Formulation of Incomplete Information and Redundant Type Structure

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Abstract

A type structure is non-redundant if no two types of a player represent the same hierarchy of belief over the given set of basic uncertainties, and it is redundant otherwise. Under a mild assumption (which is both necessary and sufficient), we show that any redundant structure can be identified with a non-redundant structure with an extended space of basic uncertainties. The belief hierarchies induced by the latter non-redundant structure, when "marginalized" onto the partial space of basic uncertainties, coincide with the hierarchies induced by the former structure. We argue that the analyst must make use of a non-redundant structure unless he believes that he misspecified the players' space of basic uncertainties, and redundant structures can provide different equilibrium predictions only in so far as they reflect the idea that there are hidden uncertainties entertained by players but unspecified by the analyst.

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1 Introduction

Players' beliefs and higher order beliefs are often important in an interactive strategic situation. Harsanyi (1967-68) proposes that a probabilistic type structure implicitly describes players' belief hierarchies over a set of payoff-relevant parameters, which enables extending the concept of strategic equilibrium to games with incomplete information.

Harsanyi's idea is formulated by Mertens and Zamir (1985)[1]. Under certain topological assumptions, they show that any *coherent* hierarchy of belief can be modelled as a type in a *non-redundant* universal type structure—where non-redundancy roughly says that two different types of each player should specify different hierarchies of beliefs over the given set of parameters.

There is a class of type structures (including one non-redundant structure) that could be used to model a given set of coherent hierarchies of belief. Researchers have observed (see Battigalli and Siniscalchi (2003), Ely and Peski (2006), and Dekel et al.(2006)) that two different type structures within this class may yield different equilibrium predictions. Let us first look at the following example which simplifies the leading example in Ely and Peski (2006).

Consider the following game[2], where Alice chooses rows $\{U, D\}$, Bob chooses columns $\{L, R\}$, and Nature chooses matrices $\{s_1, s_2\}$:

s_1	L	R
U	1,1	0,0
D	0,0	1,1

s_2	L	R
U	0,0	1,1
D	1,1	0,0

Figure 1: Coordination Under Uncertainty

Write $S = \{s_1,\ s_2\}$. First consider the type structure $\Lambda = \langle S; T^a, T^b; \lambda^a, \lambda^b \rangle$: for each

[1]See also Brandenburger and Dekel (1993), Heifetz (1993), and Mertens et al. (1994).

[2]I am grateful to an anonymous referee for providing the succinct description and discussion of the example.

player i, take the type set $T^i = \{t^i\}$ as a singleton and $\lambda^i : T^i \to \Delta(S \times T)$ with

$$\lambda^i(t^i)(s_1, t^i, t^{-i}) = \lambda^i(t^i)(s_2, t^i, t^{-i}) = \frac{1}{2}.$$

That is, type t^i of player i assigns equal probability to (s_1, t^i, t^{-i}) and (s_2, t^i, t^{-i}). Any Bayesian equilibrium associated with this type structure yields each player an expected payoff of $\frac{1}{2}$. Next consider the structure $\Phi = \langle S; U^a, U^b; \phi^a, \phi^b \rangle$: for each player i, take $U^i = \{u^i, v^i\}$ and $\phi^i : U^i \to \Delta(S \times U)$ with

$$\phi^i(u^i)(s_1, u^i, u^{-i}) = \phi^i(u^i)(s_2, u^i, v^{-i}) = \frac{1}{2},$$

$$\phi^i(v^i)(s_1, v^i, v^{-i}) = \phi^i(v^i)(s_2, v^i, u^{-i}) = \frac{1}{2}.$$

In this case, there exists a Bayesian equilibrium, namely (σ^a, σ^b), with $\sigma^a(u^a) = U$, $\sigma^a(v^a) = D$, $\sigma^b(u^b) = R$, and $\sigma^b(v^b) = L$. Under this equilibrium, the two players each have an expected payoff of 1.

In both structures, each player has exactly the same hierarchies of belief about the matrix nature chooses. For example, Alice knows (believes with probability 1) that each matrix is chosen with probability $\frac{1}{2}$, she knows that Bob knows this, and she knows that Bob knows that she knows this, and so on ad infinitum. Yet, there is a prediction in the second model that is not a prediction in the first.

This seems peculiar. Players reason within a particular language and this language captures their hierarchies of belief. We—the analysts—use a type structure as a model of this language—i.e. as a model of the hierarchies. We then go on to make a prediction based on this model. Yet, the prediction we provide depends on the model we choose.

In the example above, Λ satisfies the non-redundancy condition, while Φ does not since the two types of each player have the same belief hierarchies. *Technically*, the new prediction associated with Φ arises from a particular *correlation* between redundant types and payoff-relevant parameters: for instance, "type" u^a of Alice will know s_1 is chosen by Nature were she to know that Bob's true "type" is v^b. However, this technical explanation in terms of correlation doesn't resolve the conceptual question. Alice's single hierarchy summarizes all the information that Alice has about Nature's move and Bob's beliefs about it. How, then, can Alice distinguish the two "types" of Bob if she cannot describe their difference using her own language? In effect, Alice cannot even distinguish her own two types.

This observation suggests that the new prediction associated with the redundant structure Φ utilizes more information than the players themselves actually have and hence we should restrict attention to a non-redundant type structure to conduct Bayesian equilibrium analysis. However, in conjunction with the Bayesian equilibrium given above, the redundant type structure Φ reflects an interesting belief held by Alice: type u^a of Alice assigns probability $\frac{1}{2}$ to (i) Nature choosing s_1 and Bob playing R and (ii) Nature choosing s_2 and Bob playing L. The Bayesian equilibrium framework, following the natural problem-solution paradigm, first formulates the "game situation" by modelling the interactive beliefs over the set of payoff-relevant parameters via a type structure, and then solves the game by imposing the solution concept. But if we restrict attention to the non-redundant type structure in the problem-formulating stage, as in the example above, we necessarily preclude "correlated beliefs" in the problem-solving stage.

This seems puzzling: redundant type structures are conceptually problematic but Bayesian framework based on non-redundant structures excludes certain interesting strategic situations. One solution to this problem is to account for the redundancy and hence for the predictions based on it. Since the redundancy results from the formulation of players' language in the Bayesian framework, it should be understood independently of solution concepts.[3] This paper argues that the redundancy reflects some basic uncertainties entertained by players but unspecified by the analyst, and hence a redundant type structure corresponds to the non-redundant structure with an extended space of basic uncertainties.

[3] See also section 4 for the discussion of related papers.

Let us consider the motivating example again. Suppose the payoff-relevant parameters s_1 and s_2 indicate next week's gasoline price: high/low. This price can be nailed down if information from both the demand side and the supply side are known, but signals from one side alone are insufficient. If Alice and Bob learn signals only from the supply side, say they both work for a monopolistic oil company, then Λ is a reasonable type structure to model this situation. However, if Alice sees a private signal from the demand side and Bob receives a private signal from the supply side, then Φ, instead of Λ, is a reasonable type structure to apply to this situation.

If the analyst knows only the payoff structures, and he is unaware of (or unable to specify) some other variables reasoned about by players but he is aware of his unawareness (or misspecification), then a redundant type structure is a "safe" modelling choice: players "reason" within a redundant structure *as if* they reason about some parameters unknown to the analyst. In other words, the analyst should not make use of a redundant structure unless he believes that he misspecified the players' space of basic uncertainties.

This observation might be relevant in practice. It is perhaps sufficient for an auction designer to specify only the payoff structures for an *anonymous online* bidding game, but other variables that are hard to specify or easy to neglect could be strategically important in an *auction house*: for example, whether a bidder shows up in a dress with particular colors may impact the auction results.

This paper presents and proves the following results. With one mild assumption (which is both necessary and sufficient), Theorem 1 shows that any redundant type structure can be identified with some non-redundant structure with an expanded set of underlying uncertainties; the hierarchies generated by the latter, when "marginalized" onto the partial space of underlying uncertainties, coincide with the hierarchies generated by the former. Theorem 2 further shows that the two structures in Theorem 1 yield the same Bayesian equilibrium predictions. So the predictions based on a redundant structure are indeed interesting in that they reflect the equilibria on a full parameter space.

The Bayesian equilibrium framework, as a normative theory, states that a player's action choice should be derived from the game situation. Seen from some descriptive models, what *eventually* matters for a player's decision besides his uncertainties about payoff structures is his belief about the opponents' action choices, even though the player may arrive at such a belief after *a long way of thinking*. This suggest that, technically, a set of variables parameterizing both actions and payoffs is rich enough to capture all strategically relevant uncertainties for Bayesian equilibrium analysis. This intuition is confirmed by Theorem 3.

The rest of the paper is organized as follows. In section 2, we set up the framework, prove several basic properties and present the main theorems; in section 3, we provide proofs for the theorems we offer, and in section 4 we review the related literature and situate the current work in a discussion. Proofs not central to the paper are relegated to the appendix.

