Dynamic Contracting under Adverse Selection and Renegotiation

[JOB MARKET PAPER]

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Abstract

We study an infinitely repeated principal-agent model with adverse selection and renegotiation. The model is suitable for studying procurement, labor market relationships, and monopoly pricing, which is our main interpretation. A monopolist faces a consumer who has private information regarding his preferences for quality. The monopolist can offer long-term contracts, promising to deliver a certain quality in each future period. The terms of the contract must be honored unless both parties agree to renegotiate.

The monopolist chooses to gradually screen consumers who have high taste. During the screening process, these consumers balance the desire to immediately start consuming the high-quality good against the wish to pool themselves with consumers who have low taste and purchasing the high-quality good at a lower cost later. The model predicts an expected wedge between the marginal benefit for each kind of consumer and the marginal cost for the monopolist. As the parties renegotiate, this wedge gradually decreases and eventually disappears. We show that a version of the Coase conjecture holds for the monopoly model even in the case of a nondurable good. The possibility of renegotiation decreases the monopolist's ability to extract rents from consumers. This effect becomes more severe when the time periods shrink and, as a result, the allocation converges to an efficient allocation.

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Contents

1 Introduction

Long-term relationships under adverse selection are pervasive in economics. In these relationships, the party holding the bargaining power (the principal) wishes to know the private information held by the other party (the agent). For example, consider a company that offers career plans for its employees. The employee is typically better informed about how costly exerting effort is for him or how productive he is. The company wants to know who the most skillful employees are in order to assign the most important tasks to them. As a second example, consider a government that hires the same military contractor for many years. The government is interested in how advanced is the technology of its military contractor in order to request more advanced products from more efficient suppliers. To give a third and final example, consider a consumer who signs a long-term contract for an internet service. The internet provider wishes to know how important a quick connection is to the consumer in order to offer faster connections to consumers who value them more.

Unfortunately for the principal, information is not free. When a company learns that an employee is more productive, a higher effort may be demanded from the employee. When the government learns that the contractor is more efficient, less lucrative terms of trade may be offered in the next negotiation. And when the internet provider becomes aware that the consumer demand is inelastic, she may increase the price. As a result, the outcome in these relationships balances the desire of the principal to learn important information with the reluctance of the agent to reveal it.

In seminal papers, Freixas, Guesnerie, and Tirole (1985) and Laffont and Tirole (1988) developed models to study these types of relationships under the assumption that the parties cannot contract in the current period upon actions in future periods. In the internet provider example, this may be interpreted as a restriction wherein the firm cannot offer contracts with a duration longer than one month. In these cases, when the internet provider learns that the consumer has a high willingness to pay, she immediately changes the terms of trade. Therefore, these models give rise to the ratchet effect, that is, once the principal learns some information, she uses her bargaining power to extract future rents from the agent. The ratchet effect severely limits the amount of information that can be transmitted in a relationship. Hence, there is very little separation of types of agents.

The ratchet effect can be avoided if the parties write long-term contracts, which is natural in many situations. Under commitment, the optimal long-term contract entails two important features (Baron and Besanko (1985)). First, some agents immediately reveal their private information. Second, the principal offers inefficient allocations to some agents as a device to extract rents from others. In the internet provider example, this corresponds to the very familiar idea of second-degree price discrimination: the provider offers plans with lower than efficient speed for consumers with low willingness to pay in order to charge more from consumers with high willingness to pay when purchasing a plan with a faster connection.

In many important situations, the parties involved in a relationship cannot commit in the current period to not renegotiating the terms of trade later (when it becomes obvious how to exploit mutually beneficial opportunities). In the internet provider example, as soon as the taste of the consumer is learned, the internet provider has an incentive to renegotiate with consumers with low willingness to pay, offering them an opportunity to upgrade their plan for another with a faster speed. Therefore, the optimal contract under commitment is not robust to the possibility of renegotiation.

Motivated by this observation, we incorporate renegotiation into a dynamic principal-agent model with adverse selection, considering the price-discrimination example as our main interpretation. We assume that a monopolist faces a consumer who has private information about his marginal willingness to pay for a specific quality $q \in [0,1]$ of a certain good. The consumer's marginal valuation can be low (henceforth low type) or high (henceforth high type) and is persistent. The monopolist can offer long-term contracts, promising to deliver a particular quality in each future period. The terms of the contract must be honored unless both parties agree to renegotiate.

We show that allowing for renegotiation leads to a set of novel predictions about the equilibrium paths of prices and qualities. The monopolist begins with a screening phase during which she offers two kinds of contracts in each period: a pooling contract and a revealing contract. Low-type consumers choose only the pooling contract, while high-type consumers randomize between both contracts. In the pooling contract, the current period's quality is set below the efficient quality to the low type to extract higher rents from the high type. In the revealing contract, the efficient quality to the high type is offered in every future period.

The later the high-type consumer migrates from the pooling to the revealing contract, the higher the discount he receives once he eventually buys the revealing contract. The high-type consumer balances the desire to start consuming the high-quality good immediately against the discount he would receive from pooling with the low-type consumer for an additional period.

The expected discounted quality provided to the low-type consumer starts below the efficient level for his type and increases over time. A high-type consumer who pools with the low type in a certain period also expects a higher discounted quality in the future. In this sense, the model predicts an expected wedge between the marginal benefit for each type of consumer and the marginal cost for the monopolist. As the parties renegotiate, this wedge gradually decreases and eventually disappears. Consequently, we show that the conjecture of long-run efficiency posited by Battaglini (2007) is true in our model.

The distortion of the quality consumed by the low type in a particular period is proportional to the probability that a high type chooses the revealing contract in that period. Using this result, we obtain a lower bound on the amount of information rent of the high type. This lower bound increases as the time periods shrink and implies that the monopolist's ability to extract rents from the high type by offering an inefficient allocation to the low type is severely reduced as time periods become short. In response, the monopolist chooses to offer menus inducing to fast separation of types. Accordingly, we show that a version of the Coase conjecture holds for the monopoly model even in the case of a non-durable good. The possibility of renegotiation decreases the monopolist's ability to extract rents from consumers. This effect becomes more severe as the parties become more patient and, as a result, the allocation converges to an efficient allocation.

Finally, we explore some implications of our model for a given discount factor. We show that, as the initial prior on the consumer being a high type increases towards 1, the number of periods it takes the monopolist to screen the high type completely increases without bound. Concurrently, the information rent obtained by the high type shrinks to zero. As a result, the non-commitment costs that are due to renegotiation are non-monotonic on the monopolist's prior: they are arbitrarily small when the monopolist is very optimistic or very pessimistic about the consumer's type.

2 Literature Review

This paper belongs to the literature on dynamic adverse selection and renegotiation. Laffont and Tirole (1990) is the closest paper to ours. They also study a dynamic price-discrimination model with two types of consumer under long-term contracts and renegotiation. The main difference between their paper and ours is that they analyze a two-period model, while we analyze an infinitely repeated game. In their two-period model, the surplus of the relationship is bounded away from efficiency for any weight of the second period with respect to the first. Hence, our main result does not hold in their model. Battaglini (2007) extends the two-period model to the case in which tastes are stochastic. Battaglini (2005) studies a similar model with infinite periods and stochastic valuations under commitment. For some parameter configurations (which exclude persistent types), he shows that the optimal commitment contract is robust to the possibility of renegotiation.

Hart and Tirole (1988) study a finite horizon monopoly model with two types of buyers under the assumption that quantities are in $\{0, 1\}$, so the monopolist either sells the good in a certain period or does not. The assumption that we have a continuum of qualities in our model allows us to capture the idea that it is efficient to offer a different quality to different types of consumers (more generally, different actions are efficient for different types of agents). Accordingly, many arguments present in our analysis are a consequence of the desire of the monopolist to improve efficiency by separating types. Furthermore, the models present very different dynamics. In Hart and Tirole's model, all types consume the same quality in the long-run, while in our model the monopolist sequentially separates the consumers, selling a strictly higher quality to the high type in the long-run.

This paper is also related to the large literature on durable goods monopolists and the Coase conjecture. For seminal papers see Stokey (1981), Bulow (1982), Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986). For a comprehensive survey of the literature see Ausubel, Cramton and Deneckere (2000). These papers differ from ours since they focus on durable-goods monopolists.

3 The Model

We study a dynamic principal-agent model with adverse selection and the possibility of renegotiation. We use the benchmark model of Mussa and Rosen (1978), recently explored by Battaglini (2005), to study long-term contracting. The model can have other interpretations, such as procurement or agency in the labor market. For concreteness, we interpret the model as the relationship between a monopolist and a consumer.

A monopolist (she) and a consumer (he) meet at $t = 0, 1, ...$ In each period, the monopolist produces a quality $q \in [0, 1]$ of a non-durable good. Producing quality q costs $c(q)$.

We assume that c is a strictly increasing, strictly convex C^2 function with $c(0) = 0$ and $\min_{q\in[0,1]} c''(q) > 0$.

A consumer has a type $\theta \in \Theta = \{\theta_L, \theta_H\}$ which identifies his taste for the good. A low type (type θ_L) is identified by L and has value $\theta_L q$ for a quality q of the nondurable good. A high type (type θ_H) is identified by H and has value $\theta_H q$ for a quality q of the nondurable good ($0 < \theta_L < \theta_H$). We write $x_t \in [-X, X]$ (where X is a large positive real number) for the net transfer made from the consumer to the monopolist in period t. The monopolist's realized profit is given by:

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t\left[x_t - c\left(q_t\right)\right],$$

and the type $i \in \{L, H\}$ consumer's realized rent is given by:

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t\left[\theta_i q_t - x_t\right], \tag{1}$$

where $\delta \in (0,1)$ is the common discount factor. Both players are risk-neutral expected utility maximizers.

We assume that the optimal quality to the low type is interior:

$$\begin{aligned} 0 &< q_L^* \triangleq \arg\max_q \theta_L q - c(q) \\ &< q_H^* \triangleq \arg\max_q \theta_H q - c(q). \end{aligned}$$

A feasible contract in period t, ψ_t, specifies a transfer from the consumer to the monopolist, x_t, and a promise from the monopolist to the consumer for each future period $\{q_\tau\}_{\tau\geq t} \in [0,1]^\infty$. Thus, the set of feasible contracts in period t is

$$\mathcal{A}_t \triangleq \left\{\left(x_t, \{q_\tau\}_{\tau\geq t}\right) : x_t \in [-X, X],\ \ \{q_\tau\}_{\tau\geq t} \in [0,1]^\infty\right\}.$$

The assumption of quasilinear utilities with respect to money implies that changing the timing of payments has no impact on the marginal rate of substitution between quality and money for each consumer. Hence, changing the amount that the consumer owes to the monopolist in future periods relocates the set of allocations that are preferred to the original allocation by the same monetary amount for both types of consumer. This implies that this modification has no impact on renegotiation incentives. Therefore, we assume that all transfers are made in the current period.[1]

The set of menus available in period t, $\mathcal{M}_t$, is the set of closed subsets[2] of $\mathcal{A}_t$. In the first period, the monopolist offers to the consumer a menu $m_0 \in \mathcal{M}_0$. The consumer chooses an element from $m_0 \cup \{\emptyset\}$. If the consumer chooses $\emptyset$ then the consumer makes no payment to the monopolist, nothing is produced, and period 0 ends. If the consumer chooses $\psi_0 = \left(x_0, \{q_\tau\}_{\tau \geq 0}\right) \in m_0$, he pays x_0 to the monopolist, consumes q_0, the period ends and he starts the next period endowed with the promises $\{q_\tau\}_{\tau \geq 1}$.

Consider a period $t > 0$ in which the consumer starts with a promise $\{q_\tau\}_{\tau \geq t}$. The monopolist offers a renegotiation menu m_t. The consumer either chooses one contract from the menu, or rejects them all and remains with the promise $\{q_\tau\}_{\tau \geq t}$. If the consumer chooses to remain with the the promises $\{q_\tau\}_{\tau \geq t}$, he consumes q_t in the current period and starts the next period with the promises $\{q_\tau\}_{\tau \geq t+1}$. If the consumer chooses a contract $\left(x_t^i, \{q_\tau^i\}_{\tau \geq t}\right)$ from the menu m_t, he makes a transfer $x_t^i \in \Re$ to the monopolist, consumes q_t^i and the period ends. Then the next period starts with the promises $\{q_\tau^i\}_{\tau \geq t+1}$.

A history of length t, h^t, contains: i) all the offered menus in the previous periods $\{m_0, ..., m_{t-1}\}$; ii) the contracts selected by the consumer in the previous periods $\left\{\psi_0, ..., \psi_{t-1}\right\}$. We write H^t for the set histories of length t. Write $H = \cup_{\tau \geq 0} H^\tau$ for the set of all histories.

A pure strategy for the monopolist σ^M specifies a menu function for each period t, $\sigma_t^M : H^t \to \mathcal{M}_t$. A pure strategy is $\sigma^M = \cup_{t \geq 0} \sigma_t^M$ and with abuse of notation a mixed strategy σ^M is a mixture over pure strategies.

A pure strategy for the type $i \in \{L, H\}$ consumer specifies a menu selection function for every t, $\sigma_t^i : H^t \times \mathcal{M}_t \to \mathcal{M}_t \cup \{\emptyset\}$, with the restriction that for all $h^t \times m_t$ we have $\sigma_t^i \left(h^t \times m_t\right) \in m_t \cup \{\emptyset\}$. A pure strategy for type $i \in \{L, H\}$ is $\sigma_t^i = \cup_{t \geq 0} \sigma_t^i$ and with abuse of notation a mixed strategy σ^i is a mixture over pure strategies.[3]

[1] The set of allocations would be essentially the same (same distribution over qualities and discounted transfers) if the set of feasible promises $\mathcal{A}_t$ were replaced with $\left\{\left(\{x_\tau\}_{\tau \geq t}, \{q_\tau\}_{\tau \geq t}\right) : \{x_\tau\}_{\tau \geq t} \in [-X, X]^\infty \in \Re,\ \{q_\tau\}_{\tau \geq t} \in [0,1]^\infty\right\}$. We use $\mathcal{A}_t$ for simplicity of notation.

[2] Take the metric d on $\mathcal{A}_t$ such that for $\psi^1 = \left(x_t^1, \{q_\tau^1\}_{\tau \geq t}\right)$ and $\psi^2 = \left(x_t^2, \{q_\tau^2\}_{\tau \geq t}\right)$,

$d\left(\psi^1, \psi^2\right) = \left|x_t^1 - x_t^2\right| + \sum_{\tau \geq t} \delta^{\tau - t} \left|q_t^1 - q_t^2\right|$. Consider the topology induced by this metric and let $\mathcal{B}(\mathcal{A}_t)$ be the Borel sigma-field. We consider the Hausdorff metric topology on the collection of closed sets of $\mathcal{B}(\mathcal{A}_t)$ and we write $\mathcal{B}(\mathcal{M}_t)$ for the Borel sigma-field. For every t, $(\mathcal{A}_t, d)$ is compact. Hence, for every t, $\mathcal{M}_t$ is also compact (see Aliprantis and Border (2005), Theorem 3.85).

[3] We endow products of measurable space with the product sigma-algebra. A behavior strategy

We write P_σ for the probability measure over histories in H which is generated by the strategy profile σ.

4 Renegotiation Refinement: Optimal Contracting with Imperfect Commitment

We will study strategy profiles $\left(\sigma^M, \sigma^L, \sigma^H\right)$ which constitute a Bayesian Nash equilibrium (BNE) and satisfy an optimal-contracting-with-imperfect-commitment renegotiation refinement. For every Bayesian Nash equilibrium satisfying our refinement there is an outcome equivalent perfect Bayesian equilibrium.

Our refinement captures the idea that in each period the monopolist can offer a renegotiation proposal that induces the consumer to choose a continuation equilibrium which maximizes the monopolist's profit. This refinement is consistent with the monopolist having the bargaining power and, in this sense, being able to propose the way that the relationship should proceed. In addition, this refinement captures our view that under renegotiation there is no reason to treat asymmetrically different periods of the relationship. These requirements are standard in the literature which studies adverse selection under renegotiation (Hart and Tirole (1988), Laffont and Tirole (1990) and Battaglini (2007)).

Bester and Strausz (2001) provide a refinement with a recursive formulation capturing the concept presented in the paragraph above. For the class of finitely repeated games considered in Bester and Strausz (2001), the renegotiation requirement imposes that, in the last period, a continuation play that maximizes the expected payoff of the principal (monopolist in our model) is selected. This generates the set of credible payoffs for the last period. In the penultimate period, we select a continuation play which maximizes the expected payoff of the principal, imposing that continuation payoffs in the last period are credible. By applying this procedure finitely many times, we obtain the set of allocations satisfying the refinement. A notable result obtained by the authors is that there is always a payoff equivalent equilibrium in which the principal offers menus with the same cardinality

for the monopolist is a sequence of probability kernels σ_t^M from $\left(H^{t-1}, \mathcal{B}\left(H^{t-1}\right)\right)$ into $\left(\mathcal{M}_t, \mathcal{B}\left(\mathcal{M}_t\right)\right)$. A behavior strategy for the type $i \in \{L, H\}$ consumer is a sequence of probability kernels σ_t^i from $\left(H^{t-1} \times \mathcal{M}_t, \mathcal{B}\left(H^{t-1} \times \mathcal{M}_t\right)\right)$ into $\left(\mathcal{A}_t, \mathcal{B}\left(\mathcal{A}_t\right)\right)$ with the restriction that $\sigma_t^i\left(\left(h^{t-1}, m_t\right)\right) \in m_t$ for all $\left(h^{t-1}, m_t\right)$ (where w.l.o.g. we assume that the monopolist offers the status quo contract). Since action spaces are compact (hence Polish) the Kolmogorov Extension Theorem (Aliprantis and Border (2005), Corollary 15.27) guarantees that mixed strategies can be well defined in the infinite product sigma-algebra.

as the type space of the agent.

Here we extend the authors' solution concept to infinitely repeated games. Hence, as suggested by Bester and Strausz (2001), we use a fixed-point to define the set of continuation payoffs. Moreover, mimicking their argument, one can show that it is without loss to assume that the monopolist offers menus with two contracts. Therefore, assume that $|\mathcal{M}_t| = 2$ for the remaining of this paper.

Consider the state variables $(p, \{q_\tau\}) \in [0,1] \times [0,1]^\infty$, where p represents the belief held by the monopolist that the consumer is a high type, and $\{q_\tau\}$ is the set of promises at the beginning of the period. Interpret $V^M(p, \{q_\tau\}) \in \Re$ as the value for the monopolist and $\Phi(p, \{q_\tau\}) \subseteq \Re^2$ as the set of possible continuation payoffs for the consumer, given the state variables. We will impose that, for each state $(p, \{q_\tau\})$, the monopolist maximizes her profit by choosing an optimal continuation equilibrium. We call values $\left(V^M, \Phi\right)$ satisfying the conditions explained above profit maximizing:

Definition 1 $\left(V^M, \Phi\right)$ *are profit maximizing values if:*

I) For all $(p, \{q_\tau\}) \in [0,1] \times [0,1]^\infty$ *the program below achieves a maximum*

$$V^M(p, \{q_\tau\}) = \max_{m, a_L, a_H, \left(v_L^j, v_H^j\right)_{j=1}^2} \left\{ \sum_{j=1}^{2} \mu\left(a_L^j, a_H^j\right) \left[(1-\delta)\left[x^j - c\left(q^j\right)\right] + \delta V^M\left(p^j, \{q_\tau^j\}\right)\right] \right\}, \tag{2}$$

where $m = \left(\left(x^1, \{q_\tau^1\}\right), \left(x^2, \{q_\tau^2\}\right)\right)$ *is a menu,* $a_i = \left(a_i^1, a_i^2\right) \in \Delta(\{0,1\})$ *with* $a_i^j \in [0,1]$ *being the probability that type* $i \in \{L, H\}$ *chooses contract* j *from menu* m*,* v_i^j *is a promised future expected rent to be delivered to consumer* i *if he chooses contract* j*, and* $\mu\left(a_L^j, a_H^j\right) = pa_H^j + (1-p)a_L^j$ *is the probability that contract* $\left(x^j, \{q_\tau^j\}\right)$ *is accepted.*

The maximization (2) is subject to the following constraints:

Incentive compatibility:

$$(1-\delta)\left[\theta_i q^k - x^k\right] + \delta v_i^k \geq (1-\delta)\left[\theta_i q^j - x^j\right] + \delta v_i^j \quad \textit{if } a_i^k > 0, \tag{3}$$

for $i \in \{L, H\}$ *and for all* k, j*.*

Rationality to the high type:

$$\max_k \left[(1-\delta)\left[\theta_H q^k - x^k\right] + \delta v_H^k\right] \geq \sum_{\tau \geq 0} (1-\delta)\, \delta^\tau \theta_H q_\tau. \tag{4}$$

Rationality to the low type:

$$\max_k \left[(1-\delta)\left[\theta_L q^k - x^k\right] + \delta v_L^k\right] \geq \sum_{\tau \geq 0} (1-\delta)\,\delta^\tau \theta_L q_\tau. \tag{5}$$

Promise keeping:

$$\left(v_L^j, v_H^j\right) \in \Phi\left(p^j, \left\{q_\tau^j\right\}_{\tau>0}\right) \quad \text{for all } j. \tag{6}$$

Bayes rule:

$$p^j = \left(\frac{pa_H^j}{pa_H^j + (1-p)a_L^j}\right) \quad \text{if } \mu\left(a_L^j, a_H^j\right) > 0. \tag{7}$$

II) $(v_L, v_H) \in \Phi(p, \{q_\tau\})$ *if and only if there exists a (Borel-measurable) probability distribution* β *over the argmax of (2) such that:*

$$v_i = \int \max_k \left[(1-\delta)\left[\theta_i q^k - x^k\right] + \delta v_i^k\right] d\beta$$

for $i = L, H$.

We can interpret condition (2) as the requirement that, for each state, we select a continuation equilibrium that maximizes the monopolist's expected profit. For that, we select a menu m with 2 contracts, a randomization for each $i \in \{L, H\}$ consumer over each contract $a_i \in \Delta(\{0,1\})$, and a credible promised future expected rent, v_i^j, delivered for consumer i if he chooses contract j. The maximization program is restricted by four constraints.

First, we have incentive compatibility: (3). If the type i consumer chooses contract $j \in \{1, 2\}$, his expected rent can be divided into two elements: the current period's rent $(1-\delta)(\theta_i q^j - x^j)$ and his future expected rent δv_i^k. The constraint (3) requires that if type i consumer chooses contract j with positive probability then the expected rent from this choice must be weakly greater than the expected rent from choosing contract $j' \neq j$.

Second, we have the rationality constraint to the high type (4), and the one to the low type (5). These constraints require that the rent of type i consumer from accepting to renegotiate has to be weakly greater than the rent obtained from refusing to renegotiate and consuming his outside option (see the right side of (4) and (5)).

Third, we have the promise keeping constraint (6). The set $\Phi(p, \{q_\tau\}) \subseteq \Re^2$ can be interpreted as the set of all possible continuation rents for each type of consumer when a period starts with promises $(p, \{q_\tau\})$. Thus, if contract j gives rise to a next period state

$(p^j, \{q^j_\tau\})$, the constraint (6) insists that the continuation rent promised to each type of consumer in the next period, when contract j is chosen, is credible: $(v^j_L, v^j_H) \in \Phi\left(p^j, \{q^j_\tau\}\right)$.

Fourth, we have Bayes' rule (7). This constraint assures that the updated belief upon the acceptance of contract j is calculated from Bayes' rule whenever possible.

Part *II* of Definition 1 defines the set of credible promises when the state is $(p, \{q_\tau\})$: $\Phi(p, \{q_\tau\})$. A continuation rent for each type of consumer (v_L, v_H) belongs to this set if and only if there exists a (mixed) solution to (2) subject to (3)-(7) which delivers these rents for the consumers.

Definition 2 *A profit maximizing renegotiation equilibrium is a strategy profile* $\left(\sigma^{M*}, \sigma^{L*}, \sigma^{H*}\right)$ *such that:*

a) $\sigma^* = \left(\sigma^{M*}, \sigma^{L*}, \sigma^{H*}\right)$ *is a Bayesian Nash equilibrium;*

b) There exists profit maximizing values $\left(V^M, \Phi\right)$ *such that the continuation play is consistent with* $\left(V^M, \Phi\right)$. *That is, we require that* ***a.s.***$[P_{\sigma^*}]$ *every history* $\tilde{h}^t$ *is associated with a belief* $\tilde{p}_t \in [0,1]$ *of the consumer being a high type satisfying Bayes' rule. We also require that* ***a.s.***$[P_{\sigma^*}]$ *each menu* $\tilde{m}_t$ *offered by the monopolist at* $\tilde{h}^t$*, the randomization of the consumer over its elements, and the respective continuation rents for the consumer solve (2) subject to (3)-(7).*

Definition 2 states that a profit-maximizing continuation equilibrium is a Bayesian Nash equilibrium for which the play is consistent with some fixed-point $\left(V^M, \Phi\right)$. That is, we use Bayes' rule to calculate the belief p_t that the consumer is a high type for every history h^t on the equilibrium path. If $\{q_\tau\}$ are the current periods' promises in h^t the state is $(p_t, \{q_\tau\})$. We insist that, for the each menu offered with positive probability by the monopolist at h^t, the randomization over its elements, and the respective continuation rent for each type of consumer solve (2) subject to (3)-(7) when the state is $(p_t, \{q_\tau\})$. In Proposition 7 (in the appendix) we show that there exists an equilibrium.

