. To identify the involvement of phantoms we have the following measure:

Definition 8.9. *The* ***absolute phantom measure*** $\Xi : \mathcal{V}_\Omega \to \mathbb{R}$ *for visions is defined as*

$$\Xi(v) := \sum_{\omega\in\Omega} \left| \operatorname{ph}\left(f^2(\omega)\mathcal{P}(\omega)\right) \right|, \qquad \text{for any } v = (f, \mathcal{P}).$$

Ξ provides the total **phantom degree** latent in vision v, and is 0 if and only if all the arguments are reals. Since ambiguity is a special case of phantoms, definitions 8.8 and 8.9 identify the degree of ambiguity accommodated in visions.

9 Applications

This section presents two examples to show how problems of decision making under ambiguity (the Ellsberg paradox and an insurance dilemma) are carried naturally in phantom theory.

9.1 The Ellsberg paradox

We begin by framing the ***Ellsberg paradox*** using the phantom framework. We show that although the vNM independent axiom and the Savage P2 axiom are violated by selections as demonstrated by the Ellsberg experiment, such selections coincide with our axiomatization. First, we recall the Ellsberg experiment:

Experiment 9.1. *There is an urn with 90 colored balls, 30 of them red and the others either black or yellow. The prize for a correct bet is \$9. The experiment consists of two parts. In the first part a DM has to choose between the two alternative bets:*

(R) *drawing a red ball* *or* (B) *drawing a black ball.*

Then in the second part the DM has to choose between betting on:

(RY) *drawing a red or yellow ball* *or* (BY) *drawing a black or yellow ball.*

Behavioral experiments have demonstrated that individuals usually prefer (R) over (B) and (BY) over (RY). Let $\#B$ and $\#Y$ denote, respectively, the unknown number of black balls and yellow balls; in particular $\#B + \#Y = 60$. Since the prizes are the same, preferring (R) over (B) implies that the DM believes that drawing a red ball is more likely than drawing a black ball. That is $\#B < 30$ and thus $\#Y > 30$. Similar considerations show that preferring (BY) over (RY) implies that the DM believes that drawing a red or yellow ball is less likely than drawing a black or yellow ball. That is, $\#Y + 30 < 60$ and thus $\#Y < 30$. This provides the Ellsberg paradox, which shows that DMs usually violate the axioms of expected utility theory (vNM, Savage, and Anscombe and Aumann).

The phantom framing of the first part of this experiment is provided by the probability measure $\mathcal{P}$ defined as follows (whose conjugate is written on the right):

$$\begin{aligned} \mathcal{P}(R) &:= \tfrac{1}{3}; & \overline{\mathcal{P}}(R) &:= \tfrac{1}{3};\\ \mathcal{P}(B) &:= \wp\,\tfrac{2}{3}; & \overline{\mathcal{P}}(B) &:= \tfrac{2}{3} - \wp\,\tfrac{2}{3};\\ \mathcal{P}(Y) &:= \tfrac{2}{3} - \wp\,\tfrac{2}{3}, & \overline{\mathcal{P}}(Y) &:= \wp\,\tfrac{2}{3}. \end{aligned}$$

For example, the probability of drawing a yellow ball is $\frac{\#Y}{90}$, which can vary between 0 and $\frac{60}{90}$, and is thus recorded by $\wp\,\frac{2}{3}$ (or equivalently by $\frac{2}{3} - \wp\,\frac{2}{3}$).

Note that $\mathcal{P}$ and $\overline{\mathcal{P}}$ satisfy Axiom 2.2, i.e. the phantom terms of their images sum to 0 while the sum of their real terms is 1. Using $\mathcal{P}$ we can construct the visions $v_{(R)} = (f_{(R)}, \mathcal{P})$ and $v_{(B)} = (f_{(B)}, \mathcal{P})$, whose acts are given by sending R and B, respectively, to \$9, while the other events are sent to \$0.

The second part of the experiment involves two events RY and BY, and their measure is given as follows (the complements are written on the right)

$$\begin{array}{rclcrcl} \mathcal{P}(RY) & := & 1 - \wp\,\frac{2}{3}; & \qquad & \mathcal{P}(B) & := & \wp\,\frac{2}{3}; \\ \mathcal{P}(BY) & := & \frac{2}{3}, & & \mathcal{P}(R) & := & \frac{1}{3}. \end{array}$$

The corresponding acts are clear, i.e. $f_{(RY)}(RY) = f_{(BY)}(BY) = \$9$ and $f_{(RY)}(B) = f_{(BY)}(R) = \$0$. As before, we define the visions $w_{(RY)} = (f_{(RY)}, \mathcal{P})$ and $w_{(BY)} = (f_{(BY)}, \mathcal{P})$. Putting all this together, the preference of most DMs regarding the Ellsberg experiment are formulated in vision language as $v_{(R)} \gtrsim v_{(B)}$ and $w_{(BY)} \gtrsim w_{(RY)}$.

The expected payoffs and variances of these visions are as follows:

	$v_{(R)}$	$v_{(B)}$	$w_{(RY)}$	$w_{(BY)}$
Expectation	3	$\wp\,6$	$9 - \wp\,6$	6
Variance	18	$\wp\,18$	$\wp\,18$	18.

Table 1: **Ellsberg**

For example, the expected payoff from $w_{(RY)}$, is computed as

$$\mathrm{E}\left(w_{(RY)}\right) = 9\left(1 - \wp\,\frac{2}{3}\right) + 0\left(\wp\,\frac{2}{3}\right) = 9 - \wp\,6,$$

while the expected payoff computed with respect to conjugate probabilities is

$$\mathrm{E}\left(\overline{w}_{(RY)}\right) = 9\left(\overline{1 - \wp\,\frac{2}{3}}\right) + 0\left(\overline{\wp\,\frac{2}{3}}\right) = 9\left(\frac{1}{3} + \wp\,\frac{2}{3}\right) + 0\left(\frac{2}{3} - \wp\,\frac{2}{3}\right) = 3 + \wp\,6.$$

In view of Table (1), it can be easily seen that the preferences $v_{(R)} \gtrsim v_{(B)}$ and $w_{(BY)} \gtrsim w_{(RY)}$ imply that the DM prefers certain expected payoffs (i.e. pure real) over vague expected payoffs (i.e. phantom values), even though the variances of the inferior alternatives are "better", since the phantom components are negative and are thus considered as less uncertain. Recall that the uncertainty measure is applicable for ranking visions having identical expectation, while here this not the case.

