The accuracy of numerical solutions for dynamic GEI models*

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September 2, 2003

Abstract

This paper develops theoretical foundations for the computation of competitive equilibria in dynamic stochastic general equilibrium models with heterogeneous agents and incomplete financial markets. While there are several algorithms which compute prices and allocations for which agents' first order conditions are approximately satisfied ('approximate equilibria'), there are few results on how to interpret the errors in these candidate solutions and how to relate the computed allocations and prices to exact equilibrium allocations and prices. Following Postlewaite and Schmeidler (1981) we interpret approximate equilibria as equilibria for close-by economies, i.e. for economies with close-by individual endowments and preferences.

In order to conduct an error analysis in dynamic stochastic general equilibrium models, we define an ϵ-equilibrium to be a set of endogenous variables which consists of the finite support of an approximate equilibrium process. Given an ϵ-equilibrium we show how to derive bounds on perturbations in individual endowments and preferences which ensure that the ϵ-equilibrium approximates an exact equilibrium for the perturbed economy.

We give a detailed discussion of the error analysis for two models which are commonly used in applications, an OLG model with stochastic production and an asset pricing model with infinitely lived agents. We illustrate the analysis with some numerical examples. It is shown that in these examples the derived bounds are not more than one order of magnitude higher than maximal errors in Euler equations.

*We thank seminar participants at Stanford University, the Illinois Conference in Economic Theory, the European General Equilibrium Conference in Bielefeld, the 2003 SED conference in Paris, the 2003 SAET conference on Rhodes, the 2003 SITE workshop on computational economics and especially Peter Hammond, Martin Hellwig, Ken Judd, Mordecai Kurz, Alvaro Sandroni and Manuel Santos for discussions and very useful comments.

1 Introduction

The computation of equilibria in dynamic stochastic general equilibrium models with heterogeneous agents has become increasingly important in finance, macroeconomics and public finance. Many economic insights can be obtained through analyzing quantitative features of realistically calibrated models (prominent examples in the literature include e.g. Rios-Rull (1996), Heaton and Lucas (1996) or Krusell and Smith (1997)).

Unfortunately there are often no theoretical foundations for algorithms which claim to compute competitive equilibria in models with incomplete markets or overlapping generations. In particular, since all computation suffers from truncation and rounding errors it is obviously not possible (as some applied researchers claim) to numerically verify that the optimality and market clearing conditions are satisfied and that a competitive equilibrium is found. The concept of an exact solution is meaningless and the fact that the equilibrium conditions are approximately satisfied generally does not imply anything about how well the computed solution approximates an exact equilibrium. Computed allocations and prices could be arbitrarily far from competitive equilibrium allocations and prices.

In this paper we develop an error analysis for the computation of competitive equilibria in models with heterogeneous agents where equilibrium prices are infinite dimensional. We define an ϵ-equilibrium as a collection of finite sets of choices and prices such that there exists a (measurable with respect to the stochastic structure of the model) process of prices and choices which takes values exclusively in this set and for which the relative errors in agents' Euler equations and the errors in market clearing conditions are below ϵ at all times.

Existing algorithms for the computation of equilibria in dynamic models can be interpreted as computing ϵ-equilibria and the finiteness of ϵ-equilibria allows us to computationally verify if a given collection of endogenous variables (i.e. a candidate solution) constitutes an ϵ-equilibrium. In order to give an economic interpretation of the concept we follow (at least in spirit) Postlewaite and Schmeidler's (1981) analysis for finite economies and interpret ϵ- equilibria as approximating exact equilibria of a close-by economy.

In finite economies the problem of interpreting ϵ-equilibria is easiest illustrated in a standard Arrow Debreu exchange economy. Scarf (1967) proposes a method which 'approximates' equilibria for any given finite economy in the following sense: Given individual endowments e^i for individuals $i = 1, ..., I$ and an aggregate excess demand function $\xi(p, (e^i))$, and given an $\epsilon > 0$, the methods finds a $\bar{p}$ such that $\|\xi(\bar{p}, (e^i))\| < \epsilon$. As Richter and Wong (1999) point out[1] this does not imply that it is possible to find a $\tilde{p}$ such that $\|\tilde{p} - p^*\| < \epsilon$ for some exact equilibrium price vector p^*.

However, if individual endowments are interior and if the value of the excess demand

[1]They examine the problem of the computation of equilibria from the viewpoint of computable analysis as developed by Turing (1936) and point out that while Scarf's algorithm generates a sequence of values converging to a competitive equilibrium knowing any finite initial sequence might shed no light at all on the limit.

function at $\bar{p}$,$\|\xi(\bar{p},(e^h))\|$, is small, $\bar{p}$ is an equilibrium price for a close-by economy. Homogeneity of aggregate excess demand implies trivially that if $\bar{p}\cdot\xi(\bar{p},(e^h)) = 0$ then $\|(\bar{p},(e^h)) - (p^*,(\tilde{e}^h))\| < \epsilon$ with $\xi(p^*,(\tilde{e}^h)) = 0$. It is possible that $\bar{p}$ is not a good approximation for the equilibrium price of the given economy. However researchers rarely know the exact individual endowments of agents anyway, and if close-by specifications of exogenous variables lead to vastly different equilibria it will be at least useful to know one possible equilibrium for one realistic specification of endowments. As Postlewaite and Schmeidler (1981)[2] put it, "If we don't know the characteristics, but rather, we must estimate them, it is clearly too much to hope that the allocation would be Walrasian with respect to the estimated characteristics even if it were Walrasian with respect to the true characteristics."

From a viewpoint of computational mathematics this has been well understood for a long time. In general, sources of errors in computations can be classified in three categories:

1. Errors due to the theory: The economic model contains many idealizations and simplifications.

2. Errors due to the specification of exogenous variables: The economic model depends on parameters which are themselves computed approximately, the results of experimental measurements or the results of statistical procedures.

3. Truncation and rounding errors: each limiting process must be broken off at some finite stage, computers usually use floating point arithmetic resulting in round-off errors.

In contrast to standard error analysis, which aims to bound the distance of the approximate solution to the exact solution, 'backward error analysis' exploits a tradeoff between 2 and 3 and examines how much the given problem would have to be perturbed in order for the calculated solution to be an exact solution of the perturbed problem (see e.g. Chaitin-Chatelin and Frayssé (1996) or Higham (1996)). While in the applied economic literature which uses computations there is a large debate about the trade-off between 1 and 3, there is surprisingly little about a possible trade-off between 2 and 3. This paper explores how this latter tradeoff can be used to interpret approximate solutions to dynamic general equilibrium models via backward error analysis.

We examine two concrete applications where we take as given that standard algorithms compute values for the endogenous variables for any possible sequence of exogenous shocks. We describe a method to construct a ϵ-equilibrium from these variables. Although our definition of recursive equilibrium is discrete, it turns out to be very useful to use continuous algorithms to compute the ϵ-equilibria in practice. In particular we examine algorithms which assume that approximate policy and pricing functions are smooth. We show that

[2] While their definition of an ϵ-equilibrium is different than the one in this example, the spirit of their analysis is the same.

in these applications our methods lead to reasonable and economically meaningful error bounds.

For models with a single agent Santos and his co-authors have developed such sufficient conditions and give explicit error bounds both on policy functions and on allocations (Santos and Vigo (1998), Santos (2000), and Santos and Peralta-Alva (2002)). While even in their framework these conditions do not hold for all interesting specifications of the model, in applications, the conditions can often be verified. Under these conditions, error bounds on allocations can be derived from Euler equation residuals. However, most of these results do not generalize to models with heterogeneous agents and incomplete markets. No sufficient conditions are known which allow the derivation of error bounds on computed equilibrium prices and allocations in the models considered in this paper.

Backward error analysis is a standard tool in numerical analysis that was developed in the late 1950s and 1960s. It is surprising that it has, to the best of our knowledge, not been widely used in economics. For example, Judd's textbook (1998) mentions backward error analysis and provides a citation from the numerical analysis literature but never applies the concept to an economic problem. The only somewhat related concept in economics is "backsolving" which was introduced by Sims (1989) for solving nonlinear, stochastic systems. Ingram (1990) describes backsolving from an econometric viewpoint. The endogenous variables in a stochastic dynamic optimization problem are affected by random shocks. Instead of taking a distribution of shocks as given and then solving for the distribution of the endogenous variables, backsolving begins by specifying a convenient or intuitive distribution for some of the endogenous variables and then attempts to find underlying distributions of random shocks and other variables that would yield the assumed distributions of the endogenous variables. Note that this approach is different from backward error analysis because it does not address the question how far away the exogenous distribution is from some desired or estimated one. In backward error analysis exogenous parameters are given, then an approximate solution is computed, and then the necessary perturbations in exogenous parameters are determined. Clearly, we always would like to have very small backward errors. In fact, the focus of our analysis of popular models in Sections 4 and 5 of this paper is the calculation of backward errors. Due to the nature of economic problems we cannot perform "pure" backward error analysis and only perturb exogenous parameters. Instead, we will compute bounds on perturbations of both exogenous parameters and endogenous equilibrium values.

The paper is organized as follows. In Section 2 we outline an abstract dynamic model and define what we mean by close-by economies. In Section 3 we develop the theoretical foundations of our method. In Section 4 we apply these methods to a model with overlapping generations and production. In Section 5 we apply the methods to a version of Lucas (1978) asset pricing model with heterogeneous agents.

2 A general model

In this section we fix the main ideas in an abstract framework which encompasses both economies with overlapping generations and economies with infinitely lived agents as well as economies with and without production. In Sections 4 and 5 below we consider two standard models and show how to apply the methods developed in this and the next section.

2.1 The abstract economy

Time and uncertainty are represented by a countably infinite tree Σ. Each node of the tree, $\sigma \in \Sigma$, is a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ for a given initial shock s_0. The process of shocks (s_t) is assumed to be a Markov chain with finite support $\mathcal{S}$. If $s^{t'}$ is a successor of s^t we write $s^{t'} \succ s^t$. The number of elements in $\mathcal{S}$ is S. Given an $S \times S$ transition matrix Π, we define probabilities for each node by $\pi(s_0) = 1$ and $\pi(s^t) = \Pi(s_t|s_{t-1})\pi(s^{t-1})$ for all $t \geq 1$.

There are L commodities, $l \in \mathcal{L}$, at each node. As it is commonly done in the dynamic GEI literature (see for example in Magill and Quinzii (1994)) we take the commodity space to be

$$\ell_\infty(\Sigma, \mathcal{L}) = \{((x_1(\sigma), ..., x_L(\sigma)) : \sup_{(\sigma,l) \in \Sigma \times \mathcal{L}} |x_l(\sigma)| < \infty\}.$$

There are countably many individuals $i \in \mathcal{I}$ and countably many firms $k \in \mathcal{K}$. An individual $i \in \mathcal{I}$ is characterized by his consumption set X^i, his individual endowments $e^i \in X^i \subset \ell_\infty$, his preferences $P^i \subset X^i \times X^i$ (where $P^i = \{(x, y) \in X^i \times X^i : x \succeq^i y\}$) and trading constraints. A firm $k \in \mathcal{K}$ is characterized by its production set Y^k. An economy $\mathcal{E}$ is characterized by a demographic structure, assets, technologies and preferences, endowments and trading constraints. In the concrete models below we describe $\mathcal{E}$ explicitly.