2 Type Structures and Incomplete Information

2.1 Preliminaries

We carry out the analysis in a purely measure-theoretic setup: all spaces and all functions involved are required to be measurable. Measurability is the basic requirement of a probabilistic type structure, which allows us to see the essence of the results by avoiding the interference of topological properties. The topological assumptions are important in the literature for (i) proving that any coherent hierarchies can be modeled by types and (ii) proving the existence of Bayesian equilibria. However, neither of these concerns is central to the current paper: we take type structures as given and compare equilibria on different structures. More importantly, any topological property imposed on a redundant type structure which we know little about results in some loss of generality.[4]

Heifetz and Samet (1998) studied the topology-free formalism. Let (X, Σ) be a measurable

[4] For example, the universal type space based on a Polish parameter space is Polish (Brandenburger and Dekel (1993)), but a redundant type space can have different topology.

space with a σ-field Σ. Let $\Delta(X)$ denote the measurable space of all countably additive probability measures on X with the σ-field Σ_Δ generated by all sets of the form $\beta^p(E) = \{\mu \in \Delta(X) : \mu(E) \geq p\}$ where $E \in \Sigma$ and $0 \leq p \leq 1$. We also write $\Sigma(X)$ as the σ-field of X for any given measurable space X when the context is clear. For $x \in X$, we denote by $\delta_x \in \Delta(X)$ the *evaluation* at x : for any $E \in \Sigma(X)$, $\delta_x(E) = 1$ if $x \in E$. We consider any product of measurable spaces with the product σ-field and any subspace of a measurable space with the relative σ-field.

For a given measure $\mu \in \Delta(X)$ and a measurable map $f : X \to Y$, let μf^{-1} be the image measure of μ under f. That is, $\mu f^{-1} \in \Delta(Y)$ with $\mu f^{-1}(A) = \mu(f^{-1}(A))$ for any measurable subset A of Y. If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, then (f_1, f_2) is the map $f : X_1 \times X_2 \to Y_1 \times Y_2$ with $f(x_1, x_2) = (f(x_1), f(x_2))$, $x_1 \in X_2$ and $x_2 \in X_2$.

A measurable space (X, Σ) is *separative* if for every pair of distinct points in X there is a measurable set containing one and not the other. In this case, we call Σ a separative σ-field.[5]

The concept of separativity does not appear in the economics literature before but turns out to be crucial in this paper. Instead of imposing this property on all measurable spaces, I invoke it only when necessary to examine its implications. Here are some useful facts for further reference.

Lemma 1 *(1)* $(\Delta(X), \Sigma_\Delta)$ *is separative (even though* X *may not be).*

(2) (X, Σ) *is separative if and only if* $\delta_x \neq \delta_y$ *for any distinct points* x *and* y*.*

(3) Any product of separative measurable spaces is separative.

(4) If $f : (X, \Sigma(X)) \to (Y, \Sigma(Y))$ *is a measurable injective map and* $(Y, \Sigma(Y))$ *is separative, then* $(X, \Sigma(X))$ *is separative.*

Proof. (1) Consider two distinct probability measures $\mu_1, \mu_2 \in \Delta(X)$. Since $\mu_1 \neq \mu_2$, there exists $E \in \Sigma$ and $0 \leq p \leq 1$ such that $\mu_1(E) \geq p > \mu_2(E)$. This implies that $\mu_1 \in \beta^p(E)$ and $\mu_2 \notin \beta^p(E)$. Thus μ_1 and μ_2 can be separated by the measurable set $\beta^p(E) \in \Sigma_\Delta$.

[5] Separativity is rather a weak property. The following observation related to topological spaces is helpful: any Borel space induced by a topology satisfying the T_1 separation axiom (e.g. Royden (1988)), including the Hausdorff topology, is separative.

$(\Delta(X), \Sigma_\Delta)$ is separative. (2) If (X, Σ) is separative, then for any distinct points x and y there exists $E \in \Sigma$ such that $x \in E$ and $y \notin E$. Thus $\delta_x(E) = 1$ and $\delta_y(E) = 0$. $\delta_x \neq \delta_y$. Conversely, if $\delta_x \neq \delta_y$, then there exists $E \in \Sigma$ such that $\delta_x(E) = 1$ and $\delta_y(E) = 0$, and hence $x \in E$ and $y \notin E$. That is, (X, Σ) is separative. (3) follows from the definition of product σ-field. (4) Consider two distinct points x and x' in X. $f(x) \neq f(x')$ since f is injective. By the separativity of $(Y, \Sigma(Y))$, there exists measurable subset $E \subset Y$ such that $f(x) \in E$ and $f(x') \notin E$. $f^{-1}(E)$ is a measurable subset of X containing x and not x'. Therefore, $(X, \Sigma(X))$ is separative. ■

2.2 Type Structures

Let $I = \{1, 2, ..., n\}$ be a finite set of n individuals.

Definition 1 *An X-based type structure is a collection $\langle X, (T_i)_{i \in I}, (m_i)_{i \in I} \rangle$, or $\langle X, T, m \rangle$ for short, such that*

(1) X and T_i, for $i \in I$, are measurable spaces.

(2) For each $i \in I$, m_i is a measurable function $m_i : T_i \to \Delta(X \times T)$.

(3) For each $i \in I$ and $t_i \in T_i$, the marginal of $m_i(t_i)$ on T_i is δ_{t_i}.

A type structure $\langle X, T, m \rangle$ is separative if T_i is separative for each $i \in I$.

Separative type structures impose separativity restrictions on the type spaces T_i's but not on the parameter space X. From Lemma 1, $\delta_{t_i} \neq \delta_{t_i'}$ for any distinct player i types t_i and t_i' if and only if T_i is separative. In a sense, if $\delta_{t_i} = \delta_{t_i'}$ for some $t_i' \neq t_i$ then player i cannot distinguish t_i and t_i'. Thus separativity characterizes the notion that "a player knows his own type."[6] Without separativity, condition (3) above only indicates a weaker notion of "self-conscious." This paper treats separative type structures as a strict subclass of all type structures and studies their implications.

[6] I am grateful to a referee for his/her question that led to the investigation of the role of separativity. Again, "knowledge" is treated as "belief with probability 1" here.

One special form of type structures $\langle S \times C, U, \phi \rangle$ is particularly interesting. S is interpreted as a partial description of the full parameter space $S \times C$ and we will refer to it as a *partial parameter space*. Let $\langle S \times C, (U_i)_{i \in I}, (\phi_i)_{i \in I} \rangle$ and $\langle S \times C', (U'_i)_{i \in I}, (\phi'_i)_{i \in I} \rangle$ be two type structures. We define maps from one type structure to the other that preserve the structure of beliefs over the common partial parameter space S.

Definition 2 *Let $\tau_i : U_i \to U'_i$ be a measurable function for each i. The map $\tau = (\tau_1, \cdots, \tau_n)$ from U to U' is an S-based type morphism if for each $i \in I$ and $u_i \in U_i$,*

$$marg_{S \times U'} \phi'_i(\tau_i(u_i)) = (marg_{S \times U} \phi_i(u_i))(id_S, \tau)^{-1},$$

where (id_S, τ) is the natural map from $S \times U$ to $S \times U'$.[7] *τ is an S-based type isomorphism if τ is in addition an isomorphism.*

The following diagram illustrates the definition. If both C and C' are singleton sets, i.e., S is identified with the full parameter space, then an S-based type morphism is the standard concept of type morphism defined via the commutative diagram (e.g. Heifetz and Samet (1998)). In the general case, we consider the marginals of ϕ_i and ϕ'_i.

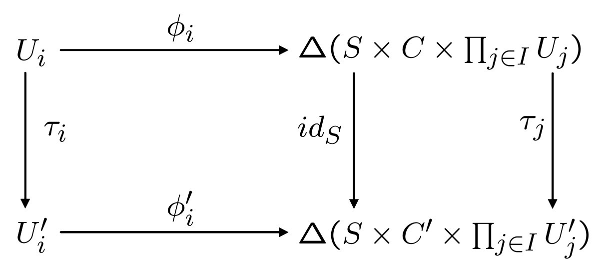

Figure 2: S-based type morphism.

An X-based type structure $\langle X, T^*, m^* \rangle$ is *universal*, if for every X-based type structure $\langle X, T, m \rangle$, there is a unique X-*based* morphism from T to T^*.

[7] (id_S, τ) is a well-defined jointly measurable function by the following result, which is an easy adaptation of Aliprantis and Border (1999, Lemma 4.48): Let $(X_1, \Sigma(X_1))$, $(X_2, \Sigma(X_2))$, $(Y_1, \Sigma(Y_1))$ and $(Y_2, \Sigma(Y_2))$ be measurable spaces, and let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be measurable. Then the induced map $f = (f_1, f_2)$ from $(X_1 \times X_2, \Sigma(X_1) \otimes \Sigma(X_2))$ to $(Y_1 \times Y_2, \Sigma(Y_1) \otimes \Sigma(Y_2))$ is jointly measurable.