Claim 9.2. *A pessimist DM prefers $v_{(R)}$ over $v_{(B)}$ and prefers $w_{(BY)}$ over $w_{(RY)}$. That is, under the conditions of Theorem 5.10 the preferences $v_{(R)} \gtrsim v_{(B)}$ and $w_{(BY)} \gtrsim w_{(RY)}$ coincide.* [14]

Proof. Pessimism implies $\left[\mathrm{E}\left(v_{(B)}\right)\right] \gtrsim \mathrm{E}\left(v_{(B)}\right)$, but for the former we have $\left[\mathrm{E}\left(v_{(B)}\right)\right] = \mathrm{E}\left(v_{(R)}\right)$, cf. Table (1). Thus, $\mathrm{E}\left(v_{(R)}\right) \gtrsim \mathrm{E}\left(v_{(B)}\right)$ which implies $v_{(R)} \gtrsim v_{(B)}$. By the same consideration, starting with $\left[\mathrm{E}\left(w_{(RY)}\right)\right] \gtrsim \mathrm{E}\left(w_{(RY)}\right)$, one can see that $w_{(BY)} \gtrsim w_{(RY)}$. Clearly, the independence of Axiom 4.1 is followed. This shows that selections as in Ellsberg's experiment can be explained by pessimism. ☐

[14]The same considerations can be applied to phantom lover with $v_{(R)} \lesssim v_{(B)}$ and $w_{(BY)} \lesssim w_{(RY)}$.

The claim shows that in circumstances such as Ellsberg's example decision making can be modeled in the phantom sense. That is, contrary to the vNM and Savage models, which can not justify such decisions, our phantom model can.

The motivation for such decisions can be attributed to pessimism; the DM is afraid that once she chooses the odds play against her. That is, she believes that she "has no luck". In the numerical example, the computations show that the relative phantom degree of $v_{(R)}$ and $w_{(BY)}$ is 0 while it is 9 for $v_{(B)}$ and $w_{(RY)}$. Next, we suggest a functional representation for such preferences.

Suppose that the DM's utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ exhibits a constant relative risk aversion,

$$\mathrm{u}(x) = \frac{x^{1-\gamma}}{1-\gamma}, \qquad \gamma = \frac{1}{2},$$

and her attitude toward phantoms is formulated by the piecewise linear function:

$$\psi(x = a + \wp b) = \begin{cases} a + \kappa b, & b \geq 0, \\ a + (1-\kappa)b, & b \leq 0, \end{cases} \qquad 0 < \kappa < 1.$$

For the case of phantom (ambiguity) aversion we take the coefficient of phantom aversion to be $\kappa = \frac{1}{4}$, and calculate the choice values of visions:

$$\begin{aligned}
\mathrm{C}(v_{(R)}) &= \psi(\mathrm{E}(v_{(R)})) &&= \psi\left(\tfrac{9^{1-\frac{1}{2}}}{1-\frac{1}{2}}\tfrac{1}{3}\right) &&= \psi(2) &&= 2, \\
\mathrm{C}(v_{(B)}) &= \psi(\mathrm{E}(v_{(B)})) &&= \psi\left(\tfrac{9^{1-\frac{1}{2}}}{1-\frac{1}{2}}\left(\wp\tfrac{2}{3}\right)\right) &&= \psi(\wp 4) &&= 1, \\
\mathrm{C}(w_{(RY)}) &= \psi(\mathrm{E}(w_{(RY)})) &&= \psi\left(\tfrac{9^{1-\frac{1}{2}}}{1-\frac{1}{2}}\left(1-\wp\tfrac{2}{3}\right)\right) &&= \psi(6-\wp 4) &&= 3, \\
\mathrm{C}(w_{(BY)}) &= \psi(\mathrm{E}(w_{(BY)})) &&= \psi\left(\tfrac{9^{1-\frac{1}{2}}}{1-\frac{1}{2}}\tfrac{2}{3}\right) &&= \psi(4) &&= 4.
\end{aligned}$$

Accordingly, $\mathrm{C}(v_{(R)}) > \mathrm{C}(v_{(B)})$ and $\mathrm{C}(w_{(BY)}) > \mathrm{C}(w_{(RY)})$, and thus justify $v_{(R)} \succsim v_{(B)}$ and $w_{(BY)} \succsim w_{(RY)}$.

Notice that in this example, since the likelihood of winning in the second part of the experiment is higher than in the first part, the choice values are respectively higher. Notice also that here risk aversion has no influence on the decision; that is, the visions will be ranked the same even if the DM becomes risk seeking with a negative risk aversion coefficient. However, if the DM becomes a phantom lover (i.e. $\kappa > \frac{1}{2}$), the ranking will be reversed, $\mathrm{C}(v_{(R)}) \leq \mathrm{C}(v_{(B)})$ and $\mathrm{C}(w_{(BY)}) \leq \mathrm{C}(w_{(RY)})$, which coincides with $v_{(R)} \precsim v_{(B)}$ and $w_{(BY)} \precsim w_{(RY)}$. For example, if $\kappa = \frac{3}{4}$, one gets $\mathrm{C}(v_{(R)}) = 2$, $\mathrm{C}(v_{(B)}) = 3$, $\mathrm{C}(w_{(RY)}) = 5$, and $\mathrm{C}(w_{(BY)}) = 4$.

9.2 The insurance dilemma

The nature of the insurance market, with its infrequent events (for instance earthquakes), which means that the estimation of probabilities is not accurate, makes it an important example in which uncertainty plays a major role. Without going too much into detail, we use this example to present the flavor of our model.

Assume a DM whose income level is subject to some uncertainty: when things go well her income will be \$1000 while in the case of disaster she will lose between \$500 and \$700. She estimates the chances of a disaster at 10%, but is unsure about this estimation. An insurance company aims to price the premium δ that the DM would be willing to pay against this loss. [15]

[15] Izhakian and Benninga (2008) analyzed the uncertainty premium under the KMM (2005) model.