5 Equilibrium

5.1 Commitment Allocation

We start our analysis by reviewing the optimal allocation under commitment. It is well known that the optimal allocation under commitment is the repetition of the optimal static allocation in each period (Baron and Besanko, 1984). We can use the revelation principle and write $\left(x_i^C, q_i^C\right)$ for the optimal contract designed for type $i \in \{L, H\}$ consumer. The monopolist's maximization problem can be written as:

$$\max_{\left\{\left(q_L^C, x_L^C\right),\left(q_H^C, x_H^C\right)\right\}} p_0 \left[x_H - c\left(q_H\right)\right] + \left(1-p_0\right)\left[x_L - c\left(q_L\right)\right]$$
$$s.t.$$
$$\theta_L q_L^C - x_L^C \geq 0 \qquad IR_L$$
$$\theta_H q_H^C - x_H^C \geq \theta_H q_L^C - x_L^C, \qquad IC_H$$
$$0 \leq q_L^C \leq q_H^C \leq 1 \qquad M$$

where the first constraint assures that it is rational to the low type to participate, the second precludes the high type from announcing being a low type and the third guarantees monotonicity and feasibility. Notice that the rent of the high type will be given by $\Delta\theta q_L = \left(\theta_H - \theta_L\right) q_L$. For $i \in \{L, H\}$ we define $\pi_i\left(q\right) = \theta_i q - c(q)$. Thus, the problem above can be rewritten as:

$$\max_{q_L^C, q_H^C} p_0 \left[\pi_H\left(q_H^C\right) - \Delta\theta q_L^C\right] + (1-p_0)\pi_L\left(q_L^C\right).$$

If the virtual valuation of the low type $\left[\theta_L - \Delta\theta\left(\frac{p_0}{1-p_0}\right)\right]$ is greater than the marginal cost at 0, $c'(0)$, then the optimal quality offered to the low type q_L^C satisfies $c'\left(q_L^C\right) = \left[\theta_L - \Delta\theta\left(\frac{p_0}{1-p_0}\right)\right]$. Otherwise, the monopolist offers $q_L^* = 0$. Hence, the monopolist distorts the low type's quality in order to extract a higher rent from the high type. The quality offered to the high type is efficient: $q_H^C = q_H^*$.

5.2 Efficient Allocations

The commitment allocation does not satisfy the renegotiation requirement because it leads to total separation of types in the first period and to a persistent distortion of the quality offered to the low type. Hence, the monopolist has an incentive to renegotiate the contract accepted by the low type in the second period.

What kind of allocations does not leave scope for renegotiation? Obvious candidates are incentive-compatible and individually-rational separating allocations which provide the efficient quality for each type of consumer from the first period on. We start our analysis by deriving the most profitable allocation in this class. As a by-product, we obtain a lower bound on the monopolist's profit.

Consider the initial period, which starts with the promise[4] $\{\mathbf{0}\}$. The monopolist can

[4] For $g \in [0,1]$ we write $\mathbf{g}$ for $\{g, g, ...\}$.

offer a menu with a contract designed exclusively to the low type: $\left(\left(\frac{\theta_L q_L^*}{1-\delta}\right), \{\mathbf{q}_L^*\}\right)$ and another designed exclusively to the high type: $\left(\left(\frac{\theta_H q_H^*}{1-\delta}\right) - \left(\frac{\Delta\theta q_L^*}{1-\delta}\right), \{\mathbf{q}_H^*\}\right)$. The contract designed for each type of consumer offers the efficient quality in each period. The low type's payment is set such that his rationality constraint binds. The high type's payment is set such that he is indifferent between his contract and the low type's contract. It is easy to check that a menu with these two contracts satisfies the constraints (3)-(7) when the state is $(p, \{\mathbf{0}\})$.[5] This menu yields a profit for the monopolist equal to:

$$p\left[\pi_H\left(q_H^*\right) - \Delta\theta q_L^*\right] + (1-p)\pi_L\left(q_L^*\right). \tag{8}$$

In this case, the information rent of the high type is given by $\Delta\theta q_L^*$, which is strictly higher than the commitment information rent. Can the monopolist extract more rents from the high type? How does the set of equilibria outcomes compare with the outcome described above? These questions are investigated in the sequel.

5.3 Screening Dynamics

Assume that at some history h^t on the equilibrium path the monopolists posts a menu $m_t = \left(\psi_t^1, \psi_t^2\right)$ and that each type of consumer randomizes over its elements according to the strategy profile σ^*. We can use Bayes' rule to calculate the belief p_{t+1}^i that the consumer is a high type conditional on the contract ψ_t^i being accepted. In this sense, we say that contract ψ_t^i is *associated with the belief* p_{t+1}^i.

We start by studying the belief evolution on the equilibrium path. A simple belief evolution is the following: either the belief decreases, in which case the monopolist becomes more pessimistic about the consumer, or the belief jumps to one, in which case the monopolist learns that the consumer is a high type. We refer to this belief evolution as sequential separating dynamics. Formally:

Definition 3 *An equilibrium presents sequential separating dynamics if, for any history associated with an interior belief, the monopolist offers a menu containing only pooling contracts, i.e., contracts associated with lower beliefs, and revealing contracts, i.e., contracts associated with belief 1.*

Definition 3 describes simple equilibrium dynamics. In each history associated with an interior belief the monopolist offers a menu with a pooling contract, i.e., a contract accepted

[5] It is straightforward to verify that $(0, \Delta\theta q_L^*) = \inf \Phi(0, \{\mathbf{0}\})$.

by the low type with probability one, which leads to a lower belief, and a revealing contract, i.e., a contract accepted exclusively by the high type.

In Proposition 1.1, we show that every equilibrium presents sequential separating dynamics. This result simplifies the equilibrium characterization significantly and is central to all the analysis that follows. This result is somewhat reminiscent of the cream-skimming property present in many bargaining games (see Fudenberg and Tirole, 1991, chapter 10). According to the cream-skimming property, a monopolist who sells a durable good trades earlier with higher-value consumers. In bargaining environments, this behavior is an immediate consequence of a single-crossing condition: a consumer who values a good more prefers buying it earlier and paying a higher price for it. This is because in these bargaining models the way that the monopolist updates her belief conditional on the consumer buying a good is irrelevant since this choice ends the game. In our model, when a consumer purchases a contract he takes into consideration the effect of this decision on the belief of the monopolist since it may affect the next period's terms of trade. This additional strategic effect adds complexity to the analysis.

We define a simple equilibrium as an equilibrium in which the monopolist does not make promises for future periods in pooling contracts and promises $\{\mathbf{q}_H^*\}$ in all revealing contracts. We say that two equilibria are outcome equivalent if each type of consumer faces the same distribution over sequence of qualities and discounted payments. Proposition 1.2 shows that for every equilibrium there exists an outcome-equivalent simple equilibrium. Finally, Proposition 1.3 shows that in all simple equilibria the price paid for the revealing contract is decreasing from period 1 on.

Below we provide a sketch of the proof. Details are in the appendix. The reader less interested in technical details may skip this sketch and proceed to Section 5.5. There we provide intuition for this result.

Proposition 1

1.1 Every equilibrium presents sequential separating dynamics.

1.2 For every equilibrium, there exists an outcome equivalent simple equilibrium.

1.3 In all simple equilibria the price paid for the revealing contract is decreasing from period 1 on.

5.4 Sketch of the Proof of Proposition 1

We call the problem of maximizing (2) subject to (3),(4),(5),(6) and (7) the monopolist's problem.

5.4.1 Proposition 1.1: A Simple Case

The argument establishing that all equilibria present sequential separating dynamics is by contradiction. We show that if a putative equilibrium does not present this property then we can find a history on the equilibrium path inducing to continuation play which does not solve the monopolist's problem. This violates our renegotiation refinement specified in Definition 2.

Due to the absence of commitment, very little structure is imposed, *a priori,* on the continuation values (V, Φ). To circumvent this difficulty, we use the idea of reachable history distributions. Take a history h^t associated with a state $(p_t, \{q_\tau\})$. For each $i = 1, ..., L$, consider a tuple $\left\{m^i, \left(a_L^i, a_H^i\right), \left(v_L^{j,i}, v_H^{j,i}\right)_{j=1,2}\right\}$ consisting of menu m^i, randomizations for each type of consumer $\left(a_L^i, a_H^i\right)$, and promises $\left(v_L^{j,i}, v_H^{j,i}\right)_{j=1,2}$, which satisfy the constraints (3)-(7), given the state $(p_t, \{q_\tau\})$ (but are not necessarily optimal). Consider a randomization of the monopolist over the tuples $\left\{m^i, \left(a_L^i, a_H^i\right), \left(v_L^{j,i}, v_H^{j,i}\right)_{j=1,2}\right\}_{i=1}^{L}$. This generates a reachable distribution $\left\{F_j^{t+1}\right\}_{j=1}^{N_{t+1}}$over histories $\left\{h_j^{t+1}\right\}_{j=1}^{N_{t+1}}$ of length $t+1$ from history[6] h^t. Since for ever i the tuple $\left\{m^i, \left(a_L^i, a_H^i\right), \left(v_L^{j,i}, v_H^{j,i}\right)_{j=1,2}\right\}$ is feasible the equilibrium requires that

$$\bar{V}^M\left(h^t\right) \geq \sum_{j=1}^{N_{t+1}} F_j^{t+1} \bar{V}^M\left(h_j^{t+1}\right),$$

where $\bar{V}^M\left(h^v\right)$ denotes the expected profit of the monopolist at $t=0$, conditional on the information that history h^v is reached in period v.

Proceeding analogously at each history h_j^{t+k}, we generate a reachable distribution $\left\{F_z^{t+k+1}\right\}_{z=1}^{N_{t+k+1}}$ over histories of length $t+k+1$, $\{h_z^{t+k+1}\}_{z=1}^{N_{t+k+1}}$. The equilibrium requires that

$$\bar{V}^M\left(h^t\right) \geq \sum_{z=1}^{N_{t+k+1}} F_z^{t+k+1} \bar{V}^M\left(h_z^{t+k+1}\right).$$

[6] Since each menu has two contracts we have $N_{t+1} = 2L$. This will not play a role and will be omitted for simplicity.

Hence, if we find $s > 0$ and a distribution over histories of length $t+s$ that is reachable from h^t under a putative equilibrium such that

$$\bar{V}^M\left(h^t\right) < \sum_{z=1}^{N_{t+s}} F_z^{t+s} \bar{V}^M\left(h_z^{t+s}\right),$$

then we reach a contradiction.

We say that a continuation play is viable if it is generated by a reachable distribution over histories. A continuation play is consistent with an equilibrium only if it yields for the monopolist a higher expected profit than the one obtained in any viable continuation play.

We use these ideas to show that all equilibria present sequential separating dynamics. First, it can be easily shown that the monopolist never offers a menu with one contract associated with an interior belief, and another associated with the belief zero[7], that is, the monopolist screens the high type not the low type. Therefore, we must show that in any history h^t on the equilibrium path (associated with an interior belief, $p_t \in (0,1)$) the monopolist never offers a menu with one contract associated with a higher but interior belief, and another one associated with a lower belief. Here, we prove that the monopolist has a profitable deviation from such a menu under a special case with four main simplifications:

i) The monopolist starts the current period with no promises, and the monopolist makes no promises in contracts associated with interior beliefs.

ii) In period $t+2$ the belief belongs to $[0, p_t] \cup \{1\}$ with probability 1 under the putative equilibrium.

iii) The monopolist plays a pure strategy.

iv) The quality in each pooling contract is different from the efficient quality to the high type.

In section 7 we prove Proposition 1 without these restrictions.

Assume that in period t the monopolist offers a menu with two contracts $\left(\psi_t^1, \psi_t^2\right)$. Contract 1, $\psi_t^1 = \left(\theta_L q_t^1, \{q_t^1, 0, 0, ...\}\right)$, is associated with the belief $p_{t+1}^1 < p_t$ and contract 2, $\psi_t^2 = \left(\theta_L q_t^2, \{q_t^2, 0, ..., \}\right)$, is associated with the belief $p_{t+1}^2 \in (p_t, 1)$. Requirement iv) above implies $q_t^2 \neq q_H^*$. Assume that the probability that contract $i \in \{1, 2\}$ is chosen is $\mu_t^i > 0$. The monopolist's expected profit at the beginning of period t is:

[7] This strategy is strictly dominated by a strategy in which both types of consumers are screened with probability 1 in the current period.

$$\mu_t^1 \left[(1-\delta) \pi_L \left(q_t^1 \right) + \delta V^M \left(p_{t+1}^1 \right) \right] \\ + \mu_t^2 \left[(1-\delta) \pi_L \left(q_t^2 \right) + \delta V^M \left(p_{t+1}^2 \right) \right], \tag{9}$$

where $\pi_L(q) = \theta_L q - c(q)$, $V^M(p) = V^M(\{\mathbf{0}\}, p)$, and we used the fact that the low type's rationality constraint binds.[8]

According to ii, in period $t+1$, conditional on the contract ψ_t^2 being accepted in period t, the monopolist offers 2 contracts: a pooling contract $\left(\theta_L q_{t+1}^2, \left\{q_{t+1}^2, 0, ..., \right\}\right)$ associated with the belief $p_{t+2}^2 < p_t$ and a revealing contract $\left(x_{t+1}^2, \{\mathbf{q}_H^*\}\right)$. The probability that the pooling (revealing) contract is chosen is given by μ_{t+1}^2 (resp. $(1-\mu_{t+1}^2)$). Thus, the monopolist's expected profit under the putative equilibrium at the beginning of period t is given by:

$$\mu_t^1 \left[(1-\delta) \pi_L \left(q_t^1 \right) + \delta V^M \left(p_{t+1}^1 \right) \right] \\ + \mu_t^2 \left[(1-\delta) \pi_L \left(q_t^2 \right) + \delta \left[\begin{array}{c} \mu_{t+1}^2 \left[(1-\delta) \pi_L \left(q_{t+1}^2 \right) + \delta V^M \left(p_{t+2}^2 \right) \right] \\ \left(1-\mu_{t+1}^2\right) \left[(1-\delta) x_{t+1}^2 - c\left(q_H^*\right) \right] \end{array} \right] \right]. \tag{10}$$

By construction, the high type is randomizing in period t. This implies that the high type's rent from accepting contract ψ_t^1 is equal to:

$$v_{H,t} = (1-\delta) \Delta\theta q_t^2 + \delta \left[\theta_H q_H^* - (1-\delta) x_{t+1}^2 \right]. \tag{11}$$

Notice that the belief p_{t+1}^2 can be written as a convex combination of p_t and 1 :

$$p_{t+1}^2 = \beta \times p_t + (1-\beta) \times 1. \tag{12}$$

We will use (12) above to propose a deviation in which the set of reached beliefs in $t+1$ is $\left\{p_{t+1}^1, p_t, 1\right\}$.

Proposed deviation: The monopolist randomizes in period t.

a) With probability γ the monopolist offers a menu with the contract $\psi_t^1 = \left(\theta_L q_{t+1}^1, \left\{q_{t+1}^1, 0, ..., \right\}\right)$ associated with a belief p_{t+1}^1 and the revealing contract $\psi_t^R = \left(\theta_H q_H^* - \frac{v_{H,t}}{1-\delta}, \{\mathbf{q}_H^*\}\right)$. Contract ψ_t^1 $\left(\psi_t^R\right)$ is chosen with probability $\mu\left(p_t, p_{t+1}^1\right)$ (resp. $\left(1-\mu\left(p_t, p_{t+1}^1\right)\right)$), where

$$\mu\left(p, p'\right) \triangleq \left(\frac{1-p}{1-p'}\right)$$

[8] This is established in Lemma 1 for the case in which the period starts with no promises. Lemma 7 establishes that it is always true along the screening process (even if a history starting with non-trivial promises is reached).

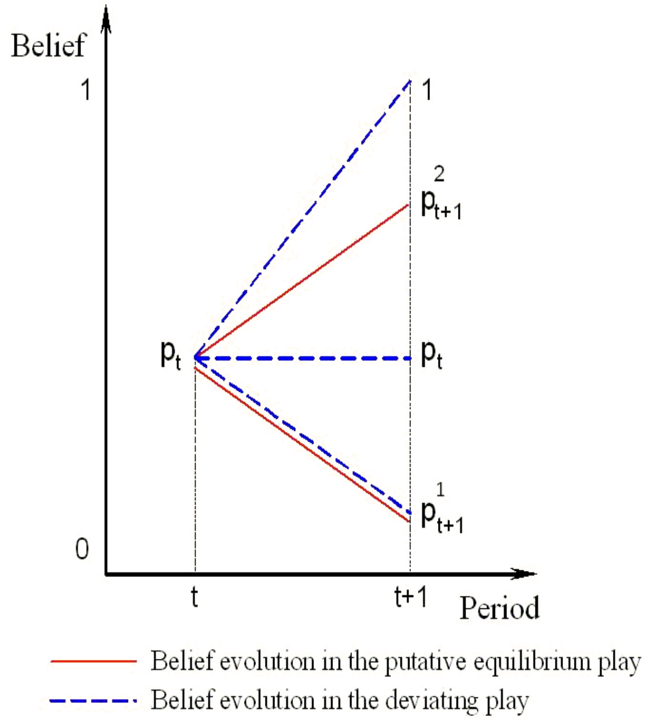

Figure 1

is the probability of acceptance of a pooling contract associated with belief $p' < p$ in a period starting with belief p, conditional on the other contract being revealing.

b) With complimentary probability $(1-\gamma)$, the monopolist offers a menu containing only the contract $\psi_t^2 = (\theta_L q_t^2, \{q_t^2, 0, ...,\})$ in period t. Both types choose this contract with probability 1.

We take γ such that the final probability that contract ψ_t^1 is accepted is the same as the putative equilibrium play's probability:

$$\gamma\mu\left(p_t, p_{t+1}^1\right) = \mu_t^1. \tag{13}$$

The belief dynamics under the original and the deviating play are illustrated in Figure 1.

Under the proposed deviation, the monopolist mimics the putative equilibrium play strategy conditional on the contract ψ_t^2 being accepted: she offers the pooling contract $\left(\theta_L q_{t+1}^2, \left\{q_{t+1}^2, 0, ...,\right\}\right)$ associated with a belief $p_{t+1}^2 < p_t$ and the revealing contract $\left(x_{t+1}^2, \{\mathbf{q}_H^*\}\right)$. Hence, her expected profit under the proposed deviation is:

$$\gamma\left[\begin{array}{c}\mu\left(p_t,p_{t+1}^1\right)\left[(1-\delta)\,\pi_L\left(q_t^1\right)+\delta V^M\left(p_{t+1}^1\right)\right]\\+\left(1-\mu\left(p_t,p_{t+1}^1\right)\right)\left[\pi_H\left(q_H^*\right)-v_{H,t}\right]\end{array}\right] \tag{14}$$
$$+\left(1-\gamma\right)\left[\left(1-\delta\right)\pi_L\left(q_t^2\right)+\delta\left(\begin{array}{c}\mu\left(p_t,p_{t+2}^2\right)\left[(1-\delta)\,\pi_L\left(q_{t+1}^2\right)+\delta V^M\left(p_{t+2}^2\right)\right]\\+\left(1-\mu\left(p_t,p_{t+2}^2\right)\right)\left[(1-\delta)\,x_{t+1}^2-c\left(q_H^*\right)\right]\end{array}\right)\right].$$

First, notice that by construction we have $\gamma\mu\left(p_t,p_{t+1}^1\right)=\mu_t^1$. Furthermore, using the martingale property of beliefs, it is straightforward to verify that $(1-\gamma)\,\mu\left(p_t,p_{t+2}^2\right)=\mu_t^2\mu_{t+1}^2$. Hence, in order to compare (10) and (14), it is sufficient to compare their strings leading to revealing contracts. We claim that the distribution over strings leading to revealing contracts in (14) yields a strictly higher profit for the monopolist. For that it is enough to show that:

$$\begin{array}{rl}&\pi_H\left(q_H^*\right)-v_{H,t}\\>&(1-\delta)\,\pi_L\left(q_t^2\right)+\delta\left[(1-\delta)\,x_{t+1}^2-c\left(q_H^*\right)\right].\end{array} \tag{15}$$

Using (11) and the fact that $\pi_L(q)=\theta_L q-c(q)$ it follows that (15) is equivalent to:

$$\begin{array}{rl}&(1-\delta)\,\pi_H\left(q_H^*\right)\\>&(1-\delta)\,\pi_L\left(q_t^2\right)+(1-\delta)\,\Delta\theta q_t^2\\=&\pi_H\left(q_t^2\right),\end{array} \tag{16}$$

which holds because $q_t^2\neq q_H^*=\arg\max_q\pi_H(q)$. Therefore, under the conditions i)-iv), plays which violate sequential separating dynamics lead to inefficiently slow separation. Optimal renegotiation implies that these plays cannot be part of an equilibrium.

5.4.2 Proposition 1.2

Next, we comment on the proof of 1.2 of Proposition 1. When the high type is being screened in the current period, the monopolist does not like promising a higher information rent for the next period because this implies charging less for a revealing contract. Moreover, she does not like being constrained to delivering a minimum information rent to the high type. Therefore, we can always find an optimal menu in which the monopolist sets $\{q_\tau\}_{\tau\geq t}=\{\mathbf{0}\}$ in pooling contracts.[9]

[9] There always exists an optimal menu in which the promises $\{q_\tau\}_{\tau\geq t}$ are strictly positive, but close to zero, so that the additional constraint can be ignored.

Consider a history h^t on the equilibrium path associated with the state $(p_t, \{q_\tau\})$, with $p_t \in (0,1)$ and $\{q_\tau\} \neq \mathbf{0}$. Write $m_t = \left(\psi_t^1, \psi_t^2\right)$ for a menu which is offered by the monopolist at h^t and assume (w.l.o.g.) that the low type chooses the contract $\psi_t^1 = \left(x_t^1, \{q_\tau^1\}_{\tau \geq t+1}\right)$ with positive probability, and that this contract leads to the belief p_{t+1}^1. If the rationality constraint of the low type holds with equality then:

$$-(1-\delta)\,x_t^1 + (1-\delta)\,\theta_L q_t^1 + \delta v_{L,t+1} = (1-\delta)\sum \delta^{\tau-t}\theta_L q_\tau, \tag{17}$$

where $v_{L,t+1} \in \Phi_L\left(p_{t+1}^1, \{q_\tau^1\}_{\tau \geq t+1}\right)$ is the rent promised to the low type in period $t+1$.

Because the contract ψ_t^1 satisfies the rationality of the high type:

$$-(1-\delta)\,x_t^1 + (1-\delta)\,\theta_H q_t^1 + \delta v_{H,t+1} \geq (1-\delta)\sum \delta^{\tau-t}\theta_H q_\tau, \tag{18}$$

where $v_{H,t+1} \in \Phi_H\left(p_{t+1}^1, \{q_\tau^1\}_{\tau \geq t+1}\right)$ is the rent promised to the high type[10] in period $t+1$.

Hence, using (17) and (18) we must have:

$$\begin{aligned} & (1-\delta)\,\Delta\theta q_t^1 + \delta\left[v_L\left(p_{t+1}^1, \{q_\tau^1\}_{\tau \geq t+1}\right) - v_H\left(p_{t+1}^1, \{q_\tau^1\}_{\tau \geq t+1}\right)\right] \\ \geq\; & (1-\delta)\sum \delta^{\tau-t}\Delta\theta q_\tau. \end{aligned} \tag{19}$$

Therefore (19) delivers a lower bound on the information rent of the high type

$$(1-\delta)\sum \delta^{\tau-t}\Delta\theta q_\tau,$$

which is minimized if $\{q_\tau\}_{\tau \geq t} = \{\mathbf{0}\}$. Hence, the monopolist is weakly better-off by setting $\{q_\tau\}_{\tau \geq t} = \{\mathbf{0}\}$.

Notice that the argument above explains the first statement in 1.2, that there exists an outcome-equivalent equilibrium in which all pooling contracts involve no promises for future periods. Next, we turn to the second statement in 1.2, that is, there exists an outcome equivalent equilibrium in which the monopolist offers $\{\mathbf{q}_H^*\}$ in revealing contracts. A standard single-crossing argument implies that the low type strictly prefers the pooling contract to the revealing one. Therefore, it is optimal for the monopolist to offer the efficient quality to the high type in each period.

[10] In fact we need the stronger condition that $(v_{L,t+1}, v_{H,t+1}) \in \Phi\left(p_{t+1}^1, \{q_\tau^1\}_{\tau \geq t+1}\right)$. The weaker restriction in the text is enough to illustrate our argument.