2.3 Belief Hierarchies

Given a measurable space X, the space of X-based belief hierarchies is defined inductively. The space of the first order beliefs for player i is $H_i^1(X) = \Delta(X)$. For each $k \geq 1$, denote $H^k(X) = \Pi_{i\in I} H_i^k(X)$. The space of $k+1$th order beliefs for player i is $H_i^{k+1}(X) = \Delta(X \times \Pi_{l=1}^k H^l(X))$. Hence, the space of X-based belief hierarchies is $H_i(X) = \Pi_{k=1}^{\infty} H_i^k(X)$. $H_i(X)$ with the product σ-field is separative (Lemma 1 (1) and (3)).

For an $(S \times C)$-based type structure $\Phi = \langle S \times C, U, \phi\rangle$, we are interested in two different belief hierarchies, the $(S\times C)$-based hierarchy—the full hierarchy based on the full parameter space $S\times C$—and the S-based hierarchy—the partial hierarchy based on the partial parameter space S.

Full Hierarchy. In the "full hierarchy" map $\widetilde{h} = (\widetilde{h})_{i\in I}$, $\widetilde{h}_i : U_i \to H_i(S \times C)$ associates a hierarchy of belief over $S \times C$ with each type of player i. The first order belief map for player i, $\widetilde{h}_i^1 : U_i \to H_i^1(S \times C)$, is given by

$$\widetilde{h}_i^1(u_i) = \phi_i(u_i)(\text{proj}_{S\times C})^{-1},$$

where $\text{proj}_{S\times C}$ is the projection operator from $S \times C \times U$ to $S \times C$. Thus for any $u_i \in U_i$ and any measurable set $E \subset S \times C$,

$$\widetilde{h}_i^1(u_i)(E) = \phi_i\,(u_i)(E \times U)\,.$$

Let us write $\widetilde{h}^k = (\widetilde{h}_i^k)_{i\in I}$ for $k \geq 1$. Define map $\widetilde{p}^k : S\times C\times U \to S\times C\times \Pi_{l=1}^k H^l(S\times C)$ with $\widetilde{p}^k((s,c,u)) = (s, c, \widetilde{h}^1(u), ..., \widetilde{h}^k(u))$. That is, for each "state of the world" $(s,c,u) \in S \times C \times U$, $\widetilde{p}^k$ specifies the "state of nature" (s,c) and the beliefs up to order k for type profile u. Inductively, define the $k+1$th order belief map for player i, $\widetilde{h}_i^{k+1} : U_i \to H_i^k(S\times C)$, as

$$\widetilde{h}_i^{k+1}(u_i) = \phi_i(u_i)(\widetilde{p}^k)^{-1}. \tag{1}$$

(1) is well defined since $\widetilde{p}^k$'s are measurable (this, together with the measurability of $\widetilde{h}_i$, is shown in Lemma 5 in the appendix). Finally, define $\widetilde{h}_i = (\widetilde{h}_i^1, \widetilde{h}_i^2, \cdots)$.

Partial Hierarchy. The "partial hierarchy" map $h_i : U_i \to H_i(S)$, which generates the hierarchies of belief over the partial parameter space S, is defined similarly by replacing $H_i^k(S \times C)$ with $H_i^k(S)$ and $\text{proj}_{S\times C}$ with proj_S in the definition above:

The first order belief map for player i, $h_i^1 : U_i \to H_i^1(S)$, is given by

$$h_i^1(u_i) = \phi_i(u_i)(\text{proj}_S)^{-1}.$$

Define function $p^k : S \times C \times U \to S \times \Pi_{l=1}^k H^l(S)$ with $p^k((s, c, u)) = (s, h^1(u), ..., h^k(u))$. Inductively,

$$h_i^{k+1}(u_i) = \phi_i(u_i)(p^k)^{-1}. \tag{2}$$

Finally $h_i = (h_i^1, h_i^2, \cdots)$. Intuitively, the S-based hierarchies are obtained by "marginalizing" the $(S \times C)$-based hierarchies onto the partial space S.

The following result says that an S-based type morphism preserves S-based hierarchies. Its proof can be found in the appendix.

Proposition 1 *Suppose $\Phi = \langle S \times C, U, \phi\rangle$ and $\Phi' = \langle S \times C', U', \phi'\rangle$ are two type structures with S-based belief hierarchy maps h and h', respectively, and $\tau : U \to U'$ is an S-based type morphism, then $h_i' \circ \tau_i = h_i$ for each $i \in I$.*

By this result, Φ and Φ' will have the same set of S-based belief hierarchies if there is an S-based type isomophism between them.

2.4 Redundant and Non-Redundant Structures

For a type structure $\langle X, T, m\rangle$, let $\sigma(\widetilde{h}_i)$ be the smallest σ-field of subsets of T_i for which the X-based hierarchy map $\widetilde{h}_i$ is measurable. The next definition formalizes the notion of non-redundancy.

Definition 3 *A type structure* $\langle X, T, m \rangle$ *is non-redundant if for each* $i \in I$, $\sigma(\widetilde{h}_i)$ *is a separative* σ*-field. A type structure is redundant if it is not non-redundant.*

The following result links the formal definition with our familiar intuition: non-redundancy means no two types of a player have the same belief hierarchies over the full parameter space.

Proposition 2 *(1)* $\langle X, T, m \rangle$ *is non-redundant if and only if* $\widetilde{h}_i : T_i \to H_i(X)$ *is injective. (2) A non-redundant type structure is separative. (3) An* X*-based universal type structure* $\langle X, T^*, m^* \rangle$ *is separative.*

Proof. $\sigma(\widetilde{h}_i)$ consists exactly of sets $(\widetilde{h}_i)^{-1}(E)$ for any measurable set E in $H_i(X)$ (e.g. Billingsley (1995, Theorem 20.1)). Thus $\sigma(\widetilde{h}_i)$ is separative if and only if for any distinct points t_i and t_i' in T_i there exists a measurable subset E in $H_i(X)$ such that $t_i \in (\widetilde{h}_i)^{-1}(E)$ and $t_i' \notin (\widetilde{h}_i)^{-1}(E)$. That is, $\sigma(\widetilde{h}_i)$ is separative if and only if $\widetilde{h}_i$ is injective. (1) then follows from Definition 3. If $\langle X, T, m \rangle$ is non-redundant, then the σ-field on T_i is finer than $\sigma(\widetilde{h}_i)$. $\sigma(\widetilde{h}_i)$ is separative by Definition 3 and hence T_i is separative, as required by (2). In a universal structure, T_i^* is identified with a subspace of $H_i(X)$ and thus a universal structure is separative by (1). ■

Remark. Definition 3 and (1) in Proposition 2 are similar to, but different from, those in Mertens and Zamir (1985, Definition 2.4 and Proposition 2.5) for topological belief spaces. The definitions of non-redundancy are similar in that they have the same intuitive motivation, i.e. the injectivity of the hierarchy maps. The definitions differ in that Mertens and Zamir's (1985) treatment imposes stronger requirements: speaking in the context of topology-free product type structures, Mertens and Zamir also require X to be separative in a non-redundant structure. So the converse of their Proposition 2.5 which links the formal definition and the motivation doesn't hold. ■

We are now ready to introduce the central concept in this paper.

Definition 4 *Fix an S-based type structure $\Lambda = \langle S, T, \lambda \rangle$. $\Phi = \langle S \times C, U, \phi \rangle$ is called an expansion of Λ via C if Φ is non-redundant and there is an S-based type isomorphism $\tau : T \to U$.*

In conjunction with Proposition 1, this definition implies that if Φ is an expansion of Λ via C, then the S-based hierarchies induced by Φ are identified with the S-based hierarchies induced by Λ through τ.

It follows from the definition that separativity is necessary for a type structure to have an expansion.

Lemma 2 *If Λ has an expansion, then Λ is separative.*

Proof. If Φ is an expansion of Λ, then there is a measurable isomorphism $\tau_i : T_i \to U_i$. Since Φ is non-redundant, U_i is separative by Proposition 2(2), and hence T_i is separative (separativity is preserved by the measurable isomorphism τ_i^{-1}—Lemma 1 (4)). ■

It turns out that separativity is also sufficient for a type structure to have an expansion.

Theorem 1 *For any S-based separative type structure $\Lambda = \langle S, T, \lambda \rangle$, there is a set C such that $\Phi = \langle S \times C, U, \phi \rangle$ is an expansion of Λ via C.*

By this theorem, any S-based type structure Λ that is redundant in terms of S-based hierarchies can be interpreted as a non-redundant type structure based on an enlarged set of parameters $S \times C$. This solves the conceptual difficulties of redundant structures: they capture certain "hidden variables" involved in reasoning by the players.