The phantom framing of this problem is given by the real number $w = 1000$, which stands for the income, and $d = 500 + \wp 200$, which stands for the potential loss. Letting ϵ denote the misspecification parameter, the ambiguous probability of disaster is given by $\pi_d = 0.1 - \epsilon + \wp 2\epsilon$ while the probability of a peaceful period is $1 - \pi_d = 0.9 + \epsilon - \wp 2\epsilon$. The DM is characterized by a constant relative risk aversion preference, modeled by the utility function $\mathrm{u} : \mathbb{PH} \rightarrow \mathbb{PH}$ of the form

$$\mathrm{u}\left(x\right) = \frac{x^{1-\alpha}}{1-\alpha}, \qquad \alpha \in \mathbb{Q}.$$

Her attitude toward phantoms takes the constant relative form:

$$\psi\left(x = a + \wp b\right) = \left[\frac{x^{1-\lambda} - 1}{1-\lambda}\right], \qquad \lambda \in \mathbb{Q}.$$

The DM will be willing to pay the uncertainty premium δ according to

$$\psi\left(\mathrm{u}\left(w - \delta\right)\right) = \psi\left(\pi_d \mathrm{u}\left(w - d\right) + \left(1 - \pi_d\right) \mathrm{u}\left(w\right)\right).$$

Table 2 presents some comparative statics of phantom (ambiguity) degree, phantom attitude, and risk attitude. Assume first that the probability misspecification is $\epsilon = 0.025$ and the coefficient of phantom aversion is $\kappa = 0.5$. Panel (a) illustrates that just as in classical expected utility theory when risk aversion increases the uncertainty premium that a DM is willing to pay is enlarged. Fixing the risk aversion coefficient $\alpha = 0.5$, panel (b) depicts that increasing phantom aversion results in a higher uncertainty premium. But, the magnitude of these changes is much smaller than the changes derived from increasing risk aversion. Panel (c) demonstrates the effect of phantoms on the uncertainty premium, focusing on increasing probability misspecification for phantom aversion $\kappa = 0.5$ and risk aversion $\alpha = 0.5$. Increasing misspecification has a positive effect on uncertainty indicators: risk ρ, relative phantom ξ, relative uncertainty κ, and absolute phantom Ξ. Therefore, as expected, the uncertainty premium is also increasing.

Risk aversion		**Phantom aversion**		**Phantom degree**					
α	δ	κ	δ	ϵ	ρ	ξ	κ	Ξ	δ
0.0	62.666	0.0	76.961	0.00	33300	10800	35007	16000	73.161
0.1	65.235	0.1	76.991	0.01	34223	13748	36881	32600	74.733
0.2	67.954	0.2	77.023	0.02	35072	16672	38833	49200	76.318
0.3	70.834	0.3	77.053	0.03	35847	19572	40842	65800	77.915
0.4	73.885	0.4	77.085	0.04	36548	22448	42891	82400	79.527
0.5	77.115	0.5	77.115	0.05	37175	25300	44967	101000	81.151
0.6	80.537	0.6	77.146	0.06	37728	28128	47059	124400	82.788
0.7	84.160	0.7	77.177	0.07	38207	30932	49158	147800	84.438
0.8	87.997	0.8	77.208	0.08	38612	33712	51258	171200	86.102
0.9	92.062	0.9	77.239	0.09	38943	36468	53352	194600	87.778
(a)		**(b)**		**(c)**					

Table 2: **Uncertainty premium**

10 Conclusions

The known results of past efforts at modeling decision making are two main disciplines: expected utility and subjective expected utility. The critics of these models mainly attack their ability to model decision making in real life. This criticism has been supported by behavioral experiments that have shown systematic violations of theories that are based on vNM or Savage results. The attempts to provide a suitable representation to contend with these problems started even earlier with the work of Knight (1921). Most of these attempts were centered around the vNM context and included modification of their underlying assumptions, replacement of functional representations, or use of a different probability measure (like a non-additive measure, or a set of priors).

This paper presents a new model of decision making that is based on phantom probability theory as introduced by Izhakian and Izhakian (2009). A significant key feature of phantom probability theory, which generalizes classical probability, is the necessity of preserving the known attributes of classical probability theory. This setting allows the formulation of a decision making model for situations that are much closer to reality and include vague consequences accompanied by ambiguous probabilities. The model addresses two main issues: modeling subjective variation of objective probabilities, yet preserving the classical principles of probability theory, and providing a solid framework encoding not only risk effects but also phantom (ambiguity) effects.

Our axiomatization generalizes the vNM axiomatization to phantoms, and allows for three sources of uncertainty: not-knowing which of the events will be realized, ambiguous estimation of odds of events occurring, and a-priori unclear outcomes of possible events. Our model distinguishes between phantom degree, derived from the available information and subjective beliefs of the DM, and phantom attitude, drawn from the DM's preferences. We introduce a consolidated uncertainty measure and refine it to risk measure and phantom measure. We show that decision making under ambiguity is a special case of our model in which probabilities are vague but outcomes of events are clearly forecasted, yielding a separation between ambiguity degree and ambiguity attitude. The applicability of our model is demonstrated for the Ellsberg paradox, showing that unlike expected utility models here it coincides with the independence axiom. A second example shows that the insurance premium is affected by phantoms.

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A Appendix

A.1 Phantom spaces: special attributes

The special properties of the phantom framework are explored in this section, supported by new definitions. Recalling that the division of a non-negative real number by a greater positive number lies in the interval $[1, 0]$, we present a phantom analog, given with respect to Λ:

Claim A.1. *Suppose $z_2 \gg z_1$ are respectively pseudo positive and pseudo non-negative, then $\frac{z_1}{z_2} \in \Lambda$.*

Proof. Write the fraction $\frac{z_1}{z_2} = \frac{a_1}{a_2} + \wp \left(\frac{\hat{z}_1}{\hat{z}_2} - \frac{a_1}{a_2} \right)$, then $0 \leq \frac{a_1}{a_2} \leq 1$ and $0 \leq \frac{\hat{z}_1}{\hat{z}_2} \leq 1$. Since $\frac{\hat{z}_1}{\hat{z}_2} = \widehat{(\frac{z_1}{z_2})}$, then $-\frac{a_1}{a_2} \leq \widehat{(\frac{z_1}{z_2})} - \frac{a_1}{a_2} \leq 1 - \frac{a_1}{a_2}$, implying that $\frac{z_1}{z_2} \in \Lambda$. □

Later, when studying phantom affinity and convexity, we also need the following:

Claim A.2. *Let $z_1, z_2, z_3 \in \mathbb{PH}$, then there exists $\lambda \in \mathbb{PH}$ such that $z_2 = \lambda z_1 + (1 - \lambda) z_3$. If $z_1 \widehat{\neq} z_3$ and at least one of them is a nonzero divisor then λ is unique.*

Proof. Let $z_i = a_i + \wp b_i$, $i = 1, 2, 3$, and $\lambda = \alpha + \wp \beta$. Expanding the product $\lambda z_1 + (1 - \lambda) z_3$, we have $\alpha a_1 + (1 - \alpha) a_3 + \wp (\alpha (b_1 - b_3) + \beta (a_1 + b_1 - a_3 - b_3) + b_3)$. This shows that each term is linear in α and β, and there exists λ satisfying the requirement. By the same token, when $z_1 \widehat{\neq} z_3$, i.e. $a_1 \neq a_3$, then α is unique, and if z_1 or z_2 is a nonzero divisor then β is also unique. □

The definitions below parallel the known definitions over the real numbers.

