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Infinitely Repeated Games with Public Monitoring and Monetary Transfers

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Abstract

In this paper, we study infinitely repeated games with imperfect public monitoring and the possibility of monetary transfers. We develop an efficient algorithm to compute the set of pure strategy public perfect equilibrium payoffs for each discount factor. We also show how all equilibrium payoffs can be implemented with a simple class of stationary equilibria that use stick-and-carrot punishments.

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1 Introduction

The theory of infinitely repeated games is used to address a wide range of topics in economics and social sciences, like employment relations, international agreements, or cartels. Results that help to find equilibria in these games and numerical procedures to quickly calculate examples are therefore of great importance. Although several theoretical breakthroughs on how to compute equilibrium value sets have been made, so far no exact algorithm exists to generally characterize the set of sequential equilibrium payoffs in infinitely repeated games, even if attention is restricted to public monitoring and pure strategies. In this paper, we present an algorithm to exactly compute the set of pure strategy equilibrium payoffs for arbitrary discount factors in infinitely repeated games with monetary transfers and imperfect public monitoring.[1]

Developing methods to compute the set of equilibrium payoffs for general stage games and arbitrary discount factors has been the focus of a small literature including Abreu, Pearce and Stacchetti (1990, henceforth APS), Judd, Yeltekin and Conklin (2003, henceforth JYC) as well as Cronshaw and Luenberger (1994) for strongly symmetric equilibria. APS develop a conceptual algorithm to compute the payoff sets for repeated games with imperfect monitoring and arbitrary discount factors. They show that the set of perfect public equilibrium payoffs is a fixed point of a monotone operator applied on candidates for the sets of equilibrium payoffs. One can iteratively apply this operator to compute the payoff set. In each iteration, one has to solve a series of static problems with enforceable continuation payoffs taken from the current candidate set of equilibrium payoffs. Yet, as JYC point out, the general method of APS is not directly implementable on a computer because it requires approximation of arbitrary sets.

JYC analyze the special case of perfect monitoring. In addition, they augment the stage game by a public randomization device, which allows to restrict attention to convex sets of continuation payoffs. They develop a method to compute upper and lower approximations for the set of pure strategy subgame perfect equilibrium payoffs and to construct strategy profiles that can support payoffs from the lower approximation. The method of JYC is still limited in so far that finding fine

[1] A software package, programmed in R, that implements the algorithms is available on the second author's website http://www.wiwi.uni-bonn.de/kranz/software.htm

For a description of the software and several examples, see Kranz (2010).

approximations for the equilibrium payoff sets for several discount factors remains computationally expensive, it is restricted to games with perfect monitoring, and does not provide much guidance for finding analytical closed-form solutions.

In the present paper, we allow for actions that can only be imperfectly monitored, but make the assumption that observable monetary transfers can be conducted. Our analysis is therefore only applicable to those economic environments in which monetary transfers are plausible, which is true for many interactions. Repeated games with monetary transfers have been used to study employment relations (Levin 2002, 2003, Malcomson and MacLeod, 1989), sovereign lending (Atkeson, 1991, Kletzer and Wright, 2000), team production (Doornik 2006, Rayo 2007), cartels[2] (Harrington and Skrzypacz, 2007, and also Athey and Bagwell, 2001) or other business to business relationships (Baker, Gibbons and Murphy, 2002).

Most of these articles consider stationary equilibria in which a single action profile is repeated in every period and any deviation from a required payment will be punished by an infinite reversion to a Nash equilibrium of the stage game. Levin (2003) shows that stationay equilibria are indeed optimal in a class of principal agent games. Our paper extends this result by showing for a general class of games that all public perfect equilibrium payoffs can be implemented by stationary equilibria that use stick-and-carrot punishments, in which a deviation from a required monetary transfer is punished by playing a punishment action profile for one period. We derive this result for the case that money burning is possible. We also establish a related result for the case that players cannot burn money but use a public correlation device.

The algorithm to compute the set of public perfect equilibrium payoffs boils down to finding optimal action profiles for the equilibrium path and for the punishment of each player. Similar to the algorithms of APS and JYC, our algorithm solves several static linear optimization problems for all relevant action profiles. In APS and JYC these optimization problems have to be repeated for different candidate sets of continuation payoffs and the whole algorithm has to be repeated for different discount factors. In our framework, we show that a single number, which has a natural interpretation as the totally available liquidity in a setting with enforcable payments, already contains all relevant information about the set

[2]Harrington and Skrzypacz (2007) explain how the Lysine and Citric Acid Cartells implemented monetary transfers via sales between the cartel members.

of continuation payoffs. Standard re-optimization techniques allow to quickly solve the static problems for all relevant levels of liquidity. One implication is that our algorithm directly computes payoff sets for the whole interval of discount factors and exactly characterizes the critical discount factors at which optimal equilibrium and punishment action profiles change.

For the special case of perfect monitoring, we obtain closed-form solutions for all static problems.[3] To compute the sets of equilibrium payoffs for all discount factors, one essentially has to calculate stage-game best-reply payoffs and sort the stage game action profiles. The resulting characterization of all pure strategy subgame perfect equilibrium payoffs is almost as simple as the one in Cronshaw and Luenberger (1994), who provide a characterization of the set of strongly symmetric subgame perfect equilibria in repeated games with perfect monitoring and a public randomization device.

For arbitrary games with imperfect public monitoring, there is no general closed-form solution for the static problems. Yet, we illustrate for a noisy prisoners' dilemma game with a non-degenerate signal structure how analytical solutions for the set of pure strategy public perfect equilibrium payoffs can be obtained. The example also illustrates how, due to monitoring imperfections, money burning can be optimal on the equilibrium path.

Money burning is a very explicit way of modeling inefficiencies that may optimally arise in an equilibrium following a signal that indicates a deviation. Other forms of inefficient continuation play can of course serve the same function. To better understand the role of money burning, we characterize the payoff set in repeated games in which players do not burn money but have access to a public correlation device. In this framework, every equilibrium payoff can be implemented by a modification of stationary equilibria: with some probability, which can depend on the realized signal, there will be a transition to a collective punishment state. We show how the equilibrium payoff set for the case without money burning can be computed by considering stationary equilibria that allow for money burning but satisfy an additional constraint on the maximal amount of money burning. In general, the set of equilibrium payoffs can shrink if money burning is not possible. If, however, the stage game has a Nash equilibrium that gives each player her min-max payoff, the possibility of money burning does not enlarge

[3] See also Kranz and Ohlendorf, (2009), where we derive a related result for two player games with perfect monitoring in order to study renegotiation-proofness.

the equilibrium payoff set of the repeated game. For games with perfect monitoring, money burning can only be neccesary to implement a Pareto dominated equilibrium payoff.

The remainder of the paper is organized as follows: Section 2 describes the model and stationary strategy profiles. Section 3 derives the main results. In Section 4, we show how the results simplify for games with perfect monitoring and illustrate the resulting algorithm with a simple Cournot game. Section 5 illustrates for a noisy prisoners' dilemma game how closed-form analytical solutions can be obtained for games with imperfect public monitoring. In Section 6, we explore the case without money burning. Section 7 briefly concludes.

2 Model and Stationary Strategy Profiles

2.1 The game

We consider an infinitely repeated n-player game with imperfect public monitoring and common discount factor $\delta \in [0, 1)$. The timing in each period is as follows: at the beginning of a period, there is a payment stage in which the players have the opportunity to make nonnegative monetary transfers to each other or to burn money. In a subsequent action stage, the players play a simultaneous move stage game, and then there is again a payment stage in which they can make monetary transfers.[4]

The stage game played in the action stage has the following structure. Each player i has a finite action space A_i.[5] The set of stage game action profiles is given by $A = A_1 \times ... \times A_n$. After an action profile $a \in A$ is chosen, nature draws a commonly observed signal y from a finite signal space Y. The probability distribution of signals depends on the selected action profile a, and is given by a function $\phi(y|a)$ with

$$\begin{aligned} \phi(y|a) &\geq 0 \text{ for all } y \in Y, a \in A \\ \sum_{y \in Y} \phi(y|a) &= 1 \text{ for all } a \in A \end{aligned}$$

[4]That we allow two payment stages emphasizes that players can make transfers at any point in the game, and it simplifies some formulae. However, the set of equilibrium payoffs stays the same if payments can be made only at the beginning of a period.

[5]Many of our results extend to action spaces that are compact subsets of $\mathbb{R}^m$.

Stage game payoffs of player i depend only on the signal y and the action a_i that player i has chosen. They are given by a function $\widehat{g}_i : Y \times A_i \to \mathbb{R}$. Let

$$g_i(a) = \sum_y \widehat{g}_i(y, a_i)\phi(y|a)$$

denote the expected payoff of player i given an action profile a. The joint payoff from an action profile a is denoted by

$$G(a) = \sum_{i=1}^{n} g_i(a)$$

The best reply or *cheating* payoff of player i is denoted by

$$c_i(a) = \max_{\tilde{a}_i} g_i(\tilde{a}_i, a_{-i}).$$

In contrast to the action choices, we assume that all transfers are commonly observable. All players choose their monetary transfers simultaneously. We also allow the players to burn money (one can think of the possibility to give money to charity or any other non-interested third party). To have a bounded action space, we assume for convenience that there exists an upper bound on a player's transfers. However, this upper bound shall be sufficiently large, so that we essentially consider a situation of unlimited liability. Players are risk-neutral and utility is linear in money and stage game payoffs. Thus, a player's payoff in a period where action profile a has been played and signal y has been realized is given by $\widehat{g}_i(y, a_i)$ minus the sum of the net payments that player i has made in the two payment stages.

A public history h of the repeated game is a list of all monetary transfers and public signals that have occurred before a given point in time. A (pure) public strategy σ_i of player i in the repeated game maps every public history that ends before the action stage into an action $a_i \in A_i$, and every public history that ends before a payment stage into a vector of monetary transfers. A public perfect equilibrium is a profile of public strategies that constitutes mutual best replies after every public history. We will restrict attention to pure strategies and public perfect equilibria.[6]

[6] The restriction to public perfect equlibria is without loss of generality once mixed strategies are excluded. The set of pure strategy PPE payoffs is the same as the set of pure strategy sequential equilibrium payoffs.