In later proofs of the main results, I construct type structures by deriving new probability measures from probability measures on some semirings of measurable rectangles via the Caratheodory Extension Procedure. The following lemma is applied to verify the measurability of functions whose ranges are probability measures. Heifetz and Samet (1998, Lemma 4.5) have a result for fields.

Lemma 3 *Let (X, Σ) be a measurable space and $\mathcal{S}$ be a semiring that generates $\Sigma : \sigma(\mathcal{S}) = \Sigma$. Let $\mathcal{S}_\Delta$ be the σ-field on $\Delta(X)$ generated by sets of the form $\beta^p(E) = \{\mu \in \Delta(X) : \mu(E) \geq p\}$ for $E \in \mathcal{S}$ and $0 \leq p \leq 1$. Then $\Sigma_\Delta = \mathcal{S}_\Delta$.*

Proof. $\mathcal{S}_\Delta \subset \Sigma_\Delta$ by definition. To prove the other direction, consider $\mathcal{L} = \{E \in \Sigma : \beta^p(E) \in \mathcal{S}_\Delta, 0 \leq p \leq 1\}$. $\Sigma_\Delta \subset \mathcal{S}_\Delta$ will follow if $\Sigma \subset \mathcal{L}$. Notice that $\mathcal{S} \subset \mathcal{L}$ and $\mathcal{S}$ is a π-system. It suffices to show $\mathcal{L}$ is a λ-system, as $\Sigma = \sigma(\mathcal{S}) \subset \mathcal{L}$ by Dynkin's π-λ theorem (e.g. Billingsley (1995, Theorem 3.2)).

(1) $X \in \mathcal{L}$ is obvious.

(2) Suppose $E \in \mathcal{L}$. Then $\beta^p(E) \in \mathcal{S}_\Delta$ for each $0 \leq p \leq 1$.

$$\{\mu : \mu(E) > 1 - p\} = \bigcup_{n=1}^{\infty} \{\mu : \mu(E) \geq 1 - p + 1/n\} \in \mathcal{S}_\Delta.$$

Therefore $\beta^p(E^c) = \{\mu : \mu(E^c) \geq p\} = \{\mu : \mu(E) \leq 1-p\} = \Delta(X) - \{\mu : \mu(E) > 1-p\} \in \mathcal{S}_\Delta$. We have $E^c \in \mathcal{L}$.

(3) Consider a finite sequence $\{E_n\}_{n=1}^m$ of pairwise disjoint sets in $\mathcal{L}$. By countable additivity,

$$\beta^p(\bigcup_{n=1}^{m} E_n) = \{\mu : \sum_{n=1}^{m} \mu(E_n) \geq r\}.$$

The latter can be written as

$$\bigcup_{q_1,\ldots,q_{m-1} \text{ are rationals}} \{\mu : \mu(E_1) \geq q_1\} \cap \ldots \cap \{\mu : \mu(E_{m-1}) \geq q_{m-1}\} \cap \{\mu : \mu(E_m) \geq r - \sum_{n=1}^{m-1} q_n\} \quad (3)$$

(3) is a countable union of finite intersections of measurable sets, and hence is measurable. Thus $\beta^p(\bigcup_{n=1}^m E_n) \in \mathcal{S}_\Delta$. We then have $\bigcup_{n=1}^m E_n \in \mathcal{L}$.

(4) Consider a pairwise disjoint sequence $\{E_n\}$ in $\mathcal{L}$. By countable additivity and (3),

$$\beta^p(\bigcup_{n=1}^{\infty} E_n) = \{\mu : \sum \mu(E_n) \geq p\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{\mu : \sum_{n=1}^{m} \mu(E_n) \geq p - 1/k\} \in \mathcal{S}_\Delta.$$

Therefore $\bigcup_{n=1}^{\infty} E_n \in \mathcal{L}$.

It follows immediately from (1), (2) and (4) that $\mathcal{L}$ is a λ-system. The results then follows from Dynkin's π-λ theorem. ■

Proof of Theorem 1. Set $C = T$ and $U_i = T_i$. For each $t_i \in T_i$, define on the semiring $\mathcal{S} = \Sigma(S) \times \Sigma(T) \times \Sigma(T)$ a set function as follows: $\phi_i(t_i)(A \times E \times F) = \lambda_i(t_i)(A \times (E \cap F))$ for a measurable rectangle $A \times E \times F \in \mathcal{S}$. For each $t_i \in T_i$, $\phi_i(t_i)$ defines a countably additive probability measure on the semiring $\mathcal{S}$. To verify the countable additivity, consider a pairwise disjoint sequence $\{A_n \times E_n \times F_n\}$ in $\mathcal{S}$ such that

$$\bigcup_{n=1}^{\infty} A_n \times E_n \times F_n = A \times E \times F \in \mathcal{S}.$$

Then $\{A_n \times (E_n \cap F_n)\}$ is a pairwise disjoint sequence in $\Sigma(S \times T)$ and

$$A \times (E \cap F) = \bigcup_{n=1}^{\infty} A_n \times (E_n \cap F_n).$$

Therefore, by definition and countable additivity of $\lambda_i(t_i)$,

$$\begin{aligned} \phi_i(t_i)(A \times E \times F) &= \lambda_i(t_i)(A \times (E \cap F)) \\ &= \sum_{n=1}^{\infty} \lambda_i(t_i)(A_n \times (E_n \cap F_n)) \\ &= \sum_{n=1}^{\infty} \phi_i(t_i)(A_n \times E_n \times F_n). \end{aligned}$$

It is straightforward to check other conditions for $\phi_i(t_i)$ to be a probability measure on $\mathcal{S}$. By the Caratheodory Extension Theorem (e.g. Aliprantis and Border (1999, Theorems 9.22)), $\phi_i(t_i)$ extends uniquely to a probability measure on $\Sigma(S \times T \times T)$, which is still written as $\phi_i(t_i)$ with some abuse of notations.

Let us verify that $\Phi = \langle S \times T, (T_i)_{i \in I}, (\phi_i)_{i \in I} \rangle$ defines a type structure. To avoid confusing "T" in the role of the parameter space with "T" in the role of the space of types, write

$\Phi = \langle S \times C, (T_i)_{i\in I}, (\phi_i)_{i\in I}\rangle$ with $C_i = T_i$ and denote $c_i : T_i \to C_i$ as the identity.

(i) $\phi_i : T_i \to \Delta(S \times C \times T)$ is measurable with respect to the natural σ-fields.

Since $\Sigma(S \times C \times T)$ is generated by the semiring $\mathcal{S} = \Sigma(S) \times \Sigma(C) \times \Sigma(T)$, by Lemma 3 we only need to check the following: For any measurable rectangle $A \times E \times F \in \mathcal{S}$ and $0 \le p \le 1$,

$$\phi_i^{-1}(\{\mu : \mu(A \times E \times F) \ge p\}) \tag{4}$$

is measurable in T_i.

Rewrite (4) as $\{t_i : \phi_i(t_i)(A \times E \times F) \ge p\}$. Hence by the definition of ϕ_i, (4) is precisely the set $\{t_i : \lambda_i(t_i)(A \times (E \cap F)) \ge p\}$, which is a measurable subset of T_i by the measurability of λ_i.

(ii) The marginal of $\phi_i(t_i)$ on T_i is δ_{t_i}.

For any $t_i \in T_i$ in the type structure Λ, we have $\text{marg}_{T_i}\lambda_i(t_i) = \delta_{t_i}$. That is, for any measurable subset $E \subset T_i$ with $t_i \in E$, $\lambda_i(t_i)(S \times T_{-i} \times E) = 1$. Thus by definition of ϕ_i,

$$\phi(t_i)(S \times C \times T_{-i} \times E) = \lambda_i(t_i)(S \times T_{-i} \times E) = 1.$$

That is, for any $t_i \in T_i$ in the type structure Φ, $\text{marg}_{T_i}\phi_i(t_i) = \delta_{t_i}$ as required.

This proves that $\Phi = \langle S \times C, (T_i)_{i\in I}, (\phi_i)_{i\in I}\rangle$ is a type structure. It is still necessary to verify that Φ is an expansion of Λ.

(a) Φ is non-redundant. It suffices to show $\widetilde{h}_i : T_i \to H_i(S \times C)$ is injective by Proposition 2. Consider two distinct types t_i and t_i'. By the separativity of T_i (and hence C_i), we can find a measurable set $K \subset C_i$ such that $c_i(t_i) \in K$ and $c_i(t_i') \notin K$ (recall that $c_i : T_i \to C_i$ is the identity). Therefore,

$$\widetilde{h}_i^1(t_i)(S \times C_{-i} \times K) = \phi_i(t_i)(S \times C_{-i} \times K \times T) = \lambda_i(t_i)(S \times T_{-i} \times K) = 1. \tag{5}$$

The last equality of (5) follows from the fact that the marginal of $\lambda_i(t_i)$ on T_i is δ_{t_i} and $t_i = c_i(t_i) \in K$. On the other hand,

$$\widetilde{h}_i^1(t_i')(S \times C_{-i} \times K) = \phi_i(t_i')(S \times C_{-i} \times K \times T) = \lambda_i(t_i')(S \times T_{-i} \times K) = 0.$$

Therefore, $\widetilde{h}_i$ is injective as required.