5.4.3 Proposition 1.3

The proof of the claim made in 1.3 that the price paid for the revealing contracts is strictly decreasing from period 1 on is by contradiction. Consider an equilibrium in which the monopolist makes no promises in pooling contracts and consider a history h^t $(t \geq 1)$ on the equilibrium path with a belief $p_t \in (0,1)$. Let m_t be the menu offered by the monopolist[11] at h^t, with the revealing contract $\psi_t^R = \left(x_t^R, \{\mathbf{q}_H^*\}\right)$, and with a pooling contract leading to the belief $p_{t+1} \in (0,1)$. Suppose that following the acceptance of the pooling contract the monopolist offers a revealing contract $\psi_{t+1}^R = \left(x_{t+1}^R, \{\mathbf{q}_H^*\}\right)$ in the next period such that $x_t^R \leq x_{t+1}^R$, that is, prices are not strictly decreasing. This implies that the information rent of the high type in period t, $v_{H,t}$, is weakly grater than his information rent in period $t+1, v_{H,t+1}$:

$$\begin{aligned} v_{H,t} &= \theta_H q_H^* - (1-\delta)\, x_t^R \\ &\geq \theta_H q_H^* - (1-\delta)\, x_{t+1}^R = v_{H,t+1}. \end{aligned} \tag{20}$$

This can be used to show that the monopolist can profitably deviate by "speeding-up" separation in period $t-1$. That is, she can offer a pooling contract which induces the belief $p_{t+1} < p_t$. By doing so, she increases the surplus of the relationship by screening a larger measure of consumers[12] in the current period and from (20) she offers a lower information rent to the high type. Therefore, this deviation yields a strictly higher profit for the monopolist.

■

5.5 Discussion of Proposition 1

In 1.1 of Proposition 1 we showed that the high type is sequentially screened. The main challenge lies in showing that the monopolist never offers two contracts, one associated with a lower belief, and another associated with a higher but interior belief. The efficient quality to the low type, q_L^*, being different from the efficient quality to the high type, q_H^*, is key for our argument. Suppose that the monopolist starts period t with belief $p_t \in (0,1)$ and offers a menu with contracts 1 and 2. Contract 1 is associated with a strictly lower

[11] For simplicity we assume here that the monopolist plays a pure strategy.

[12] This intuition emphasizes the economic driving forces. The formal argument (a corollary of Lemma 6) is slightly more involving.

belief $p^1_{t+1} < p_t$, while contract 2 is associated with a strictly higher, but interior belief $p^2_{t+1} \in (p_t, 1)$. In the proof, we show that the monopolist could profitably deviate by screening the high type in period t with positive probability, so that a contract leading to a higher but interior belief is not offered. The main insight is establishing that plays involving sequential separating dynamics (hence not reaching higher, but interior beliefs) lead to earlier separation and preserve the same rent for each type of consumer. Hence, since the monopolist's profit is equal to the difference between the surplus of the relationship and the rent of the consumer, the renegotiation requirement implies that there exists a profitable deviation inducing sequential separation.

1.2 of Proposition 1 established that for every equilibrium there exists another outcome-equivalent equilibrium in which all pooling contracts involve no promises for future periods (i.e. the monopolist promises $\{\mathbf{0}\}$), and all revealing contracts involve promising the efficient quality to the high type, q^*_H, for every future period. We restrict attention to equilibria with these properties for the remaining of this paper.

1.3 of Proposition 1 shows that the monopolist repeatedly offers price cuts for the revealing contract, while the price per quality remains constant in the pooling contract (so the rationality of the low type holds with equality). Hence, the model predicts that the later the high-type consumer migrates from the pooling to the revealing contract, the higher the discount he receives once he eventually buys the revealing contract. The high-type consumer balances the desire to start consuming the high-quality good immediately against the discount he would receive from pooling with the low-type consumer for an additional period before purchasing the revealing contract.

5.6 Quality Dynamics

We use our finding that the price in revealing contracts decreases over time to obtain predictions about quality dynamics. Because the high type is always indifferent between the pooling and the revealing contract, the statement that prices are strictly decreasing is equivalent to expected discounted qualities being strictly increasing in pooling contracts, or

$$E^*_{\sigma}\left[(1-\delta)\sum_{\tau\geq t+1}\delta^{\tau-t-1}\Delta\theta q_\tau \mid \theta_L, h^t\right] > E^*_{\sigma}\left[(1-\delta)\sum_{\tau\geq t}\delta^{\tau-t}\Delta\theta q_\tau \mid \theta_L, h^t\right], \tag{21}$$

where we write $E_{\sigma}[Z \mid Y]$ for the conditional expectation of any (measurable) $r.v.$ Z given Y, according to the probability measure induced by the strategy profile σ.

Proposition 2 *For every $t \geq 1$ the expected discounted quality of type θ, $E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t}\delta^{\tau}q_{\tau} \mid \theta, h^{t}\right]$, is a submartingale. Moreover, if h^t is a history associated with an interior belief then:*

$$E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t+1}\delta^{\tau-t-1}q_{\tau} \mid \theta_L, h^t\right] > E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t}\delta^{\tau-t}q_{\tau} \mid \theta_L, h^t\right].$$

If there exists a subsequent history h^{t+1} associated with an interior belief reached with positive probability, then the strict inequality is also true for $\theta = \theta_H$. Finally, if the history h^t puts probability 1 on type θ, then the monopolist offers the efficient quality in each future period.

Proof. For the low type the inequality (21) implies the result for any history h^t associated with an interior belief. For histories starting with a belief $p_t \in \{0, 1\}$, the renegotiation refinement implies that the efficient quality is provided in each future period. Therefore, it is sufficient to prove the inequality above to the high type.

It can be shown that the monopolist promises the minimum continuation rent to the high type for each belief[13] in order to charge more for revealing contracts. Hence, the realized continuation information rent of the high type from period t on is **a.s.**$[P_{\sigma^*}]$ equal to a constant[14], which we label $v_{H,t}$. Consider a history h^t reaching with positive probability in the next period a history h^{t+1} associated with a belief $p_{t+1} \in (0, 1)$. The history h^t induces to a distribution of continuation qualities to the high type, with support given by $\mathcal{Q} \subseteq [0, 1]^{\infty}$. Consider the partition of $\mathcal{Q}$ indicating the period in which the high type accepts a revealing contract. We write

$$\mathcal{Q}^s \triangleq \left\{\{q_\tau\}_{\tau\geq t} \in \mathcal{Q} : q_\tau^\lambda = q_H^* \iff \tau \geq s\right\}.$$

Hence[15] $\mathcal{Q} = \cup_{s\geq t}\mathcal{Q}^s$. Thus:

[13] For each $t \geq 1$ the continuation rent of the high type is $v_{H,t} = \min \Phi_H(p_t, \{\mathbf{0}\})$.
[14] This technical observation is also true in the sequel, but is omited for brevity.
[15] It can be shown that $P_{\sigma^*}(\mathcal{Q} \setminus \cup_{s\geq t}\mathcal{Q}^s) = 0$.

$$
\begin{aligned}
& E_{\sigma^*}\left[(1-\delta)\sum_{\tau\geq t}\delta^{\tau-t}q_\tau \mid \theta_H, h^t\right] \\
= \; & P_{\sigma^*}\left(\mathcal{Q}^t \mid \theta_H, h^t\right) q_H^* \\
& +\sum_{s>t}^{\infty} P_{\sigma^*}\left(\mathcal{Q}^s \mid \theta_H, h^t\right) E_{\sigma}^{*}\left[\begin{array}{c}(1-\delta)\sum_{t\leq\tau<s}\delta^{\tau-t}q_\tau \\ +\delta^{s-t}q_H^* \mid \theta_H, h^t, \{q_\tau\}_{\tau\geq t}\in\mathcal{Q}^s\end{array}\right].
\end{aligned} \tag{22}
$$

Next, notice that if $\{q_\tau\}_{\tau\geq t}\in\mathcal{Q}^s$ $(s>t)$ there exists a sequence of continuation qualities to the low type $\{q_{\tau,L}\}_{\tau\geq s}$ such that:

$$
v_{H,t}=(1-\delta)\sum_{t\leq\tau<s}\delta^{\tau-t}\Delta\theta q_\tau+(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t}\Delta\theta q_{\tau,L},
$$

where we used the fact that the continuation realized information rent of the high type from period t on is $v_{H,t}$.

Hence (22) is equivalent to:

$$
\begin{aligned}
& P_{\sigma^*}\left(\mathcal{Q}^t \mid \theta_H, h^t\right) q_H^* \\
& +\sum_{s>t}^{\infty} P_{\sigma^*}\left(\mathcal{Q}^s \mid \theta_H, h^t\right) E_{\sigma}^{*}\left[\begin{array}{c}(1-\delta)\sum_{t\leq\tau<s}\delta^{\tau-t}q_\tau+(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t}q_{\tau,L} \\ +(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t}\left(q_H^*-q_{\tau,L}\right) \mid \theta_H, h^t, \{q_\tau\}_{\tau\geq t}\in\mathcal{Q}^s\end{array}\right] \\
= \; & \sum_{s=t}^{\infty} P_{\sigma^*}\left(\mathcal{Q}^s \mid \theta_H, h^t\right) E_{\sigma}^{*}\left[\left(\frac{v_{H,t}}{\Delta\theta}\right)+(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t}\left(q_H^*-q_{\tau,L}\right) \mid \theta_H, h^t, \{q_\tau\}_{\tau\geq t}\in\mathcal{Q}^s\right].
\end{aligned}
$$

Similarly, we can write

$$
\begin{aligned}
& E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t+1}\delta^{\tau-t-1}q_\tau \mid \theta_H, h^t\right] \\
= \; & P_{\sigma^*}\left(\mathcal{Q}^t \mid \theta_H, h^t\right) q_H^* \\
& +\sum_{s>t}^{\infty} P_{\sigma^*}\left(\mathcal{Q}^s \mid \theta_H, h^t\right) E_{\sigma}^{*}\left[\left(\frac{v_{H,t+1}}{\Delta\theta}\right)+(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t-1}\left(q_H^*-q_{\tau,L}\right) \mid \theta_H, h^t, \{q_\tau\}_{\tau\geq t}\in\mathcal{Q}^s\right].
\end{aligned}
$$

Since $v_{H,t+1}\geq v_{H,t}$ and $(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t-1}\left(q_H^*-q_{\tau,L}\right)>(1-\delta)\sum_{\tau\geq s}\delta^{\tau-t}\left(q_H^*-q_{\tau,L}\right)$ (because $q_{\tau,L}\in[0,q_L^*]$) we have

$$
E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t+1}\delta^{\tau-t-1}q_\tau \mid \theta_H, h^t\right]>E_{\sigma}^{*}\left[(1-\delta)\sum_{\tau\geq t+1}\delta^{\tau-t-1}q_\tau \mid \theta_H, h^t\right],
$$

which proves the Proposition. ■

Propositions 1 and 2 predict the pattern of qualities and prices. The monopolist repeatedly offers price cuts for high-quality goods, while the price per quality remains constant in the pooling contract. The expected discounted quality provided to the low-type consumer starts below the efficient level for his type and increases over time. A high-type consumer who pools with the low type in a certain period also expects a higher discounted quality in the future. In this sense, the model predicts an expected wedge between the marginal benefit for each type of consumer and the marginal cost for the monopolist. As the parties renegotiate, this wedge gradually decreases and eventually disappears. In Section 5.8, Proposition 3 establishes that there exists an upper bound on the number of periods taken by the monopolist to screen the high type. Hence, the relationship becomes efficient in finite time. But how inefficient is the equilibrium allocation? In order to answer this question we must understand how the monopolist chooses the quality in pooling contracts.

5.7 Pooling Contract Quality

During the screening process, while the current period's belief p_t is interior, the monopolist offers two contracts, one associated with a lower belief $p_{t+1} \in [0, p_t)$ and another associated with the belief one. The monopolist always promises the lowest credible information rent to the high type (for a given belief), since otherwise the monopolist could charge more for revealing contracts. With some abuse of notation, we write $V_H(p)$ for this rent (for a belief p) and we write $V^M(p)$ for $V^M(p, \{\mathbf{0}\})$. Therefore, the monopolist's problem can be written as:

$$\max_{\substack{\tilde{p} \in [0,p] \\ q_L \in [0,1]}} \left\{ \begin{array}{c} \mu(p, \tilde{p}) \left[(1-\delta) \pi_L(q_L) + \delta V^M(\tilde{p}) \right] \\ +(1-\mu(p,\tilde{p})) \left[\pi_H(q_H^*) - (1-\delta) \Delta\theta q_L - \delta V_H(\tilde{p}) \right] \end{array} \right\}. \tag{23}$$

In (23) the monopolist chooses the future belief in the pooling contract $\tilde{p} \in [0, p]$ and the current period's quality $q_L \in [0, 1]$. We used Bayes' rule to calculate the probability that the contract associated with belief $\tilde{p}$ is reached by a menu presenting sequential separating dynamics when the initial prior is p :

$$\mu(p, \tilde{p}) \triangleq \left(\frac{1-p}{1-\tilde{p}} \right).$$

Offering q_L yields $\pi_L(q_L) \triangleq \theta_L q_L - c(q_L)$ if the pooling contract is accepted and leads to the information rent to the high type $\Delta\theta q_L$ when the revealing contract is accepted. Hence, renegotiation implies the following optimality condition with respect to q_L :

$$\begin{array}{ll} \pi_L'(q_L) - \left(\frac{p-\tilde{p}}{1-p}\right)\Delta\theta = 0 & \text{if } q_L > 0 \\ \pi_L'(q_L) - \left(\frac{p-\tilde{p}}{1-p}\right)\Delta\theta \leq 0 & \text{if } q_L = 0. \end{array} \tag{24}$$

From (24) we can immediately see that the quality in the pooling contract should be distorted downwards: $q_L < q_L^*$. This distortion is an increasing function of $\Delta\theta\left(\frac{p-\tilde{p}}{1-p}\right)$, where $p - \tilde{p}$ is the measure of high types screened in the period. To develop further intuition, assume that the monopolist offers a menu that screens all high-type consumers in one period. In this case, the current period's quality should be set at the same level as in the dynamic problem with commitment. However, since the monopolist can renegotiate, in every future period she would offer the efficient quality to the low type. Thus, the rent of the high type would be given by $\Delta\theta q_L^C(1-\delta) + \delta\Delta\theta q_L^*$ (where we write q_L^C for the commitment quality). This information rent is very close to $\Delta\theta q_L^*$ when the discount factor is high, which suggests that the monopolist cannot extract large rents from the high type when the parties are patient. This conjecture is investigated next.

5.8 Renegotiation Implies Efficiency

In this section we prove that when the discount factor is high the high type's information rent is close to $\Delta\theta q_L^*$. Moreover, it is also shown that, as the time periods shrink, the equilibrium allocation converges to the most profitable individually-rational, incentive-compatible, and efficient allocation. These two results are corollaries of Proposition 3 (proof in the appendix), which shows that the number of periods taken by the monopolist to screen the high type has an upper bound that is independent of the discount factor.

Proposition 3 *For every $p \in (0,1)$ there exists $T(p) \in \mathbb{N}$ such that if the initial prior p_0 belongs to $(0,p)$ then, for any $\delta \in (0,1)$, the high type is screened with probability 1 in at most $T(p)$ periods.*

This result shows that the renegotiation forces lead to efficiency in finite time, independent of patience. We obtain a version of the Coase Conjecture for a nondurable-goods monopolist as a Corollary of this result.

Corollary 1: A Version of the Coase Conjecture for a Nondurable Goods Monopoly *The surplus of the game converges to the efficient surplus as the time periods shrink. Formally, for every* $\varepsilon > 0$ *there exists a* $\delta' \in (0,1)$ *such that if* $\delta > \delta'$ *then*

$$\left| E_{\sigma^*(\delta)} \left[\sum_{t=0}^{\infty} (1-\delta)\, \delta^t q_t \mid \theta_i \right] - q_i^* \right| < \varepsilon$$

for $i = L, H$.

To develop an intuition about the driving forces of the model, in Section 5.9 we prove two weaker results that imply Corollary 1. Section 5.10 discusses these results.

5.9 Coasian Forces

In Proposition 4 we prove that, as the time periods shrink, the amount of information rent obtained by the high type converges to $\Delta\theta q_L^*$. For the remainder of this section, we associate an equilibrium $\sigma^*(\delta)$ to each discount factor δ and, for simplicity of notation, we assume that the monopolist plays a pure strategy.[16]

Proposition 4 *As the time periods shrink the information rent obtained by the high type converges to* $\Delta\theta q_L^*$. *Formally, for every* $\varepsilon > 0$ *there exists* $\delta' \in (0,1)$ *such that if* $\delta > \delta'$ *then the high type receives an information rent at least as large as* $\Delta\theta q_L^* - \varepsilon$ *in any equilibrium.*

Proof. Since the high type is always indifferent between revealing his type and imitating the low type, we can write his rent as

$$v_{H,0} = \sum_{t=0}^{\infty} (1-\delta)\, \delta^t \Delta\theta q_t^{\sigma^*(\delta)}, \tag{25}$$

where $\left\{ q_t^{\sigma^*(\delta)} \right\}_{t \geq 0}$ is the continuation consumption of the low type. Write $\left\{ p_t^{\sigma^*(\delta)} \right\}_{t \geq 0}$ for the (decreasing) sequence of beliefs associated with the pooling contracts and, for simplicity of notation, ignore the dependence on $\sigma^*(\delta)$ in the rest of this proof.

[16] Since no assumption is imposed on the selection $\sigma^*(\delta)$ the following asymptotic results hold for all equilibria.

Let $\varepsilon > 0$. Take $\zeta = \frac{\varepsilon}{2\Delta\theta} > 0$. Notice that from (24), a necessary condition for $q_t < q_L^* - \zeta$ is

$$\pi_L' \left(q_L^* - \zeta\right) \leq \left(\frac{\Delta p_{t+1}}{1 - p_t}\right) \Delta\theta$$

where $\Delta p_{t+1} = p_t - p_{t+1}$. Since we have sequential separating dynamics, $\{p_t\}$ is decreasing:

$$\Delta p_{t+1} \geq \left(\frac{\pi' \left(q_L^* - \zeta\right)(1 - p_0)}{\Delta\theta}\right). \tag{26}$$

Writing $\mathbf{1}$ for an indicator function, from (25) and (26) we have the following lower bound on rent of the high type

$$\sum_{t=0}^{\infty} (1-\delta)\,\delta^t \Delta\theta \left[\begin{array}{c} \mathbf{1}_{\left\{\Delta p_{t+1} < \pi'\left(q_L^* - \zeta\right)(1-p_0)\Delta\theta\right\}} \times \left(q_L^* - \zeta\right) \\ +\mathbf{1}_{\left\{\Delta p_{t+1} \geq \pi'\left(q_L^* - \zeta\right)(1-p_0)\Delta\theta\right\}} \times 0 \end{array} \right].$$

Since there exists a maximum number of periods $M(\zeta)$ such that $\Delta p_{t+1} \geq (\Delta\theta)^{-1} \pi_L' \left(q_L^* - \zeta\right)(1 - p_0)$, we obtain the following lower bound on the high type's rent:

$$\delta^{M(\varepsilon)} \Delta\theta \left(q_L^* - \frac{\varepsilon}{2\Delta\theta}\right).$$

Thus there exists δ' such that if $\delta > \delta'$ we have $\delta^{M(\varepsilon)} \Delta\theta \left(q_L^* - \frac{\varepsilon}{2\Delta\theta}\right) > \Delta\theta q_L^* - \varepsilon$ which proves the Proposition. ■

Next, we use Proposition 4 to ask how fast separation occurs in equilibrium. Consider the random variable $\hat{T}(\sigma^*(\delta))$ indicating the last period (possibly infinite) such that the high type pools with the low type. Accordingly, $\delta^{\hat{T}(\sigma^*(\delta))}$ measures the (discounted) time in which both types are separated. In Proposition 5 below we show that when the discount factor is high screening occurs very early with high probability so the probability that $\delta^{\hat{T}(\sigma^*(\delta))}$ is close to 1 is very high.

Proposition 5 *The expected total time in which the high type pools with the low type converges to zero as the time periods shrink. Formally, for every* $\xi \in (0,1)$ *there exists* $\delta' \in (0,1)$ *such that if* $\delta > \delta'$ *then* $P_{\sigma^*}\left[\delta^{\hat{T}(\sigma^*(\delta))} > \xi \mid \theta_H\right] > \xi$.

Proof. Take $\xi \in (0,1)$. Assume that the high type still pools with the low type with probability $\alpha \geq (1-\xi)$ at $\hat{T}(\sigma^*(\delta))$ such that $\delta^{\hat{T}(\sigma^*(\delta))} < \left(\frac{\xi+1}{2}\right)$. We obtain the following upper bound to the surplus obtained from the high type:

$$\xi\left[\theta_H q_H^* - c\left(q_H^*\right) - V_H\left(p_0\right)\right]$$
$$+\left(1-\xi\right)\left[\left(\frac{1-\xi}{2}\right)\left(\theta_H q_L^* - c\left(q_L^*\right)\right) + \left(\frac{1+\xi}{2}\right)\left(\theta_H q_H^* - c\left(q_H^*\right)\right) - V_H\left(p_0\right)\right].$$

Since $\pi_L\left(q_L^*\right)$ is an upper bound to the surplus obtained from the low type, we have the following upper bound on the monopolist's surplus:

$$p_0\left\{\begin{array}{c}\xi\left[\theta_H q_H^* - c\left(q_H^*\right) - V_H\left(p_0\right)\right] \\ +\left(1-\xi\right)\left[\left(\frac{1-\xi}{2}\right)\left(\theta_H q_L^* - c\left(q_L^*\right)\right) + \left(\frac{1+\xi}{2}\right)\left(\theta_H q_H^* - c\left(q_H^*\right)\right) - V_H\left(p_0\right)\right]\end{array}\right\} \tag{27}$$
$$+\left(1-p_0\right)\pi_L\left(q_L^*\right).$$

Now, assume that $V_H\left(p_0\right) \geq \Delta\theta q_L^* - \varepsilon$. Inserting this in (27) we obtain the following upper bound to the monopolist's surplus:

$$\begin{array}{l} p_0\left(\theta_H q_H^* - c\left(q_H^*\right) - \Delta\theta q_L^*\right) + \left(1-p_0\right)\pi\left(q_L^*\right) \\ -p_0\left[\left(\frac{\left(1-\xi\right)^2}{2}\right)\left(\begin{array}{c}\theta_H q_H^* - c\left(q_H^*\right) \\ -\left(\theta_H q_L^* - c\left(q_L^*\right)\right)\end{array}\right) - \varepsilon\right].\end{array} \tag{28}$$

Thus, take δ' such that if $\delta > \delta'$ then $V_H\left(p_0\right) > \Delta\theta q_L^* - \varepsilon'$ with $\varepsilon' = \left(\frac{(1-\xi)^2\left[\left(\theta_H q_H^* - c\left(q_H^*\right)\right) - \left(\theta_H q_L^* - c\left(q_L^*\right)\right)\right]}{4}\right)$. Replacing ε with ε' in (28) we conclude that the monopolist's profit is strictly less than:

$$p_0\left(\theta_H q_H^* - c\left(q_H^*\right) - \Delta\theta q_L^*\right) + \left(1-p_0\right)\pi_L\left(q_L^*\right). \tag{29}$$

But in 5.2 we showed that (29) is a lower bound to the monopolist's profit (see (8)). This proves the result for all $\delta > \delta'$. ■

According to Proposition 5, any equilibrium exhibits almost no delay in separation for high discount factors. From Proposition 4, the information rent of the high type is very close to the rent that he would obtain if the monopolist were to screen all high types in the first period. If there were a real delay in separation, then the high type would pool with the low type and would consume a quality no greater than $\theta_L q_L^*$ for a long horizon. Conditional on the consumer being a high type, the surplus falls short of the efficient one by at least $\int_{q_L^*}^{q_H^*}\left(\theta_H - c'\left(s\right)\right)ds > 0$ in each period that there is pooling. Since the rent of the high type is close to $\Delta\theta q_L^*$, the bulk of this loss in surplus is subtracted from the monopolist's profit. This argument can be used to show that screening the high type in the first period would be a profitable deviation for the monopolist. Below we use Propositions 4 and 5 to prove Corollary 1.

Proof of Corollary 1 A direct inspection of the proof of Proposition 4 shows that the average quality consumed by the low type must be close to the efficient quality for high discount factors. Proposition 5 shows that there is almost no delay in separating to the high type. Since after the high type separates he consumer q_H^* in each period the Claim follows. ■

5.10 Discussion

Section 5.2 derived a lower bound for the monopolist's profit (8). The payments in that lower bound were derived from the following constraints: a) each type consumes the efficient quality in each period; b) the rationality constraint of the low type holds with equality; c) the high type is indifferent between his contract and the low-type's one. From Propositions 4 and 5 and Corollary 1 we conclude that, as the time periods shrink, the monopolist's expected profit, the expected payment, and the expected quality of each type converge to the values attained in that menu. Therefore, the possibility of renegotiation severely decreases the monopolist's profits.

We have shown that the allocation is approximately efficient when the discount factor is high. This shows a fundamental difference between our model and the two-period model of Laffont and Tirole (1990). Assume that the weight of the first period is β in their model, while the weight of the second is $(1-\beta)$. When β is close to 1 the model is essentially static and, hence, the monopolist obtains a profit close to the commitment one. Similarly, the same conclusion is obtained when β is close to zero. Although less drastic, the monopolist always has a significant degree of commitment for intermediate values of β since at least one of the two periods has a weight of at least $\left(\frac{1}{2}\right)$. This enables the monopolist to limit the information rents of the high type and precludes efficiency in two-period models. The degree of monopolist's commitment is $(1-\delta)$ in our infinite horizon model. As δ increases to 1 this measure converges to zero and efficiency is approximately restored.