Payoffs and continuation payoffs of the repeated game are defined as average discounted payoffs, i.e. as the discounted sum of future payoffs multiplied by $(1-\delta)$. We denote by $u^0(\sigma)$ the vector of payoffs in the repeated game given a strategy profile σ.

2.2 Stationary strategy profiles

In this section, we introduce a class of stationary strategy profiles that allow a simple characterization of PPE payoffs for every discount factor. These stationary strategy profiles have the feature that the same action profile is played in every period on the equilibrium path and punishments have a simple stick-and-carrot structure.

While a strategy is supposed to specify gross amounts $\tilde{p}_{ij}$ that player i pays to player j, where $j = 0$ means that the money is being burned, for convenience we will describe all monetary transfers in stationary strategy profiles in form of net payments. For any net payment, i.e. any vector $p = (p_1, ..., p_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n p_i \geq 0$, one can find corresponding gross monetary transfers $\tilde{p}_{ij}$ with

$$p_i = \sum_{j=0}^{n} \tilde{p}_{ij} - \sum_{j=1}^{n} \tilde{p}_{ji},$$

and with the property that there is no player who at the same time makes and receives positive monetary transfers.[7]

A stationary strategy profile is characterized by $n+2$ states. Play starts in the *up-front payment state*, in which players are required to make up-front payments p^0. Afterwards play can be in one of $n+1$ states, which we index by $k \in K = \{e, 1, 2, ..., n\}$. We call the state $k = e$ the *equilibrium state* and $k = i \in \{1, ..., n\}$ the *punishment state* of player i. A stationary strategy profile specifies for each state $k \in K$ an action profile $a^k \in A$ that will be played in the action stage. Furthermore, it specifies for each state $k \in K$ a payment function $p^k : Y \to \mathbb{R}^n$ that maps the signal y from the preceding action stage into a required vector of

[7]Concretely, we can assume that gross monetary transfers from player i to j are given by

$$\tilde{p}_{ij} = \begin{cases} p_i \frac{|p_j|}{\sum_{j \in I} |p_j|} & \text{if } i \in I^+ \text{ and } j \in I^- \\ 0 & \text{otherwise.} \end{cases}$$

where $I^+ = \{i \mid p_i > 0\}$ is the set of net payers, $I^- = \{i \mid p_i \leq 0\} \cup \{0\}$ be the set of net receivers (including the sink for burned money).

payments. Payments in the beginning of the period only occur in the upfront state in the first period, but not in the equilibrium state or in a punishment state.

The state transitions are as follows: If no player unilaterally deviates from a required payment, the new state becomes the equilibrium state: $k = e$. If player i unilaterally deviates from a required payment, the new state becomes the punishment state of player i, i.e. $k = i$. In all other situations the state does not change.

A stationary strategy profile σ is completely characterized by a vector of upfront payments p^0, its *action plan* $(a^k)_{k\in K}$ that specifies one action profile for every state k and its *payment plan* $(p^k)_{k\in K}$ that specifies a payment function for every state k. For a given discount factor δ, we call a stationary strategy profile σ a *stationary equilibrium* if σ constitutes a public perfect equilibrium of the repeated game. We denote by $(a^k, p^k)_{k\in K}$ a stationary strategy-profile without up-front payments and by Σ^0 the set of stationary equilibria without up-front payments.

The following definitions are useful for the characterization of stationary equilibria. For any payment function p, we let

$$E[p_i|a] = \sum_y p_i(y)\phi(y|a)$$

denote the expected payments of player i if the action profile a is played. For any stationary strategy profile player i's payoff at the beginning of a period in the equilibrium state is

$$u_i(\sigma) = g_i(a^e) - E[p_i^e|a^e].$$

Whenever the equilibrium in question is clear from the context, we will suppress the dependence on σ. Similarly, the joint equilibrium state payoff is given by

$$U(\sigma) = G(a^e) - \sum_{i=1}^n E[p_i^e|a^e],$$

where the sum on the right hand side denotes the expected amount of money that is burned on the equilibrium path. Player i's continuation payoff at the beginning of his punishment state is denoted by

$$v_i(\sigma) = (1-\delta)(g_i(a^i) - E[p_i^i|a^i]) + \delta u_i.$$

We call v_i player i's *punishment payoff.* We denote the sum of punishment payoffs by

$$V(\sigma) = \sum_{i=1}^n v_i.$$

3 Main results

3.1 Conditions for stationary equilibria

Using the one shot deviation principle, we now establish the constraints that a stationary strategy profile without up-front payments $\sigma = (a^k, p^k)_k$ has to satisfy to be a stationary equilibrium. There are three types of constraints, which we call payment constraints, budget constraints, and action constraints.

Payment constraints Given that player i has an equilibrium payoff of u_i and a punishment payoff of v_i, he is never willing to make a higher payment than $\frac{\delta}{1-\delta}(u_i - v_i)$. A stationary equilibrium thus must satisfy the following payment constraints for all states $k \in K$:

$$p_i^k(y) \leq \frac{\delta}{1-\delta}(u_i - v_i) \text{ for all } i, y. \qquad \text{(PC-k)}$$

Budget constraints Even though players can burn money, they cannot get any outside funding. In every state k, the following budget constraints must therefore be satisfied:

$$\sum_{i=1}^{n} p_i^k(y) \geq 0 \quad \text{for all } y \qquad \text{(BC-k)}$$

Action constraints There are no incentives to deviate from the prescribed action profiles in state $k \in K$ if and only if

$$g_i(a^k) - E[p_i^k | a^k] \geq g_i(a_i, a_{-i}^k) - E[p_i^k | a_i, a_{-i}^k] \text{ for all } i \text{ and } a_i \in A_i. \qquad \text{(AC-k)}$$

Next, we describe how the possibility of up-front payments transforms the set of feasible payoffs. Up-front payments are incentive compatible if they do not exceed $\frac{\delta}{1-\delta}(u_i - v_i)$ for any player. Incentive compatible up-front payments allow any distribution of the joint equilibrium payoff that guarantees every player at least his punishment payoff. This leads to the following straightforward result:

Proposition 1 *If there exists a stationary equilibrium σ with joint equilibrium payoffs U and punishment payoffs v then every payoff in the simplex*

$$\mathcal{U}^0(\sigma) = \{u^0 \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i^0 \leq U \text{ and } u_i^0 \geq v_i \text{ for all } i\} \qquad (1)$$

can be achieved by some stationary equilibrium that differs from σ only in the up-front payments.

Proof. Straightforward. ■

Note that the payoffs below the Pareto frontier of $\mathcal{U}^0(\sigma)$ can be implemented by burning some money up-front.

We say a payment plan $(p^k)_k$ is optimal for a given action plan $(a^k)_k$ if it maximizes the difference between joint equilibrium payoffs and total punishment payoffs, $U - V$, subject to the equilibrium constraints. An optimal payment plan thus solves the following linear program:

$$\max_{\{p^k\}_{k\in K}} U - V \qquad \text{(LP-OPS)}$$
$$\text{s.t.} \quad \text{(PC-k), (BC-k), (AC-k), for all states } k \in K.$$

Proposition 2 *Every payoff of stationary equilibria with action plan $(a^k)_k$ can be implemented by stationary equilibria whose payment plan is optimal for $(a^k)_k$. If the linear program (LP-OPS) has no solution then there does not exist a stationary equilibrium with action plan $(a^k)_k$.*

Proof. Let $\bar{U}$ denote the highest joint payoff and $\bar{v}_i$ the lowest punishment payoff of player i of all stationary equilibria with action plan $(a^k)_k$. We will construct a stationary equilibrium with action plan $(a^k)_k$ that has joint equilibrium payoffs $\bar{U}$ and at the same time punishment payoffs $\bar{v}_i$ for each player i, which implies an optimal payment plan. Let σ^e be a stationary equilibrium with action plan $(a^k)_k$, some payment plan $(p^{k,e})_k$, and joint equilibrium payoff $\bar{U}$. Similarly, let σ^i be a stationary equilibrium with action plan $(a^k)_k$, payment plan $(p^{k,i})_k$ and punishment payoff $\bar{v}_i$ for player i.

We define the payment functions

$$\begin{aligned} p^e(y) &= p^{e,e}(y) \\ p^i(y) &= p^{i,i}(y) + \frac{\delta}{1-\delta}(u_i(\sigma^e) - u_i(\sigma^i)) \text{ for all } i = 1,..,n. \end{aligned}$$

The stationary strategy profile $\sigma \in \Sigma^0$ defined by action plan $(a^k)_k$ and payment plan $(p^k)_k$ has joint equilibrium payoff $\bar{U}$ and punishment payoffs

$$v_i = (1-\delta)(g_i(a^i) - E[p^{i,i}|a^i]) - \delta(u_i(\sigma^e) - u_i(\sigma^i)) + \delta u_i(\sigma^e) = \bar{v}_i.$$

It is clear that the action constraints (AC) of σ hold, since all payment functions p^k are only shifted by a constant from $p^{k,k}$. To show that the budget constraints hold we use the fact $\bar{U} \geq U(\sigma^i)$, which implies that $\sum_{i=1}^n p_i^i(y) \geq 0$. The payment constraints in the equilibrium state have to hold since $\bar{v}_i \leq v_i(\sigma^e)$. The payment constraints in player i's punishment state are

$$p^{i,i}(y) + \frac{\delta}{1-\delta}(u_i(\sigma^e) - u_i(\sigma^i)) \leq \frac{\delta}{1-\delta}\left(u_i(\sigma^e) - \bar{v}_i\right),$$

which are equivalent to the payment constraints for player i's punishment state in σ^i. Thus, σ is a stationary equilibrium. ■

We say an action plan $(a^k)_k$ is optimal for a given discount factor if no other action plan can achieve a higher value of $U - V$. An optimal stationary equilibrium has an optimal action plan and an optimal payment plan. We can now state one key result:

Theorem 1 *All public perfect equilibrium payoffs can be implemented with a set of optimal stationary equilibria that only differ by their up-front payments.*