(b) The type isomorphism. Consider τ_i as the identity mapping on T_i. We want to show τ is a type morphism between $\Lambda = \langle S, (T_i)_{i\in I}, (\lambda_i)_{i\in I}\rangle$ and $\Phi = \langle S\times C, (T_i)_{i\in I}, (\phi_i)_{i\in I}\rangle$. Namely,

$$\mathrm{marg}_{S\times T}(\phi_i \circ \tau_i)(t_i) = \lambda_i(t_i)(id_S, \tau)^{-1}. \tag{6}$$

By definition, $(\phi_i \circ \tau_i)(t_i)(E \times C \times F) = \phi_i(t_i)(E \times C \times F) = \lambda_i(t_i)(E\times F)$ for any measurable rectangle $E \times C \times F$ with $E \in \Sigma(S)$ and $F \in \Sigma(T)$. Notice that fixing C, $\phi_i(t_i)(C \times \cdot)$ defines a measure on $\Sigma(S \times T)$ and it agrees with $\lambda_i(t_i)(\cdot)$ on the semiring $\Sigma(S) \times \Sigma(T)$. Therefore, by Billingsley (1995, Theorem 10.3), $\phi_i(t_i)(B \times C) = \lambda(t_i)(B)$ for any measurable subset $B \in \Sigma(S \times T)$. This proves (6).

Thus Φ constructed above is an expansion of Λ. ■

3 Games of Incomplete Information

Consider a game $G = (S; (A_i)_{i\in I}; (g_i)_{i\in I})$ with a measurable parameter space S, measurable strategy space $A = \prod_{i\in I} A_i$, and bounded *jointly* measurable payoff functions $g_i : S\times A \to \mathbb{R}$. Append to the game a type structure $\Lambda = \langle S, T, \lambda\rangle$. Write $G[\Lambda]$ for the Bayesian game associated with G and Λ.

Let $(\beta_i)_{i\in I}$ be a tuple of (behavioral) *strategies* $\beta_i : T_i \times \Sigma(A_i) \to [0, 1]$ such that (1) for every $t_i \in T_i$, $\beta_i(t_i, \cdot) : \Sigma(A_i) \to [0, 1]$ is a probability measure and (2) for every $E \in \Sigma(A_i)$, $t_i \to \beta_i(\cdot, E)$ is measurable. $(\beta_i)_{i\in I}$ is a *Bayesian equilibrium* of $G[\Lambda]$ if for any $i \in I$ and any $\widehat{a}_i \in A_i$,

$$\int_{S\times T}\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)\ d\lambda_i(t_i) \geq \int_{S\times T}\int_A g_i(s,\widehat{a}_i,a_{-i})\prod_{j\in I\ \setminus\{i\}}\beta_j(t'_j,da_j)\ d\lambda_i(t_i).$$

In the expression above, $\int_{S\times T}\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)d\lambda_i(t_i)$ is player i's expected payoff from $G[\Lambda]$ when players play according to $(\beta_i)_{i\in I}$. $\int_A g_i(s,a)\prod_{j\in I}\beta_j(t_j,da_j)$ is the shorthand for

$$\int_{A_1}\cdots[\int_{A_n} g_i(s,a)\beta_1(t_1,da_1)]\cdots\beta_n(t_n,da_n)\quad .$$

These notations are justified by a Fubini-type theorem (Proposition 3).

Let $G=(S;(A_i)_{i\in I};(g_i)_{i\in I})$ be a game with parameter space S and let Φ be an expansion of Λ via C. Let $\bar{G}=(S\times C;(A_i)_{i\in I};(\bar{g}_i)_{i\in I})$ be such that $\bar{g}_i(s,c,a)=g_i(s,a)$ for all $(s,a)\in S\times A$. Then write $G[\Phi]$ for the game $\bar{G}[\Phi]$ and call $G[\Phi]$ an expanded game.

We have the following results about equilibria. This theorem says that any equilibrium associated with a redundant type structure can be obtained from its non-redundant expansions, and vice versa. In conjunction with Theorem 1, this result says that the set of predictions obtained from a redundant structure is indeed interesting in that it reflects the set of equilibria based on a full parameter space.

Theorem 2 *Let $\Phi=\langle S\times C,U,\phi\rangle$ be an expansion of $\Lambda=\langle S,T,\lambda\rangle$ via C, and $\tau:T\to U$ is the associated S-based type isomorphism. For any Bayesian equilibrium (if it exists) $(\beta_i)_{i\in I}$ of $G[\Lambda]$, the strategy profile $(\gamma_i)_{i\in I}$, defined via $\gamma_i(\cdot,\cdot)=\beta_i(\tau_i^{-1}(\cdot),\cdot)$ for each i, is a Bayesian equilibrium of the expanded game $G[\Phi]$. Conversely, for any Bayesian equilibrium (if it exists) $(\gamma_i)_{i\in I}$ of $G[\Phi]$, the strategy profile $(\beta_i)_{i\in I}$, defined via $\beta_i(\cdot,\cdot)=\gamma_i(\tau_i(\cdot),\cdot)$ for each i, is a Bayesian equilibrium of $G[\Lambda]$*

Proof. For a Bayesian equilibrium $(\beta_i)_{i\in I}$ of $G[\Lambda]$, let us write

$$f_i(s,t):=\int_A g_i(s,a)\prod_{j\in I}\beta_j(t_j,da_j).$$

and

$$f_i^{\widehat{a}_i}(s,t) := \int_A g_i(s, \widehat{a}_i, a_{-i}) \prod_{j \in I \setminus \{i\}} \beta_j(t_j, da_j).$$

$f_i(s,t)$ is player i's payoff when the payoff-relevant state is s, the type profile is t, and players follow the equilibrium strategy profile $(\beta_i)_{i\in I}$. $f_i^{\widehat{a}_i}(s,t)$ is player i's payoff eveluated at (s,t) when player i plays $\hat{a}_i$ and other players follow the given equilibrium strategies.

Consider the strategy profile $(\gamma_i)_{i\in I}$ on $G[\Phi]$ defined by $\gamma_i(\cdot,\cdot) = \beta_i(\tau_i^{-1}(\cdot),\cdot)$ for each i, we have

$$\int_A \overline{g}_i((s,c),a) \prod_{j\in I} \gamma_j(u'_j, da_j) = \int_A g_i(s,a) \prod_{j\in I} \beta_j(\tau_j^{-1}(u'_j), da_j) = (f_i \circ (id_S, \tau)^{-1})(s,u') \quad (7)$$

That is, $(f_i \circ (id_S,\tau)^{-1})(s,u')$ is i's payoff eveluated at (s,u') when players follow $(\gamma_i)_{i\in I}$.

Similarly, $\int_A \overline{g}_i((s,c), \widehat{a}_i, a_{-i}) \prod_{j\in I\setminus\{i\}} \gamma_j(u'_j, da_j) = (f_i^{\widehat{a}_i} \circ (id_S,\tau)^{-1})(s,u')$. $(f_i^{\widehat{a}_i} \circ (id_S,\tau)^{-1})(s,u')$ is player i's payoff when i takes $\widehat{a}_i$ and other players follow the proposed strategies. With some abuse of notation, let us treat $f_i^{\widehat{a}_i} \circ (id_S,\tau)^{-1}$ and $f_i \circ (id_S,\tau)^{-1}$ as the functions on $S \times C \times U$ because C consists of payoff-irrelevant parameters.