It is interesting to use the findings above to compare the behavior in our model with the model of Hart and Tirole (1988), in which the monopolist sells qualities in $\{0,1\}$. Under this unitary supply assumption, the authors show that the equilibrium outcome is the same whether trade-opportunities are durable (selling) or subject to renegotiation (rental).

Intuitively, when $q_t \in \{0,1\}$ the monopolist has only one device to screen the high type: the time. Hence, the well understood Coasian forces drive down the monopolist's profit when the time periods shrink. When $q_t \in [0,1]$, as in our model, the monopolist has two instruments to screen the high type: time and quality. Wang (1998) showed that when trade opportunities are durable (as in Hart and Tirole) these two instruments lead to the commitment profit. As shown here, the possibility of renegotiation drastically limits this rent extraction.

The result that as the time periods shrink the equilibrium allocation becomes (approximately) efficient can be interpreted as a version of the Coase Conjecture for a nondurable goods monopoly. One difference between our result and the original version of the Coase conjecture is that, in our model, the monopolist still makes more profits from consumers with higher valuation. Therefore, the lack of commitment reduces the monopolist's market power only to the extent necessary to bring back efficiency.

5.11 Renegotiation and Initial Prior

Next, we consider a given discount factor and ask how the non-commitment costs that are due to renegotiation depend on the initial prior. In Proposition 6 (proof in the appendix) we show that this cost is not monotonic on the initial prior. In fact, for very high and very low priors the cost is negligible, being more significant for intermediate types. Write $\Pi(p_0)$ for the commitment profit and $V^M(p_0)$ for the profit in the equilibrium with renegotiation.

Proposition 6 *For every $\varepsilon > 0$ there exists $\eta > 0$ such that if $p_0 \in [0,\eta] \cup [1-\eta,1]$ then $\left|\Pi^*(p_0) - V^M(p_0)\right| < \varepsilon$.*

Renegotiation is thus more costly if the prior lies in an intermediate range. For small priors, renegotiating does not involve large costs since the likelihood of finding a high type is very low. For large priors, the monopolists can credibly offer a small quality pattern to the low type. This enables her to extract large rents from the high type and screen him with high probability in the first period. Corollary 2 follows from the proof of Proposition 6.

Corollary 2 *For every $T \in \mathbb{N}$ there exists $p_0 \in (0,1)$ such that if the initial prior is* *larger than p_0 then it takes at least T periods for the monopolist to screen the high type completely.*

According to Corollary 2, as the prior increases, the number of periods the monopolist takes to screen the high type increases without bound. When it takes many periods to screen the consumer, we observe a sharp difference between the price paid for the revealing contract in the beginning of the relationship with the one paid after many periods of pooling with the low type. The price of the revealing contract starts very high, close to $\theta_H q_H^*$, and gradually decreases to a value belonging to $[\theta_H q_H^* - \Delta\theta q_L^*, \theta_H q_H^* - \delta\Delta\theta q_L^*]$ in the last period of the screening process. Therefore, the model predicts large discounts.

6 Concluding Remarks

We study a dynamic principal-agent model with adverse selection and the possibility of renegotiation. We consider a dynamic-monopoly-pricing model as our main interpretation. We characterize the dynamics of the screening process and the evolution of quality and prices. We show that, as the time periods shrink, the outcome of the game converges to the most profitable among all individually-rational, incentive-compatible, and efficient allocations.

We consider simple payoff functions. This is done for simplicity and tractability. Our main results, Propositions 1,4 and 5 (and Corollary 1) extend naturally to more general quasilinear utility functions satisfying a standard single-crossing condition.

The solution of the model for an arbitrarily long finite horizon again yields outcomes presenting sequential separating dynamics. Analogues of Propositions 4 and 5 also hold. That is, if the discount factor δ is close to one and the horizon is sufficiently long then the information rent of the high type is close to $\Delta\theta q_L^*$, and the outcome is approximately efficient.

An extension is the case in which the consumer has arbitrarily many (or a continuum of) types. How much rent can the monopolist extract from each type? How efficient are the equilibrium outcomes? These interesting questions are left for future research.

7 Appendix A

For expositional convenience, we divide the analysis into 2 parts. Appendix A we prove Proposition 1.1 for the case in which Condition 1 below holds, that is, the graph $(p, \Phi(p, \{q_\tau\}))$ is closed. This rules out cases in which set of feasible tuples for each $(p, \{q_\tau\})$ $\left(\left(m, a_L, a_H, \left(v_L^j, v_H^j\right)_{j=1}^2\right)\right.$ satisfying (3)-(7)) is not upper-semicontinuous. In Appendix B, we complete the proof of Proposition 1 for the general case.

Condition 1: *The graph $(p, \Phi(p, \{q_\tau\}))$ is closed.*

Preliminaries

Given the state variables, we call the problem of maximizing (2) subject to (3),(4),(5),(6) and (7) the monopolist's problem. A tuple $\varphi = \left(m, a_L, a_H, \left(v_L^j, v_H^j\right)_{j=1}^2\right)$ is called a solution to the monopolist's problem if it solves (2) subject to (3),(4),(5),(6) and (7).

We say that a tuple φ is feasible when the state is $(p, \{q_\tau\})$ if is it satisfies (3),(4),(5),(6) and (7).

We write $\hat{\Phi}(p, \{q_\tau\})$ for the set of continuation rents net of the outside option of the low type $\sum (1-\delta)\,\delta^\tau \theta_L q_\tau$. Formally:

$$\hat{\Phi}(p, \{q_\tau\}) \triangleq \Phi(p, \{q_\tau\}) - \left\{\sum (1-\delta)\,\delta^\tau \theta_L q_\tau\right\}.$$

We write $\Phi_i(p, \{q_\tau\})$ for the set of continuation values for consumer i when the state is $(p, \{q_\tau\})$.

For $(v_L, v_H) \in \Phi(p, \{q_\tau\})$ the difference $v_H - v_L$ is the information rent of the high type.

7.1 Proposition 1

7.1.1 Lemma 1

Lemma 1 *Assume that the state is $(p, \{\mathbf{0}\})$ for some $p \in (0,1)$. Then any solution to the monopolist's problem and any solution to (2) subject to (3),(5),(6) and (7) achieve the same value, that is, the rationality constraint of the high type can be ignored. Furthermore, in any solution to the monopolist's problem the constraint (5) binds.*

Proof. The first result follows because the rationality of the high type is implied by the rationality of the low type when $\{q_\tau\} = \{\mathbf{0}\}$. For the second claim, assume towards a contradiction that φ solves the monopolist's problem but

$$\max_k \left[(1-\delta)\left[\theta_L q^k - x^k\right] + \delta v_L^k\right] = \varepsilon > 0.$$

Then consider φ' differing from φ only in that payments in each contract are increased by $\left(\frac{\varepsilon}{1-\delta}\right)$. This new tuple is feasible and provides a higher profit for the monopolist, a contradiction. ■

7.1.2 Lemma 2

For $A, B \subset \Re^n$ we write $A \leq B$ if $b \in B$ implies that there exists $a \in A$ such that $a \leq b$.

Lemma 2 *For any $p \in (0,1)$ and for all $\{q_\tau\} \in [0,1]^\infty$ we have $\hat{\Phi}(p, \{q_\tau\}) \geq \hat{\Phi}(p, \{\mathbf{0}\})$.*

Proof. Consider $(v_L, v_H) \in \hat{\Phi}(p, \{\mathbf{0}\})$ and $(v'_L, v'_H) \in \hat{\Phi}(p, \{q_\tau\})$. Since $\hat{\Phi}(p, \{\mathbf{0}\}) = \Phi(p, \{\mathbf{0}\})$ from Lemma 1 $v_L = 0$. From (5) we must have $v'_L \geq 0$.

Thus, it is enough to show that for each $(v'_L, v'_H) \in \hat{\Phi}(p, \{q_\tau\})$ there exists $(v''_L, v''_H) \in \hat{\Phi}(p, \{\mathbf{0}\})$ such that $v''_H \leq v'_H$. This result is established by contradiction. If not, there is a tuple φ' solving the monopolist's problem when the state is $(p, \{q_\tau\})$ yielding $(v'_L, v'_H) \in \hat{\Phi}(p, \{q_\tau\})$ with $v''_H > v'_H$ for all $(v''_L, v''_H) \in \hat{\Phi}(p, \{\mathbf{0}\})$. But then there exists a tuple φ'' solving the monopolist's problem when the state is $(p, \{\mathbf{0}\})$ yielding a strictly higher information rent $(v''_H - v''_L)$ to the high type which dominates any feasible tuple which gives an information rent $(v'_H - v'_L)$ to the high type when the state is $(p, \{\mathbf{0}\})$. Therefore offering a tuple $\tilde{\varphi}''$ differing from φ'' in that the transfers are discounted by $\sum \delta^\tau \theta_L q_\tau$ is feasible when the state is $(p, \{q_\tau\})$ and it is a profitable deviation for the monopolist. ■

7.1.3 Lemma 3

Lemma 3 *Let φ be a solution to the monopolist's problem when the state is $(p, \{q_\tau\})$ with $\sum \delta^\tau q_\tau \leq q_H^*$. Assume that the offered menu in φ is $m = (\psi^1, \psi^2)$ and the contract ψ^1 is associated with a belief $p^1 \in (0,1)$ and to continuation rents $(v_{L,1}, v_{H,1}) \in \Phi(p, \{q_\tau^1\})$. Then $v_{H,1} - v_{L,1} < \sum \delta^\tau \Delta\theta q_L^*$.*

Proof. First, if $\sum \delta^\tau q_\tau \in [q_L^*, q_H^*]$ then the monopolist can offer a menu with contracts $(\tilde{\psi}^1, \tilde{\psi}^2)$, $\tilde{\psi}^1 = \left(\{\mathbf{q}_L^*\}, \left(\frac{\theta_L q_L^* - (1-\delta)\sum \delta^\tau \theta_L q_\tau}{1-\delta}\right)\right)$ and $\tilde{\psi}^2 = \left(\{\mathbf{q}_H^*\}, \left(\frac{\theta_H q_H^* - (1-\delta)\sum \delta^\tau \theta_H q_\tau}{1-\delta}\right)\right)$. Contract $\tilde{\psi}^1$ ($\tilde{\psi}^2$) offers the efficient quality to the low (high) type consumer in each future period and its payments are set such that the rationality constraint of the consumer holds with equality. Furthermore, it is easy to see that the strategy in which the low (high) type consumer chooses contract $\tilde{\psi}^1$ ($\tilde{\psi}^2$) is incentive-compatible. Hence, this menu is optimal for the monopolist. Furthermore, it is easy to see that any other optimal menu induces the same outcome.

Next, consider the case in which $\sum \delta^\tau q_\tau \in [0, q_L^*)$. Write $v_{H,0}(\psi^1) - v_{L,0}(\psi^1)$ for the information rent of the high type. Clearly if $v_{H,0}(\psi^1) - v_{L,0}(\psi^1) \geq \Delta\theta q_L^*$ then the monopolist could profitably deviate by offering the cheapest individually-rational, incentive-compatible, and efficient menu. Hence, assume $v_{H,0}(\psi^1) - v_{L,0}(\psi^1) < \Delta\theta q_L^*$. In this case, the monopolist can replace contract ψ^1 with two contracts: a contract associated with belief 0 and another associated with belief 1. The contract associated with belief 0 offers promises $\{\tilde{q}, q_L^*, q_L^*, ...\}$ and sets transfers to satisfy the rationality of the low type with equality, where $\tilde{q}$ is set to keep the information rent of the high type at $v_{H,0}(\psi^1) - v_{L,0}(\psi^1)$:

$$(1-\delta)\Delta\theta\tilde{q} + \delta\Delta\theta q_L^* = v_{H,0}(\psi^1) - v_{L,0}(\psi^1).$$

The revealing contract offers promises $\{\mathbf{q}_H^*\}$ and set transfers to make the high type indifferent between both contracts. By construction, the rent of each type of consumer is the same under the deviating menu and under the original contract, but the surplus is greater under the deviating menu, which establishes a contradiction and proves the Lemma. ■

7.1.4 Lemma 4

Here we prove certain properties of the set of continuation rents.

Lemma 4 *The set of continuation rents Φ satisfies the following properties:*

i) For each state $(p, \{q_\tau\})$, the set $\Phi(p, \{q_\tau\})$ is convex;

ii) For any sequence of beliefs $\{p_n\}_{n=1}^{\infty}$ with $p_n > 0$ and $p_n \to 0$ we have $\Delta\theta q_L^* \in \lim \Phi_H(p_n, \{\mathbf{0}\})$;

iii) The set $\mathcal{V}(v'_H) \triangleq \{p : \min \Phi_H(p, \{\mathbf{0}\}) \leq v'_H\}$ is closed. Furthermore if $v'_H < \Delta\theta q_L^*$ then $v'_H \in \Phi(\inf \mathcal{V}(v'_H), \{\mathbf{0}\})$.

Proof. i) Follows from II in Definition 2.

ii) A standard argument shows that the average quality of the low type converges to q_L^* as $p \to 0$. Thus for any sequence $(p_n) \downarrow 0$ if $v_H^n \in \Phi_H(p_n, \{\mathbf{0}\})$ we must have $\limsup v_H^n \geq \Delta\theta q_L^*$. If the inequality is strict one can find n such that the monopolist has a profitable deviation by offering the cheapest efficient and incentive-compatible separating menu.

iii) Follows from ii) and the assumption that the graph of $(p, \Phi(p, \{\mathbf{0}\}))$ is closed imposed by Condition 1. ■

7.1.5 Lemma 5

Lemma 5 *Consider a state $(p_0, \{q_\tau^0\})$ and the event that the contract ψ is accepted with positive probability. ψ involves the current period's quality q, the next period's state $(p, \{q_\tau\}_{\tau>0})$ and transfers yielding a rent α to the low type. Consider the event that the alternative contract ψ' is accepted. ψ' involves the current period's quality q, is associated with the state $(p, \{\mathbf{0}\})$ and also yields the rent α to the low type. The acceptance of the contract ψ' yields a weakly higher profit to the monopolist. Furthermore, if there is no solution to the monopolist's problem when the state is $(p, \{\mathbf{0}\})$ leading to an equivalent distribution over next-period outcomes as some solution to the monopolist's problem when the state is $(p, \{q_\tau\}_{\tau>0})$, then the acceptance of ψ' yields a strictly higher profit than the acceptance of ψ.*

Proof. With some abuse of notation, write h^1 for the event that contract ψ is accepted. Write G for the distributions over menus m' resulting from the monopolist randomization at h^1 in the subsequent period. The profit of the monopolist is given by:

$$\int \bar{V}^M \left(h^1, m'\right) dG(m').$$

Let $\bar{V}_L\left(h^1\right)$ be the realized expected rent of the low type. We have $\bar{V}_L\left(h^1\right) = \int \bar{V}_L\left(h^1, m'\right) dG(m')$. Thus $0 = \int \left[\bar{V}_L\left(h^1, m'\right) - \bar{V}_L\left(h^1\right)\right] dG(m')$, which implies:

$$\begin{aligned} & \int \bar{V}^M \left(h^1, m'\right) dG(m') \\ = & \int \left[\bar{V}^M \left(h^1, m'\right) + \left[\bar{V}_L\left(h^1, m'\right) - \bar{V}_L\left(h^1\right)\right]\right] dG(m'). \end{aligned} \tag{30}$$

That is, the expected profit of the monopolist is the same if a random payment which ensures that the rent of the low type is $\bar{V}_L\left(h^1\right)$ is transferred from the monopolists to the consumer after every menu.

Next, consider the contract ψ'. Write $\tilde{h}^1$ for the event associated with the contract ψ' being accepted in the previous period. From Lemma 1, the rationality of the high type is implied by the rationality of the low type one if $\{q_\tau\}_{\tau>0} = \{\mathbf{0}\}$. Thus consider the event that menu m' is offered at h^1. For that menu, there exists another menu m'' differing from m' in that the transfers are set to satisfy the rationality of the low type with equality at $\tilde{h}^1$. Write $\tilde{G}$ for the correspondent distribution and notice that the profit of the monopolist under this strategy satisfy:

$$\begin{aligned} & \int \bar{V}^M \left(\tilde{h}^1, m''\right) d\tilde{G}(m'') \\ = & \int \left[\bar{V}^M \left(h^1, m'\right) + \left[\bar{V}_L\left(h^1, m'\right) - \bar{V}_L\left(h^1\right)\right]\right] dG(m'), \end{aligned}$$

which shows that the monopolist is weakly better-off. The second statement follows from a standard revealed preference argument and *II)* in Definition 1. ■

7.1.6 Lemma 6

Lemma 6 *Assume that the history h^t is associated with state $\left(p_t, \{q_\tau\}_{\tau\geq t}\right)$, with $\sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}q_\tau < q_L^*$ and $p_t \in (0,1)$. Consider a solution to the monopolist's problem at h^t in which she offers a contract $\psi_t = \left(x_t^1, \{q_\tau^1\}_{\tau\geq t+1}\right)$ associated with a belief p_{t+1}^1 and to a promise $v_{H,t+1}^1 \in \hat{\Phi}_H\left(p_{t+1}^1, \{q_\tau^1\}_{\tau\geq t+1}\right)$. Then, for all $p < p_{t+1}^1$ and $v_H \in \Phi_H(p, \{\mathbf{0}\})$, we have $v_{H,t+1}^1 < v_H$.*

Proof. Assume towards a contradiction that there exists $p < p_{t+1}^1$ and $v_H \in \Phi_H(p, \{\mathbf{0}\})$ with $v_{H,t+1}^1 \geq v_H$. We will show that there exists 2 other contracts $\left(\tilde{\psi}_t^1, \tilde{\psi}_t^2\right)$ and a randomization of each type of consumer such that the contract $j \in \{1,2\}$ is accepted with probability $\tilde{\mu}_t^j$ and is associated with the belief $\tilde{p}_{t+1}^j$, the feasibility constraints ((3),(4),(5),(6) and (7)) are satisfied, each type of consumer receives the same rent and the monopolist receives a strictly higher profit. This contradicts the assumption that the monopolist's problem is being solved.

Step 0. *Profit under the putative equilibrium play*

We write h_1^{t+1} for the history associated with the acceptance of the contract ψ_t^1 under the putative equilibrium play. Let q_t^1 be the current period's quality in that contract and without loss assume[17] $q_t^1 \leq q_L^*$.

We write $v_{L,t+1}^1$ ($v_{H,t+1}^1$) for the continuation rent of the low (high) type consumer at h_1^{t+1}.

Step 1 *Deviating strategy.*

From Lemma 3 we have $v_{H,t+1}^1 - v_{L,t+1}^1 < \Delta\theta q_L^*$. From ii) and iii) in Lemma 4 there exists $\tilde{p}_{t+1}^1 \in \left(0, p_{t+1}^1\right)$ such that $v_{H,t+1}^1 \in \Phi_H\left(\tilde{p}_{t+1}^1, \{\mathbf{0}\}\right)$.

The deviation will contain a pooling and a revealing contract.

Pooling contract: $\left(\frac{-v_{L,t+1}^1+(1-\delta)q_t^1}{1-\delta}, \left\{q_t^1, \mathbf{0}\right\}\right)$.

Revealing contract: $\left(\frac{-v_{H,t+1}^1+q_H^*}{1-\delta}, \{\mathbf{q}_H^*\}\right)$.

The pooling (revealing) contract is accepted with probability $\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)$ (resp. $\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right)$) and leads to the history $\tilde{h}_1^{t+1}$ (resp. $\tilde{h}_1^{t+1*}$). We must show:

$$\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right) \bar{V}^M\left(\tilde{h}_1^{t+1}\right) + \left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right) \bar{V}^M\left(\tilde{h}_1^{t+1*}\right) > \bar{V}^M\left(h_1^{t+1}\right). \tag{31}$$

Step 2 *Writing expected realized rents as martingales*

Consider the history h_1^{t+1}. Notice that we make no pure strategy assumption, thus we will have to consider the possibility that the monopolist randomizes over menus and the expected realized rent of each type of consumer varies according to this randomization. A fact which will be key to our proof is that expected realized rents are martingales.

Following the acceptance of h_1^{t+1}, the monopolist is indifferent among all menus which she chooses.[18] Also, the expected rent of the consumer before this randomization is an element of $\Re^2$. From Lemma 4, the set Φ is convex and from Condition 1 Φ is closed, hence we can apply Caratheodory's Theorem to write any vector of expected rents as a convex combination of 3 of its extreme points. Thus assume (w.l.o.g.) that the monopolist randomizes among

[17] From the argument which will be presented in this Lemma it will be clear how to construct a profitable deviation for the case in which $q_1 > q_L^*$. In this case the monopolist could replace that contract with a pair of contracts in which the pooling contract offers a current period quality q_L^*. This would be less attractive for the high type consumer. In order to maintain his information rent the monopolist can associate the pooling contract with a lower belief. This deviation increases her profit even more.

[18] Formally all the menus yield the same expected profit ($a.s.$). We omit this observation in the remaining of this paper.

$\{m^1_{t+1,1}, m^1_{t+1,2}, m^1_{t+1,3}\}$ with probabilities $\{\gamma^1_{t+1,1}, \gamma^1_{t+1,2}, \gamma^1_{t+1,3}\}$. We write $\bar{V}^M\left(h_1^{t+1}, m^1_{t+1,j}\right)$ (resp. $\bar{V}_L\left(h_1^{t+1}, m^1_{t+1,j}\right), \bar{V}_H\left(h_1^{t+1}, m^1_{t+1,j}\right)$) for the monopolist (resp. low-type and high type) expected realized profit (resp. rent) conditional on the menu $m^1_{t+1,j}$ being offered. Let[19] $\left(\xi^1_{L,t+1,1}, \xi^1_{L,t+1,2}, \xi^1_{L,t+1,3}\right)$ (respectively $\left(\xi_{H,t+1,1}, \xi_{H,t+1,2}, \xi_{H,t+1,3}\right)$) represent the innovation on the expected realized rent of the low type (respectively high type) when the menu $m^1_{t+1,i}$ is proposed. Formally, for $i \in \{L, H\}$ and $j \in \{1,2,3\}$ we define

$$\xi^1_{i,t+1,j} \triangleq \bar{V}_i\left(h_1^{t+1}, m^1_{t+1,j}\right) - \bar{V}_i\left(h_1^{t+1}\right).$$

Since expected expected realized rents are martingales:

$$0 = \sum_{j=1}^{3} \gamma^1_{t+1,j} \xi^1_{i,t+1,j}. \tag{32}$$

Step 3 *Summing martingales to realized payoffs without changing their (ex-ante) expected value.*

Using (32), the (LHS) of (31) is equal to:

$$\sum_{j=1}^{3} \gamma^1_{t+1,j} \left\{ \mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right) \bar{V}^M\left(\tilde{h}_1^{t+1}\right) + \left(1 - \mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\right) \left[\bar{V}^M\left(\tilde{h}_1^{t+1*}\right) + \xi^1_{L,t+1,j} - \xi^1_{H,t+1,j}\right] \right\}. \tag{33}$$

While the (RHS) of (31) is equal to:

$$\sum_{j=1}^{3} \gamma^1_{t+1,j} \left[\bar{V}^M\left(h_1^{t+1}, m^1_{t+1,j}\right) + \xi^1_{L,t+1,j}\right]. \tag{34}$$

The innovation $\xi^1_{L,t+1,j}$ is a transfer from the consumer to the monopolist which is added upon the introduction of contract $m^1_{t+1,j}$. Its goal is to keep the rent of the low type constant along the play. Although this may change the feasibility of the allocation, for example, it may violate the indifference of the monopolist among the three menus, it does not change the monopolist *ex-ante* expected profit. This suffices to show the existence of a profitable deviation. Claim 1 below is straightforward:

Claim 1 *If $\xi^1_{L,t+1,j}$ is transferred from the consumer to the monopolist when the menu $m^1_{t+1,j}$ is offered, then the expected realized rent of the low type remains constant:*

$$\bar{V}_L\left(h_1^{t+1}, m^1_{t+1,j}\right) - \xi^1_{L,t+1,j} = \bar{V}_L\left(h_1^{t+1}\right). \tag{35}$$

The goal of transferring $\xi^1_{L,t+1,j} - \xi^1_{H,t+1,j}$ from the consumer to the monopolist in the deviating play associated with histories in which the monopolist holds the belief 1 is to keep the high

[19] It is easy to show that in the continuation play the high type consumer is always indifferent between the two contracts offered by the monopolist.

type's rent the same under the putative equilibrium play and under a revealing history in the deviating play (after transferring $\xi^1_{L,t+1,j}$ to the monopolist). Notice that, by construction, we have $\bar{V}_H\left(\tilde{h}_1^{t+1*}\right)+\xi^1_{H,t+1,j}-\xi^1_{L,t+1,j}=\bar{V}_H\left(h_1^{t+1},m^1_{t+1,j}\right)-\xi^1_{L,t+1,j}$. Hence,

Claim 2 *Assume that upon the introduction of $m^1_{t+1,j}$ the high type transfers $\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}$ to the monopolist at $\tilde{h}_1^{t+1*}$ and he transfers $\xi^1_{L,t+1,j}$ to the monopolist at at h_1^{t+1}. Then the high type's rent is equal in these two scenarios.*

Step 4 *Dividing the putative equilibrium play into 4 possible cases*

The continuation play under the original strategy profile after the offer of $m^1_{t+1,j}$ can be divided into 4 different cases. Case 1 (2,3,4) will be analyzed in Step 4.1 (4.2, 4.3 and 4.4).