The original economy is assumed to be Markov. The number of agents active in markets at a given node is finite and time-invariant but might depend on the underlying shock, agents maximize time and state-separable utility, firms only make decisions on spot markets and all individual endowments, payoffs of assets, production sets of firms and spot utility functions of individuals are time-invariant functions of the shock, s, alone. Since there are finitely many shocks, this allows us to describe the economy by finitely many spot utility functions, production sets, endowment vectors and asset payoffs.

2.2 Close-by economies

As explained in the introduction we are interested in analyzing equilibria of economies $\mathcal{E}'$ which are close-by to an original Markovian economy $\mathcal{E}$ in the sense that all individuals' endowments and preferences are close-by. In order to formalize this, we index economies by preferences and endowments, i.e. we write $\mathcal{E} = (P^\mathcal{I}, e^\mathcal{I})$, where $P^\mathcal{I}$ denotes the profile of preferences across agents and $e^\mathcal{I}$ denotes the profile of individual endowments.

We need to define a metric on economies, i.e. distances for preferences and for endowments. Throughout the paper, for a vector $x \in I\!R^n$, $\|x\|$ denotes the sup-norm,

$$\|x\| = \max\{|x_1|, \ldots, |x_n|\}.$$

For an element of the commodity space $x \in \ell_\infty$ we define

$$\|x\| = \sup_{(\sigma,l)\in\Sigma\times\mathcal{L}} \|x_l(\sigma)\|.$$

Following Postlewaite and Schmeidler (1981) and Debreu (1969) we use the Hausdorff distance to define closeness of two preferences P and P':

$$d^H(P, P') = \max\left\{\sup_{x\in P}\left(\inf_{y\in P'}\|x-y\|\right), \sup_{x\in P'}\left(\inf_{y\in P}\|x-y\|\right)\right\}.$$

We define a distance between the economies

$$d(\mathcal{E}, \mathcal{E}') = \max_{i\in\mathcal{I}}\left(\max\left\{\|e^i - e^{i\prime}\|, d^H(P^i, P^{i\prime})\right\}\right).$$

We want to parameterize economies by node dependent perturbations $o(\sigma) \in \mathcal{O} \subset I\!R^N$ and write $\mathcal{E}(o(\sigma)_{\sigma\in\Sigma})$ for a given (possibly non-stationary) perturbed economy. In the original economy $o(\sigma) = 0$ for all $\sigma \in \Sigma$.

In many applications we are interested in examining close-by economies with identical preferences. In these cases $o(\sigma)$ are additive perturbations of endowments of individuals which are active in markets at node σ. While small differences in individual endowments are easy to interpret, small differences in preferences are much harder to quantify. However, in some cases (e.g. when endowments are specified to lie on the boundary and we do not want to consider interior endowments) we need to perturb preferences.

2.2.1 Perturbations of preferences

One has to carefully model how node-dependent perturbations in utility functions lead to preferences which are close by the original preferences as defined above. For simplicity we focus on the case where agents are infinitely lived and their consumption sets are infinite dimensional. The case of finitely lived agents follows immediately from this case.

We assume throughout the paper that preferences can be represented by a time-separable expected utility function. For an infinitely lived agent i there exists a Bernoulli function $u^i : I\!R^L_+ \times \mathcal{S} \to I\!R$ as well as beliefs Π^i and a discount factor β_i such that if we define

$$U^i(x) = \sum_{t=0}^{\infty} \beta_i^t \sum_{s^t} \pi^i(s^t) u^i(x(s^t), s^t),$$

we have

$$(x, y) \in P^i \text{ if and only if } U^i(x) \geq U^i(y).$$

In the original economy, Bernoulli utilities only depend on the current shock, i.e. $u^i(x, s^t) = u^i(x, s_t)$. We assume that each agents' consumption set X^i is bounded[3] and define

$$\bar{c} = \sup_{x \in X^i, i \in \mathcal{I}} \|x\|.$$

We assume that each u^i is continuously differentiable, strictly increasing and concave in x and that $\|D_x u^i(x, s)\|$ is bounded below by $\bar{m} > 0$ for all $x \in I\!R^L$, $\|x\| \leq \bar{c}$, and all $s \in \mathcal{S}$.

We can restrict attention to perturbations in Bernoulli utilities, as long as we want the perturbed preferences to also be time- and state-separable. Perturbations in probabilities $\pi^i(s^t)$ or node-dependent perturbations in the discount factor, β, just result in a multiplicative perturbation of $u^i(., s^t)$. Note that small perturbations of conditional probabilities at each node might propagate to large perturbations of $\pi^i(s^t)$ for large t.

As is often done in general equilibrium analysis (see e.g. Mas-Colell (1985)) we consider linear perturbations of utility functions. Given $u^i(x, s_t)$ and $o^i(s^t) \in \mathbb{R}^L$, the perturbed Bernoulli utility is

$$\tilde{u}^i(x, s^t) = u^i(x, s^t) + o^i(s^t) \cdot x$$

If P^i denotes the original preferences, we denote the implied perturbed preferences by $\tilde{P}^i$ which we represent by the utility function

$$\tilde{U}^i(x) = \sum_{t=0}^{\infty} \beta_i^t \sum_{s^t} \pi^i(s^t)(u^i(x(s^t), s^t) + o^i(s^t) \cdot x(s^t)).$$

The following lemma gives bounds on $d^H(P^i, \tilde{P}^i)$.

Lemma 1 *Given perturbations $(o^i(\sigma))_{\sigma \in \Sigma}$, define $\bar{\omega} = \sup_\sigma \|o^i(\sigma)\|$. Then a bound on the distance between original and perturbed preferences is as follows.*

$$d^H(P^i, \tilde{P}^i) \leq L\bar{c}\frac{\bar{\omega}}{\bar{m}}.$$

The proofs of lemmas are collected in the appendix.

It is clear that as $\|(o^i(\sigma))_{\sigma \in \Sigma}\| \to 0$ we have that $d^H(P^i, \tilde{P}^i) \to 0$. Moreover, the bound in the lemma is invariant to affine transformations in u^i.

2.3 Equilibrium

A competitive equilibrium for the economy $\mathcal{E}(o(\sigma))_{\sigma \in \Sigma})$ is a process of endogenous variables $(z(\sigma))_{\sigma \in \Sigma}$ with $z(\sigma) \in \mathcal{Z} \subset \mathbb{R}^M$, which solve agents' optimization problems and clear markets. We refer to the collection of the economy and the endogenous variables, $(\mathcal{E}((o(\sigma))_{\sigma \in \Sigma}), (z(\sigma))_{\sigma \in \Sigma})$, as an 'economy in equilibrium'.

[3] In the model we consider aggregate endowments are always bounded. While it is true that an agent may contemplate consumption boundles that exceed aggregate endowments, it simplifies the analysis considerably to restrict preferences to be defined over bounded sets.

2.3.1 The expectations correspondence

For the computation of competitive equilibria it is important that equilibrium conditions can be summarized in a set of inequalities which relate current period exogenous and endogenous variables to endogenous and exogenous variables one period ahead. Duffie et al. (1994) describe this relation via an *expectations correspondence*. We use their terminology but slightly alter the concept for our specific purposes.

We restrict attention to economies where a time invariant expectations correspondence can encompass all conditions for agents' optimality and market clearing. A competitive equilibrium can then be characterized by an expectations correspondence which maps endogenous variables today to possible (i.e. consistent with individuals' Euler equations and market clearing) endogenous variables and perturbations of the fundamentals at the S possible shocks next period. That is we want to be able to define a correspondence

$$H : \mathcal{S} \times \mathcal{Z} \rightrightarrows \bigotimes_{s \in \mathcal{S}} (\mathcal{O} \times \mathcal{Z}),$$

where $(z(\sigma))_{\sigma \in \Sigma}$ is an equilibrium for $\mathcal{E}((o(\sigma))_{\sigma \in \Sigma})$ if for all $s^t \in \Sigma$,

$$(o(s^t 1), z(s^t 1), \ldots, o(s^t S), z(s^t S)) \in H(s_t, z(s^t)).$$

We assume furthermore that elements in the graph of the expectations correspondence can be characterized as (part of) a solution to a system of equations, i.e. we assume that there exists a set $\mathcal{K} \subset \mathbb{R}^K$, and a function

$$h : \mathcal{S} \times \mathcal{Z} \times \mathcal{K} \times (\bigotimes_{s \in \mathcal{S}} (\mathcal{O} \times \mathcal{Z})) \to I\!R^L$$

such that $(o_1, z_1, \ldots, o_S, z_S) \in H(\hat{s}, \hat{z})$ if and only if there exists $\kappa \in \mathcal{K}$ such that

$$h(\hat{s}, \hat{z}, \kappa, o_1, z_1, \ldots, o_S, z_S) = 0.$$

In this formulation the variables $\kappa \in \mathcal{K}$ should be thought of representing slack variables in inequalities or Kuhn-Tucker multipliers. In the applications below the functions h consist of individuals' intertemporal Euler equations, market clearing equations and first order conditions for spot optimality. We refer to h as the equilibrium equations.

3 Approximate equilibria and their interpretation

As mentioned in the introduction, we want to give conditions which allow us to interpret the results of algorithms used in practice. It is therefore useful to define a notion of ϵ-equilibrium which is general enough that it exists in most interesting specifications of the model and that is tractable in the sense that actual approximations in the literature can be interpreted as such ϵ-equilibria (or at least that ϵ-equilibria can be constructed fairly easily from the output of commonly used algorithms).

DEFINITION 1 *An ϵ-equilibrium is a finite sets $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_S$, $\mathcal{F}_s \subset \mathcal{Z}$ for all $s = 1, ..., S$, such that for all $\hat{s} \in \mathcal{S}$ and all $\hat{z} \in \mathcal{F}_{\hat{s}}$ there exist $(z_1, \ldots, z_S) \in \mathcal{F}$ such that*

$$\min_{\kappa \in \mathcal{K}} \|h(\hat{s}, \hat{z}, \kappa, , z_1, ..., , z_S)\| < \epsilon. \tag{1}$$

In most interesting models one can show existence of ϵ-equilibria for all $\epsilon > 0$ (the existence of a competitive equilibrium is often a sufficient but not a necessary condition for the existence of ϵ-equilibria). We define an ϵ-equilibrium as a finite collection of points because we want to be able to verify whether a candidate solution constitutes an ϵ-equilibrium. Evidently this only involves checking finitely many inequalities.

For readers familiar with computational recursive methods it may be useful to interpret an ϵ-equilibrium as a generalization of an approximate policy function of a 'recursive equilibrium'.

3.1 Recursive methods

The applied computational literature often refers to recursive equilibria. These equilibria are characterized by policy functions which map the current 'state' of the economy into choices and prices and by transition functions which map the state today into a probability distribution over the next period's state. While in dynamic GEI models, recursive equilibria do not always exist and no non-trivial assumptions are known which guarantee the existence of recursive equilibria (for counterexamples to existence see e.g. Hellwig (1982), Kubler and Schmedders (2002) and Kubler and Polemarchakis (2003)) recursive methods are useful for computational purposes. Recursive ϵ-equilibria exist whenever ϵ-equilibria exist and formulating these ϵ-equilibria recursively faciliates the notation and the error analysis. We therefore now define a recursive ϵ-equilibrium formally.