Definition A.3. *A phantom-valued function $f : \mathbb{PH} \to \mathbb{PH}$ is **affine** if it is of the form*

$$f(\alpha_1 z_1, \alpha_2 z_2, \ldots, \alpha_n z_n) = \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_n z_n + \beta,$$

for some $\alpha_1, \ldots, \alpha_n, \beta \in \mathbb{PH}$.

The class of affine functions has subclasses of functions $f : \mathbb{PH} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. (Continuity and differentiability of phantom functions are discussed in Section 1.7 of I&I (2009))

Lemma A.4. ***(Phantom affinity)** If the function $f : \mathbb{PH} \to \mathbb{PH}$ is affine, then for any $\lambda \in \Lambda$*

$$f(\lambda z_1 + (1 - \lambda) z_2) = \lambda f(z_1) + (1 - \lambda) f(z_2).$$

Proof. By affinity, $f(\lambda z_1 + (1 - \lambda) z_2) = \lambda (\alpha z_1 + \beta) + (1 - \lambda)(\alpha z_2 + \beta) = \lambda f(z_1) + (1 - \lambda) f(z_2)$. □

Lemma A.5. *Let $f : \mathbb{PH} \to \mathbb{R}$ be a continuous function. For any real $\alpha \in [0, 1]$ there exists $\lambda \in \bar{\Lambda}$ such that*

$$\alpha f(x) + (1 - \alpha) f(y) = f(\lambda x + (1 - \lambda) y). \tag{3}$$

Conversely, when f is monotonic, for any $\lambda \in \bar{\Lambda}$, there exists a real $\alpha \in [0, 1]$ for which (3) is satisfied.

Proof. Viewing f as a function $f : \mathbb{R}^2 \to \mathbb{R}$, the proof is obtained by continuity of functions over $\mathbb{R}^2$. □

In order to compare phantom numbers one needs to equip $\mathbb{PH}$ with a weak order $\succsim_{\text{wk}}$, usually assumed to be given with the phantom structure. In our case, to be seen later, this order should correspond to the DM's preferences, which, as the representation theorem in Section 5.2 proves, has an order-preserving function.

Assumption A.6. *The order $\succsim_{\text{wk}}$ is taken to be compatible with the familiar order of the reals; that is, if $x \succsim_{\text{wk}} y$ for $x, y \in \mathbb{R}$, then $x \geq y$.*

Definition A.7. *A function $f : \mathbb{PH} \to \mathbb{PH}$ is:*

*(i) **Monotonic** (also monotonically increasing, or non-decreasing) if $f(z_1) \succsim_{\text{wk}} f(z_2)$ for all $z_1 \succsim_{\text{wk}} z_2$, and **monotonically decreasing** (also decreasing, or non-increasing) if $f(z_1) \precsim_{\text{wk}} f(z_2)$;*

(ii) ***Strictly increasing*** *if $f(z_1) \succ_{\mathrm{wk}} f(z_2)$ for all $z_1 \succ_{\mathrm{wk}} z_2$, and* ***strictly decreasing*** *if $f(z_1) \prec_{\mathrm{wk}} f(z_2)$.*

In the special case when $f : \mathbb{PH} \to \mathbb{R}$, the inequality between evaluation of functions is the standard inequality on $\mathbb{R}$. This leads us to the convexity of the phantom function:

Definition A.8. *A continuous function $f : \mathbb{PH} \to \mathbb{PH}$ is* ***convex*** *on a set $X \subseteq \mathbb{PH}$ if for any two (nonzero divisor) points $z_1, z_2 \in X$ and any $\lambda \in \Lambda$*

$$f(\lambda z_1 + (1-\lambda) z_2) \precsim_{\mathrm{wk}} \lambda f(z_1) + (1-\lambda) f(z_2).$$

f is said to be ***concave*** *on a set X if $f(\lambda z_1 + (1-\lambda) z_2) \succsim_{\mathrm{wk}} \lambda f(z_1) + (1-\lambda) f(z_2)$.*

In the restricted case, when the target of f is $\mathbb{R}$, the weight λ is taken in the real interval $[0,1]$. Having the phantom convexity, a version of Jensen's inequality is obtained for functions having finite supports.

Lemma A.9. *Let $f : \mathbb{PH} \to \mathbb{PH}$ be a continuous convex function on $X \subseteq \mathbb{PH}$ and let $\lambda_i \in \Lambda$, $i = 1, 2, \dots, n$, be phantom weights such that $\sum \lambda_i = 1$. Then,*

$$f\left(\sum \lambda_i z_i\right) \precsim_{\mathrm{wk}} \sum \lambda_i f(z_i),$$

assuming z_i is a nonzero divisor for all i. In the case when $f : \mathbb{PH} \to \mathbb{R}$ and $\lambda_i \in [0,1]$ are real weights, $f(\sum \lambda_i z_i)) \leq \sum \lambda_i f(z_i)$, assuming each z_i is a nonzero divisor.

Proof. Assuming f has a finite support, we prove the lemma by induction on n. When $n = 2$, the convexity of f implies $f(\lambda_1 z_1 + \lambda_2 z_2) \precsim_{\mathrm{wk}} \lambda_1 f(z_1) + \lambda_2 f(z_2)$.