Proof. We rely on the recursive structure of public perfect equilibria and compactness of the equilibrium value set (see e.g. the result in APS, which straightforwardly extend to our setting). Let $\bar{U}$ denote the highest joint payoff that can be implemented with some PPE and $\bar{v}_i$ the lowest payoff for player i that can be implemented with some PPE. There must exist a PPE σ^e without payments in the first payment stage whose joint payoffs are given by

$$\sum_{i=1}^{n} u_i^0(\sigma^e) = \bar{U}.$$

Furthermore, for every player $i = 1, ..., n$, there exists a PPE σ^i without payments in the first payment stage that gives player i a payoff of

$$u_i^0(\sigma^i) = \bar{v}_i.$$

For all $k \in K$ let a^k be the first action profile played on the equilibrium path of σ^k. Let $w^k(y)$ be the vector of continuation payoffs of σ^k in the first period after signal y has been realized (but before the second payment stage), i.e. we have

$$u_i^0(\sigma^k) = (1-\delta)g_i(a^k) + E[w_i^k|a^k].$$

We define

$$p_i^k(y) = \frac{\delta u_i^0(\sigma^e) - w_i^k(y)}{1-\delta},$$

and will show that the stationary strategy profile σ defined by action plan $(a^k)_k$ and payment plan $(p^k)_k$ is a stationary equilibrium. The budget constraints of σ are equivalent to

$$\delta \bar{U} \geq \sum_{i=1}^{n} w_i^k(y),$$

which holds due to the definition of $\bar{U}$ as the highest possible sum of payoffs and the fact that the sum of payments cannot be negative. Second, for the action constraints, we have to show that

$$g_i(a^k) - E[p_i^k|a^k] \geq g_i(a_{-i}^k, a_i) - E[p_i^k|a_{-i}^k, a_i],$$

for all $i \in \{1, ..., n\}, a_i \in A_i$. This condition is equivalent to

$$g_i(a^k)(1-\delta) + E[w_i^k|a^k] \geq g_i(a_{-i}^k, a_i)(1-\delta) + E[w_i^k|a_{-i}^k, a_i],$$

which is the incentive constraint for playing a^k in the first period of σ^k. Third, for the payment constraints we have to show that

$$p_i^k(y) \leq \delta(g_i(a^e) - E[p_i^e|a^e] - g_i(a^i) + E[p_i^i|a^i]).$$

With our definition of payments $p_i^k(y)$ this reads

$$\delta u_i^0(\sigma^e) - w_i^k(y) \leq \delta(g_i(a^e)(1-\delta) + E[w_i^e|a^e] - g_i(a^i)(1-\delta) - E[w_i^i|a^i]),$$

which is equivalent to

$$w_i^k(y) \geq \delta \bar{v}_i.$$

Because $\bar{v}_i$ is the lowest player i payoff in the action stage, this condition obviously holds if player i receives a net payment after signal y in the corresponding continuation equilibrium of σ^i. It also holds for signals which require player i to make a net transfer, because otherwise player i would have an incentive not to make the payment and σ^i would not be a PPE. Player i's expected payoff in the stationary equilibrium σ is

$$g_i(a^e) - \frac{1}{1-\delta} E[\delta \bar{u}_i^e - w_i^k(y)|a^e] = u_i(\sigma^e),$$

and his punishment payoff is

$$g_i(a^i)(1-\delta) - E[\delta \bar{u}_i^e - w_i^k(y)|a^i] + \delta \bar{u}_i^e = \bar{v}_i.$$

It then follows from Proposition 1 that we can define incentive compatible up-front payments for σ to implement any PPE equilibrium payoff. ■

Hence, the essential step to find the set of PPE payoffs is to find an optimal action plan. For finite stage games, there is a simple brute force algorithm: Go through all possible (n+1)-tuples of action profiles $(a^k)_k \in A^{n+1}$ and calculate the corresponding maximum value of $U - V$ by solving the linear program (LP-OPS). We will now develop a quicker algorithm that relies on an explicit link of the constraints for stationary equilibria with a series of static problems with enforceable payments.

3.2 A characterization using static problems with enforceable payments

Consider the following static problem. The stage game is played once and there exist enforceable contracts that specify for each player $i = 1, .., .n$ and every signal $y \in Y$ a vector of gross monetary transfers to other players and an amount of money burning. From an incentive perspective, only net payments are relevant. We therefore write an enforceable contract as a payment function $p(.)$ that specifies the net payments $p_i(y)$ of player i if signal y realizes.

The possible payments that player i can make shall be bounded by an exogenously given liquidity constraint $\lambda_i L \geq 0$, with $L \geq 0$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. This means the totally available liquidity across all players is given by L and λ denotes the liquidity distribution.

We say that an action profile $a \in A$ can be implemented with a payment function $p(.)$ in the static problem given liquidity allocation λL, if the following payment, budget and action constraints hold:

$$p_i(y) \leq \lambda_i L \quad \text{for all } i, y. \tag{PC}$$

$$\sum_{i=1}^{n} p_i(y) \geq 0 \text{ for all } y \tag{BC}$$

$$g_i(a) - E[p_i|a] \geq g_i(a_i', a_{-i}) - E[p_i|a_i', a_{-i}] \text{ for all } i, a_i' \in A_i. \tag{AC}$$

Whether an action profile a can be implemented with some payment function and how much money needs to be burned, does not depend on the liquidity distribution λ, but only on the total liquidity L. More precisely, we have the following straightforward result:

Lemma 1 *If the payment function p can implement an action profile a for the liquidity allocation λL then the payment function $\widetilde{p}$ with*

$$\widetilde{p}_i(y) = p_i(y) + \left(\widetilde{\lambda}_i - \lambda_i\right) L \tag{2}$$

can implement a for the liquidity allocation $\widetilde{\lambda} L$.

We define the liquidity requirement $L(a)$ of an action profile a as the minimum total liquidity L that is necessary to implement a in the static problem. Because of Lemma 1, the liquidity requirement is independent of the actual liquidity distribution λ, and given as the solution to the following linear program:

$$L(a) = \min_{p(.),L\geq 0} L \text{ s.t. (PC), (BC), (AC).} \tag{LP-L}$$

To find closed-form solutions for $L(a)$ in specific examples, it will often be convenient to solve (LP-L) with a liquidity distribution that gives all liquidity to a single player or distributes liquidity equally across players. The liquidity requirement of an action profile a is 0 if and only if a is a Nash equilibrium of the stage game.

For a given value of total liquidity $L \geq L(a)$, we denote by $U^e(L,a)$ the maximum expected joint payoff that can be implemented with action profile a:

$$U^e(L,a) = \max_{p(.)} \left(G(a) - \sum_{i=1}^{n} E[p_i|a] \right) \text{ s.t. (PC),(BC), (AC).} \tag{LP-e}$$

Lemma 1 implies that the solution to the linear program (LP-e) is independent of the chosen liquidity distribution λ. Observe that $U^e(L,a)$ is bounded, weakly increasing and concave in L, and since we assumed a finite action space, it is piece-wise linear with a finite number of kinks.[8] Appendix A explains a method that exploits these attributes in order to quickly compute $U^e(L,a)$. We denote by $\bar{L}^e(a)$ the lowest liquidity level for which $U^e(L,a)$ attains its maximum value.

We now define a punishment payoff for player i in the static problem. For any given action profile a, liquidity $L \geq L(a)$ and some arbitary liquidity distribution λ, we define

$$v^i(L,a) = \min_{p(.)} \left(g_i(a) + \lambda_i L - E[p_i|a]\right) \text{ s.t. (PC),(BC), (AC).} \tag{LP-i}$$

Again, because of Lemma 1, $v^i(L,a)$ is independent of the liquidity distribution λ. Note that $v^i(L,a)$ is the lowest expected payoff that can be imposed on player i in the static problem if no liquidity is given to player i.

[8] That $U^e(L|a)$ is weakly increasing and bounded is obvious. Concavity and piece-wise linearity follows from standard results on linear optimization.

Lemma 2 *It holds true that* $v^i(L,a) \geq c_i(a)$*, and if* $g_i(a) = c_i(a)$*, then* $v^i(L,a) = g_i(a)$*.*

Proof. Since $p_i(y) \leq \lambda_i L$, the action constraint (AC) for player i implies

$$g_i(a) + \lambda_i L - E[p_i|a] \geq g_i(a_i', a_{-i}) \text{ for all } a_i' \in A_i,$$

which implies $v^i(L,a) \geq c_i(a)$. In the case $g_i(a_i) = c_i(a_i)$ one can take $\lambda_i = 0$ and $p_i(y) = 0$ for all y to implement a. ■

Similar to $U^e(L,a)$, the function $v^i(L,a)$ is bounded, weakly decreasing, convex and piece-wise linear in L (with a finite number of kinks); efficient computation techniques are also described in Appendix A. We denote by $\bar{L}^i(a)$ the lowest liquidity level at which $v^i(L,a)$ attains its minimum.

We now show how the solutions of the static problems are linked to stationary equilibria of the repeated game.

Definition 1 *We say that a liquidity* L *can be generated by action plan* $(a^k)_{k\in K}$ *given discount factor* δ *if*

$$\max_{k\in K} L(a^k) \leq L \leq \frac{\delta}{1-\delta}\left(U^e(L,a^e) - \sum_{i=1}^n v^i(L,a^i)\right). \tag{3}$$

The left hand side denotes the minimal liquidity that is required to implement all action profiles of action plan $(a^k)_{k\in K}$ in the separate static problems. The right hand side can be interpreted as the maximum endogenous total liquidity that the action plan can generate given that liquidity L is available. If some liquidity can be generated by an action plan $(a^k)_{k\in K}$, there must exist a largest liquidity L^* that can be generated, and it satisfies

$$L^* = \frac{1-\delta}{\delta}\left(U^e(L^*,a^e) - \sum_{i=1}^n v_i(L^*,a^i)\right). \tag{4}$$

That is because $U^e(L,a^e) - \sum_{i=1}^n v_i(L,a^i)$ is bounded, weakly increasing and continuous in L.

If an action plan $(a^k)_{k\in K}$ can generate some liquidity given δ, we say that $(a^k)_{k\in K}$ is regular if the condition $v_i(L^*,a^i) \leq v_i(L^*,a^e)$ is satisfied, where L^* is the maximum liquidity that can be generated.