The relevant endogenous state space $\Theta \subset \mathbb{R}^D$ is a subset of $\mathcal{Z}$ and depends on the underlying model - it is determined by the payoff-relevant variables; that is, by variables, endogenous as well as exogenous, sufficient for the optimization of individuals at every date-event. If $\Theta \subset \mathcal{Z}$ is the 'endogenous state space' there must exist sets $\mathcal{Z}_1^*, ..., \mathcal{Z}_S^* \subset \mathbb{R}^{M-D}$ such that for all $s \in \mathcal{S}$, $\mathcal{F}_s = \Theta \times \mathcal{Z}_s^*$. The value of the state variables $(s_0, \theta_0) \in \mathcal{S} \times \Theta$ in period 0 is called 'initial condition' and is part of the description of the economic model. A recursive ϵ-equilibrium is then defined as follows.

DEFINITION 2 *Given an ϵ-equilibrium $\mathcal{F}$ and a state space Θ, a recursive ϵ equilibrium consists of a policy function $\rho : \mathcal{S} \times \Theta \to \mathbb{R}^{M-D}$ such that*

$$\mathcal{F}_s = \textit{graph}(\rho_s) \text{ for all } s \in \mathcal{S}.$$

By the discrete nature of ϵ-equilibria it is clear that these can always be described as the graph of a policy function. In the following we will always assume that a given ϵ-equilibrium also has a recursive representation.

In the contexts of recursive equilibria den Haan and Marcet (1994) and Judd (1998) suggest to evaluate the quality of a candidate solution using these Euler-equation residuals. In these methods relative maximal errors in 'Euler-equations' of ϵ usually imply that the solution describes an ϵ-equilibrium.

3.2 Error analysis

However, ϵ-equilibria defined by condition (1) are very difficult to interpret. What does an ϵ-equilibrium describe for, say $\epsilon = 0.001$? Should this be regarded as a good approximation or as a bad one ? Several authors (e.g. Judd (1998)) justify ϵ-equilibria (or maximal relative errors in Euler equations) as a measure of quality of a solution via a bounded rationality argument. They argue that economic agents have bounded computational capacity and can only find approximately optimal choices. Any improvement over their choices results in an extra gain of at most ϵ. However, in dynamic general equilibrium models with rational expectations there is a tension between assuming that agents have rational expectations about future prices and assuming that they make errors in choosing their consumptions given the correct forecasts for prices. Furthermore, since markets clear exactly, future prices must already reflect the agents' optimization errors. We therefore want to move away from a bounded rationality justification and interpret ϵ-equilibria as approximating exact equilibria of a close-by economy.

In order to formalize such an interpretation we first need to discuss what it means to approximate an infinite dimensional equilibrium by a finite set. Given an initial value of the shock, s_0, initial values for the endogenous state, θ_0, and implied initial endogenous variables $z^\epsilon(s_0) \in \mathcal{F}_{s_0}$ we can define an[4] ϵ-equilibrium process $(z^\epsilon(\sigma))_{\sigma\in\Sigma}$ by

$$(z^\epsilon(s^t1), \ldots, z^\epsilon(s^tS)) \in \arg \min_{(z(s^t1),\ldots,z(s^tS))\in\mathcal{F}} \left[\min_{\kappa\in\mathcal{K}} \|h(s_t, z^\epsilon(s^t), \kappa, , z(s^t1), ..., , z(s^tS))\| \right].$$

This definition assigns a value of endogenous variables to any node in the infinite event tree. Note that the resulting process depends on the function h in a cardinal way. While multiplying an equation in h by a large constant or re-normalizing an equation (e.g. using an absolute versus a relative expression) does not affect the exact solution this clearly has a significant impact on the approximate solution and thus the resulting ϵ-equilibrium. Recursive methods often impose on the ϵ equilibrium process that choices in this period uniquely determine the probability distribution over the endogenous state next period (for example, beginning-of-period portfolio holdings are equal to portfolios chosen last period). This implies that equations in h which link last period's choice to the current endogenous state hold with equality and often pins down the equilibrium process uniquely. In our formulation this is equivalent to multiplying the relevant equations by large constants.

[4]In exceptional cases there might be several processes associated with one ϵ-equilibrium – this turns out to be insignificant for our analysis below and we usually refer to 'the' ϵ-equilibrium process as a generic one.

Ideally one would hope that this process is close-by to a competitive equilibrium for a close-by economy, i.e. that one can find small perturbations of endowments and preferences of the original economy such that the perturbed economy has a competitive equilibrium which is well approximated by the ϵ-equilibrium process at *each* node of the event tree. This leads to a very strong definition of approximate equilibrium.

DEFINITION 3 *An ϵ-equilibrium process $(z^\epsilon(\sigma))_{\sigma\in\Sigma}$ for an economy $\mathcal{E}$ path-approximates an economy in equilibrium with precision δ if there exists a close-by economy in equilibrium, $(\mathcal{E}', (\tilde{z}(\sigma))_{\sigma\in\Sigma})$, $\mathcal{E}' = \mathcal{E}((o(\sigma))_{\sigma\in\Sigma})$, with*

$$d(\mathcal{E}, \mathcal{E}') < \delta \text{ and } \sup_{\sigma\in\Sigma} \|z^\epsilon(\sigma) - \tilde{z}(\sigma)\| < \delta.$$

The following simple example illustrates that there may be ϵ-equilibria which do not path-approximate an economy in equilibrium.

EXAMPLE 1 *Consider an infinite horizon exchange economy with two infinitely lived agents, a single commodity and no uncertainty. Suppose that agents have identical initial endowments $e_t^i = 0.5$ for all t and identical preferences with $u_i(c_t) = \log(c_t)$ and with $\beta = 0.95$. There is a consol in zero net supply which pays 1 unit of the consumption good each period. The price of the consol is q_t, portfolios are θ_t^i. Each agent i, $i = 1, 2$, faces a short sale constraint $\theta_t^i \geq -0.5$ for all t. The initial conditions are $\theta_0^i = 0$, $i = 1, 2$.*

Even though the example is very simple, it is useful to explicitly spell out the equilibrium equations. Let the endogenous variables be $z = ((\theta_-^i, \theta^i, c^i, m^i)_{i=1,2}, q)$. Admissable perturbations are $o = (o_P^i, o_e^i)_{i=1,2} \in \mathbb{R}^4$, i.e. we allow for perturbations in endowments as well as in preferences. The expectations correspondence is characterized by $h(\bar{z}, \kappa, z, o) = 0$ where $h = (h^1, ..., h^6)$ with

$$\begin{array}{lll}
h^1 = & -1 + \frac{\beta(1+q)m^i}{\bar{q}\bar{m}^i} + \kappa^i & i = 1,2 \\
h^2 = & \kappa^i(\bar{\theta}^i + 0.5) & i = 1,2 \\
h^3 = & c^i - \theta_-^i(q+1) + \theta^i q - (e^i + o_e^i) & i = 1,2 \\
h^4 = & R \cdot (\theta_-^i - \bar{\theta}^i) & i = 1,2 \\
h^5 = & m^i - (u_i'(c^i) + o_P^i) & i = 1,2 \\
h^6 = & \theta^1 + \theta^2 &
\end{array}$$

The constant $R > 0$ is assumed to be sufficiently large to ensure that for an ϵ-equilibrium, the unique equilibrium process can be found by minimizing the error in h^4.

Obviously the unique exact equilibrium is no trade in the consol with each agent consuming 0.5 each period and in the absence of bubbles the price of the consol being $q_t = \frac{\beta}{1-\beta} = 19$ for all $t \geq 0$.

Now consider an economy where agents have different time preference $\beta_1 = 0.95 + \delta$, $\beta_2 = 0.95$. As long as $0.05 > \delta > 0$ the unique equilibrium has the property that the price of the consol is always above 19 and converges to $\frac{\delta+0.95}{0.05-\delta}$ The consumption of agent 1 converges to 1 while the consumption of the second agent converges to 0. This is an ϵ-equilibrium path for the original economy but it is not close by to an exact equilibrium of an economy with near-by endowments or Bernoulli utilities. Since this is an important example which largely motivates our analysis in Section 5 below we now examine it more formally. For a given $\epsilon > 0$ and a given large R, we construct an ϵ-equilibrium, $\mathcal{F}$, by truncating the exact equilibrium for the economy with different discount factors after a large but finite number of periods.

Let $\delta = \frac{\beta\epsilon}{1-\epsilon}$, define $\theta_0^1 = 0$, $\theta_0^2 = 0$ and recursively,

$$\theta_n^1 = \frac{(\beta+\delta/2)\theta_{n-1}^1 + \delta/4}{\beta + \delta\theta_{n-1}^1 + \delta/2}, \quad \text{if } \frac{\delta(0.25 - (\theta_{n-1}^1)^2)}{\beta + \delta\theta_{n-1}^1 + \delta/2} > \frac{\epsilon}{R}.$$

If $\frac{\delta(0.25-(\theta_{n-1}^1)^2)}{\beta+\delta\theta_{n-1}^1+\delta/2} \leq \frac{\epsilon}{R}$, define $\theta_n = \theta_{n-1}$ and let $N = n$. For all $n = 1, ..., N$ let $\theta_n^2 = -\theta_n^1$, let $q_n = \frac{\beta+\delta/2+\delta\theta_n^1}{1-\beta-\delta/2-\theta_n^1\delta}$ and let $c_n^i = e^i + \theta_{n-1}^i(q_n+1) - \theta_n^i q_n$. Collect in $\mathcal{F}$

$$z_n = ((\theta_{n-1}^i, \theta_n^i, c_n^i, \frac{1}{c_n^i})_{i=1,2}, q_n) \text{ for } n = 1, ..., N.$$

While it is easy to verify that $\mathcal{F}$ is an ϵ-equilibrium, it does not path approximate any economy in equilibrium that is obtained through small perturbations in individual endowments and Bernoulli utilities as long as the constant R in the equilibrium equations is sufficiently large. The unique ϵ equilibrium process is characterized by $h^4 = 0$ for all $t < N$. With this, the errors in Equations $h^2 - h^6$ in the equilibrium equations are equal to zero while the errors in h^1 are always below ϵ. If R is very large, for sufficiently large t, consumption of agent 2 in the ϵ-equilibrium is close to 0 while in any perturbed economy equilibrium consumption must be close to 0.5.

Obviously this is a very simple example. For small R, the constructed ϵ-equilibrium does path approximate an equilibrium for a close-be economy, the true equilibrium has finite support and can therefore be computed easily with different methods. However, unfortunately the simple example turns out to be a general problem in models with infinitely lived agents. ϵ-equilibria will often not path-approximate an economy in equilibrium. It is very difficult to rule out problems like the one in the above example without making very strong assumptions on the exact equilibrium (see e.g. Santos and Peralta-Alva (2002) for such assumptions for representative agent models). It also often does not help to consider other perturbations in preferences than the ones considered in this paper.

As mentioned in the introduction, we want to develop methods which need as few assumptions as possible on the exact equilibrium. Therefore we need a weaker definition of approximate equilibrium. Instead of requiring that the exact equilibrium process is well approximated by the ϵ-equilibrium process we merely require that for each node $s^t \in \Sigma$ and value of the exact equilibrium $z(s^t)$ there is some $\tilde{z} \in \mathcal{F}_{s_t}$ which is close to $z(s^t)$. Of course this would be a vacuous condition if there are many $\tilde{z} \in \mathcal{F}_{s_t}$ which do not approximate any equilibrium values. These considerations lead us to the following definition.