For $n+1$, write $\sum_{i=1}^{n+1} \lambda_i = \mu + \lambda_{n+1}$. Then $\sum_{i=1}^{n+1} \lambda_i z_i = \mu(\sum_{i=1}^{n} \frac{\lambda_i}{\mu} z_i) + \lambda_{n+1} z_{n+1}$, and by convexity

$$f\left(\lambda_{n+1} z_{n+1} + \mu \sum_{i=1}^{n} \frac{\lambda_i}{\mu} z_i\right) \precsim_{\mathrm{wk}} \lambda_{n+1} f(z_{n+1}) + \mu f\left(\sum_{i=1}^{n} \frac{\lambda_i}{\mu} z_i\right).$$

The induction step shows that the left side is $\lambda_{n+1} f(z_{n+1}) + \mu \sum_{i=1}^{n} \frac{\lambda_i}{\mu} f(z_i) = \sum_{i=1}^{n+1} \lambda_i f(z_i)$. □

A.2 Preferences: properties

This section studies additional properties of preferences $\succsim$; cf. Axiom 4.1, on visions in $\mathcal{V}_\Sigma$ denoted by v, w, u.

Lemma A.10. ***(Intermediate)*** *If $v \succ w$, and $\alpha \in \Lambda$ then $v \succsim \alpha v + (1-\alpha) w \succsim w$. When $\alpha \neq 0$, the relation is a strict relation.*

Proof. Since $v \succ w$, the independence condition implies that $v = \alpha v + (1-\alpha) v \succsim \alpha v + (1-\alpha) w \succsim \alpha w + (1-\alpha) w = w$. □

Lemma A.11. ***(Mixture monotonicity)*** *Suppose that $v \succ w$ with $\alpha, \beta \in [0,1]$, then*

$$\beta v + (1-\beta) w \succ \alpha v + (1-\alpha) w \iff 1 > \beta > \alpha > 0.$$

Proof. ($\Leftarrow$) Assume that $\beta > \alpha$. Define $\gamma \in [0,1]$ to be $\gamma = \frac{\beta-\alpha}{1-\alpha}$ and write $\beta v + (1-\beta) w = \gamma v + (1-\gamma)(\alpha v + (1-\alpha) w)$. Lemma A.10 implies that $\gamma v + (1-\gamma)(\alpha v + (1-\alpha) w) \succ \alpha v + (1-\alpha) w$, since $v \succ w$, for any $\alpha \in [0,1] \subseteq \Lambda$. So we conclude that $\beta v + (1-\beta) w \succ \alpha v + (1-\alpha) w$.

($\Rightarrow$) Suppose that $\beta \leq \alpha$ and $\beta v + (1-\beta) w \succ \alpha v + (1-\alpha) w$. Interchanging α and β in the previous argument, one has $\alpha v + (1-\alpha) w \succsim \beta v + (1-\beta) w$ – a contradiction. □

Lemma A.12. ***(Unique solvability)*** *For $v \succsim w \succsim u$ there exists a unique $\alpha^* \in [0,1]$ such that $w \backsim \alpha^* v + (1-\alpha^*) u$.*

Proof. When $v \backsim w$ (reps. $w \backsim u$), take $\alpha^* = 1$ (reps. $\alpha^* = 0$). Otherwise, consider the sets

$$\mathcal{H}_A = \left\{\alpha \in [0,1] \;\middle|\; w \succ \alpha v + (1-\alpha) u\right\} \quad \text{and} \quad \mathcal{H}_B = \left\{\alpha \in [0,1] \;\middle|\; \alpha v + (1-\alpha) u \succ w\right\}.$$

These are disjoint nonempty open sets (due to the continuity axiom), which do not cover $[0,1]$. Thus there exists $\alpha^* \in [0,1]$ for which $w \backsim \alpha^* v + (1-\alpha^*) u$, which is unique by mixture monotonicity (Lemma A.11). □

A.3 Proofs

Proof of Theorem 5.7.
Proposition 5.6 shows that the set of consequences $\mathcal{X} \subset \mathbb{PH}$ is equipped with an order induced by that of $\mathcal{V}_\Sigma$. Assumption 3.3 together with the Archimedean feature (Axiom 4.1) provides the convexity of $\mathcal{X}$, implying that $\mathcal{X}$ is connected. Since $\mathcal{X}$ is a subset of the topological space $\mathbb{PH}$, realized as the Cartesian product $\mathbb{R} \times \mathbb{R}$ and the rational numbers are dense in $\mathbb{R}$, then $\mathcal{X}$ contains a countable dense subset; that is $\mathcal{X}$ is a separable. Thus, by Debreu's (1954) Theorem I [16] , there is a function $\psi : \mathcal{X} \to \mathbb{R}$ that preserves $\succsim$. Since $\succsim$ assumes coinciding with the order of the reals, any strictly increasing transformation of the value function ψ preserves $\succsim$. [17] $\square$

Proof of Theorem 5.10 (Representation Theorem).
Recall that Σ is assumed to be finite and $\mathrm{C} : \mathcal{V}_\Sigma \to \mathbb{R}$, is given by $\mathrm{C} : (f, \mathcal{P}) \mapsto \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)\right)$, i.e. it is a composition of the utility function $\mathrm{u} : \mathcal{X} \to \mathbb{PH}$ and the value function $\psi : \mathbb{PH} \to \mathbb{R}$.

($\Longrightarrow$) The proof is done in two steps. First we prove the assertion for a closed bounded (by a worst and a best vision) subset of visions $\mathcal{V}_\Sigma$ and then extend this result for the whole $\mathcal{V}_\Sigma$. [18] We are assisted mainly by the lemmas of Section 4.

I) Assume v^* and v_*, with $v^* \succsim v_*$, are a best vision and a worst vision, respectively. Note that if $v^* \precsim v_*$, then $v \backsim w$ for all $v, w \in \mathcal{V}_\Sigma$ and we accomplish the proof by taking a constant value function. Let $(v^*)^{\mathrm{c}}$ and $(v_*)^{\mathrm{c}}$ be constant visions, identified with the consequences $x^* \in \mathcal{X}$ and $x_* \in \mathcal{X}$, equivalent to v^* and v_* respectively, cf. Assumption 3.3.