Theorem 2 *Fix a discount factor* δ*. Only if an action plan* $(a^k)_{k\in K}$ *can generate some liquidity, there exists a stationary equilibrium with action plan* $(a^k)_{k\in K}$*. If*

$(a^k)_{k\in K}$ *can generate some liquidity and is also regular, there exists a stationary equilibrium with action plan* $(a^k)_{k\in K}$*. Then, if* L^* *denotes the largest generated liquidity, an optimal payment plan yields joint equilibrium payoffs* $U^e(L^*, a^e)$ *and punishment payoffs* $v^i(L^*, a^i)$*.*

Proof. First, if there is a stationary equilibrium σ with action plan $(a^k)_{k\in K}$ and optimal payment plan $(p^k)_{k\in K}$, then the scalar $L = \frac{\delta}{1-\delta}(U(\sigma) - V(\sigma))$ is a liquidity that can be generated by $(a^k)_{k\in K}$, since every action profile a^k can be implemented with p^k given L and $\lambda_i = \frac{u_i - v_i}{U-V}$.

Let us now assume that there exists a liquidity L generated by $(a^k)_k$, and hence also a largest such liquidity L^*. In the following, we construct a liquidity distribution λ^* and a stationary equilibrium σ with action plan $(a^k)_{k\in K}$ that satisfies $\lambda_i^* L^* = \frac{\delta}{1-\delta}(u_i(\sigma) - v_i(\sigma))$. The payment constraints (PC-k) in σ and the payment constraints in the static problem (LP-k) given liquidity λ^* will coincide for all states k.

Let λ be an arbitrary liquidity distribution and let $\tilde{p}^k$ be a payment function that solves the static problem (LP-k) given liquidity allocation λL^*. Consider the vector λ^* defined by

$$\lambda_i^* = \delta\left(\lambda_i + \frac{g_i(a^e) - E[\tilde{p}_i^e|a^e] - v^i(L^*, a^i)}{L^*}\right).$$

It is straightforward to check that λ_i^* is a liquidity distribution if for all players $v^i(L^*, a^i) \leq v^i(L^*, a^e)$, which holds due to regularity of $(a^k)_{k\in K}$. It follows from Lemma 1 that the payment function

$$p^k = \tilde{p}^k + (\lambda^* - \lambda)\, L^*$$

then solves the static problem for state k with liquidity distribution λ^*. For the strategy profile $\sigma \in \Sigma^0$ defined by action plan $(a^k)_k$ and payment plan $(p^k)_k$ it holds that

$$\lambda_i^* L^* = \frac{\delta}{1-\delta}(u_i(\sigma) - v^i(L^*, a^i)).$$

Furthermore, it holds that $v^i(L^*, a^i) = v_i(\sigma)$ and $U^e(L^*, a^e) = U(\sigma)$. By construction the payment, budget and action constraints of σ are satisfied. ■

While the action and budget constraints are the same in the static problem of implementing all action profiles a^k and the dynamic problem of finding a stationary equilibrium with action plan $(a^k)_k$, the payment constraints differ. In the static problem we can choose arbitrary liquidity distributions for every state but in a stationary equilibrium player i's maximal payments are limited in every state by an endogenous bound that depends on equilibrium payoffs $u_i(\sigma)$ and punishment payoffs $v_i(\sigma)$. In the proof, we construct payments and a particular liquidity distribution λ^* such that also the payment constraints coincide in the two problems.

The steps to find an optimal payment structure and corresponding payoffs given a regular action plan $(a^k)_{k\in K}$ are as follows: First, calculate for all states the liquidity requirements $L(a^k)$, as well as $U^e(L, a^e)$ and all $v^i(L, a^i)$ using some convenient liquidity distributions. Second, solve equation (4) to find L^* (which is typically unique since $U^e(L, a^e) - \sum_{i=1}^n v^i(L, a^i)$ is concave in L). We then already know the joint equilibrium payoffs $U^e(L^*, a^e)$ and punishment payoffs $v^i(L^*, a^i)$ and can obtain the set of equilibrium payoffs (see Proposition 1). The proof of Theorem 2 also explains how an optimal payment plan can be derived from the solutions of the static problems.

The following result establishes an upper bound on the size of required transfers in an optimal payment plan, which guarantees that payments needed to implement a given action plan do not go to infinity as the discount factor goes to 1. Note, however, that in order to achieve every payoff that can be implemented with the action plan, up-front payments may have to exceed the bound.

Proposition 3 *If there exists a stationary equilibrium with action plan $(a^k)_k$, there exists an optimal payment plan in which no player makes payments above $L^o = \min\{L^*, \max_k\{\bar{L}^k(a^k)\}\}$.*

Proof. Similar to the proof of Theorem 2, we take a payment function $\tilde{p}^e$ that implements a^e with minimal money burning given an arbitrary liquidity allocation λL^0 and define

$$\lambda_i^* = \delta\left(\lambda_i + \frac{g_i(a^e) - E[\tilde{p}_i^e|a^e] - v^i(L^*, a^i)}{L^*}\right).$$

It is straightforward to check that λ_i^* is a liquidity distribution. We define

$$p^e = \tilde{p}^e + (\lambda^* - \lambda)\, L^o.$$

Because $L^0 \leq L^*$, it then holds that

$$\lambda_i^* L^o \leq \delta\left(\lambda_i^* L^o + g_i(a^e) - E[p_i^e|a^e] - v^i(L^*, a^i)\right),$$

hence

$$\lambda_i^* L^o \leq \frac{\delta}{1-\delta}\left(g_i(a^e) - E[p_i^e|a^e] - v^i(L^*, a^i)\right).$$

Let p^i be the payment function that leads to a punishment payoff $v^i(L^o, a^i)$ in the problem (LP-i) given liquidity allocation $\lambda^* L^0$. Then $(p^k)_k$ is the payment plan we were looking for. ■

3.3 Finding optimal action profiles

There is a natural procedure to find an optimal action plan and the payoff set for all discount factors. While the results in the previous section took the action plan as given, we are interested in optimal action profiles and corresponding payoffs in order to find the set of PPE payoffs. We denote the upper envelope of all $U^e(L, a)$ functions by

$$\bar{U}^e(L) = \max_{a \in A | L(a) \geq L} U^e(L, a),$$

and by $\bar{a}^e(L)$ an optimal action profile that solves this problem given liquidity L. We denote the lower envelope of player i's punishment payoffs by

$$\bar{v}^i(L) = \min_{a \in A | L(a) \geq L} v^i(L, a),$$

and a corresponding optimal punishment profile by $\bar{a}^i(L)$. If the stage game is symmetric then $\bar{v}^i(L)$ is identical for all players and optimal punishment profiles $\bar{a}^i(L)$ are given by the corresponding permutation of $\bar{a}^1(L)$, i.e. it suffices to characterize the punishment state for player 1.

To determine these envelopes and optimal action profiles, it is often not necessary to calculate the values $U^e(L, a)$ and $v^i(L, a)$ for all action profiles a. For example, if the joint equilibrium payoff $G(a)$ of an action profile a is lower than the joint payoff of a stage game Nash equilibrium, a is clearly not an optimal equilibrium state profile and we can dismiss it without any further calculation. In Appendix A, we discuss several heuristics that speed up the calculation of $\bar{U}^e(L)$ and $\bar{v}^i(L)$.

We define the largest liquidity that can be generated with any action plan for a given discount factor δ as

$$\bar{L}(\delta) = \max\{L \mid L = \frac{\delta}{1-\delta}\left(\bar{U}^e(L) - \sum_{i=1}^{n} \bar{v}^i(L)\right)\}.$$

The liquidity $\bar{L}(\delta)$ can be generated by the action plan $(\bar{a}^k(\bar{L}(\delta)))_k$ and it follows from Theorem 2 that $(\bar{a}^k(\bar{L}(\delta)))_k$ is an optimal action plan given δ. Together with Theorem 1 this implies

Corollary 1 *Given discount factor δ, the set of public perfect payoffs is given by*

$$\mathcal{U}^0(\delta) = \{u^0 \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i^0 \leq \bar{U}^e(\bar{L}(\delta)) \text{ and } u_i^0 \geq \bar{v}^i(\bar{L}(\delta))\}. \tag{5}$$

To calculate closed-form solutions for $\bar{L}(\delta)$ and to determine the critical discount factors δ where $\bar{U}^e(\bar{L}(\delta))$ and $\bar{v}^i(\bar{L}(\delta))$ have a kink or jump, it is often convenient to work with discount rates $r = \frac{1-\delta}{\delta}$. We denote by

$$r^*(L) = \frac{\bar{U}^e(L) - \sum_{i=1}^{n} \bar{v}^i(L)}{L} \tag{6}$$

and $\delta^*(L) = \frac{1}{1+r^*(L)}$ the discount rate and discount factor that correspond to some liquidity level L.[9] The numerator on the right hand side of (6) is a piece-wise linear function in L and by piece-wise inverting this function, we can obtain the largest liquidities $\bar{L}(\delta)$ that can be generated for any discount factor. We illustrate this procedure in Sections 4 and 5.

4 Perfect monitoring

With perfect monitoring, the played action profile is perfectly observable by all players. This means that we have a game with perfect monitoring if the signal space is equal to the action space, i.e. $Y = A$ and the signal distribution is

$$\phi(y|a) = \begin{cases} 1 & \text{if } y = a \\ 0 & \text{if } y \neq a \end{cases} .$$

To implement an action profile a in the static problem, one can use a payment function $\hat{p}$ that requires each player i to pay $c_i(a) - g_i(a)$ following any signal (a_i', a_{-i}) with $a_i' \neq a_i$, and to pay nothing otherwise. The liquidity requirement is given by

$$L(a) = \sum_{i=1}^{n} (c_i(a) - g_i(a)) . \tag{7}$$

[9] We define $r^*(L) = \infty$ if $L = 0$.

That this liquidity suffices to implement a can be seen by considering the liquidity distribution $\lambda_i = \frac{c_i(a)-g_i(a)}{L(a)}$. That this liquidity is necessary follows from summing up the action and payment constraints over all players.