DEFINITION 4 *An ϵ-equilibrium $\mathcal{F}$ for the economy $\mathcal{E}$ weakly approximates an economy in equilibrium with precision δ if there exists a close-by economy in equilibrium, $(\mathcal{E}', (\tilde{z}(\sigma))_{\sigma\in\Sigma})$, $\mathcal{E}' = \mathcal{E}((o(\sigma))_{\sigma\in\Sigma})$, with $d(\mathcal{E}, \mathcal{E}') < \delta$, such that*

$$\min_{z^\epsilon \in \mathcal{F}_{s_t}} \|z^\epsilon - z(s^t)\| < \delta \text{ for all } s^t$$

and such that for all s, $z^\epsilon \in \mathcal{F}_s$ there exists a $\sigma \in \Sigma$ such that

$$\|z(\sigma) - z^\epsilon\| < \delta.$$

This condition is much weaker than requiring that the ϵ equilibrium process strongly approximates an economy in equilibrium since it only implies that there exists some process with values in $\mathcal{F}$ which approximates the exact equilibrium but does not explicitly state how to construct this process.

In the applications below we give one example with finitely lived agents where the ϵ equilibrium process can be shown to path approximate an economy in equilibrium. We give another example with infinitely lived agents, where we can only show that the ϵ equilibrium weakly approximates an economy in equilibrium. In this example, for each T, one can construct a process $(z^\epsilon(s^t))_{t\leq T}$ such that this process approximates an exact equilibrium at all s^t, $t \leq T$. However, as T gets larger, the construction of this process becomes more and more computationally intensive.

It is easy to see that the ϵ-equilibrium in example 1 approximates the exact equilibrium very well, for the initial conditions $\theta_0^i = 0$, $z_1 \in \mathcal{F}$ as defined above is very close to the exact equilibrium $\theta^i = 0$, $q = 19$ and $c^i = 0.5$. Definition 4 is satisfied. However, without knowing the exact equilibrium it is not entirely obvious how to verify the definition from the ϵ-equilibrium alone. In order to derive a sufficient condition we need to construct a process which is an exact equilibrium for a close-by economy and which is close to values in $\mathcal{F}$. Initial conditions dictate that the value of endogenous variables in the first period are equal to z_1. How can we find values for the next period which are close to an element of $\mathcal{F}$ and for which the equilibrium equations hold exactly? One possible way to proceed is to search over $\mathcal{F}$ for consumption values which make the error in h^1 identical across agents. Evidently this is obtained if next period's individual consumption is set equal to this period's consumption. In order for $h^1 = 0$ to hold, one now has to adjust prices. In this simple example, it is clear that setting $q = 19$ in *both* periods results in zero errors in h^1. Of course these perturbations lead to errors in h^3. One has to perturb individual endowments in order for h^3 to hold with equality. As a concrete example, suppose $\epsilon = 10^{-4}$ and consider the ϵ-equilibrium resulting from the above construction with $\delta = 0.000095$. With this $c_1^1 = 0.499525$, $\theta_1^1 = 0.000025$ and $q_1 = 19.019$. In order to satisfy the budget constraint in period 0 with the new prices one has to add $0.019 \cdot \theta_1^1 = 0.0095$ to agent 1's individual endowments and subtract it to agent 2's. In order for the budget constraint next period to hold one has to subtract $0.499525 - 0.5 - \theta_1^1 = 0.0050045$ from agent 1's endowment and add it to agent 2's. This last step has to be repeated every period in order to obtain an economy in equilibrium with equilibrium price of the consol being $q = 19$ and equilibrium consumption being $c_t^1 = 0.499525$ and $c_t^2 = 0.500475$ for all t.

In general, of course, verifying that an ϵ-equilibrium satisfies Definition 4 will not be as straightforward as in this example. We now describe how to generalize the idea of the simple example to more interesting models.

3.3 A simple sufficient condition

A general sufficient condition on an ϵ-equilibrium which guarantees that it weakly approximates an economy in equilibrium can be derived as follows.

Suppose that for all $\bar{z} \in \mathcal{F}$ we can find values of endogenous variables which are close-by to elements in $\mathcal{F}$ and small perturbations at all nodes over the next T periods which satisfy the equilibrium equations exactly (i.e. could be part of an economy in equilibrium). If the values of endogenous variables at all nodes at T are exactly equal to variables in $\mathcal{F}$ we have verified Definition 4. However, in general exact equality never holds for any finite T. Therefore we require that for all initial endogenous variables in a small neighborhood (within some $\delta_0 < \delta$) of $\bar{z}$ perturbations and values of endogenous variables can be found such that at all nodes at T the endogenous variables again lie in the same δ_0 neighborhood of some value in $\mathcal{F}$. If this is the case, we say that the ϵ equilibrium is (T, δ) balanced.

We refer to abstract histories of shocks of length $t \geq 1$ which are not associated with a node in the tree as

$$\vec{s}(t) = (s_1, ..., s_t) \in \mathcal{S}^t,$$

where $\mathcal{S}^t$ denotes the t-fold Cartesian product of $\mathcal{S}$. We denote by $\vec{s}(0)$ a given shock $s \in \mathcal{S}$. An ϵ-equilibrium $\mathcal{F}$ is (T, δ) balanced for some $T \in \mathbb{N}$ if there exist non-negative numbers $\delta_1, ..., \delta_M < \delta$ such that for all $\bar{s}$, $\bar{z} \in \mathcal{F}_{\bar{s}} \subset I\!R^M$, all $z_m(\vec{s}(0))$ with $|z_m(\vec{s}(0)) - \bar{z}_m| \leq \delta_m$ and all $\vec{s}(t) \in \mathcal{S}^t$, $1 \leq t \leq T$, there exist $z(\vec{s}(t))$, $o(\vec{s}(t))$ with

$$\min_{z^\epsilon \in \mathcal{F}} \|z^\epsilon - z(\vec{s}(t))\| < \delta \text{ and } \|o(\vec{s}(t))\| < \delta \text{ for } t \leq T,$$

with

$$\|z_m^\epsilon - z_m(\vec{s}(T))\| \leq \delta_m, \quad m = 1, ..., M \text{ for some } z^\epsilon \in \mathcal{F}_{s_T},$$

and with

$$\min_{\kappa \in \mathcal{K}} \|h(s_t, z(\vec{s}(t)), \kappa, o(\vec{s}(t), 1), z(\vec{s}(t)1), \ldots, o(\vec{s}(t)S), z(\vec{s}(t)S))\| = 0$$

for all $0 \leq t < T$.

Evidently, if an ϵ-equilibrium is (T, δ) balanced for some T then it weakly approximates an economy in equilibrium with precision δ. The condition is only interesting because it can often be verified in applications that a candidate ϵ-equilibrium is (T, δ) balanced for $T = 1$.

We give a more elaborate example of balancedness in Section 5, where we consider the Lucas model with heterogeneous agents.

3.4 Construction of ϵ-equilibria

This paper does not develop explicit algorithms to compute ϵ-equilibria. In fact, it is usually not feasible to compute them directly since their discrete nature does not allow directly for the application of standard methods in numerical analysis (which usually assume smoothness). Below we will examine some existing algorithms in detail. These algorithms are assumed to output $z(s^t)$ for any finite sequence of shocks. We want to construct recursive ϵ-equilibria from this output.

Fix a small $\delta > 0$. Starting from the root node s_0 collect all pairs of shocks and endogenous variables $(s_t, z(s^t))$ in a set $\mathcal{Y}_t$. Define the set of states rounded within δ by

$$\mathcal{Y}_t^\delta = \{(s, z^\delta) : \frac{|z_m^\delta|}{\delta} \in \mathbb{N} \text{ for } m = 1, \ldots, M \text{ and there exists } (s, z) \in \mathcal{Y}_t \text{ with } \|z - z^\delta\| < \delta\}.$$

We collect all rounded states which have occurred up to time t in $\mathcal{Y}^{\delta t}$, i.e.

$$\mathcal{Y}^{\delta t} = \cup_{t'=0}^{t} \mathcal{Y}_{t'}^\delta.$$

It is obvious that with $\delta > 0$ the number of elements of $\mathcal{Y}^{\delta t}$ remains bounded. Furthermore, if the outputs of the computations $z(s^t)$ constitute an exact equilibrium and if for some t^*, $\mathcal{Y}_{t^*}^\delta \subset \mathcal{Y}^{\delta(t^*-1)}$ it must be the case that $\mathcal{Y}^{\delta(t^*-1)}$ is an ϵ equilibrium for some $\epsilon > 0$, $\epsilon \to 0$ as $\delta \to 0$. The actual ϵ can be computed by evaluating the error in the expectations correspondence at all s , $z \in \mathcal{F}_s$.

Of course, this procedure might be hopelessly inefficient if for a given δ the number of elements in $\mathcal{F}$ turns out to be very large or the resulting ϵ is not sufficiently small. However, the examples below show that in many economic applications this is feasible even when the dimension of the endogenous state space is fairly large.

In many applications researchers verify errors only along one randomly determined path. The implicit assumption is that simulating the economy along one sample path suffices to verify the accuracy of computations. However, the example in Kubler and Schmedders (2003) shows that this is often not sufficient. In particular, for a given finite sample path, it is obviously impossible to infer the maximal error from the error along the path.

4 A model with overlapping generations and production

As the first application of our methods, we consider a model of a production economy with overlapping generations and several commodities. This is a generalization of models frequently used in macroeconomics and public finance (see e.g. Rios-Rull (1996)) and of the overlapping generations model analyzed in Duffie et al. (1994).

We show that in this model the ϵ-equilibrium process actually path approximates an economy in equilibrium and we derive bounds on the distance between the close-by economy in equilibrium and the specified economy. These bounds are constructed from the ϵ equilibrium $\mathcal{F}$ using simple linear algebra.

While the model is fairly standard, we describe it in some detail to fix notation. At each date-event a single individual commences his economic life; he lives for N dates. An individual is identified by the date event of his birth, (s^t). He consumes at the date-event $s^t, \ldots, s^{t+N-1}$; the age of an individual is $a = 1, \ldots, N$.

There are L physical commodities and one representative firm at each date-event, s^t. The firm produces in spot markets using a constant returns to scale technologies which depends on the current shock alone. In order to simplify the error analysis we assume that commodities $1, ..., K$, $K < L$, are always used as inputs to spot production and commodities $K+1, ..., L$ are always outputs. We assume that the technology can be described by a function $f(.,s) : I\!R^K_+ \to I\!R^{L-K}_+$. A production plan $y \in I\!R^L$ is feasible for the firm at shock s if and only if $(y_{K+1}, ..., y_L) - f((-y_1, ..., -y_K), s) \leq 0$.

Households have access to a risky intertemporal technology which, for simplicity, is assumed to be linear. For each shock s we define an $L \times L$ matrix, $D(s)$, where the element $d_{l'l}$ denotes how much of commodity l' is produced from one unit of commodity l used as input in the previous period. We denote by the vector $\phi^\sigma(s^t) \in I\!R^L_+$ the bundle of commodities invested by individual σ into the technology at date event s^t, the output at s^{t+1} is given by $D(s_{t+1})\phi^\sigma(s^t) \in I\!R^L_+$. In order to distinguish between spot production of the firm and intertemporal household production, we refer to the latter as storage.