We first show, inductively, that each vision $v \in \mathcal{V}_\Sigma$ (not necessarily constant) is equivalent to a compound vision constructed by x^* and x_*. Suppose $v : \mathcal{E}_i \mapsto (x_i, \mathfrak{p}_i)$, $i = 1, \dots, n$, then for the sequence of the constant visions ${v_i}^{\mathrm{c}} = (\{x_i\}, \mathcal{P})$, we have

$$v \backsim \mathfrak{p}_1 {v_1}^{\mathrm{c}} + \mathfrak{p}_2 {v_2}^{\mathrm{c}} + \dots + \mathfrak{p}_n {v_n}^{\mathrm{c}}, \qquad \mathfrak{p}_i \in \Lambda, \quad \sum_i \mathfrak{p}_i = 1.$$

In its turn, by the Archimedean feature (Axiom 4.1), ${v_i}^{\mathrm{c}} \backsim \alpha_i x^* + (1 - \alpha_i) x_*$ for some $\alpha_i \in \Lambda$. Therefore

$$v \backsim \sum_i \mathfrak{p}_i(\alpha_i x^* + (1 - \alpha_i)x_*) = x^* \sum_i \mathfrak{p}_i \alpha_i + x_*(1 - \sum_i \mathfrak{p}_i \alpha_i).$$

Recalling that $x_i = f(\mathcal{E}_i)$, we set $\alpha_i = \mathrm{u}\left(f(\mathcal{E}_i)\right)$, a phantom value, and thus

$$v \backsim x^*(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)) + x_*(1 - \sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)),$$

i.e. $\mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right) \in \Lambda$. On the other hand, by Lemma A.12 (unique solvability), there is a unique (real) $\alpha^* \in [0, 1]$ such that

$$v \backsim \alpha^* x^* + (1 - \alpha^*)\, x_*.$$

and by Theorem 5.7 we have $\psi\left(\alpha^*\right) = \alpha^* = \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)\right) \in [0, 1]$. Hence, the converse direction of Theorem 5.7 implies

$$v \backsim \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)\right) x^* + \left(1 - \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)\right)\right) x_*. \tag{4}$$

To complete this part of the proof, suppose $v \succsim w$ are two visions in $\mathcal{V}_\Sigma$, where $v = (f, \mathcal{P})$ and $w = (g, \mathcal{Q})$. By the considerations developed above, for w we have (see Equation (4))

$$w \backsim \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(g(\mathcal{E})\right)\mathcal{Q}\left(\mathcal{E}\right)\right) x^* + \left(1 - \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(g(\mathcal{E})\right)\mathcal{Q}\left(\mathcal{E}\right)\right)\right) x_*.$$

[16] Debreu's Theorem: If $\succsim$ is a complete order over a connected, separable topological space X such that for some z, $\left\{x \in X \mid x \succsim z\right\}$ and $\left\{y \in X \mid z \succsim y\right\}$ are both closed for all x in X, then there exists a continuous real-valued function ψ such that $x \succsim y \iff \psi\left(x\right) \geq \psi\left(y\right)$.

[17] Theorem 3.1 at Fishburn (1970) suggests an alternative proof.

[18] Our proof follows the steps used in Mas-Colell, Whinston, and Green (1995) and others, in proving classical expected utility theory.

Then Lemma A.11 (mixture monotonicity) completes the proof of this part:

$$v \succsim w \iff \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(f(\mathcal{E})\right)\mathcal{P}\left(\mathcal{E}\right)\right) \geq \psi\left(\sum_{\mathcal{E}\in\Sigma} \mathrm{u}\left(g(\mathcal{E})\right)\mathcal{Q}\left(\mathcal{E}\right)\right).$$

II) Next we extend the result of part I for the case when there is no best or worst vision. If there is no best or worst vision, we take an arbitrary pair of constant visions $v_0 \succ w_0$ in $\mathcal{V}_\Sigma$ and define the set of constant visions $\mathcal{T}_0 = \left\{u \in \mathcal{V}_\Sigma \middle| v_0 \succsim u \succsim w_0\right\}$. If $v_0 \backsim w_0$, for all v_0, w_0 in $\mathcal{V}_\Sigma$ then the conclusion holds trivially. Otherwise, following the reasoning above, we can obtain a functional representation C_0 for all the visions in $\mathcal{T}_0$ (preference-wise).

Next, consider visions $v_1 \succ v_0$ and $w_0 \succ w_1$, i.e. they are outside $\mathcal{T}_0$, and as before define the set of visions $\mathcal{T}_1 = \left\{u \in \mathcal{V}_\Sigma \middle| v_1 \succsim u \succsim w_1\right\}$. Clearly $\mathcal{T}_0 \subset \mathcal{T}_1$, and therefore the choice function C_0 on $\mathcal{T}_0$ agrees with the choice function represents the preferences on $\mathcal{T}_1$, that is $\mathrm{C}_0(u) = \mathrm{C}_1(u)$ for every $u \in \mathcal{T}_1 \cap \mathcal{T}_0$. Proceeding inductively, we obtain a sequence $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots \subset \mathcal{V}_\Sigma$ of sets of visions, each having a preference representative C_i with $\mathrm{C}_i(u) = \mathrm{C}_{i+1}(u) = \mathrm{C}_{i+2}(u) = \ldots$ for all $u \in \mathcal{T}_i$, and thereby construct a choice function C whose domain is $\mathcal{V}_\Sigma$.

($\Longleftarrow$) Assuming $\mathrm{C} : \mathcal{V}_\Sigma \to \mathbb{R}$ is a choice function representing $\succsim$ we need to prove that Axiom 4.1 is satisfied. As before, since each vision assumes having an equivalence constant vision (Assumption 3.3), we may consider only constant visions which are identified with consequences $x \in \mathcal{X}$. Therefore, C can be viewed as a function sending $\mathcal{X} \subset \mathbb{PH}$ to $\mathbb{R}$, and is thus assumed continuous. ***Completeness*** and ***Transitivity*** are immediate by Remark 5.9 and the fact that $\mathcal{V}_\Sigma$ is contained in the domain of C.

Archimedean: By representation $v \succsim w \succsim u$ implies $\mathrm{C}(v) \geq \mathrm{C}(w) \geq \mathrm{C}(u)$, which are reals, and thus there is a $\lambda \in (0,1)$ for which $\lambda\mathrm{C}(v) + (1-\lambda)\mathrm{C}(u) \geq \mathrm{C}(w)$. By Lemma A.5, there is $\alpha \in \Lambda$ such that $\lambda\mathrm{C}(v) + (1-\lambda)\mathrm{C}(u) = \mathrm{C}(\alpha v + (1-\alpha)u)$, hence $\mathrm{C}(\alpha v + (1-\alpha)u) \geq \mathrm{C}(w)$, which by representation (converse direction) yields $\alpha v + (1-\alpha)u \succsim w$. The same argument shows that $w \succsim \beta v + (1-\beta)u$ for some $\beta \in \Lambda$.