With the payment function $\hat{p}$, no money will be burned on the equilibrium path. Thus, for all $L \geq L(a)$ we find that the maximal implementable joint payoffs are equal to the joint stage game payoffs:

$$U^e(L, a) = G(a).$$

To calculate the minimal punishment payoffs $v^i(L, a)$ for player i and an action profile a, consider a liquidity distribution λ that gives no liquidity to player i, i.e. $\lambda_i = 0$. It follows from Lemma 1 that a can then be implemented with the payment function $p + \lambda L(a) - (c(a) - g(a))$. We thus find that

$$v^i(L, a) = c_i(a),$$

i.e., player i's minimal punishment payoff is always equal to his stage game cheating payoff under his punishment profile a^i. Given the derived closed-form solutions for $U^e(L, a)$ and $v(L, a)$, Theorem 2 translates into the following result:

Proposition 4 *Under perfect monitoring there exist a stationary equilibrium with action plan $(a^k)_{k \in K}$ if and only if for every state $k \in K$*

$$(1-\delta)\sum_{i=1}^{n}\left(c_i(a^k) - g_i(a^k)\right) \leq \delta\left(G(a^e) - \sum_{i=1}^{n} c_i(a^i)\right) \qquad \text{(PM-k)}$$

Optimal payment structures then implement joint equilibrium payoff $G(a^e)$ and for each player i a punishment payoff $c_i(a^i)$.

4.1 Finding optimal action structures for every discount rate

We now describe a simple and quick algorithm that finds optimal action plans for every discount factor if the stage game has finitely many action profiles. We illustrate the algorithm for a simplified Cournot game taken from Abreu (1988). Two firms simultaneously choose either low (L), medium (M), or high (H) output and stage game payoffs are given by the following matrix:

		Firm 2		
		L	M	H
	L	$10, 10$	$3, 15$	$0, 7$
Firm 1	M	$15, 3$	$7, 7$	$-4, 5$
	H	$7, 0$	$5, -4$	$-15, -15$

The algorithm consists of different steps.

Step 1: The first step is to create a list of candidates for optimal equilibrium action profiles. We order all action profiles $a \in A$ decreasingly in their joint payoff $G(a)$ and break ties by putting action profiles with a lower liquidity requirement $L(a)$ first. Then we remove all action profiles from the list that do not have a strictly lower liquidity requirement than all earlier action profiles in the list. In the example, we get the following list:

No.	a^e	$G(a^e)$	$L(a^e)$
1.	(L,L)	20	10
2.	(L,M)[10]	18	4
3.	(M,M)	14	0

Note that if the stage game has at least one Nash equilibrium then the last profile of the list is always the Nash equilibrium with the highest joint payoffs.

Step 2: In a similar way, we create for each punishment state $i = 1, ..., n$ a list of action profiles. We order action profiles increasingly in player i's cheating payoff $c_i(a)$. We break ties by putting those profiles with a lower liquidity requirement $L(a)$ first. We remove action profiles that do not have a strictly lower liquidity requirement than all earlier action profiles. In the example, we get the following list for the punishment state of player 1:

No.	a^1	$c_1(a^1)$	$L(a^1)$
1.	(M,H)	0	6
2.	(M,M)	7	0

If the stage game is symmetric, as in our example, the lists of punishment profiles for the other players will simply consist of the correspondingly permuted action profiles.

[10] Alternatively, we could pick the profile (M, L) as second element of the list.

Step 3: The first action profiles in each list form our initial action plan. In the example, we have $(a^e = (L, L), a^1 = (M, H), a^2 = (H, M))$. Proposition 4 allows us to find the minimal discount factor for which a stationary equilibrium with this action plan exists. As noted in the end of Section 3, it is convenient to reformulate those conditions as a single condition on the discount rate $r = \frac{1-\delta}{\delta}$: There exists a stationary equilibrium with an action plan $(a^k)_{k\in K}$ if and only if the discount rate r satisfies

$$r \leq r^* \equiv \frac{G(a^e) - \sum_{i=1}^{n} c_i(a^i)}{\max_k\{L(a^k)\}}. \tag{8}$$

where we assume $r^* = \infty$ if all action profiles are Nash equilibria of the stage game. The critical discount rate in our example is given by

$$r^* = \frac{20}{\max\{10, 6\}} = 200\%.$$

This corresponds to a critical discount factor of $\delta^* = \frac{1}{1+r^*} = \frac{1}{3}$. Thus, by varying the up-front payments, we can implement for every discount factor $\delta \in [\frac{1}{3}; 1]$ every (weakly) individually rational distribution of the maximum joint stage game payoff of 20 as sequential equilibrium payoff of the repeated game.

It is straightforward that for any finite stage game, the minimal discount factor δ^* for which every individually rational distribution of the maximum joint stage game payoff can be implemented is always strictly below 1. This result is a folk theorem for games with side payments. For games without side payments, it generally only holds true that every feasible and *strictly* individually rational payoff can be implemented for sufficiently large discount factors.[11]

Step 4: In the next step, we replace the action profile a^k that has the highest liquidity requirement $L(a^k)$ by the next action profile in the list for state k. If several action profiles of the action plan have the highest liquidity requirement, we replace all those action profiles. In our example, we replace the equilibrium action profile a^e, so that the new action plan becomes $a^e = (L, M), a^1 = (M, H), a^2 = (H, M)$. Using again formula (8), we find that this action plan can be implemented whenever

$$r \leq r^* = \frac{18}{\max\{4, 6\}} = 300\%.$$

[11] Furthermore, in games with more than 2 players, the folk theorem without side payments only holds under a regularity condition. See Fudenberg and Maskin (1986).

Correspondingly, for every discount rate $\delta \in [\frac{1}{4}, \frac{1}{3})$ the actual action plan is optimal and the set of subgame perfect equilibrium payoffs is given by all (u_1, u_2) with $u_1 + u_2 \leq 18$ and $u_1, u_2 \geq 0$.

We repeat step 4 until we reach the end of the list of action profiles in every state k. The final action plan only consists of Nash equilibria of the stage game. In the example, we find the following critical discount factors, payoffs and action plans:

Step	δ^*	U^e	v_1	v_2	a^e, a^1, a^2
1	1/3	20	0	0	(L,L),(M,H),(H,M)
2	1/4	18	0	0	(L,M),(M,H),(H,M)
*3	1/2	18	7	7	(L,M),(M,M),(M,M)
4	0	14	7	7	(M,M),(M,M),(M,M)

Note that the critical discount factor δ^* does not necessarily decrease in every step. If δ^* it is not lower than in all previous steps, we simply ignore the corresponding action plan. This is the case in step 3 of our example.

The algorithm always delivers a list of all critical discount factors, corresponding payoff sets and optimal action structures. When using a heap sort algorithm to create the $n + 1$ ordered lists, which each have a maximal length of $|A|$ action profiles, the computational complexity of our algorithm in terms of elementary calculations and comparisons is of just log-linear order $\mathcal{O}(n|A| \log |A|)$. Even large stage games with more than a 100000 action profiles can be solved in less than a second.

Kranz (2010) explains how to use the software implementation of our algorithm and gives several examples. It is also illustrated how methods of adaptive grid refinement and random sampling of action profiles allow to effectively compute inner approximations to the sets of SPE payoffs for continuos stage games with high dimensional action spaces (like oligopolies with 10 or more firms).[12]

[12] For games with perfect monitoring, Theorem 1 and Proposition 4 will also hold for stage games with compact action spaces $A \subset \mathbb{R}^m$ and continous payoff functions. If one can provide closed-form solutions of the cheating payoffs of the continous stage game, one can calculate the liquidity requirement $L(a)$ for any action profile $a \in A$.

To compute inner approximations of the sets of SPE payoffs, we can draw a finite random sample of action profiles in order to calculate lower bounds of the functions $\bar{U}^e(L)$ and $\bar{v}^i(L)$ in a similar way we calculated the step functions above. As the sample size grows large, these lower bounds converge in probability to the true functions.

In comparison, we can note that allowing for monetary transfers allows much faster computation of the set of equilibrium payoffs than in the framework studied by Judd, Yeltekin and Conklin (2003) with public randomization. That is because without monetary transfers no general closed-form solutions for the static problems could be obtained, and in each iteration the algorithm of JYC has to solve several linear programs.[13]

5 A Noisy Prisoners' Dilemma game

In this example we derive closed form solutions for the set of pure strategy PPE payoffs in a repeated noisy prisoners' dilemma game with imperfect public monitoring. There are two players. In the stage game, a player can either cooperate C or defect D. Expected payoffs $g(a)$ are given by the following normalized payoff matrix:

	C	D
C	$1,1$	$-s,1+d$
D	$1+d,-s$	$0,0$

with $d,s>0$ and $d-s<1$. Players do not publicly observe the played action profile, but only a realized signal y that can take four different values: y_C, y_D, y_1 and y_2. The signal distribution is as follows:

$\phi(\mathbf{y}\|\mathbf{a})$	**CC**	**CD**	**DC**	**DD**
$\mathbf{y}_C$	$1-\alpha_A-2\alpha_P$	$1-\alpha_A-\beta_A-2\alpha_P-\beta_P$	$1-\alpha_A-\beta_A-2\alpha_P-\beta_A$	0
$\mathbf{y}_D$	α_A	$\alpha_A+\beta_A$	$\alpha_A+\beta_A$	1
$\mathbf{y}_1$	α_P	α_P	$\alpha_P+\beta_P$	0
$\mathbf{y}_2$	α_P	$\alpha_P+\beta_P$	α_P	0

with $0<\alpha_A\leq\alpha_A+\beta_A$ and $0<\alpha_P\leq\alpha_P+\beta_P$ and $1-\alpha_A-\beta_A-2\alpha_P-\beta_P\geq 0$. To interpret the signal structure, assume that mutual cooperation CC shall be implemented.[14] The signal y_D is an anonymous indicator for defection: y_D becomes

The practical issue is to sample action profiles in a way that achieves relatively quick convergence for most stage games. Different methods are implemented in the software package and work well in examples.