Note that $(\gamma_i)_{i\in I}$ is a Bayesian equilibrium of $G[\Phi]$ if and only if for any $i \in I$, $u_i \in U_i$, and $\widehat{a}_i \in A_i$,

$$\int_{S\times C\times U} f_i \circ (id_S,\tau)^{-1}\, d\phi_i(u_i) \geq \int_{S\times C\times U} f_i^{\widehat{a}_i} \circ (id_S,\tau)^{-1}\, d\phi_i(u_i). \quad (8)$$

By Definition 4, for any $u_i \in U_i$, $\text{marg}_{S\times U}\phi_i(u_i) = \lambda_i(\tau_i^{-1}(u_i))(id_S,\tau)^{-1}$. That is, for any $u_i \in U_i$, and any $O \in \Sigma(S \times U)$,

$$\phi_i(u_i)(O \times C) = \lambda_i(\tau_i^{-1}(u_i))(id_S,\tau)^{-1}(O).$$

Therefore, for $F_i = f_i$ or $f_i^{\widehat{a}_i}$,

$$\int_{S\times C\times U} F_i \circ (id_S, \tau)^{-1}\, d\phi_i(u_i) = \int_{S\times U} F_i \circ (id_S, \tau)^{-1}\, d\lambda_i(\tau_i^{-1}(u_i))(id_S, \tau)^{-1}. \qquad (9)$$

By the change of variables theorem (Billingsley (1995, Theorem 16.13)), the right hand side of (9) can be rewritten as

$$\int_{S\times T} F_i\, d\lambda_i(\tau_i^{-1}(u_i)) \qquad (10)$$

Therefore, (8) is equivalent to

$$\int_{S\times T} f_i\, d\lambda_i(\tau_i^{-1}(u_i)) \geq \int_{S\times T} f_i\, d\lambda_i(\tau_i^{-1}(u_i))$$

On the other hand, $(\beta_i)_{i\in I}$ is a Bayesian equilibrium of $G[\Lambda]$ if and only if for any $i \in I$, $t_i \in T_i$, and $\widehat{a}_i \in A_i$,

$$\int_{S\times T} f_i\, d\lambda_i(t_i) \geq \int_{S\times T} f_i^{\widehat{a}_i}\, d\lambda_i(t_i). \qquad (11)$$

The theorem follows immediately by comparing (11) and (10). ■

In Theorem 1 and Theorem 2, the set of "hidden variables" C can be very large and could depend on the type structure being considered. The next result says that it is possible to construct a single C that depends only on the size of the game from which the Bayesian equilibrium outcomes of any S-based type structures can be obtained. In particular, this set can be taken as the action set: $C = A$.

Theorem 3 *Let $(\beta_i)_{i\in I}$ be a Bayesian equilibrium of $G[\Lambda]$ for the type structure $\Lambda = \langle S, (T_i)_{i\in I}, (\lambda_i)_{i\in I}\rangle$. Then there is an $(S\times A)$-based non-redundant type structure $\Phi = \langle S\times A, (U_i)_{i\in I}, (\phi_i)_{i\in I}\rangle$ with a Bayesian equilibrium $(\gamma_i)_{i\in I}$ of $G[\Phi]$ and an S-based surjective type morphism $\tau : T \to U$ such that $\gamma_i(\tau_i(t_i), \cdot) = \beta_i(t_i, \cdot)$ for each $i \in I$ and $t_i \in T_i$.*

To prove the result, we need the following generalized version of Fubini's theorem, whose proof is in the appendix.

Proposition 3 *Let* $f : S \times A_1 \times A_2 \times T_1 \times T_2 \to R^+$ *be a jointly measurable function. For* $i = 1, 2$, *let* $\kappa_i(t_i, da) : T_i \times \Sigma(A_i) \to [0, 1]$ *be a stochastic kernel, and* $\pi_i : T_i \to \Delta(S \times T)$ *be measurable.*

(1) $F(s, t) = \int_{A_1} \int_{A_2} f(s, a_1, a_2, t_1, t_2)\kappa_2(t_2, da_2)\ \kappa_1(t_1, da_1)$ *is jointly measurable on* $S \times T$.

(2) There exists a unique map $P_i : T_i \to \Delta(S \times T \times A)$ *such that for any* $E \in \Sigma(S \times T)$ *and* $F_i \in \Sigma(A_i)$, $i \in \{1, 2\}$,

$$P_i(t_i)(E \times F_1 \times F_2) = \int_E \kappa_1(t'_1, F_1)\kappa_2(t'_2, F_2)\ d\pi_i(t_i)$$

(3) $P_i : T_i \to \Delta(S \times T \times A)$ *is measurable.*

(4) $\int_{S\times T}[\int_{A_1} \int_{A_2} f(s, a, t)\kappa_2(t'_2, da_2)\kappa_1(t'_1, da_1)]d\pi_i(t_i) = \int_{S\times A\times T} f(s, a, t)dP_i(t_i)$.

Lemma 4 *Given a type structure* $M = \langle X, T, m\rangle$, *then* $M' = \langle X, T', m'\rangle$ *defined as follows is a type structure,*

(1) $T'_i \subset T_i$ *with the relative* σ*-field:* $\Sigma(T'_i) = \{T'_i \cap E : E \in \Sigma(T_i)\}$.

(2) For each $i \in I$ *and* $t_i \in T'_i$, $m_i(t_i)$ *has a support in* $X \times T'$ *and for each measurable set* $G \subset X \times T$, $m'_i(t_i)((X \times T') \cap G) = m_i(t_i)(G)$.

We call M' *a sub-structure of* M.

Proof. It is easy to show that the product σ-field on $X \times T'$ when T'_i inherits a σ-field from T_i is the relative σ-field $\{(X \times T') \cap G : G \in \Sigma(X \times T)\}$ that $X \times T'$ inherits from $X \times T$. Therefore,

$$\{t_i : m'_i(t_i)((X \times T') \cap G) \geq p\} = T_i \cap \{t_i : m_i(t_i)(G) \geq p\} \in \Sigma(T'_i).$$

This shows that m'_i is a well defined measurable function. It is straightforward to verify that M' satisfies other conditions in Definition 1. ■

Proof of Theorem 3. The idea of proof is as follows. First construct an $(S\times A)$-based type structure $\Psi = \langle S\times A, V, \psi\rangle$ by augmenting Λ "via the given equilibrium $(\beta_i)_{i\in I}$." (Step 1 and

Step 2). Ψ is mapped onto a sub-structure $\Phi = \langle S \times A, U, \phi \rangle$ in the $(S \times A)$-based universal type structure $\langle S \times A, U^*, \phi^* \rangle$. Φ is a type structure satisfying the required conditions. (Step 3 and Step 4).

Step 1: For a given type structure $\Lambda = \langle S, (T_i)_{i \in I}, (\lambda_i)_{i \in I} \rangle$, denote by $\mathcal{S}$ the semiring of measurable rectangles $\Sigma(S) \times \prod_{j \in I} \Sigma(A_j) \times \Sigma(T)$. Define a type structure $\Psi = \langle S \times A, V, \psi \rangle$ as follows. For each $i \in I$, let $V_i = T_i$ and $\psi_i(t_i)$ be the unique probability measure such that for any measurable rectangles $E \times F \times G = E \times \prod_{j \in I} F_j \times G$ in $\mathcal{S}$

$$\psi_i(t_i)(E \times F \times G) = \int_{E \times G} \prod_{j \in I} \beta_j(t'_j, F_j) \, d\lambda_i(t_i). \tag{12}$$

The existence and measurability of ψ_i is guaranteed by Proposition 3. Furthermore, by taking $E = S$, $F = A$, $G_j = T_j$ for $j \neq i$ in (12), for any $t_i \in T_i$,

$$\psi_i(t_i)(S \times A \times T_{-i} \times G_i) = \lambda_i(t_i)(S \times T_{-i} \times G_i).$$

Therefore, $\text{marg}_{T_i} \psi_i(t_i) = \text{marg}_{T_i} \lambda_i(t_i) = \delta_{t_i}$. Thus, $\Psi = \langle S \times A, V, \psi \rangle$ is a well-defined type structure.

Step 2: Define for each player $i \in I$ in the Bayesian game $G[\Psi]$ a strategy $\rho_i : T_i \times \Sigma(A_i) \to [0,1]$ by

$$\rho_i(t_i, \cdot) = \operatorname*{marg}_{A_i} \psi_i(t_i). \tag{13}$$

Therefore, by taking $E = S$, $G = T$ and $F_j = A_j$ for $j \neq i$ in (12),

$$\rho_i(t_i, \cdot) = \int_{S \times T} \beta_i(t'_i, \cdot) \, d\lambda_i(t_i) = \beta_i(t_i, \cdot). \tag{14}$$

Step 3: Given $\Psi = \langle S \times A, T, \psi \rangle$, we define an $(S \times A)$-based non-redundant type structure $\Phi = \langle S \times A, U, \phi \rangle$ as follows. According to the construction of Heifetz and Samet (1998), Ψ is mapped into an $(S \times A)$-based universal type structure $\langle S \times A, U^*, \phi^* \rangle$ under the unique $(S \times A)$-based type morphism τ^*. Let $U_i = \tau_i^*(T_i) \subset U_i^*$ with the relative σ-field, and for any

$i \in I$, $u_i \in U_i^*$ and $G \in \Sigma(S \times A \times U^*)$, let

$$\phi_i(u_i)((S \times A \times U) \cap G) = \phi_i^*(u_i)(G).$$

Then $\langle S \times A, U, \phi \rangle$ is a well-defined non-redundant type structure by Lemma 4.

Step 4: Define a strategy for each player i in the game $G[\Phi]$, $\gamma_i : U_i \times \Sigma(A_i) \to [0,1]$, by

$$\gamma_i(u_i, \cdot) = \operatorname*{marg}_{A_i} \phi_i(u_i). \tag{15}$$

$\gamma_i(\cdot, \cdot)$ is a well defined behavioral strategy for player i. We will show that $(\gamma_i)_{i \in I}$ is a Bayesian equilibrium for $G[\Phi]$.