Independence is then obtained by the Archimedean axiom, the continuity of C and its monotonicity. □

Proof of Theorem 6.1.
We prove (a); the other two cases are proven by similar considerations. $(i) \Longrightarrow (ii)$. Assume that $\succsim$ exhibits risk aversion represented by u, that is $\mathrm{E}_{\mathcal{P}}(v) = \sum_{\mathcal{E}_i\in\Sigma} f(\mathcal{E}_i)\mathcal{P}(\mathcal{E}_i) \succsim v$. The latter can be written as a compound vision $\oplus_i \alpha_i v_i$ of constant visions (cf. Remark 3.2), $v_1, \ldots, v_k$, each identified with a constant consequence $x_i \in \mathcal{X}$. By Theorem 5.10, $\mathrm{u}(\mathrm{E}_{\mathcal{P}}(v)) \geq_\psi \sum_{i=1}^{n} \alpha_i \mathrm{u}(v_i)$, where $\alpha_i = \mathcal{P}(\mathcal{E}_i)$, and thus, by Lemma A.9, u is concave . $(ii) \Longrightarrow (iii)$: Lemma A.9. $(iii) \Longrightarrow (i)$: If u is concave then, by Lemma A.9, $\mathrm{u}(\mathrm{E}_{\mathcal{P}}(f)) \geq_\psi \mathrm{E}_{\mathcal{P}}(\mathrm{u}(f))$ thus $\mathrm{E}(v) \succsim v$. □

Proof of Theorem 7.13.
Recall that since DMs are assumed to have an identical probability measures and identical risk attitudes, we can use the language of consequence in which a vision $v \in \mathcal{V}_\Sigma$ is represented by its expected utility $x \in \mathbb{PH}$; in particular a real vision ℓ is represented by $r \in \mathbb{R}$. In parts of the proof we use this notation.
$(ii) \Longrightarrow (i)$: If $v \succsim_A \ell$ then $\mathrm{rce}_A(v) \succsim_A \mathrm{rce}_A(\ell)$, that is $\mathrm{rce}_A(x) \geq \mathrm{rce}_A(\ell)$, since $\succsim_A$ is compatible with the order of $\mathbb{R}$. Then, (ii) implies that $\mathrm{rce}_B(v) \geq \mathrm{rce}_A(\ell)$. Note that $\mathrm{rce}_A(\ell) = \mathrm{rce}_B(\ell)$; thus, by Theorem 5.7 applied to rce, $\mathrm{rce}_B(v) \succsim_B \mathrm{rce}_A(\ell)$ by transitivity $v \succsim_B \ell$.
$(iii) \Longrightarrow (ii)$: Recall that $\psi : \mathbb{PH} \to \mathbb{R}$ is assumed to be monotonically increasing with respect to an order that coincides with the order on $\mathbb{R}$. Therefore, ψ^{-1}, restricted to the real subdomain $\mathbb{R}$, is well defined. Denote this restriction as $\psi|_\mathbb{R}$

By concavity, $\frac{\psi_B(x)+\psi_B(\overline{x})}{2} = \psi_B(\mathrm{rce}_B(x))$, i.e. $\mathrm{rce}_B(x) = \psi_B^{-1}\left(\frac{\psi_B(x)+\psi_B(\overline{x})}{2}\right)$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined to be $g = \psi_A|_\mathbb{R} \circ \psi_B^{-1}|_\mathbb{R}$, which is concave by (iii). Since g is concave, by the Jensen inequality we have

$$\begin{aligned} \psi_A(\mathrm{rce}_B(x)) &= \psi_A\left(\psi_B^{-1}\left(\frac{\psi_B(x)+\psi_B(\overline{x})}{2}\right)\right) = \\ g\left(\frac{\psi_B(x)+\psi_B(\overline{x})}{2}\right) &\geq \frac{g(\psi_B(x)) + g(\psi_B(\overline{x}))}{2} = \frac{\psi_A(x)+\psi_A(\overline{x})}{2} = \psi_A(\mathrm{rce}_A(x)). \end{aligned}$$

The monotonicity of ψ_A implies that $\text{rce}_B \geq \text{rce}_A$.
$(i) \implies (iii)$: Suppose $x, y, z \in \mathbb{PH}$, such that $\psi_B(x) < \psi_B(y) < \psi_B(z)$. Let $\lambda \in (0,1)$ satisfy $\psi_B(y) = \lambda\psi_B(x) + (1-\lambda)\psi_B(z)$. We have to show that $\psi_A(y) \geq \lambda\psi_A(x) + (1-\lambda)\psi_A(z)$.

Assume that $\psi_A(y) < \lambda\psi_A(x) + (1-\lambda)\psi_A(z)$. Then, by monotonicity and continuity, there is $y' \succ y$ close enough to y for which $y' \prec_A \lambda x \oplus (1-\lambda)z$ and $y' \succ_B \lambda x \oplus (1-\lambda)z$ – a contradiction to (i). Therefore $y' \succsim_A \lambda x \oplus (1-\lambda)z$ and, by Theorem 5.10, $\psi_A(y) \geq \lambda\psi_A(x) + (1-\lambda)\psi_A(z)$. Then for $g = \psi_A|_{\mathbb{R}} \circ \psi_B^{-1}|_{\mathbb{R}}$ it follows that $g(\psi_B(y)) \geq \lambda g(\psi_B(x)) + (1-\lambda)g(\psi_B(z))$; that is, g is concave. □

Proof of Corollary 7.15.
Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\psi_A = g \circ \psi_B$. Thus $\partial_\wp \psi_A = g'(\psi_B)\partial_\wp \psi_B$ and $\partial_{\wp\wp}\psi_A = g''(\psi_B)(\partial_\wp \psi_B)^2 + g'(\psi_B)\partial_{\wp\wp}\psi_B$. Since ψ_A and ψ_B are strictly increasing then $g'(\psi_B) = \frac{\partial_\wp \psi_A}{\partial_\wp \psi_B} > 0$. Putting all this together:

$$\vartheta_A = -\frac{\partial_{\wp\wp}\psi_A}{\partial_\wp \psi_A} = -\frac{g''(\psi_B)}{g'(\psi_B)}\partial_\wp \psi_B - \frac{\partial_{\wp\wp}\psi_B}{\partial_\wp \psi_B} = -\frac{g''(\psi_B)}{g'(\psi_B)}\partial_\wp \psi_B + \vartheta_B$$