Step 4.1 *Play leading to a contract associated with a belief in $\left(\tilde{p}^1_{t+1},p^1_{t+1}\right)$ and to another associated with the belief 1.*

Suppose that menu $m^1_{t+1,j}$ consists of 2 contracts, one associated with the belief $p^1_{t+2,j,1}\in\left(\tilde{p}^1_{t+1},p^1_{t+1}\right)$ (history $h^{t+2}_{j,1}$) and another associated with the belief 1 (history $h^{t+2*}_{j,1}$). Thus we have:

$$\begin{aligned}
&\bar{V}^M\left(h_1^{t+1},m^1_{t+1,j}\right)+\xi^1_{L,t+1,j}\\
=\;&\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\left[\bar{V}^M\left(h^{t+2}_{j,1}\right)+\xi^1_{L,t+1,j}\right]\\
&+\left(1-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\right)\left[\bar{V}^M\left(h^{t+2*}_{j,1}\right)+\xi^1_{L,t+1,j}\right]. \qquad (36)
\end{aligned}$$

Now, consider the play under the deviating strategy. Conditional on the history $\tilde{h}_1^{t+1}$, the monopolist offers only one contract $\left(\{q^*_L,\mathbf{0}\},\theta_L q^*_L\right)$ and have both types of consumers choosing that contract with probability 1 (history $\tilde{h}^{t+2}_{j,1}$). Conditional on the history $\tilde{h}_1^{t+1*}$, the monopolist does not propose any renegotiation (history $\tilde{h}_1^{t+1*}$). This yields the following lower bound to the monopolist's profit:

$$\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right)+\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]. \qquad (37)$$

We will proceed to show that (37) is strictly greater than (36).

Notice that since $p^1_{t+2,j,1}\in\left(\tilde{p}^1_{t+1},p^1_{t+1}\right)$ there exists $\lambda\in(0,1)$ such that:

$$\left(1-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\right)=(1-\lambda)\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right). \qquad (38)$$

Subtracting (36) from (37) we have:

$$\left\{\begin{array}{c}
\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right)+\lambda\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]\\
-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\left[\bar{V}^M\left(h^{t+2}_{j,1}\right)+\xi^1_{L,t+1,j}\right]\\
+\left(1-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\right)\left[\begin{array}{c}\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\\ -\left(\bar{V}^M\left(h^{t+2*}_{j,1}\right)+\xi^1_{L,t+1,j}\right)\end{array}\right]
\end{array}\right\}. \qquad (39)$$

Consider the last line of (39), i.e., the term multiplied by $\left(1-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\right)$. From Claim 2 the rent of the high type is the same under both strings (the one in the deviating play and the one in the putative equilibrium play) which yield this term. However, under the deviating play the high type consumes the efficient quality q^*_H from period t on. Thus, this term is strictly positive. We have proved:

Fact 1: *The last term under square brackets in (39) is strictly positive.*

For future reference, denote $\zeta_{t+1,j}(1)$ the term inside the brackets in the last line of (39) and let $\mu\left(\zeta_{t+1,j}(1)\right)=\gamma^1_{t+1,j}\left(1-\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\right)$ represent the probability that this term is reached (where (i) identifies Step 4.i).

Therefore, we are left with showing that

$$\begin{aligned}\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right)+\lambda\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]\\ \geq\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)\left[\bar{V}^M\left(h^{t+2}_{j,1}\right)+\xi^1_{L,t+1,j}\right]\end{aligned}. \tag{40}$$

Notice that, by construction we are comparing 2 elements with the same measure:

$$\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)+\lambda\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)=\mu\left(p^1_{t+1},p^1_{t+2,j,1}\right)$$

and with the same measure of high types:

$$\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\times\tilde{p}^1_{t+1}+\lambda\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\times 1=\mu\left(p^1_{t+2},p^1_{t+1,j,1}\right)\times p^1_{t+2}.$$

In the next period, the monopolist randomizes over 3 menus and we can add innovations and proceed similarly. Next we move to Step 4.2.

Step 4.2 *Play leading to 2 contracts, both associated with beliefs in* $\left(\tilde{p}^1_{t+1},1\right)$.

In this case, we assume that the menu $m^1_{t+1,j}$ is associated with 2 contracts, one leading to the belief $p^1_{t+2,j,1}\in\left(\tilde{p}^1_{t+1},1\right)$ and another leading to the belief $p^1_{t+2,j,2}\in\left(\tilde{p}^1_{t+1},1\right)$. The contracts are accepted with probabilities $\mu^1_{t+1,j,1}$ and $\mu^1_{t+1,j,1}$ respectively. Write $h^{t+2}_{j,k}$ for the history associated with a contract $k\in\{1,2\}$. The monopolist's profit is:

$$\mu^1_{t+1,j,1}\left[\bar{V}^M\left(h^{t+2}_{j,1}\right)+\xi^1_{L,t+1,j}\right]+\mu^1_{t+1,j,2}\left[\bar{V}^M\left(h^{t+2}_{j,2}\right)+\xi^1_{L,t+1,j}\right]. \tag{41}$$

As in Step 4.1, consider the play under the deviating strategy. Conditional on the history $\tilde{h}^{t+1}_1$, the monopolist offers only one contract $(\theta_L q^*_L,\{q^*_L,\mathbf{0}\})$, which is chosen by both types of consumer with probability 1. Conditional on the history $\tilde{h}^{t+1}_1$, the monopolist does not propose any renegotiation. This yields the same continuation as in (37).

Subtracting (41) from (37):

$$\begin{aligned}&\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right)+\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]\\ &-\mu^1_{t+1,j,1}\left[\bar{V}^M\left(h^{t+2*}_{j,1}\right)+\xi^1_{L,t+1,j}\right]-\mu^1_{t+1,j,2}\left[\bar{V}^M\left(h^{t+2*}_{j,2}\right)+\xi^1_{L,t+1,j}\right].\end{aligned} \tag{42}$$

There exists $(\vartheta, \nu) \in (0,1)^2$ such that (42) is equal to:

$$\left[\begin{array}{c} \vartheta \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right) \bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right) \\ +\nu\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right)\left[\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi_{L,t+1,j}^1-\xi_{H,t+1,j}^1\right] \\ -\mu_{t+1,j,1}^1\left[\bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t+1,j}^1\right] \end{array}\right] + \left[\begin{array}{c} (1-\vartheta) \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right) \bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right) \\ +(1-\nu)\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right)\left[\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi_{L,t+1,j}^1-\xi_{H,t+1,j}^1\right] \\ -\mu_{t+1,j,2}^1\left[\bar{V}^M\left(h_{j,2}^{t+2*}\right)+\xi_{L,t+1,j}^1\right] \end{array}\right],$$

where measures are preserved:

$$\begin{aligned} \vartheta \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)+\nu\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right) &= \mu_{t+1,j,1}^1 \\ (1-\vartheta) \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)+(1-\nu)\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right) &= \mu_{t+1,j,2}^1 \end{aligned}$$

and so are high type's measures:

$$\begin{aligned} \vartheta \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right) \times \tilde{p}_{t+1}^1+\nu\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right) \times 1 &= \mu_{t+1,j,1}^1 \times p_{t+2,j,1}^1 \\ (1-\vartheta) \mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right) \times \tilde{p}_{t+1}^1+(1-\nu)\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right) \times 1 &= \mu_{t+1,j,2}^1 \times p_{t+2,j,2}^1 \end{aligned}$$

Thus, we can divide the putative equilibrium continuation play and the deviating continuation play into 2 separate continuation plays and proceed analogously in the next period.

Step 4.3 *Play leading to 2 contracts, one associated with the belief $p_{t+2,j,1}^1$ in $[0, \tilde{p}_{t+1}^1]$ and another associated with the belief 1.*

Under the putative equilibrium play the monopolist offers a pooling contract and a revealing contract. The pooling contract $\psi_{t+1,j,1}^1 = \left(x_{t+1,j,1}^1, \left\{q_{t+1,j,1}^1\right\}_{\tau \geq t+1}\right)$ leads to the belief $p_{t+2,j,1}^1 \in [0, \tilde{p}_{t+1}^1]$, is accepted with probability $\mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right)$ and is associated with the history $h_{j,1}^{t+2}$. The revealing contract leads to the belief 1, is accepted with probability $\left(1-\mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right)\right)$ and is associated with the history $h_{j,1}^{t+2*}$. Hence, the monopolist's profit is

$$\mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right)\left[\bar{V}^M\left(h_{j,1}^{t+2}\right)+\xi_{L,t+1,j}^1\right]+\left(1-\mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right)\right)\left[\bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t,j}\right]. \quad (43)$$

Now, consider the play under the deviating strategy. Conditional on the history $\tilde{h}_1^{t+1}$, the monopolist offers a menu with a pooling contract and a revealing contract. The pooling contract $\tilde{\psi}_{t+1,j,1}^1 = \left(\tilde{x}_{t+1,j,1}^1, \left\{q_{t+1,j,1}^1\right\}_{\tau \geq t+1}\right)$ leads to the state $\left(p_{t+2,j,1}^1, \left\{q_{t+1,j,1}^1\right\}_{\tau \geq t+2}\right)$ and its transfers $\tilde{x}_{t+1,j,1}^1$ are set to satisfy the rationality of the low type consumer with equality. It is associated with the history $\tilde{h}_{j,1}^{t+2}$. The revealing contract promises the lowest rent to the high type subject to incentive compatibility. It leads to the history $\tilde{h}_{j,1}^{t+2*}$. Conditional on the history $\tilde{h}_1^{t+1*}$, the monopolist does not propose any renegotiation (history $\tilde{h}_1^{t+1*}$). Using the fact that if $0 \leq p' < p'' \leq p'''$ then $\mu(p''', p'')\mu(p'', p') = \mu(p''', p')$, we have the following lower bound to the monopolist's profit:

$$\left\{ \begin{array}{c} \mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right) \bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right) \\ +\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\left(1-\mu\left(\tilde{p}_{t+1}^1, p_{t+2,j,1}^1\right)\right) \bar{V}^M\left(\tilde{h}_{j,1}^{t+2*}\right) \\ +\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right)\left(\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi_{L,t+1,j}^1-\xi_{H,t+1,j}^1\right) \end{array} \right\}. \tag{44}$$

Next, consider the strings leading to the value $\bar{V}^M\left(h_{j,1}^{t+2}\right)+\xi_{L,t+1,j}^1$ and to the value $\bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right)$. From construction, both strings lead to the same quality pattern and from Claim 2 the rent of the low type is equal in both strings. Hence, trivially we have:

$$\bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right) \geq \left[\bar{V}^M\left(h_{j,1}^{t+2}\right)+\xi_{L,t+1,j}^1\right]. \tag{45}$$

Using the observation above and the fact that the high type is indifferent between the revealing and the pooling contract[20] at h_1^{t+1} and at $\tilde{h}_1^{t+1}$:

$$\begin{aligned} \bar{V}_H\left(\tilde{h}_{j,1}^{t+2*}\right) &= \bar{V}_H\left(\tilde{h}_{j,1}^{t+2}\right) \\ &= \bar{V}^M\left(h_{j,1}^{t+2}\right)+\xi_{L,t+1,j}^1. \end{aligned} \tag{46}$$

Using (46), it is straightforward to check that:

$$\bar{V}^M\left(\tilde{h}_{j,1}^{t+2*}\right) \geq \bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t+1,j}^1. \tag{47}$$

Using an argument analogous to the one in Step 4.1 (see Fact 1) we have:

$$\bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi_{L,t+1,j}^1-\xi_{H,t+1,j}^1 > \bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t+1,j}^1. \tag{48}$$

Hence subtracting (43) from (44) we have:

$$\left\{ \begin{array}{c} \mu\left(p_{t+1}^1, p_{t+2,j,1}^1\right)\left[\bar{V}^M\left(\tilde{h}_{j,1}^{t+2}\right)-\left(\bar{V}^M\left(h_{j,1}^{t+2}\right)+\xi_{L,t+1,j}^1\right)\right] \\ +\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\left(1-\mu\left(\tilde{p}_{t+1}^1, p_{t+2,j,1}^1\right)\right)\left[\begin{array}{c} \bar{V}^M\left(\tilde{h}_{j,1}^{t+2*}\right) \\ -\left(\bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t+1,j}^1\right) \end{array}\right] \\ +\left(1-\mu\left(p_{t+1}^1, \tilde{p}_{t+1}^1\right)\right)\left[\begin{array}{c} \bar{V}^M\left(\tilde{h}_1^{t+1*}\right)+\xi_{L,t+1,j}^1-\xi_{H,t+1,j}^1 \\ -\left(\bar{V}^M\left(h_{j,1}^{t+2*}\right)+\xi_{L,t+1,j}^1\right) \end{array}\right] \end{array} \right\}. \tag{49}$$

[20] By construction the minimal rent to the high type consumer, $\min \Phi_H\left(p_{t+2,j,1}^1, \left\{q_{t+1,j,1}^1\right\}_{\tau \geq t+2}\right)$, is offered in the deviating contract. If a higher rent is offered under the putative equilibrium the argument to be presented becomes even stronger. Hence here and hereafter we assume the monopolist offers minimal rents under the original play.

Thus from (45), (47) and (48) we have:

Fact 2: *The expression (49) is strictly positive.*

For future reference, denote $\zeta_{t+1,j}(3)$ for (49) and let $\mu(\zeta_{t+1,j}(3)) = \gamma^1_{t+1,j}$ represent the probability that the strings leading to these values are reached.

Step 4.4 *Play leading to 2 contracts, one associated with the belief* $p^1_{t+2,j,1} \in [0, \tilde{p}^1_{t+1}]$ *and another associated with the belief* $p^1_{t+2,j,2} \in (\tilde{p}^1_{t+1}, 1)$

In this case, the monopolist offers the menu $m^1_{t+1,j} = (\psi^1_{t+1,j,1}, \psi^1_{t+1,j,2})$, where $\psi^1_{t+1,j,i} = \left(x^1_{t+1,j,i}, \left\{q^1_{t+1,j,i}\right\}_{\tau \geq t+1}\right)$ for $i \in \{1,2\}$. The contract $\psi^1_{t+1,j,1}$ ($\psi^1_{t+1,j,2}$) leads to the belief $p^1_{t+2,j,1} \in [0, \tilde{p}^1_{t+1}]$ (resp. $p^1_{t+2,j,2} \in (\tilde{p}^1_{t+1}, 1)$), is accepted with probability $\mu^1_{t+1,j,1}$ (resp. $\mu^1_{t+1,j,2}$) and is associated with the history $h^{t+2}_{j,1}$ (resp. $h^{t+2}_{j,2}$). The monopolist's profit is:

$$\mu^1_{t+1,j,1}\left[\bar{V}^M\left(h^{t+2*}_{j,1}\right) + \xi^1_{L,t+1,j}\right] + \mu^1_{t+1,j,2}\left[\bar{V}^M\left(h^{t+2*}_{j,2}\right) + \xi^1_{L,t+1,j}\right]. \tag{50}$$

Now, consider the play under the deviating strategy. Conditional on the history $\tilde{h}^{t+1}_1$, the monopolist randomizes:

With probability $\alpha \in (0,1)$, the monopolist offers a menu with a pooling contract and a revealing contract. The pooling contract specifies a current period quality $q_{t+1,j,1}$, leads to the state $\left(p_{t+1,j,1}, \{q_{1,j,\tau}\}_{\tau \geq t+2}\right)$ in the next period, its transfers are set such that the rent of the low type is zero, and it is associated with the history $\tilde{h}^{t+2}_{j,1}$. The revealing contract specifies a quality q^*_H for the every future period and its transfers are set to make the high type indifferent between both contracts. It is associated with the history $\tilde{h}^{t+2*}_{j,1}$.

With probability $(1-\alpha) \in (0,1)$ the monopolist offers only one pooling contract $(\{q^*_L, \mathbf{0}\}, \theta_L q^*_L)$ which is associated with the history $\tilde{h}^{t+2}_{j,2}$.

The probability α is chosen such that the belief $p^1_{t+2,j,1}$ is reached with the same probability as under the putative equilibrium play:

$$\alpha\mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\mu\left(\tilde{p}^1_{t+1}, p_{t+1,j,1}\right) = \mu^1_{t+1,j,1}.$$

No renegotiation is proposed at history h^{t+1*}_1. Hence the monopolist's profit is:

$$\left\{\begin{array}{c} \alpha\mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\left[\mu\left(\tilde{p}^1_{t+1}, p_{t+1,j,1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right) + \left(1-\mu\left(\tilde{p}^1_{t+1}, p_{t+1,j,1}\right)\right)\bar{V}^M\left(\tilde{h}^{t+2*}_{j,1}\right)\right] \\ +(1-\alpha)\mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,2}\right) \\ +\left(1-\mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right) + \xi^1_{L,t+1,j} - \xi^1_{H,t+1,j}\right] \end{array}\right\}.$$

Using an argument analogous to the one in (45), we conclude that

$$\alpha\mu\left(p^1_{t+1}, \tilde{p}^1_{t+1}\right)\mu\left(\tilde{p}^1_{t+1}, p_{t+1,j,1}\right)\left[\begin{array}{c} \bar{V}^M\left(\tilde{h}^{t+2}_{j,1}\right) - \\ -\left(\bar{V}^M\left(h^{t+2}_{j,1}\right) + \xi^1_{L,t+1,j}\right) \end{array}\right] \geq 0. \tag{51}$$

Fact 3: *The term under brackets in (51) is weakly positive.*

For future reference denote $\zeta_{t+1,j}(4)$ for the term inside brackets in *(51)* and let $\mu\left(\zeta_{t+1,j}(4)\right) = \gamma^1_{t+1,j}\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\mu\left(\tilde{p}^1_{t+1},p_{t+1,j,1}\right)$ represent the probability that this term is accepted.

Hence we need to compare:

$$\left\{\begin{array}{c}(1-\alpha)\,\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+2}_{j,2}\right)\\ +\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]\\ +\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\left(1-\mu\left(\tilde{p}^1_{t+1},p_{t+1,j,1}\right)\right)\bar{V}^M\left(\tilde{h}^{t+2*}_{j,1}\right)\end{array}\right\} \tag{52}$$

and

$$\mu^1_{t+1,j,2}\left[\bar{V}^M\left(h^{t+2*}_{j,2}\right)+\xi^1_{L,t+1,j}\right]. \tag{53}$$

Notice that by construction the total measure and the high type's measure in the strings leading to (52) and (53) are the same. Notice also that in (52) there are values correspondent to separation in two different periods. The term $\bar{V}^M\left(\tilde{h}^{t+2*}_{j,1}\right)$ corresponds to a string in which the high type separates in period $t+2$, while the term $\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]$ corresponds to a string in which there is separation in period $t+1$. One can repeat the same procedure letting the term:

$$\left(\frac{1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)}{\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)+\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)}\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_1\right)+\xi^1_{L,t+1,j}-\xi^1_{H,t+1,j}\right]$$
$$+\left(\frac{\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)}{\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)+\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)}\right)\left(1-\mu\left(\tilde{p}^1_{t+1},p_{t+1,j,1}\right)\right)\bar{V}^M\left(\tilde{h}^{t+2*}_{j,1}\right),$$

which happens with probability $\left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)+\alpha\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)$, represent the earlier separation term. Then we can repeat the algorithm in the next period.

Step 5 *Desired Inequality*

In Step 4 we used a stopping time to compare the profit reached under the putative equilibrium play and the one under the deviating play. Each time a belief in $\left[0,\tilde{p}^1_{t+1}\right]\cup\{1\}$ was reached under the putative equilibrium the stopping time truncated both the putative equilibrium play and the deviating play. The difference in profits obtained by this truncation was called $\zeta_{t+1,j}(i)$ (under Step 4.i) and we called $\mu\left(\zeta_{t+1,j}(i)\right)$ the probability of such event.

For each $T\geq t+2$, we can use the algorithm defined in Step 4 to construct a stopping time according to which the stochastic process defined by the putative equilibrium play and the one defined by the deviating play are truncated in the first time the belief reaches $\left[0,\tilde{p}^1_{t+1}\right]\cup\{1\}$ under the putative equilibrium play or in period T if the belief does not reach $\left[0,\tilde{p}^1_{t+1}\right]\cup\{1\}$ before period T.

For every $s\geq t+1$ and $i\in\{1,2,3\}$ write Λ^i_s for the set of events in which the truncation occurs in period s under step i. Write $\omega\in\Lambda^i_s$ for a typical element, write ζ_ω for its value (the difference in the profit obtained under the deviating play and under the putative equilibrium play) and write $\mu(\zeta_\omega)$ for the probability according to which this event is reached. The argument in Step 4 implies that:

i) For all $s \geq t+1$ and for all $\omega \in \Lambda_s^1 \cup \Lambda_s^3$ we have $\zeta_\omega > 0$.

ii) For all $s \geq t+1$ and for all $\omega \in \Lambda_s^4$ we have $\zeta_\omega \geq 0$.

Write Ω_T for the set of putative equilibrium histories according to which the belief does not reach $\left[0, \tilde{p}_{t+1}^1\right] \cup \{1\}$ before period T under the putative equilibrium. Write $h_\omega^T \in \Omega_T$ for a typical history in this set. The algorithm defined in Step 4 links to each history h_ω^T a history $\tilde{h}_\omega^T$ associated with the belief $\tilde{p}_{t+1}^1$ under the deviating play and also to a set of histories under the deviating play $\left\{\tilde{h}_\omega^{z*}\right\}_{z=t+1}^T$, where z indicates the period in which the high type is screened.[21] For each $h_\omega^T \in \Omega_T$, for each $z \leq s-1$ write $\xi_{L,z,T}^\omega$ (resp. $\xi_{H,z,T}^\omega$) for the sum of innovations in the low type (resp. high type) consumer's rent in periods $t \in \{z, ..., T-1\}$ (set $\xi_{L,T,T}^\omega = 0$). Write $P_{\sigma^*}^{h^t}\left(h_\omega^T\right)$ for the probability of history h_ω^T being reached under the stopping time under the putative equilibrium play. For $h \in \left\{\tilde{h}_\omega^T, \tilde{h}_\omega^{t+1*}, ..., \tilde{h}_\omega^{T*}\right\}$ write $P_{\hat{\sigma}}^{h^t}(h)$ for the probability that history h is reached under the stopping time under the deviating play.

Hence, we must show that there exists $T \geq t+2$ such that:

$$\left\{\begin{array}{c} \sum\limits_{s=t+1,...,T-1} \sum_{i=1,3,4} \sum_{\omega \in \Lambda_s^i} \mu(\zeta_\omega)\, \zeta_\omega \\ + \sum_{h_\omega^T \in \Omega_T} \left[\begin{array}{c} P_{\hat{\sigma}}^{h^t}\left(\tilde{h}_\omega^T\right) \bar{V}^M\left(\tilde{h}_\omega^T\right) + \sum\limits_{t+1 \leq z \leq T} P_{\hat{\sigma}}^{h^t}\left(\tilde{h}_\omega^{z*}\right)\left[\bar{V}^M\left(\tilde{h}_\omega^{z*}\right) + \xi_{L,z,T}^\omega - \xi_{H,z,T}^\omega\right] \\ -P_{\sigma^*}^{h^t}\left(h_\omega^T\right)\left[\bar{V}^M\left(h_\omega^T\right) + \xi_{L,t+1,T}^\omega\right] \end{array}\right] \end{array}\right\} > 0. \tag{54}$$

Step 6 *Final Step*

Take an upper bound $A > 0$ to the sum of continuation payoffs of the monopolist and the high type.

Claim 3 *There exists $\varepsilon > 0$ and T' such that if $T > T'$ then $\bar{V}^M\left(\tilde{h}_\omega^{t+1*}\right) + \xi_{L,t+1,T}^\omega - \xi_{H,t+1,T}^\omega > \bar{V}^M\left(h_\omega^T\right) + \xi_{L,t+1,T}^\omega + \varepsilon$ for all $\omega \in \Omega_T$.*

Proof By construction the rent of the high type is the same under both strings above. However he consumes $q_t^1 \leq q_L^*$ in period t under the putative equilibrium play and q_H^* under the history $\tilde{h}_\omega^{t+1*}$ under the deviating play. This improves the surplus by at least $(1-\delta)\int_{q_L^*}^{q_H^*} \pi'_H(s)\,ds > 0$. Taking $\varepsilon = \left(\frac{(1-\delta)\int_{q_L^*}^{q_H^*} \pi'_H(s)ds}{2}\right)$ and T' such that $\delta^{T'-t} < \left(\frac{(1-\delta)\int_{q_L^*}^{q_H^*} \pi'_H(s)ds}{4A}\right)$ the Claim follows. ■

Claim 4 *For every $T \geq t$ and every $\omega \in \Omega_T$ we have $\bar{V}^M\left(\tilde{h}_\omega^t\right) \geq \bar{V}^M\left(h_\omega^T\right) + \xi_{L,t+1,T}^\omega - 2\delta^{T-t}A$.*

[21] It is (w.l.o.g.) to assume that for each $z \in \{t, ..., T-1\}$ there is a history according to which the high type consumer is screened in period z since we can always choose a history arbitrarily and associate probability zero to it.