By Definition 2 (take C as a singleton and replace S with $S \times A$),

$$\phi_j \circ \tau_j^*(t_j) = \psi_j(t_j)(id_{S \times A}, \tau^*)^{-1}. \tag{16}$$

By the definition of $(id_{S \times A}, \tau^*)$, $\operatorname{marg}_{A_j} \psi_j(t_j)(id_{S \times A}, \tau^*)^{-1} = \operatorname{marg}_{A_j} \psi_j(t_j)$. This equality together with (16) implies

$$\operatorname*{marg}_{A_j} \phi_j \circ \tau_j^*(t_j) = \operatorname*{marg}_{A_j} \psi_j(t_j),$$

and hence, $\gamma_j(\tau_j^*(t_j), \cdot) = \rho_j(t_j, \cdot)$ by the definitions of γ_j (15) and ρ_j (13). Therefore, $\gamma_j(\tau_j^*(t_j), \cdot) = \beta_j(t_j, \cdot)$ by (14). Thus, by the change of variables theorem, for any $E \in \Sigma(S \times A \times U)$ and $F \in \prod_{j \in I} \Sigma(A_j)$,

$$\int_{(id,\tau^*)^{-1}(E)} \prod_{j \in I} \beta_j(t_j')(F_j)\ d\psi_i(t_i) = \int_E \prod_{j \in I} \gamma_j(u_j')(F_j)\ d\phi_i \circ \tau_i^*(t_i) \tag{17}$$

Therefore,

$$
\begin{aligned}
& \int_{(S\times A)\times U}[\int_A \overline{g}_i((s,a'),a)\prod_{j\in I}\gamma_j(u'_j,da_j)]\,d\phi_i\circ\tau_i^*(t_i) \\
= & \int_{(S\times A)\times T}[\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)]\ d\psi_i(t_i) && (18)\\
= & \int_{S\times T}[\int_A[\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)]\ \prod_{j\in I}\beta_j(t'_j,da_j)]\ d\lambda_i(t_i) && (19)\\
= & \int_{S\times T}[\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)]\ \ d\lambda_i(t_i) && (20)
\end{aligned}
$$

(18) is by (17) and Fubini's theorem; (19) is by (12).

Similarly,

$$
\int_{S\times A\times U}\int_A \overline{g}_i((s,a'),\overline{a}_i,a_{-i})\prod_{j\in I}\gamma_j(u'_j,da_j)\,d\phi_i\circ\tau_i^*(t_i) = \int_{S\times T}[\int_A g_i(s,\overline{a}_i,a_{-i})\prod_{j\in I\ \setminus\{i\}}\beta_j(t'_j,da_j)]d\lambda_i(t_i) \tag{21}
$$

Since $(\beta_j)_{j\in I}$ is a Bayesian equilibrium $G[\Lambda]$, we have

$$
\int_{S\times T}[\int_A g_i(s,a)\prod_{j\in I}\beta_j(t'_j,da_j)]\ \ d\lambda_i(t_i) \geq \int_{S\times T}[\int_A g_i(s,\overline{a}_i,a_{-i})\prod_{j\in I\ \setminus\{i\}}\beta_j(t'_j,da_j)]d\lambda_i(t_i)
$$

This together with (20) and (21) shows that $(\gamma_j)_{j\in I}$ is a Bayesian equilibrium of $G[\Phi]$. ■

4 Discussion

Without requiring the specification of type spaces, Battigalli and Siniscalchi (2003) provide a unified framework for the analysis of both dynamic and static games with incomplete information, with actions being treated as basic uncertainties. This framework is consistent with the Bayesian equilibrium framework: with compatible restrictions on the first order beliefs, the Bayesian equilibrium outcomes in *all* type structures, including redundant structures, are exactly those predicted in their framework. They discuss the strategic relevance of redundancy (in subsection 4.2 and section 6), but leave open the conceptual interpretation of redundant type structures.

There are two other important papers that explore the strategic relevance of redundancy in more detail.

Ely and Peski (2006) show how, given a type structure, to distill all information that is relevant to the set of interim rationalizable outcomes. It turns out that this information is summarized by the hierarchies over the statements about a type's beliefs over underlying uncertainties conditional on opponents' types. But to interpret this hierarchy, they need to explicitly refer to players' action choices. Furthermore, they also observe that their notion of hierarchies cannot determine all Bayesian equilibrium outcomes.

Dekel et al. (2006) propose a new solution concept of "interim correlated rationalizability" that is invariant over the class of type structures modeling the same sets of belief hierarchies. In their definition, each player's strategy is measurable with respect to his own type; however, the player may believe that his opponents' action choices are not measurable with respect to their types; that is, he may conjecture correlations between opponents' actions and the underlying payoff parameters even though the given type structure may say that the opponents cannot tell apart the payoff parameters. This solution concept uses more information than what is modeled in the given type structure. To account for the outcomes predicted by this solution, Dekel et al. use an epistemic model in which actions are explicitly modelled as basic uncertainties.

A comparison between the two papers may be helpful to understand how each work deals with the correlations between types and payoff parameters embedded in a type structure. Both papers are solution-concept dependent. Ely and Peski (2006) adopt the concept of interim independent rationalizability, but the correlations are captured by the extended notion of hierarchy. In contrast, Dekel et al. (2006) adopt the standard notion of Mertens-Zamir belief hierarchy, but the correlation is embedded in their new solution concept "interim correlated rationalizability."

In response, Liu (2005) argues that a type structure is simply the analyst's model of the game situation, and hence the correlation embedded in the redundant type structures should be understood independently of solution concepts. Liu (2005) shows how to construct a redundant type structure from a non-redundant type structure with the same set of belief hierarchies through a system of transition probabilities which is interpreted as a state-dependent correlating mechanism. This provides a tractable way to analyze properties of the unfamiliar redundant structures within the well-known non-redundant structure. However, the characterization serves only as an analytic tool; the important conceptual issues are not addressed.

This paper provides one way to resolve the conceptual difficulties. The so-called redundancy is not redundant in that it reflects a full parameter space larger than the payoff parameters specified by the analyst. The analyst should always make use of non-redundant structures unless he believes that he has misspecified the parameter space.

A

A.1 Measurability of Hierarchy Maps in Section 2.3

Lemma 5 *If both* $f_1 : X \to \Delta(X)$ *and* $g : X \to Y$ *are measurable, then the function* $f_2 : X \to \Delta(Y)$*, defined by* $f_2(x)(E) = f_1(x)(g^{-1}(E))$ *for each* $x \in X$ *and measurable set* $E \subset Y$*, is measurable.*

Proof. We only need to show for $F = \{\mu \in \Delta(Y) : \mu(E) \geq p\}$ with $0 \leq p \leq 1$ and E a measurable subset of Y, $f_2^{-1}(F) = \{x \in X : f_1(x)(g^{-1}(E)) \geq p\}$ is a measurable subset of X. Since $g^{-1}(E) \subset X$ is measurable, $G = \{\nu \in \Delta X : \nu(g^{-1}(E)) \geq p\} \subset \Delta X$ is measurable, and hence $f_2^{-1}(F) = \{x \in X : f_1(x) \in G\} = f_1^{-1}(G)$, which is measurable by the measurability of f_1. ■

From Lemma 5, $\widetilde{h}_i^1$ is measurable (with ϕ_i in the role of f_1 and $\text{proj}_{S\times C}$ in the role of g). Inductively, we can show that $\widetilde{h}$, h, $\widetilde{p}^k$ and p^k are measurable.

A.2 Proof of Proposition 1

Proof. By Definition 2,

$$\text{marg}_{S\times U'}\phi_i' \circ \tau_i(u_i) = (\text{marg}_{S\times U}\phi_i(u_i))(id_S, \tau)^{-1}. \tag{22}$$

$\text{marg}_{S\times U'}\phi_i' \circ \tau_i(u_i) = (\text{marg}_{S\times U}\phi_i(u_i))(id_S, \tau)^{-1}$. Therefore for any $u_i \in U_i$,

$$\phi_i' \circ \tau_i(u_i)(\text{proj}_S)^{-1} = \phi_i(u_i)(\text{proj}_S)^{-1}. \tag{23}$$

From (23) and the definition of the first order belief map, $h_i'^1 \circ \tau_i = h_i^1$. Suppose $h_i'^l \circ \tau_i = h_i^l$ for each $i \in I$ and $1 \leq l \leq k$. Consider measurable sets $E_0 \subset S$ and $E_l \subset H^l(S)$ for $1 \leq l \leq k$. Then

$$(h^l)^{-1}(E_l) = (h'^l \circ \tau)^{-1}(E_l) = \tau^{-1}((h'^l)^{-1}(E_l)).$$

Recall that $p^k = (\text{proj}_S, h^1 \circ \text{proj}_U, ..., h^k \circ \text{proj}_U)$, then $(p^k)^{-1}(\prod_{l=0}^k E_l) = E_0 \times C \times \bigcap_{l=1}^k (h^l)^{-1}(E_l)$ and similarly, $(p'^k)^{-1}(\prod_{l=0}^k E_l) = E_0 \times C \times \bigcap_{l=1}^k (h'^l)^{-1}(E_l)$. Therefore,

$$\begin{aligned} \phi_i' \circ \tau_i(u_i)(p'^k)^{-1}(\prod_{l=0}^k E_l) &= \phi_i' \circ \tau_i(u_i)(E_0 \times C \times \bigcap_{l=1}^k (h'^l)^{-1}(E_l)) \\ &= \text{marg}_{S\times U'}\phi_i' \circ \tau_i(u_i)(E_0 \times \bigcap_{l=1}^k (h'^l)^{-1}(E_l)) \end{aligned} \tag{24}$$