An agent born at date event s^t has individual endowments at nodes $s^t, ..., s^{t+N-1}$ which are a function of the shock and his age alone, i.e. for all $a = 1, ..., N$, $e^{s^t}(s^{t+a-1}) = \mathbf{e}^a(s_{t+a-1})$ for some function $\mathbf{e}^a : \mathcal{S} \to \mathbb{R}^L_+$. For an agent $\sigma = s^T$ we denote his consumption choices over his lifetime by $x^\sigma = (x^\sigma(s^t))_{t=T,\ldots,T+N-1, s^t \succeq \sigma}$, and his investment choices by $\phi^\sigma = (\phi^\sigma(s^t))_{t=T,\ldots,T+N-1, s^t \succeq \sigma}$. To simplify notation we define $\phi^{s^t}(s^{t-1}) = 0$ for all s^t.

The agent has an intertemporal, von Neumann-Morgenstern utility function

$$U^{s^t}(x) = \mathrm{E}_{s^t} \sum_{a=1}^{N} u_a(x(s^{t+a-1}), s_{t+a-1}).$$

The Bernoulli utility u depends on the age and the current shock alone.

At the root node, s_0, there are individuals of all ages $s^{-1}, ..., s^{-N+1}$ with initial holdings $\phi^{s^{-a}}(s^{-1})$. These determine the 'initial condition' of the economy.

A competitive equilibrium is a collection of prices, choices of individuals and choices of the firm $(p(\sigma), (\phi^i(\sigma), x^i(\sigma))_{i \in \mathcal{I}(\sigma)}, y(\sigma))_{\sigma \in \Sigma}$ such that markets clear and agents optimize, i.e. for all nodes $s^t \in \Sigma$ we have

- Market clearing:

$$\sum_{a=1}^{N} \left(x^{s^{t-a+1}}(s^t) - \mathbf{e}^a(s_t) - D(s_t)\phi^{s^{t-a+1}}(s^{t-1}) + \phi^{s^{t-a+1}}(s^t) \right) = y(s^t).$$

- Individual's maximize utility:

$$(x^{s^t}, \phi^{s^t}) \in \arg \max_{(x,\phi) \geq 0} U^{s^t}(x) \text{ s.t.}$$

$$p(s^{t+a})\cdot\Big(x^{s^t}(s^{t+a}) - \mathbf{e}^{a+1}(s_{t+a}) + \phi^{s^t}(s^{t+a}) + D(s_{t+a})\phi^{s^t}(s^{t+a-1})\Big) \leq 0, a = 0, ..., N-1$$

Optimality conditions for initially alive agents, $s^{-1}, ..., s^{-N+1}$ are analogous.

- The firm maximizes profits,

$$(y_1(s^t), ..., y_K(s^t)) \in \arg\max_{(y_1,...,y_K)\leq 0} \sum_{l=K+1}^{L} p_l(s^t) f((-y_1, ..., -y_K), s_t) + \sum_{l=1}^{K} p_l(s^t) y_l(s^t)$$

$$(y_{K+1}(s^t), ..., y_L(s^t)) = f((-y_1(s^t), ..., -y_K(s^t)), s_t)$$

We want to characterize competitive equilibria by an expectations correspondence.

The expectations correspondence

We define the endogenous variables at some node σ to consist of investments from the previous period, $\phi_- = (\phi_-^1, ..., \phi_-^N)$, new investments, $\phi = (\phi^1, ..., \phi^N)$, consumptions, $x = (x^1, ..., x^N)$ as well as excess demands, $\xi = (\xi^1, ..., \xi^N) \in I\!R^{NL}$, and Lagrange multipliers, $\lambda = (\lambda^1, ..., \lambda^N) \in I\!R_+^N$, for all individuals alive; of spot prices, p and the firm's choice, y, i.e.

$$z = (\phi_-, \phi, x, \xi, \lambda, y, p).$$

We build trivial normalizations into the admissible endogenous variables, i.e. we only consider z for which $\phi_-^1 = 0$, $\phi^N = 0$, $\phi \geq 0$, $\phi_- \geq 0$, $c \geq 0$ and $p_1 = 1$.

We consider perturbations in individual endowments and preferences, i.e. define $o(\sigma) = (o_e^{(\sigma)}, o_P^{(\sigma)}) \in I\!R^{2LN}$ to be perturbations in endowments and preferences across all agents alive at a current node σ. As explained in Section 2 preferences are perturbed by perturbing Bernoulli utility functions node by node. We write for agent s^t's perturbed Bernoulli function at node s^{t+a}, $u^a(x, s_t, o_P(s^t)) = u^a(x, s_t) + o_P(s^t) \cdot x$.

We characterize the expectations correspondence H via the equilibrium equations, $(z_1, o_1^{\mathcal{I}}, ..., z_S, o_S^{\mathcal{I}}) \in H(\bar{s}, \bar{z})$ iff there exist $\kappa \in I\!R_+^{(N-1)L}$ such that $h(\bar{s}, \bar{z}, \kappa, , z_1, o_1^{\mathcal{I}}, ..., z_S, o_S^{\mathcal{I}}) = 0$. We define $h = (h^1, ..., h^9)$ with

$$\begin{array}{lll}
h^1 = & \phi_-^a(s) - \bar{\phi}^{a-1} & a = 2, ..., N, s \in \mathcal{S} \\
h^2 = & p(s) \cdot \xi^a(s) & a = 1, ..., N, s \in \mathcal{S} \\
h^3 = & x^a(s) - (\mathbf{e}^a(s) + o_e^a(s)) - \xi^a(s) + \phi^a(s) - D(s)\phi_-^a(s) & a = 1, ..., N, s \in \mathcal{S} \\
h^4 = & -\bar{p}\bar{\lambda}^{a-1} + \beta \mathrm{E}_{s|\bar{s}} p(s) D(s) \lambda^a(s) + \kappa^a & a = 2, ..., N \\
h^5 = & \kappa_l^a \bar{\phi}_l^{a-1} & a = 2, ..., N, l \in \mathcal{L} \\
h^6 = & D_x u^a(x^a(s), s) + o_P(s) - \lambda^a(s) p(s) & a = 1, ..., N, s \in \mathcal{S} \\
h^7 = & \sum_{l=K+1}^{L} p_l(s) \frac{\partial f_l((-y_1(s), ..., -y_K(s)), s)}{\partial y_{l'}} - p_{l'}(s) & s \in \mathcal{S}, l' = 1, ..., K \\
h^8 = & (y_{K+1}(s), ..., y_L(s)) - f(-y_1(s), ..., -y_K(s), s) & s \in \mathcal{S} \\
h^9 = & \sum_{a=1}^{N} \xi^a(s) - y(s) & s \in \mathcal{S}
\end{array}$$

We assume throughout that in h^6 derivatives are only taken with respect to commodities which enter the utility function.

Under standard assumptions on preferences and the production function which guarantee that first order conditions are necessary and sufficient, a competitive equilibrium can be characterized by these equations. Kubler and Polemarchakis (2003) prove the existence of ϵ-equilibria.

4.1 Error analysis

Throughout this section we use the following well known fact from linear algebra. For an under-determined system $Ax = b$ with a matrix A that has linearly independent rows, denote by $A^+ = A^\top(AA^\top)^{-1}$ the pseudo inverse of A (where $A^\top$ denotes the transpose of the matrix A). The unique solution of the system that minimizes $\|x\|_2$ is then given by $x_{LS} = A^+b$. We assume that A^+b can be computed without error. While this is obviously incorrect the error analysis for this problem is well understood and explicit bounds on the errors are usually very small (see e.g. Higham (1996), Chapter 20). We use the pseudo inverse below without explicitly assuming that $AA^\top$ is invertible. If $AA^\top$ is singular in our analysis below, there is no bound on errors. While we are interested in maximum errors, we use the Euclidean (or two-) norm here since it is well understood how to compute A^+b accurately. Evidently, for an $x \in I\!R^n$, we have that

$$\|x\| \leq \|x\|_2$$

and so this approach will immediately yield an upper bound on the minimal sup-norm of a solution x.

For the error analysis, we assume that there is at least one commodity l^* such that $u^a(x,s)$ is strictly increasing in x_{l^*} for all a, s and that agents' choices always satisfy $x_{l^*} > 0$. We restrict the consumption set of each agent to be bounded above in each component by twice the maximal aggregate consumption occurring in the ϵ equilibrium

$$\bar{c} = 2 \max_{s\in\mathcal{S}, z\in\mathcal{F}_s} \|\sum_{a=1}^{N} c^a\|$$

and define

$$\bar{m} = \min_{s\in\mathcal{S}, a=1,\ldots,N} \left(\min_{x\geq 0, \|x\|\leq\bar{c}} \|D_x u^a(x,s)\| \right).$$

It holds that $\bar{m} > 0$ since we assume that the utility function is strictly increasing.

Let $\mathcal{G}^1 \subset \mathcal{L}$ consist of all those commodities which are inputs of intertemporal production (storable commodities),

$$\mathcal{G}^1 = \{l \in \mathcal{L} : \exists l' \in \mathcal{L}, s \in \mathcal{S} : d_{l'l}(s) \neq 0\},$$

and let $\mathcal{G}^2 \subset \mathcal{L}$ consist of all those commodities which are output of intertemporal production (stored commodities),

$$\mathcal{G}^2 = \{l \in \mathcal{L} : \exists l' \in \mathcal{L}, s \in \mathcal{S} : d_{ll'}(s) \neq 0\}.$$

We want to distinguish between inputs and outputs of spot production which have previously been stored and inputs and outputs which can not be produced through the storage technology. In order to do so with as little notation as possible, we assume[5] that there are integers j and m, $0 \leq j \leq K \leq m \leq L$ such that

$$\mathcal{G}^2 = \{l \in \mathcal{L} : l \leq j \quad \text{or} \quad K+1 \leq l \leq m\}.$$

Given any $\bar{s}, \bar{z} \in \mathcal{F}_{\bar{s}}$, let

$$(z(1), ..., z(S)) = \arg \min_{(z_1,...,z_S) \in \mathcal{F}} \left(\min_{\kappa \in \mathcal{K}} \|h(\bar{s}, \bar{z}, \kappa, \mathbf{0}, z_1, \ldots, \mathbf{0}, z_S)\| \right).$$

Without loss of generality we can restrict attention to ϵ-equilibria for which Equations h^1, h^5, h^8 and h^9 hold with equality at $(\bar{s}, \bar{z}, z(1), ..., z(S))$ since an error in these equations can be easily put to zero by increasing the error in other equations. The other equations will generally only hold with some error. In order to facilitate the error analysis, it is useful to state explicitly the variables these functions depend on.

- $h^2 = (h^2_{a,s})_{a=1,\ldots,N, s \in \mathcal{S}}$, $h^2_{a,s}(p(s), \xi(s)) = \epsilon^2_{a,s} \in I\!R$
- $h^3 = (h^3_{a,l,s})_{a=1,\ldots,N, l \in \mathcal{L}, s \in \mathcal{S}}$, $h^3_{a,l,s}(x^a_l(s), \xi^a_l(s), \phi^a_l(s), \phi^a_-(s), o(s)) = \epsilon^3_{a,s} \in I\!R$
  We assume w.l.o.g. that $h^3_{a,l,s}$ holds with equality whenever the individual a has no endowments in commodity l and that commodity does not enter his utility function.
- $h^4 = (h^4_{a,l})_{a=2,\ldots,N, l \in \mathcal{G}^1}$,

  $$\epsilon^4_{a,l} = \min_{\kappa \geq 0} \left| h^4_{a,l}(\bar{p}_l, \bar{\lambda}^{a-1} \bar{\phi}^{a-1}_l, (p_l(s))^{s \in \mathcal{S}}_{l \in \mathcal{G}^2}, (\lambda^a(s))_{s \in \mathcal{S}}, \kappa) \right| \quad \text{s.t. } \kappa \bar{\phi}^{a-1}_l = 0$$

  Note that for all commodities which cannot be stored, i.e. $l \notin \mathcal{G}^1$, there is no equilibrium equation.
- $h^6 = (h^6_{a,s,l})_{a=1,\ldots,N, l \in \mathcal{L}, s \in \mathcal{S}}$, $h^6_{a,s,l}(x^a(s), \lambda^a(s), p_l(s)) = \epsilon^6_{a,s,l}$
- $h^7 = (h^7_s)_{s \in \mathcal{S}}$, $h^7_s(y(s), p(s)) = \epsilon^7_s \in I\!R^K$.