Proof The rent of the low type is the same under both plays by construction. Also, the qualities in $\tilde{h}^T_\omega$ are weakly more efficient for the low type. Since the the sum of continuation payoffs is bounded by A the Claim follows. ■

Claim 5 below is straightforward.

Claim 5 *For every $s \geq t+1$ and every $\omega \in \Omega_T$ we have* $\bar{V}^M\left(\tilde{h}^{s*}_\omega\right) + \xi^\omega_{L,s,T} - \xi^\omega_{L,s,T} \geq \bar{V}^M\left(h^T_\omega\right) + \xi^\omega_{L,t+1,T} - 2\delta^{T-t}A$.

Thus take T'' such that $\delta^{T''-t} < \left[\frac{(1-\mu(p^1_{t+1},\tilde{p}^1_{t+1}))\varepsilon}{4A}\right]$. Let $\hat{T} = \max\left\{T', T''\right\}$.

The analysis is divided into two cases.

Case 1: $\sum_{\omega\in\Omega_{\hat{T}}} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{\hat{T}*}_\omega\right) = \left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)$

First notice that (54) is at least:

$$\sum_{h^{\hat{T}}_\omega\in\Omega_T}\left[\begin{array}{c} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{\hat{T}}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{\hat{T}}_\omega\right) - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right] \\ +P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{t+1*}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{t*}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}} - \xi^\omega_{H,t+1,\hat{T}} - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right] \\ +\sum_{t+1<z\leq\hat{T}} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{z*}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{z*}_\omega\right) + \xi^\omega_{L,z,\hat{T}} - \xi^\omega_{H,z,\hat{T}} - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right]\end{array}\right], \quad (55)$$

From Claim 3

$$\begin{aligned}&\sum_{h^{\hat{T}}_\omega\in\Omega_T} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{t+1*}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{t+1*}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}} - \xi^\omega_{H,t+1,\hat{T}} - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right] \\ \geq\ & \left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)\varepsilon\end{aligned}$$

From Claims 4 and 5:

$$\begin{aligned}&\sum_{h^{\hat{T}}_\omega\in\Omega_T}\left[\begin{array}{c} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{\hat{T}}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{\hat{T}}_\omega\right) - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right] \\ +\sum_{t+1<z\leq\hat{T}} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{z*}_\omega\right)\left[\bar{V}^M\left(\tilde{h}^{z*}_\omega\right) + \xi^\omega_{L,z,\hat{T}} - \xi^\omega_{H,z,\hat{T}} - \left(\bar{V}^M\left(h^{\hat{T}}_\omega\right) + \xi^\omega_{L,t+1,\hat{T}}\right)\right]\end{array}\right] \\ \geq\ & -2\delta^{\hat{T}-t}A\end{aligned}$$

Hence (55) is at least $\left[\frac{(1-\mu(p^1_{t+1},\tilde{p}^1_{t+1}))\varepsilon}{4A}\right] > 0$.

Case 2: $\sum_{\omega\in\Omega_{\hat{T}}} P^{h^t}_{\hat{\sigma}}\left(\tilde{h}^{\hat{T}*}_\omega\right) < \left(1-\mu\left(p^1_{t+1},\tilde{p}^1_{t+1}\right)\right)$.

In this case let $\vartheta = \sum_{i=1,3}\sum_{\omega\in\Lambda^i_{t+1}} \mu(\zeta_\omega)\zeta_\omega > 0$ and notice that it is at most

$$\sum_{s=t+1,\ldots,\hat{T}-1}\ \sum_{i=1,3,4}\ \sum_{\omega\in\Lambda^i_s} \mu(\zeta_\omega)\zeta_\omega.$$

Take $\breve{T} > \hat{T}$ such that $\delta^{\breve{T}-t} < \left[\frac{\vartheta}{4A}\right]$. From Claims 4 and 5

$$\sum_{h_{\omega}^{\breve{T}} \in \Omega_{\breve{T}}} \left[\begin{array}{c} P_{\hat{\sigma}}^{h^t}\left(\tilde{h}_{\omega}^{\breve{T}}\right)\left[\bar{V}^M\left(\tilde{h}_{\omega}^{\breve{T}}\right) - \left(\bar{V}^M\left(h_{\omega}^{\breve{T}}\right) + \xi_{L,t+1,\breve{T}}^{\omega}\right)\right] \\ + \sum_{t+1 \leq z \leq \breve{T}} P_{\hat{\sigma}}^{h^t}\left(\tilde{h}_{\omega}^{z*}\right)\left[\bar{V}^M\left(\tilde{h}_{\omega}^{z*}\right) + \xi_{L,z,\breve{T}}^{\omega} - \xi_{H,z,\breve{T}}^{\omega} - \left(\bar{V}^M\left(h_{\omega}^{\breve{T}}\right) + \xi_{L,t+1,\breve{T}}^{\omega}\right)\right] \end{array} \right]$$
$$\geq \quad -2\delta^{\breve{T}-t}A.$$

Hence (55) is at least $\left(\frac{\vartheta}{2}\right) > 0$, which proves the Lemma. ■

7.1.7 Lemma 7

Lemma 7 *Consider a history h^t associated with the state $(p_t, \{q_\tau\})$ such that $\sum_\tau (1-\delta)\delta^\tau q_\tau < q_L^*$ and $p_t \in (0,1)$. The rationality constraint of the low type holds with equality in any menu posted at h^t which solves the monopolist's problem.*

Proof. Assume that the monopolist offers a menu $m = \left(\psi^1, \psi^2\right)$ such that the rationality constraint of the low type holds with a slack $\eta > 0$. Consider the pooling contract ψ^1 associated with the belief[22] $p \in (0,1)$ and let ψ' be the contract differing from ψ^1 in that it does not contain promises for future periods and its transfers are set such that the low type's rent is decreased by some small $\varepsilon > 0$. From Lemma ii) and iii) in Lemma 4 we can find a belief $p' \in (0,p]$ such that the monopolist can credibly promise an information rent to the high type which is larger by $\varepsilon > 0$. Hence, following the steps in Lemma 6, one can show the monopolist achieves a strictly higher profit from replacing the acceptance of contract ψ^1 with a mixture over ψ' and a revealing contract. ■

7.1.8 Lemma 8

Lemma 8 *Consider a history h^t associated with the state $(p_t, \{q_\tau\})$ such that $\sum_\tau (1-\delta)\delta^\tau q_\tau < q_L^*$. Then there is sequential separating dynamics in any menu posted at h^t which solves the monopolist's problem.*

Proof. The proof is by contradiction. There are 2 cases.

Case 1: Suppose that at history h^t the state is $(p_t, \{q_\tau\})$, with $p_t \in (0,1)$ and $\sum_{\tau \geq t}(1-\delta)\delta^{\tau-t}q_\tau < q_L^*$, and the monopolist offers a menu with 2 contracts $\left(\psi_t^1, \psi_t^2\right)$, $\psi_t^i = \left(x_t^i, \left\{q_\tau^i\right\}\right)$ for $i = 1,2$. Contract ψ_t^1 is associated with the belief $p_{t+1}^1 < p_t$ while ψ_t^2 is associated with the belief $p_{t+1}^2 \in (p_t, 1)$.

Case 2: Suppose that at history h^t the state is $(p_t, \{q_\tau\})$, with $p_t \in (0,1)$ and $\sum_{\tau \geq t}(1-\delta)\delta^{\tau-t}q_\tau < q_L^*$, and the monopolist offers a menu with 2 contracts $\left(\psi_t^1, \psi_t^2\right)$, with $\psi_t^i = \left(x_t^i, \left\{q_\tau^i\right\}\right)$ for $i = 1,2$, and both contracts are associated with the belief p_t.

[22] If the menu has a pooling contract associated to belief 0 the proof is straightforward.

Step. 1 *Case 1*

For $i = 1, 2$ let μ_t^i represent the probability that the contract ψ_t^i is accepted and write h_i^{t+1} for the history led by this event.

There are 2 possibilities:

Possibility 1: $\sum_{\tau \geq t} (1-\delta) \delta^{\tau - t} q_\tau > (1-\delta) \Delta\theta q_L^* + \delta (\min \Phi_H (p_t, \{\mathbf{0}\}))$

Possibility 2: $\sum_{\tau \geq t} (1-\delta) \delta^{\tau - t} q_\tau \leq (1-\delta) \Delta\theta q_L^* + \delta (\min \Phi_H (p_t, \{\mathbf{0}\}))$.

Deviating Strategy: We construct a proposed deviation in which the monopolist randomizes. With probability $\alpha \in (0,1)$ the monopolist offers a menu with 2 contracts: $\left(\tilde{\psi}_t^1, \tilde{\psi}_t^R\right)$.

Contract $\tilde{\psi}_t^1$: Let $\tilde{q}_t^1 = \min\left\{q_t^1, q_L^*\right\}$. The contract $\tilde{\psi}_t^1 = \left(\theta_L \tilde{q}_t^1 - \sum_{\tau \geq t} \delta^{\tau - t} q_\tau, \left\{\tilde{q}_t^1, \mathbf{0}\right\}\right)$ is a pooling contract and is associated with the minimum[23] $\tilde{p}_{t+1}^1 \in (0, p_t)$ such that

$$v_{H,t} = (1-\delta) \Delta\theta \tilde{q}_t^1 + \delta \tilde{v}_{H,t+1},$$

where $v_{H,t}$ is the rent that the high type obtains from the contract ψ_t^1.

Notice that from Lemma 2 $\tilde{p}_{t+1}^1 \leq p_{t+1}^1$.

Contract $\tilde{\psi}_t^R$: This is a revealing contract in which transfers are set to make the high type indifferent between both contracts.

We take α such that:

$$\alpha \mu \left(p_t, p_{t+1}^1\right) = \mu_t^1,$$

that is, if $p_{t+1}^1 = \tilde{p}_{t+1}^1$ then contract ψ_t^1 and contract $\tilde{\psi}_t^1$ are accepted with the same probability.

Write $\tilde{h}_1^{t+1}$ for the history led by the contract $\tilde{\psi}_t^1$ and $\tilde{h}^{t+1*}$ for the history led by the contract $\tilde{\psi}_t^R$.

With probability $(1-\alpha) \in (0,1)$ the monopolist offers a menu with 2 contracts: $\left(\tilde{\psi}_t^1, \tilde{\psi}_t^R\right)$ if we have Possibility 1 and only one contract if we have Possibility 2.

Possibility 1:

Under Possibility 1 let $\tilde{p}_{t+1}^2$ be the minimum belief p such that $\tilde{v}_{H,t+1} \in \Phi_H (p, \{\mathbf{0}\})$ and

$$\sum_{\tau \geq t} (1-\delta) \delta^{\tau - t} \Delta\theta q_\tau = (1-\delta) \Delta\theta q_L^* + \delta \tilde{v}_{H,t+1},$$

with $\tilde{v}_{H,t+1} \in \Phi_H (p, \{\mathbf{0}\})$. Notice that by assumption (Possibility 1) $\tilde{p}_{t+1}^2 \in [0, p_t)$. In this case the, monopolist offers a menu with 2 contracts: $\left(\tilde{\psi}_t^2, \tilde{\psi}_t^{2R}\right)$. The contract $\tilde{\psi}_t^2 = \left(\theta q_L^* - \sum_{\tau \geq t} \delta^{\tau - t} \Delta\theta q_\tau, \{q_L^*, \mathbf{0}\}\right)$ is a pooling contract leading to the belief $\tilde{p}_{t+1}^2$, while the contract $\tilde{\psi}_t^{2R}$ is a revealing contract with transfers set to make the high type indifferent between both contracts. Write $\tilde{h}_2^{t+1}$ for the history led by the contract $\tilde{\psi}_t^2$ and $\tilde{h}^{t+1**}$ for the history led by contract $\tilde{\psi}_t^{R2}$.

Possibility 2:

[23] Remember that we are assuming that the graph of $(p, \Phi(p, \{\mathbf{0}\}))$ is closed.

Under Possibility 2 the monopolist offers only one contract $\tilde{\psi}_t^2 = \left(\theta_L q_L^* - \sum_{\tau \geq t} \delta^{\tau-t} q_\tau, \{q_L^*, \mathbf{0}\}\right)$ and the belief is not updated. Write $\tilde{h}_2^{t+1}$ for the history led by the acceptance of the contract $\tilde{\psi}_t^2$.

Next, we must compare the profit under the putative play involving sequential separating dynamics and under the deviating play presented above. Under the putative equilibrium play, the profit of the monopolist is:

$$\mu_t^1 \bar{V}^M \left(h_1^{t+1}\right) + \mu_t^2 \bar{V}^M \left(h_2^{t+1}\right). \tag{56}$$

Under the deviating play, the profit of the monopolist[24] is:

$$\begin{aligned} &\alpha \left[\mu \left(p_t, \tilde{p}_{t+1}^1\right) \bar{V}^M \left(\tilde{h}_1^{t+1}\right) + \left(1 - \mu \left(p_t, \tilde{p}_{t+1}^1\right)\right) \bar{V}^M \left(\tilde{h}^{t+1*}\right)\right] \\ &+ (1-\alpha) \left[\mu \left(p_t, \tilde{p}_{t+1}^2\right) \bar{V}^M \left(\tilde{h}_2^{t+1}\right) + \left(1 - \mu \left(p_t, \tilde{p}_{t+1}^2\right)\right) \bar{V}^M \left(\tilde{h}^{t+1**}\right)\right]. \end{aligned} \tag{57}$$

There are 2 possible subcases

Subcase 1 $\tilde{p}_{t+1}^1 < p_{t+1}^1$.

In this case, take $\lambda = \left(\frac{\mu\left(p_t, \tilde{p}_{t+1}^1\right)}{1-\mu\left(p_t, p_{t+1}^1\right)}\right) \left(\frac{\mu\left(p_t, p_{t+1}^1\right)}{\mu\left(p_t, \tilde{p}_{t+1}^1\right)} - 1\right)$, notice that $\lambda \in (0,1)$ and write the difference between (56) and (57) as the sum of two terms:

$$\left[\begin{array}{c} \alpha\mu \left(p_t, \tilde{p}_{t+1}^1\right) \bar{V}^M \left(\tilde{h}_1^{t+1}\right) + \alpha\lambda \left(1 - \mu \left(p_t, \tilde{p}_{t+1}^1\right)\right) \bar{V}^M \left(\tilde{h}^{t+1*}\right) \\ -\mu_t^1 \bar{V}^M \left(h_1^{t+1}\right) \end{array}\right] \tag{58}$$

and

$$\left[\begin{array}{c} (1-\alpha) \left[\mu \left(p_t, \tilde{p}_{t+1}^2\right) \bar{V}^M \left(\tilde{h}_2^{t+1}\right) + \left(1 - \mu \left(p_t, \tilde{p}_{t+1}^2\right)\right) \bar{V}^M \left(\tilde{h}^{t+1**}\right)\right] \\ +\alpha (1-\lambda) \left(1 - \mu \left(p_t, \tilde{p}_{t+1}^1\right)\right) \bar{V}^M \left(\tilde{h}^{t+1*}\right) \\ -\mu_t^2 \bar{V}^M \left(h_2^{t+1}\right) \end{array}\right]. \tag{59}$$

It is straightforward to check that (as in Step 4 in Lemma 6) the measure of terms from the deviating play and from the original play in (58) and (59) are the same and so are the high type measures. One can then use an analogous procedure as in Step 4 in Lemma 6 to show that the sum of (58) and (59) is strictly positive.

Subcase 2 $\tilde{p}_{t+1}^1 = p_{t+1}^1$.

Proceeding analogously as in Subcase 1, we write the difference between (56) and (57) as the sum of two terms:

$$\left[\alpha\mu \left(p_t, p_{t+1}^1\right) \bar{V}^M \left(\tilde{h}_1^{t+1}\right) - \mu_t^1 \bar{V}^M \left(h_1^{t+1}\right)\right]$$

[24] If we have Possibility 2 then we can write $\tilde{p}_{t+1}^2 = p_t$ and $\mu\left(p_t, \tilde{p}_{t+1}^2\right) = 1$. $\tilde{h}^{t+1**}$ can be chosen arbitrarily.

and
$$\left[\begin{array}{c} (1-\alpha)\left[\mu\left(p_t,\tilde{p}^2_{t+1}\right)\bar{V}^M\left(\tilde{h}^{t+1}_2\right)+\left(1-\mu\left(p_t,\tilde{p}^2_{t+1}\right)\right)\bar{V}^M\left(\tilde{h}^{t+1**}\right)\right] \\ +\alpha(1-\lambda)\left(1-\mu\left(p_t,\tilde{p}^1_{t+1}\right)\right)\bar{V}^M\left(\tilde{h}^{t+1*}\right) \\ -\mu^2_t\bar{V}^M\left(h^{t+1}_2\right) \end{array}\right].$$

An argument similar to the one in Step 4 in Lemma 6 establishes a profitable deviation.

Step. 2 *Case 2*

If the promises are $\{\mathbf{0}\}$ or if $\sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}\Delta\theta q_\tau \leq \min\Phi_H\left(p_t,\{\mathbf{0}\}\right)$ the result is straightforward since for every solution to the monopolist's problem when the state is $\left(p_t,\{q_\tau\}\right)$ there exists a solution to the monopolist's problem when the state is $\{\mathbf{0}\}$ which differs from the first only in that current period's transfers are decreased by $\sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}\theta q_\tau$ in each contract. It is straightforward to show that the belief has to change in any such solution. Hence suppose that $\sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}\Delta\theta q_\tau > \min\Phi_H\left(p_t,\{\mathbf{0}\}\right)$. Take $\tilde{p}_{t+1} < p_t$ such that $v_{H,t+1}\in\Phi_H\left(p_{t+1},\{\mathbf{0}\}\right)$ where $v_{H,t+1}$ satisfies:

$$(1-\delta)\,\Delta\theta q^*_L + \delta v_{H,t+1} = \sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}\Delta\theta q_\tau.$$

Construct a deviation in which the monopolist offers two contracts: a pooling contract $\tilde{\psi}^1_t = \left(\theta_L q^*_L - \sum_{\tau\geq t}(1-\delta)\,\delta^{\tau-t}\theta_L q_\tau, \{q^*_L,\mathbf{0}\}\right)$ leading to a belief $\tilde{p}_{t+1}$ and a revealing contract in which the transfers are set to make the high type indifferent between both contracts. The argument in Lemma 6 implies that the deviating play leads to a strictly higher profit for the monopolist. ■

7.1.9 Lemma 9

Lemma 9 *Assume that a period starts with the state $\left(p,\{q_\tau\}\right)$ such that $\sum_\tau(1-\delta)\,\delta^\tau q_\tau < q^*_L$. Then, for every solution to the monopolist's problem, there is an outcome equivalent solution to the monopolist's problem in which she offers a menu with sequential separating dynamics and she does not make promises for future periods in pooling contracts.*

Proof. In Lemma 8 it was established that any solution to the monopolist's problem when the state is $\left(p_t,\{q_\tau\}_{\tau\geq t}\right)$ involves sequential separating dynamics. Assume that the monopolist offers a menu with contracts $\left(\psi^1_t,\psi^2_t\right)$, where $\psi^1_t=\left(x_{t,1},\left\{q^1_\tau\right\}_{\tau\geq t}\right)$ is a pooling contract associated with the belief $p_{t+1}\in[0,p_t)$, while ψ^2_t is a revealing contract. The rent that the low type obtains from ψ^1_t is:

$$v_{L,t,1}=(1-\delta)\,\theta_L q_{t,1}-(1-\delta)\,x_{t,1}+\delta v_{L,t+1,1},$$

while the rent of the high type is

$$v_{H,t,1}=(1-\delta)\,\theta_L q_{t,1}-(1-\delta)\,x_{t,1}+(1-\delta)\,\Delta\theta q_{t,1}+\delta v_{H,t+1,1}.$$

From Lemma 7

$$v_{L,t+1,1} = \sum_{\tau \geq t+1} (1-\delta)\, \delta^{\tau-(t+1)} \theta_L q_{\tau,1}.$$

Whereas from Lemma 6 the information rent of the high type is minimal and from Lemma 2 this rent is associated with[25] the promises $\{\mathbf{0}\}$. Hence

$$(0, v_{H,t+1,1} - v_{L,t+1,1}) \in \Phi\left(p_{t+1}, \{\mathbf{0}\}\right).$$

Thus, the monopolist can replace the contract ψ_t^1 with

$$\tilde{\psi}_t^1 = \left(x_{t,1} - \delta \sum_{\tau \geq t+1} \delta^{\tau-(t+1)} \theta_L q_{\tau,1}, \{q_{t,1}, \mathbf{0}\} \right),$$

and have both consumers choose the contract $\tilde{\psi}_t^1$ with the same probability as contract ψ_t^1, which is feasible and payoff equivalent for all players. This proves the Lemma. ■

7.1.10 Lemma 10

Lemma 10 *Assume that a period starts with the state $(p, \{q_\tau\})$ such that $\sum_\tau (1-\delta)\, \delta^\tau q_\tau < q_L^*$. Assume that $\tilde{m}$ is a menu solving the monopolist's problem at $(p, \{q_\tau\})$ and let $\tilde{\psi}$ be a pooling contract in this menu. Assume that $\tilde{\psi}$ is associated with the belief $\tilde{p} \in (0,1)$ and to the promises $\{\tilde{q}_\tau\}$. Then $\sum_\tau (1-\delta)\, \delta^\tau \tilde{q}_\tau \leq q_L^*$.*

Proof. First notice that from Lemma 8 the monopolist offers a menu with sequential separating dynamics. Hence we can write $\tilde{m} = \left(\tilde{\psi}, \tilde{\psi}^R\right)$ where $\tilde{\psi}^R$ is a revealing contract and $\tilde{p} < 1$ is the belief associated with $\tilde{\psi}$.

Assume towards a contradiction that $\sum_\tau (1-\delta)\, \delta^\tau \tilde{q}_\tau > q_L^*$.

Notice that from Lemma 2 we have $\hat{\Phi}\left(\tilde{p}, \{\tilde{q}_\tau\}\right) \geq \hat{\Phi}\left(\tilde{p}, \{\mathbf{0}\}\right)$. Hence there is a menu m^* offered at $(p, \{q_\tau\})$ such that the monopolist extracts a weakly higher rent from the high type by offering a pooling contract which is associated with belief $\tilde{p} < 1$ and to promises $\{\mathbf{0}\}$ and yields the same rent to the low type. We proceed to show that this deviation is profitable if $\sum_\tau (1-\delta)\, \delta^\tau \tilde{q}_\tau > q_L^*$. From the second statement in Lemma 5, it suffices to show that any solution to the monopolist's problem when the state is $(\tilde{p}, \{\tilde{q}_\tau\})$ and any solution to the monopolist's problem when the state is $(\tilde{p}, \{\mathbf{0}\})$ lead to different continuation future consumption distributions.

Case 1 $(1-\delta) \sum \delta^\tau \tilde{q}_\tau \in (q_L^*, q_H^*]$. In this case, any optimal menu at $(\tilde{p}, \{\tilde{q}_\tau\})$ is outcome equivalent to the cheapest individually-rational and incentive compatible menu. When promises are $\{\mathbf{0}\}$ the monopolist never offers an efficient menu, since she could profitably deviate by decreasing the current period's quality in the contract designed to the low type.

[25] Not necessarily only to.

Case 2 $(1-\delta)\sum\delta^{\tau}\tilde{q}_{\tau}\in(q_H^*,1]$. When promises are $\{\mathbf{0}\}$ we know from Lemma 9 that the monopolist offers a contract in which the high type consumes q_H^* in each future period. When $(1-\delta)\sum\delta^{\tau}\tilde{q}_{\tau}\in(q_H^*,1]$ it is straightforward to verify that conditionally on offering a contract in which the high type consumes q_H^* in each future period the monopolist would choose the most efficient individually-rational and incentive-compatible contract. Using the same argument as in Case 1 the Lemma follows. ■

7.1.11 Proof of Proposition 1

Proposition 1

1.1 Every equilibrium presents sequential separating dynamics.

1.2 For every equilibrium there exists an outcome equivalent simple equilibrium.

1.3 In all simple equilibria the price paid for the revealing contract is decreasing from period 1 on.

Proof. The proof of 1.1 follows from Lemmas 8 and 10, while the proof of 1.2 follows from Lemmas 8, 9 and 10. Below we prove 1.3.