And similarly,

$$\phi_i(u_i)(p^k)^{-1}(\prod_{l=0}^k E_l) = (\text{marg}_{S\times U}\phi_i(u_i))(E_0 \times \bigcap_{l=1}^k (h^l)^{-1}(E_l)). \tag{25}$$

Note that by (22), (24) and (25),

$$\phi_i' \circ \tau_i(u_i)(p'^k)^{-1}(\prod_{l=0}^{k} E_l) = \phi_i(u_i)(p^k)^{-1}(\prod_{l=0}^{k} E_l).$$

Since sets of the form $\prod_{l=0}^{k} E_l$ generates the the σ-field on $S \times \prod_{l=1}^{k} H^l(S)$, we have

$$\phi_i' \circ \tau_i(u_i)(p'^k)^{-1} = \phi_i(u_i)(p^k)^{-1}.$$

Thus, $h_i'^k \circ \tau_i = h_i^k$. ■

A.3 Proof of Proposition 3

Lemma 6 *Suppose $\pi : \Theta \to \Delta(X)$ is measurable with respect to the corresponding σ-fields. Then $\int f(x)d\pi(\theta) : \Theta \to R$ is measurable for any non-negative real-valued measurable function f with domain X.*

Proof. For a fixed measurable set $E \subset X$, $\theta \mapsto \pi(\theta)(E)$ is a real-valued measurable function since

$$\{\theta : \pi(\theta)(E) \geq r\} = \pi^{-1}\{\mu : \mu(E) \geq r\}$$

is a measurable subset of Θ by the measurability of π. For a simple function $g = \sum_{n=1}^{m} b_n 1_{E_n}$ on X, $\int g(x)d\pi(\theta) = \sum_{n=1}^{m} b_n \pi(\theta)(E_n)$ is a measurable function on Θ. Since f is a pointwise

limit of an increasing sequence $\{g_k\}$ of simple functions, $\int f(x)d\pi(\theta) = \lim_k \int g_k(x)d\pi(\theta)$ for each $\theta \in \Theta$ by the monotone convergence theorem, and therefore $\theta \mapsto \int f(x)d\pi(\theta)$ is measurable. ■

Proof of Proposition 3. (1) It suffices to show $(s,t) \mapsto \int_{A_1}\int_{A_2} 1_E(s,a,t)\,\kappa_2(t_2, da_2)$ $\kappa_1(t_1, da_1)$ is jointly measurable for every measurable subset $E \subset S \times A \times T$. The stated result will then follow from passing to the limit of an increasing sequence of non-negative simple functions. First consider $E = E^1 \times E^2 \times E^3$ where $E^1 \in \Sigma(S)$, $E_i^2 \in \Sigma(A_i)$ and $E_3 \in \Sigma(T)$. We then have

$$\int_A 1_{E^1 \times E^2 \times E^3}(s, a, t)\kappa_2(t_2, da_2)\kappa_1(t_1, da_1) = 1_{E^1}(s)\kappa_2(t_2, E_2^2)\kappa_1(t_1, E_1^2)1_{E^3}(t).$$

Consider $F^r = \{(s,t) : 1_{E^1}(s)\kappa_2(t_2, E_2^2)\kappa_1(t_1, E_2^1)1_{E^3}(t) > r\}$. $F^r = \emptyset$ for $r \geq 1$; $F^r = S \times T$ for $r < 0$; for $0 \leq r < 1$,

$$F^r = E^1 \times (\{t : \kappa_2(t_2, E_2^2)\kappa_1(t_1, E_2^1) > r\} \cap E^3);$$

Notice that

$$\{t : \kappa_2(t_2, E_2^2)\kappa_1(t_1, E_2^1) > r\} = \bigcup_{q>0 \text{ is rational}} \{t_1 : \kappa_1(t_1, E_2^1) > q\} \times \{t_2 : \kappa_2(t_2, E_2^2) > r/q\}$$

is measurable in T. Therefore F^r is measurable in $S \times T$. $(s,t) \mapsto \int_{A_1}\int_{A_2} 1_E(s,a,t)\,\kappa_2(t_2, da_2)$ $\kappa_1(t_1, da_1)$ is jointly measurable for measurable rectangles $E = E^1 \times E^2 \times E^3$. To show the result hold for any measurable E in $S \times A \times T$, consider $\mathcal{L} = \{E \in \Sigma(S \times A \times T) : (s,t) \mapsto \int_A 1_E(s,a,t)\,\kappa_2(t_2, da_2)\ \kappa_1(t_1, da_1)$ is jointly measurable $\}$. We have shown that the π-system $\Sigma(S) \times \Sigma(A_1) \times \Sigma(A_2) \times \Sigma(T)$ of measurable rectangles belongs to $\mathcal{L}$. It will follow from Dynkin's π-λ theorem that $\Sigma(S \times A \times T) = \mathcal{L}$ if $\mathcal{L}$ is a λ-system. Let us verify that $\mathcal{L}$ is a λ-system. First, it is straightforward to check that $S \times A \times T \in \mathcal{L}$ and that if $E \in \mathcal{L}$ then $E^c \in \mathcal{L}$. Secondly, consider a pairwise disjoint sequence $\{F^n\}$ in $\mathcal{L}$. We have, by the monotone convergence theorem,

$$\int_A 1_{\bigcup_{n=1}^{\infty} F^n}(s,a,t)\,\kappa_2(t_2, da_2)\kappa_1(t_1, da_1) = \sum_{n=1}^{\infty}\int_A 1_{F^n}(s,a,t)\,\kappa_2(t_2, da_2)\kappa_1(t_1, da_1).$$

From the right hand side, we see that the left hand side is jointly measurable in (s,a,t). Thus

$\bigcup_{n=1}^{\infty} F^n \in \mathcal{L}$. Therefore $\mathcal{L}$ is a λ-system.

We only sketch the proof for (2) and (4) since they are not very different from the familiar form of Fubini's theorem, see for example, Billingsley (1995, Theorem 18.3 and Exercise 18.20).

(2) For each $t_i \in T_i$, we show $\underline{P}_i(t_i)(E \times F_1 \times F_2) = \int_E \kappa_2(t_2', F_2)\kappa_1(t_1', F_1)\ d\pi_i(t_i)$ is a countably additive probability measure on the semiring $\mathcal{S} = \Sigma(S \times T) \times \Sigma(A)$ of measurable rectangles. Suppose $\{E_n \times F_n\}$ is a pairwise disjoint sequence in $\mathcal{S}$ and $\bigcup_{n=1}^{\infty} E^n \times F_1^n \times F_2^n = E \times F_1 \times F_2 \in \mathcal{S}$. It is immediate that $\bigcup_{n=1}^{\infty} E^n = E$ and $\bigcup_{n=1}^{\infty} F_i^n = F_i$, $i = 1, 2$. Therefore, for any $\omega \in S \times T$ and $a \in A$,

$$1_{E\times F_1\times F_2}((\omega, a_1, a_2)) = \sum_{n=1}^{\infty} 1_{E^n\times F_1^n\times F_2^n}((\omega, a_1, a_2)) = \sum_{n=1}^{\infty} 1_{E^n}(\omega)1_{F_1^n}(a_1)1_{F_2^n}(a_2)$$

Integrating with respect to $\kappa_2(t_2', \cdot)$, then with respect to $\kappa_1(t_1', \cdot)$, and finally with respect to $\pi_i(t_i)$, we have

$$\int_E \kappa_2(t_2', F_2)\kappa_1(t_1', F_1)\ d\pi_i(t_i) = \sum_{n=1}^{\infty} \int_{E^n} \kappa_2(t_2', F_2^n)\kappa_1(t_1', F_1^n)\ d\pi_i(t_i),$$

showing the countable additivity of $\underline{P}_i(t_i)$. By the Caratheodory Extension Theorem, $\underline{P}_i(t_i)$, for each $t_i \in T_i$, extends uniquely to a measure $P_i(t_i)$ on $\Sigma(S \times T \times A)$.

(3) By Lemma 3, we only need to show that $t_i \mapsto P_i(t_i)(E \times F_1 \times F_2)$ is measurable. This measurability follows from Lemma 6.

(4) The proof is straightforward: check the result first for indicator functions, and then for simple functions and finally for non-negative real functions. ■

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