[13] JYC report a computation time of almost 45 minutes (on a Pentium 500Mhz, PC) for the finest considered approximation for the payoff set of a discretized repeated Cournot duopoly with 15 x 15 action profiles and a given discount factor of $\delta=0.8$

[14] For notational convenience, we abbreviate action profiles (a_1,a_2) by a_1a_2.

more likely if some player unilaterally defects but its probability distribution does not depend on the identity of the deviator. The parameter α_A can be interpreted as the probability of a type-one error, i.e. the probability that y_D is observed even if no player defected. The parameter β_A measures by how much the likelihood of y_D increases if some player unilaterally deviates.

The signal y_i is an indicator for unilateral defection by player i. Like α_A, the parameter α_P can be interpreted as the probability of a type-one error, i.e. the probability to wrongly get a signal for unilateral defection of player i. Similar to β_A, the parameter β_P measures by how much the likelihood of y_i increases if player i unilaterally deviates from mutual cooperation.

To calculate the required liquidity to implement mutual cooperation in the static problem, consider an equal liquidity distribution $\lambda_1 = \lambda_2 = \frac{1}{2}$. Clearly, incentives to deviate for each player i are minimized if he is required to make the maximal payments $\frac{1}{2}L$ after signals y_D and y_i. Since the problem is symmetric, it is disadvantageous to impose on some player a payment after signal y_C. Whether player i has to make a payment or receives a payment after signal y_{-i} has no effect on his incentives to deviate in the static problem. Mutual cooperation can thus be implemented with total liquidity L if and only if

$$0 \geq d - (\beta_A + \beta_P)\frac{1}{2}L,$$

which yields a liquidity requirement of

$$L(CC) = \frac{2d}{\beta_A + \beta_P}.$$

This formula is quite intuitive. If actions could be perfectly monitored, the liquidity requirement would be $2d$. This value is divided by the increase in the likelihood to get a signal y_i or y_D if player i defects.

To minimize the amount of money burning, it is optimal that after signal y_1 player 1 transfers all of his liquidity to player 2, and vice versa. Money burning can only be optimal after signal y_D. We find that for $L \geq \frac{2d}{\beta_P}$, mutual cooperation can be implemented without any money burning and that for $L \in [L(CC), \frac{2d}{\beta_P})$, a total amount of $\frac{2d - \beta_P L}{\beta_A}$ must be burned after signal y_D. The maximal implementable joint payoffs are thus given by

$$U^e(L, CC) = \begin{cases} 2 & \text{if } L \geq \frac{2d}{\beta_P} \\ 2(1 - \frac{\alpha_A}{\beta_A}d) + \frac{\alpha_A}{\beta_A}\beta_P L & \text{if } \frac{2d}{\beta_A+\beta_P} \leq L \leq \frac{2d}{\beta_P} \end{cases}. \quad (9)$$

Let us now consider the asymmetric action profile CD. Its liquidity requirement can be most easily calculated by assuming that the whole liquidity is allocated to player 1. The minimal required payment $p_1(y_D)$ after signal y_D that removes player 1's incentives to defect satisfies

$$s + (\alpha_A + \beta_A)p_1(y_D) = p_1(y_D).$$

If after signal y_D player 1 makes that payment $p_1(y_D)$ to player 2 and no other payments are made, no player has an incentive to deviate and no money is burned. We thus find

$$L(CD) = \frac{s}{1 - \alpha_A - \beta_A}$$

and

$$U^e(L, CD) = G(CD) = 1 + d - s.$$

For the action profile DC the same results hold and for the stage game Nash equilibrium it is true that $L(DD) = 0$ and $U^e(L, DD) = 0$.

For every level of total liquidity L, the profile DD is an optimal punishment profile for both players, since the Nash equilibrium payoffs are min-max payoffs for both players. Hence, we find $\overline{v}_i(L) = 0$ for all $L \geq 0$.

Recall that in games with perfect monitoring, $\overline{U}^e(L) - \overline{V}(L)$ is always a step function. The algorithm for perfect monitoring calculates the critical discount rate $r^*(L)$ at every jump point. With imperfect monitoring, $\overline{U}^e(L) - \overline{V}(L)$ is in general an increasing piece-wise linear function with jumps. Figure 2 illustrates the function $\overline{U}^e(L) - \overline{V}(L)$ for the noisy prisoners' dilemma game for a parameter constellation that satisfies $\beta_P > 0$ and $0 < G(CD) < U^e(L(CC), CC)$.

The graph has a kink P_1 and two jump points P_2 and P_3. We can calculate the critical discount rate at every jump point, kink and increasing linear segment of $\overline{U}^e(L) - \overline{V}(L)$ by using the formula

$$r^*(L) = \frac{\overline{U}^e(L) - \overline{V}^e(L)}{L}. \tag{10}$$

For the points P_1 and P_2, we find

$$r^*\left(\frac{2d}{\beta_P}\right) = \frac{d}{\beta_P}$$

and

$$r^*\left(\frac{2d}{\beta_A + \beta_P}\right) = \frac{\beta_P + \beta_A - d\alpha_A}{d}.$$

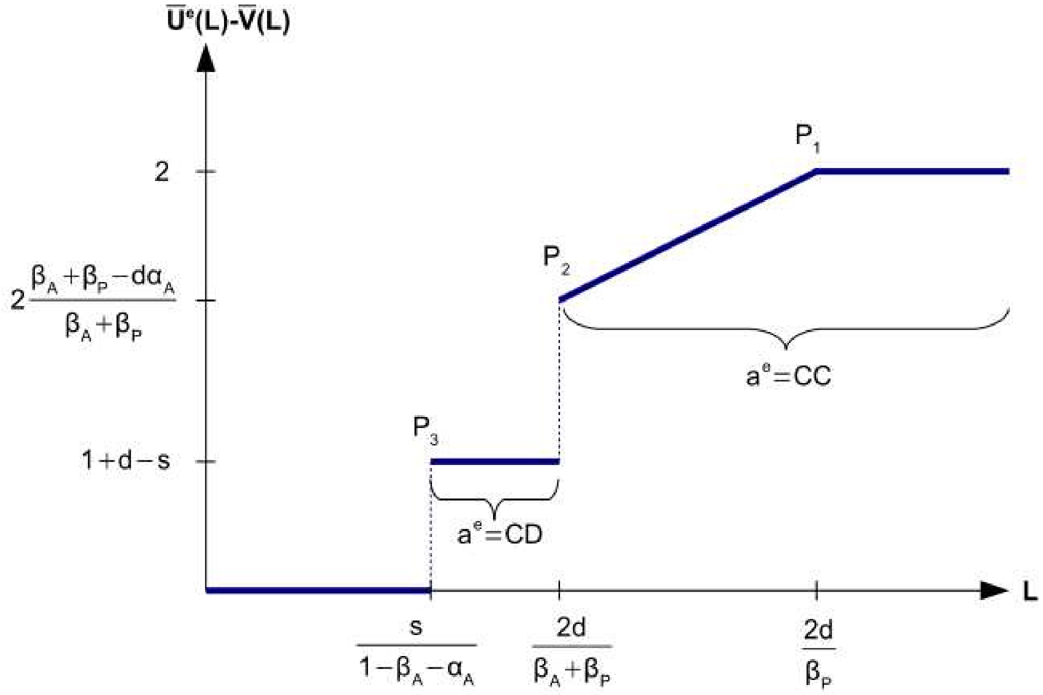

Figure 1: Optimal action profiles and payoffs of the noisy prisoners' dilemma game

On the line segment between the two points, i.e. for $L \in [\frac{2d}{\beta_A+\beta_P}; \frac{2d}{\beta_P}]$, the maximal discount rate is given by

$$r^*(L) = \frac{2(1-\frac{\alpha_A}{\beta_A}d)}{L} + \frac{\alpha_A}{\beta_A}\beta_P.$$

Money burning facilitates the implementation of CC if the maximal discount rate increases when moving from P_1 to P_2. This is the case if and only if $d \leq \frac{\beta_A}{\alpha_A}$.

Given a plot of $\overline{U}^e(L) - \overline{V}(L)$, as in Figure 1, there is a simple graphical rule to find out whether the maximal discount rate increases or decreases along a line segment. Consider the intercept at $L = 0$ of the line going through P_1 and P_2. The critical discount rate increases from P_1 to P_2 if and only if this intercept is positive. With a sharp glance, one can establish that this is indeed the case in Figure 1.

Similarly, one can check graphically whether the maximal discount rate is higher in point P_3 than in point P_2. In Figure 1, the intercept of the line through P_2 and P_3 is negative. This means that in the depicted case there is no discount rate for which CD or DC are optimal equilibrium state profiles: playing CC

with appropriate amounts of money burning yields higher payoffs and can be implemented for a larger range of discount factors.

By solving equation (10) for L and plugging into the formula for $\overline{U}^e(L)$, one can find the maximal joint equilibrium payoff $U^e(r)$ as a function of the discount rate r. For the case depicted in Figure 1, we find:

$$U^e(r) = \begin{cases} 2 & \text{if } r \leq \frac{d}{\beta_P} \\ 2(1 - \frac{\alpha_A}{\beta_A} d) \left[1 + \frac{\alpha_A}{r\beta_P\beta_A - \alpha_A}\right] & \text{if } \frac{d}{\beta_P} \leq r \leq \frac{\beta_P + \beta_A - d\alpha_A}{d} \\ 0 & \text{otherwise} \end{cases} . \tag{11}$$

Together with the fact that always zero punishment payoffs can be implemented, condition (11) characterizes the set of pure strategy sequential equilibrium payoffs for the considered case. Alternative cases, e.g. parameter constellations where CD is an optimal equilibrium state profile for some discount rates, can be characterized in a similar fashion.

6 Repeated games without money burning

In this section we explore what can be achieved in a repeated game with side-payments if money burning is not allowed. In particular, we investigate the question to what extent money burning can be replaced by the use of a public correlation device. We consider a variant of the previous set-up in which payments are required to add up to zero, and in which players observe the outcome of a public correlation device at the beginning of each period.

To characterize the set of PPE payoffs in this class of games, we extend action and payment plans by a collective punishment state, indexed with $k = b$. The public correlation device allows strategies that implement positive transition probabilities between states. The proof of Theorem 3 below shows that all PPE payoffs can be implemented by a class of stationary equilibria that put a positive probability on a transition to the collective punishment state instead of the equilibrium state if all payments are conducted.