Therefore, $\vartheta_A(x) \geq \vartheta_B(x)$ iff $g''(\psi_B)/g'(\psi_B) < 0$, i.e. g is concave. The proof is then completed by Theorem 7.13. □

Proof of Proposition 7.20.
We prove the case of pessimism; the cases of optimism and apathy are proven similarly. Take $x = a + b\wp \in \mathbb{PH}$ with $[x] = a + \frac{1}{2}b \in \mathbb{R}$. Then by Theorem 5.10, $[x] \succ x$ iff $\psi\left(a + \frac{1}{2}b\right) > \psi(a + b\wp)$. Since ψ is assumed differentiable, then

$$\psi\left(a + \frac{1}{2}b\right) \backsimeq \psi(a) + \partial_r \psi(a)\frac{1}{2}b > \psi(a) + \partial_\wp \psi(a)b \backsimeq \psi(a + b\wp).$$

If $b > 0$ then $\theta = \frac{\partial_r \psi(\cdot)}{\partial_\wp \psi(\cdot)} > 2$ and if $b < 0$ then $\theta = \frac{\partial_r \psi(\cdot)}{\partial_\wp \psi(\cdot)} < 2$ □

Proof of Theorem 8.3.
$(\Longrightarrow)$ Assume $v = (f, \mathcal{P})$ is more uncertain than $w = (g, \mathcal{Q})$ and let $x = f - \mathrm{E}(v)$, $y = g - \mathrm{E}(w)$. As random variables, by Definition 8.1, $x =_d y + \epsilon$ for some vision $u = (f_u, \mathcal{P}_u)$ in $\mathcal{V}_\Omega$ with $\mathrm{E}(u) = 0$. Considering a DM's preference $\succsim$, characterized by the utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ and the value function $\psi : \mathbb{PH} \to \mathbb{R}$, we have

$$\mathrm{E}_{\mathcal{P}}(\mathrm{u}(x)) =_\psi \mathrm{E}_{\mathcal{Q}}(\mathrm{E}_{\mathcal{P}_u}(\mathrm{u}(y+\epsilon))).$$

The risk aversion results in concavity of u. Thus by Lemma A.9:

$$\mathrm{E}_{\mathcal{P}_u}(\mathrm{u}(y+\epsilon)) \leq_\psi \mathrm{u}(\mathrm{E}_{\mathcal{P}_u}(y+\epsilon)) = \mathrm{u}(y).$$

Taking expectations gives $\mathrm{E}(\mathrm{u}(x)) \leq_\psi \mathrm{E}(\mathrm{u}(y))$. Applying the same arguments to these results, by using the value function ψ, gives $\psi(\mathrm{E}(\mathrm{u}(x))) < \psi(\mathrm{E}(\mathrm{u}(y)))$. Hence $v \prec w$.

$(\Longleftarrow)$ By contradiction assume that $v \succsim w$ but v is more uncertain than w, i.e $x =_d y + \epsilon$. But by the arguments of the first part of the proof, if the DM is uncertainty averse then $\mathrm{E}(\mathrm{u}(x)) \leq_\psi \mathrm{E}(\mathrm{u}(y))$ and therefore $v \precsim w$ – a contraindication. □

Proof of Proposition 8.4.
Write $f_v + \lambda_2 f_u = \mu(f_v + \lambda_1 f_u) + (1-\mu)f_v$, where $\mu = \frac{\lambda_2}{\lambda_1}$, which by Claim A.1 is in $\Lambda^\times$. Then

$$\mathrm{u}(f_v + \lambda_2 f_u) >_\psi \mu\mathrm{u}(f_v + \lambda_1 f_u) + (1-\mu)\mathrm{u}(f_v), \tag{5}$$

since the utility function $\mathrm{u} : \mathbb{PH} \to \mathbb{PH}$ is strictly concave, cf. Definition A.8. Taking expectations we have $\mathrm{E}(\mathrm{u}(f_v + \lambda_2 f_u)) >_\psi \mu\mathrm{E}(\mathrm{u}(f_v + \lambda_1 f_u)) + (1-\mu)\mathrm{E}(\mathrm{u}(f_v))$. Since the vision whose act is $f_v + \lambda_1 f_u$ is strictly more uncertain than v, we have $\mathrm{E}(\mathrm{u}(f_v)) >_\psi \mathrm{E}(\mathrm{u}(f_v + \lambda_1 f_u))$, which together with (5) yields $\mathrm{E}(\mathrm{u}(f_v + \lambda_2 f_u)) >_\psi \mathrm{E}(\mathrm{u}(f_v + \lambda_1 f_u))$. Then, Theorem 8.3 asserts that $f_v + \lambda_1 f_u$ is strictly more uncertain than $f_v + \lambda_2 f_u$. □

Proof of Theorem 8.7.
Recall that phantom variances , as well as their fraction, are pseudo positives (see Lemma 1.10 in I&I (2009)).
($\Longrightarrow$) Suppose v is strictly more uncertain than w. Then $f_v - \mathrm{E}(v) =_d f_w - \mathrm{E}(w) + f_u$, for some vision u with $\mathrm{E}(u) = 0$. Thus $f_v =_d f_w + f_u$ (since $\mathrm{E}(v) = \mathrm{E}(w)$). But, f_v and f_w are independent, so $\sigma_v^2 = \sigma_w^2 + \sigma_u^2$, yielding $\sigma_v^2 \gg \sigma_w^2$ since σ_u^2 is pseudo-positive.

($\Longleftarrow$) The vision w' having the act $f_{w'} = \lambda(f_w - \mathrm{E}(w))$ is symmetric with mean 0 and variance $\lambda^2\sigma_w^2$. Let $\lambda = \frac{\sigma_v^2}{\sigma_w^2}$, which is pseudo positive, and $\lambda \gg 1$ by assumption. Clearly $\sigma_v^2 = \lambda\sigma_w^2$, therefore $\lambda(f_w - \mathrm{E}(w))$ has a distribution identical to $(f_v - \mathrm{E}(v))$. By Corollary 8.5, $\lambda(f_w - \mathrm{E}(w))$ is strictly more uncertain than $(f_w - \mathrm{E}(w))$, and hence $(f_v - \mathrm{E}(v))$ is strictly more uncertain than $(f_w - \mathrm{E}(w))$. Since expectations play no role, v is strictly more uncertain than w. □