1.3 In all simple equilibria the price paid for the revealing contract is decreasing from period 1 on.

The proof is by contradiction. If not then we can find two periods on the equilibrium path, t and $t+1$ $(t\geq 1)$, (with respective beliefs $p_t>p_{t+1}>0$ and respective continuation rents to the high type $v_{H,t}$ and $v_{H,t+1}$) such that the rent of the high type is weakly greater in t than in $t+1:$ $v_{H,t}\geq v_{H,t+1}$. First notice that from Lemma 6 we have $v_{H,s}=\min\Phi_H(p_s,\{\mathbf{0}\})$ for $s=t,t+1$. Hence, Lemma 6 implies that the monopolist has a profitably deviation, which establishes a contradiction and proves the proposition. ■

7.2 Proposition 3

7.2.1 Lemma 11

Lemma 11 *There exists $p_0^*\in(0,1)$ such that, for any $\delta\in(0,1)$, if $p_0\in(0,p_0^*)$ then the high type is screened in one period in any equilibrium.*

Proof. Step 0. *The function $R(p,\delta)$.*

First, for each belief p_0 consider the function:

$$R(p_0,\delta)=\inf_{\{p_1,p_2,\ldots\}}\sum_{\tau=0}^{\infty}(1-\delta)\,\delta^{\tau}\Delta\theta q(p_{\tau},p_{\tau+1}),$$

subject to $\{p_0, p_1, ...\}$ being decreasing and $q(p_\tau, p_{\tau+1})$ satisfying:

$$\begin{array}{c} \pi'_L (q(p_\tau, p_{\tau+1})) = \Delta\theta \left(\frac{p_\tau - p_{\tau+1}}{1-p_\tau}\right) \text{ if } q(p_\tau, p_{\tau+1}) > 0 \\ \pi'_L(0) \leq \Delta\theta\left(\frac{p_\tau - p_{\tau+1}}{1-p_\tau}\right) \text{ if } q(p_\tau, p_{\tau+1}) = 0. \end{array} \tag{60}$$

Notice that $R(p_0, \delta) \in [0, \Delta\theta q_L^*]$, hence it is well defined.

Step 1 *Considering small beliefs.*

Take p'_0 such that

$$\pi'_L(0) = \Delta\theta\left(\frac{p'_0}{1-p'_0}\right)$$

and assume that $p_0 \leq \left(\frac{p'_0}{2}\right) = p_0^1$. Thus from (60) $q(p_0, p_1) > 0$.

Standard arguments can be applied to show that $R(p_0, \delta)$ is Lipschitz with constant[26] $K_1(1-\delta)$ where $K_1 = \left[\frac{\Delta^2\theta\left(1-p_0^*\right)^{-1}}{\inf_q c''(q)}\right] > 0$. Hence, it is absolutely continuous, increasing and differentiable (a.e.). We can write $R(p_0, \delta) = R(0, \delta) + \int_0^p R'(s, \delta)\, ds$, with $R'(s, \delta) \in [0, K_1(1-\delta)]$ (a.e.).

Step 2 *The result to be established.*

Fix a discount factor δ and consider the problem which gives an upper bound to the monopolist's profit:

$$\begin{aligned} \bar{V}^M(p_0, \delta) &= \max_{p_1 \in [0, p_0)} \Lambda(p_0, p_1, \delta) \\ &= \max_{p_1 \in [0, p_0)} \left\{ \begin{array}{c} (1-p_0)\left[(1-\delta)\pi_L(q(p_0, p_1)) + \delta\pi_L(q_L^*)\right] \\ +\left(\frac{p_0 - p_1}{1-p_1}\right)\left[\pi_H(q_H^*) - (1-\delta)\Delta\theta q(p_0, p_1) - \delta R(p_1, \delta)\right] \\ +\left[p_0 - \left(\frac{p_0-p_1}{1-p_1}\right)\right]\left[(1-\delta)\pi_L(q(p_0,p_1)) + \delta\left[\pi_H(q_H^*) - R(p_1, \delta)\right]\right] \end{array} \right\}. \end{aligned}$$

Writing $V^M(p_0, p_1, \delta)$ for the maximum profit obtained by the monopolist if the next period's belief in the pooling contract is p_1 when the discount factor is δ, we have:

$$\begin{array}{c} \Lambda(p_0, p_1, \delta) > V^M(p_0, p_1, \delta), \text{ if } p_1 > 0 \\ \Lambda(p_0, p_1, \delta) = V^M(p_0, p_1, \delta), \text{ if } p_1 = 0 \end{array}.$$

Hence, if we show that there exists $p_0^* \in (0,1)$ such that for every $\delta \in (0,1)$ if $p_0 \in (0, p_0^*)$ then

$$0 = \arg\max_{p_1 \in [0, p_0)} \Lambda(p_0, p_1, \delta),$$

it follows that $0 = \arg\max_{p_1 \in [0, p_0)} V^M(p_0, p_1, \delta)$, concluding that the high type will be screened in one period.

[26] The constant K_1 is such that applying the Implicit Function Theorem into the *f.o.c.* w.r.t. quality we obtain $\left|\frac{\partial q}{\partial p}\right| \leq K_1$.

For that it is enough to show that if $p \in (0, p_0^*)$ then

$$\frac{\partial \Lambda(p_0, p_1, \delta)}{\partial p_1} < 0 \text{ (a.e.)}.$$

Let p_1 be a point of differentiability of R. Differentiating $\Lambda(p_0, p_1, \delta)$ with respect to p_1 we obtain:

$$\begin{aligned}&\frac{\partial \Lambda(p_0, p_1, \delta)}{\partial p_1} \qquad (61)\\ &= \left\{ \begin{array}{c} (1-\delta)\left[\left(\frac{1-p_0}{1-p_1}\right)\pi'_L(q(p_0,p_1)) - \left(\frac{p_0-p_1}{1-p_1}\right)\Delta\theta\right]\frac{\partial q(p_0,p_1)}{\partial p_1} \\ -\delta p_0 R'(p_1,\delta) \\ -\left(\frac{1-p_0}{[1-p_1]^2}\right)\left[\begin{array}{c} [\pi_H(q_H^*) - (1-\delta)\Delta\theta q(p_0,p_1) - \delta R(p_1,\delta)] \\ -[(1-\delta)\pi_L(q(p_0,p_1)) + \delta[\pi_H(q_H^*) - R(p_1,\delta)]] \end{array}\right] \end{array} \right\}.\end{aligned}$$

We work towards showing that (61) is negative.

Step 3 *Showing that* (61) *is negative.*

Step 3.1 *The term* $|\delta p_0 R'(p_1,\delta)|$.

From Step 1 we have $|\delta p_0 R'(p_1,\delta)| \leq (1-\delta) p_0 K_1$.

Step 3.2 *The term* $\left|(1-\delta)\left[\left(\frac{1-p_0}{1-p_1}\right)\pi'_L(q(p_0,p_1)) - \left(\frac{p_0-p_1}{1-p_1}\right)\Delta\theta\right]\frac{\partial q(p_0,p_1)}{\partial p_1}\right|$.

Notice that from (60) this term is 0.

Step 3.3 *The term*

$$-\left(\frac{1-p_0}{[1-p_1]^2}\right)\left[\begin{array}{c} [\pi_H(q_H^*) - (1-\delta)\Delta\theta q(p_0,p_1) - \delta R(p_1,\delta)] \\ -[(1-\delta)\pi_L(q(p_0,p_1)) + \delta[\pi_H(q_H^*) - R(p_1,\delta)]] \end{array}\right].$$

Noticing that $\left(\frac{1-p_0}{[1-p_1]^2}\right) \geq 1-p_0$ and that $\pi_L(q(p_0,p_1)) \leq \pi_L(q_L^*)$ the term above is no more than

$$-\left[1-p_0^1\right](1-\delta)\left[\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds - \Delta\theta\left|q(p_0^1,0) - q_L^*\right|\right].$$

Next, take $p_0^* \leq p_0^1$ such that

$$\Delta\theta\left|q(p_0^*,0) - q_L^*\right| < \left(\frac{\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds}{2}\right)$$

and

$$(1-p_0^*)\left(\frac{\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds}{2}\right) - p_0^* K_1 > \left(\frac{\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds}{4}\right).$$

Step 3.4 *For* $p_0 \in (0, p_0^*)$ *and* $\delta \in (0,1)$ *we have* $\frac{\partial \Lambda(p_0,p_1,\delta)}{\partial p_1} < 0$ *(a.e).*

Using Steps 3.1 and 3.3 notice that

$$\begin{aligned}\frac{\partial\Lambda\left(p_0,p_1,\delta\right)}{\partial p_1} &< -\left(1-\delta\right)\left(\left(1-p_0^*\right)\left(\frac{\int_{q_L^*}^{q_H^*}\pi'_H\left(s\right)ds}{2}\right)-p_0^*K_1\right). \\ &< -\left(1-\delta\right)\left(\frac{\int_{q_L^*}^{q_H^*}\pi'_H\left(s\right)ds}{4}\right)<0.\end{aligned}$$

This completes the proof since p_0^* does not depend on δ. ■

7.2.2 Lemma 12

Lemma 12 *Let $\bar{\delta}\in(0,1)$ and $p\in(0,1)$. There exists T such that, for all $\delta\geq\bar{\delta}$, and for all initial priors $p_0\in(0,p)$, the high type is screened in at most T periods in any equilibrium.*

Proof. Step 0. We assume that, for all $t\leq T$, there exists $p_T^*\in(0,1)$ such that if $p\in(0,p_T^*)$ then the high type is screened in at most T periods. By the Lemma 11, this is true for $T=1$.

In **Step 1** We show that there exists $p_{T+1}^*>p_T^*$ such that if $p\in\left(0,p_{T+1}^*\right)$ then the high type is screened in at most $T+1$ periods.

In **Step 2** We show that $\lim p_T^*=1$.

Step 1 Without loss we assume that the monopolist plays a pure strategy[27]. We write $q(p_t,p_{t+1})$ for the quality supplied in the pooling contract in period t if the belief is p_s in period s.

Assume that there is $p_0\in(p_T^*+\varepsilon)$ such that after 2 periods the belief is still above p_T^* : $p_T^*+\varepsilon\geq p_0>p_1>p_2\geq p_T^*$.

The monopolist's profit is given by:

$$\begin{aligned}&\mu\left(p_0,p_2\right)\left[\left(1-\delta\right)\pi_L\left(q\left(p_0,p_1\right)\right)+\delta\left(1-\delta\right)\pi_L\left(q\left(p_1,p_2\right)\right)+\delta^2V^M\left(p_2\right)\right]\\&+\mu\left(p_0,p_1\right)\left(1-\mu\left(p_1,p_2\right)\right)\left[\left(1-\delta\right)\pi_L\left(q\left(p_0,p_1\right)\right)+\delta\left[\pi_H\left(q_H^*\right)-\left(1-\delta\right)\Delta\theta q\left(p_1,p_2\right)-\delta V_H\left(p_2\right)\right]\right]\\&+\left(1-\mu\left(p_0,p_2\right)\right)\left[\pi_H\left(q_H^*\right)-\left(1-\delta\right)\Delta\theta q\left(p_0,p_1\right)-\delta\left(1-\delta\right)\Delta\theta q\left(p_1,p_2\right)-\delta^2V_H\left(p_2\right)\right].\end{aligned}$$

We wish to show that for ε small this is less than:

$$\begin{aligned}&\mu\left(p_0,p_2\right)\left[\left(1-\delta\right)\pi_L\left(q\left(p_0,p_2\right)\right)+\delta V^M\left(p_2\right)\right]\\&+\left(1-\mu\left(p_0,p_2\right)\right)\left[\pi_H\left(q_H^*\right)-\left(1-\delta\right)\Delta\theta q\left(p_0,p_2\right)-\delta^2V_H\left(p_2\right)\right],\end{aligned}$$

so that the monopolist can profitably deviate by speeding-up the screening process.

[27] There always exists an equilibrium in which the monopolist plays a pure strategy which leads to screening in the maximum number of periods.

Step 1.1 *First Inequality*

First, notice that since the monopolist can always screen the high type in one period we have $V^{M}(p) \geq \pi_{L}(q_{L}^{*}) + p\int_{q_{L}^{*}}^{q_{H}^{*}} \pi'_{H}(s)\,ds$. Hence,

$$\begin{aligned} & (1-\delta)\,\pi_{L}(q(p_{0},p_{2})) + \delta V^{M}(p_{2}) \\ & -\left[(1-\delta)\,\pi_{L}(q(p_{0},p_{1})) + \delta(1-\delta)\,\pi_{L}(q(p_{1},p_{2})) + \delta^{2}V^{M}(p_{2})\right] \\ \geq\ & (1-\delta)\left[\pi_{L}(q(p_{0},p_{2})) - \pi_{L}(q(p_{0},p_{1}))\right] \\ & +(1-\delta)\,\delta p_{T}\int_{q_{L}^{*}}^{q_{H}^{*}} \pi'_{H}(s)\,ds. \end{aligned} \tag{62}$$

Take $\varepsilon_{1} \in \left(0, \left(\frac{1-p_{T}^{*}}{2}\right)\right)$such that if $p_{T}^{*}+\varepsilon_{1} \geq p_{0} > p_{1} > p_{2} \geq p_{T}^{*}$ then:

$$\pi_{L}(q(p_{0},p_{2})) - \pi_{L}(q(p_{0},p_{1})) < \left(\frac{\bar{\delta}p_{T}^{*}\int_{q_{L}^{*}}^{q_{H}^{*}} \pi'_{H}(s)\,ds}{2}\right).$$

Hence, (62) is at least:

$$(1-\delta)\left(\frac{\bar{\delta}p_{T}^{*}\int_{q_{L}^{*}}^{q_{H}^{*}} \pi'_{H}(s)\,ds}{2}\right).$$

Step 1.2 *Second Inequality*

Next consider the term:

$$\begin{gathered} \pi_{H}(q_{H}^{*}) - (1-\delta)\,\Delta\theta q(p_{0},p_{2}) - \delta V_{H}(p_{2}) \\ -\left[\pi_{H}(q_{H}^{*}) - (1-\delta)\,\Delta\theta q(p_{0},p_{1}) - \delta(1-\delta)\,\Delta\theta q(p_{1},p_{2}) - \delta^{2}V_{H}(p_{2})\right] \\ = (1-\delta)\left[\Delta\theta q(p_{0},p_{1}) - \Delta\theta q(p_{0},p_{2})\right] \\ -\delta(1-\delta)\left[V_{H}(p_{2}) - \Delta\theta q(p_{1},p_{2})\right]. \end{gathered}$$

Notice that $V_{H}(p_{2}) \leq \Delta\theta q_{L}^{*}$ and $q(p_{0},p_{1}) \geq q(p_{0},p_{2})$, hence the term above is more than

$$-(1-\delta)\left[\Delta\theta q_{L}^{*} - \Delta\theta q(p_{1},p_{2})\right].$$

Next, notice that there exists K_{1} such that if $p_{T}^{*}+\varepsilon_{1} \geq \tilde{p}_{0} > \tilde{p}_{1} \geq p_{T}^{*}$ then $\left|\frac{\partial q(\tilde{p}_{0},\tilde{p}_{1})}{\partial \tilde{p}_{1}}\right| \leq K_{1}\left|\tilde{p}_{0}-\tilde{p}_{1}\right|$. Thus the expression above is more than $-(1-\delta)\,K_{1}\varepsilon_{1}$.

Step 1.3 *Third Inequality*

Finally, consider the term

$$\begin{aligned} & \pi_{H}(q_{H}^{*}) - (1-\delta)\,\Delta\theta q(p_{0},p_{2}) - \delta^{2}V_{H}(p_{2}) \\ & -\left[(1-\delta)\,\pi_{L}(q(p_{0},p_{1})) + \delta\left[\pi_{H}(q_{H}^{*}) - (1-\delta)\,\Delta\theta q(p_{1},p_{2}) - \delta V_{H}(p_{2})\right]\right]. \end{aligned}$$

Since $\pi_H(q_H^*) > \pi_L(q_L^*) \geq \pi_L(q(p_0,p_1))$, it is straightforward to show that there exists $\varepsilon_2 \in \left(0, \left(\frac{1-p_T^*}{2}\right)\right)$ such that if $p_T^* + \varepsilon_2 \geq p_0 > p_1 > p_2 \geq p_T$ then the above term is weakly positive.

Step 1.4 *The result*

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and assume that $p_T^* + \varepsilon \geq p_0 > p_1 > p_2 \geq p_T^*$ then the proposed deviation increases the profit by at least:

$$\mu(p_0,p_2)(1-\delta)\left(\frac{\bar{\delta}p_T^* \int_{q_L^*}^{q_H^*} \pi'_H(s)\,ds}{2}\right)$$
$$-(1-\mu(p_0,p_2))(1-\delta)K_1\varepsilon.$$

Next, notice that $\mu(p_0,p_2) \geq (1-p_{T^*})$ and $(1-\mu(p_0,p_2)) \leq \left(\frac{\varepsilon}{1-p_{T^*}-\varepsilon}\right)$. Thus, a lower bound on the advantage of the deviation is:

$$(1-\delta)\left[\begin{array}{c} (1-p_{T^*})\left(\frac{\bar{\delta}p_T^* \int_{q_L^*}^{q_H^*} \pi'_H(s)ds}{2}\right) \\ -\left(\frac{\varepsilon}{1-p_T^*-\varepsilon}\right)K_1\varepsilon \end{array}\right]. \tag{63}$$

Finally, take $\varepsilon^* < \min\{\varepsilon_1, \varepsilon_2\}$ such that the expression inside the brackets above is at least:

$$\left[(1-p_T^*)\left(\frac{\bar{\delta}p_T^* \int_{q_L^*}^{q_H^*} \pi'_H(s)\,ds}{4}\right)\right] > 0.$$

Thus if $p_T^* + \varepsilon^* \geq p_0 > p_1 > p_2 \geq p_T^*$ and $\delta \geq \bar{\delta}$ we have a contradiction, which proves this step.

Step 2 *Showing that* $\lim p_T = 1$.

For the second step, let p_T^* be the supremum of all beliefs such that the high type is screened in at most T periods (for all $\delta \geq \bar{\delta}$). Clearly $\{p_T^*\}$ is increasing and hence convergent. Assume that $\{p_T^*\} \to p^* < 1$. By the argument in Step 1 we can find $\varepsilon > 0$ such that if $p \in (p^* - \varepsilon, p^* + \varepsilon)$ then the belief decreases by at least 2ε, which contradicts the assumption that $p^* < 1$ and concludes the proof. ■

7.2.3 Lemma 13

Lemma 13 *For any $p \in (0,1)$, there exists a discount factor $\bar{\delta} \in (0,1)$ such that if $\delta \in (0, \bar{\delta})$ and the initial prior $p_0 \in (0,p)$ then the belief drops to 0 in at most 2 periods in any equilibrium.*

Proof. From Lemma 11, we know that once the belief drops to $(0, p_0^*)$ then the belief drops to 0 in the next period. Hence, it suffices to show that if the the monopolist offers a menu in which the pooling contract is associated with a belief $p_1 \geq p_0^*$ then the monopolist can profitably deviate by screening the consumer in one period.

A menu in which the pooling contract is associated with a belief p_1 yields to the monopolist:

$$\mu(p_0,p_1)\left[(1-\delta)\pi_L(q(p_0,p_1))+\delta V^M(p_1)\right]$$
$$+(1-\mu(p_0,p_1))\left[\pi_H(q_H^*)-(1-\delta)\Delta\theta q(p_0,p_1)-\delta V_H(p_1)\right],$$

which is no more than

$$(1-p_0)\left[(1-\delta)\pi_L(q(p_0,p_1))+\delta\pi_L(q_L^*)\right] \tag{64}$$
$$+(1-\mu(p_0,p_1))\left[\pi_H(q_H^*)-(1-\delta)\Delta\theta q(p_0,p_1)-\delta V_H(p_1)\right]$$
$$+(p_0-(1-\mu(p_0,p_1)))\left[(1-\delta)\pi_L(q_L^*)+\delta\pi_H(q_H^*)\right].$$

Next, notice that the profit from screening the high type in one period is at least

$$p_0\left[(1-\delta)\left[\pi_H(q_H^*)-\Delta\theta q(p_0,p_1)\right]+\delta\left[\pi_H(q_H^*)-\Delta\theta q_L^*\right]\right] \tag{65}$$
$$+(1-p_0)\left[(1-\delta)\pi_L(q(p_0,p_1))+\delta\pi_L(q_L^*)\right].$$

The difference between (65) and (64) is bounded below by:

$$(1-\delta)\ \left[(p_0-(1-\mu(p_0,p_1)))\left[\pi_H(q_H^*)-\Delta\theta q(p_0,p_1)-\pi_L(q_L^*)\right]\right] \tag{66}$$
$$-\delta\pi_H(q_H^*).$$

Next, notice that if $p_0\in(p_0^*,p)$ and $p_1\in(p_0^*,p_0)$ then $(p_0-(1-\mu(p_0,p_1)))$ is equal to $\left(\frac{p_1}{1-p_1}\right)(1-p_0)\geq\left(\frac{p_0^*}{1-p_0^*}\right)(1-p)$.

Thus, if $\delta=0$ then (66) is at least:

$$\left(\frac{p_0^*}{1-p_0^*}\right)(1-p)\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds.$$

Hence, by continuity, there exists $\bar{\delta}\in(0,1)$ such that if $\delta\in(0,\bar{\delta})$ and $p_0\in(0,p)$ then (66) is at least:

$$\left(\frac{1}{2}\right)\left(\frac{p_0^*}{1-p_0^*}\right)(1-p)\int_{q_L^*}^{q_H^*}\pi'_H(s)\,ds>0,$$

which completes the Lemma. ■

7.2.4 Proof of Proposition 3

Proposition 3 *For every $p\in(0,1)$ there exists $T(p)\in\mathbb{N}$ such that if the initial prior p_0 belongs to $(0,p)$ then the high type is screened with probability 1 in at most $T(p)$ periods in any equilibrium.*

Proof. Consider the prior $p\in(0,1)$. From Lemma 13, there exists $\bar{\delta}\in(0,1)$ such that if $\delta\in(0,\bar{\delta})$ and $p_0\in(0,p)$ then the belief drops do 0 in at most 2 periods. From Lemma 12 there exists $T'\in\mathbb{N}$ such that if $p_0\in(0,p)$ and $\delta\in\left(\frac{1}{2}\bar{\delta},1\right)$ then the belief drops down to 0 in at most T' periods. Taking $T=T'+1$ concludes the proof. ■

7.3 Proposition 6

Proposition 6 *For every* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that if* $p_0 \in [0, \eta] \cup [1 - \eta, 1]$ *then* $\left|\Pi^* (p_0) - V^M (p_0)\right| < \varepsilon$.

Proof. Step 0. *Lower Beliefs*

The claim that there exists $\eta^1 > 0$ such that if $p_0 \in \left[0, \eta^1\right]$ then $\left|\Pi^* (p_0) - V^M (p_0)\right| < \varepsilon$ follows directly from the fact that if $p \in (0, p_0^*)$ then the high type is screened in one period (see Lemma 11). Furthermore, it is straightforward to show that in the renegotiation solution[28] as well as in the commitment solution the low type average quality $\sum_{\tau \geq 0} (1 - \delta) \delta^\tau q_t$ converges to q_L^* as the initial belief converges to 0. This establishes this result.

Step 1 *Upper limit.*

For the second result, suppose that for every $T > 0$ there exists $p_T \in (0, 1)$ such that for $p \geq p_T$ we have $V_H (p, \{\mathbf{0}\}) \leq \delta^{T-1} \Delta\theta q_L^*$.

This is clear for $T = 1$.

We claim that there exists $p_{T+1} \in (0, 1)$ such that $p \geq p_{T+1}$ implies $V_H (p, \{\mathbf{0}\}) \leq \delta^T \Delta\theta q_L^*$.

Step 1.1 *There exists a belief* $\tilde{p} \in (p_T, 1)$ *such that* $V_H (\tilde{p}, \{\mathbf{0}\}) \leq \delta^T \Delta\theta q_L^* + \left(\frac{\delta^{T-1}\Delta\theta q_L^* - \delta^T \Delta\theta q_L^*}{2}\right)$ *in any equilibrium.*

Suppose not. We will show that for some $\tilde{p} \in (p_T, 1)$ the monopolist has a profitable deviation. If the rent of the high type is at most $\delta^T \Delta\theta q_L^* + \left(\frac{\delta^{T-1}\Delta\theta q_L^* - \delta^T \Delta\theta q_L^*}{2}\right)$ an upper bound on the monopolist's profit is:

$$\begin{array}{c} \tilde{p} \left[\pi_H (q_H^*) - \delta^T \Delta\theta q_L^* - \left(\frac{\delta^{T-1}\Delta\theta q_L^* - \delta^T \Delta\theta q_L^*}{2}\right)\right] \\ + (1 - \tilde{p}) \pi_L (q_L^*) \end{array} . \tag{67}$$

Next, consider a menu with a contract leading to a belief p_T and containing the current period quality 0. It yields at least:

$$\left(\frac{\tilde{p} - p_T}{1 - p_T}\right) \left[\pi_H (q_H^*) - \delta^T \Delta\theta q_L^*\right]. \tag{68}$$

Next, notice that if $\tilde{p} = 1$ then (68) is strictly greater than (67). The existence of $\tilde{p} \in (0, 1)$ follows from a continuity argument.