We will develop a more convenient characterization of equilibrium payoffs by considering stationary equilibria that have an endogenous restriction on the amount of money burning. Consider a stationary strategy σ of the game with money burning, which is described by states k with action profile a^k and payment function p^k. We add a collective punishment state $k = b$ with action profile a^b and payment function p^b that define an additional constraint on the amount of money burning

$$\sum_{i=0}^{n} p_i^k(y) \le \frac{\delta}{(1-\delta)} \left(U(\sigma) - U^b\right) \quad \text{for all } y \tag{MBC-k}$$

with $U^b = \sum_{i=1}^{n} u_i^b$ and

$$u_i^b = (1-\delta)(g_i(a^b) - E[p_i^b|a^b]) + \delta u_i(\sigma).$$

Then we consider the following maximization problem over action plans and payment plans that are extended in this way

$$\max_{(a^k,p^k)_{k=e,b,1,\ldots,n}} \quad U - V - U^b \tag{LP-OPS-LMB}$$
$$\text{s.t.} \quad \text{(PC-k),(AC-k),(BC-k) and (MBC-k) for all } k = e, b, 1, ..., n.$$

Theorem 3 *If LP-OPS-LMB is solved by a stationary equilibrium σ of the game with money burning and collective punishment state action profile a^b and payment function p^b, it holds that the set of PPE payoffs in the game without money burning is given by*

$$\left\{ u^0 \in \mathbb{R}^n | U^b \le \sum_{i=1}^{n} u_i^0 \le U(\sigma), u_i^0 \ge v_i(\sigma) \right\} \tag{12}$$

Proof. First we show that the set described in (12) is a subset of the set of PPE payoffs without money burning.

Let $(a^k, p^k)_{k=e,b,1,\ldots,n}$ be a solution of LP-OPS-LMB. The profile $\sigma = (a^k, p^k)_{k \in K}$ is a stationary equilibrium in the game with money burning with joint equilibrium payoff U and punishment payoffs v_i. It is augmented by a collective punishment state with joint payoff U^b. We now connect σ to the collective punishment state to get a PPE $\widetilde{\sigma}$ without money burning but with the same payoffs as σ. This is done by replacing the money burning by an appropriate choice of transition probabilities between the equilibrium state and the collective punishment state. That is, the structure of the strategy $\widetilde{\sigma}$ differs from the one of σ only in so far as that if in state $k = e, b, 1, ...n$ signal y has been realized and no player deviated from the required payments $p^k(y)$, the state changes with a probability $\beta_P^k(y)$ to the collective punishment state and with probability $1 - \beta_P^k(y)$ to the equilibrium state. We define this probability as

$$\beta_P^k(y) = \frac{1-\delta}{\delta} \frac{\sum_{i=1}^{n} p_i^k(y)}{U - U^b}. \tag{13}$$

Constraints (BC-k) and (MBC-k) tell us that $\beta_P^k(y)$ indeed is a probability. Note that on the equilibrium path of $\widetilde{\sigma}$ there can be repeated stochastic transitions between the equilibrium state and the collective punishment state.

We define the payment function of the strategy $\widetilde{\sigma}$ in state $k = e, b, 1, ..., n$ by

$$\widetilde{p}_i^k(y) = p_i^k(y) - \frac{\delta}{1-\delta}\beta_P^k(y)(u_i - u_i^b).$$

Up-front transfers are set to zero. The probabilities $\beta_P^k(y)$ have been chosen such that the payments $\widetilde{p}_i^k(y), i = 1, ..., n$ add up to zero. With this definition of payments we have that

$$\begin{aligned} u_i(\widetilde{\sigma}) &= (1-\delta)(g_i(a^e) - E[\widetilde{p}^e(y)|a]) + \delta u_i(\widetilde{\sigma}) + \delta E[\beta_P^e|a^e](u_i^b(\widetilde{\sigma}) - u_i(\widetilde{\sigma})) \\ u_i^b(\widetilde{\sigma}) &= (1-\delta)(g_i(a^b) - E[\widetilde{p}^b(y)|a]) + \delta u_i(\widetilde{\sigma}) + \delta E[\beta_P^b|a^b](u_i^b(\widetilde{\sigma}) - u_i(\widetilde{\sigma})) \end{aligned}$$

reduces to

$$u_i(\widetilde{\sigma}) = u_i \text{ and } u_i^b(\widetilde{\sigma}) = u_i^b.$$

After signal y in state k, continuation payoffs in $\widetilde{\sigma}$ are equal to

$$-(1-\delta)p_i^k(y) + \delta u_i(\sigma).$$

Hence, actions in $\widetilde{\sigma}$ are incentive compatible and the individual punishment payoffs of $\widetilde{\sigma}$ are equal to $v_i(\sigma)$. It is also straightforward to show that payments are incentive compatible. By varying the up-front payments in $\widetilde{\sigma}$ all divisions of the surplus $U(\widetilde{\sigma})$ in which each player gets at least v_i can be achieved. Moreover, the correlation device can be used in the up-front payment state to achieve all joint payoffs between U and U^b.

Second, we show that the set of PPE payoffs without money burning is a subset of the set defined in (12).

Let $\bar{U}$ and $\bar{U}^b$ denote the highest and lowest joint payoff that can be implemented with some PPE in the repeated game without money burning. Similarly, let $\bar{v}_i$ denote the lowest payoff for player i that can be implemented with some PPE. Let σ^e be a PPE with $U(\sigma^e) = \bar{U}$, σ^b a PPE with $U(\sigma^b) = \bar{U}^b$ and for every player i, let σ^i denote a PPE with $u_i(\sigma^i) = \bar{v}_i$. For all $k = e, b, 1, ..., n$ let a^k be the first action profile played on the equilibrium path of σ^k. Note that it always holds true that

$$G(a^b) \leq \bar{U}^b \text{ and } \bar{U} \leq G(a^e).$$

Let $w^k(y)$ denote the vector of continuation payoffs after signal y has been realized in the first period according to $\widetilde{\sigma}^k$ and define

$$p_i^k(y) = \frac{\delta u_i(\sigma^e) - w_i^k(y)}{1-\delta}.$$

That the action, payment and budget constraints are satisfied follows as in the proof of Theorem 1. To see that money burning constraints (MBC-k) are satisfied note that

$$\sum_{i=1}^n p_i^k(y) = \frac{\delta \bar{U} - \sum_{i=1}^n w_i^k(y)}{1-\delta} \leq \frac{\delta}{1-\delta}\left(U - U^b\right).$$

Hence, $(a^k, p^k)_{k=e,b,1,\ldots,n}$ solves LP-OPS-LMB with value $\bar{U} - \Sigma_{i=1}^n \bar{v}_i - \bar{U}^b$. ■

6.1 Characterization based on static problems with enforceable payments

We can derive similar links to static problems than in games with unlimited money burning. Consider the static problem of Section 3.2, with the extra restriction that there is an upper bound $B \geq 0$ on the amount of money that is allowed to be burned after any signal y. We denote by $L(a, B)$ the liquidity requirement of an action profile with that upper bound on money burning:

$$L(a,B) = \min_{p(.)} L \text{ s.t. } (PC), (AC), (BC) \text{ and} \qquad \text{(LP-B-L)}$$

$$\sum_{i=0}^n p_i(y) \leq B \text{ for all } y \in Y \qquad \text{(MBC)}$$

Similarly, we define for all $L \geq L(a, B)$ and $0 \leq B \leq L$ the highest joint equilibrium payoff in the static problem by

$$U^e(L,B,a) = \max_{p(.)} \left(G(a) - \sum_{i=1}^n E[p_i|a] \right) \qquad \text{(LP-B-e)}$$
$$\text{s.t. } \quad \text{(PC),(BC), (AC), (MBC),}$$

the lowest collective punishment payoff by

$$U^b(L,B,a) = \min_{p(.)} \left(G(a) - \sum_{i=1}^n E[p_i|a] \right) \qquad \text{(LP-B-b)}$$
$$\text{s.t. } \quad \text{(PC),(BC), (AC), (MBC),}$$

and player i's punishment payoff by

$$\begin{aligned} v^i(L,B,a) &= \min_{p(.)} \left(g_i(a) + \lambda_i L - \sum_{i=1}^{n} E[p_i|a] \right) \qquad \text{(LP–B-i)} \\ \text{s.t.} & \quad \text{(PC),(BC), (AC) and (MBC).} \end{aligned}$$

The corresponding upper and lower envelopes over all action profiles are denoted by

$$\begin{aligned} \bar{U}^e(L,B) &= \max_{a\in A} U^e(L,B,a), \\ \bar{U}^b(L,B) &= \min_{a\in A} U^b(L,B,a), \\ \overline{v}_i(L,B) &= \min_{a\in A} v^i(L,B,a). \end{aligned}$$

The profiles at which these values are attained are denoted by $\overline{a}^k(L,B)$. We say a pair (L,B) of liquidity and bound on money burning can be generated by a discount factor δ if

$$\begin{aligned} \max_{k=e,0,1,\ldots,n} L(\overline{a}^k(L,B),B) \leq L &\leq \frac{\delta}{1-\delta}\left(U^e(L,B) - V(L,B)\right), \\ B &\leq \frac{\delta}{1-\delta}\left(U^e(L,B) - U^b(L,B)\right). \end{aligned}$$

Let (L^*,B^*) denote the (element-wise) largest pair of liquidity and bound on money burning that can be generated. If some pair (L,B) can be generated, a largest such pair must always exists, since larger levels of B allow larger consistent levels of L and vice versa.