The general strategy to derive error bounds will be as follows. We identify a set of commodities whose prices we can perturb at any node in order to ensure that $h^7(y(s), \tilde{p}) = 0$ for the perturbed $\tilde{p}$. These prices do not appear in h^4. We then give bounds on the errors caused in h^4 by perturbations in previous periods of $\bar{\lambda}$ and $\bar{p}$. The perturbations in λ and p then determine the perturbations necessary in individual endowments to satisfy the budget constraints and in Bernoulli utility functions to satisfy $h^6 = 0$.

[5] This assumption implicitly implies that all commodities are either inputs or outputs. This simplifies the notation but the analysis can also be conducted if some commodities are not part of production.

Errors in h^7

For a given $s = 1, ..., S$ and $z(s)$, let the $(L-K) \times K$ matrix J denote the Jacobian of $f(., s)$ with respect to all inputs at $y(s)$. For our analysis it is helpful to divide J into 4 submatrices. Denoting the row index by l and the column index by l' we write

$$J = \left(\frac{\partial f_l(-y_1(s), ..., -y_K(s), s)}{\partial y_{l'}} \right)_{l'=1,...,K}^{l=K+1,...,L} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \quad \text{with}$$

$$J_{11} = \left(\frac{\partial f_l}{\partial y_{l'}} \right)_{l'=1,...,j}^{l=K+1,...,m}, \qquad J_{12} = \left(\frac{\partial f_l}{\partial y_{l'}} \right)_{l'=j+1,...,K}^{l=K+1,...,m},$$

$$J_{21} = \left(\frac{\partial f_l}{\partial y_{l'}} \right)_{l'=1,...,j}^{l=m+1,...,L}, \qquad J_{22} = \left(\frac{\partial f_l}{\partial y_{l'}} \right)_{l'=j+1,...,K}^{l=m+1,...,L}.$$

Define a $(K-j)+(l-m)$ vector

$$er^1(s) = \begin{pmatrix} 0 & J_{21}^\top \\ I_{K-j} & J_{22}^\top \end{pmatrix}^+ \epsilon_s^7.$$

Recall that $\bar{s}, \bar{z}$ denote the shock and the endogenous variables from the previous period and let

$$\Delta_1(\bar{s}, \bar{z}) = \max_{s \in \mathcal{S}} \max\{ \max_{i=1,...,K-j} |\frac{er_i^1(s)}{p_{j+i}(s)}|, \max_{i=1,...,L-m} |\frac{er_{K-j+i}^1(s)}{p_{m+i}(s)}|\}.$$

The proof of Lemma 2 shows that this bound denotes the maximal relative perturbation necessary in prices to obtain equality in Equation h^7 for z. Finally let $\overline{\Delta}_1 = \max_{s,z \in \mathcal{F}_s} \Delta_1(s,z)$ denote the upper bound on this perturbation across all points in the ϵ-equilibrium. We can now ensure that there exist relative perturbations in the prices of commodities $l = j+1, ..., K$ and $l = m+1, ..., L$ (that is, for $l \in \mathcal{L} - \mathcal{G}^2$ which are uniformly bounded by $\overline{\Delta}_1$ and which guarantee that for all s, $z \in \mathcal{F}_s$, $h_s^7(\tilde{z}) = 0$. The following lemma states this formally.

LEMMA 2 *For each s, $z \in \mathcal{F}_s$, there exist $\tilde{p}_l$, $l \in \mathcal{L} - \mathcal{G}^2$ with $\|\frac{p_l - \tilde{p}_l}{p_l}\| < \overline{\Delta}_1$ such that*

$$h_s^7(y, (p_1, ..., p_j, \tilde{p}_{j+1}, ..., \tilde{p}_K, p_{K+1}, ..., p_m, \tilde{p}_{m+1}, ..., \tilde{p}_L)) = 0.$$

Note that prices of commodities which are output of the intertemporal storage technology, i.e. p_l for $l \in \mathcal{G}^2$, are not perturbed. Therefore, the performed perturbations do not affect the error in Equation h^4.

Errors in h^2

Given $\bar{s}, \bar{z}, z(1), ..., z(S)$ from above and given perturbed prices, $\tilde{p}(s)$ to ensure that h^7 holds with equality, we need to perturb $\xi^a(s)$ for all $a = 1, ..., N$ to ensure that $\tilde{p}(s)\xi^a(s) = 0$. Since we assume constant returns to scale, $\tilde{p}(s) \cdot y(s) = 0$ and since $h^9 = 0$ we have that

$\tilde{p}(s)\sum_{a=1}^{N}\xi^a(s) = 0$. Since l^* is a commodity that is desired by all agents, we have that $\tilde{p}_{l^*}(s) > 0$, therefore a sufficient perturbation would be

$$\tilde{\xi}^a_{l^*}(s) = -\frac{1}{\tilde{p}_{l^*}(s)}\sum_{l\neq l^*}\xi^a_l(s)\tilde{p}_l(s).$$

Since an upper bound for $\tilde{p}_{l^*}|\tilde{\xi}^a_{l^*} - \xi^a_{l^*}|$ is given by

$$\tilde{p}_{l^*}(s)|\tilde{\xi}^a_{l^*}(s) - \xi^a_{l^*}(s)| \leq \tilde{p}_{l^*}(s)|\tilde{\xi}^a_{l^*}(s)| + \tilde{p}_{l^*}(s)|\xi^a_{l^*}(s)| \leq p_{l^*}(s)\Delta_1|\xi^a_{l^*}(s)| + |\epsilon^2_{a,s}| + \sum_{l\neq l^*}p_l(s)\Delta_1|\xi^a_l(s)|,$$

an upper bound for the perturbation in $\xi_{l^*}(s)$ is given by

$$\Delta_2(\bar{s},\bar{z}) = \max_{s\in\mathcal{S},a=1,\ldots,N}\frac{1}{p_{l^*}(s)(1-\Delta_1(\bar{s},\bar{z}))}\left(|\epsilon^2_{a,s}| + \Delta_1(\bar{s},\bar{z})\sum_{l\in\mathcal{L}}p_l(s)|\xi^a_l(s)|\right).$$

Define

$$\overline{\Delta}_2 = \max_{s,z\in\mathcal{F}_s}\Delta_2(s,z).$$

Errors in h^3

Given $\bar{s},\bar{z},z(1),...,z(S)$ and $\Delta_2(\bar{s},\bar{z})$ from above, recall that h^3 is assumed to hold with equality for all commodities which do not enter the utility function and in which the agent has zero endowments. For commodities which do not enter the utility function, adjust $o^a_e(s)$ to ensure equality. These adjustments all lie within $\|\epsilon^3\|$ since by construction, the excess demands ξ in these commodities were not perturbed in the previous step.

For commodities which enter the utility function, adjust $x^a(s)$ to ensure equality. These commodities include l^* for which ξ_{l^*} has been perturbed. Define

$$\nu^a_s = \max\{\Delta_2(\bar{s},\bar{z}) + |\epsilon^3_{l^*}|, \max_{l\neq l^*}|\epsilon^3_l|\}.$$

An upper bound on the necessary perturbations in consumptions is therefore given by

$$\Delta_3(\bar{s},\bar{z}) = \max_{a=1,\ldots,N,s\in\mathcal{S}}\nu^a_s.$$

Define $\overline{\Delta}_3 = \max_{s\in\mathcal{S},z\in\mathcal{F}_s}\Delta_3(s,z)$.

For commodities that do enter the utility function, this perturbation may increase the error in h^6. In order to capture this effect, define

$$\Delta_6(\bar{s},\bar{z}) = \max_{a=1,\ldots,N,s\in\mathcal{S}}\left(|\epsilon^6_{as}| + \|D_xu^a(x^a(s)-\nu_s\mathbf{1},s) - D_xu^a(x^a(s),s)\|\right).$$

Define $\overline{\Delta}_6 = \max_{s\in\mathcal{S},z\in\mathcal{F}_s}\Delta_6(s,z)$. We must take this error into account when we examine Equation h^6.

Errors in h^4

Given $\bar{s}, \bar{z}, z(1), ..., z(S)$ from above define a payoff matrix A by

$$A = \left(\pi(s|\bar{s}) \sum_{l \in \mathcal{G}^2} d_{ll'}(s) p_l(s) \right)_{l's}, \quad l' \in \mathcal{G}^1, s \in \mathcal{S}.$$

Define $er^2(a) = A^+(\epsilon^4_{al})_{l \in \mathcal{G}^1}$ and let $\Delta_4(\bar{s}, \bar{z}) = \max_{a=2,...,N} \max_{s \in \mathcal{S}} |er^2_s(a)/\lambda^a_s|$. Define

$$\overline{\Delta}_4 = \left((1 + \max_{s,z \in \mathcal{F}_s} \Delta_4(s, z))(1 + \overline{\Delta}_1) \right)^N .$$

It is straightforward to see that this imposes an upper bound on necessary relative perturbations in λ^a which ensure that h^4 holds with equality, given the perturbations in prices for h^7 and given the errors in h^4.

Errors in h^6: Necessary perturbations in Bernoulli utilities

Finally we need to perturb Bernoulli utilities in order to impose equality on the first order conditions for individuals' spot optimality. A bound on the necessary (linear) perturbations is given by

$$\Delta^P = \overline{\Delta}_6 + \overline{\Delta}_4 \max_{s,(\phi_-,\phi,x,\xi,\lambda,y,p) \in \mathcal{F}_s} \left(\max_{a=1,...,N} p_l \lambda^a \right).$$

The following theorem summarizes the above discussion and uses Lemma 1 to give bounds on the overall perturbations in endowments and preferences necessary.

Theorem 1 *Given an ϵ equilibrium $\mathcal{F}$ for an economy $\mathcal{E}$, with $\overline{\Delta}_j$, $j = 1, \ldots, 4$ and Δ^P as defined above, there exists an economy $\mathcal{E}'$ with $d(\mathcal{E}, \mathcal{E}') < \Delta^P \frac{\bar{c}}{\bar{m}}$ and with a competitive equilibrium $(z(\sigma))_{\sigma \in \Sigma}$ such that the ϵ equilibrium process $z^\epsilon = (\phi^\epsilon_-, \phi^\epsilon, x^\epsilon, \xi^\epsilon, \lambda^\epsilon, y^\epsilon, p^\epsilon)$ satisfies for all $\sigma \in \Sigma$,*

$$\begin{aligned}
\phi^\epsilon_-(\sigma) - \phi_-(\sigma) &= 0 \\
\phi^\epsilon(\sigma) - \phi(\sigma) &= 0 \\
y^\epsilon(\sigma) - y(\sigma) &= 0 \\
\|x^\epsilon(\sigma) - x(\sigma)\| &\leq \overline{\Delta}_3 \\
\|\lambda^\epsilon(\sigma) - \lambda(\sigma)\| &\leq \overline{\Delta}_4 \|\lambda^\epsilon(\sigma)\| \\
\|p^\epsilon(\sigma) - p(\sigma)\| &\leq \overline{\Delta}_1 \|p^\epsilon(\sigma)\| \\
\|\xi^\epsilon(\sigma) - \xi(\sigma)\| &\leq \overline{\Delta}_2
\end{aligned}$$

Note that the portfolios $(\phi^\epsilon_-, \phi^\epsilon)$ and the firm's output were not perturbed.