Step 1.2 *There exists a belief* $p_{T+1} \in (\tilde{p}, 1)$ *such that if* $p' \in (p_{T+1}, 1)$ *then* $V_H (\{\mathbf{0}\}, p') \leq \delta^T \Delta\theta q_L^*$ *in any equilibrium.*

Suppose not. Consider any belief $p \in (\tilde{p}, 1)$. If the rent of the high type is at most $\delta^T \Delta\theta q_L^*$, an upper bound on the monopolist's profit is:

$$\begin{array}{c} p \left[\pi_H (q_H^*) - \delta^T \Delta\theta q_L^*\right] \\ + (1 - p) \pi_L (q_L^*) \end{array} . \tag{69}$$

[28] See Lemma 4.

Next, consider a menu with a contract leading to a belief $\tilde{p}$, and containing current period quality 0. It yields at least:

$$\left(\frac{p-\tilde{p}}{1-p}\right)\left[\pi_H\left(q_H^*\right)-\delta\left[\delta^T\Delta\theta q_L^*+\left(\frac{\delta^{T-1}\Delta\theta q_L^*-\delta^T\Delta\theta q_L^*}{2}\right)\right]\right],$$

which is greater than $\pi_H\left(q_H^*\right)-\delta^T\Delta\theta q_L^*$ at $p=1$ since $\delta\left[\delta^T\Delta\theta q_L^*+\left(\frac{\delta^{T-1}\Delta\theta q_L^*-\delta^T\Delta\theta q_L^*}{2}\right)\right]<\delta^T\Delta\theta q_L^*$. The argument follows by continuity.

Step 1.3 *Final Step.*

For the final step, notice that there exists $\eta^2\in(0,1)$ such that if $p_0\in\left[1-\eta^2,1\right]$ then $\Pi^*\left(p_0\right)=p_0\pi_H\left(q_H^*\right)$. Take T' such that $\delta^{T'-1}\Delta\theta q_L^*<\left(\frac{\varepsilon}{2}\right)$. If $p>p_{T'}$ then a lower bound to the monopolist's profit is:

$$\left(\frac{p-p_T}{1-p_T}\right)\left[\pi_H\left(q_H^*\right)-\left(\frac{\varepsilon}{2}\right)\right].$$

Hence, by continuity, we can find $\eta^3\in\left(0,\eta^2\right)$ such that if $p_0\in\left[1-\eta^3,1\right]$ then $\Pi^*\left(p_0\right)-V^M\left(p_0\right)\leq\varepsilon$. Taking $\eta=\min\left\{\eta^1,\eta^3\right\}$ completes the proof. ■

7.4 Proposition 7

Proposition 7 *There exists an equilibrium.*

Proof. In this proof, we must show the existence of $\left(V^M,\Phi\right)$ and of a Bayesian Nash equilibrium consistent with $\left(V^M,\Phi\right)$. The second task is straightforward, so we focus on the first here.

In Proposition 1, it was established that if there are profit-maximizing values $\left(V^M,\Phi\right)$ consistent with an equilibrium then there exists an outcome equivalent play presenting sequential separating dynamics in which the monopolist does not make promises for future periods. Remember that Proposition 1 is a statement about equilibrium, while we are looking for a characterization of values $\left(V^M,\Phi\right)$ here. Nonetheless, slightly modifying arguments leading to the Proposition 1's findings one can prove the following facts:

Fact 4:

Construct a fixed-point $\left(V^M,\Phi\right)$ for states $(p,\{q_\tau\})\in\mathcal{C}^1$, with

$$\mathcal{C}^1\triangleq\left\{(p,\{q_\tau\})\in[0,1]\times[0,1]^\infty:p\in(0,1)\Rightarrow\{q_\tau\}=\{\mathbf{0}\}\right\}.$$

That is, restrict the promises to $\{q_\tau\}=\{\mathbf{0}\}$ in all pooling contracts and impose that the monopolist cannot make promises for future periods in pooling contracts and that every menu presents sequential separating dynamics. Next, consider a state $(p,\{\mathbf{0}\})$ with $p\in(0,1)$ and any solution φ of the monopolist's problem at $(p,\{\mathbf{0}\})$ in which she is not restricted to offering menus with sequential separation dynamics, but she is still restricted to making no promises in pooling contracts. Then φ presents sequential separating dynamics.[29]

Fact 5:

Construct values $\left(V^M, \Phi\right)$ for the case in which $\{q_\tau\}$ are such that $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [q_L^*, q_H^*]$ by imposing that the monopolist is restricted to offering efficient menus. Next, construct a fixed point $\left(V^M, \Phi\right)$ for states $(p, \{q_\tau\}) \in \mathcal{C}^2$, with

$$\mathcal{C}^2 \triangleq \left\{ (p, \{q_\tau\}) \in [0,1] \times [0,1]^\infty : p \in (0,1) \Rightarrow \sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [0, q_H^*] \right\}$$

by imposing that in pooling contracts the promises can be set either to $\{\mathbf{0}\}$ or to $\{q_\tau\}$ such that $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [q_L^*, q_H^*]$. There always exists a solution of the unrestricted problem in which the monopolist always makes no promises in pooling contracts[30] when the state is $(p, \{q_\tau\})$ such that $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [0, q_L^*)$ and the monopolist offers efficient menus when the state is $(p, \{q_\tau\})$ and $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [q_L^*, q_H^*]$.

Using Facts 4 and 5, one can construct a fixed-point for the case in which promises $\{q_\tau\}$ are restricted to satisfy $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in [0, q_H^*]$.

Next, consider the case in which[31] $\sum_\tau (1-\delta)\,\delta^\tau q_\tau \in (q_H^*, 1]$. Notice that if $(1-\delta)\sum \delta^\tau q_\tau \in (q_H^*, 1]$ efficiency implies that the monopolist would like to "buy back" a different quality from each consumer. The monopolist would like to buy back $(1-\delta)\sum \delta^\tau (q_\tau - q_H^*)$ from the high type and $(1-\delta)\sum \delta^\tau (q_\tau - q_L^*)$ from he low type. Consider a menu implementing this allocation in which the rationality constraint of each type of consumer binds with equality: i) high type's contract $\left(-\theta_H \sum \delta^\tau (q_\tau - q_H^*), \{\mathbf{q}_H^*\}\right)$; ii) low type's contract $\left(-\theta_L \sum \delta^\tau (q_\tau - q_L^*), \{\mathbf{q}_L^*\}\right)$. If the high type chooses the contract designed for him, he obtains a rent of $(1-\delta)\,\theta_H \sum \delta^\tau q_\tau$, hence his rationality constraint binds with equality. It is also straightforward to see that choosing the contract designed to the low type would make him strictly worse off. If the low type chooses the contract designed for him, he obtains a rent of $(1-\delta)\,\theta_L \sum \delta^\tau q_\tau$. Hence, his rationality constraint binds with equality. On the other hand, from choosing the contract designed to the high type he obtains a rent of

$$\begin{aligned} & (1-\delta)\,\theta_L \sum \delta^\tau q_\tau + (1-\delta)\,\Delta\theta \sum \delta^\tau (q_\tau - q_H^*) \\ > \; & (1-\delta)\,\theta_L \sum \delta^\tau q_\tau . \end{aligned}$$

Thus, the low type could profitably deviate by imitating the high type. In order to make this contract incentive-compatible, the monopolist would have to compensate the low type by

[29] The steps in Lemma 8 can be easily adapted to prove this Fact.

[30] The argument mimicks the one in Lemma 9.

[31] If $q_H^* = 1$ the proof will be done at this step.

paying him an information rent of $\Delta\theta \sum \delta^{\tau} \left(q_{\tau} - q_H^*\right)$, which is decreasing in the quality of the high type. Hence, if the monopolist could commit to a quality for each future period she would set a quality $q_H^c > q_H^*$ to the high type in order to extract a higher rent from the low type. Using this observation, it is easy to use analogues to Facts 4 and 5 to construct a fixed-point for the case in which $(1-\delta) \sum \delta^{\tau} q_{\tau} \in [q_L^*, 1]$. For that, one can show that when the state is $(p, \{q_{\tau}\})$ such that $(1-\delta) \sum \delta^{\tau} q_{\tau} \in (q_H^*, 1]$ in each period, the monopolist offers (w.l.o.g.) a menu with a pooling contract containing a current period quality in $[q_H^*, 0]$ and a promise $\{\mathbf{1}\}$. In the first period in which the low type is totally screened, the monopolist offers to renegotiate to $\{\mathbf{q}_H^*\}$.

Therefore, the argument above (details below) show how to construct a fixed-point for the case in which:

i) If $(p, \{q_{\tau}\})$ is such that $p \in (0,1)$ and $\{q_{\tau}\}$ satisfies $(1-\delta) \sum \delta^{\tau} q_{\tau} \in [0, q_L^*]$ then the monopolist never makes a promise $\{q_{\tau}\}$ such that $(1-\delta) \sum \delta^{\tau} q_{\tau} \in (q_H^*, 1]$ in pooling contracts.

ii) If $(p, \{q_{\tau}\})$ is such that $p \in (0,1)$ and $\{q_{\tau}\}$ satisfies $(1-\delta) \sum \delta^{\tau} q_{\tau} \in (q_L^*, 1]$ then the monopolist never makes a promise $\{q_{\tau}\}$ such that $(1-\delta) \sum \delta^{\tau} q_{\tau} \in [0, q_L^*)$ in pooling contracts.

Next, applying an argument analogous to the one developed in Lemma 5 and in Lemma 10 one can show that this is a fixed-point when the program is not restricted in any state.

Since the cases in which $(1-\delta) \sum \delta^{\tau} q_{\tau} \in [0, q_L^*)$ and $(1-\delta) \sum \delta^{\tau} q_{\tau} \in (q_H^*, 1]$ are analogous[32], we present the first case here.

Step 1 *Construction of* $\left(V^M, \Phi\right)$ *for* $\{q_{\tau}\} = \{\mathbf{0}\}$.

In Step 1.1 We construct values $\left(V_1^M, \Phi_1\right)$ imposing separation in one period.

Step 1.1 *Construction of* $\left(V_1^M, \Phi_1\right)$*: separation in one period.*

In this Step we assume that the monopolist screens the high type in one period. Hence, her problem is:

$$\begin{aligned} V_1^M(p) &\triangleq \max_{\tilde{q} \in [0,1]} \Gamma_1(p, 0, \tilde{q}) \qquad (70) \\ &= \max_{\tilde{q} \in [0,1]} \left\{ \begin{array}{c} (1-p)\left[(1-\delta)\pi_L(\tilde{q}) + \delta \pi_L(q_L^*)\right] \\ p\left[\pi_H(q_H^*) - (1-\delta)\Delta\theta\tilde{q} - \delta\Delta\theta q_L^*\right] \end{array} \right\}. \end{aligned}$$

Notice that (70) defines a function $q_1(p) = \arg\max_q \Gamma_1(p, 0, \tilde{q})$. Hence, we write $\Phi_{H,1}(p)$ for the information rent of the high type if separation occurs in one period:

$$\Phi_{H,1}(p) \triangleq \left\{(1-\delta)\Delta\theta q_1(p) + \delta\Delta\theta q_L^*\right\}.$$

In **Step 1.2** We show how to use $\left(V_n^M, \Phi_n\right)$ to construct $\left(V_{n+1}^M, \Phi_{n+1}\right)$.

First, for all n we set $\Phi_{H,n}(0) = [\Delta\theta q_L^*, \Delta\theta]$ and $\Phi_{H,n}(1) = 0$.

[32] The difference being in the promises the monopolist makes in pooling contracts (minimal in the first and maximal in the second) and whom the monopolist screens: the low type in the first and the high type in the second.

The function $V_{n+1}^{M}(p)$ is defined by:

$$\begin{aligned} V_{n+1}^{M}(p) &\triangleq \max_{\tilde{p}\in[0,p],\tilde{q}\in[0,1]} \Gamma_{n+1}(p,\tilde{p},\tilde{q}) \\ &= \max_{\tilde{p}\in[0,p],\tilde{q}\in[0,1]} \left\{ \begin{array}{c} \mu(p,\tilde{p})\left[(1-\delta)\pi_L(\tilde{q})+\delta V_n^M(\tilde{p})\right] \\ (1-\mu(p,\tilde{p}))\left[\pi_H(q_H^*)-(1-\delta)\Delta\theta\tilde{q}-\delta v_{H,n}(\tilde{p})\right] \end{array} \right\}, \end{aligned}$$

where we set $v_{H,n}(\tilde{p}) = \min \Phi_{H,n}(p)$ [33] because unnecessary rents to the high type do not solve the monopolist's problem.

Next we define $\Phi_{H,n}(p)$ by:

$$\Phi_{H,n}(p) = co \left\{ \begin{array}{c} v_{H,n+1}(p) : v_{H,n+1}(p) = (1-\delta)\Delta\theta\tilde{q} + v_{H,n}(\tilde{p}), \\ (\tilde{p},\tilde{q}) \text{ solve } V_{n+1}^{M}(p) \end{array} \right\}.$$

Notice that using an argument similar to the one in Lemma 11, there exists $p_1^* \in (0,1)$ such that for all $p \in [0,p_1^*]$ the high type is screened in one period. Hence, we have $\left(V_n^M(p), \Phi_{H,n}(p)\right) = \left(V_1^M(p), \Phi_{H,1}(p)\right)$ for all $n \in \mathbb{N}$ and all $p \in [0,p_1^*]$. In general, we can find (see Lemma 12) a sequence of beliefs $\{p_T^*\}_{T=1}^{\infty}$such that $\left(V_n^M(p), \Phi_{H,n}(p)\right) = \left(V_T^M(p), \Phi_{H,T}(p)\right)$ for all $n \geq T$ and $p \in [0,p_T^*]$. Thus, we define for all $p \in [0,1]$

$$\left(V^M(p), \Phi_H(p)\right) = \lim_n \left(V_n^M(p), \Phi_{H,n}(p)\right). \tag{71}$$

Finally, we define for all $p \in [0,1]$, $V^M(p,\{\mathbf{0}\}) \triangleq V^M(p,\{\mathbf{0}\})$ and $\Phi(p,\{\mathbf{0}\}) \triangleq \{(0,v_H) : v_H \in \Phi_H(p)\}$.

Step.2 The *Construction of* $\left(V^M,\Phi\right)$ *for* $\{q_\tau\} \neq \{\mathbf{0}\}$.

We separate the state space $(p,\{q_\tau\})$ into 3 subspaces. In **Step 2.1**.we consider the case in which $(1-\delta)\sum\delta^\tau q_\tau \in [0,q_L^*)$. In **Step 2.2** the case in which $(1-\delta)\sum\delta^\tau q_\tau \in [q_L^*,q_H^*]$ and in **Step 2.3** we analyze the case in which $(1-\delta)\sum\delta^\tau q_\tau \in (q_H^*,1]$.

Step 2.1 $(1-\delta)\sum\delta^\tau q_\tau \in [0,q_L^*)$.

In Step 1 we analyzed de case in which $\{q_\tau\} = \{\mathbf{0}\}$. Consider $\{q_\tau\}$ such that $\{q_\tau\} \neq \{\mathbf{0}\}$. One can use an argument similar to the one in Lemma 9 to show that it is without loss to assume that the monopolist offers a menu in which there is sequential separation dynamics and the monopolist never makes promises for future periods (setting $\{q_\tau\} = \{\mathbf{0}\}$). Hence, we define for all $p \in (0,1)$

$$\begin{aligned} & V^M(p,\{q_\tau\}) \\ \triangleq\ & \max_{\tilde{p}\in[0,p],\tilde{q}\in[0,1]} \left\{ \begin{array}{c} \mu(p,\tilde{p})\left[(1-\delta)\pi_L(\tilde{q})+\delta V^M(\tilde{p})-(1-\delta)\sum\delta^\tau\theta_L q_\tau\right] \\ (1-\mu(p,\tilde{p}))\left[\pi_H(q_H^*)-(1-\delta)\Delta\theta\tilde{q}-\delta v_H(\tilde{p})-(1-\delta)\sum\delta^\tau\theta_L q_\tau\right] \end{array} \right\} \\ s.t.\ :\ & (1-\delta)\Delta\theta\tilde{q}+\delta v_H(\tilde{p}) \geq (1-\delta)\sum\delta^\tau\Delta\theta q_\tau, \end{aligned}$$

[33] It is straightforward to verify that $\Phi_{H,n}(p)$ is compact for every n that and for every p. Also, it is straightforward to verify that $v_{H,n}(\tilde{p})$ is lower-semicontinuous so that the monopolist's problem is upper-semicontinuous.

where $v_H(\tilde{p}) = \min \Phi_H(p)$.

Furthermore, we define:

$$\Phi_H(p, \{q_\tau\}) \triangleq co \left\{ \begin{array}{c} v_H : v_H = (1-\delta)\Delta\theta\tilde{q} + \delta v_H(\tilde{p}) + (1-\delta)\sum \delta^\tau \theta_L q_\tau \\ (\tilde{p}, \tilde{q}) \text{ solve } V^M(p, \{q_\tau\}) \end{array} \right\},$$

and $\Phi(p, \{q_\tau\}) \triangleq \{((1-\delta)\sum \delta^\tau q_\tau, v_H) : v_H \in \Phi_H(p, \{q_\tau\})\}$.

We set[34]

$$\begin{array}{rcl} \Phi(0, \{q_\tau\}) & = & \left\{ \begin{array}{c} ((1-\delta)\sum \delta^\tau \theta_L q_\tau, v_H) : \\ v_H \in [\Delta\theta q_L^* + (1-\delta)\sum \delta^\tau \theta_L q_\tau, \Delta\theta + (1-\delta)\sum \delta^\tau \theta_L q_\tau] \end{array} \right\} \\ \Phi(1, \{q_\tau\}) & = & \left\{ \begin{array}{c} (v_L, (1-\delta)\sum \delta^\tau \theta_H q_\tau) : \\ v_L \in [(1-\delta)\sum \delta^\tau \theta_L q_\tau, (1-\delta)\sum \delta^\tau \theta_H q_\tau] \end{array} \right\}. \end{array}$$

Step 2.2 $(1-\delta)\sum \delta^\tau q_\tau \in [q_L^*, q_H^*]$.

In this case, consider the separating menu containing the contract $(\{q_L^*\}, \sum \delta^\tau \theta_L (q_L^* - q_\tau))$ designed exclusively to the low type and the contract $(\{q_H^*\}, \sum \delta^\tau \theta_H (q_H^* - q_\tau))$ designed exclusively to the high type It is straightforward to show that: i) the rationality of each type of consumer binds with equality; ii) each type of consumer weakly prefers his contract. Hence, this menu is optimal. Furthermore, in any optimal menu there should be separation in one period. Hence, for every $p \in (0,1)$ we set:

$$\begin{array}{rl} & V^M(p, \{q_\tau\}) \\ = & \left\{ \begin{array}{c} p[\pi_H(q_H^*) - (1-\delta)\sum \delta^\tau \theta_H q_\tau] \\ +(1-p)[\pi_L(q_L^*) - (1-\delta)\sum \delta^\tau \theta_L q_\tau] \end{array} \right\} \end{array}$$

and

$$\begin{array}{rl} & \Phi(p, \{q_\tau\}) \\ = & \left((1-\delta)\sum \delta^\tau \theta_L q_\tau, (1-\delta)\sum \delta^\tau \theta_H q_\tau\right). \end{array}$$

For $p \in \{0,1\}$ we set $\Phi(p, \{q_\tau\})$ as in **Step 2.2.**

Step 2.3 $(1-\delta)\sum \delta^\tau q_\tau \in (q_L^*, q_H^*]$. The profit values for this case can be generated using an algorithm analogous to the one used for Steps 1.1-2.1, with the difference that the monopolist makes maximal promises ($\{q_\tau\} = \mathbf{1}$) in pooling contracts and she screens the low type. We omit it for brevity. ■

[34] In order to generate the values $((1-\delta)\sum \delta^\tau \theta_L q_\tau, (1-\delta)\sum \delta^\tau \theta_L q_\tau + \Delta\theta)$ when the state is $(0, \{q_\tau\})$ the monopolist offers a contract associated to the current period's consumption 1, to future period's promises $\{\mathbf{1}\}$, the belief is updated to 0 and the monopolist extracts all the continuation rent from each type in each future period.

In order to generate the values $((1-\delta)\sum \delta^\tau \theta_H q_\tau, (1-\delta)\sum \delta^\tau \theta_H q_\tau)$ when the state is $(1, \{q_\tau\})$ the monopolist offers a contract associated to the current period's consumption 0, to future period's promises $\{\mathbf{0}\}$, the belief is updated to 1 and the monopolist extracts all the continuation rent from each type in each future period.

8 Appendix B

Here we prove for the case in which the graph $(p, \Phi(p, \{q_\tau\}))$ is not closed. This requirement was advantageous because it established that when the graph $(p, \Phi(p, \{q_\tau\}))$ is closed the problem is well behaved and we used those properties for our existence proof.[35]

When the graph $(p, \Phi(p, \{q_\tau\}))$ is not closed iii) in Lemma 4 does not hold. Hence Lemma 8 (and the argument in Lemma 6 indirectly) cannot be used because the deviation in these Lemmas required that we find (for certain high-type rent v_H) a belief p' in which $v_H \in \Phi(p', \{\mathbf{0}\})$. This is not possible if the graph $(p, \Phi(p, \{\mathbf{0}\}))$ is not closed. However, that constitutes a difficulty only if the current period's promises $\{q_\tau\}$ are large, in the sense that large promises imply a large value for $\sum (1-\delta)\delta^\tau \Delta\theta q_\tau$. That is, the argument in Lemma 8 could be used to obtain a profitable deviation when the promises are $\{\mathbf{0}\}$. Hence, consider the first period t in which we do not have sequential separating dynamics in a putative equilibrium. Notice that since $\{q_\tau\} \neq 0$ we have $t > 0$. Hence at $t-1$ there is sequential separating dynamics. Hence at $t-1$ the monopolist foresees this difficulty at t, which incentivizes her to offer promises $\{\mathbf{0}\}$ to period t. Also since from Lemma 2 larger promises $\{q_\tau\}$ lead to (weakly) larger information rents, the monopolist is also motivated to deviate by promising $\{\mathbf{0}\}$ at t. Therefore we conclude that period t is not reached. This argument is formalized in Lemma 14 below, where we prove that the claim 1.1 of Proposition 1 holds (where we use indirectly Lemma 8). The claim 1.2 of Proposition 1 will follow from Lemma 9 and the claim 1.3 of Proposition 1 will follow from the proof of Proposition 1.1 which is given in Appendix A.

Lemma 14 *In every history under the equilibrium path the monopolist offers a menu m presenting sequential separating dynamics.*

Proof. Suppose towards a contradiction that the claim is not true. Then there exists an equilibrium in which the monopolist does not offer a menu involving sequential separating dynamics. Let $t \in \mathbb{N}$ indicate the shortest period in which this happens and consider a history h^t in which the monopolist offers a menu not presenting sequential separating dynamics. Notice that by assumption the monopolist is offering menus with sequential separating dynamics in all periods $s \in \{0, ..., t-1\}$. Let $\mathcal{H}^t = \{h^s, h^t \succ h^s, s > 0\}$ be the set of predecessor of h^t (disregarding h^0) and notice that in each $h^s \in \mathcal{H}^t$ all menus should present sequential separating dynamics, otherwise the minimality of the period t would be contradicted.

For each $h^s \in \mathcal{H}^t$ let $(p(h^s), \{q_\tau(h^s)\})$ be the state at h^s and let $h^r \in \mathcal{H}^t$ be such that $\{q_\tau(h^z)\} \neq \mathbf{0}$ and if $h^z \prec h^r$ then $\{q_\tau(h^z)\} = \mathbf{0}$. Writing h^{r-1} for the predecessor of h^r we let $m(h^{r-1})$ be the menu offered at h^{r-1} such that if the pooling contract is accepted the play reaches history h^r. Let $(x^P_{r-1}, \{q_{r-1}, q_r, ...\})$ be the pooling contract in $m(h^{r-1})$.

[35] In fact, the only requirement used to establish Lemma 6-10 was that the graph of $\Phi(p, \{\mathbf{0}\})$ is closed.

Consider the replacement of the menu $m\left(h^{r-1}\right)$ with a menu containing a revealing contract and a pooling contract $(\theta_L q_{r-1}, \{q_{r-1}, \mathbf{0}\})$, which induces the belief $p(h^r)$. From Lemma 2 $\hat{\Phi}(p(h^r), \{q_\tau(h^r)\}) \geq \hat{\Phi}(p(h^s), \{\mathbf{0}\})$, then without loss assume that the monopolist offers the same revealing contract. From Lemma 5 the deviating menu leads to a weakly higher profit. Hence we work towards showing that the monopolist has a deviation at h^{r-1}. Since for all h^n such that $h^t \succ h^n \succ h^r$ the monopolist offers a menu with sequential separating dynamics we can proceed analogously and construct an alternative play in which the monopolist never makes promises to future periods in pooling contracts in all periods[36] $s < t$. Hence assume without loss that there exists an equilibrium in which the monopolist offers a menu in which there is no sequential separating dynamics at h^0. One can then follow the same steps as in Lemma 8 and obtain a contradiction. ■

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[36] Consider a period t starting with promises $\{\mathbf{0}\}$ and assume the monopolist offers a menu containing sequential separating dynamics. Lemmas 2 and 5 imply for every continuation outcome generated by promises $\{q_\tau\}$ in a pooling contract at t, there exists an equivalent continuation outcome in which the monopolist offers $\{\mathbf{0}\}$ in the pooling contract in period t.

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