Proposition 5 *Let (L^*,B^*) be the largest consistent liquidity and bound on money burning given discount factor δ. The set of equilibrium payoffs that can then be implemented are*

$$\left\{ u \in \mathbb{R}^n \mid \bar{U}^b(L^*,B^*) \leq \sum_{i=1}^{n} u_i \leq \bar{U}^e(L^*,B^*) \text{ and } u_i \geq v^i(L^*,B^*) \text{ for all } i \right\}. \tag{14}$$

Proof. The proof proceeds similarly as the proof of Theorem 2 and is therefore omitted. ■

To compute the functions $U^e(L,B,a)$, $U^b(L,B,a)$ and $v^i(L,B,a)$ for all $L \geq L(a,B)$ and $B \leq L$ one can exploit the fact that their surface is described by a finite number of planar segments, which can be characterized by methods of parametric linear programming and sensitivity analysis (see, e.g., Gal and Nedoma, 1972). The computations can take considerably longer than computing the one-dimensional functions for the case of unlimited money burning. Still, one may be able to obtain closed-form solutions for simple signal structures.[15] Once $\bar{U}^e(L, B)$ and $v^i(L, B)$ are fully characterized, optimal action structures for all discount factors can be very quickly obtained.

A sufficient condition for the equilibrium payoff set not to be affected by the possibility to burn money, is that a single stage game Nash equilibrium a^b is an optimal punishment profile for all players. Both the collective punishment payoff U^b and the sum of individual punishment payoffs V are then equal to $G(a^b)$ and the payment constraints imply the money burning constraints. Hence, our characterization of the payoff sets in the noisy prisoners' dilemma game remains valid even if no money burning is allowed. In addition, we have already found that the restriction not to burn money does not shift the Pareto frontier of the set of equilibrium payoffs in games with perfect monitoring.

7 Conclusion

In this paper, we presented a characterization of equilibrium payoff sets for infinitely repeated games with public monitoring and monetary transfers. Monetary transfers are a realistic assumption and at the same time greatly simplify the analysis. Our results can be used to numerically compute the equilibrium payoff sets for any finite stage game and they also facilitate the finding of closed-form analytical solutions.

One interesting direction for future work is to study to which extend monetary transfers, in conjunction with communication, allow a tractable characterization of payoff sets for games with private monitoring or for the set of mixed strategy equilibrium payoffs in games with public monitoring. The problem becomes considerably more complicated, since it is not necessarily optimal to use a payment plan that induces full information revelation in every period (see, e.g. Fuchs, 2007, for an analysis in a principal agent framework).

Another direction for future research is to study optimal renegotiation-proof

[15] For example, in the noisy prisoner's dilemma game and the action profile $a = CC$, we find $L(a, B) = \frac{1}{\beta_P}(2d - B\beta_A)$ and $U^e(L, B, a)$ is given as in equation (9).

equilibria in a framework with monetary transfers and imperfect public monitoring. If we would only consider stationary equilibria, a natural, minimal renegotiation-proofness requirement is that after no history there shall be money burning. An interesting question is whether there is a concept of renegotiation-proofness for which every renegotiation-proof payoff can be implemented with a stationary equilibrium without money burning.

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Appendix A: Computing $U^e(L,a)$ and $\overline{U}^e(L)$

This appendix illustrates how $U^e(L,a)$ and $\overline{U}^e(L)$ can be exactly computed and describes heuristics to reduce computation time. Similar methods can be applied to the computation of $v_i(L,a)$ and $\overline{v}_i(L)$.

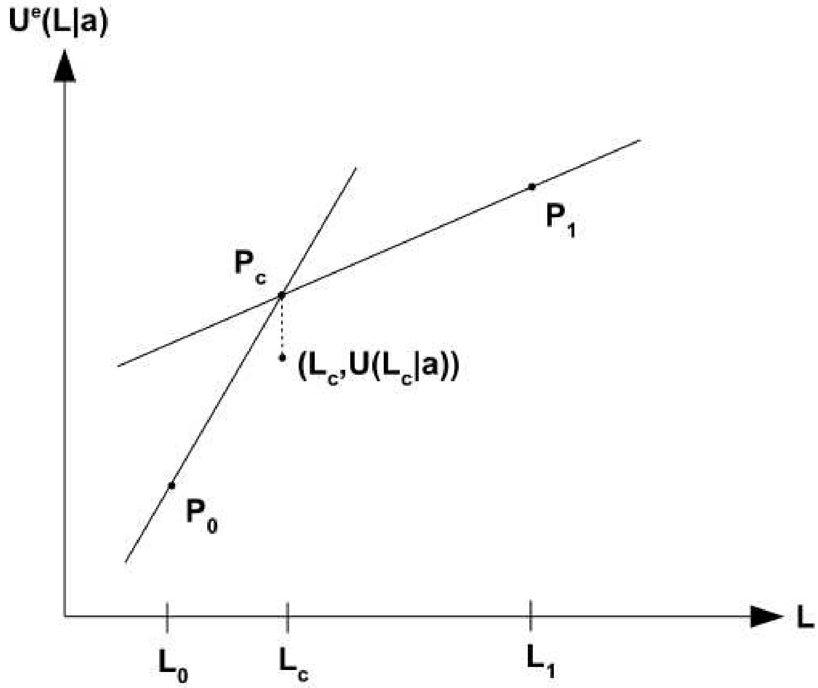

Figure 2: Constructing $U^e(L|a)$

Calculating $\mathbf{U}^e(\mathbf{L}, \mathbf{a})$ Assume that we have calculated $U^e(L, a)$ at two different levels $L_0 < L_1$ illustrated by the points P_0 and P_1 in Figure 2. We describe a procedure that fully computes $U^e(L, a)$ on the interval $[L_0, L_1]$. From the dual values of the solution of the problem (LP-e) we can get the slope of $U^e(L, a)$ at L_0 and L_1.[16] Figure 2 illustrates the corresponding tangents. The two tangents either coincide or have a cut point $P_c = (L_c, U_c)$ with $L_0 < L_c < L_1$ and $U_0 < U_c < U_2$. In the first case, $U^e(L, a)$ is given on the interval $[L_0, L_1]$ by the line $\overline{P_0 P_1}$. In the second case the line $\overline{P_0 P_c P_1}$ constitutes an upper bound on $U^e(L, a)$. We calculate $U^e(L_c, a)$. If $U^e(L_c, a) = U_c$ then $U^e(L, a)$ coincides with this upper bound $\overline{P_0 P_c P_1}$. Otherwise, we proceed recursively by calculating $U^e(L, a)$ on the two intervals $[L_0, L_c]$ and $[L_c, L_1]$. If there are $n_k \geq 2$ kinks between L_0 and L_1, this procedure fully characterizes the function $U^e(L, a)$ on the interval by solving

[16] If $U^e(L|a)$ has a kink at L, it depends on the way the linear program is set up, whether the dual values delivers the right hand or left hand slope. It is no problem to calculate, the correct slope, however.

at most $2(n_k - 1) + 1$ times the linear program (LB-e). To quickly solve (LP-e) at different levels of L, one can use standard re-optimization techniques, e.g. based on the dual simplex algorithm.[17]

The lowest possible level of L is given by the liquidity requirement $L(a)$. The right hand starting point of our procedure is given by the minimal liquidity $\overline{L}^e(a)$ above which $U^e(L, a)$ does not anymore increase in L. We can calculate $\overline{L}^e(a)$ by adding a restriction on the maximal allowed expected amount of money burning in the problem (LP-L).[18]

Calculating the upper envelope $\overline{U}^e(L)$ For the calculation of the upper envelope $\overline{U}^e(L)$, let us define by

$$U^e(L, \widetilde{A}) = \max_{a \in \widetilde{A}} U^e(L, a)$$

the upper envelope with respect to a subset of action profiles $\widetilde{A} \subseteq A$. Hence, we have

$$U^e(L, \widetilde{A} \cup \{a\}) = \max\{U^e(L, \widetilde{A}), U^e(L, a)\}.$$

We can calculate $\overline{U}^e(L)$ by subsequently adding all action profiles to the set $\widetilde{A}$. To calculate the new envelope $U^e(L, \widetilde{A} \cup \{a\})$, it is often not necessary to compute the whole function $U^e(L, a)$. Recall, that the method to calculate $U^e(L, a)$ delivers in each step an upper bound on $U^e(L, a)$. It suffices to proceed the calculation of $U^e(L, a)$ only for those values of L where the upper bound exceeds $U^e(L, \widetilde{A})$.

If an upper bound of $U^e(L, a)$ lies everywhere below $U^e(L, \widetilde{A})$, we can immediately dismiss the action profile a. Since $U^e(L, a)$ is bounded by $G(a)$, a sufficient condition to dismiss a is that $G(a) \leq U^e(L(a), \widetilde{A})$. A weaker sufficient condition is $G(a) \leq U^e(\widetilde{L}(a), \widetilde{A})$, where $\widetilde{L}(a) \equiv \sum_{i=1}^{n} (c_i(a) - g_i(a))$ is the liquidity requirement under perfect monitoring, which always satisfies $\widetilde{L}(a) \leq L(a)$. The last

[17] Moreover, using a simplex algorithm, the case $U^e(L_c, a) = U_c$ can sometimes be verified without the need of solving the linear program (LP-e) at L_c. A sufficient condition for $U^e(L_c, a) = U_c$ is that the the optimal (dual) basis of the solved problem at L_0 (or L_1) remains an optimal basis at L_c. This condition can be checked with standard formulas used to calculate sensitivity bounds. However, it can happen that the optimal basis changes between L_0 and L_c even though the function $U^e(L|a)$ has no kink between L_0 and L_c.

[18] If the full-dimensionality condition of the folk theorem by Maskin, Fudenberg and Levine (1994) holds we must impose zero money burning to calculate L. Otherwise, we first have to solve the problem (LB-e) with unlimited liquidity to calculate the minimally required amount of money burning.

condition can be checked very quickly since no linear program has to be solved for a.

The order in which action profiles are added to $\widetilde{A}$ can influence the total computation time, because action profiles can be more quickly dismissed if $U^e(L, \widetilde{A})$ is already large. One should first add all Nash equilibria of the stage game, which satisfy $U^e(L, a) = G(a)$ for all $L \geq 0$. An educated guess about which optimal action profiles are likely to be optimal, e.g. symmetric ones, can be furthermore helpful.

Punishment states Similar methods can be used to calculate $v_i(L, a)$ and $\overline{v}_i(L)$. For the computation of $\overline{v}_i(L)$, it is helpful to first add to $\widetilde{A}$ all those action profiles a where a_i is a best-reply to a_{-i}, since these action profiles satisfy $v_i(L, a) = g_i(a)$ for all $L \geq L(a)$.