4.2 Simple parametric examples

We want use a simple example to illustrate that the bounds in Theorem 1 are fairly tight and that methods which lead to low maximal errors in Euler equations (even if these are only measured along one simulated path) usually approximate an economy in equilibrium very well.

Suppose that there are only three commodities: Capital, k, labor, l, and a consumption good, c. Agents have access to a risk-less storing technology, which transforms one unit of the consumption good at node s^t into 1 unit of capital at each node $s^{t+1} \succ s^t$. The risky spot production function is Cobb-Douglas

$$f(k, l, s) = \eta(s)k^{\alpha}l^{1-\alpha} + (1 - \delta(s))k^{\alpha}$$

for shocks η, δ. Agents live for 9 periods and only derive utility from the consumption good. An agent born at shock s^t has utility function

$$U^{s^t} = \mathrm{E}_{s^t} \sum_{a=1}^{N} \beta^{a-1} u(c(s^{t+a-1})).$$

We assume that there are 4 shocks which are iid with $\pi_s = 0.25$ for $s = 1, ..., 4$. Bernoulli utilities are of the CRRA form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{2}$$

with a coefficient of relative risk aversion $\gamma = 3$. Suppose that $\beta = 0.8$ and that individual endowments are deterministic and given by

$$(\mathbf{e}^1, ..., \mathbf{e}^N) = (1, 1, 1, 1, 1, 1, 0.5, 0, 0)$$

We consider 4 different specifications for the shocks to production.

Table 1: Specifications for Shocks

	State 1	State 2	State 3	State 4
Case 1: η	0.95	1.05	0.95	1.05
Case 1: δ	0.7	0.7	0.7	0.7
Case 2: η	0.85	1.15	0.85	1.15
Case 2: δ	0.7	0.7	0.7	0.7
Case 3: η	0.95	1.05	0.95	1.05
Case 3: δ	0.5	0.5	0.9	0.9
Case 4: η	0.85	1.15	0.85	1.15
Case 4: δ	0.5	0.5	0.9	0.9

We do *not* impose that individual investment has to be non-negative. This is done to simplify computations. It is easy to see that the above error analysis remains valid even without the non-negativity restriction.

4.2.1 Computation

By the finite nature of recursive ϵ-equilibria it should be possible to derive a globally convergent algorithm which computes an ϵ-equilibrium for any given $\epsilon > 0$ and any given specification of preferences and endowments. However, since agents live for 9 periods the endogenous state space Θ is of dimension 7 and any discrete algorithm will be hopelessly inefficient.

Krueger and Kubler (2003) develop an algorithm to approximate equilibria in OLG models where agents live for several periods using polynomial approximations. The algorithm assumes that pricing and policy functions which describe a recursive ϵ-equilibrium are defined over a compact set Θ and that these functions exhibit a high degree of smoothness. They approximate them by polynomials, using Smolyak's method to avoid a curse of dimensionality. The unknown polynomial coefficients are solved for through a time iteration algorithm.

Given the discussion above, using this algorithm to obtain recursive ϵ-equilibria might seem odd: By definition recursive ϵ-equilibria are a finite collection of points and there is no guarantee that the functions in the definition can be extended to smooth functions over a compact set. However, in practice the algorithm has been proven to converge very well and it is clear that it would be infeasible to compute a recursive ϵ-equilibrium directly for a 7-dimensional state space. Our concept of recursive ϵ-equilibrium is not meant to imply anything for the actual computation of equilibria. It merely provides a method to assess the quality of a candidate solution.

We use the method described above to construct a recursive ϵ-equilibrium from the computed values of the algorithm. We set $\delta = 1/300000$. This results in a set $\mathcal{F}$ with around 2 million elements. All states are visited within the first 24 periods, i.e. the algorithm terminates at $\mathcal{Y}_t^\delta$ for $t = 24$.

The numéraire commodity is taken to be capital, its price is always 1.

4.2.2 Error analysis in practice

In this simple example, most of the above steps reduce to a single calculation. The error in Euler equations does propagate over time, but this is the only source of high overall errors. Since neither labor nor capital enters individual's utility functions, the errors in h^7 are around machine-precision (10^{-15} on the machine used for this computation). The necessary perturbations in spot prices and consumptions are then smaller than 10^{-13}. Only errors in h^4 are significantly higher, around 10^{-4}, these then lead to the higher overall errors.

The following table shows maximal errors in Euler equations along a simulated path as well as the maximal error in the constructed ϵ equilibrium and the maximal perturbations.

Table 2: Errors				
	Case 1	Case 2	Case 3	Case 4
simerr	8.3 (-5)	2.5 (-4)	8.2 (-4)	1.3 (-3)
ϵ	9.2 (-4)	4.2 (-4)	1.3 (-3)	2.5 (-3)
$\bar{\Delta}^P$	8.2 (-3)	8.3 (-3)	9.9 (-3)	1.3 (-2)

The table reports the errors along one simulated path of length 100000, simerr, the ϵ which resulted from taking $\delta = 1/300000$ in the above discretization procedure as well as the maximum perturbation necessary in individual Bernoulli utilities.

5 The Lucas model with several agents

As a second application we consider the model of Duffie et al. (1994, Section 3). This model is a version of the Lucas (1978) asset pricing model with finitely many heterogeneous agents.

There are I infinitely lived agents, $i \in \mathcal{I}$, and a single commodity in a pure exchange economy. Each agent $i \in \mathcal{I}$ has endowments $e^i(\sigma) > 0$ at all nodes $\sigma \in \Sigma$ which are time-invariant functions of the shock alone, i.e. there exist functions $\mathbf{e}^i : \mathcal{S} \to I\!R_+$ such that $e(s^t) = \mathbf{e}(s_t)$. Agent i has von Neumann-Morgenstern utility over infinite consumption streams

$$U^i(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_i(c_t)$$

for a differentiable, strictly increasing and concave Bernoulli function u_i, where E_0 denotes the expectation under some set of positive probabilities.

There are J infinitely lived assets in unit net supply. Each asset j pays shock dependent dividends $d_j(s)$, we denote its price at node s^t by $q_j(s^t)$. Agents trade these assets but are restricted to hold non-negative amounts of each asset. We denote portfolios by $\theta^i \geq 0$. At the root node s_0 agents hold initial shares $\theta^{\mathcal{I}}(s^{-1})$ which are assumed to be identical across agents and sum up to 1.

A competitive equilibrium is a collection $((c^i(\sigma), \theta^i(\sigma))_{i\in\mathcal{I}}, q(\sigma))_{\sigma\in\Sigma}$ such that market clear,

$$\sum_{i\in\mathcal{I}} \theta^i_j(\sigma) = 1 \text{ for all } \sigma \in \Sigma, j \in \mathcal{J},$$

and such that agents optimize

$$\begin{aligned} c^i \in \arg\max\nolimits_{c\geq 0} U^i(c) \quad \text{s.t.} \quad & \forall s^t \in \Sigma \\ & c^i(s^t) = e^i(s^t) + \theta^i(s^{t-1})(q(s^t) + d(s_t)) - \theta^i(s^t)q(s^t), \\ & \theta^i(s^t) \geq 0. \end{aligned}$$

5.1 The expectations correspondence

Following Kubler and Schmedders (2003) it is useful to include as an endogenous variable individual shares of total financial wealth

$$w^i(s^t) = \frac{\theta^i(s^{t-1}) \cdot (q(s^t) + d(s^t))}{\sum_{j=1}^{J} q_j(s^t) + d_j(s^t)}$$

Note that $w^{\mathcal{I}} = (w^1, ..., w^I) \in \Delta^{I-1}$, the $(I-1)$-dimensional simplex in $I\!R^I$. We define the current endogenous variables to consist of wealth shares, asset prices, individuals' consumption and portfolios,

$$z = (w^{\mathcal{I}}, q, c^{\mathcal{I}}, \theta^{\mathcal{I}}).$$

As before, we built trivial normalizations into the state space, i.e. we assume that $\theta^i \geq 0$, $c^i \geq 0$ for all $i \in \mathcal{I}$ and that $w^{\mathcal{I}} \in \Delta^{I-1}$.

Since, we want to perturb individual endowments only (perturbing preferences does not simplify the analysis), we take perturbations to be I-vectors, $o^{\mathcal{I}} = (o^1, ..., o^I) \in I\!R^I$. The equilibrium equations are then $h(\bar{s}, \bar{z}, \kappa, z_1, ..., z_S) = 0$ with $h = h^1, ..., h^5$ and

$$\begin{array}{rcl}
h_i^1 & = & -\bar{q}u_i'(\bar{c}^i) + \beta E_{\bar{s}}\left[(q(s) + d(s))u_i'(c^i(s))\right] + \kappa^i \\
h_{is}^2 & = & w^i(s) - \frac{\bar{\theta}^i \cdot (q(s)+d(s))}{\sum_{j=1}^{J} q_j(s)+d_j(s)} \\
h_{is}^3 & = & c^i(s) - w^i(s)\sum_{j\in\mathcal{J}}(q_j(s) + d_j(s)) + \theta^i(s) \cdot q(s) - (e^i(s) + o^i(s)) \\
h_{ij}^4 & = & \bar{\theta}_j^i \kappa_j^i \\
h_{js}^5 & = & \sum_{i\in\mathcal{I}} \theta_j^i(s) - 1
\end{array}$$

Kubler and Schmedders (2003) show that under standard assumptions on preferences and endowments competitive equilibria can be characterized by the expectations correspondence and that recursive ϵ equilibria always exist. Let $\rho = (\rho_q, (\rho_{c^i}, \rho_{\theta^i})_{i\in\mathcal{I}})$ denote the policy function associated with a recursive ϵ equilibrium.

5.2 Error analysis

The main problem in the error analysis is that, because agents are infinitely lived, the necessary perturbations to correct for errors in h^1 may propagate without bounds. For example, a small reduction in consumption in a given period may result in a bigger reduction in the subsequent period which in turn result to a further reduction in the third period and so on. As a result the perturbed values may move far away from the ϵ-equilibrium. Therefore, as we explained in Section 3, we are no longer able to show that the ϵ-equilibrium path approximates an economy in equilibrium. Instead, we need to perturb the distribution of wealth $w^{\mathcal{I}}$. As we move from period to period we must allow for small perturbation in the state of the economy in order to maintain closeness between all perturbed and ϵ-equilibrium values.

Given a (recursive) ϵ equilibrium $\mathcal{F}$ and given any $\bar{s}$, $\bar{z} \in \mathcal{F}_{\bar{s}}$, let

$$(\tilde{z}(1), ..., \tilde{z}(S)) = \arg \min_{(z_1,...,z_S)\in\mathcal{F}} \left(\min_{\kappa\in\mathcal{K}} \|h(\bar{s}, \bar{z}, \kappa, \mathbf{0}, z_1, ..., \mathbf{0}, z_S)\| \right)$$

As in Section 4, we can assume w.l.o.g. that Equations h^4 and h^5 hold with equality given $\tilde{z}_1, ..., \tilde{z}_S$.

In order for h^1 to hold with equality, we need to perturb marginal utilities through $c_1, ..., c_S$. Define the $S \times J$ payoff matrix

$$A(\bar{s}, \bar{z}) = \left(\beta\pi(s|\bar{s})(\tilde{q}_j(s) + d_j(s))\right)_{js}.$$

Given an agent i, define $\epsilon^i = (\epsilon^i_1, ..., \epsilon^i_J)$ with

$$\epsilon^i_j = \min_{\kappa \geq 0} |-\bar{q}_j u_i'(\bar{c}^i) + \beta E_{\bar{s}}\left[(\tilde{q}_j(s) + d_j(s))u_i'(\tilde{c}^i(s))\right] + \kappa| \text{ s.t. } \kappa\bar{\theta}^i_j = 0$$

Let $er^1(\bar{s}, \bar{z}) = \max_{i \in \mathcal{I}} \|A^+\epsilon^i\|$ and $\delta = \max_{s,z \in \mathcal{F}_s} er^1(s, z)$.

This denotes the maximum necessary perturbation of marginal utilities over the entire set $\mathcal{F}$ to obtain equality in h^1 given that marginal utilities last period were not perturbed. However, obviously this only constitutes a lower bound on total necessary perturbations.

Using the fact that we consider a recursive ϵ equilibrium, we can write current endogenous variables as functions of $w^{\mathcal{I}}$ alone and define

$$M(w^{\mathcal{I}}(1), \cdots, w^{\mathcal{I}}(S)) = \left(\beta\pi(s|\bar{s})(\rho_{q_j}(w^{\mathcal{I}}(s), s) + d_j(s))\right)_{js}.$$

Note that at $\tilde{w}$, $M(\tilde{w}) = A(\bar{s}, \bar{z})$.

We use the concept of balancedness as defined above to derive an upper bound. We now determine perturbations in next period's wealth distribution which guarantee that necessary perturbations in marginal utilities next period are within 2δ, given that perturbations of marginal utilities this period are within 2δ. For any $\bar{\mu} \in I\!R^I$ with $|\bar{\mu}_i - u_i'(\bar{c^i})| < 2\delta$, $\forall i$, we want to find $w^{\mathcal{I}}_1, ..., w^{\mathcal{I}}_S$ such that there exist $\tilde{\mu}_1, ..., \tilde{\mu}_S \in I\!R^I$ with $|\tilde{\mu}^i_s - u_i'(\rho_{c^i}(w^{\mathcal{I}}_s, s))| < 2\delta$ and with

$$-\bar{q}\bar{\mu}^i + \beta \sum_{s \in \mathcal{S}} \pi(s|\bar{s})[\rho_q(w^{\mathcal{I}}_s, s) + d_s]\tilde{\mu}^i_s = 0 \text{ for all } i \in \mathcal{I}.$$

Since the true wealth distribution is determined through $\bar{\theta}$ which cannot be perturbed in order to achieve equality in h^2, $\tilde{w}^{\mathcal{I}}$ as it appears in h^3 is fixed. However, the wealth distribution which supports the new consumptions and prices generally differ from $\tilde{w}^{\mathcal{I}}$. We then need to perturb endowments in h^3 in order to be able to support $\rho_q(w)$ and $\rho_c(w)$ at the predetermined wealth distribution $\tilde{w}$. We want to find $w^{\mathcal{I}}_1, ..., w^{\mathcal{I}}_S$ which minimize these perturbations in the endowments, i.e. which minimizes $|w^i_s \sum_{j \in \mathcal{J}}(\rho_{q_j}(w^{\mathcal{I}}_s, s) + d_j(s)) - \bar{\theta}^i \cdot (\rho_q(w^{\mathcal{I}}_s, s) + d(s))|$ at all $s \in \mathcal{S}$.

In order to ensure that this is possible for all $\bar{\mu}$ which are within 2δ from $u_i'(\bar{c}^i)$ it suffices to check that at the 3^I distinct points, $(u_i'(\bar{c}^i)(1 + 2\delta n_i))_{i \in \mathcal{I}}$ for all $n \in \{-1, 0, 1\}^I$, this can be achieved with the perturbations next period lying within one times δ, i.e. with

$$|\tilde{\mu}^i_s - u_i'(\rho_{c^i}(w^{\mathcal{I}}_s, s))| < \delta.$$

Since $\mathcal{F}$ is finite one can use grid search[6] to determine the following number. Given $n \in \{-1, 0, 1\}^I$ define

$$err^2(n) = \min_{w=(w_1^{\mathcal{I}},\ldots,w_S^{\mathcal{I}})} \left(\max_{i\in\mathcal{I}, s\in\mathcal{S}} |w_s^i \sum_{j\in\mathcal{J}} (\rho_{q_j}(w_s^{\mathcal{I}}, s) + d_j(s)) - \bar{\theta}^i \cdot (\rho_q(w_s^{\mathcal{I}}, s) + d(s))| \right)$$

subject to for all $i \in \mathcal{I}$,

$$\min_{\kappa\geq 0, \kappa\bar{\theta}^i=0} \left\| (M(w))^+ \left[\bar{q} u_i'(\bar{c}^i)(1 + 2n_i\delta) - M(w) \begin{pmatrix} u_i'(\rho_{c^i}(w(1), 1)) \\ \vdots \\ u_i'(\rho_{c^i}(w(S), S)) \end{pmatrix} - \kappa \right] \right\| < \delta.$$

Define

$$\Delta_1(\bar{s}, \bar{z}) = \max_{n\in\{-1,0,1\}^I} err^2(n) \text{ and } \overline{\Delta}_1 = \max_{s,z\in\mathcal{F}_s} \Delta_1(s, z).$$

The discussion above shows that this is an upper bound on perturbations in individual endowments necessary to offset the perturbations in the wealth distribution.

In addition we need to perturb individual endowments in order to obtain the 'correct' marginal utilities. For this, we define

$$\Delta_2(\bar{s}, \bar{z}) = \max_{i\in\mathcal{I}, s\in\mathcal{S}} \max\{u_i'^{-1}\left(u_i'(\bar{c}^i(s))(1 + 2\delta)\right) - \bar{c}^i(s), |u_i'^{-1}\left(u_i'(\bar{c}^i(s))(1 - 2\delta)\right) - \bar{c}^i(s)|\}$$

and $\overline{\Delta}_2 = \max_{s,z\in\mathcal{F}_s} \Delta_s(s, z)$.

The following theorem now summarizes the above discussion.

THEOREM 2 *The ϵ-equilibrium $\mathcal{F}$ weakly approximates an economy in equilibrium with precision* $\overline{\Delta}_1 + \overline{\Delta}_2$.

Note that in order to achieve $(1, \delta)$ balancedness, it seems crucial that there are sufficiently many states compared to the number of assets. In particular, with this construction will generally not be possible when markets are complete. We now turn to several examples which illustrate that when markets are incomplete, this construction often results in reasonable error bounds.

5.3 Simple parametric example

We consider two examples to illustrate the analysis above. In both examples there are two agents with identical CRRA utility and a coefficient of risk aversion of 2. There are 4 shocks which are i.i.d. and equi-probable. Dividends and endowments in two cases are as follows

1. There is a single tree. Dividends are $d(s) = 1$ for all $s = 1, ..., 4$. Individual endowments are $\mathbf{e}^1 = (2, 5, 2, 5), \mathbf{e}^2 = (5, 2, 5, 2)$.

[6] In the example below this issue is dealt with more sophisticatedly.

2. There are two trees. Dividends are $d_1(1) = d_1(2) = 1$, $d_1(3) = d_1(4) = 2$ and $d_2(1) = d_2(3) = 1$, $d_2(2) = d_2(4) = 2$. Individual endowments are $\mathbf{e}^1 = (1,2,1,2), \mathbf{e}^2 = (2,1,2,1)$.

Since there are only two agents, the endogenous state space for the recursive ϵ equilibrium simply consists of the interval $[0,1]$. We use the algorithm described in Kubler and Schmedders (2003) and discretize the state space into 10^8 possible wealth levels to obtain an ϵ-equilibrium. The resulting maximal error lies around 10^{-3} in both examples. The necessary (relative) perturbations in individual endowments lie around 4.3×10^3 in the first specification and around 1.2×10^{-2} for the second specification. The maximal error in Euler equations along one simulated path for 20000 simulated periods lies around 10^{-5}. This is a large discrepancy, but it is mainly caused by the fact that along a simulated path many areas of the state space are not visited and some errors are simply 'missed'. On the other hand, a maximal perturbation of 1.2 percent might still be viewed as acceptable if one is only interested in moments of asset prices.

Appendix: Proofs of lemmas

Proof of Lemma 1. For a given (y, x) on the boundary of P, $(y, x) \in \partial P$, we need to find $(z, x) \in \partial \tilde{P}$ which minimizes the distance to (y, x).

For any $x \in X^i$, let $\omega \cdot x = \sum_{s^t \in \Sigma} \beta^t \pi(s^t)\omega(s^t)x(s^t)$. By definition, we have that

$$U(z) + \omega \cdot z = U(x) + \omega \cdot x, \quad U(x) = U(y)$$

and so,

$$U(z) + \omega \cdot z = U(y) + \omega \cdot x.$$

(Note that we can freely rearrange terms in all series since all of them are absolutely convergent.) By Taylor's Theorem

$$\sum_{s^t \in \Sigma} \beta^t \pi(s^t)(u(z(s^t)) - u(y(s^t))) = \sum_{s^t \in \Sigma} \beta^t \pi(s^t) D_x u(\xi(s^t))(z(s^t) - y(s^t))$$

for some $\xi(s^t) \in I\!R^L$, $\|\xi(s^t)\| \leq \bar{c}$ for all $s^t \in \Sigma$. Therefore

$$\sum_{s^t \in \Sigma} \beta^t \pi(s^t) D_x u(\xi(s^t))(z(s^t) - y(s^t)) = \sum_{s^t \in \Sigma} \beta^t \pi(s^t)\omega(s^t)(x(s^t) - z(s^t)).$$

Since for all s^t, $|\omega(s^t)(z(s^t) - x(s^t))| \leq L\bar{\omega}\bar{c}$, and with the bound on marginal utility, $\bar{m}$, we obtain

$$\|z - y\| \leq L\frac{\bar{\omega}}{\bar{m}}\bar{c},$$

which proves the lemma. $\square$

Proof of Lemma 2. For each s, z_s, the system of Equations h^7 in linear in prices. We write it as follows

$$-(p_1 \cdots p_j \cdots p_K) + (p_{m+1} \cdots p_n) \begin{pmatrix} J_{21} & J_{22} \end{pmatrix} = \epsilon - (p_{K+1} \cdots p_m) \begin{pmatrix} J_{11} & J_{12} \end{pmatrix}$$

Equivalently

$$-(p_{j+1} \cdots p_K, p_{m+1} \cdots p_n) \begin{pmatrix} 0 & I \\ J_{21} & J_{22} \end{pmatrix} = \epsilon - (p_1 \cdots p_j, p_{K+1} \cdots p_m) \begin{pmatrix} I & 0 \\ J_{11} & J_{12} \end{pmatrix}$$

Using the definition of the pseudo inverse implies the lemma. □

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